# 2D Geometric Transformations

# Introduction

- Changes in orientation, shape, and size are accomplished by geometric transformations that alters the coordinates description of objects.
- Basic geometric transformation are:
  - Translation
  - Rotation
  - Scaling
- Other transformations are:
  - Reflection
  - Shear

# Types

- **Rigid body transformation** (transformation without deformation in shape.)
  - *Translation*
  - *Rotation*
  - *Reflection.*
- - **Non rigid body transformation** (transformation with change in shape.)
  - *Scaling*
  - *Shearing.*

# Translation

- A translation is applied to an object by repositioning it along a straight line path from one co-ordinate location to another.
- We translate a two-dimensional point by adding *translation distances*, $t_x$ *and* $t_{y,}$ to the respective co-ordinate values of original co-ordinate $(x, y)$ position to move the point to a new position as:

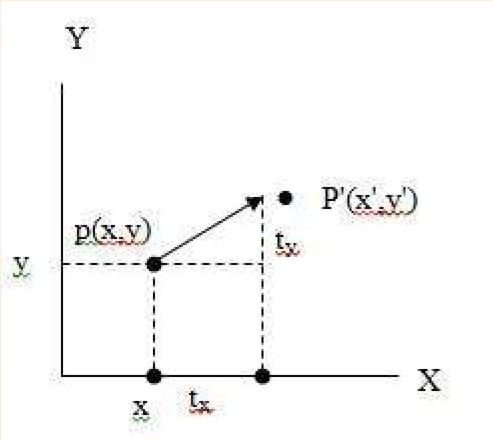
$$x' = x + t_x \qquad y' = y + t_{y.}$$

- The translation distance pair $(t_x, t_{y,})$ is called **translation vector** or **shift vector.**

$$x' = x + t_x, \qquad y' = y + t_y$$

$$P = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad p' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

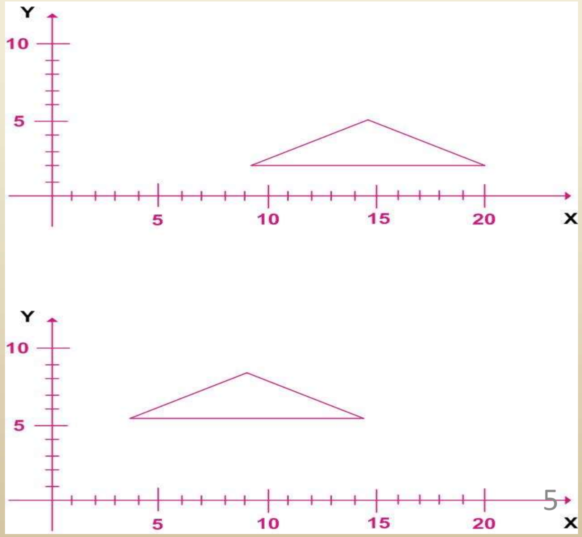
$$P' = P + T$$




- Some times matrix transformation are represented by co-ordinate rows vector instead of column vectors as :

$$P = [x, y] \qquad T = [t_x, t_y] \qquad P' = P + T$$

- For translation of any object in Cartesian plane, we transform the distinct co-ordinates by the translation vector and re-draw image at the new transformed location



# 2DRotation

- The 2D rotation is applied to re-position the object along a circular path in XY-plane. To generate rotation, we specify a rotation angle $\theta$, and the position ($x_r$ , $y_r$) *of the rotation point ( or pivot point)* about which the object is rotated.
- Rotation can be made by angle either clockwise or anticlockwise direction. The positive value of $\theta$ rotates object in anti-clockwise direction while the negative value of $\theta$ rotates the object in clock-wise direction.

- Let P($x$, $y$) is a point in XY-plane which is to be rotated with angle $\theta$ . Also let OP = r (is constant distance from origin. Let r makes angle $\phi$ with positive X – direction as shown in figure.
- When OP is rotated through angle $\theta$ taking origin as pivot point for rotation, then OP' makes angle $\theta + \phi$ with X-axis.

Now , $x' = r\cos(\phi + \theta) = r\cos\phi.\cos\theta - r\sin\phi.\sin\theta$

$y' = r\sin(\phi + \theta) = r\cos\phi.\sin\theta + r\sin\phi.\cos\theta$

And, original coordinates in polar form are:

$$x = r\cos\phi, \qquad y = r\sin\phi$$

Then,

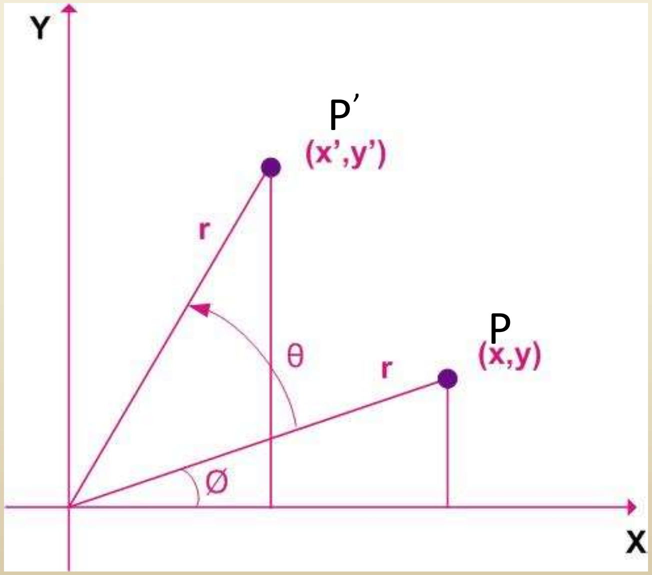
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

- Rotation Equation in matrix form:

$$P' = R.P$$

Where rotation matrix is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



- If coordinates are represented with row vectors, then the matrix product equation in rotation is transposed so that the transformed row coordinates vector [$x'$ $y'$] is calculated as:

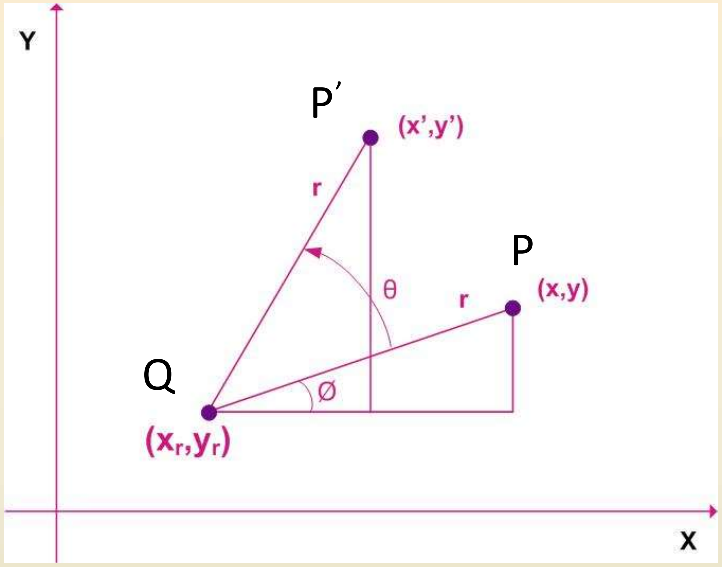
$$P'^T = (R.P)^T$$
$$= P^T.R^T$$

- Where $\mathbf{P^T}$ = [$x$ $y$] and the transpose $\mathbf{R^T}$ of matrix **R** is obtained by interchanging row and column. For a rotation matrix, the transpose is obtained by simply changing sign of the sine terms.

# General Pivot Rotation

- The transformation equations for rotation of a point about any arbitrary pivot position $(x_r, y_r)$ is given by:

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$
$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$



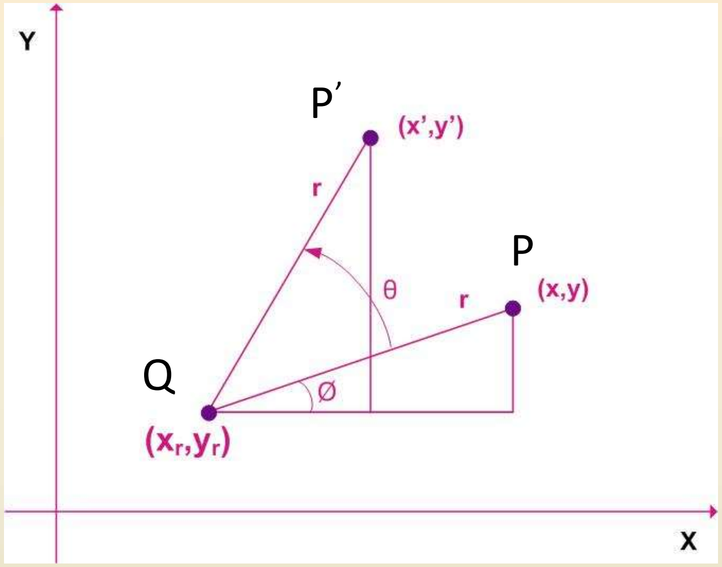
- Let $Q(x_r, y_r)$ is pivot point for rotation.
- $P(x, y)$ is co-ordinate of point to be $\theta$ rotated by angle .
- Let $\phi$ is the angle made by QP with X-direction. .
- Then angle made by QP' with X-direction is $\theta + \phi$
- Hence,

$\cos(\theta + \phi) = (x' - x_r)/r$

*or,* $r\cos(\theta + \phi) = r\cos\theta\cos\phi - r\sin\theta\sin\phi$

*since* $r\cos\phi = x - x_r$ and $r\sin\phi = y - y_r$

$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$ .............. (1)




Similarly,

$\sin(\theta + \phi) = (y' - y_r)/r$

*or,* $r\sin(\theta + \phi) = (y' - y_r)$

*or,* $y' - y = r\sin\theta\cos\phi + r\sin\phi\cos\theta$

*since,* $r\cos\phi = x - x_r$ and $r\sin\phi = y - y_r$

$y' = y_r + (x - x_r)\sin\theta - (y - y_r)\cos\theta$ ............. (2)

These equations (1) and (2) are the equations for rotation of a point (x,y) with angle $\theta$ taking pivot point $(x_r, y_r)$.

The translation about pivot point ($x_r$, $y_r$) can be achieve by sequence of translation, rotation about origin and reverse translation.

- Translate the point ($x_r$ , $y_r$) and P(x ,y) by translation vector

  ($-x_r$ , $-y_r$) *which translate the* pivot point to origin and

  P(x , y) to ($x-x_r$ , $y-y_r$).

- Now apply the rotation equations when pivot is at origin to rotate the translated point ($x-x_r$ , $y-y_r$) as:

  $x_1 = (x-x_r) \cos\theta - (y-y_r) \sin\theta$

  $y_1 = (x-x_r) \sin\theta - (y-y_r) \cos\theta$

- Re-translate the rotated point ($x_1$ , $y_1$) with translation vector ($x_r$ , $y_r$) which is reverse translation to original translation. Finally we get the equation after successive translation as:

- $x' = x_r + (x-x_r) \cos\theta - (y-y_r) \sin\theta$
- $y' = y_r + (x-x_r) \sin\theta - (y-y_r) \cos\theta$

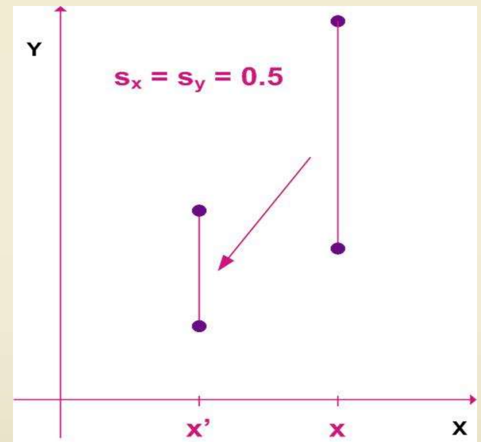
# 2D Scaling

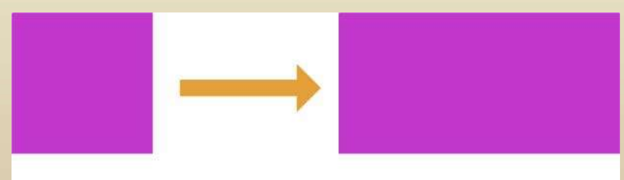
- A scaling transformation alters the size of the object.
- The operation can be carried bout for polygon by multiplying the co-ordinates values (x , y) of each vertex by scaling factor $s_x$ and $s_y$ to produce transformed co-ordinates ($x'$ , $y'$ ).
- i.e. $x' = x.s_x$ and $y' = y.s_y$ .
- The matrix equation for scaling is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = S.P$$

- Where **S** is the 2 by 2 scaling matrix.



