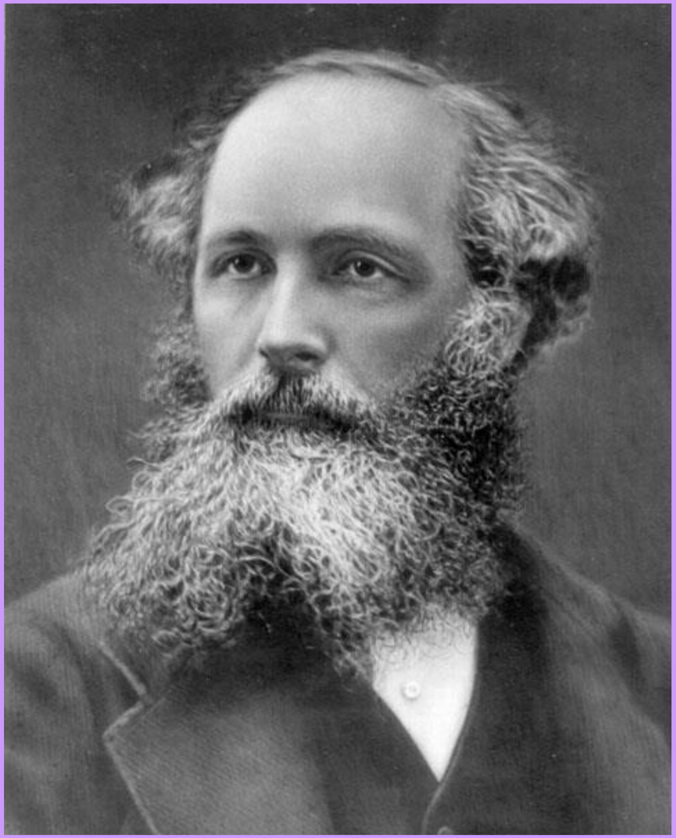
ENGINEERING ELECTROMAGNETICS



MAXWELL

&

HERTZ

THEORY

OF EM

James Clerk Maxwell

Heinrich Rudolf Hertz

"POLITEHNICA" University Of Timişoara, ROMANIA

ENGINEERING ELECTROMAGNETICS

LESSON 1 - REMINDER

MATH FOR ENGINEERING

COORDINATE SYSTEMS

VECTORS AND SCALARS

DIFERENTIAL (INFINITE SMALL) LENGTH, AREA AND VOLUME

INTEGRALS

DEL OPERATOR

GRADIENT, DIVERGENCE, CURL

LAPLACIAN OPERATOR

James Clerk Maxwell

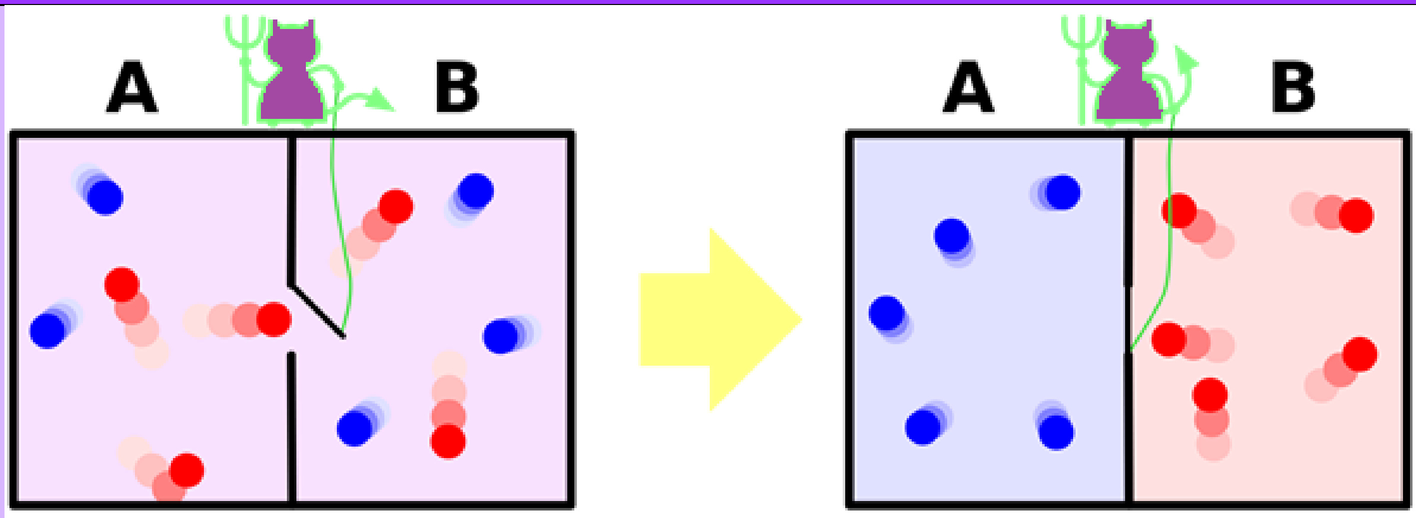
Born 13 June 1831 Edinburgh, Scotland
Died 5 November 1879 (aged 48) Cambridge, England
Citizenship United Kingdom **Nationality** Scottish
Fields Physics and Mathematics
Academic advisors William Hopkins

Known for

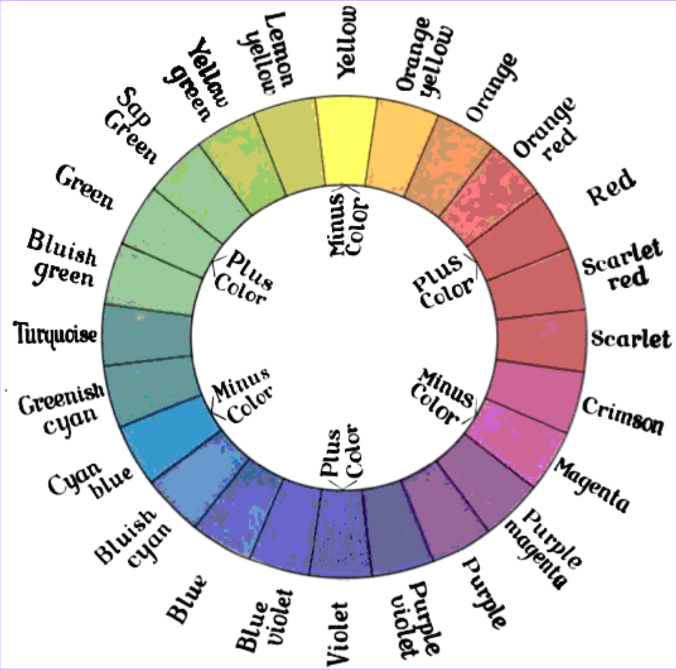
- **Maxwell's equations** and **Displacement current**
- **Maxwell distribution** (particle's speed in gases)
- **Maxwell's demon** ("VIOLATION" OF SECOND PRINCIPLE OF THERMODINAMICS)
- **Maxwell's discs** (colors combinations)
- **Maxwell's theorem** (in probability theory)
- **Generalized Maxwell model** (linear model of viscoelasticity)

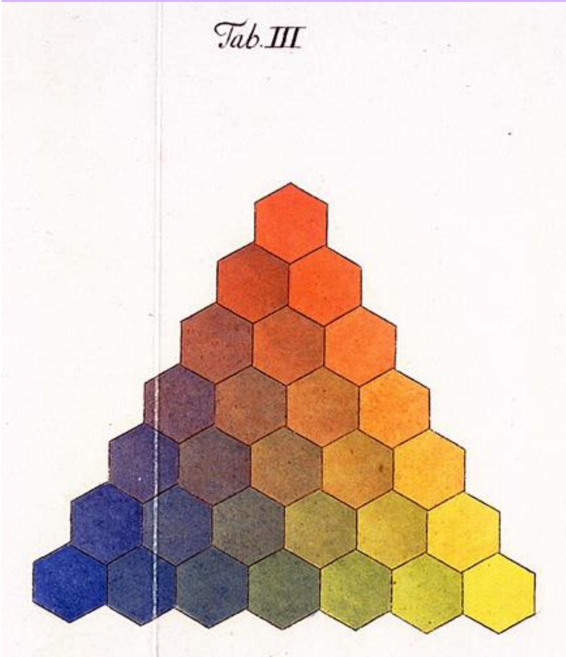
James Clerk Maxwell

Maxwell's demon



Maxwell's discs of colors





Heinrich Rudolf Hertz

Born February 22, 1857 Hamburg, Germany

Died January 1, 1894 (aged 36) Bonn, Germany

Residence Germany **Nationality** German

Fields Physics
Electronic Engineering

Doctoral advisor Hermann von Helmholtz

Known for **Electromagnetic radiation**
Photoelectric effect

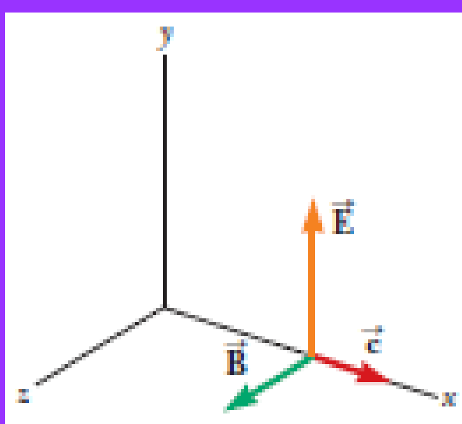
German physicist who showed that Scottish physicist James Clerk Maxwell's theory of electromagnetism was correct and that light and heat are electromagnetic radiations.

