

A sample computation

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1 Notation

- $p > 2$ a prime number.
- $\mathbf{SL}_3(\mathbb{Z})$ the special linear group over \mathbb{Z} .
- Γ_p the p th congruence subgroup of $\mathbf{SL}_3(\mathbb{Z})$.
- $Z_G(x)$ the centralizer of $x \in G$.

•

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -p^2 \end{pmatrix} \in \mathbf{SL}_3(\mathbb{Z}) \quad (1)$$

•

$$\tilde{A} = \mathbf{Id} + pA = \begin{pmatrix} 1 & 0 & p \\ p & 1 & 0 \\ 0 & p & 1 - p^3 \end{pmatrix} \in \Gamma_p \quad (2)$$

- If C is a matrix, $\chi_C(\lambda) := \det(\lambda \mathbf{Id} - C)$ is the characteristics polynomial.
- $f(t) := \chi_A(t) = t^3 + p^2 t^2 - 1$.

2 Intro

In this note we establish the following results:

1. The regulator of \tilde{A} grows as $\approx 3 \ln^2(p)$.
That is, $\text{Reg}(\mathbb{Q}(\alpha)/\mathbb{Q}) \approx 3 \ln^2(p)$ for α a root of $\chi_{\tilde{A}}$.
2. The index of the centralizers $[Z_{\text{SL}_3(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] = 3$ does not depend on p .

This might be interesting.

3 Regulators

We mimic the proofs from a Keigh Conrad's write-up on the Dirichlet unit theorem [1].

Let $\mathbb{Q}(\alpha)$ be the number field obtain by attaching the root of $f(t)$ to \mathbb{Q} .

Proposition 3.1. $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a totally real number field of degree 3.

Proof. First, $f(t)$ is irreducible as it has no rational roots: $f(1) = p^2 \neq 0$, $f(-1) = p^2 - 2 \neq 0$.

The discriminant of $f(t)$ equals

$$\text{disc}_f(p) = 4p^6 - 27.$$

The discriminant is always positive $\text{disc}_f(p) > 0$, thus the number field is totally real. \square

Proposition 3.2. $\mathbb{Z}[\alpha]^* = \{\pm \alpha^a (1 + p\alpha)^b \mid a, b \in \mathbb{Z}\}$.

Note that $\alpha, 1 + p\alpha$ are not necessarily fundamental units in $\mathbb{Q}(\alpha)/\mathbb{Q}$ as we don't claim that the ring of integers of $\mathbb{Q}(\alpha)/\mathbb{Q}$ coincides with $\mathbb{Z}[\alpha]$.

Proof. We have

$$\alpha(\alpha^2 + p^2\alpha) = 1 \tag{3}$$

$$(1 - p\alpha)(1 + p\alpha) = \alpha^3 \tag{4}$$

It shows that $\alpha, 1 + p\alpha$ are indeed units.

Let $\alpha_1 < \alpha_2 < \alpha_3$ be the three different roots of f . We shall compute them approximately.

$$\alpha_1 = -p^2 + O(p^{-4}),$$

$$\alpha_2 = -p^{-1} + O(p^{-4}),$$

$$\alpha_3 = p^{-1} + O(p^{-4}).$$

Remark 3.3. A computation shows that for $p = 100$ we have

$$\begin{aligned} \alpha_1 &= -9999.99999999000, \\ \alpha_2 &= -0.01000000500000625, \\ \alpha_3 &= 0.00999999500000625. \end{aligned}$$

It is not important that p is not a prime in this case as the estimate works for any sufficiently large p .

By the definition of the regulator we have

$$\begin{aligned} \text{Reg}(\alpha, p\alpha + 1) &= \begin{vmatrix} \ln |\alpha_1| & \ln |\alpha_3| \\ \ln |p\alpha_1 + 1| & \ln |p\alpha_3 + 1| \end{vmatrix} \\ &\approx \begin{vmatrix} \ln |-p^2| & \ln |\frac{1}{p}| \\ \ln |-p^3 + 1| & \ln |2| \end{vmatrix} = \ln(p^2) \ln 2 + \ln(p^3 - 1) \ln(p) \\ &\approx \ln(p)(3 \ln(p) + 2 \ln 2). \quad (5) \end{aligned}$$

Therefore $\text{Reg}(\alpha, p\alpha + 1) > 0$ for all prime p .