Square to Rectangle ( $s_x = 2$, $s_y = 1$ )

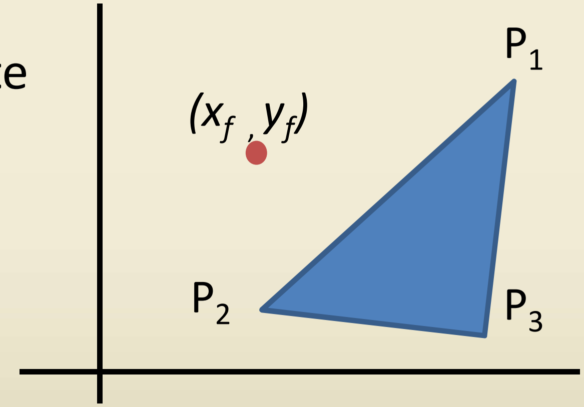
- Scaling factor $s_x$ scales object in x-direction and $s_y$ scales in y-direction.
- If the scaling factor is less than 1, the size of object is decreased and if it if greater than 1 the size of the object is increased.
- The scaling factor =1 for both direction does not change the size of the object.
- If the scaling factors have same value then the scaling factor is know as uniform scaling.
- If the value of $s_x$ and $s_y$ are different, then the scaling is known as differential scaling. The differential scaling is used mostly used in the graphical package to change the shape of the object.

# Fixed Point Scaling

- To control the size of a scaled object, we choose the position called fixed point.
- Let the co-ordinate of fixed point be $(x_f, y_f)$. It can be the vertices, the object centroid, or any other position.
- A polygon is then scaled relative to $(x_f, y_f)$ by scaling the distance from each vertex to the fixed point.
- For a vertex with co-ordinate, the scaled co-ordinate $(x', y')$ are calculated as:

  $x' = x_f + (x - x_f)s_x$

  $y' = y_f + (y - y_f)s_y$




Equivalently,

$x' = x.s_x + (1 - s_x)x_f$

$y' = y.s_x + (1 - s_x)y_f$

*where* $(1-s_y)x_f$ *and* $(1-s_y)y_f$ *are constants for all point in object.*

# Homogeneous co-ordinate representation of 2D Transformation

- The homogeneous co-ordinate system provide a uniform frame-work for handling different geometric transformations, simply as multiplication of matrices.
- Its extension to 3D is straight forward which also helps to produce perspective projections by use of matrix multiplication. We simply add a third co-ordinate to 2D point i.e.

$(x, y) = (x_h, y_h, h)$ where $x = x_h/h$ , $y = y_h/h$ where h is 1 usually for 2D case.

- By using this homogeneous co-ordinate system a 2D point would be (x,y,1).

- Expressing positions in homogeneous coordinates allows us to represent all geometric transformation equations as matrix multiplication.
- Coordinates are represented with three-element column vectors, and transformation operations are written by 3 by 3 matrices.

- In homogeneous representation if position P = (x, y) is translated to new position p'= (x', y') then:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = T(t_x, t_y).P$$

- In homogeneous co-ordinate, rotation equations about the coordinate origin is written as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = R(\theta).P$$

- In homogeneous co-ordinate scaling equations about the coordinate origin is written as:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = S(s_x, s_y).P$$

# Composite Transformation

- Composite, or concatenation of matrices refers to the forming of product of individual transformations matrices.
- We form composite matrices in order from right to left i.e. each successive transformation matrix premultiplies the product of the preceding transformation matrices.

# Successive Translation

- If two successive Translation vectors ($tx_1$,$ty_1$) and ($tx_2$,$ty_2$) are applied to a coordinate position **P**, the final transformed location **P’** is calculated as:

$$P' = T(t_{x2}, t_{y2}).\{T(t_{x1}, t_{y1}).P\}$$
$$= \{T(t_{x2}, t_{y2}).T(t_{x1}, t_{y1})\}.P$$

- Where **P** and **P’** are represented as homogenous coordinate column vector.

Successive Translations are additive

$$\begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(t_{x2}, t_{y2}).T(t_{x1}, t_{y1}) = T(t_{x1} + t_{x2}, t_{y1} + t_{y2})$$

# Successive Rotation

- For two successive rotations applied to the point **P** through $\theta_1$ and $\theta_2$

$$P' = R(\theta_2).\{R(\theta_1).P\}$$
$$= \{R(\theta_2).R(\theta_1)\}.P$$

- By multiplying the two rotation matrices, we can verify the two successive rotations are additive:

$R(\theta_2).R(\theta_1) = R(\theta_1 + \theta_2)$

Therefore,

**P'** $= R(\theta_1 + \theta_2)$.**P**

# Successive Scaling

- Successive scaling are multiplicative

$$\begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x1}.s_{x2} & 0 & 0 \\ 0 & s_{y1}.s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S(s_{x2}, s_{y2}).S(s_{x1}, s_{y1}) = S(s_{x1}.s_{x2}, s_{y1}.s_{y2})$$

**Q**. What will be the final size of the object when we triple the size of the object twice successively?

# General Pivot-Point Rotation

- We can generate rotation of a point or an object about any selected point ($x_r$ , $y_r$) by performing following sequence of translate-rotate-translate operations:

1. Translate object so as to coincide pivot to origin
2. Rotate object about the origin
3. Translate object back so as to return pivot to original position