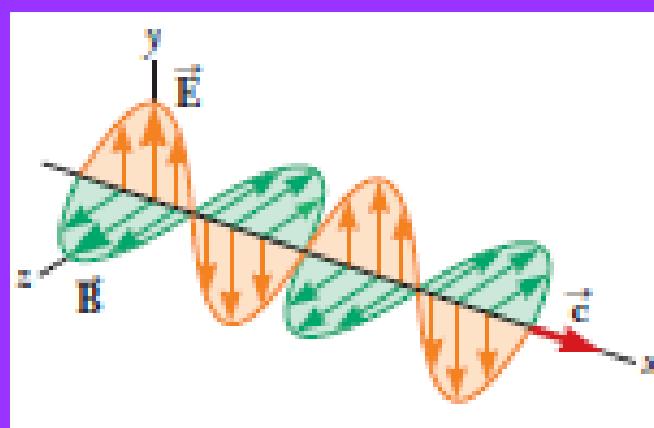
Heinrich Rudolf Hertz

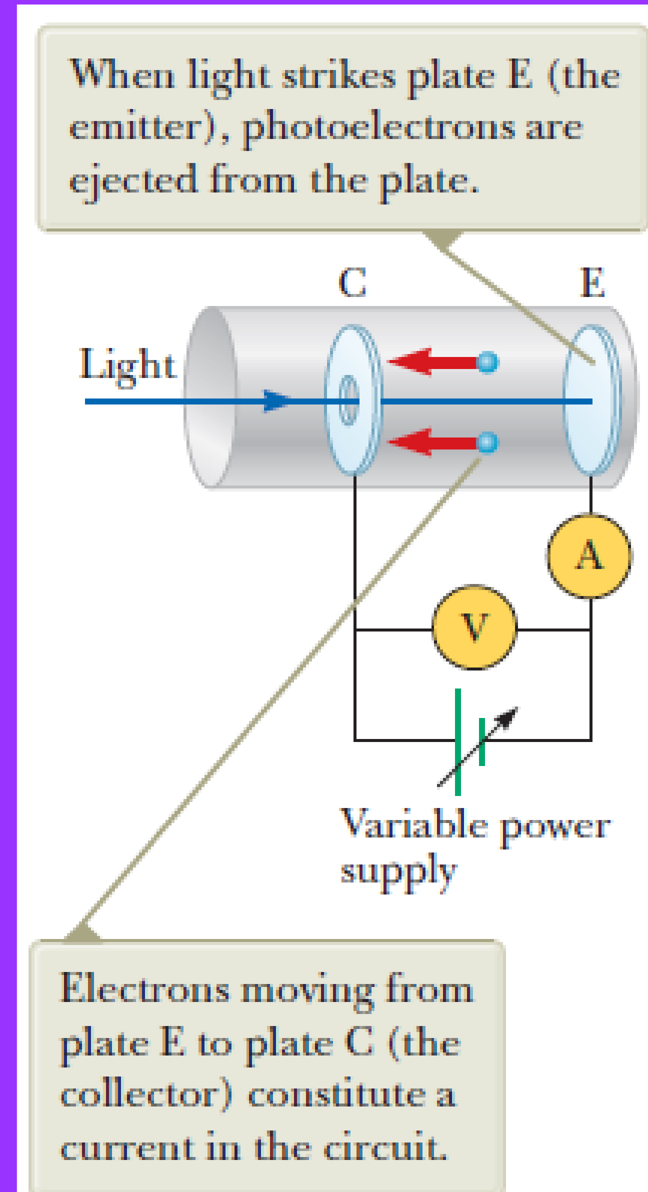
Hertz helped establish the photoelectric effect (which was later explained by Albert Einstein) when he noticed that a charged object loses its charge more rapidly when illuminated by ultraviolet light.

Hertz found that the speed of an electromagnetic wave was the same as that of light and that the electromagnetic waves, like light waves, could be reflected, refracted, and diffracted.

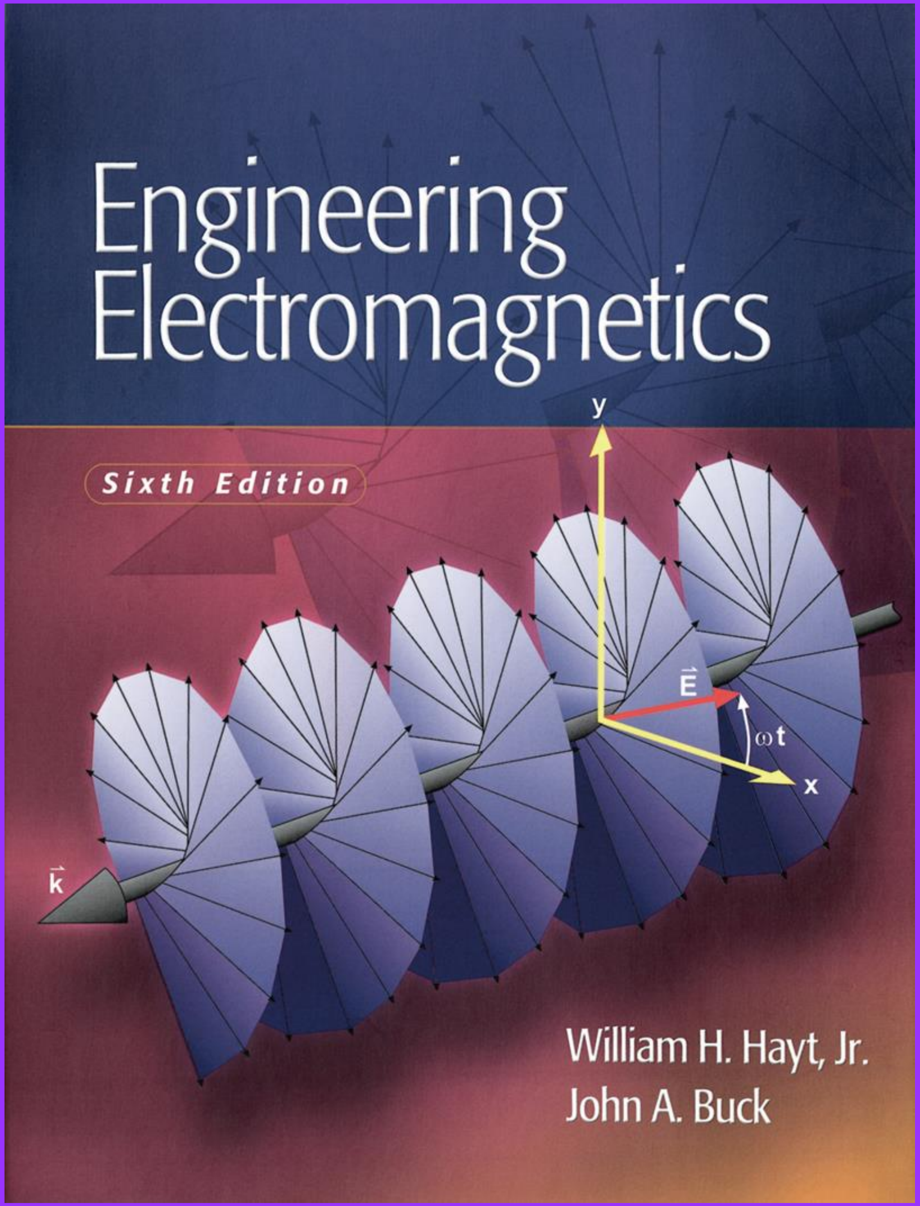
The hertz, [Hz] equal to one complete vibration or cycle per second, is named after him.