Hence $\alpha, p\alpha + 1$ are independent units.

Remark 3.4. For example, for $p = 73$ SageMath computes the regulator to be approximately $\text{Reg} = 61.1719663782187$.

```
p = Primes().unrank(20) # p = 73
R.<x> = PolynomialRing(QQ)
P = x^3 + p^2 * x^2 - 1
K.<a> = QQ.extension(P)
print(K.regulator())
```

Listing 1: SageMath code

Whereas the above estimate gives 61.1719663782957. These numbers coincide up to 10^{-11} .

Another example, for $p = 547$ SageMath computes the regulator to be approximately $\text{Reg} = 127.978045931846$.

```
p = Primes().unrank(100) # p = 547
R.<x> = PolynomialRing(QQ)
P = x^3 + p^2 * x^2 - 1
K.<a> = QQ.extension(P)
print(K.regulator())
```

Listing 2: SageMath code

Whereas the above estimate gives 127.97804593184651. These numbers coincide up to 10^{-12} .

It is left to prove that they are fundamental units in $\mathbb{Z}[\alpha]$. By Corollary 5.9 from Conrad it is sufficient to check

$$\frac{16 \text{Reg}(\alpha, p\alpha + 1)}{(\ln(\text{disc}_f/4))^2} < 2.$$

Substituting, we obtain

$$\frac{16 \text{Reg}(\alpha, 2\alpha + 1)}{(\ln(\text{disc}_f/4))^2} \approx \frac{16 \ln(p)(3 \ln(p) + 2 \ln 2)}{(\ln((4p^6 - 27)/4))^2}.$$

Asymptotically, the latter equals

$$\xrightarrow{p \rightarrow \infty} \frac{48 \ln(p)^2}{(\ln(p^6))^2} = \frac{4}{3}.$$

Therefore it is < 2 for big enough p , QED. \square

Remark 3.5. *In fact, the function*

$$\frac{16 \ln(p)(3 \ln(p) + 2 \ln 2)}{(\ln((4p^6 - 27)/4))^2}$$

is smaller than 2 for all $p > 2.6$. It monotonously decreases to the asymptotic value $\frac{4}{3}$.

The first several values of $g(p) \stackrel{\text{def}}{=} \frac{16 \text{Reg}(\alpha, 2\alpha+1)}{(\ln(\text{disc}_f/4))^2}$ are presented in the following

p	$g(p)$
2	2.3005757200277737
3	0.948774985707828
5	1.7162543373046009
7	0.824986119068219
11	1.5902803808431747
13	1.573545207927126
17	1.5508005143276846
19	1.5425857167054255
23	1.529835319790773
29	1.5163082966042292
31	1.5127547443392193
37	1.5039633069049658
41	1.4992465878896728
43	1.4971456295557026
47	1.4933611793019335
53	1.4885186045113814
59	1.484436992267399
61	1.483211643289589
67	1.4798674282030506
71	1.4778740553564413
73	1.4769381932859227
79	1.4743421857047407
83	1.4727660220987417
89	1.4705979224908206
97	1.4680152465868979
101	1.4668359850881518
103	1.466271166102171
107	1.4651872592939545
109	1.4646667672702098
113	1.4636655275160453
127	1.4605230591156473

Table 1: Values of p and $g(p)$

which we computed using SageMath

```
for i, p in enumerate(Primes()):
    if i > 30:
        break
    R.<x> = PolynomialRing(QQ)
    P = x^3 + p^2 * x^2 - 1
    K.<a> = QQ.extension(P)
    rg = K.regulator()
    print(p, float(16 * rg * ln(p^6 - 27/4)^(-2)))
```

Listing 3: SageMath code

Remark 3.6. We just proved that the regulator is approximately

$$\ln(p)(3 \ln(p) + 2 \ln 2),$$

which is close to $3 \ln^2 p$ we wanted from the beginning.