- The composite transformation matrix for this sequence is obtained with the concatenation

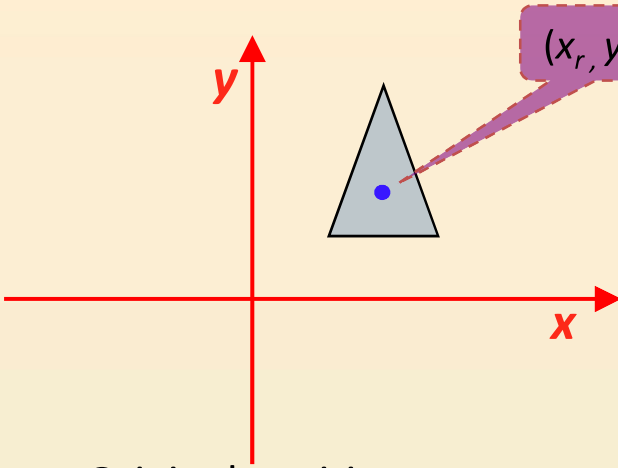
$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta)+y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta)-x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

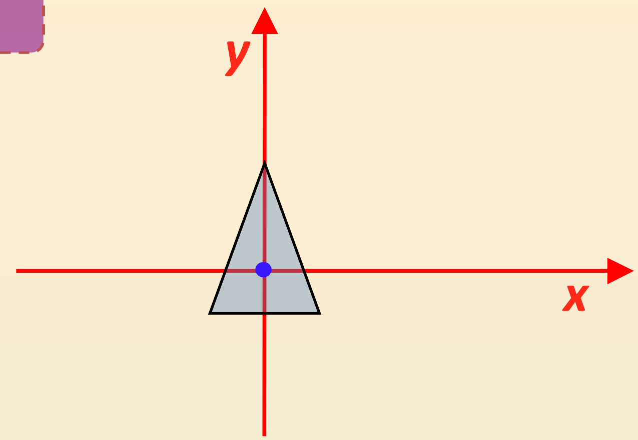
$$T(x_r, y_r).R(\theta).T(-x_r, -y_r) = R(x_r, y_r, \theta)$$

Where,

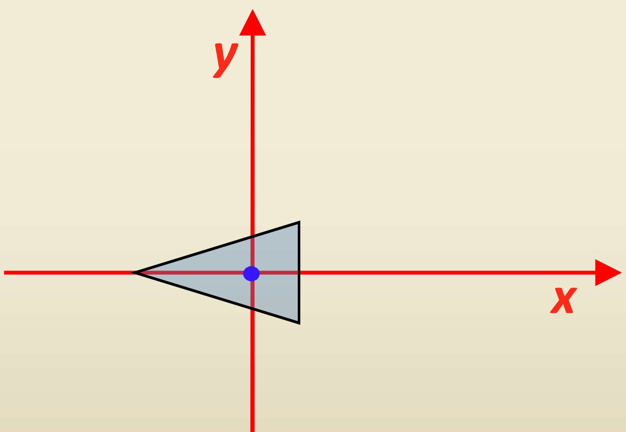
$$T(-x_r, -y_r) = T^{-1}(x_r, y_r)$$



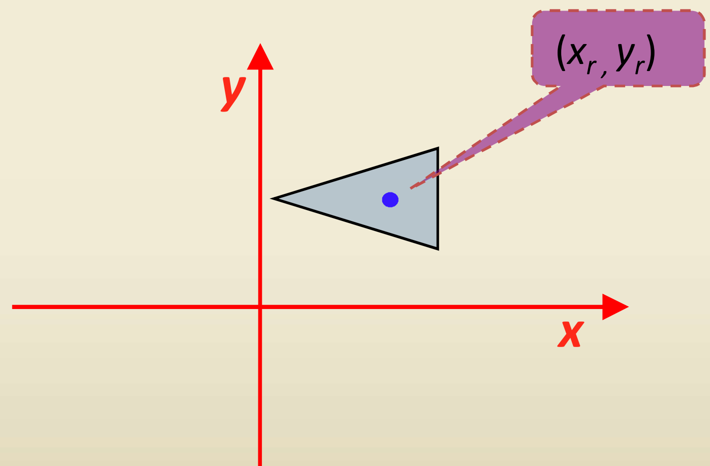

Original position
and Pivot Point




Translate Object so that
Pivot Point is at origin




Rotation about origin




Translate object so that Pivot Point
is return to position $(x_r, y_r)$

# General Fixed-Point Scaling

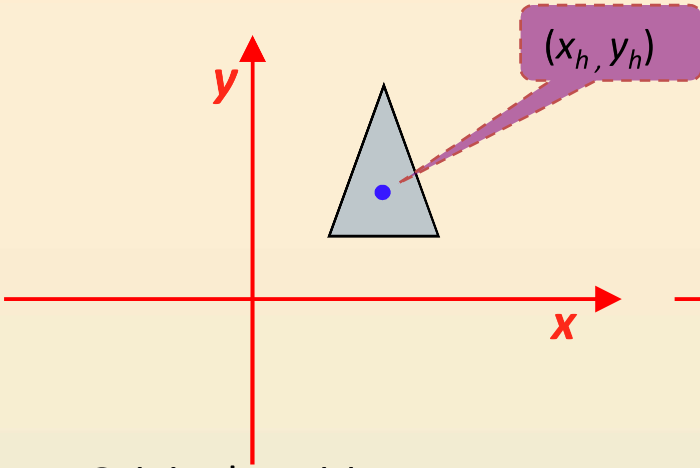
- A object scaled about a fixed point ($x_f$ , $y_f$ ) can be obtained as follows:

1. Translate the object so that fixed point coincide with the origin.
2. Scale the object with respect to the origin.
3. Use the inverse translation to return the object its original position.

