3D Object Representation

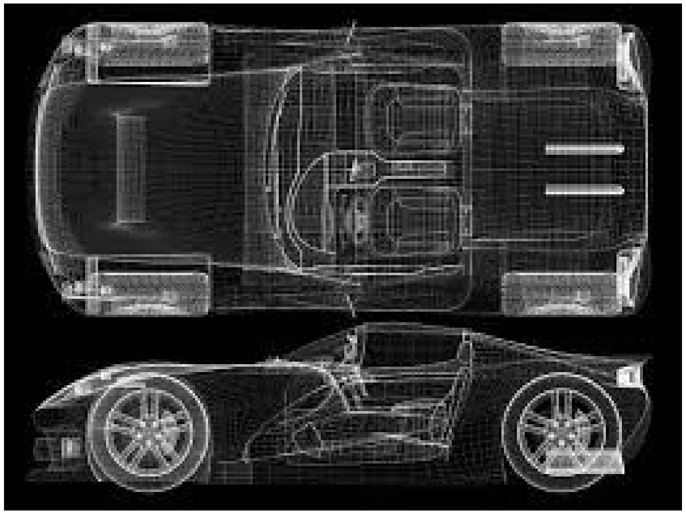
Remaining parts

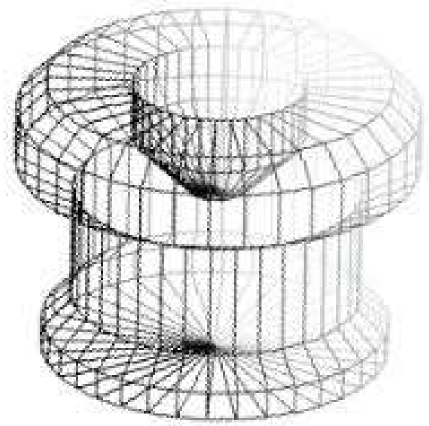
Topics

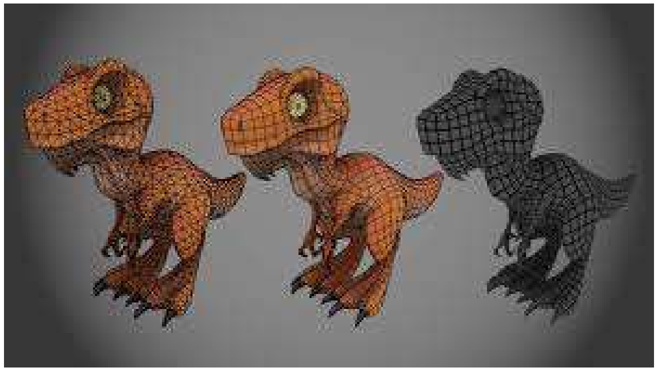
- Curves and surfaces
- Parametric Cubic Curves
  - Spline Representation,
  - Hermite Curve,
  - Bezier Curve and surfaces
- Quadratic Surfaces (Sphere, Ellipsoid)

Wireframe Representation

- Visual representation of a 3-Dimesional or physical representation in 3D computer graphics.
- Created by specifying each edge of the physical object.
- Shows the skeletal structure of the objects.
- Represents the shape of a solid object with its characteristic lines and points







Quadric Surfaces

- Regular curved surfaces can be generated as Quadric Surfaces.
- Frequently used class of objects, which are described with second degree equations (quadratics)
- Examples:
  - Spheres
  - Ellipsoids,
  - Tori,
  - Paraboloids
  - hyperboloids etc.

Quadric Surfaces include:

1. Sphere: For the set of surface points (x, y, z) the spherical surface is represented as: $x^2+y^2+z^2 = r^2$, with radius r and centered at co-ordinate origin.
2. Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where (x, y, z) is the surface point and a, b, c are the radii on X, Y and Z directions respectively.
3. Elliptic parboiled: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$
4. Hyperbolic parboiled : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$
5. Elliptic cone : $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$
6. Hyperboloid of one sheet: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
7. Hyperboloid of two sheet: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

Blobby Objects

- Non-rigid objects that don't maintain a fixed shape but change their surface characteristics in certain motions are referred to as blobby objects. Blobby objects are used to generate irregular surfaces.
- E.g. molecular structures, water droplets, melting objects, muscle shaped in human body, cloth, rubber, liquids, water droplets, etc.

