Brownian motion and fractional Brownian motion

QUANTITATIVE FINANCIAL MODELING

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1 Introduction

Stochastic processes are essential in finance because they provide ways to describe how random variables change over time. These processes are used to model important financial factors like prices, interest rates, and exchange rates, which are affected by various unpredictable events such as market demand, political happenings, and natural disasters. Using accurate stochastic models helps us understand how financial markets work, predict future trends, and manage risks effectively.

One of the most basic continuous-time stochastic processes in finance is Brownian motion, also called the Wiener process. This was discovered by British botanist Robert Brown in 1827 and it is known for its randomness and continuous nature.

Geometric Brownian motion (GBM) is a variation of Brownian motion designed for financial modeling. It ensures prices stay positive and can represent the exponential growth of asset prices, which fits real financial data better.

This paper looks at the definition, properties, and financial importance of GBM, and then discusses the generalization of fractional Brownian motion (fBM). We will see how fBM addresses some of the limitations of GBM and Brownian motion, its unique features, and how it is used in financial modeling. Additionally, the paper will cover the challenges of using these models with real financial data.

2 Brownian motion

2.1 Standard Brownian motion

Standard Brownian motion, also known as the Wiener process, is a continuous-time stochastic process characterized by several key properties:

1. Initial Condition: The process starts at zero, almost surely, denoted as W(0)=0. This initial condition is derived from the limiting behavior of the process.

2. With probability 1, the function W(t) is continuous at every t. Despite this continuity, the function is almost nowhere differentiable. This implies that the path of Brownian motion is highly irregular and rough.

3. Stationary and Independent Increments: The increments of Brownian motion are both stationary and independent. This means that the statistical properties of the increments are consistent over time and that the increments over non overlapping intervals are independent of each other.

4. The difference $W(t_1) - W(t_2)$ between two points in time is normally distributed with a mean of 0 and a variance equal to the time difference, $t_2 - t_1$. Mathematically, $W(t_2)–W(t_1) \sim \mathcal{N}(0, t_2–t_1)$.

5. Over an interval of finite length, the quadratic variation of Brownian motion is exactly the length of the interval. Specifically, if we consider the interval [0,t] the quadratic variation is t. This property highlights that the total variation of Brownian motion over any finite interval is infinite, emphasizing its highly unpredictable nature.

2.2 Geometric Brownian motion (GBM)

Definition Geometric Brownian Motion is a non-negative variation of Brownian motion and can be defined as a stochastic-process in continuous time where the logarithm of a randomly changing amount of interest follows a Brownian motion with drift. Geometric Brownian motion was first observed by the British botanist Robert Brown in 1827 to describe the trajectory of particles in water. Its path shows the characteristics of a random walk, that is the position of the particles in space changes with time, but the displacement at any point in time obeys a normal distribution.

A r.v. S(t) is said to follow a Geometric Brownian Motion if $S(t) = S(0)e^{X(t)}$ where $S(0) > 0$ is the initial value and $X(t) = \mu t + \sigma W(t)$ is a Brownian motion with drift μ and variance $\sigma^2 t$.

Moreover, we can define the Geometric Brownian motion as the solution of the stochastic differential equation (SDE): $dS_t = \mu S_t dt + \sigma S_t dW_t, s.t. S_0 > 0$ where the solution represented by the GBM is $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$

Properties As every model, also Geometric Brownian Motion has pros and cons. For pros we can indicate some properties of it, that are what makes this model a valuable tool in financial modeling. First of all, we know that GBM produces prices that have a log-normal distribution. Because of the positive skewness of it, prices have limits from below by zero, but can rise significantly. The prices' logarithm, which is helpful for modeling asset prices, has a normal distribution. As already stated, we have acknowledged that GBM is a continuous-time process, that signifies that the price of the asset moves gradually over time, with no intervals. About the increments, we know that they are stationary over non-overlapping intervals, also they are independent of each other. GBM follows the Markov property, which makes GBM a memoryless process, allowing the future to be independent of the past, given the present. For cons we can state some limitations of the model that are to be considered in the application in the real world. First of all, the two constants drift and volatility are actually not constant in reality, this of course causes some trouble and inaccuracy during predictions. As we said before, we know that a feature of GBM is not having to deal with sudden jumps, but these actually take place in the market, caused either by announcements or political events all over the world. This discrepancy between reality and GBM applies also when talking about log-normally distributed prices, which are not part of reality in extreme market conditions, which leads us also to say that the market is not always as efficient as assumed by GBM. Ultimately, we can say that the assumption of continuous-time modeling doesn't always reflect reality, since financial data are frequently taken discontinuously.

