A sample computation

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Contents

1 Notation

- $p > 2$ a prime number.
- $\mathbf{SL}_3(\mathbb{Z})$ the special linear group over $\mathbb{Z}$.
- Γ_p the pth congruence subgroup of $\mathbf{SL}_3(\mathbb{Z})$.
- $Z_G(x)$ the centralizer of $x \in G$.
- $$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -p^2 \end{pmatrix} \in \mathbf{SL}_3(\mathbb{Z}) \tag{1}$$
- $$\tilde{A} = \mathbf{Id} + pA = \begin{pmatrix} 1 & 0 & p \\ p & 1 & 0 \\ 0 & p & 1-p^3 \end{pmatrix} \in \Gamma_p \tag{2}$$
- If C is a matrix, $\chi_C(\lambda) \coloneqq \det(\lambda\mathbf{Id} - C)$ is the characteristics polynomial.
- $f(t) \coloneqq \chi_A(t) = t^3 + p^2t^2 - 1$.

2 Intro

In this note we establish the following results:

1. The regulator of $\tilde{A}$ grows as $\approx 3\ln^2(p)$.
   That is, $\mathrm{Reg}(\mathbb{Q}(\alpha)/\mathbb{Q}) \approx 3\ln^2(p)$ for α a root of $\chi_{\tilde{A}}$.
2. The index of the centralizers $[Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] = 3$ does not depend on p.

This might be interesting.

3 Regulators

We mimic the proofs from a Keigh Conrad's write-up on the Dirichlet unit theorem [Con].

Let $\mathbb{Q}(\alpha)$ be the number field obtain by attaching the root of $f(t)$ to $\mathbb{Q}$.

Proposition 3.1. *$\mathbb{Q}(\alpha)/\mathbb{Q}$ is a totally real number field of degree 3.*

Proof. First, $f(t)$ is irreducible as it has no rational roots: $f(1) = p^2 \neq 0, f(-1) = p^2 - 2 \neq 0$.

The discriminant of $f(t)$ equals

$$\mathrm{disc}_f(p) = 4p^6 - 27.$$

The discriminant is always positive $\mathrm{disc}_f(p) > 0$, thus the number field is totally real. □

Proposition 3.2. $\mathbb{Z}[\alpha]^* = \{\pm\alpha^a(1+p\alpha)^b \mid a, b \in \mathbb{Z}\}$.

Note that $\alpha, 1 + p\alpha$ are not necessarily fundamental units in $\mathbb{Q}(\alpha)/\mathbb{Q}$ as we don't claim that the ring of integers of $\mathbb{Q}(\alpha)/\mathbb{Q}$ coincides with $\mathbb{Z}[\alpha]$.

Proof. We have

$$\alpha(\alpha^2 + p^2\alpha) = 1 \tag{3}$$

$$(1 - p\alpha)(1 + p\alpha) = \alpha^3 \tag{4}$$

It shows that $\alpha, 1 + p\alpha$ are indeed units.

Let $\alpha_1 < \alpha_2 < \alpha_3$ be the three different roots of f. We shall compute them approximately.

$$\alpha_1 = -p^2 + O\left(p^{-4}\right),$$
$$\alpha_2 = -p^{-1} + O\left(p^{-4}\right),$$
$$\alpha_3 = p^{-1} + O\left(p^{-4}\right).$$

Remark 3.3. *A computation shows that for $p = 100$ we have*

$$\alpha_1 = -9999.99999999000,$$
$$\alpha_2 = -0.01000000500000625,$$
$$\alpha_3 = 0.00999999500000625.$$

It is not important that p is not a prime in this case as the estimate works for any sufficiently large p.

By the definition of the regulator we have

$$
\begin{aligned}
\operatorname{Reg}(\alpha, p\alpha+1) &= \begin{vmatrix} \ln|\alpha_1| & \ln|\alpha_3| \\ \ln|p\alpha_1+1| & \ln|p\alpha_3+1| \end{vmatrix} \\
&\approx \begin{vmatrix} \ln|-p^2| & \ln|\frac{1}{p}| \\ \ln|-p^3+1| & \ln|2| \end{vmatrix} = \ln(p^2)\ln 2 + \ln(p^3-1)\ln(p) \\
&\approx \ln(p)(3\ln(p)+2\ln 2). \qquad (5)
\end{aligned}
$$

Therefore $\operatorname{Reg}(\alpha, p\alpha+1) > 0$ for all prime p.

