3D Geometric Transformation

3D Translation

- Repositioning an object along a straight line path from one co-ordinate location to another

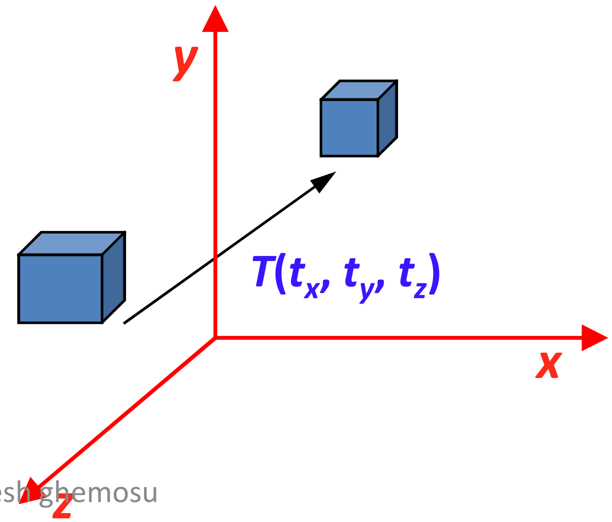
$$(x, y, z) \longrightarrow (x', y', z')$$

To translate a 3D position, we add translation distances t_x, t_y and t_z to the original coordinates (x,y,z) to obtain the new coordinate position (x',y',z')

$$x' = x + t_x \;,\; y' = y + t_y \;,\; z' = z + t_z$$

Matrix form (4 × 4)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

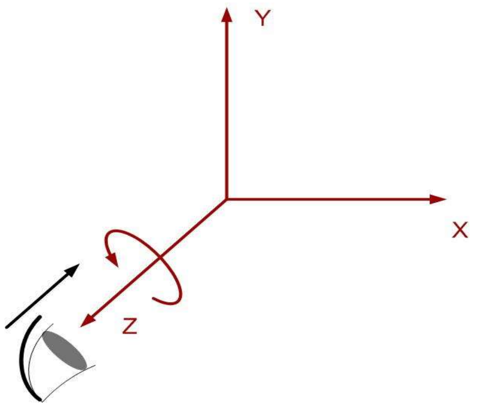


$$P' = T(t_x, t_y, t_z)P$$

3D Rotation about the coordinate axes

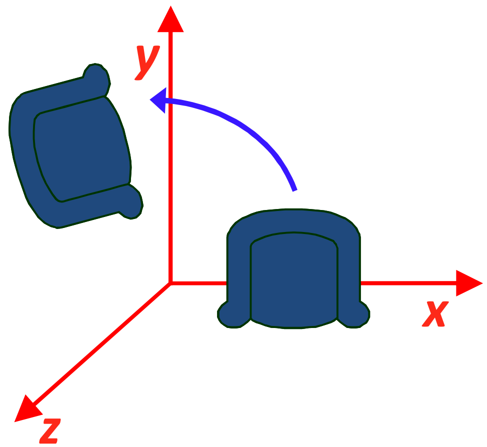
- z-axis
  - The 2D z-axis rotation equations are extended to 3D.

$$
\begin{aligned}
x' &= x\cos\theta - y\sin\theta \\
y' &= x\sin\theta + y\cos\theta \\
z' &= z
\end{aligned}
$$



Matrix form

$$
\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
$$



$$P' = R_z(\theta)P$$

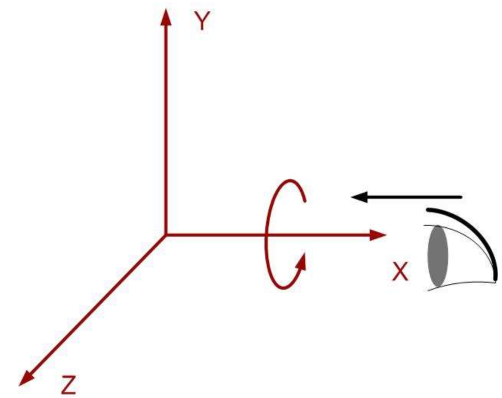
Note

Transformation equations for rotations about the other two coordinate axes can be obtained with a cyclic permutation of the coordinates parameters x, y, and z in above equations. that is, we use the replacements