Financial meaning In the financial field, Brownian motion and its derivatives are of great significance in asset pricing, risk management, and derivatives pricing. Brownian motion is widely used to model stochastic fluctuations in asset prices and provides the basis for the pricing of many financial instruments, such as options, futures, and other derivatives. Specifically, we know that Geometric Brownian motion other than eliminating the Brownian Motion's negativity issue can be supported from fundamental economic theories as a plausible model for stock prices in an "ideal" non-arbitrage context. This process is often used to model certain phenomena in financial markets; since it is a positive process, it is used to represent the positive nature of prices. GBM can capture the randomness and the volatility of stock prices, which is a key feature of financial markets. GBM can also be used to estimate the expected return and volatility of a stock, which are important inputs for financial decision making. GBM can also be used to simulate future stock prices. Simulations can be used to estimate the probability of different outcomes, such as the probability of a stock price reaching a certain level. The application of GBM in financial markets can be traced back to the proposal of the Black-Scholes option pricing model, which uses Brownian motion as the basic assumption of asset prices, such as these following examples:

1. Asset Price Simulation Price Process: GBM models the evolution of asset prices assuming the logarithmic returns are independently and normally distributed. This means the logarithmic returns of asset prices are random, with known mean and variance. This makes GBM a common model for simulating the price movements of financial assets like stocks, currencies, and more. In the GBM model, the asset price St is simulated using the formula

   $$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

   where S_0 is the initial price, μ is the expected return of the asset, σ is the volatility of the asset, and W_t is the standard Brownian motion.

2. Option Pricing: Black-Scholes Model: The Black-Scholes option pricing model is based on GBM and is a classic model used for pricing European options. The model derives an option pricing formula by assuming that asset prices follow a geometric Brownian motion.

3. Other Pricing Models: In addition to the Black-Scholes model, many other option pricing models (such as binomial tree models and Monte Carlo simulations) have been developed based on the theory of GBM. They are widely used in financial markets for pricing options, futures, and other derivatives.

Construction We construct the Geometric Brownian motion starting from the Stochastic differential equation. In the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t, s.t. S_0 > 0$$

we identify dS_t as the change in the asset price, S_t is the value of the asset price at time t, W_t represents the process of Brownian motion, μ is the percentage drift and it represents the expected return of the stock per unit of time, σ is the percentage volatility, that measures the standard deviation of the assets' returns, both these last two elements are constants. Higher volatility makes options more valuable because there is more chance that the stock price will change substantially by the expiration date. The compounded interest rate is higher than the fixed- income interest rate, so it's more convenient to invest in stocks. This model is used to price options. The SDE describes the random changes in asset prices over continuous time, where the first term $(\mu S_t dt)$ represents the trend part of the asset price, and the second term $(\sigma S_t dW_t)$ represents the random fluctuation part of the asset price. Stochastic differential equations of geometric Brownian motion can be solved by analytical methods or numerical methods. The analytical method uses the techniques of stochastic integration and probability theory to obtain the analytical expression of asset prices; while the numerical method generates simulated asset price paths by simulating a large number of random paths and based on the characteristics and initial conditions of stochastic differential equations. Commonly used numerical values Methods include Monte Carlo simulations. In practical applications, the parameters in the model need to be estimated based on historical data or market conditions. Common methods include maximum likelihood estimation, regression analysis, and historical data fitting. Through parameter estimation, parameters such as return rate and volatility in the model can be determined, allowing for model application and analysis. These steps constitute the basic construction and application process of geometric Brownian motion.

2.3 Fractional Brownian motion (fBM)

Fractional Brownian motion, introduced in 1968 by Mandelbrot and Van Ness, is an extension of Brownian motion obtained by adding one parameter, called Hurst parameter, which can take on a value between 0 and 1. The transition from Brownian Motion (BM) to Fractional Brownian Motion (fBM) occurs to address certain limitations of BM in modeling real-world phenomenons. Brownian motion is defined by the independence of its increments, implying that the future progression of the process does not depend on its historical values. Additionally, the variance of increments in BM is proportional to time. This linear variance is not suitable for modeling situations where the variance changes in a more complex manner. These characteristics fail to capture many real-world phenomena that exhibit temporal dependencies or long-term memory.

From BM to fBM In this paper, we will specifically address these inconsistencies, starting by relaxing the assumption of independence of increments. First, we must consider the well-known covariance function for Brownian motion, which is as follows:

$$\mathbb{E}(B_tB_s) = Cov(B_t, B_s) = min(t,s) = \frac{1}{2}(|t| + |s| - |t-s|) = \frac{1}{2}(|t|^1 + |s|^1 - |t-s|^1)$$

The generalization of Brownian motion (fBM) can be achieved by relaxing the assumption of independence. This is done by introducing the parameter H and defining the process to have a specific covariance function:

$$\mathbb{E}(B_t^H B_s^H) = Cov(B_t^H, B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

where 0¡H¡1 ensures that the process preserves Gaussianity. The idea behind generalizing BM into fractional Brownian motion (fBM) is to start with BM and weight the increments of BM by a function dependent on H.