Hence $\alpha, p\alpha+1$ are independent units.

Remark 3.4. *For example, for $p = 73$ SageMath computes the regulator to be approximately* $\operatorname{Reg} = 61.1719663782187$.

```
p = Primes().unrank(20) # p = 73
R.<x> = PolynomialRing(QQ)
P = x^3 + p^2 * x^2 - 1
K.<a> = QQ.extension(P)
print(K.regulator())
```

Listing 1: SageMath code

Whereas the above estimate gives 61.1719663782957. *These numbers coincide up to* 10^{-11}.

Another example, for $p = 547$ SageMath computes the regulator to be approximately $\operatorname{Reg} = 127.978045931846$.

```
p = Primes().unrank(100) # p = 547
R.<x> = PolynomialRing(QQ)
P = x^3 + p^2 * x^2 - 1
K.<a> = QQ.extension(P)
print(K.regulator())
```

Listing 2: SageMath code

Whereas the above estimate gives 127.97804593184651. *These numbers coincide up to* 10^{-12}.

It is left to prove that they are fundamental units in $\mathbb{Z}[\alpha]$. By Corollary 5.9 from Conrad it is sufficient to check

$$\frac{16\operatorname{Reg}(\alpha, p\alpha+1)}{(\ln(\operatorname{disc}_f/4))^2} < 2.$$

Substituting, we obtain

$$\frac{16\operatorname{Reg}(\alpha, 2\alpha+1)}{(\ln(\operatorname{disc}_f/4))^2} \approx \frac{16\ln(p)(3\ln(p)+2\ln 2)}{(\ln((4p^6-27)/4))^2}.$$

Asymptotically, the latter equals

$$\xrightarrow{p\to\infty} \frac{48\ln(p)^2}{(\ln(p^6))^2} = \frac{4}{3}.$$

Therefore it is < 2 for big enough p, QED. $\square$

Remark 3.5. *In fact, the function*

$$\frac{16\ln(p)(3\ln(p)+2\ln 2)}{(\ln((4p^6-27)/4))^2}$$

is smaller then 2 *for all* $p > 2.6$. *It monotonously decreases to the asymptotic value* $\frac{4}{3}$.

The first several values of $g(p) \stackrel{\text{def}}{=} \frac{16\,\mathrm{Reg}(\alpha,2\alpha+1)}{(\ln(\mathrm{disc}_f/4))^2}$ *are presented in the following*

p	$g(p)$
2	2.3005757200277737
3	0.948774985707828
5	1.7162543373046009
7	0.824986119068219
11	1.5902803808431747
13	1.573545207927126
17	1.5508005143276846
19	1.5425857167054255
23	1.529835319790773
29	1.5163082966042292
31	1.5127547443392193
37	1.5039633069049658
41	1.4992465878896728
43	1.4971456295557026
47	1.4933611793019335
53	1.4885186045113814
59	1.484436992267399
61	1.483211643289589
67	1.4798674282030506
71	1.4778740553564413
73	1.4769381932859227
79	1.4743421857047407
83	1.4727660220987417
89	1.4705979224908206
97	1.4680152465868979
101	1.4668359850881518
103	1.466271166102171
107	1.4651872592939545
109	1.4646667672702098
113	1.4636655275160453
127	1.4605230591156473

Table 1: Values of p and $g(p)$

which we computed using SageMath

```
for i, p in enumerate(Primes()):
    if i > 30:
        break
    R.<x> = PolynomialRing(QQ)
    P = x^3 + p^2 * x^2 - 1
    K.<a> = QQ.extension(P)
    rg = K.regulator()
    print(p, float(16 * rg * ln( p^6 - 27/4 )^(-2)))
```

Listing 3: SageMath code

Remark 3.6. *We just proved that the regulator is approximately*

$$\ln(p)(3\ln(p) + 2\ln 2),$$

which is close to $3\ln^2 p$ *we wanted from the beginning.*

4 Centralizers

Proposition 4.1. *The centralizer of* $\tilde{A}$ *in* $\mathbf{SL}_3(\mathbb{Z})$ *is generated by* $A, pA + \mathbf{Id}$.

$$Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) = \{A^a(pA + \mathbf{Id})^b \mid a, b \in \mathbb{Z}\}.$$