4 Centralizers

Proposition 4.1. The centralizer of \tilde{A} in $\mathbf{SL}_3(\mathbb{Z})$ is generated by $A, pA + \mathbf{Id}$.

$$Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) = \{A^a(pA + \mathbf{Id})^b \mid a, b \in \mathbb{Z}\}.$$

Proof. Since \tilde{A} is regular, its centralizer in $\mathbf{Mat}_3(\mathbb{C})$ is $\mathbb{C}\langle \mathbf{Id}, A, A^2 \rangle$. Now,

$$\mathbb{C}\langle \mathbf{Id}, A, A^2 \rangle \cap \mathbf{SL}_3(\mathbb{Z}) \subset \mathbb{Z}\langle \mathbf{Id}, A, A^2 \rangle = \mathbb{Z}[A].$$

Indeed,

$$\mathbf{Id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -p^2 \end{pmatrix}, A^2 = \begin{pmatrix} 0 & 1 & -p^2 \\ 0 & 0 & 1 \\ 1 & -p^2 & p^4 \end{pmatrix}, \quad (6)$$

Considering the first matrix column we see that if a complex combination has integer coefficients, it is in fact integer combination.

Moreover, the centralizer of \tilde{A} is a group, therefore it lies inside the multiplicative group of $\mathbb{Z}[A]$

$$Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) \subset \mathbb{C}\langle \mathbf{Id}, A, A^2 \rangle \cap \mathbf{SL}_3(\mathbb{Z}) \subset \mathbb{Z}[A]^*.$$

There is an isomorphism of \mathbb{Z} -algebras $\mathbb{Z}[A] \simeq \mathbb{Z}[x]/(f(x)) = \mathbb{Z}[\alpha]$. Applying Proposition 3.2 end the proof

$$Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) \subset \mathbb{Z}[A]^* = \{\pm A^a(pA + \mathbf{Id})^b \mid a, b \in \mathbb{Z}\}.$$

□

We are to study the centralizer of \tilde{A} in Γ_p .

$$Z_{\Gamma_p}(\tilde{A}) \subset Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) \cong \mathbb{Z}^2.$$

Proposition 4.2. The index of the one centralizer inside the other does not depend on p and equals 3.

$$[Z_{\mathbf{SL}_3(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] = 3.$$

Proof. As $A \pmod{p}$ is equivalent to the cyclic permutation matrix, it follows that $A^3 \in \Gamma_p$.

Thus the smaller centralizer is generated by A^3, \tilde{A}

$$Z_{\Gamma_p}(\tilde{A}) = \{A^{3a}(pA + \mathbf{Id})^b \mid a, b \in \mathbb{Z}\},$$

and it is clear that the index is 3. □

5 Computing (co)homology

Christophe Soulé [4] computed the cohomology of $\mathbf{SL}_3(\mathbb{Z})$ to be

n	0	1	2	3	4	...
$H^n(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z})$	\mathbb{Z}	0	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2$...

The universal coefficient formula gives us the homology.

n	0	1	2	3	...
$H_n(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z})$	\mathbb{Z}	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2$...

Recall the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -p^2 \end{pmatrix}, B = \mathbf{Id} + pA = \begin{pmatrix} 1 & 0 & p \\ p & 1 & 0 \\ 0 & p & 1 - p^3 \end{pmatrix}.$$

We denote \tilde{A} by B for convenience.

As A, B commute they define a map

$$f : \mathbb{Z}^2 \rightarrow \mathbf{SL}_3(\mathbb{Z}),$$

$$(k, l) \mapsto A^k B^l.$$

The goal for us it to understand the map

$$H_2(\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z} \rightarrow H_2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Consider the natural embedding

$$i : \mathbf{SL}_3(\mathbb{Z}) \hookrightarrow \mathbf{SL}_3(\mathbb{R}).$$

Proposition 5.1. *The pullback*

$$i^* : H^2(B\mathbf{SL}_3(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \rightarrow H^2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

is an embedding.

Proof. Consider a pair of matrices

$$C_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathbf{SL}_3(\mathbb{Z}).$$

They give rise to a principle $\mathbf{SL}_3(\mathbb{Z})$ bundle E on \mathbb{T}^2 with non-zero w_2 , the second Stiefel-Whitney class.

Indeed, by the splitting principle we have

$$w(E) = w(E_1)w(E_2)w(E_3) = (1+x)(1+x+y)(1+y) = 1+2(x+y)+(x^2+xy+xy+xy+y^2)+x(x+y)y = 1+xy,$$

where x, y denote the generators of $H^1(\mathbb{T}^2, \mathbb{Z}/2\mathbb{Z})$. Therefore the composition

$$H^2(B\mathbf{SL}_3(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \rightarrow H^2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow H^2(\mathbb{T}^2, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})$$

is nonzero and thus the first map is an embedding. \square

Thus w_2 is one of the generators of the second cohomology of $\mathbf{SL}_3(\mathbb{Z})$. Consider the principal $\mathbf{SL}_3(\mathbb{Z})$ -bundle E on \mathbb{T}^2 associated with A, B .