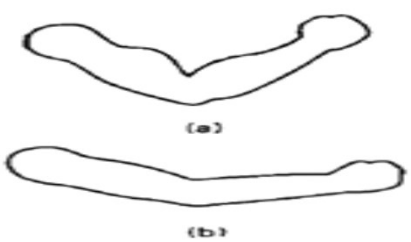


Figure: Blobby muscle shapes in a human arm

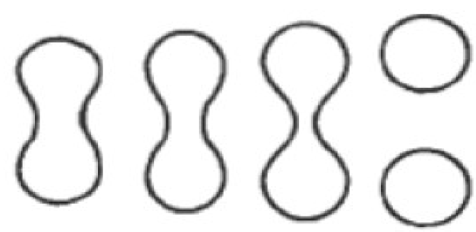


Figure: Molecular bonding. As two molecules move away from each other, the surface shapes stretch, snap, and finally contract into spheres

- Among several techniques, combination of Gaussian density functions, or bumps is used.
- A surface function is defined as:

$$f(x, y, z) = \sum_k b_k\, e^{-a_k r_k^2} - T = 0$$

$$\text{Where } r_k = \sqrt{x_k^2 + y_k^2 + z_k^2}\,,\quad T = \text{Threshold,}$$

- a and b are used to adjust amount of blobbiness of the individual objects.

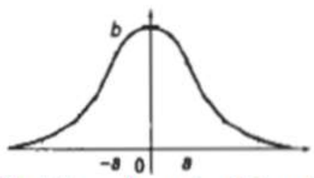


Fig: A three-dimensional Gaussian bump centered at position 0, with height band standard deviation a.

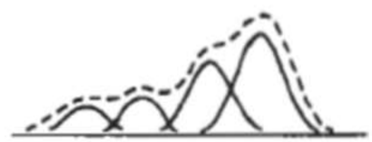


Fig: A composite blobby object formed with four Gaussian bumps

Other method for generating blobby objects uses quadratic density function as:

$$f(r) = \begin{cases} b(1-3r^2/d^2), & \text{if } 0 < r \le d/3 \\ (3/2)b(1-r/d)^2, & \text{if } d/3 < r \le d \\ 0, & \text{if } r > d \end{cases}$$

- **Advantages**
- Can represent organic, blobby or liquid line structures.
- Suitable for modeling natural phenomena like water, human body.
- Surface properties can be easily derived from mathematical equations.

- **Disadvantages**
- Requires expensive computation.
- Requires special rendering engine.
- Not supported by most graphics hardware.

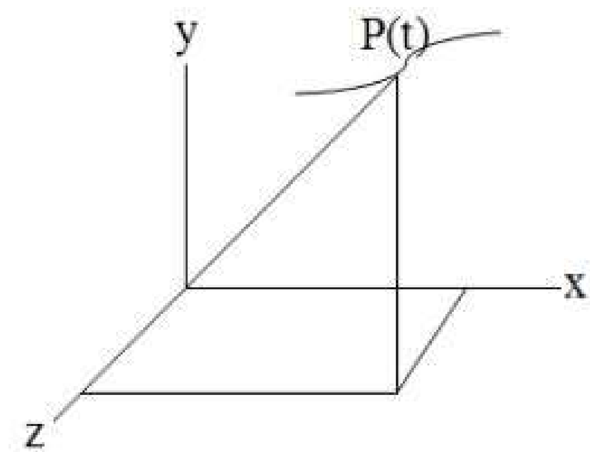
Curves and surfaces

- Curves and surface equations can be expressed in either a parametric or a non parametric form.
- A parametric surface is defined by equations that generate vertex coordinates as a function of one or more free variables.
- In the one-dimensional case it is customary to define parametric curves (e.g. Bezier, Lissajous, or any of several other types) of curves using free variable *t* often defined on the interval [0,1] which can be thought of as a sort of fractional arc length. An equation is specified which generates each coordinate value as a function of *t*. As a result, the curve can be rendered to arbitrary precision by evaluating as many vertex points as desired along the defined interval of *t* values.
- The alternative is a nonparametric curve which is simply defined as a specific set of vertices which are generally connected with straight lines. Curves defined nonparametrically don't hold up well to scaling and zooming as eventually the limitations of the defining geometry become apparent.
- Parametric surfaces are the higher-dimensional equivalents of parametric curves, where two or more free variables and corresponding functions define the vertices of a mesh.