Proof. Since $\tilde{A}$ is regular, its centralizer in $\mathbf{Mat}_3(\mathbb{C})$ is $\mathbb{C}\left\langle \mathbf{Id}, A, A^2 \right\rangle$. Now,

$$\mathbb{C}\left\langle \mathbf{Id}, A, A^2 \right\rangle \cap \mathbf{SL}_3(\mathbb{Z}) \subset \mathbb{Z}\left\langle \mathbf{Id}, A, A^2 \right\rangle = \mathbb{Z}[A].$$

Indeed,

$$\mathbf{Id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -p^2 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 1 & -p^2 \\ 0 & 0 & 1 \\ 1 & -p^2 & p^4 \end{pmatrix}, \tag{6}$$

Considering the first matrix column we see that if a complex combination has integer coefficients, it is in fact integer combination.

Moreover, the centralizer of $\tilde{A}$ is a group, therefore it lies inside the multiplicative group of $\mathbb{Z}[A]$

$$Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) \subset \mathbb{C}\left\langle \mathbf{Id}, A, A^2 \right\rangle \cap \mathbf{SL}_3(\mathbb{Z}) \subset \mathbb{Z}[A]^*.$$

There is an isomorphism of $\mathbb{Z}$-algebras $\mathbb{Z}[A] \simeq \mathbb{Z}[x]/(f(x)) = \mathbb{Z}[\alpha]$. Applying Proposition 3.2 end the proof

$$Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) \subset \mathbb{Z}[A]^* = \{\pm A^a(pA + \mathbf{Id})^b \mid a, b \in \mathbb{Z}\}.$$

□

We are to study the centralizer of $\tilde{A}$ in Γ_p.

$$Z_{\Gamma_p}(\tilde{A}) \subset Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) \cong \mathbb{Z}^2.$$

Proposition 4.2. *The index of the one centralizer inside the other* does not depend on p *and equals* 3.

$$[Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] = 3.$$

Proof. As $A \pmod p$ is equivalent to the cyclic permutation matrix, it follows that $A^3 \in \Gamma_p$.

Thus the smaller centralizer is generated by $A^3, \tilde{A}$

$$Z_{\Gamma_p}(\tilde{A}) = \{A^{3a}(pA + \mathbf{Id})^b \mid a, b \in \mathbb{Z}\},$$

and it is clear that the index is 3. ☐

5 Computing (co)homology

Christophe Soulé [Sou78] computed the cohomology of $\mathbf{SL}_3(\mathbb{Z})$ to be

n	0	1	2	3	4	...
$H^n(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z})$	$\mathbb{Z}$	0	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2$	...

The universal coefficient formula gives us the homology.

n	0	1	2	3	...
$H_n(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z})$	$\mathbb{Z}$	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2$	...

Recall the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -p^2 \end{pmatrix}, B = \mathbf{Id} + pA = \begin{pmatrix} 1 & 0 & p \\ p & 1 & 0 \\ 0 & p & 1-p^3 \end{pmatrix}.$$

We denote $\tilde{A}$ by B for convenience.

As A, B commute they define a map

$$f : \mathbb{Z}^2 \to \mathbf{SL}_3(\mathbb{Z}),$$

$$(k, l) \mapsto A^k B^l.$$

Theorem 5.1. *The pushforward map in the second homology*

$$f_* : H_2(\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z} \to H_2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

is zero.

The goal of this section is to prove this theorem.

The first observation is that one can pass from the homology to the cohomology with coefficients $\mathbb{Z}/2\mathbb{Z}$.

Lemma 5.2. *The pushforward*

$$f_* : H_2(\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z} \to H_2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

is zero iff the pullback

$$f^* : H^2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \to H^2(\mathbb{Z}^2, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

is zero.

Proof. As the first homology

$$H_1(\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z}^2, H_1(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}) \cong 0$$

are torsion-free, the universal coefficien formula gives perfect $\mathbb{Z}/2\mathbb{Z}$-pairings between homology and cohomology

$$H^2(\mathbb{Z}^2, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_2(\mathbb{Z}^2, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}),$$

$$H^2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}).$$

Therefore the pullback in cohomology is the transposed map of the pushforward in homology and one is zero iff the other is zero. □

Next, let us find a nicer description for $H^2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$.

Claim 5.3.