$x \rightarrow y \rightarrow z \rightarrow x$

- *x*-axis

$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$
$$x' = x$$



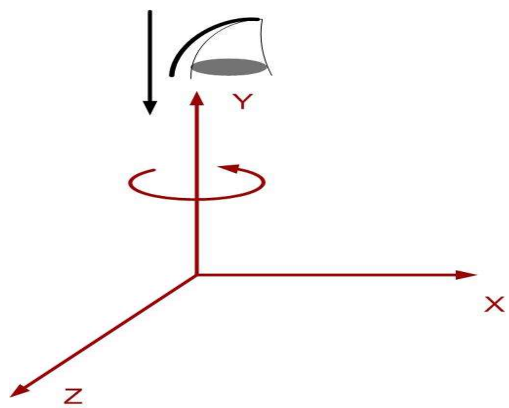
Matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$P' = R_x(\theta)P$$

- *y*-axis

$$
\begin{aligned}
z' &= z\sin\theta - x\cos\theta \\
x' &= z\sin\theta + x\cos\theta \\
y' &= y
\end{aligned}
$$



Matrix form

$$
\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
$$

$$P' = R_y(\theta)P$$

Rotation about axis parallel to co-ordinate axis

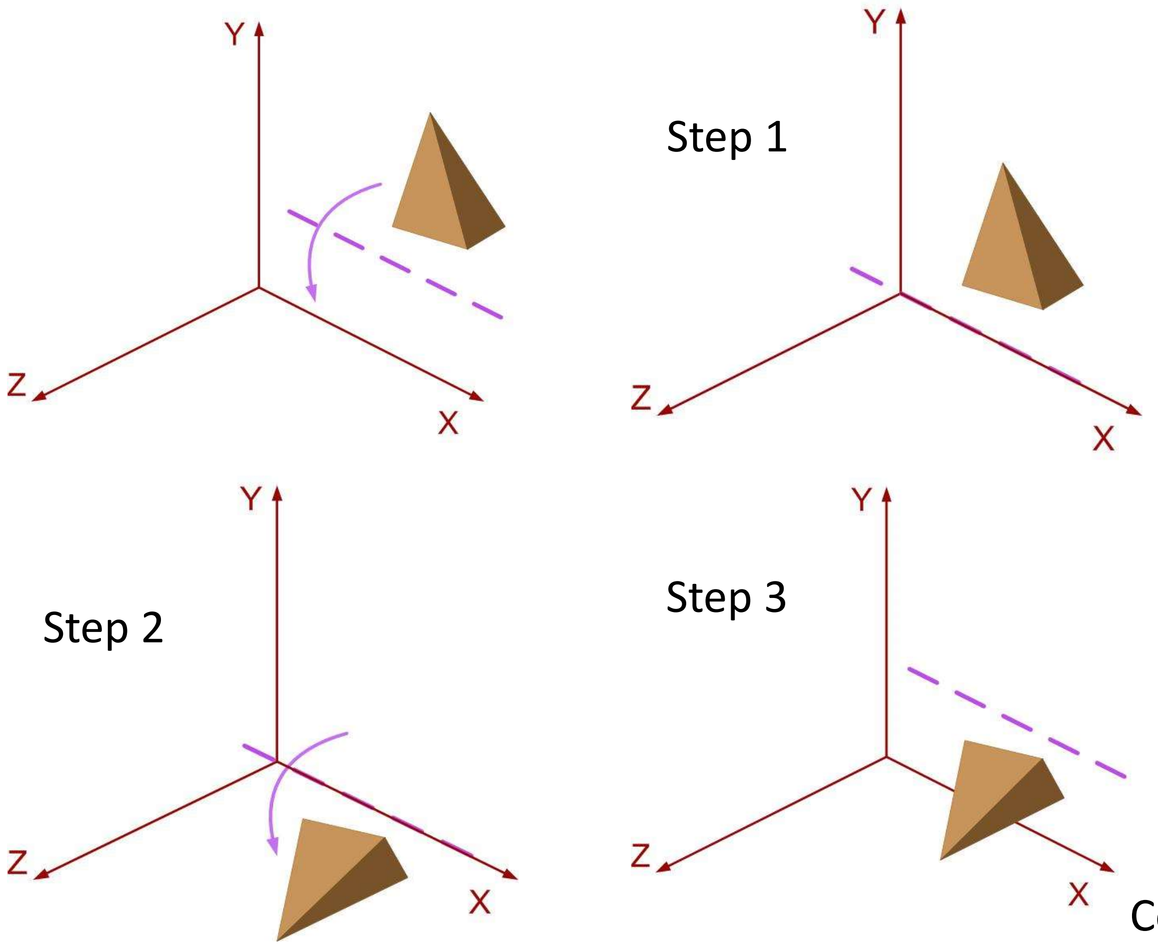


Figure: Sequence of transformation for rotating an object about an axis that is parallel to the axis