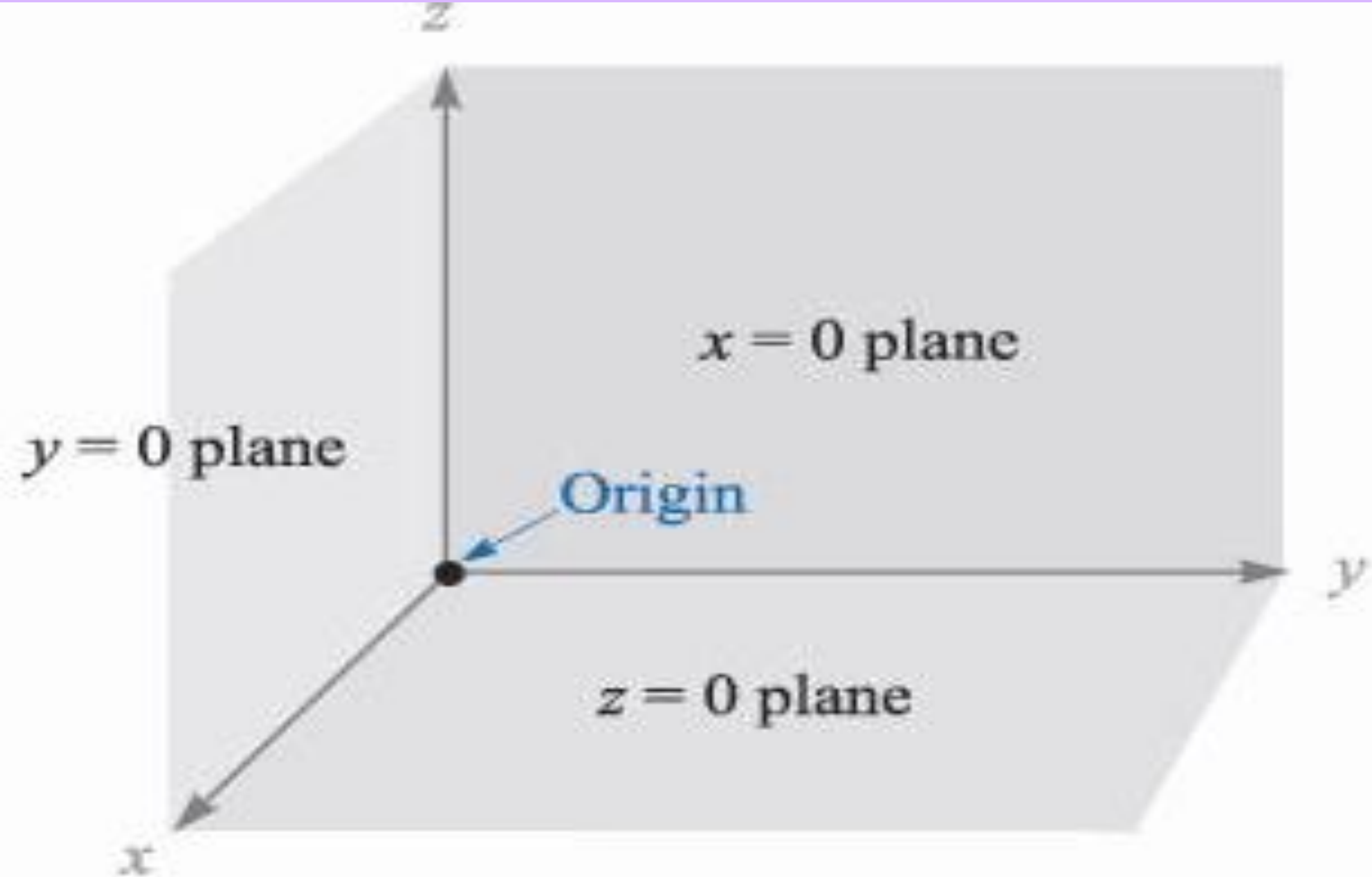




REFFERENCES

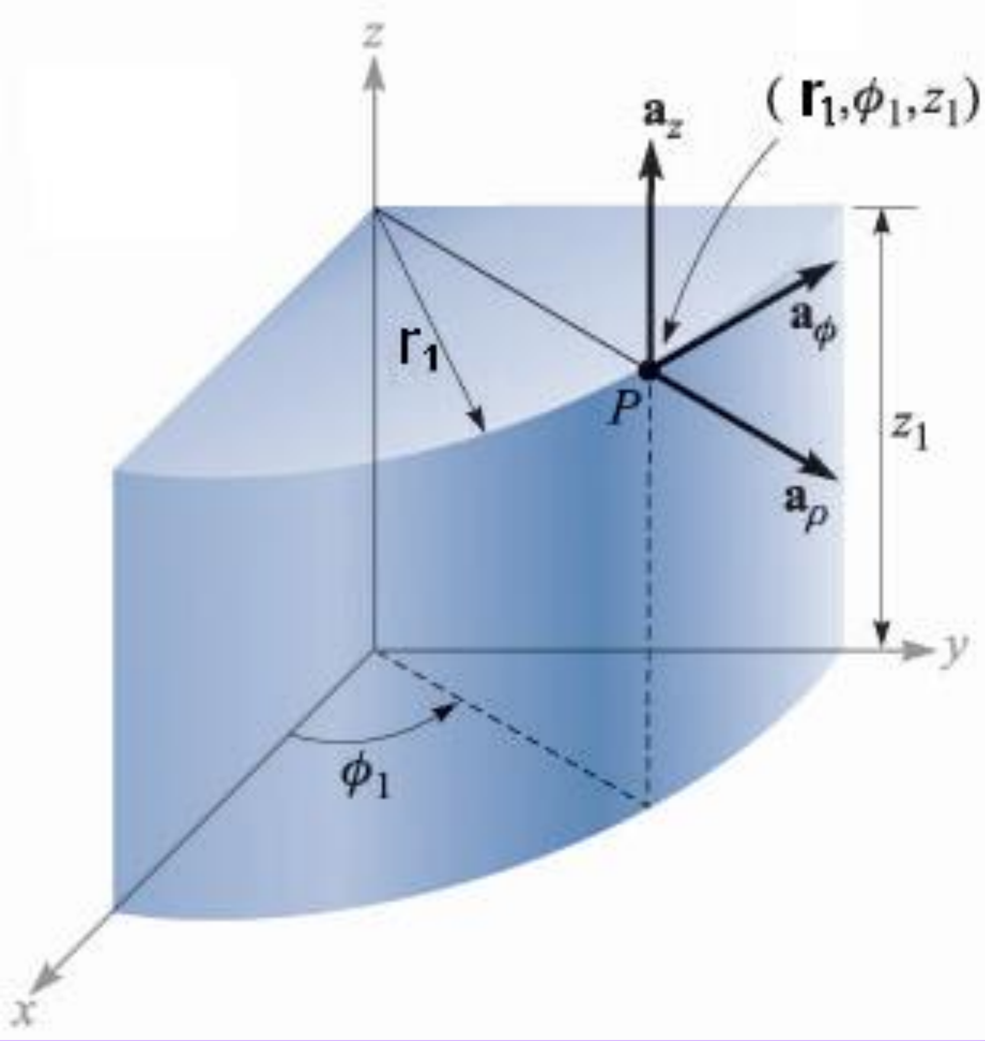


COORDINATE SYSTEMS



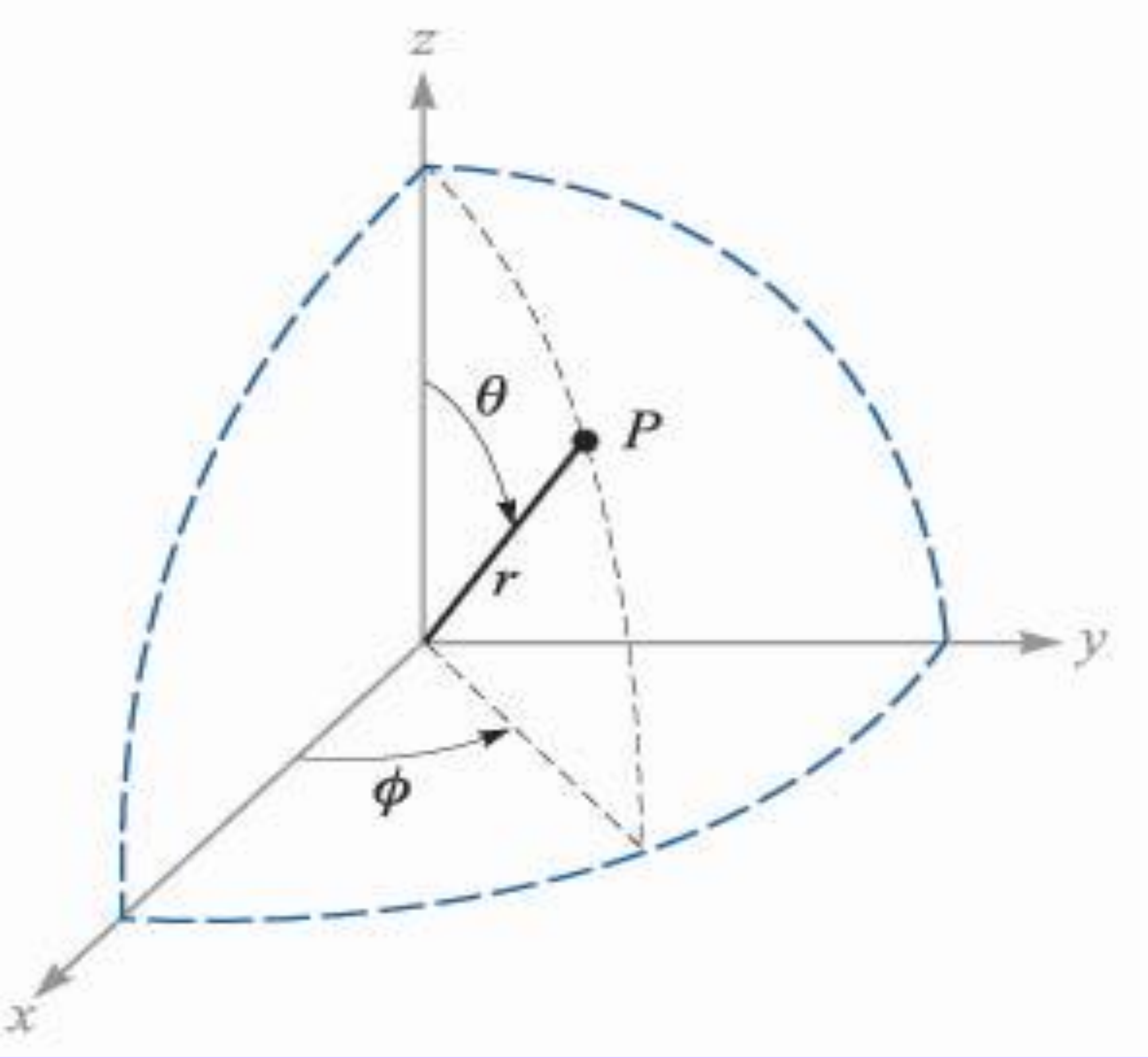
COORDINATE SYSTEMS

Cylindrical system



COORDINATE SYSTEMS

Spherical system



SCALARS AND VECTORS

The volume of water in a swimming pool might be 500 cubic meters, or the winning time of a race could be 11.3 seconds. In cases like these, only the size of the numbers matters. A sign, + or -, and units are required also (see temperatures).

Volume and time are examples of scalar quantities.

A ***scalar quantity*** is one that can be described with a single number (including any units) giving its size or magnitude. Some other common scalars are temperature (e.g., 20 °C, or 68 °F, or 293.15K) , mass (e.g., 85 kg) or mass density (e.g. 1 kg/dm^3).

The information (number, sign and units) for a scalar are quite enough to describe completely the situation quantified by the scalar.

SCALARS AND VECTORS

A scalar is a quantity that has only magnitude, measure unite and a reference.

Some other examples of scalar quantities are: distance, pressure, electric potential (voltage), current (NOT current density!) etc.

If it is known the distribution in a domain of a scalar quantity this can be called as scalar field.

A **vector** is a quantity that has both magnitude and direction; such as : force, velocity (speed), gradient of a scalar, etc.

In order to have information about a vector field we must know the magnitude (module) and the direction of the vector quantity in each point of the domain. It is also possible that a time variation of the vector quantity occurs.