- *The second cohomology $H^2(G, M)$ classify the extensions $0 \to M \to \tilde{G} \to G \to 0$;*
- *In particular, $H^2(G, \mathbb{Z}/2\mathbb{Z})$ correspond to the extensions of degree 2;*
- *Given an extension $0 \to M \to \tilde{G} \to G \to 0$ and a principle G-bundle over X we get its class in $H^2(X, M)$ via the boundary (in other words, Bockstein) homomorphism $H^1(X, G) \to H^2(X, M)$. In details, the Bockstein goes by taking the 1-cocycle defining the $G-$bundle to $\tilde{G}$ and apply the Cech differential;*
- *$H^2(\mathbf{SL}_3(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \cong H^2(\mathbf{SL}_3(\mathbb{F}_2), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$;*
- *The corresponding unique extensions of degree two are $\widetilde{\mathbf{SL}_3}(\mathbb{R})$ and $\mathbf{SL}_2(\mathbb{F}_7)$. We make use of the exceptional isomorphism $\mathbf{PSL}_2(\mathbb{F}_7) \cong \mathbf{SL}_3(\mathbb{F}_2)$, see, for example, [Mer].*
- *If E is a principle $\mathbf{SL}_3(\mathbb{R})$-bundle on X the corresponding class in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ is called $w_2(E)$, the second Stiefel-Whitney class;*
- *If E is a principle $\mathbf{SL}_3(\mathbb{F}_2)$-bundle on X let us denote the corresponding class in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ by $u_2(E)$.*

This is all very well known. For the cohomology of $\mathbf{SL}_3(\mathbb{F}_2)$ see, for example, [Ade06].

Now, consider the natural mappings

$$i : \mathbf{SL}_3(\mathbb{Z}) \hookrightarrow \mathbf{SL}_3(\mathbb{R}),$$

$$\pi : \mathbf{SL}_3(\mathbb{Z}) \hookrightarrow \mathbf{SL}_3(\mathbb{F}_2).$$

Proposition 5.4. *The pullback*

$$i^* \oplus \pi^* : H^2(B\mathbf{SL}_3(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \oplus H^2(B\mathbf{SL}_3(\mathbb{F}_2), \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \to H^2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

is an isomorphism.

Proof. It is enough to show that a principal $\mathbf{SL}_3(\mathbb{Z})$ bundle E on $\mathbb{T}^2$ can have an arbitrary combination of $(w_2(E), u_2(E)) \in (\mathbb{Z}/2\mathbb{Z})^2$.

Plan: we shall construct principal bundles E_1, E_2 on $\mathbb{T}^2$ with $(w_2(E_1), u_2(E_1)) = (1, 0)$ and $u_2(E_2) = 1$. This will prove the proposition.

The first bundle E_1 is defined via the monodromy matrices C_1, C_2

$$C_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathbf{SL}_3(\mathbb{Z}).$$

By the splitting principle we have

$$w(E_1) = (1 + x)(1 + x + y)(1 + y) = 1 + xy,$$

where x, y denote the generators of $H^1(\mathbb{T}^2, \mathbb{Z}/2\mathbb{Z})$. Therefore $w_2(E_1) = 1$. As modulo 2 the matrices C_1, C_2 are the identity the bundle E_1 mod 2 is trivial; thus $u_2(E_1) = 0$.

Define $D_1' = \left(\begin{smallmatrix} 0 & 6 \\ 1 & 0 \end{smallmatrix}\right), D_2' = \left(\begin{smallmatrix} 2 & 3 \\ 3 & 5 \end{smallmatrix}\right) \in \mathbf{SL}_2(\mathbb{F}_7)$. Project the matrices to $\in$ $\mathbf{PSL}_2(\mathbb{F}_7) \cong\in \mathbf{SL}_3(\mathbb{F}_2)$. Pick a pair of commuting lifts $D_1, D_2 \in \mathbf{SL}_3(\mathbb{Z})$. One can take them to be

$$D_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{7}$$

Define the bundle E_2 corresponding to the monodromy matrices D_1, D_2. We claim that $u_2(E_2) = 1$.

Indeed, we shall compute the Bockstein morphism $H^1(\mathbb{T}^2, \mathbf{SL}_3(\mathbb{F}_2) \to H^2(\mathbb{T}^2, \mathbb{Z}/2\mathbb{Z})$ by lifting the cocycles.

One checks that $D_1'D_2' = -D_2'D_1'$; thus the image of the Bockstein map is nontrivial.

Thus we have showed that E_1, E_2 have the right characteristic classes and we are done. □

Proof of the Theorem 5.1. In view of the Lemma 5.2 we shall show that the pullback map in second $\mathbb{Z}/2\mathbb{Z}$-cohomology vanishes.

