Damped Spring Model

R. Andrei Romero Alvarez
https://github.com/romxz

In physics, a particle (i.e. a point) is completely described via its position vector and its velocity vector. This is called the particle's "state".

Suppose that a particle named $Alice$ with position vector $p = (p_x, p_y, p_z)$ is attracted to another particle named Bob (the "attractor") with its own position vector $q = (q_x, q_y, q_z)$.

For simplicity, write the velocity of the first particle $Alice$ as:

$$v = (v_x, v_y, v_z) := dp/dt = (dp_x/dt, dp_y/dt, dp_z/dt) ,$$

and the velocity of the second attracting particle Bob as:

$$u = (u_x, u_y, u_z) := dq/dt = (dq_x/dt, dq_y/dt, dq_z/dt)$$

If you know the particle $Alice$'s starting state, $\{p_0, v_0\}$, as well as the attracting particle Bob's starting state $\{q_0, u_0\}$, you can calculate $Alice$'s next state $\{p_1, v_1\}$ after some small time step Δt if you know the nature of the interaction between the two.

In this case, you can estimate/model the interaction with the following parameters:

m : $Alice's$ resistance to acceleration, ie. its inertia
k : The strength of the attraction, which makes $Alice$ accelerate towards Bob
b : The damping factor, what prevents $Alice$ from moving too fast towards Bob

These are just some numbers you can play around with. In that case, given that we know p_0 , v_0 , as well as q_1 and u_1 , then after a time step of Δt seconds, we have:

$$p_1 = c_0 p_0 + c_1 q_0 + c_2 v_0 + c_3 u_0$$
$$v_1 = c_4 p_0 + c_5 q_0 + c_6 v_0 + c_7 u_0$$

Where the above c_i "constants" (i.e. only dependent on the length of the time step as well as the above parameters) are given by:

$$c_0 = 1 - k(\Delta t)^2/2m ; \qquad c_1 = k(\Delta t)^2/2m ;$$
$$c_2 = \Delta t - b(\Delta t)^2/2m ; \qquad c_3 = b(\Delta t)^2/2m ;$$
$$c_4 = -(k\Delta t/m) ; \qquad c_5 = (k\Delta t/m) ;$$

$c_6 = 1 - (b\Delta t/m)$; $c_7 = (b\Delta t/m)$

These “constants” can be calculated once, if you set what the parameters m, b, k and Δt are. This way, if Bob is externally driven (i.e. its position and velocity are determined independently), $Alice's$ position and velocity can be calculated this way

The above equations are vector equations: Just apply it individually to each vector component:

$$p_{1x} = c_0 p_{0x} + c_1 q_{0x} + c_2 v_{0x} + c_3 u_{0x}$$
$$p_{1y} = c_0 p_{0y} + c_1 q_{0y} + c_2 v_{0y} + c_3 u_0 y$$
$$p_{1z} = c_0 p_{0z} + c_1 q_{0z} + c_2 v_{0z} + c_3 u_{0z}$$

And likewise for v_1:

$$v_{1x} = c_4 p_{0x} + c_5 q_{0x} + c_6 v_{0x} + c_7 u_{0x}$$
$$v_{1y} = c_4 p_{0y} + c_5 q_{0y} + c_6 v_{0y} + c_7 u_{0y}$$
$$v_{1z} = c_4 p_{0z} + c_5 q_{0z} + c_6 v_{0z} + c_7 u_{0z}$$

A suggestion to use for the c_i constants (just as a starting point, before you tweak it), given that you know ~ what Δt is to use:

$m \simeq 1000$
$k \simeq 0.6$
$b \simeq 35$

This should give you a damping ratio of ~0.7, which is underdamped. This should look ok assuming you update every $\Delta t \approx 2ms$ [Working on it, not 100% sure]

You should also however, set a maximum velocity for all particle’s, otherwise you’ll get funny stuff. Start with $v_0 = 0$;

How to “drive” $Alice$:

To move $Alice$, you need to specify the state of Bob as a pair of vectors $\{q, u\}$ in such a way that u remains the velocity of q. I.e. q and u are not independent.

So, if you prescribe q through some function, then you need to calculate u, while if you prescribe u with some function, you need to calculate q.

This is actually very easy to do, and you can do whichever way is more convenient. You need a starting state $\{q_0, u_0\}$, which can be anything you want at first. Then:

If you prescribe q_1, then $u_1 := (q_1 - q_0)/\Delta t$
If you prescribe u_1, then $q_1 := q_0 + u_1 \Delta t$

Derivation (can completely skip if you want):

One way to model this dynamically is by supposing that this attraction is dependent on the distance (vector) $s := (p - q)$ between these two points, as well as the relative velocity (vector) $ds/dt = (v - u)$ between them.

If this attraction is like a damped spring, then the interactive force will be negatively proportional to s as well as to ds/dt, i.e.:

$$F = -ks - b(ds/dt)$$

By Newton's second law, this force will produce an acceleration on the location of $Alice$ via:

$$m(d^2p/dt^2) = F = -ks - b(ds/dt)$$

Writing the acceleration as $a := (d^2p/dt^2)$, for simplicity, rewriting the above have:

$$a = -(k/m)s - (b/m)(ds/dt)$$

$a = -(k/m)(p-q) - (b/m)(v-u)$

This is the acceleration of point p. With this, if we know what $\{p, v\}$ and $\{q, u\}$ are at time $t = 0$, we can also approximate what $\{p, v\}$ will be after time step Δt through the Taylor expansion:

$$p_1 = p_0 + v_0\Delta t + (1/2)a_0(\Delta t)^2$$
$$p_1 = p_0 + v_0\Delta t - (1/2m)[k(p_0 - q_0) + b(v_0 - u_0)](\Delta t)^2$$

Which can be rewritten as:

$$p_1 = c_0 p_0 + c_1 q_0 + c_2 v_0 + c_3 u_0$$

Where:
$c_0 = 1 - k(\Delta t)^2/2m$; $\quad c_1 = k(\Delta t)^2/2m$;
$c_2 = \Delta t - b(\Delta t)^2/2m$; $\quad c_3 = b(\Delta t)^2/2m$

Likewise, we can approximate how v will change after the current time step Δt, also through the Taylor expansion:

$$v_1 = v_0 + a_0\Delta t$$
$$v_1 = v_0 - (k/m)(p_0 - q_0)\Delta t - (b/m)(v_0 - u_0)\Delta t$$

Which can be rewritten as:

$$v_1 = c_4 p_0 + c_5 q_0 + c_6 v_0 + c_7 u_0$$

Where:
$c_4 = -(k\Delta t/m)$; $\quad c_5 = (k\Delta t/m)$;
$c_6 = 1 - (b\Delta t/m)$; $\quad c_7 = (b\Delta t/m)$