Proposition 5.2. $w_2(E) = 0$.

Proof. By the splitting principle

$$w(E) = w(E_1)w(E_2)w(E_3) = (1 + x + y)(1 + x + y)(1) = 1.$$

□

There are perfect pairings

$$H_2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}) \times H^2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z},$$

$$H_2(\mathbb{T}^2, \mathbb{Z}) \times H^2(\mathbb{T}^2, \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}.$$

Combining them, we get that the map

$$H_2(\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z} \rightarrow H_2(\mathbf{SL}_3(\mathbb{Z}), \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

has one coordinate equal to zero.

6 From $\mathbf{SL}_3(\mathbb{Z})$ to $\mathbf{SL}_n(\mathbb{Z})$

Consider

- $p > 2$ a prime number;
- $\mathbf{SL}_n(\mathbb{Z})$ the n -th special linear group over \mathbb{Z} ;
- Γ_p the p th congruence subgroup of $\mathbf{SL}_n(\mathbb{Z})$;
- $Z_G(x)$ the centralizer of $x \in G$;
- $f(t) \stackrel{\text{def}}{=} t^n - (1 + pt)(1 + 2pt) \dots (1 + (n - 1)pt)$;
- If C is a matrix, $\chi_C(\lambda) := \det(\lambda \mathbf{Id} - C)$ is the characteristics polynomial.
- Let A be the irreducible operator with $\chi_A(t) = f(t)$;

For example, when $n = 4$ A takes a form

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 6p \\ 0 & 1 & 0 & 11p^2 \\ 0 & 0 & 1 & 6p^3 \end{pmatrix} \in \mathbf{SL}_4(\mathbb{Z}); \quad (7)$$

- Define $\tilde{A} \stackrel{\text{def}}{=} \mathbf{Id} + pA$.

For example, when $n = 4$ \tilde{A} takes a form

$$\tilde{A} = \mathbf{Id} + pA = \begin{pmatrix} 1 & 0 & 0 & p \\ p & 1 & 0 & 6p^2 \\ 0 & p & 1 & 11p^3 \\ 0 & 0 & p & 1 + 6p^4 \end{pmatrix} \in \Gamma_p. \quad (8)$$

7 Intro2

In this note we establish the following results:

1. The regularor of \tilde{A} grows as $O(\ln^{n-1}(p))$;
2. If n is prime, the index of the centralizers $[Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] < C$ is bounded by some constant C and does not depend on p .

This might be interesting.

8 Regulators

We mimic the proofs from a Keigh Conrad's write-up on the Dirichlet unit theorem [1].

There are more general estimates due to Duke [2] (see Section 3 Proposition 1).

Let $K \stackrel{\text{def}}{=} \mathbb{Q}(\alpha)$ be the number field obtain by attaching the root of $f(t)$ to \mathbb{Q} .

Proposition 8.1. K/\mathbb{Q} is a totally real number field of degree n .

Proof. First, $f(t)$ is irreducible as it has no rational roots:

$$f(1) = 1 - (1+p)(1+2p)\dots(1+(n-1)p) \neq 0, \quad (9)$$

$$f(-1) = (-1)^n - (1-p)(1-2p)\dots(1-(n-1)p) \neq 0. \quad (10)$$

Let's compute the roots of $f(t)$ approximately and show they are real.

Notice that for $|t| \ll 1$ all but one roots of $f(t)$ are close to the roots of

$$(1+pt)(1+2pt)\dots(1+(n-1)pt),$$

them being $p^{-1}, \frac{1}{2}p^{-1}, \dots, \frac{1}{(n-1)}p^{-1}$.

As all but one roots are real, the remaining one is also real. \square

The next goal is to tackle the unit group \mathcal{O}_K^* of K .

The Dirichlet unit theorem implies that \mathcal{O}_K^* is an abelian group of rank $n-1$ (there might be some finite torsion of roots of unity. It is bounded universally in p). A basis of \mathcal{O}_K^* modulo roots of unity is called a system of fundamental units. Finding a system of fundamental units in general is a complicated task.

Instead, we shall find a finite index subgroup $U \subset \mathcal{O}_K^*$. Then we shall bound above the regulator of K with the regulator of U .