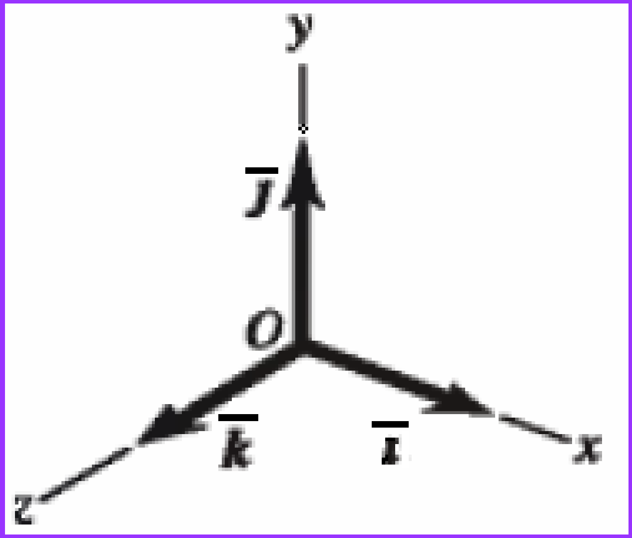
UNIT VECTORS

Vector quantities are often expressed in terms of unit vectors.

A unit vector is a dimensionless vector with a magnitude of 1 and is used to specify a given direction.

Unit vectors have no other physical significance. They are used when describing a direction in space.

The 3 unit vectors form a set of mutually perpendicular unit vectors as shown in figure.



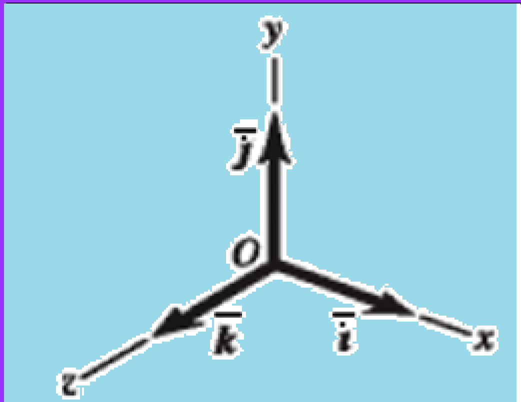
$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\bar{A} = A_x\bar{\imath} + A_y\bar{\jmath} + A_z\bar{k}$$

$$\bar{B} = B_x\bar{\imath} + B_y\bar{\jmath} + B_z\bar{k}$$

UNIT VECTORS



For unit vectors as well as for vectors we shall use the **boldface** notation or the over line notation.

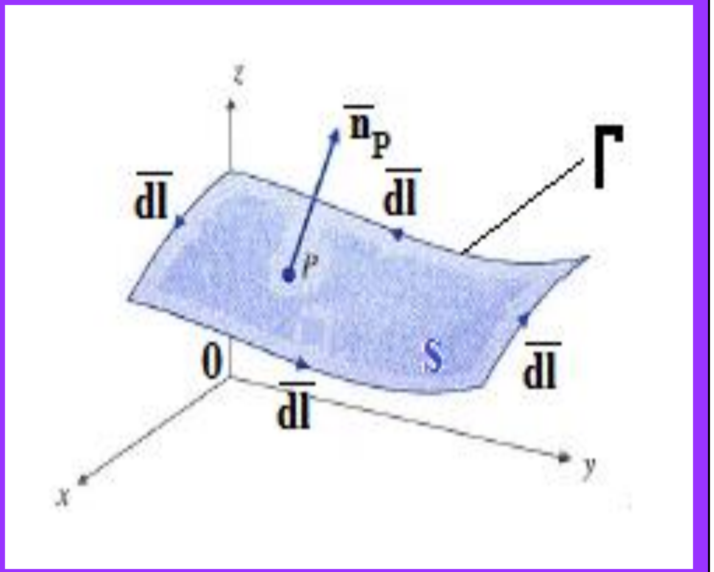
Most common unit vector notation for the Cartesian system is (bold) **i**, **j** and **k**.

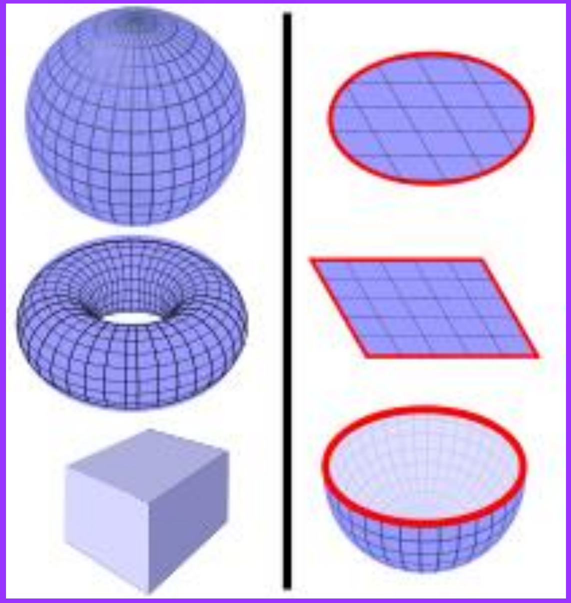
Some books use also (bold or over line) symbols $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, or $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$ for unit vectors along x, y and z axis.

The unit vector direction is in the direction of increasing values of the coordinates.

The unit vector **n** is normal (orthogonal) in a point on the surface S. We will use for elementary surface vector the notation $\mathbf{dS} = dS \cdot \mathbf{n}$, where dS is a scalar equal to the area of the elementary surface . The direction of **n** is chosen by us (some restrictions shall be presented lately).