Parametric Cubic curve

A parametric cubic curve is defined as $P(t) = \sum_{i=0}^{3} a_i t^i \qquad 0 <= t <= 1$ ------ (i)

Where, P (t) is a point on the curve



Expanding equation (i) yield

$P(t) = a_3t^3 + a_2t^2 + a_1t + a_0$------------------------------ (ii)

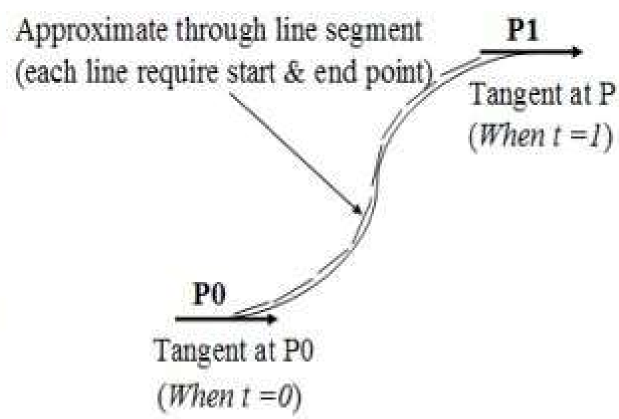
This equation is separated into three components of P (t)

$x(t) = a_{3x}t^3 + a_{2x}t^2 + a_{1x}t + a_{0x}$

$y(t) = a_{3y}t^3 + a_{2y}t^2 + a_{1y}t + a_{0y}$

$z(t) = a_{3z}t^3 + a_{2z}t^2 + a_{1z}t + a_{0z}$ ---------------------(iii)

- To be able to solve (iii) the twelve unknown coefficients a_{ij} (algebraic coefficients) must be specified
- From the known end point coordinates of each segment, six of the twelve needed equations are obtained.
- The other six are found by using tangent vectors at the two ends of each segment
- The direction of the tangent vectors establishes the slopes (direction cosines) of the curve at the end points
- This procedure for defining a cubic curve using ***end points*** and ***tangent vector*** is one form of **Hermite interpolation**
- Each cubic curve segment is parameterized from 0 to 1 so that known end points correspond to the limit values of the parametric variable t, that is P(0) and P(1)
- Substituting $t = 0$ and $t = 1$ the relationship between two end point vectors and the algebraic coefficients are found



$$\mathbf{P}\ (0) = \mathbf{a_0} \qquad\qquad \mathbf{P}\ (1) = \mathbf{a_3} + \mathbf{a_2} + \mathbf{a_1} + \mathbf{a_0} \text{ --------------------- (IV)}$$

- To find the tangent vectors equation (ii) must be differentiated with respect to t

$$\mathbf{P'}\ (t) = 3a_3t^2 + 2a_2t + a_1$$

- The tangent vectors at the two end points are found by substituting t = 0 and t = 1 in this equation

$$\mathbf{P'}\ (0) = a1 \qquad\qquad \mathbf{P'}\ (1) = 3a_3 + 2a_2 + a_1 \text{---------------------} \ \text{(V)}$$

- The algebraic coefficients 'a_i' in equation (ii) can now be written explicitly in terms of boundary conditions – endpoints and tangent vectors are

$$a_0 = P\ (0) \qquad\qquad a_1 = P'\ (0)$$

$$a_2 = -3\ P\ (0) - 3\ P\ (1) - 2\ P'\ (0) - P'\ (1) \qquad\qquad a3 = 2\ P\ (0) - 2\ P\ (1) + P'\ (0) + P'\ (1)$$

(Note: - The value of a2 & a3 can be determined by solving the equation IV & V)