Moreover, using a Silverman's lower bound on the regulator of K we bound above the index $[\mathcal{O}_K^* : U] < C_1$ with C_1 independent on p .

Define a subgroup

$$U = (1+p\alpha)^{\mathbb{Z}} \dots (1+(n-1)p\alpha)^{\mathbb{Z}} \subset \mathcal{O}_K^*.$$

Proposition 8.2. $U \subset \mathcal{O}_K^*$ is a finite index subgroup. The regulator Reg_U is bounded above by $C_2 \log(p)^{n-1}$ where C_2 is a constant independent on p .

Proof. We have

$$\alpha^n - (1+p\alpha)(1+2p\alpha)\dots(1+(n-1)p\alpha) + 1 = 1, \quad (11)$$

$$(1+p\alpha)(1+2p\alpha)\dots(1+(n-1)p\alpha) = \alpha^n. \quad (12)$$

It shows that $1+p\alpha, \dots, 1+(n-1)p\alpha$ are indeed units.

Let $\alpha_1 < \dots < \alpha_{n-1} < 0$ be the negative roots of f . We shall compute them approximately.

Lemma 8.3.

$$\alpha_k = \frac{-1}{k}p^{-1} + \frac{(-1)^{k+1}}{k \cdot k!(n-k+1)!}p^{-n-1} + O(p^{-2n-1}).$$

Proof. In the first approximation $\alpha_k = \frac{-1}{k}p^{-1} + o(p^{-1})$. Let's compute the residue term more persisely. Let $\alpha_k = \frac{-1}{k}p^{-1}(1+\epsilon)$. Substituting to $f(t)$, we

get

$$\begin{aligned}
(1 + p\alpha_k)(1 + 2p\alpha_k) \dots (1 + (n-1)p\alpha_k) &= \alpha_k^n, \\
-(-kp\alpha_k - k)(-2kp\alpha_k - k) \dots (-(n-1)kp\alpha_k - k) &= (-k\alpha_k)^n, \\
-(1 - \epsilon - k)(2 - 2\epsilon - k) \dots ((n-1) - (n-1)\epsilon - k) &= p^{-n}(1 + \epsilon)^n, \\
-(1 - k)(2 - k) \dots (-1)k\epsilon 1 \dots ((n-1) - k) + O(\epsilon^2) &= p^{-n} + O(\epsilon p^{-n}), \\
(-1)^k k!(n - k - 1)! \epsilon + O(\epsilon^2) &= p^{-n} + O(\epsilon p^{-n}), \\
\epsilon &= \frac{(-1)^k p^{-n}}{k!(n - k - 1)!} + O(\epsilon p^{-n}) + O(\epsilon^2), \\
\epsilon &= \frac{(-1)^k p^{-n}}{k!(n - k - 1)!} + O(p^{-2n}).
\end{aligned}$$

□

By the definition of the regulator we have

$$\begin{aligned}
\text{Reg}_U &= \begin{vmatrix} \ln |1 + p\alpha_1| & \ln |1 + p\alpha_2| & \dots & \ln |1 + p\alpha_{n-1}| \\ \ln |1 + 2p\alpha_1| & \ln |1 + 2p\alpha_2| & \dots & \ln |1 + 2p\alpha_{n-1}| \\ \vdots & \vdots & \ddots & \vdots \\ \ln |1 + (n-1)p\alpha_1| & \ln |1 + (n-1)p\alpha_2| & \dots & \ln |1 + (n-1)p\alpha_{n-1}| \end{vmatrix} \\
&\approx \begin{vmatrix} \ln \left(\frac{p^{-n}}{n!} \right) & \ln \left(1 - \frac{1}{2} \right) & \dots & \ln \left(1 - \frac{1}{n-1} \right) \\ \ln |1 - 2| & \ln \left(\frac{p^{-n}}{2 \cdot 2! (n-1)!} \right) & \dots & \ln \left(1 - \frac{2}{n-1} \right) \\ \vdots & \vdots & \ddots & \vdots \\ \ln |1 - (n-1)| & \ln |1 - \frac{n-1}{2}| & \dots & \ln \left(\frac{p^{-n}}{(n-1) \cdot (n-1)! 2!} \right) \end{vmatrix} \\
&\approx n^{n-1} \log(p)^{n-1}. \quad (13)
\end{aligned}$$

□

From the proof of the Proposition above one finds that the regulator of U $\text{Reg}_U > 0$ is positive for big enough prime p . Hence the generators $(1 + p\alpha), \dots, (1 + (n-1)p)$ of U are multiplicatively independent, and $U \subset \mathcal{O}_K^*$ is a finite index subgroup.