ORIENTED SURFACE

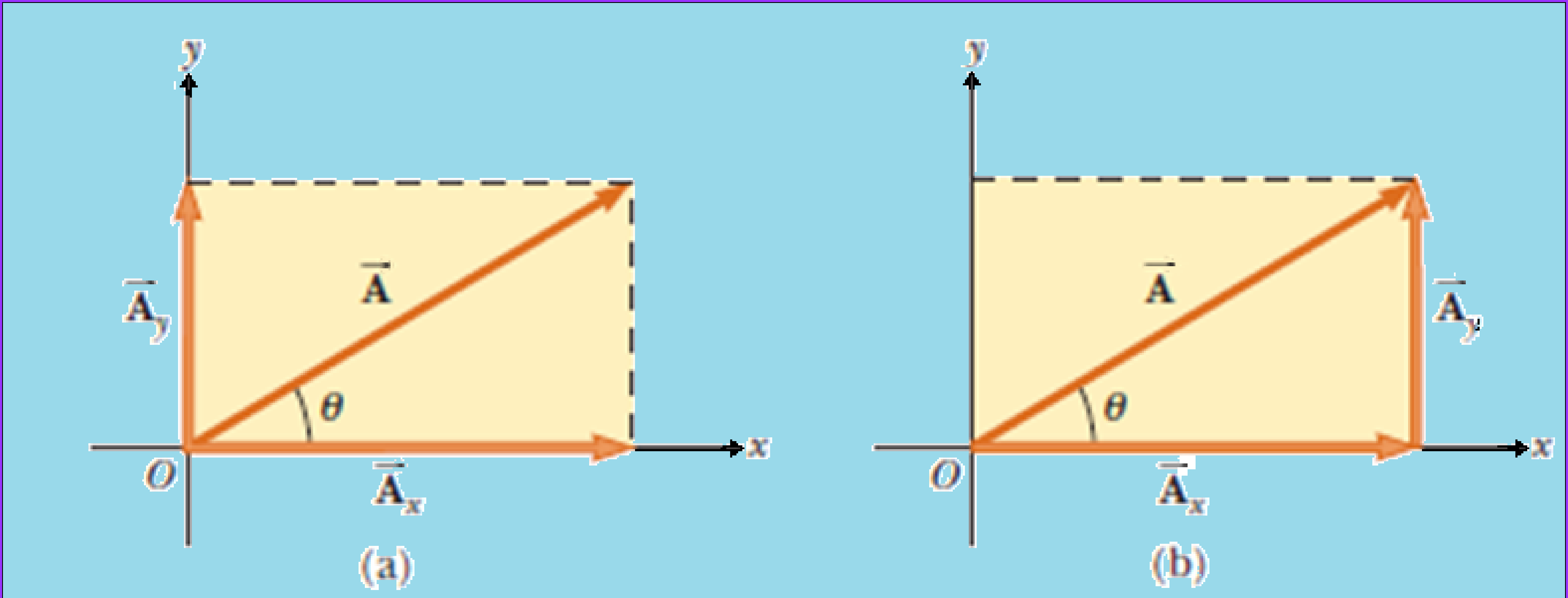




For the open surface S bordered by the contour Γ the unit vector in point P is $\mathbf{n_P}$. The direction of $\mathbf{n_P}$ might be upwards or downward, is connected by the right hand screw rule with **dl**. This is important for the calculations of the flux integrals.

For a closed surface the normal direction is always outwards, as for sphere, torus, cube, cylinder or "potato shape". For open surface the direction of surface unit vector is correlated with the bordering curve (red colored in the drawing).

COMPONENTS OF A VECTOR



The components of the vector can be used for vector's expression using unit vectors:

$$\mathbf{A} = A_x \cdot \mathbf{i} + A_y \cdot \mathbf{j}$$

$$\mathbf{A} = \mathbf{A_x} + \mathbf{A_y} \qquad \mathbf{A_x} = A_x \cdot \mathbf{i} \qquad \mathbf{A_y} = A_y \cdot \mathbf{j}$$

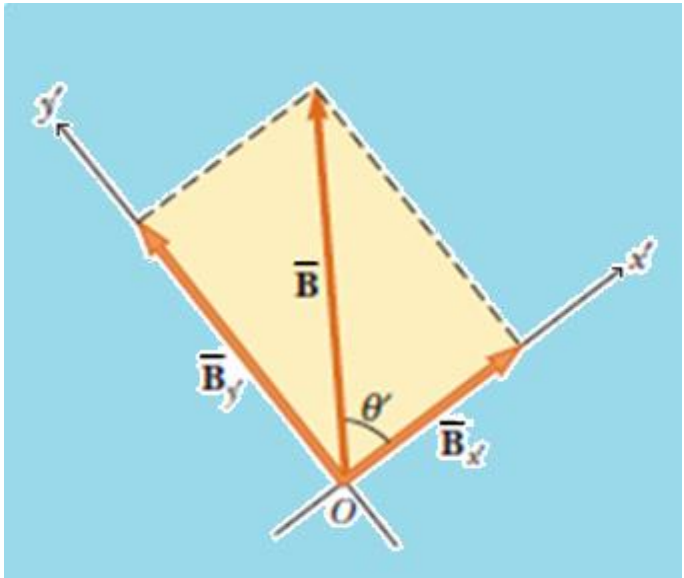
COMPONENTS OF VECTORS

When using these component equations, θ must be measured counterclockwise from the positive x axis. From the triangle, it follows that the magnitude of the vector and its direction are related to its components through the Pythagorean theorem and the definition of the tangent function:

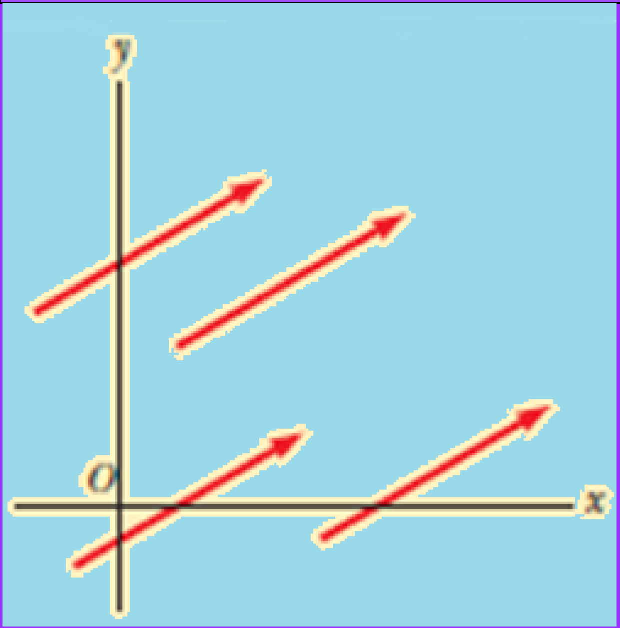
$$A = \sqrt{{A_x}^2 + {A_y}^2} \qquad \tan\theta = \frac{A_y}{A_x}$$

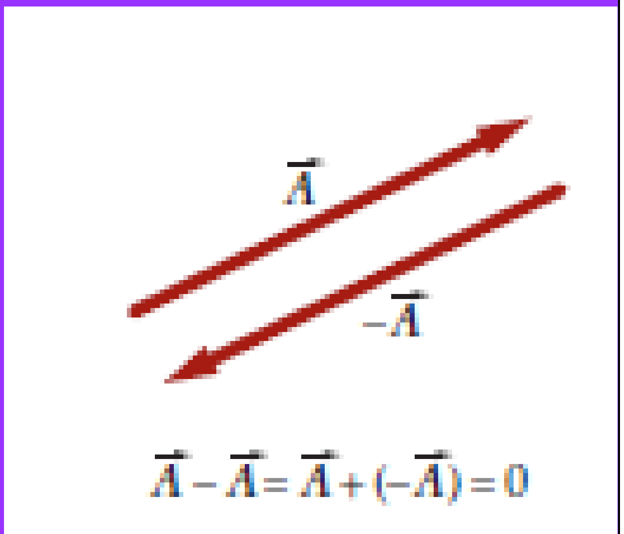
If you choose reference axes or an angle other than those shown previously, the components of the vector must be modified accordingly.

In many applications, it is more convenient to express the components of a vector in a coordinate system having axes that are not horizontal and vertical, but are still perpendicular to each other.



VECTOR PROPERTIES





These four representations of vectors are equal because all four vectors have the same magnitude and point in the same direction.

The difference between them is made by the starting point of the vector.