Hurst exponent The parameter H plays a crucial role in determining the properties of the process: For H>0.5, the trajectory is smoother, and observations exhibit strong dependency, indicative of long-term memory. This occurs because H influences the covariance, leading to greater weight being given to past realizations of the process. Consequently, the autocovariance function decreases more slowly, meaning that the process retains memory of events even over long periods. For H<0.5, the trajectory becomes rougher. Here, the process displays ant persistent behavior, where increases are likely to be followed by decreases, and vice versa. For H=0.5, the process reduces to standard Brownian motion, which lacks memory, meaning that past values have no influence on future values. Therefore, Memory for Fractional Brownian Motion can be categorized based on the value of H:

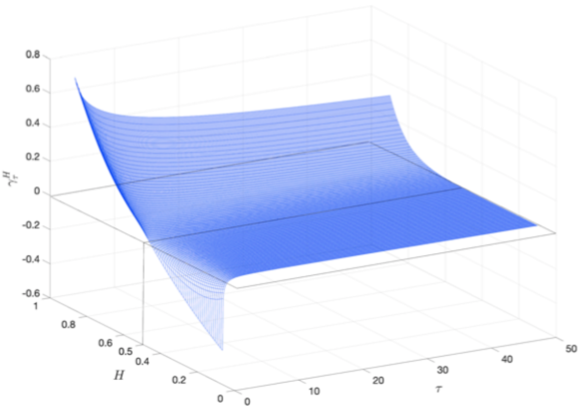
- *Long-term Memory (H>0.5)*: the sum of the autocovariances does not converge, the process exhibits long-term memory. This means that it retains a significant correlation between distant observations over time.

- *Short-term Memory (H<0.5)*: the sum of the autocovariances converges, the process has short-term memory. In this case, the memory is anti-persistent, meaning that the process tends to revert to the mean. When a process is mean-reverting, values tend to return to the average value over time.

Autocovariance function of the increments After discussing the concepts of long-term memory and short-term memory, it is important to delve into the autocovariance function of the increments in the fractional Brownian motion (fBM). This function provides deeper insights into the dependencies present in the process. The autocovariance function of the τ-lagged stationary discrete sequence of increments $\Delta B^H_{t,t-1}$ is given by $\gamma^H_\tau := \mathbb{E}(\Delta B^H_{t+\tau,t-1+\tau} \Delta B^H_{t,t-1})$.

This can be further expanded as: $\gamma^H_\tau = \frac{1}{2}[(\tau+1)^{2H} - 2\tau^{2H} + (\tau-1)^{2H}]$.

This function is crucial for understanding how the increments of the process relate to each other over different time lags and how this relationship changes with different values of H. Presented below is a three-dimensional plot of the autocovariance function of the increments as a function of the lag τ and the Hurst parameter H. This visualization illustrates the complex interplay between these variables.



Covariance function γ_τ^H

The three-dimensional plot provides several key insights into the behavior of the autocovariance function with respect to different values of the Hurst parameter H and the lag τ:

- For H=0.5, the autocovariance function lies on the plane with height 0, indicating that the process is standard Brownian motion. This is expected as Brownian motion has no memory, meaning past increments do not influence future increments.
- For H>0.5, the autocovariance function increases with respect to H but decreases very slowly with respect to the lag τ. This slow decay is significant because it results in the non-summability of the autocovariances. In other words, the autocovariance decreases so gradually that summing the autocovariances over all τ does not lead to convergence. This characteristic is indicative of long-term memory, where past values have a prolonged influence on the process.
- For H<0.5, the autocovariance function is below zero, reflecting a negative autocovariance. This occurs because the process is mean reverting. Additionally, the autocovariance function converges to zero quickly, which is typical of short-term memory processes.

Considering the previous concepts, the principal idea is that one of the significant implications of considering fBM is the relaxation of the assumption of independence in the data. Traditional Brownian motion assumes that increments are independent; however, fBM introduces dependence while maintaining the Gaussian nature of the process. This makes fBM a powerful reference model for scenarios where "Gaussianity" is required but independence is too restrictive. In financial modeling, for example, asset returns may exhibit dependencies that cannot be captured by independent models. fBM allows for the incorporation of such dependencies, providing a more accurate and flexible framework for modeling financial time series. The ability to model both long-term and short-term memory processes enables better risk assessment and strategy development, reflecting more realistic market behaviors.

Properties Building on the understanding of the autocovariance function and the role of the Hurst parameter H, it is essential to highlight the key properties of fractional Brownian motion (fBM):

1. *Continuity and Differentiability*: It is a continuous process that is almost nowhere differentiable.