Steps

- Translate the object so as to coincide rotation axis to parallel co-ordinate axis
- Perform the rotation about the axis
- Translate back the object so as to move rotation axis to original position

$$P' = T^{-1} \cdot R_x(\theta) \cdot T \cdot P$$

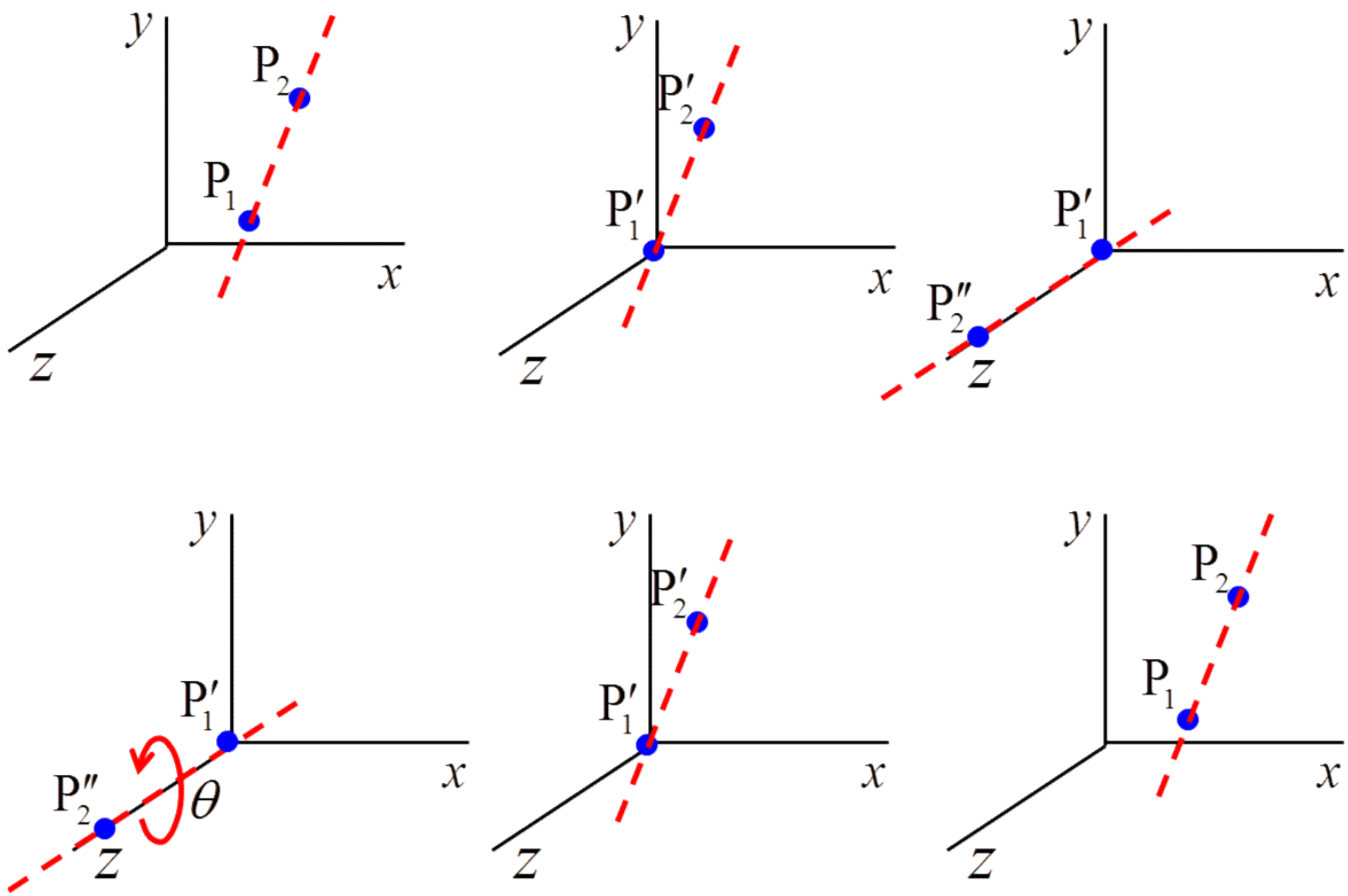