Consider the principal $\mathbf{SL}_3(\mathbb{Z})$-bundle E on $\mathbb{T}^2$ associated with A, B. We should show that $w_2(E) = 0$ and $u_2(E) = 0$.

By the splitting principle using that the roots of the characteristic polynomial 3.3 satisfy

$$\alpha_1, 1 + p\alpha_1 < 0, \alpha_2, 1 + p\alpha_2 < 0, \alpha_3, 1 + p\alpha_3 > 0$$

we get the Stiefel-Whitney class to be

$$w(E) = w(E_1)w(E_2)w(E_3) = (1 + x + y)(1 + x + y)(1) = 1,$$

and thus $w_2(E) = 0$ indeed does vanish.

Modulo 2 one easily spots that $B = A^5 \pmod 2$, thus their lifts to $\mathbf{SL}_2(\mathbf{F}_7)$ do commute and the image of the Bockstain morphism vanishes; thus $u_2(E) = 0$. □

6 From $\mathbf{SL}_3(\mathbb{Z})$ to $\mathbf{SL}_n(\mathbb{Z})$

Consider

- $p > 2$ a prime number;
- $\mathbf{SL}_n(\mathbb{Z})$ the n-th special linear group over $\mathbb{Z}$;
- Γ_p the pth congruence subgroup of $\mathbf{SL}_n(\mathbb{Z})$;
- $Z_G(x)$ the centralizer of $x \in G$;
- $f(t) \stackrel{\text{def}}{=} t^n - (1+pt)(1+2pt)\dots(1+(n-1)pt)$;
- If C is a matrix, $\chi_C(\lambda) := \det(\lambda\mathbf{Id} - C)$ is the characteristics polynomial.
- Let A be the irreducible operator with $\chi_A(t) = f(t)$;

  For example, when $n = 4$ A takes a form

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 6p \\ 0 & 1 & 0 & 11p^2 \\ 0 & 0 & 1 & 6p^3 \end{pmatrix} \in \mathbf{SL}_4(\mathbb{Z}); \tag{8}$$

- Define $\tilde{A} \stackrel{\text{def}}{=} \mathbf{Id} + pA$.

  For example, when $n = 4$ $\tilde{A}$ takes a form

$$\tilde{A} = \mathbf{Id} + pA = \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & 0 & 6p^2 \\ 0 & p & 1 & 11p^3 \\ 0 & 0 & p & 1+6p^4 \end{pmatrix} \in \Gamma_p. \tag{9}$$

7 Intro2

In this note we establish the following results:

1. The regularor of $\tilde{A}$ grows as $O(\ln^{n-1}(p))$;
2. If n is prime, the index of the centralizers $[Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] < C$ is bounded by some constant C and does not depend on p.

This might be interesting.

8 Regulators

We mimic the proofs from a Keigh Conrad's write-up on the Dirichlet unit theorem [Con].

There are more general estimates due to Duke [Duk03] (see Section 3 Proposition 1).

Let $K \stackrel{\text{def}}{=} \mathbb{Q}(\alpha)$ be the number field obtain by attaching the root of $f(t)$ to $\mathbb{Q}$.

Proposition 8.1. *$K/\mathbb{Q}$ is a totally real number field of degree n.*

Proof. First, $f(t)$ is irreducible as it has no rational roots:

$$f(1) = 1 - (1+p)(1+2p)\dots(1+(n-1)p) \neq 0, \tag{10}$$
$$f(-1) = (-1)^n - (1-p)(1-2p)\dots(1-(n-1)p) \neq 0. \tag{11}$$

Let's compute the roots of $f(t)$ approximately and show they are real.

Notice that for $|t| \ll 1$ all but one roots of $f(t)$ are close to the roots of

$$(1+pt)(1+2pt)\dots(1+(n-1)pt),$$

them being $p^{-1}, \frac{1}{2}p^{-1}, \dots, \frac{1}{(n-1)}p^{-1}$.

As all but one roots are real, the remaining one is also real. □

The next goal is to tackle the unit group $\mathcal{O}_K^*$ of K.

The Dirichlet unit theorem implies that $\mathcal{O}_K^*$ is an abelian group of rank $n-1$ (there might be some finite torsion of roots of unity. It is bounded universally in p). A basis of $\mathcal{O}_K^*$ modulo roots of unity is called a system of fundamental units. Finding a system of fundamental units in general is a complicated task.