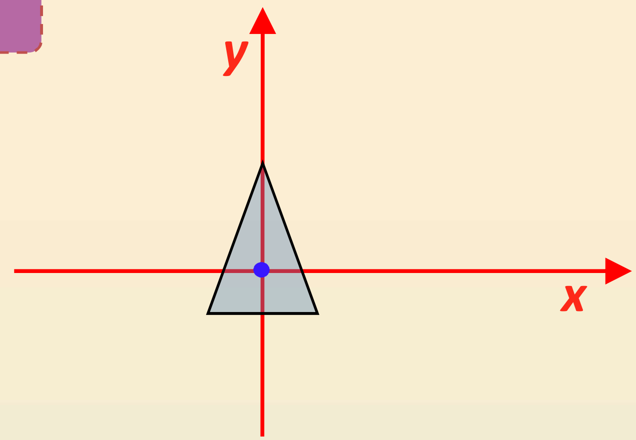
$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

*or*

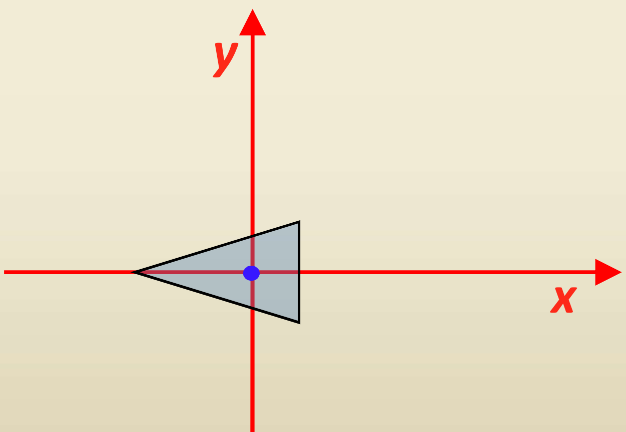
$$T(x_f, y_f).S(s_x, s_y).T(-x_f, -y_f) = S(x_f, y_f, s_x, s_y)$$



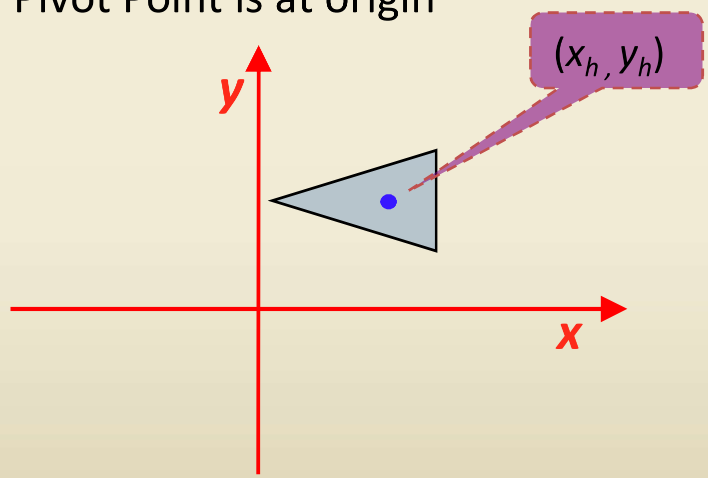

Original position
and Pivot Point




Translate Object so that
Pivot Point is at origin




Scale on about origin




Translate object so that Pivot Point
is return to position $(x_h , y_h)$

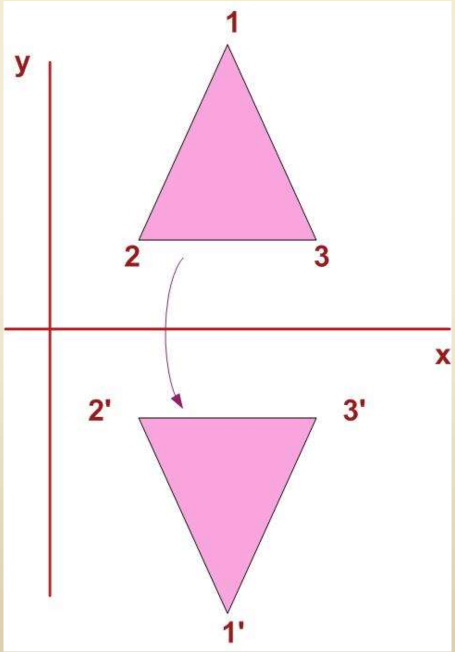
# Reflection

- A transformation that produces a mirror image of an object.
- A mirror image for 2D reflection is generated relative to the axis of reflection. It is same as rotating the object $180^0$ about the reflection axis.

# Reflection about x-axis

- The line representing x-axis is y=0.
- The reflection of a point P(x, y), changes the y-coordinate sign i.e. reflection about x-axis the reflected point of P(x, y) will be $P^{|}$ (x , -y).
- The transformation equation is:

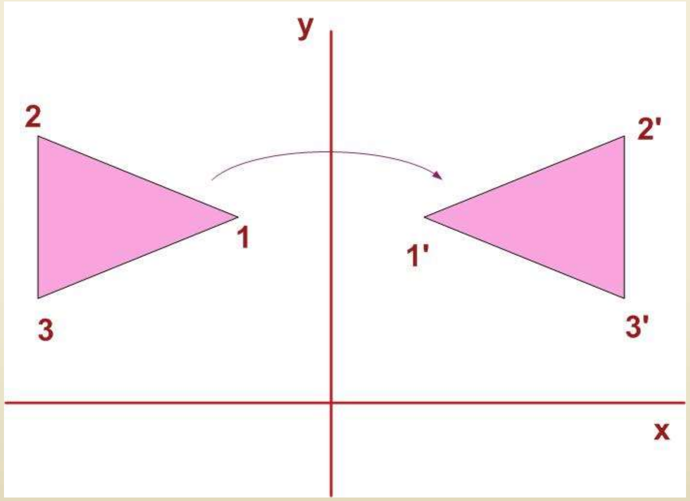
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



# Reflection about y-axis

- The line representing y-axis is x=0.
- The reflection of a point P(x, y), changes the x-coordinate sign i.e. reflection about y-axis the reflected point of P(x, y) will be P| (-x , y).
- The transformation equation is:

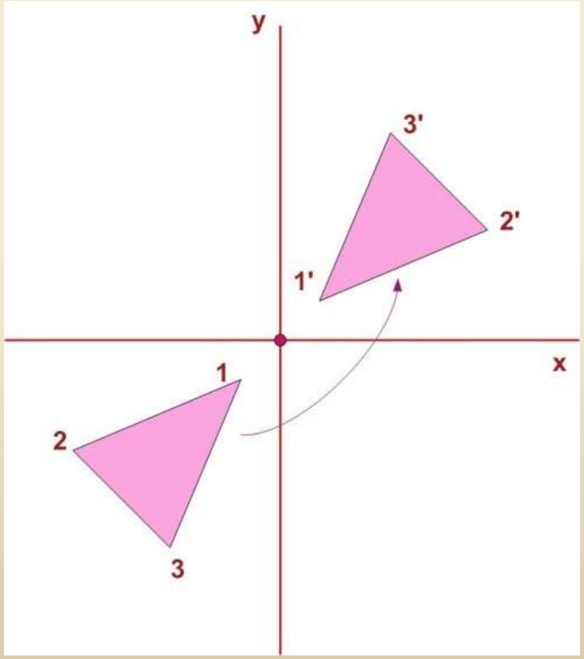
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



# Reflection About Origin

- Both the x and y coordinates of point are flipped by reflecting relative to an *xy* axis i.e. perpendicular to the *xy* plane and that passes through the origin.
- So the sign of x and y coordinates are changed.
- The transformation matrix is:

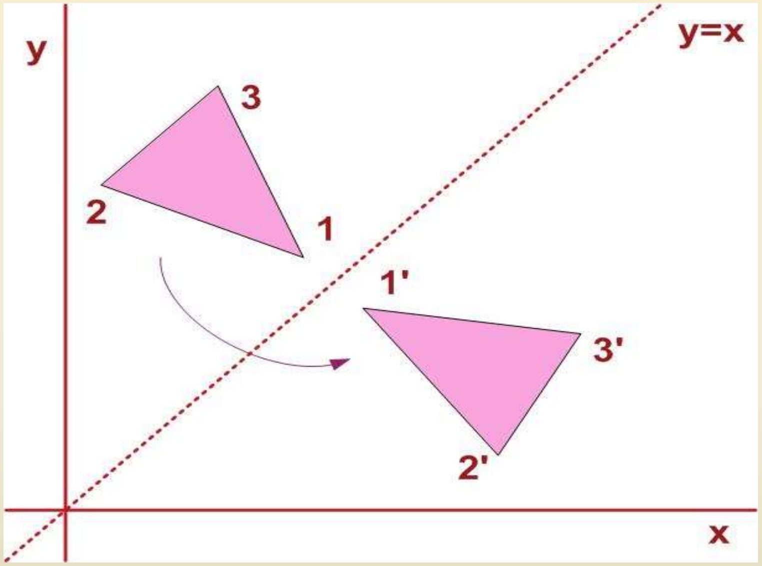
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



# Reflection about the line y=x

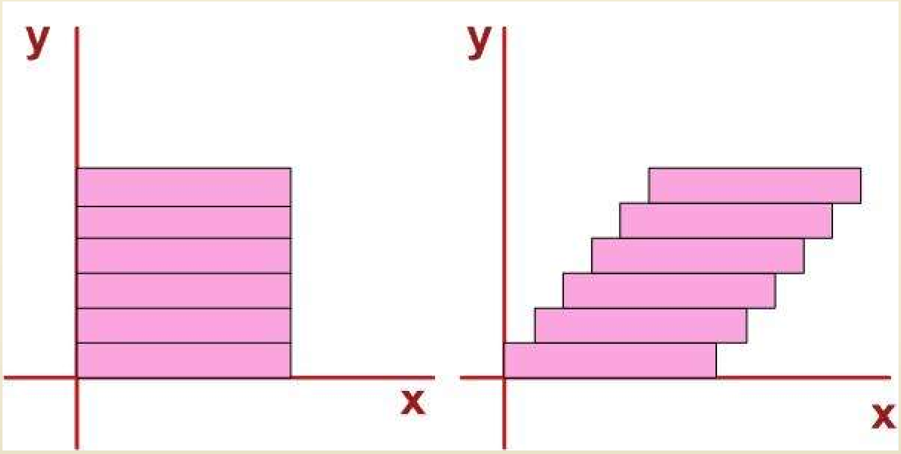
- The transformation equation is:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



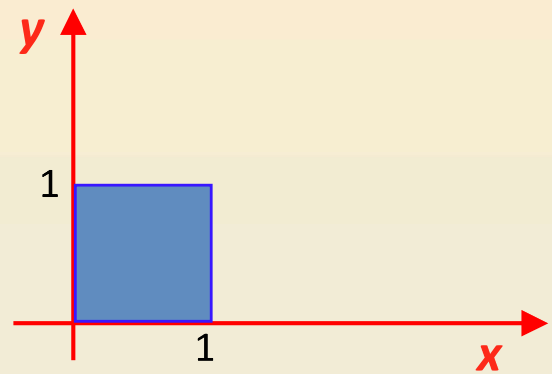
# Shearing

- A transformation that distorts the shape of the object such that the transformed shape appears as if the object were composed of internal layers.



