ESE 545: Final Project

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1 Logistic Population Growth Model and Analysis

The logistic growth model is an ordinary differential equation (ODE) used to describe simple population growth, using a growth rate r and a carrying capacity K, in which population growth far away from the carrying capacity is approximately exponential, and growth approaching the carrying capacity becomes flat.

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \tag{1}$$

In the real world, population growth will never exactly follow this model. Not only is nature inherently stochastic, but the use of a continuous time equation to model a discrete process will create the appearance of noise at a large scale. With that in mind, we can modify the equation to a stochastic form, introducing a process noise term represented by brownian motion with intensity σ.

$$dP_t = rP_t(1 - \frac{P_t}{K})dt + \sigma P_t dW_t \tag{2}$$

Real-world populations also often exhibit (or are subject to) a range of other dynamics. One is the Allee effect, experimentally observed in some populations, which describes a direct relationship between population density and the growth rate. In particular, the *strong Allee effect* suggests the existence of a threshold under which the population will decline to extinction. We can model this by introducing the Allee threshold S.

$$dP_t = [\, rP_t(1 - \frac{P_t}{K})(\frac{P_t}{S} - 1)]\, dt + \sigma P dW_t \tag{3}$$

Another dynamic of great concern is harvesting. This is especially important is fish farming for instance, in which populations are hard to measure and overfishing has historically been a problem. We assume that harvest amounts are approximately proportional to the population, and thus represent harvesting by introducing a proportional term as shown.

$$dP_t = [\, rP_t(1 - \frac{P_t}{K}) - \lambda P_t]\, dt + \sigma P dW_t \tag{4}$$

We could further combine these two effects to obtain the following population model.

$$dP_t = [\, rP_t(1 - \frac{P_t}{K})(\frac{P_t}{S} - 1) - \lambda P_t] \, dt + \sigma P_t dW_t \tag{5}$$

This population model represents an interesting case study in both stochastic estimation and control. For the former, population estimation is a very noisy, infrequent, sampling process. Accordingly, the measurement process is best represented by the following equation,

$$y_{t_k} = h(P_{t_k}) + \nu_{t_k} \tag{6}$$

where ν_{t_k} is a zero-mean gaussian random variable with variance σ_ν. This means that the evolution of the population's probability density over time will be governed by two rules: the first is the Fokker-Planck equation for the model, which will be derived below, and Bayes' Rule, which describes how the conditional density updates from t_{k^-} to t_{k^+} based on y_{t_k}. The inherent nonlinearity of the system makes implementing this computationally quite difficult, but we can represent it simply at a mathematical level. The model also poses an opportunity for control design and analysis. While the nonlinearity presents a significant hurdle for estimation, the Allee effect introduces equilibria that we can take advantage of for control design.

2 Estimation

The relevant starting place is the Fokker-Planck equation (FPE) for the one-dimensional case (8), which describes how the probability density of a stochastic process evolves over time, given its dynamics:

$$dx_t = f(x_t, t)dt + g(x_t, t)dw_t \tag{7}$$

$$\frac{\partial \rho}{\partial t} = -\nabla(f\rho) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2\rho) \tag{8}$$

Starting with the simple logistic model, the FPE becomes

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial P_t}((rP_t - \frac{rP_t^2}{K})\,\rho) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 P_t^2 \rho) \tag{9}$$

Applying the product rule and simplifying gives

$$\frac{\partial \rho}{\partial t} = (r - 2r\frac{P_t}{k} + \sigma^2)\rho + (rP_t - r\frac{P_t^2}{K} + 2\sigma^2 P_t)\frac{\partial \rho}{\partial P_t} + \frac{1}{2}\sigma^2 P_t^2 \frac{\partial^2 \rho}{\partial P_t^2} \tag{10}$$

The same procedure can be followed after adding in the Allee effect (11) and then the harvest term (12).