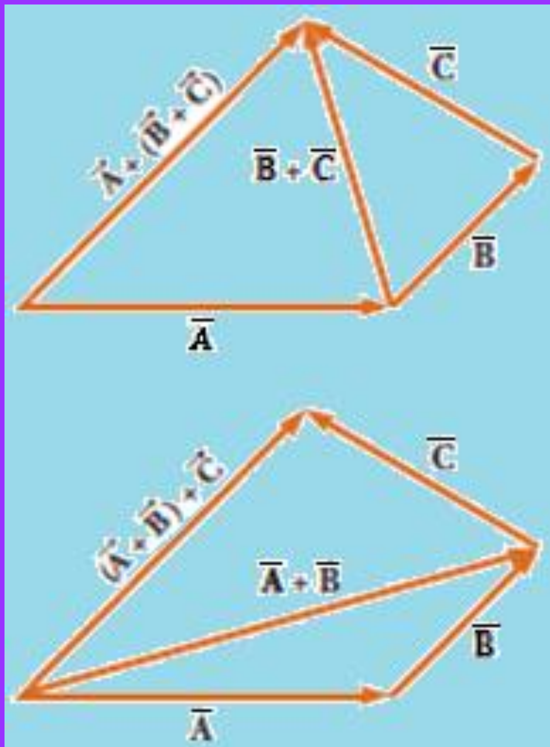
Two vectors are equal only if they have the same module and the same direction.

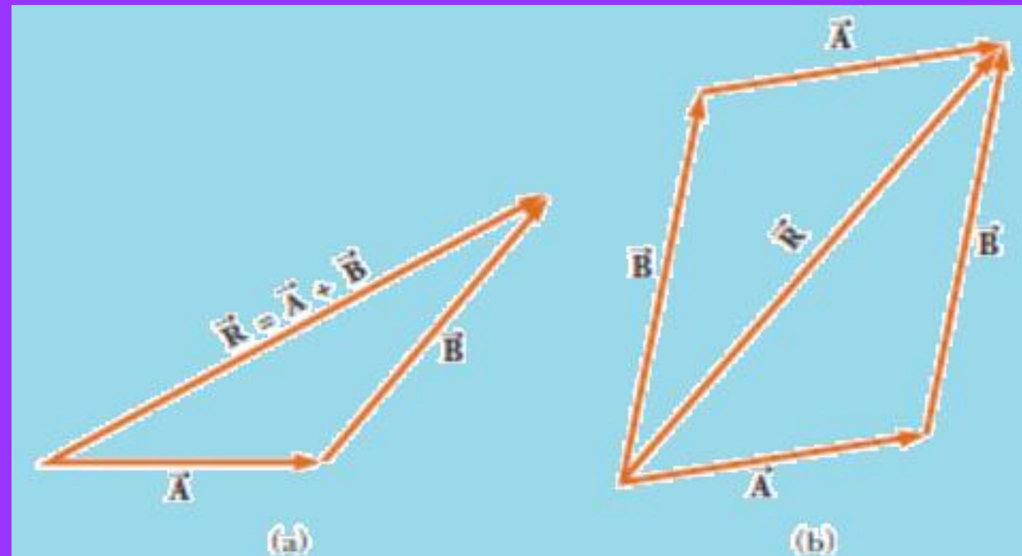
Two parallel vectors of the same magnitude are NOT EQUAL if they have opposite direction.

$$\mathbf{A} + \mathbf{A} = 2\mathbf{A}, \quad \mathbf{A} + (-\mathbf{A}) = 0$$

Adding a vector to its self gives a vector of the same direction but double in module.

VECTOR ADDITION





$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

For the addition of 2 vectors they may commute (change their position in the sum) and the result is the same, **R**.

The addition of the vectors is also associative.

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

MULTIPLICATION OF VECTORS

Multiplication of a Vector by a Scalar

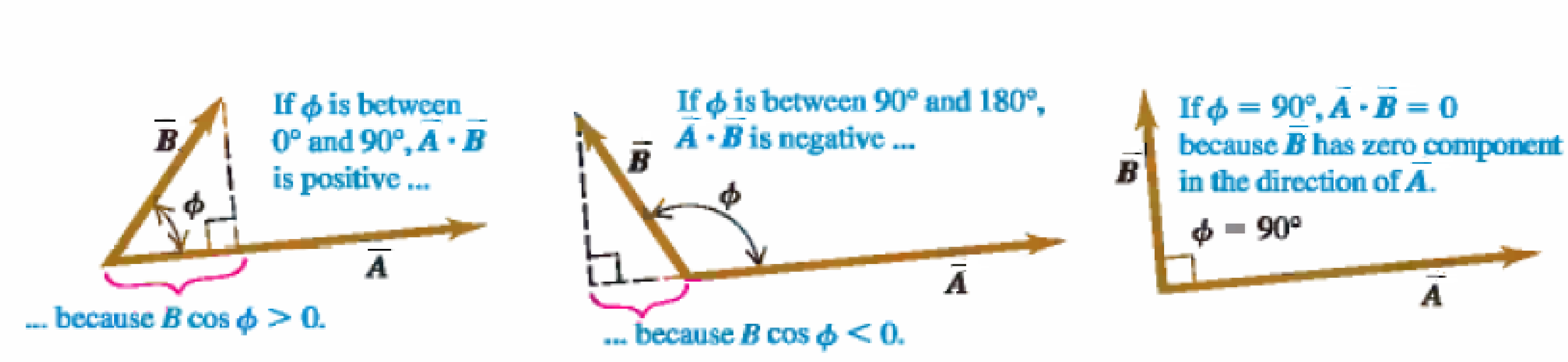
If a vector **F** is multiplied by a positive scalar quantity β, the product β **F** is a vector that has the same direction and the magnitude $|\beta \mathbf{F}|$. If β is a negative scalar quantity, the vector obtained by multiplication with β is directed opposite to the initial vector (because of the negative sign) but has the same module.

Multiplication of Two Vectors

Two vectors can be multiplied in two different ways to produce either a scalar or a vector quantity.

The **scalar product** (or dot product) is a scalar quantity equal to $\mathbf{A}\cdot\mathbf{B} = AB\cos\alpha$, where α is the angle between the two vectors .

SCALAR (DOT) PRODUCT



$$\bar{\imath}\cdot\bar{\imath} = \bar{\jmath}\cdot\bar{\jmath} = \bar{k}\cdot\bar{k} = (1)(1)\cos 0° = 1$$

$$\bar{\imath}\cdot\bar{\jmath} = \bar{\imath}\cdot\bar{k} = \bar{\jmath}\cdot\bar{k} = (1)(1)\cos 90° = 0$$

$$\bar{A}\cdot\bar{B} = A_xB_x + A_yB_y + A_zB_z$$

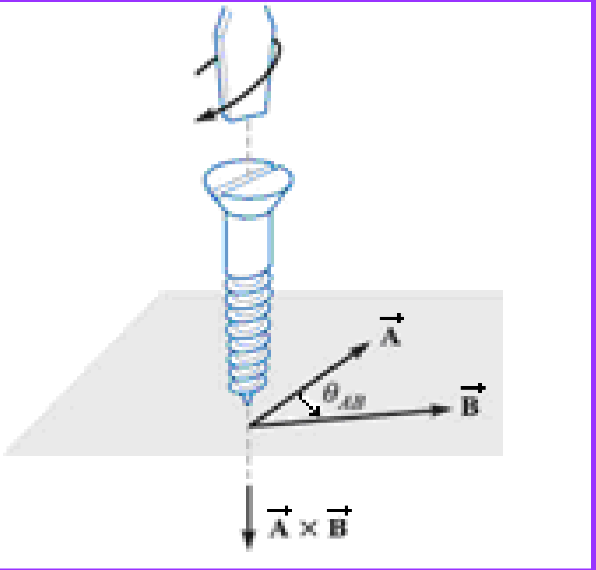
MULTIPLICATION OF VECTORS

The **vector product** (or cross product) is a vector quantity whose magnitude is equal to $AB \sin \theta_{AB}$.

The direction of **A** x **B** is in the direction of advance of a right-handed screw as **A** is turned into **B**.

Reversing the order of the vectors **A** and **B** results a unit vector in the opposite direction. We see that the cross product is not commutative, the resulting vector having the same module, but opposite direction.

A x **B** = - **B** x **A**



$$\overline{\mathbf{C}} = \overline{\mathbf{A}} \times \overline{\mathbf{B}}$$

$$C = |\overline{\mathbf{C}}| = AB \sin\theta$$

$$\overline{\mathbf{C}}' = \overline{\mathbf{B}} \times \overline{\mathbf{A}}$$

$$\overline{\mathbf{C}}' = -\overline{\mathbf{C}}$$

$$|\overline{\mathbf{C}}| = |\overline{\mathbf{C}}'| = AB \sin\theta$$

VECTOR (CROSS) PRODUCT

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0$$

$$\bar{i} \times \bar{j} = -\bar{j} \times \bar{i} = \bar{k}$$
$$\bar{j} \times \bar{k} = -\bar{k} \times \bar{j} = \bar{i}$$
$$\bar{k} \times \bar{i} = -\bar{i} \times \bar{k} = \bar{j}$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\bar{A} \times \bar{B} = (A_y B_z - A_z B_y)\,\bar{i} + (A_z B_x - A_x B_z)\bar{j} + (A_x B_y - A_y B_x)\bar{k}$$

The result of the cross product of two vectors is also a vector.