Instead, we shall find a finite index subgroup $U \subset \mathcal{O}_K^*$. Then we shall bound above the regulator of K with the regulator of U.

Moreover, using a Silverman's lower bound on the regulator of K we bound above the index $[\mathcal{O}_K^* : U] < C_1$ with C_1 independent on p.

Define a subgroup

$$U = (1+p\alpha)^{\mathbb{Z}} \dots (1+(n-1)p\alpha)^{\mathbb{Z}} \subset \mathcal{O}_K^*.$$

Proposition 8.2. *$U \subset \mathcal{O}_K^*$ is a finite index subgroup. The regulator Reg_U is bounded above by $C_2 \log(p)^{n-1}$ where C_2 is a constant independent on p.*

Proof. We have

$$\alpha^n - (1+p\alpha)(1+2p\alpha)\dots(1+(n-1)p\alpha) + 1 = 1, \tag{12}$$
$$(1+p\alpha)(1+2p\alpha)\dots(1+(n-1)p\alpha) = \alpha^n. \tag{13}$$

It shows that $1+p\alpha, \dots, 1+(n-1)p\alpha$ are indeed units.

Let $\alpha_1 < \ldots < \alpha_{n-1} < 0$ be the negative roots of f. We shall compute them approximately.

Lemma 8.3.

$$\alpha_k = \frac{-1}{k}p^{-1} + \frac{(-1)^{k+1}}{k \cdot k!(n-k+1)!}p^{-n-1} + O(p^{-2n-1}).$$

Proof. In the first approximation $\alpha_k = \frac{-1}{k}p^{-1} + o(p^{-1})$. Let's compute the residue term more persisely. Let $\alpha_k = \frac{-1}{k}p^{-1}(1+\epsilon)$. Substituting to $f(t)$, we get

$$
\begin{gathered}
(1+p\alpha_k)(1+2p\alpha_k)\dots(1+(n-1)p\alpha_k)=\alpha_k^n,\\
-(-kp\alpha_k-k)(-2kp\alpha_k-k)\dots(-(n-1)kp\alpha_k-k)=(-k\alpha_k)^n,\\
-(1-\epsilon-k)(2-2\epsilon-k)\dots((n-1)-(n-1)\epsilon-k)=p^{-n}(1+\epsilon)^n,\\
-(1-k)(2-k)\dots(-1)k\epsilon 1\dots((n-1)-k)+O(\epsilon^2)=p^{-n}+O(\epsilon p^{-n}),\\
(-1)^k k!(n-k-1)!\epsilon+O(\epsilon^2)=p^{-n}+O(\epsilon p^{-n}),\\
\epsilon=\frac{(-1)^k p^{-n}}{k!(n-k-1)!}+O(\epsilon p^{-n})+O(\epsilon^2),\\
\epsilon=\frac{(-1)^k p^{-n}}{k!(n-k-1)!}+O(p^{-2n}).
\end{gathered}
$$

□

By the definition of the regulator we have

$$
\begin{aligned}
\mathrm{Reg}_U=&\begin{vmatrix}
\ln|1+p\alpha_1| & \ln|1+p\alpha_2| & \dots & \ln|1+p\alpha_{n-1}|\\
\ln|1+2p\alpha_1| & \ln|1+2p\alpha_2| & \dots & \ln|1+2p\alpha_{n-1}|\\
\vdots & \vdots & \ddots & \vdots\\
\ln|1+(n-1)p\alpha_1| & \ln|1+(n-1)p\alpha_2| & \dots & \ln|1+(n-1)p\alpha_{n-1}|
\end{vmatrix}\\
\approx&\begin{vmatrix}
\ln\left(\frac{p^{-n}}{n!}\right) & \ln\left(1-\frac{1}{2}\right) & \dots & \ln\left(1-\frac{1}{n-1}\right)\\
\ln|1-2| & \ln\left(\frac{p^{-n}}{2\cdot 2!(n-1)!}\right) & \dots & \ln\left(1-\frac{2}{n-1}\right)\\
\vdots & \vdots & \ddots & \vdots\\
\ln|1-(n-1)| & \ln|1-\frac{n-1}{2}| & \dots & \ln\left(\frac{p^{-n}}{(n-1)\cdot(n-1)!2!}\right)
\end{vmatrix}\\
&\approx n^{n-1}\log(p)^{n-1}. \qquad (14)
\end{aligned}
$$

□

From the proof of the Proposition above one finds that the regulator of U $\mathrm{Reg}_U>0$ is positive for big enough prime p. Hence the generators $(1+p\alpha),\dots,(1+(n-1)p)$ of U are multiplicatively independent, and $U\subset\mathcal{O}_K^*$ is a finite index subgroup.