2. *Gaussian Nature*: The fBM is a Gaussian process. Its increments are Gaussian random variables. However, unlike standard Brownian motion, the variance of the increments scales with t^{2H}, reflecting the influence of the Hurst parameter.

3. *Zero Mean*: The fBM is a zero mean stochastic process, which implies that $\mathbb{E}[B^H(t)]$ for all t.

4. *Starts at Zero*: The fBM starts at zero, meaning $B_0^H = 0$ almost surely. This initial condition ensures that the process begins from a well-defined point.

5. *Stationary Increments*: One of the defining characteristics of fBM is that it has stationary increments. This means that the statistical properties of the increments do not change over time; they are normally distributed with a variance that depends on the length of the time interval and the Hurst parameter. Specifically, $B_t^H – B_s^H = B_{t-s}^H$.

6. *Self-Similarity*: The fBM is a self-similar process with parameter H. This property implies that for any positive constant a, the rescaled process has the same distribution as the original process). This self-similarity extends to the increments of the process as well, indicating that $\Delta B_{t+a,t}^H = a^H \Delta B_{t+1,t}^H$.

7. *P-variation*: For H $\neq \frac{1}{2}$, the order p-variation of fBM is given by:

$$V_{n,p} = \sum_{j=1}^{2^n} |B_{j/2^n}^H – B_{(j-1)/2^n}^H|^p \to \begin{cases} 0, \text{ if } p > 1/H \\ \infty, \text{ if } p < 1/H \end{cases}$$

It converges to 0 if p>1/H and diverges to ∞ if $p < 1/H$.

This result highlights that for H> $\frac{1}{2}$, the quadratic variation (second-order variation) is 0. Conversely, for H< $\frac{1}{2}$, the quadratic variation is infinite. This is problematic for modeling processes as martingales, as it requires the first-order variation to be infinite and the second-order variation to be finite but different from zero, like standard Brownian motion. By the Lévy's Characterization theorem, the fact that has respectively zero and infinite quadratic variation when H> $\frac{1}{2}$ and H< $\frac{1}{2}$ suffices to state that it is not a semi martingale. Intuitively, values of H in (0, $\frac{1}{2}$) indicate trajectories that are too wild compared to those of a martingale, leading to infinite quadratic variation. Conversely, values of H in ($\frac{1}{2}$,1) indicate trajectories that are too regular.

3 Limitations and Conclusion

While fractional Brownian motion (fBM) offers a versatile and powerful framework for modeling a variety of stochastic processes, it is important to acknowledge and understand its limitations. This understanding is crucial for accurately assessing the contexts in which fBM is applicable and for identifying potential areas where alternative models may be more appropriate.

One significant limitation of the fBM model is observed in the property of p-variation explained above. In addition, notable issues arise when considering the dynamics of real-world data. The statistical properties of fBM are influenced by the lenght of the data series and the sampling frequency. Inadequate data lenght or improper sampling can result in biased parameter estimates and incorrent interpretation of the process. Moreover, if Brownian motion were a perfect model for price dynamics, the estimated Hurst parameter H from real data should consistently be 0.5. However, empirical observations reveal that it tends to fluctuate around 0.5, with statistically significant deviations. These fluctuations suggest that the parameter is not constant over time. This variability indicates that the assumption of a constant H in fBM may be too restrictive for accurately capturing the true dynamics of financial markets. Consequently, a more general model might be required. One such model is the multifractional Brownian motion, which allows the Hurst parameter to vary with time, providing a more flexible and realistic representation of market behavior.

In conclusion, while GBM and fBM are indispensable tools in financial modeling, their limitations highlight the need for continuous advancements in stochastic modeling. Multifractional Brownian motion represents a significant step forward in capturing the complexities of real-world market behavior.

References

Sergio Bianchi "*Basics from Stochastic processes*" slides, a.y. 2023/2024

Brown, R. (1828). *A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies*. Philosophical Magazine, 4(21), 161-173

Black, F., Scholes, M. (1973). "*The pricing of options and corporate liabilities*". Journal of Political Economy, 81(3), 637-654

Mandelbrot, B.B., and Van Ness, J.W. (1968). "*Fractional Brownian Motions, Fractional Noises and Applications*". SIAM Review, 10(4), 422-437

W Farida Agustini et al 2018 J. Phys.: Conf. Ser. 974 012047

Karl Sigman, *Notes on GBM*, Columbia edu

Steven P. Sanderson II, MPH, *Geometric Brownian Motion with healthyR.ts*

Alejandro Ribeiro, classes of *Stochastic Systems Analysis and Simulation*, University of Pennsylvania

Georgiy Shevchenko, *Fractional Brownian motion in a nutshell*

Sun, L., Chen, J. Lu, X. *Parameter estimation for discretized geometric fractional Brownian motions with applications in Chinese financial markets*. Adv Cont Discr Mod 2022, 69 2022