Height bounds over quaternion algebras

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January 10, 2013

Siegel's Lemma and Cassels' Theorem

Let $A = (a_{ij})$ be an $M \times N$ matrix with integer entries. Consider the system of equations

$$Ax = 0. \tag{1}$$

If $M < N$ then there are non-trivial integral solutions.

Theorem (Siegel,1929)

There is a solution $0 \neq z = (z_1, \ldots, z_N) \in \mathbb{Z}^N$ *to* (1) *with*

$$\max_i \{|z_i|\} \leq (N \max_{i,j} \{|a_{ij}|\})^{M/(N-M)} + 1.$$

Siegel's Lemma and Cassels' Theorem

Theorem (Cassels, 1955)

Let $F(x) = \sum_{i,j=1}^{N} f_{ij} x_i x_j \in \mathbb{Z}[x_1, \ldots, x_N]$ *be a quadratic form in* N *variables. If there exists a point* $0 \neq z \in \mathbb{Z}^N$ *such that* $F(z) = 0$*. Then there exists such a point with*

$$\max_i \{|z_i|\} \leq (3 \sum_{i,j=1}^{N} |f_{ij}|)^{\frac{N-1}{2}}.$$

Generalizations of Siegel's Lemma and Cassels' Theorem

Question

When are theorems like Siegel's Lemma and Cassels' theorem possible?

Almost never!

- Work by Matijasevich indicates a negative answer to Hilbert's 10th problem.
- It was proved by J. P. Jones (1980) that the question of whether a single Diophantine equation of degree four or larger has a solution in positive integers is already undecidable.

Generalizations of Siegel's Lemma and Cassels' Theorem

Some directions that have been explored include:

- Analogous results over number fields (using height functions)
- Additional algebraic conditions

In this talk we will discuss such generalizations but in a non-commutative setting.

Height Functions for Number Fields

Let K be a a number field of degree d. We will define height functions

$$H : K^N \to \mathbb{R}_{\geq 0}.$$

- $M(K) =$ the set of places of K.
- $K_\nu =$ the completion of K with respect to the metric induced by $\nu \in M(K)$. We also define, for each $\nu \in M(K)$, the local degree $d_\nu = [K_\nu : \mathbb{Q}_\nu]$.

Height Functions for Number Fields

Let $a \in K$. For each $\nu \in M(K)$, we choose an absolute value $|\ |_\nu$ so that the **Artin-Whaples product formula** is satisfied, i.e.

$$\prod_{\nu \in M(K)} |a|_\nu^{d_\nu} = 1. \tag{2}$$

Definition

The **projective height** of $x = (x_1, \ldots, x_N) \in K^N$ is defined as

$$H(x) = \prod_{\nu \in M(K)} \max_i \{|x_i|_\nu\}^{d_\nu/d}$$

And we define an inhomogeneous height $h(x) := H(1, x)$.

Height Functions for Number Fields

We can also define the height of a subspace $W \subseteq K^N$, the Schmidt height.

- Let $w_1, \ldots, w_L$ be a basis for W.
- $H(W)$ is the height of the vector $w_1 \wedge w_2 \wedge \ldots \wedge w_L$ when viewed as a vector in $K^{\binom{N}{L}}$. Here the standard Grassmann coordinates are used.

Non-commutative Setting

Let K be a totally real number field. A positive definite quaternion algebra is a 4-dimensional K-vector space D with basis $\{1, i, j, k\}$ satisfying the relations

$$i^2 = \alpha,\ j^2 = \beta,\ ij = -ji = k,\ k^2 = -\alpha\beta$$

for some $\alpha, \beta \in \mathcal{O}_K$ that are totally negative.
Write each element $x \in D$ as

$$x = x(0) + x(1)i + x(2)j + x(3)k$$

Non-commutative Setting

We have the following vector space isomorphism

$$\begin{aligned} [\ ] : D &\to K^4 \\ x &\mapsto [x] = (x(0), x(1), x(2), x(3)). \end{aligned} \tag{3}$$

For $N \geq 1$, we extend $[\ ]$ to the module D^N, i.e.

$$\begin{aligned} [\ ] : D^N &\to K^{4N} \\ x &\mapsto [x] = ([x_1], \ldots, [x_N]). \end{aligned} \tag{4}$$

It is clear that $[\ ]$ is an isomorphism. We denote its inverse by $[\ ]^{-1}$.