We shall bound the index $[\mathcal{O}_K^*:U]$ from above via the following

Theorem 8.4 (Silverman, [Sil84])**.** *Let K be a number field of degree d with regulator R, and absolute discriminant D_K. Let $r(K)$ be the rank of the unit group in K, and let $p(K)$ be the maximum of $r(k)$ as k ranges over proper subfields of K. We prove*

$$R_K>c_d(\log\gamma_d D_K)^{r(K)-p(K)}$$

for constants c_d,γ_d depending only on d.

It seems it's actually due to Robert Remak in "Über Größenbeziehungen zwischen Diskriminante und Regulator eines algebraischen Zahlkörpers"

Proposition 8.5. *Let n be a prime. Then the index $[\mathcal{O}_K^*:U]<C_3$ is bounded from above by a constant C_3 independent on p.*

Proof. The discriminant of K is the discriminant of the polynomial $f(t)$, which is approximately

$$\operatorname{disc}_K = \operatorname{disc}_f = O(p^{2(n-1)^2} p^{-(n-1)(n-2)}) = O(p^{n(n-1)}).$$

More precisely,

$$\operatorname{disc}_K = \operatorname{disc}_f \approx 1!^2 \dots (n-1)!^2 p^{n(n-1)}.$$

Given that n is prime and so there are no nontrivial subfield of K the Silverman's theorem implies

$$\operatorname{Reg}_K > C_4 (\log \operatorname{disc}_K)^{n-1} \approx n^{n-1}(n-1)^{n-1} C_4 \log(p)^{n-1} > C_5 \log(p)^{n-1}.$$

It is easy to see that

$$\operatorname{Reg}_U = [\mathcal{O}_K^* : U] \operatorname{Reg}_K .$$

Thus we have

$$C_2 \log(p)^{n-1} > \operatorname{Reg}_U = [\mathcal{O}_K^* : U] \operatorname{Reg}_K > [\mathcal{O}_K^* : U] C_5 \log(p)^{n-1}.$$

It follows that the index is bounded from above

$$[\mathcal{O}_K^* : U] < \frac{C_2}{C_5},$$

as desired. ◻

9 Centralizers

Proposition 9.1. *Let n be a prime. Then the index of the centralizers*

$$[Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] < C_3$$

is bounded from above by a constant independent of p.

Proof. We shall exploit the properties of the previously introduced number field $K = \mathbb{Q}(\alpha)$, where α is a root of $f(t) = t^n - (1+pt)\dots(1+(n-1)pt)$.

Since A is regular, there is an isomorphism of $\mathbb{Z}$-algebras

$$\Phi : \mathbb{Z}[A] \xrightarrow{\sim} \mathbb{Z}[\alpha], A \mapsto \alpha.$$

It identifies the stabilizer of $\tilde{A}$ in $\mathbf{SL}_n(\mathbb{Z})$ with the multiplicative group of the order $\mathbb{Z}[\alpha] \subset \mathcal{O}_K$.

$$Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) = Z_{\mathbf{SL}_n(\mathbb{Z})}(A) = \mathbb{Z}[A]^* \overset{\Phi}{\simeq} \mathbb{Z}[\alpha]^*.$$

The isomorphism Φ identifies the subgroup $U \subset \mathbb{Z}[\alpha]^*$ with

$$\Phi^{-1}(U) = \tilde{A}^{\mathbb{Z}}(2\tilde{A} - \mathbf{Id})^{\mathbb{Z}} \dots ((n-1)\tilde{A} - (n-2)\mathbf{Id})^{\mathbb{Z}} \subset Z_{\Gamma_p}(\tilde{A}).$$

Thus, there is a chain of inclusions

$$\Phi^{-1}(U) \subset Z_{\Gamma_p}(\tilde{A}) \subset Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) = \Phi^{-1}\mathbb{Z}[\alpha]^*.$$

Therefore, there is a bound

$$[Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] \leq [\mathbb{Z}[\alpha]^*, U] \leq [\mathcal{O}_K^*, U] < C_3.$$

◻

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