- Substituting these values of 'a_i' in equation (ii) and rearranging the terms yields

$$P(t) = (2t^3 - 3t^2 + 1)\ P(0) + (-2t^3 + 3t^2)\ P(1) + (t^3 - 2t^2 + t)\ P'(0) + (t^3 - t^2)\ P'(1)$$

Spline Representations

- A Spline is a flexible strip used to produce smooth curve through a designated set of points. A curve drawn with these set of points is spline curve.
- Spline curves are used to model 3D object surface shape smoothly.
- Mathematically, spline are described as piece-wise cubic polynomial functions satisfying certain boundary condition. In computer graphics, a spline surface can be described with two set of orthogonal spline curves.
- Splines are used in graphics applications to design curve and surface shapes, to digitizes drawings for computer storage, and to specify animation paths for the objects or camera in a scene.
- Typical CAD application for spline includes the design of automobile bodies, aircraft and spacecraft surface etc.

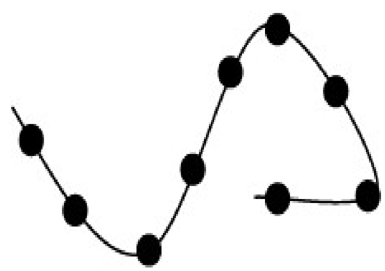


Figure: A spline curve through nine control points

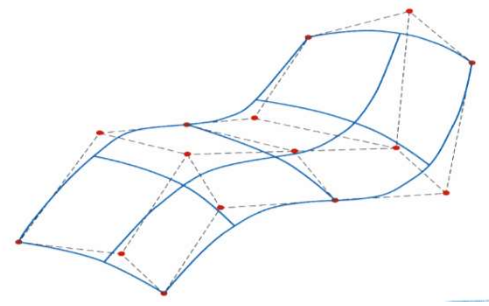


Figure: A spline surface through 15 control points

Control points

- We specify a spline curve by giving a set of coordinate positions, called control points, which indicates the general shape of the curve. These control points are then fitted with piecewise continuous parametric polynomial functions in one of two ways.
- A spline curve is defined, modified, and manipulated with operations on the control points
- **Interpolation curve**: the polynomial sections are fitted by passing the curve through each control points. Interpolation curves are commonly used to digitize drawings or to specify animation paths.
- **Approximation curve**: The polynomials are fitted to the general control-points path without necessarily passing through any control point, approximation curves are primarily used as design tools to structure object surfaces.

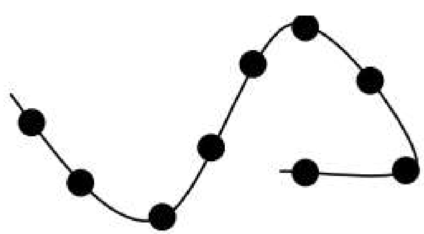


Fig: *A* set of nine control point **interpolated** with piecewise continuous polynomial sections.

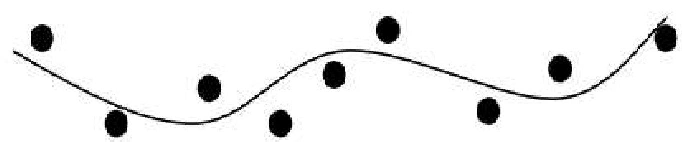


Fig: *A* set of nine control points **approximated** with the piecewise continuous polynomial sections

Convex Hull

- The convex polygon boundary that encloses a set of control points
- Think of an elastic band stretched around the control points
- Provide a measure for the deviation of a curve or surface from the region bounding the control points