Composite Transformation Matrix is

$$R(\theta) = T^{-1} \cdot R_x(\theta) \cdot T$$

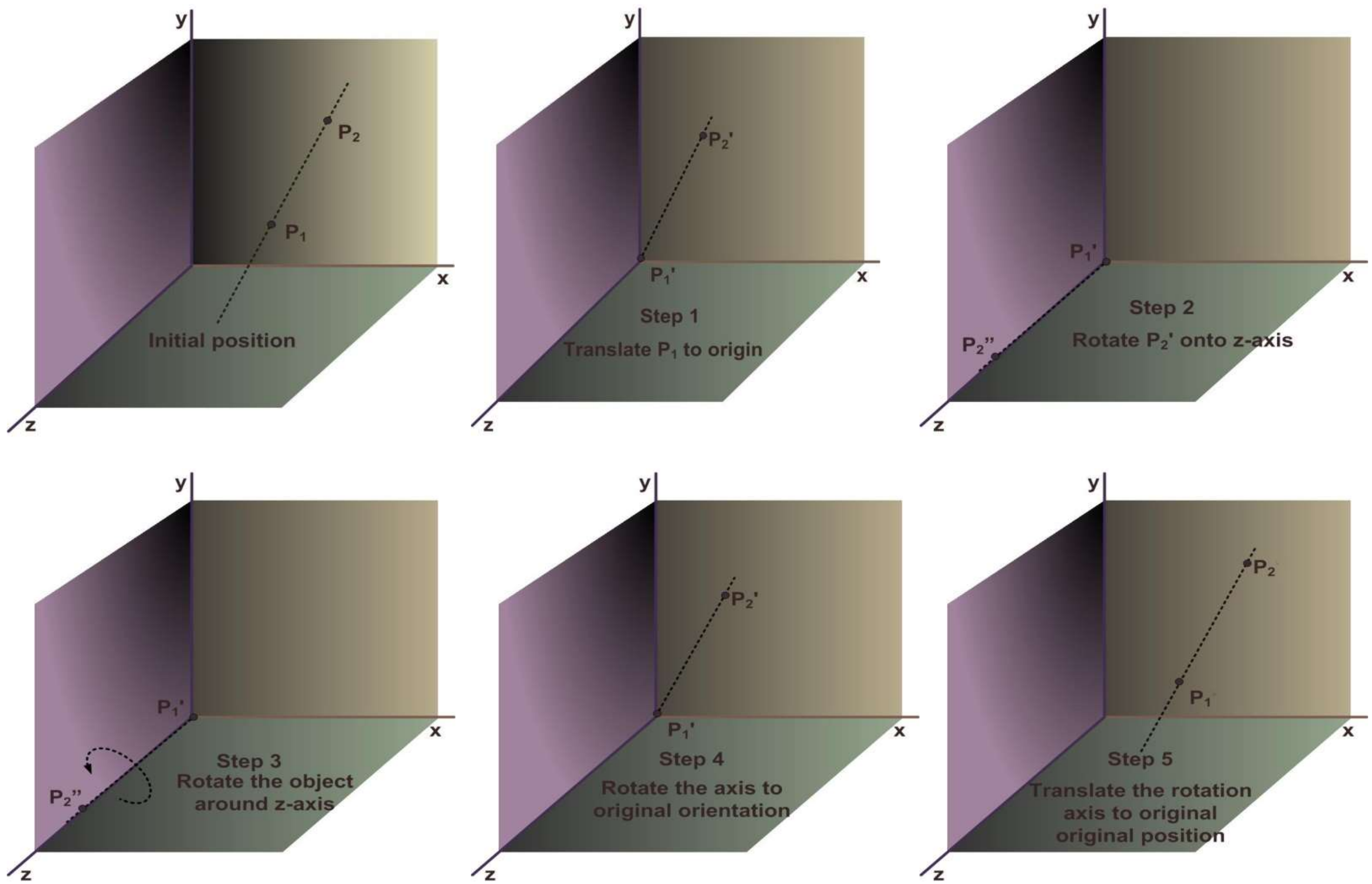
General 3D Rotation

When an object is to be rotated about an axis that is not parallel to one of the coordinates axes, we acquire required rotation in five steps:

1. Translate the object so that the rotation axis passes through the coordinate origin.
2. Rotate the origin so that the axis of rotation coincides with one of the coordinate axis.
3. Perform the specified rotation about that coordinate axis.
4. Apply inverse rotations to bring the rotation axis back to its original orientation.
5. Apply the inverse translation to bring the rotation back to its original position.



y
x
z
P1
P2
P′1
P′2
P″2
θ


y
x
z
P2
P1
Initial position
P1'
P2'
Step 1
Translate P1 to origin
Step 2
Rotate P2' onto z-axis
P2''
Step 3
Rotate the object
around z-axis
Step 4
Rotate the axis to
original orientation
Step 5
Translate the rotation
axis to original
original position

3D scaling

- Scaling about the origin

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

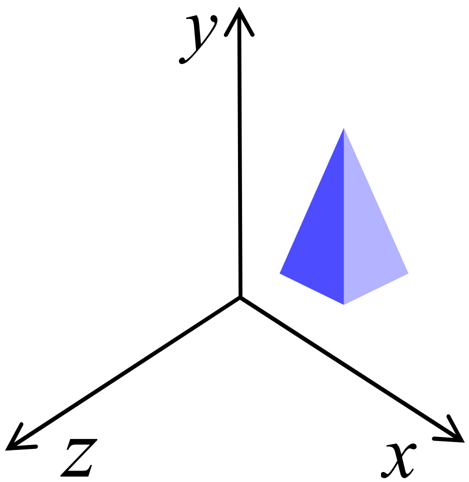
$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Fixed Point Scaling

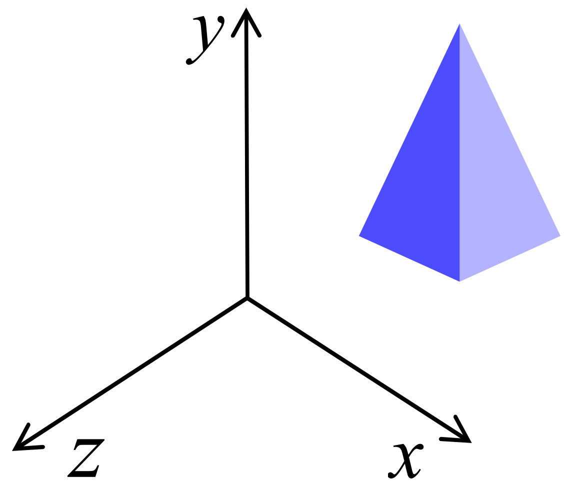
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$S(x_f, y_f, z_f, s_x, s_y, s_z) = T(x_f, y_f, z_f).S(s_x, s_y, s_z).T(-x_f, -y_f, -z_f)$$