# X-direction shear

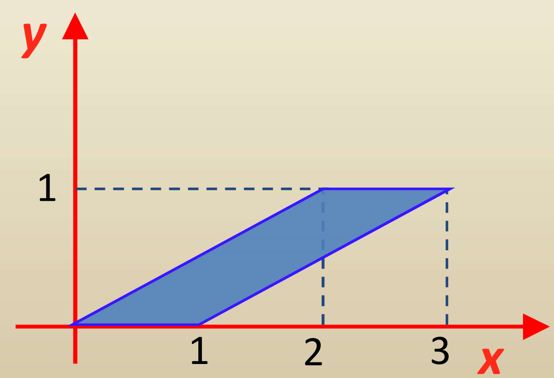
$$x' = x + sh_x \cdot y$$
$$y' = y$$




Initial object

**Matrix form**

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



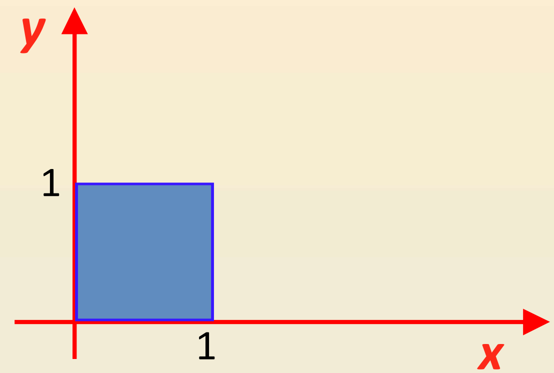

$sh_x = 2$

# Y-direction Shear

- ***y*-direction shear**

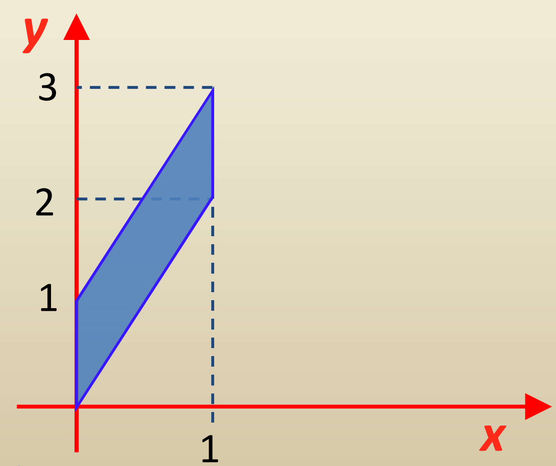
$$x' = x$$

$$y' = y + sh_y \cdot x$$




Initial object

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

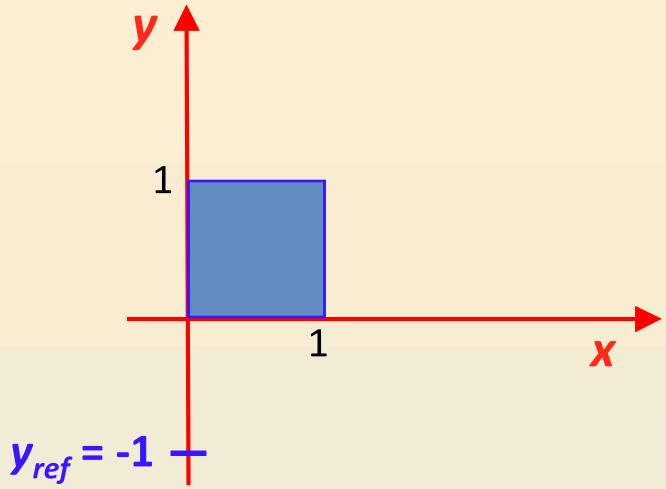



$sh_y = 2$

- ***x*-direction relative to other reference line**

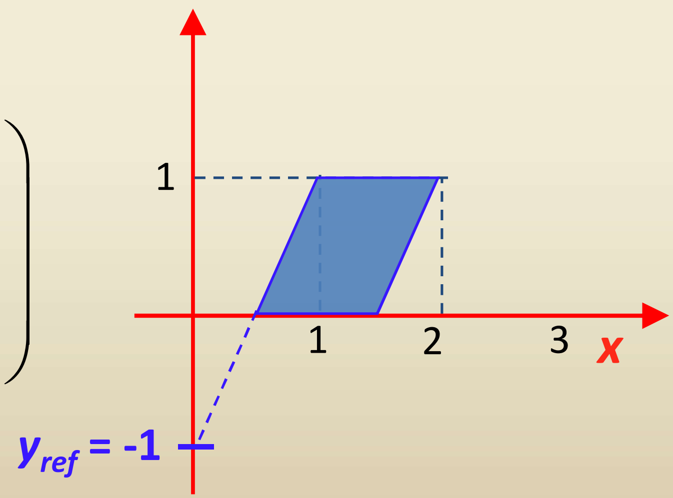
$$x' = x + sh_x(y - y_{ref})$$

$$y' = y$$




**Matrix form**

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

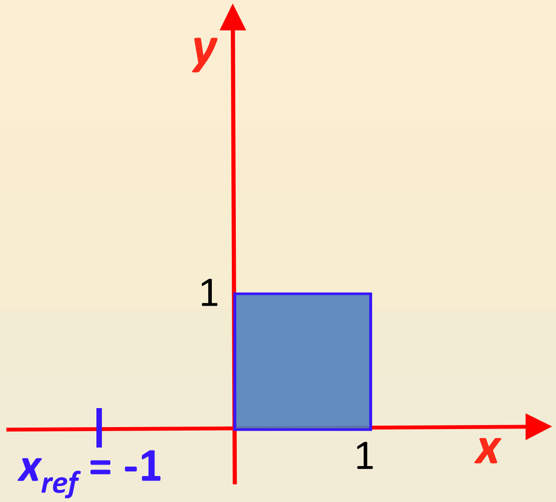



$sh_x = ½$, $y_{ref} = -1$

- ***y*-direction relative to other reference line**

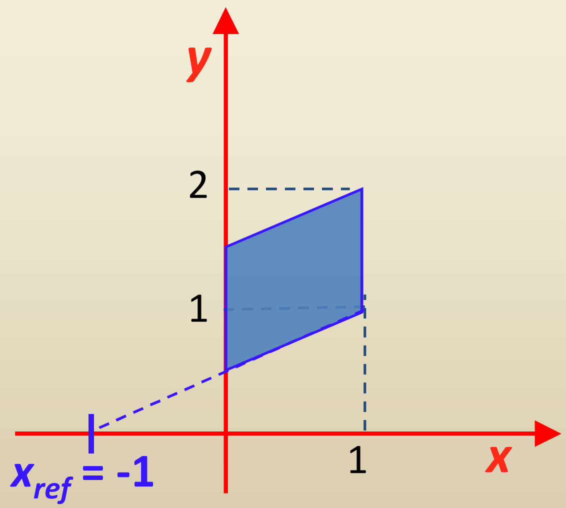
$$x' = x$$

$$y' = y + sh_y(x - x_{ref})$$



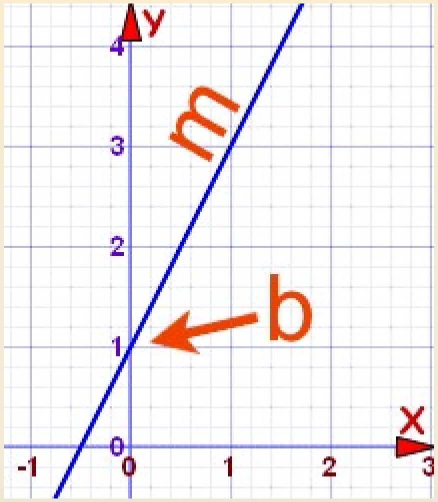

**Matrix form**

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ sh_x & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



# Reflection about a line y=mx+b

Reflection about any line *y= mx + b* can be accomplished with a combination of translate-rotate-reflect transformations.




**Steps are as follows**

1. Translate (0, -b) so that the line passes through the origin
2. Rotate the line about the x-axis by $-\theta^0$ about the origin.
3. Reflect object about the x-axis.
4. Rotate back the line by $\theta^0$ about the origin.
5. Translate back (0,b)

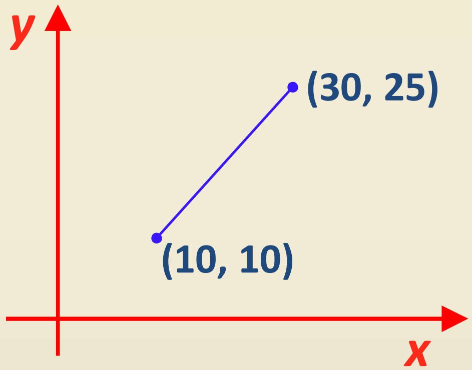
$$[T]=[T_t][T_r]R[T_r]^{-1}[T_t]^{-1}$$