The relations above show the calculation of the resulting vector after a cross product, using vector's components.

CONVENTIONS

For a vector quantity the symbol for a vector, lets say force, can be: **F**, the boldface type.

In most of the references recommended during this course, vectors will be indicated in text by boldface type, as **F** in the previous example, and sometimes (in equations or in drawings, for example) with symbol of the letter over lined.

The simple letter, symbol of the vector quantity ("unbolded" or "not over lined") means the module of that vector quantity. The module of the vector is called also magnitude.

If **a** is the vector acceleration, as **a** = 3**i** + 4**j**, the magnitude of this acceleration is a = $5 m/s^2$ and the components are

$a_x = 3m/s^2$ and $a_y = 4m/s^2$

Differential length, aria and volume

Differential displacement:

$$\mathbf{dl} = dx\mathbf{a_x} + dy\mathbf{a_y} + dz\mathbf{a_z}$$

Differential volume (scalar)

$$dv = dx \cdot dy \cdot dz$$

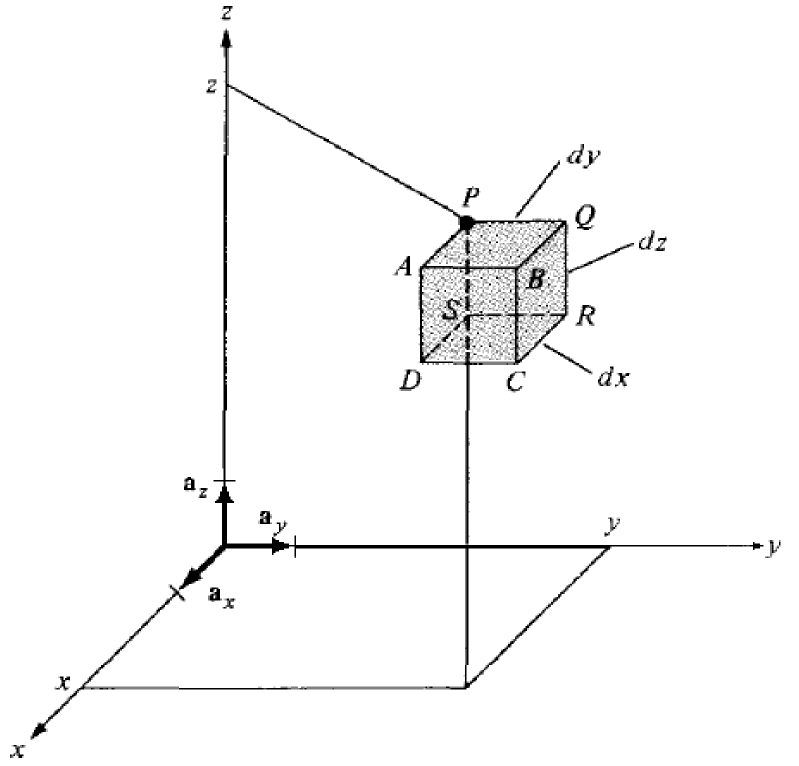
Differential normal areas

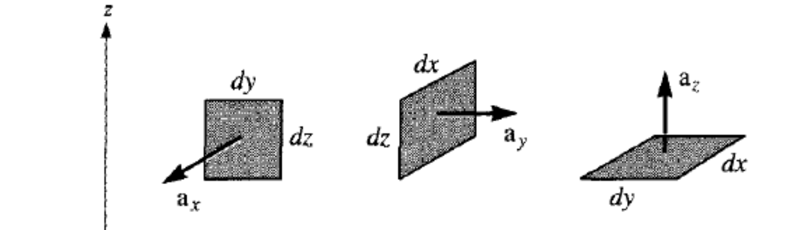
$$d\mathbf{S} = dydz\mathbf{a_x}$$

or $\quad dxdz\mathbf{a_y}$

or $\quad dxdy\mathbf{a_z}$

$$\mathbf{dS} = dS \cdot \mathbf{a_n}$$





(a) (b) (c)

INTEGRALS

The line integral of vector **A** along a path *C* is given by

$$\int_C \overline{A} \cdot \overline{d}\overline{l} \qquad \textbf{or} \qquad \int_{\mathbf{C}} \mathbf{A} \cdot \mathbf{dl} \; .$$

If the path is closed, the line integral becomes the circulation of **A** around the closed curve Γ, $\oint_{\Gamma} \overline{A} \cdot d\overline{l}$

The flux or surface integral of a vector **A** across a surface *S* is defined as $\int_s \mathbf{A} \cdot \boldsymbol{dS}$.

When the surface *S* is closed, the surface integral becomes the net outward flux of **A** across the closed surface Σ .

$$\oint_{\Sigma} \overline{A} \cdot d\overline{S} = \int_{\Sigma} \mathbf{A} \cdot \boldsymbol{dS}$$

The volume integral of a scalar ϱ_v over a volume v is defined as $\int_{\mathrm{v}} \varrho_v dv$.

DEL operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

The vector differential operator is not a vector itself, but when it operates on a scalar function results a vector and when operates (as dot product) on a vector, results a scalar.

Below you can see four results by using (operating with) the "del" operator:

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector $\mathbf{A}$, written as $\nabla \cdot \mathbf{A}$
3. The curl of a vector $\mathbf{A}$, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

Gradient of a scalar

$$\left.\frac{dV}{dl}\right|_{\max} = \frac{dV}{dn} = G$$

where dV/dn is the normal derivative.

G (vector) has its magnitude G and direction is that of the maximum rate of change of V.

By definition **G,** vector, is the gradient of V:

$$\text{grad } V = \nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

Gradient of a scalar

1. The magnitude of ∇V equals the maximum rate of change in V per unit distance.
2. ∇V points in the direction of the maximum rate of change in V.
3. ∇V at any point is perpendicular to the constant V surface that passes through that point
4. The projection (or component) of ∇V in the direction of a unit vector $\mathbf{a}$ is $\nabla V \cdot \mathbf{a}$ and is called the *directional derivative* of V along $\mathbf{a}$. This is the rate of change of V in the direction of $\mathbf{a}$. Thus the gradient of a scalar function V provides us with both the direction in which V changes most rapidly and the magnitude of the maximum directional derivative of V.
5. If $\mathbf{A} = \nabla V$, V is said to be the scalar potential of $\mathbf{A}$.

Given $\Phi = xy + yz + xz$, find gradΦ at point (1, 2, 3).

Answer: $\quad \text{grad}\Phi(1,2,3) = 5\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$

Note that is not $\Phi(1,2,3)$, but grad$\Phi(1,2,3)$!!!

Divergence of a vector

The divergence of a vector **A** at a given point P is the outward flux per unit volume as the volume shrinks about P.

$$\operatorname{div} \bar{\mathbf{A}} = \nabla \cdot \bar{\mathbf{A}} = \lim_{\Delta v \to 0} \frac{\oint_{\Sigma} \bar{\mathbf{A}} \cdot \overline{d\mathbf{S}}}{\Delta v}$$

Divergence of a vector

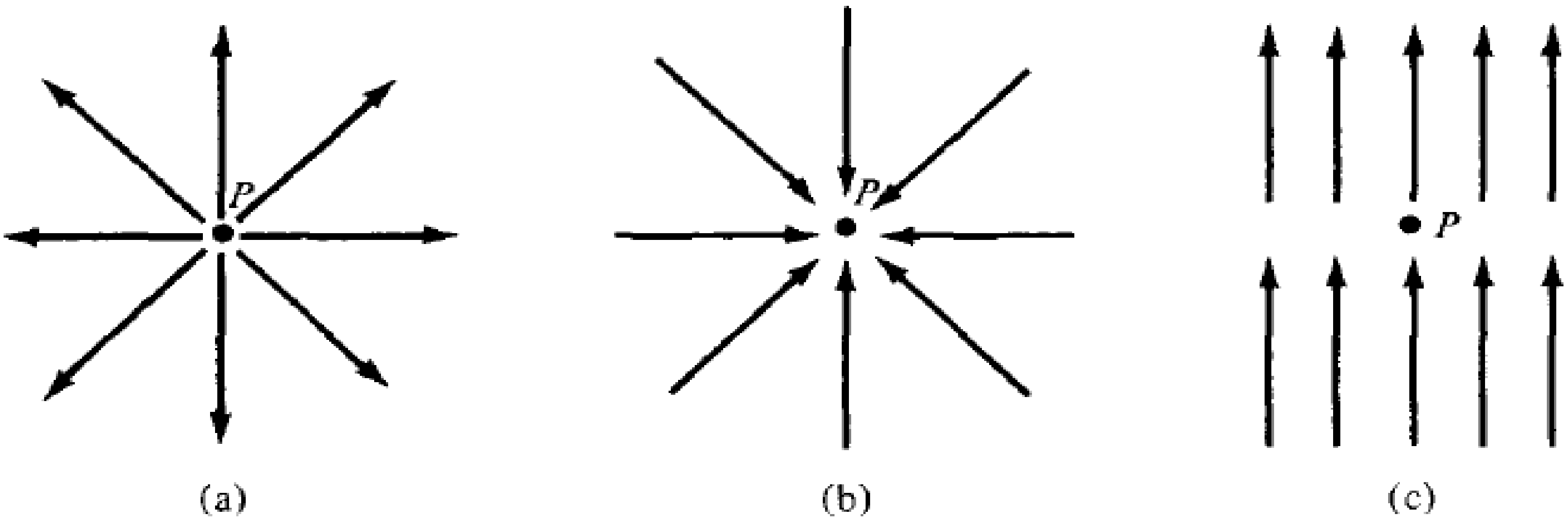


Figure Illustration of the divergence of a vector field at P; **(a)** positive divergence, **(b)** negative divergence, **(c)** zero divergence.

