

# Interior eigenvectors of symmetric matrices are saddle points\*

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Every eigenvector of a symmetric matrix is a critical point of the Rayleigh quotient

$$R(\mathbf{A}, \mathbf{x}) = \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}, \quad \|\mathbf{x}\|_2^2 \neq 0. \quad (1)$$

In fact, this relationship can be used to *define* matrix eigenvalues, with the critical point condition on the Rayleigh quotient being the eigenpair equation  $\mathbf{A} \mathbf{x} = \lambda \mathbf{x}$ .<sup>1</sup>

Observe from the eigenpair equation that the eigenvector magnitude is unimportant so long as it is nonzero, which motivates the common choice to 2-normalize eigenvectors. If one does this, all eigenvectors of  $\mathbf{A} \in \mathbb{R}^{n \times n}$  lie on the unit  $(n-1)$ -sphere, and the extreme value theorem can be used to prove that all symmetric matrices with  $n \geq 2$  must have at least two eigenvectors, corresponding to the maximum and minimum of the Rayleigh quotient. However, this only classifies  $\mathcal{O}(1)$  critical points. What can be said of the other  $n-2$  critical points comprising the interior of the spectrum?

To answer this question, consider an interior eigenpair  $(\mathbf{x}_i, \lambda_i)$  of a symmetric matrix  $\mathbf{A}$  with  $n > 2$  and some scaled vector  $\beta \mathbf{x}_j$  in the direction of eigenvector  $\mathbf{x}_j$ . Let the respective norms of  $\mathbf{x}_i, \mathbf{x}_j$  be  $\alpha_i, \alpha_j$ .<sup>2</sup> The Rayleigh quotient at  $\mathbf{x}_i + \beta \mathbf{x}_j$  is

$$R(\mathbf{A}, \mathbf{x}_i + \beta \mathbf{x}_j) = \frac{(\mathbf{x}_i + \beta \mathbf{x}_j)^T \mathbf{A} (\mathbf{x}_i + \beta \mathbf{x}_j)}{(\mathbf{x}_i + \beta \mathbf{x}_j)^T (\mathbf{x}_i + \beta \mathbf{x}_j)} \quad (2)$$

$$= \frac{\mathbf{x}_i^T \mathbf{A} \mathbf{x}_i + 2\beta \mathbf{x}_j^T \mathbf{A} \mathbf{x}_i + \beta^2 \mathbf{x}_j^T \mathbf{A} \mathbf{x}_j}{\mathbf{x}_i^T \mathbf{x}_i + 2\beta \mathbf{x}_j^T \mathbf{x}_i + \beta^2 \mathbf{x}_j^T \mathbf{x}_j} \quad (3)$$

$$= \frac{\lambda_i \alpha_i^2 + \lambda_j \beta^2 \alpha_j^2}{\alpha_i^2 + \beta^2 \alpha_j^2} \quad (4)$$

where we've expanded and used the symmetry of  $\mathbf{A}$  between the first two steps. Between the last two steps we utilized  $\mathbf{x}_i$  and  $\mathbf{x}_j$  being eigenvectors of  $\mathbf{A}$  and the fact that eigenvectors of symmetric matrices are mutually orthogonal. The change in Rayleigh quotient from the original critical point  $\mathbf{x}_i$  is then

$$R(\mathbf{A}, \mathbf{x}_i + \beta \mathbf{x}_j) - R(\mathbf{A}, \mathbf{x}_i) = \frac{(\lambda_j - \lambda_i) \beta^2 \alpha_j^2}{\alpha_i^2 + \beta^2 \alpha_j^2} \quad (5)$$

Note that the sign of the quantity above depends only on  $\lambda_j - \lambda_i$  since all other quantities are defined to be real numbers. So for an interior eigenpair  $\lambda_i$  there exist at least two unique values of  $j$  such that  $\lambda_j - \lambda_i < 0$  and  $\lambda_j - \lambda_i > 0$ , for concreteness one can choose  $j = 1, n$  for spectrum  $\lambda_1 < \dots < \lambda_n$ .<sup>3</sup> Therefore, any interior eigenvector  $\mathbf{x}_i$  has an arbitrarily close point (we placed no magnitude restrictions on  $\beta$ ) that is larger in Rayleigh quotient and another point that is smaller. This condition defines a saddle point.

<sup>1</sup>The gradient of the Rayleigh quotient is  $2 \frac{\mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} - 2 \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{(\mathbf{x}^T \mathbf{x})^2} \mathbf{x}$ . Critical points are defined by a zero gradient, and using the nonzero norm condition on  $\mathbf{x}$  one finds  $\mathbf{A} \mathbf{x} - R(\mathbf{A}, \mathbf{x}) \mathbf{x} = \mathbf{0}$ . Recognizing that the Rayleigh quotient is a scalar (call it  $\lambda$ ), we recover the familiar  $\mathbf{A} \mathbf{x} = \lambda \mathbf{x}$ . At first, it may appear replacing the Rayleigh quotient by some arbitrary scalar could define different conditions if there exists  $\lambda$  such that  $\mathbf{A} \mathbf{x} = \lambda \mathbf{x}$ , but  $\lambda \neq R(\mathbf{A}, \mathbf{x})$ . However, this is not possible, which one can prove by taking the inner product of the of the eigenpair equation with eigenvector  $\mathbf{x}$  and rearranging to show that any scalar satisfying the eigenpair equation is precisely the Rayleigh quotient defined by the matrix and eigenvector.

<sup>2</sup>Although employing the extreme value theorem requires a compact domain like the  $(n-1)$ -sphere, one can classify interior eigenpairs without such a closed domain. Furthermore, it is trivial to reformulate this proof to work on the  $(n-1)$ -sphere.

<sup>3</sup>\* This enforces algebraic multiplicity one for all all eigenvalues. When the spectrum has algebraic multiplicity greater than one at the edges (i.e.  $\lambda_1 = \lambda_2 < \dots$  for the lower end), it is possible to show that this cluster of eigenpairs are all local minima or maxima by observing that the eigenvectors form a basis for  $\mathbb{R}^n$ , and so there exist no directions that decrease the Rayleigh quotient, respectively. Similar arguments apply for degenerate maxima at the top of the spectrum.