3D Scaling about the origin



$$x' = x s_x$$
$$y' = y s_y$$
$$z' = z s_z$$

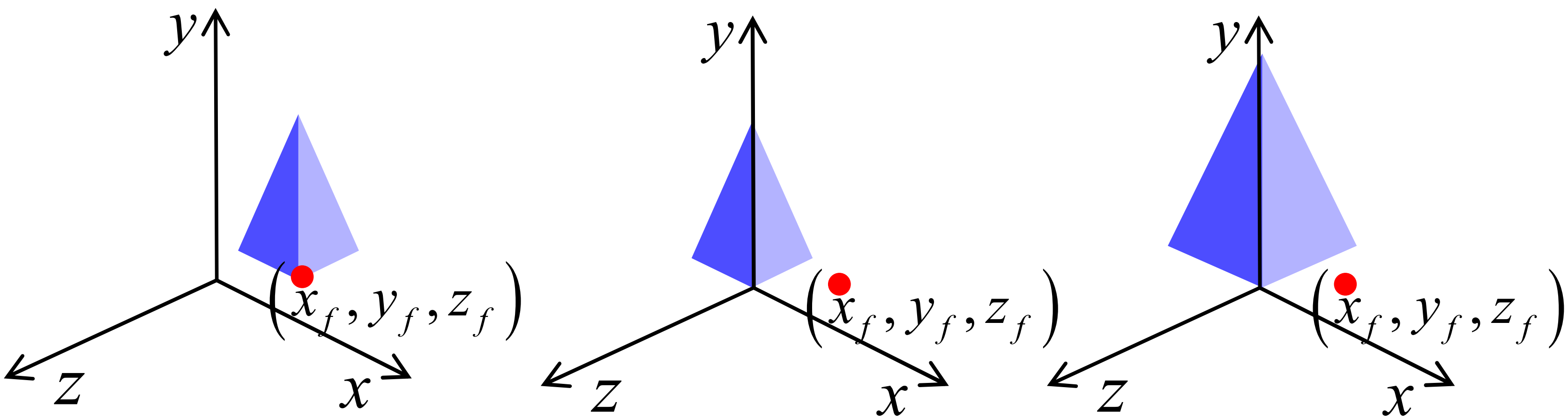


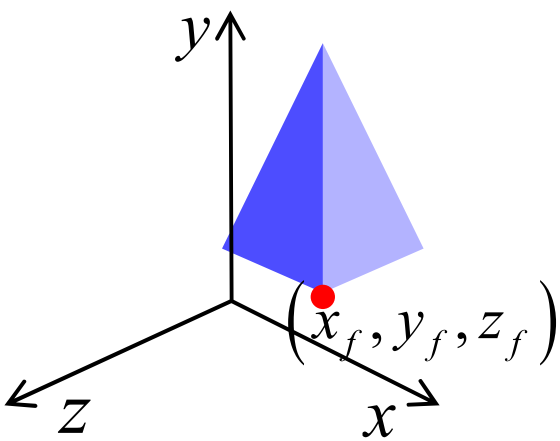
Matrix form

$$\mathbf{P}' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{S} \cdot \mathbf{P}$$

3D fixed point Scaling

- Translate the fixed point to the origin
- Scale the object relative to the coordinate origin
- Translate the fixed point back to its original position.





$$\mathbf{T}\cdot\mathbf{S}\cdot\mathbf{T}^{-1} = \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_f \\ 0 & s_y & 0 & (1-s_y)y_f \\ 0 & 0 & s_z & (1-s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$S(x_f, y_f, z_f, s_x, s_y, s_z) = T(x_f, y_f, z_f).S(s_x, s_y, s_z).T(-x_f, -y_f, -z_f)$$

3D Reflection

- Performed relative to a reflection axis or reflection plane
- Axis reflection → equivalent to 180 degree rotation about the axis in 3-D space
- Plane reflection → equivalent to 180 degree rotation in 4-D space
  - 4-D space ?? → not visualized in Euclidian space
- Reflection about a plane converts right handed co-ordinate system to left handed co-ordinate system and vice versa

- Reflection in xy plane

$$x' = x$$
$$y' = y$$
$$z' = -z$$

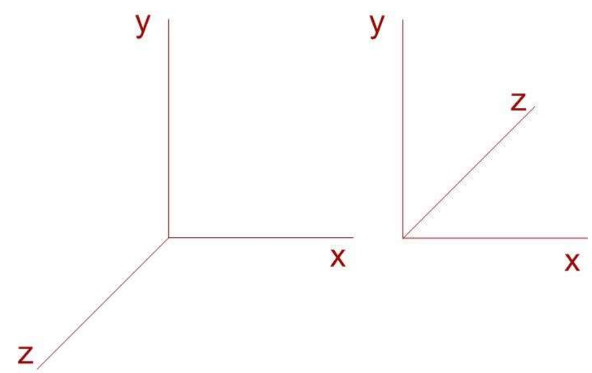


Figure: Conversion of coordinate specification from right-handed to a left handed system can be carried out with reflection transformation

Matrix Representation relative to *xy* plane is:

$$RF_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Shears

- Can be used to modify the object shapes.
- Also used in 3D viewing for obtaining general projection transformation.
- Z-axis shear

x' = x + a.z

y' = y + b.z

z' = z

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$SH_z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$