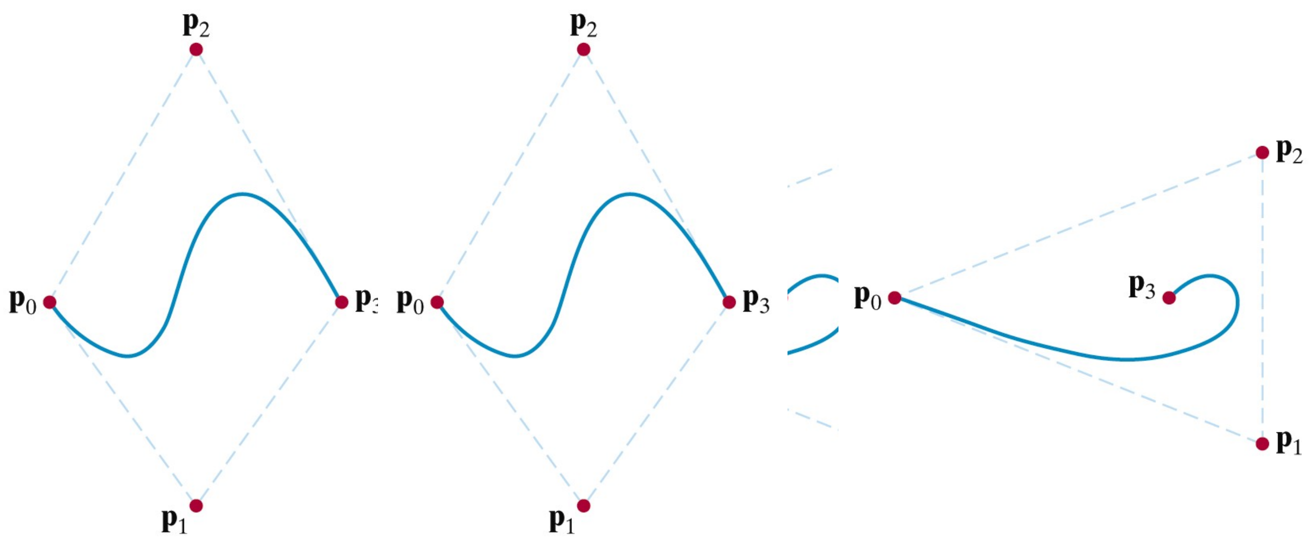


Figure: convex hull shapes (dashed lines) for two sets of control points

Control graph

- A polyline connecting the control points in order is known as a **control graph**
- Usually displayed to help designers keep track of their splines

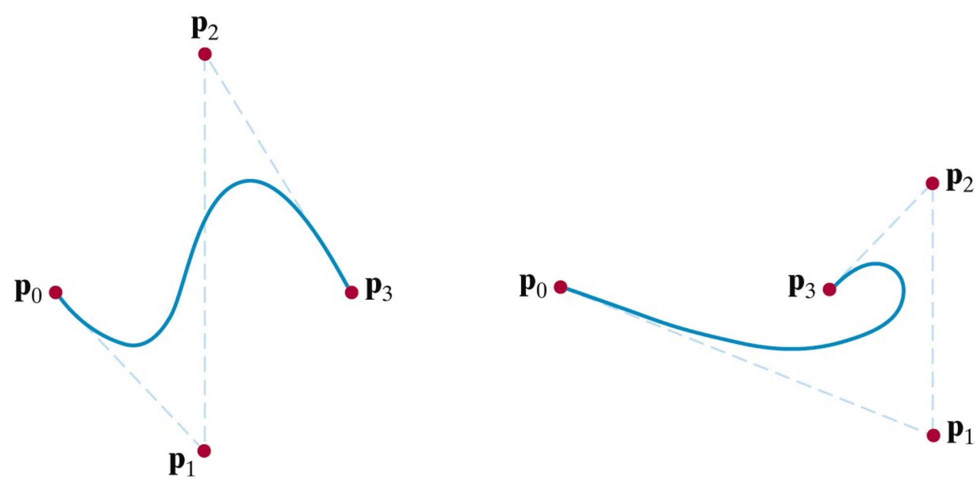


Figure: Control graphs shapes(dashed lines) for two different sets of control points

Spline Specifications

- Three methods for specifying a particular spline representation:
  1. Boundary Conditions
  2. Characterizing Matrix
  3. Blending Functions or Basis Functions

Boundary Conditions

- We can state the set of boundary conditions that are imposed on the spline.
- Suppose we have parametric cubic polynomial representation specified by the following set of equations:
  - $x = x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$ $\quad 0 \le u \le 1.$
  - $y = y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
  - $z = z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$
- Boundary conditions for this curve might be set, for example, on the endpoint coordinates $x(0)$ and $x(1)$ and on the parametric first derivatives at the endpoints $x'(0)$ and $x'(1)$. These four boundary conditions are sufficient to determine the values of the four coefficients a_x, b_x, c_x, and d_x.
- Similar approach can be used to determine values for y and z coordinate