Divergence of a vector

In Cartesian system of coordinates:

$$\nabla \cdot \bar{\mathbf{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The divergence is applied to a vector field and the result is a scalar , normally as result of a dot product.

$$\mathrm{div}(\mathbf{A}+\mathbf{B}) = \mathrm{div}\mathbf{A} + \mathrm{div}\mathbf{B}$$

$$\mathrm{div}(V\mathbf{A}) = V\mathrm{div}\mathbf{A} + \mathbf{A} \cdot \mathrm{grad}V$$

Gauss – Ostrogradsky theorem

$$\oint_{\Sigma} \bar{\mathbf{A}} \cdot \overline{d\mathbf{S}} = \int_{v} \nabla \cdot \bar{\mathbf{A}}\, dv$$

The divergence theorem is also known as the Gauss – Ostrogradsky theorem.

The divergence theorem states that the total outward flux of a vector field **A** through the closed surface ∑ is the same as the volume integral (on the volume bordered by the closed surface ∑) of the divergence of **A**.

Application

Determine the divergence of the vector field

$$\mathbf{P} = x^2yz\boldsymbol{i} + xz\boldsymbol{k}$$

$$\nabla \cdot \mathbf{P} = \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$

$$\nabla \cdot \mathbf{P} = 2xyz + x$$

Determine the divergence and evaluate in a given point:

$$\mathbf{A} = yz\boldsymbol{i} + 4xy\boldsymbol{j} + y\boldsymbol{k} \text{ at } (1,-2,3)$$

$$\text{div}\mathbf{A} = 0 + 4x + 0 = 4x$$

$$\text{div}\mathbf{A}\ (1,-2,3) = 4\cdot 1 = 4$$

Note that it is not div**A**(1,-2,3) but div**A**(1,-2,3) !!!!

Curl of a vector

The curl of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area a when the area is orientated so as to make the circulation on Γ maximum.

Some authors use rot**A** instead of curl**A** .

$$\operatorname{curl} \overline{\mathbf{A}} = \nabla \times \overline{\mathbf{A}} = \left(\lim_{\Delta S_\Gamma \to 0} \frac{\oint_\Gamma \overline{\mathbf{A}} \cdot \overline{d\mathbf{l}}}{\Delta S_\Gamma} \right)_{\max} \overline{\mathbf{a}}_n$$

Curl of a vector

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

The curl of a vector is another vector field.

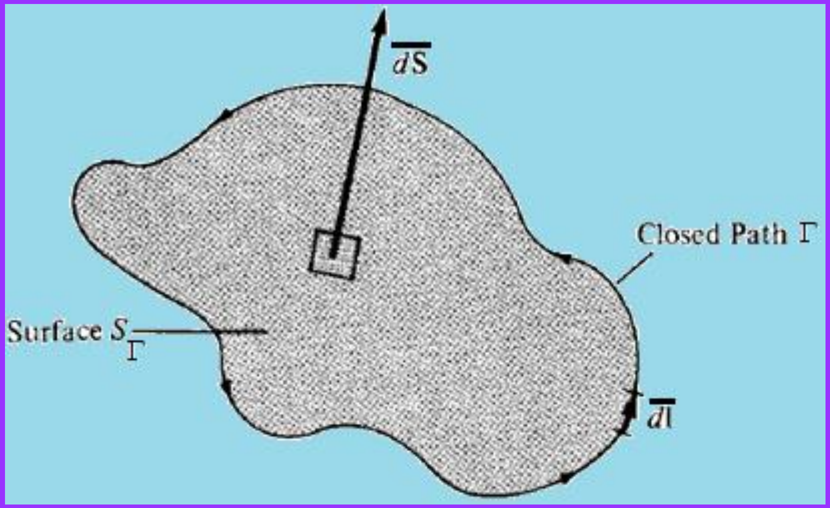
curl(**A**+**B**) = curl**A** + curl**B**

curl(V**A**) = Vcurl**A** + gradV x **A**

div(curl**A**) = 0

curl(gradV) = 0

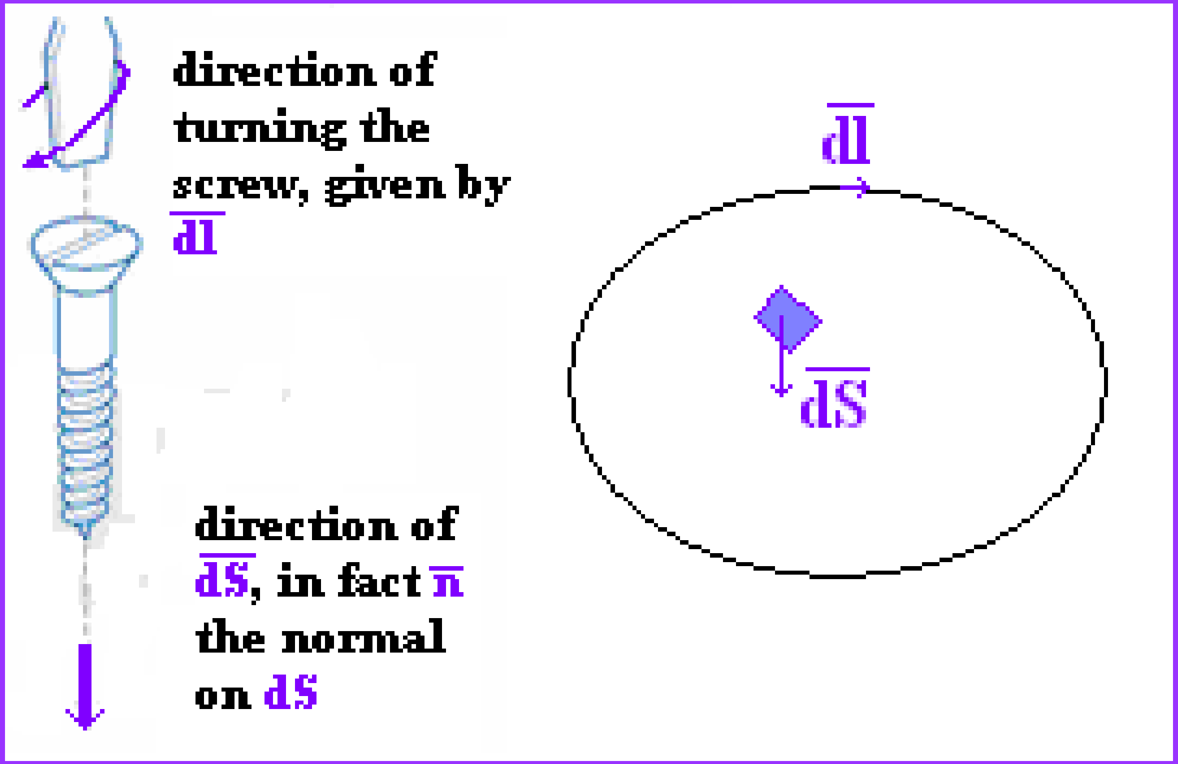
Stoke's theorem



$$\oint_\Gamma \mathbf{A} \cdot d\mathbf{l} = \int_{S_\Gamma} (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Stoke's theorem states that the circulation of a vector field **A** around a closed path Γ is equal to the surface integral of the curl of **A** over the open surface S bounded by Γ, but only when both **A** and curl**A** are continuous on S.

Connection of **dl** with **dS**



The direction of **dl** and **dS** must be chosen using the right-handed screw rule.

dS has the same direction as that normal on the surface having the same direction as the vertical movement of the screw, turned according to the direction of **dl**.

Laplacian of a scalar

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Usually the operation $\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared")

For practical reasons was introduced a single operator which is the composite of gradient and divergence. It is known as Laplacian.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Since the Laplacian operator is a scalar operator it is also possible to define the Laplacian of a vector.

$$\nabla^2 \mathbf{A} \equiv \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z$$