We shall bound the index $[\mathcal{O}_K^* : U]$ from above via the following

Theorem 8.4 (Silverman, [3]). *Let K be a number field of degree d with regulator R , and absolute discriminant D_K . Let $r(K)$ be the rank of the unit group in K , and let $p(K)$ be the maximum of $r(k)$ as k ranges over proper subfields of K . We prove*

$$R_K > c_d (\log \gamma_d D_K)^{r(K) - p(K)}$$

for constants c_d, γ_d depending only on d .

It seems it's actually due to Robert Remak in "Über Größenbeziehungen zwischen Diskriminante und Regulator eines algebraischen Zahlkörpers"

Proposition 8.5. *Let n be a prime. Then the index $[\mathcal{O}_K^* : U] < C_3$ is bounded from above by a constant C_3 independent on p .*

Proof. The discriminant of K is the discriminant of the polynomial $f(t)$, which is approximately

$$\text{disc}_K = \text{disc}_f = O(p^{2(n-1)^2} p^{-(n-1)(n-2)}) = O(p^{n(n-1)}).$$

More precisely,

$$\text{disc}_K = \text{disc}_f \approx 1!^2 \dots (n-1)!^2 p^{n(n-1)}.$$

Given that n is prime and so there are no nontrivial subfield of K the Silberman's theorem implies

$$\text{Reg}_K > C_4 (\log \text{disc}_K)^{n-1} \approx n^{n-1} (n-1)^{n-1} C_4 \log(p)^{n-1} > C_5 \log(p)^{n-1}.$$

It is easy to see that

$$\text{Reg}_U = [\mathcal{O}_K^* : U] \text{Reg}_K.$$

Thus we have

$$C_2 \log(p)^{n-1} > \text{Reg}_U = [\mathcal{O}_K^* : U] \text{Reg}_K > [\mathcal{O}_K^* : U] C_5 \log(p)^{n-1}.$$

It follows that the index is bounded from above

$$[\mathcal{O}_K^* : U] < \frac{C_2}{C_5},$$

as desired. \square

9 Centralizers

Proposition 9.1. *Let n be a prime. Then the index of the centralizers*

$$[Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] < C_3$$

is bounded from above by a constant independent of p .

Proof. We shall exploit the properties of the previously introduced number field $K = \mathbb{Q}(\alpha)$, where α is a root of $f(t) = t^n - (1 + pt) \dots (1 + (n-1)pt)$.

Since A is regular, there is an isomorphism of \mathbb{Z} -algebras

$$\Phi : \mathbb{Z}[A] \xrightarrow{\sim} \mathbb{Z}[\alpha], A \mapsto \alpha.$$

It identifies the stabilizer of \tilde{A} in $\mathbf{SL}_n(\mathbb{Z})$ with the multiplicative group of the order $\mathbb{Z}[\alpha] \subset \mathcal{O}_K$.

$$Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) = Z_{\mathbf{SL}_n(\mathbb{Z})}(A) = \mathbb{Z}[A]^* \xrightarrow{\Phi} \mathbb{Z}[\alpha]^*.$$

The isomorphism Φ identifies the subgroup $U \subset \mathbb{Z}[\alpha]^*$ with

$$\Phi^{-1}(U) = \tilde{A}^{\mathbb{Z}} (2\tilde{A} - \mathbf{Id})^{\mathbb{Z}} \dots ((n-1)\tilde{A} - (n-2)\mathbf{Id})^{\mathbb{Z}} \subset Z_{\Gamma_p}(\tilde{A}).$$

Thus, there is a chain of inclusions

$$\Phi^{-1}(U) \subset Z_{\Gamma_p}(\tilde{A}) \subset Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) = \mathbb{Z}[\alpha]^*.$$

Therefore, there is a bound

$$[Z_{\mathbf{SL}_n(\mathbb{Z})}(\tilde{A}) : Z_{\Gamma_p}(\tilde{A})] \leq [\mathbb{Z}[\alpha]^*, U] \leq [\mathcal{O}_K^*, U] < C_3.$$

\square

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