Characterizing Matrix

- We can state the matrix that characterizes the spline.
- From the boundary condition, we can obtain the characterizing matrix for spline. Then the matrix representation for the x-coordinate can be written as;

$$x(u) = [u^3\ u^2\ u\ 1] \begin{bmatrix} a_x \\ b_x \\ c_x \\ d_x \end{bmatrix} = \boldsymbol{U} \cdot \boldsymbol{C}$$

  - Where U is the row matrix of powers of parameter u, and C is the coefficient column matrix.
- Similar approach for y and z coordinate.

Blending Functions or Basis Functions

- We can state the set of blending functions (or basis functions) that determine how specified geometric constraints (boundary conditions) on the curve are combined to calculate positions along the curve path.
- Polynomial representation for coordinate x in terms of the geometric constraint parameters:

$$x(u) = \sum_{0}^{3} x_i . B_i(u)$$

  - Where x_i is the x-coordinate of the control point and $B_i(u)$ is i^{th} blending function which can be obtained by using Lagrange interpolation method.
- Similar approach for y and z coordinate, that is;

$$y(u) = \sum_{0}^{3} y_i . B_i(u)$$

$$z(u) = \sum_{0}^{3} z_i . B_i(u)$$

Bezier Curves

- A spline approximation method developed by the French engineer Pierre Bézier for use in the design of Renault car bodies.
- A Bézier curve can be fitted to any number of control points – although usually 4 are used.
- A set of characteristics polynomial approximating functions called as Beizer Blending functions are used.
- They are called so because as we bend the control points to produce a Bezier curve segment.

- Consider the case of $n+1$ control points denoted as $p_k=(x_k, y_k, z_k)$ where k varies from 0 to n
- The coordinate positions are blended to produce the position vector $P(u)$ which describes the path of the Bézier polynomial function between p_0 and p_n

$$P(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u), \qquad 0 \le u \le 1$$

- The Bézier blending functions $BEZ_{k,n}(u)$ are the *Bernstein polynomials*

$$BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$$

where parameters $C(n,k)$ are the binomial coefficients

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

- So, the individual curve coordinates can be given as follows

$$x(u) = \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$$

$$y(u) = \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$$

$$z(u) = \sum_{k=0}^{n} z_k BEZ_{k,n}(u)$$

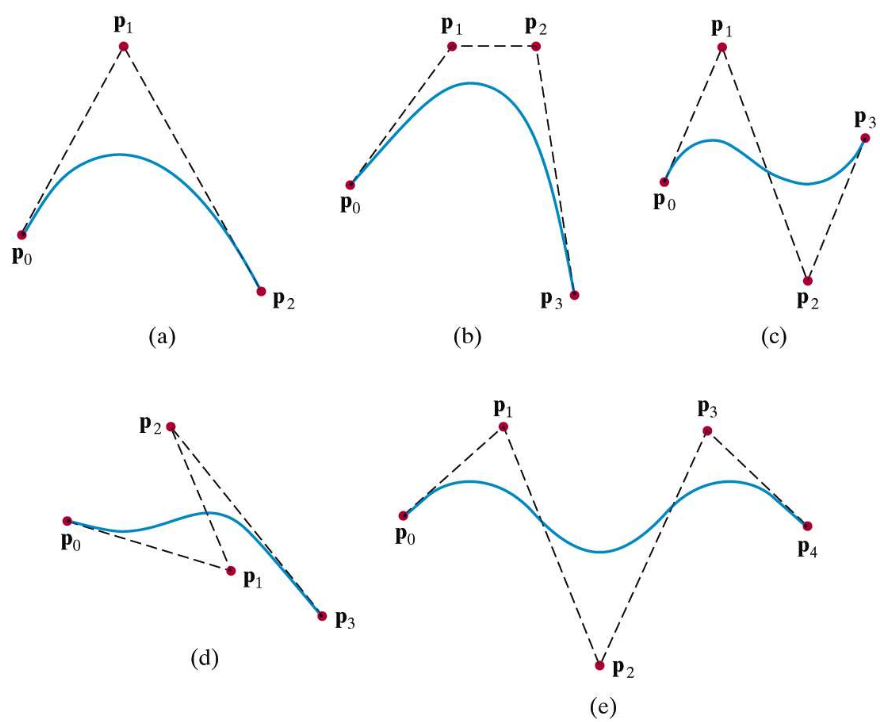


Figure: Examples of two dimensional Bezier curves generated from three, four, and five control points. Dashed lines connect the control-point positions.