Non-Commutative setting

We use height functions on D^N as defined by C. Liebendörfer (2004).

- The homogenous height $H : D^N \to \mathbb{R}$ is defined with respect to an order $\mathcal{O}$ of D.
- The inhomogeneous height $h : D^N \to \mathbb{R}$ is independent of chosen order.

Non-Commutative setting

Basic strategy:

1. Search for a point $y \in K^{4N}$ so that $[y]^{-1} \in D^N$ satisfies the desired algebraic conditions;
2. Use the height comparison lemmas (Due to W.K. Chan and L. Fukshansky) to bound the height of $[y]^{-1}$.

Limitations:

We often can't transfer algebraic conditions in D^N to manageable conditions in K^{4N}.

Results

- Let $N \geq 2$ be an integer
- Let $Z \subseteq D^N$ be an L-dimensional right D-subspace, $1 \leq L \leq N$.
- Let $U_1, \ldots, U_M \subseteq D^N$ be proper right D-subspaces.
- Let $G_1(X, Y), \ldots, G_J(X, Y) \in D[X, Y]$ be hermitian forms in $2N$ variables.
- Let $W_i = \{x \in D^N : G_i(x) = G_i(x, x) = 0\}$ for each $1 \leq i \leq J$.
- Let $\mathcal{O}$ be an order in D.

Results: Small Zeros of Hermitian Forms with Additional Conditions.

Theorem

Let $F(X,Y) \in D[X,Y]$ be a hermitian form in $2N$ variables. Assume that F is isotropic on Z. Suppose that there exists a zero of $F(X) := F(X,X)$ in $Z \setminus \left(\left(\cup_{m=1}^{M} U_m\right) \cup \left(\cup_{i=1}^{J} W_i\right)\right)$. Then there exists such a zero y with

$$H(y) \ll_{K,\mathcal{O},L,M,J,\alpha,\beta} H_{inf}(F)^{\frac{9L+11}{2}} H(Z)^{4(9L+12)}.$$

Furthermore, there exists a zero $z \in D^N \setminus \left(\cup_{m=1}^{M} U_m\right)$ such that

$$H(z) \ll_{K,N,M,\alpha,\beta} H_{inf}(F)^{\frac{N+1}{2}}. \tag{5}$$

Results: Counting Points of Bounded Height

Theorem

Let $R > 0$ be a real number and consider the set

$$S_{D,N}(R) := \{x \in D^N : h(x) \leq R\}. \tag{6}$$

Then

$$R^{4N+1} \ll_{K,N,\alpha,\beta} |S_{D,N}(R)| \ll_{K,N,\alpha,\beta} R^{(4N+1)d}.$$

To obtain our lower bound we are transferring a bound by Schmidt (1993) in the number field setting. To obtain the upper bound we are using a bound from Loher-Masser (2004). These results are generalizations of Schanuel's famous asymptotic estimate (1967).

Results: Counting Integral Points of Bounded Height

Let

- $N \geq 2$,
- $R \in \mathbb{R}$,
- $\mathcal{O}$ be an order in D,
- $Z \subseteq D^N$ be an L-dimensional right D-subspace, $1 \leq L \leq N$.
- $\mathcal{N}_{\mathcal{O}}(Z, R) := |\{x \in Z \cap \mathcal{O}^N : h(x) \leq R\}|$

Theorem

Assume $R \geq cH(Z)^{4d}$ where c is an explicit constant depending on $D, \mathcal{O}$, and K. Then,

$$\mathcal{N}_{\mathcal{O}}(Z, R) \gg_{K,L,\alpha,\beta,\mathcal{O}} R^{4Ld} H(Z)^{-4d} \tag{7}$$

Thanks!

Thank You!