$$\theta=\tan^{-1}(m)$$

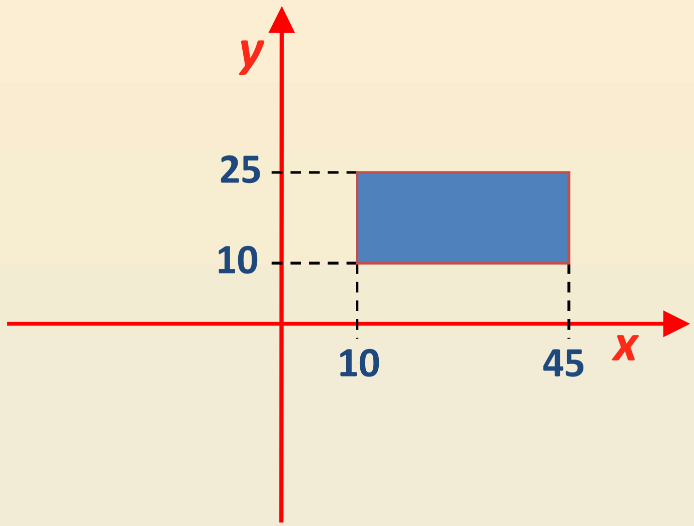
# Numerical Problems

1. Write 2*2 transformation matrix for each of the following rotation about the origin
   - Counter clockwise by π
   - Counter clockwise by π/2
   - Counter anticlockwise by π
   - Counter anticlockwise by π/2

- Consider the line with endpoints (10, 10) and (30, 25). Translate it by $t_x = -20$, $t_y = -10$ and then rotate it by $\vartheta = 90º$.

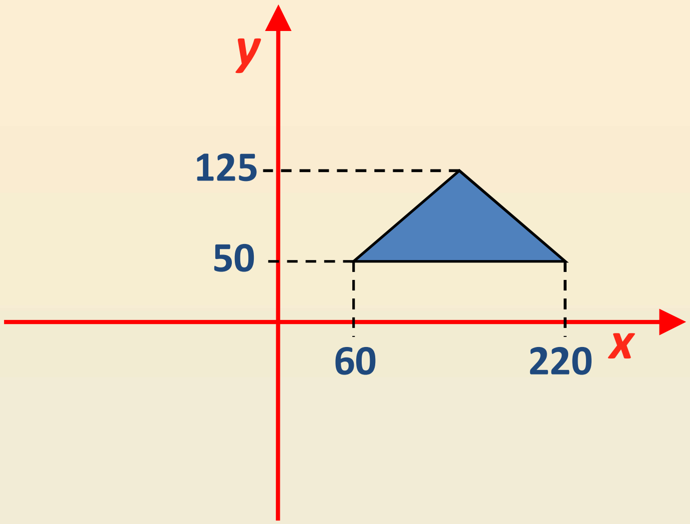


# Consider the following object:




1. Apply a rotation by 60° on the Pivot Point (-10, 10) and display it.
2. Apply a rotation by 30° on the Pivot Point (45, 10) and display it.
3. Apply a rotation by 270° on the Pivot Point (10, 0) and then translate it by $t_x = -20$ and $t_y = 5$. Display the final result.

# Consider the following object:




1. Scale it by $s_x = 2$ and $s_y = ½$ relative to the fixed point (140, 125) and display it.
2. Apply a rotation by 90° on the Pivot Point (50, 60) and then scale it by $s_x = s_y = 2$ relative to the Fixed Point (0, 200). Display the result.
3. Scale it $s_x = s_y = ½$ relative to the Fixed Point (50, 60) and then rotate it by 180° on the Pivot Point (50, 60). Display the final result.

- A triangle with vertices A(4,5), B(2,1), C(6,1) is required to be enlarged twice its initial size about any fixed point(4,2). Find the final coordinates.
- A triangle is defined with coordinates A(2,2), B(4,2) and C(4,4) .Find the transformation coordinates after the following transformation

i) $90^0$ rotation about the origin

ii) reflection about line y = -x