- The first and last control points are the first and last point on the curve
  - $P(0) = p_0$
  - $P(1) = p_n$
- The curve lies within the convex hull as the Bézier blending functions are all positive and sum to 1

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

- Many graphics packages restrict Bézier curves to have only 4 control points (i.e. $n = 3$)
- The blending functions when $n = 3$ are simplified as follows:

$$BEZ_{0,3} = (1-u)^3$$

$$BEZ_{1,3} = 3u(1-u)^2$$

$$BEZ_{2,3} = 3u^2(1-u)$$

$$BEZ_{3,3} = u^3$$

Properties of Beizer Curve

- The Bezier curve always passes through the first and last control points.
- The blending function is always a polynomial one degree less than the number of control points.
- The curve is always contained within the convex Hull of the control points. It means the curve will never oscillate away from the control points.
- The curve can be translated and rotated by applying transformation on the control points.
- A circle cannot be exactly represented with a Bezier curve.

Bezier Surface

- A Bezier Surface is formed as the Cartesian product of the blending functions of two orthogonal Bezier curves.
- Two sets of orthogonal Bezier curves can be used to design an object surface by specifying an input mesh of control points. That is,

- $P(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} . BEZ_{i,m}(u). BEZ_{j,n}\ (v)$

Where, $P_{i,\,j}$ specify the location of the (m + 1) by (n + 1) control points.

Numerical

- Construct the enough point on the Beizer curve whose control points ae $P_0(4, 2)$, $P_1(8, 8)$ and $P_2(16, 4)$ to draw accurate sketch.

i. What is the degree of the curve? → degree = order – 1

ii. What are the coordinates at u = 0.5.

- The control points are connected by dashed lines, and the solid lines show curves of constant u and constant v.
- Each curve of constant u is plotted by varying v over the interval from 0 to 1, with u fixed at one of the values in this unit interval. Curves of constant v are p

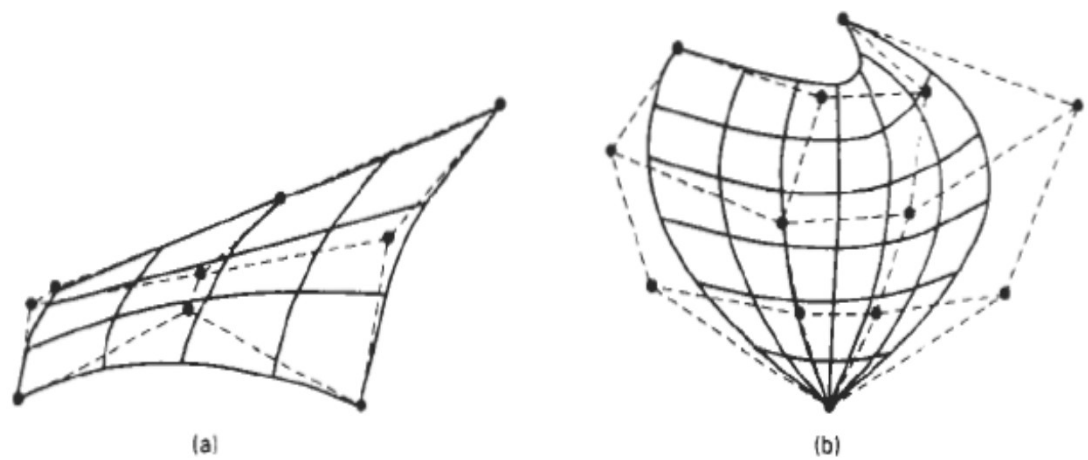


Figure: Bezier surfaces constructed tor (a) m = 3, n = 3, and (b) m = 4, n = 4. Dashed lines connect the control points.

Hermite Interpolation

- See yourself