$$\frac{\partial \rho}{\partial t} = (r - 2r\frac{P}{k} + \sigma^2)\rho + (rP_t - r\frac{P_t^2}{K} + 2\sigma^2 P_t)\frac{\partial \rho}{\partial P_t} + \frac{1}{2}\sigma^2 P_t^2 \frac{\partial^2 \rho}{\partial P_t^2} \tag{11}$$

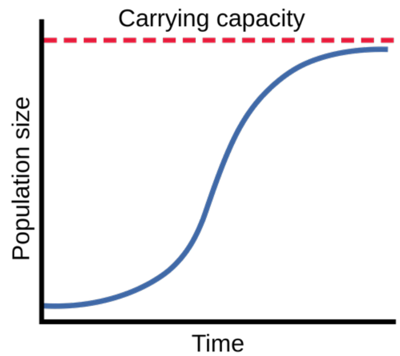

Figure 1: Graph G corresponding to (A,B)

$$\frac{\partial\rho}{\partial t} = (\frac{3rP_t^2}{KS} + r + \sigma^2 - \frac{2rP_t}{S} - \frac{2rP_t}{K} + \lambda)\rho$$
$$+(\frac{rP_t^3}{KS} + rP_t + 2\sigma^2 P_t - \frac{rP_t^2}{S} - \frac{rP_t^2}{K} + \lambda P_t)\frac{\partial\rho}{\partial P_t} \tag{12}$$
$$+\frac{1}{2}\sigma^2 P_t^2 \frac{\partial^2\rho}{\partial P_t^2}$$

This is a very complicated partial differential equation, that is in general quite difficult to solve analytically. In more simple systems we have asssumed a gaussian form of $\rho(x,t) = \exp(a(t)x^2 + b(t)x + c(t))$, and then derived the forms of (a, b, c). In the model of (5), it is not at all obvious that the density should be gaussian. Further, if we consider the solution form of the basic logistic growth model (shown in Figure 1), it should be clear that some asymmetry will be introduced near the asymptotic equilibria of the zero-level and carrying capacity. Beyond that, it is not obvious what a good guess for the solution form would be.

What we can do is investigate the (time) asymptotic form of the density, by assuming $\frac{\partial\rho}{\partial t} \to 0$ at some point.

$$0 = (\frac{3rP_t^2}{KS} + r + \sigma^2 - \frac{2rP_t}{S} - \frac{2rP_t}{K} + \lambda)\rho$$
$$+(\frac{rP_t^3}{KS} + rP_t + 2\sigma^2 P_t - \frac{rP_t^2}{S} - \frac{rP_t^2}{K} + \lambda P_t)\frac{d\rho}{dP_t} \tag{13}$$
$$+\frac{1}{2}\sigma^2 P_t^2 \frac{d^2\rho}{dP_t^2}$$

This now ordinary differential equation can be solved using the power series approach. It will yield a very complicated recurrence relation that is itself difficult to analyze, but we can get a good sense of the character of the solution by inspection.

Consider the case with no harvesting (3) and zero noise. The long run density will be piecewise initial condition dependent. If the population starts below the Allee threshold, it will tend to zero. If it starts above S, it will tend to the capacity K. Upon adding back noise, the density will be similarly initial condition dependent, but now with some spread about the carrying capacity equilibrium. We may imagine that the long-run density for initial populations greater than S would be gaussian with mean K, but this will be complicated by the additional probability of noise-induced phase transitions for initial conditions near S, which will be highly noise dependent.

This paints the picture of a non-trivial, but still understandable probability density as $t \to \infty$. Adding back in a nonzero harvest term complicates this picture to degree which makes in hard to picture, but we can expect that an immediate affect of small harvest levels is to begin to change the unstable equilibrium of the Allee threshold, and that at a certain sufficiently high harvest level, all trajectories would be sent to zero, regardless of initial condition.

Returning to the time-dependent case, it is useful to assume that have a representation for $\rho(x,t)$ at every time t, whether one is somehow derived analytically, or by solving the PDE numerically. Then we can show how the density would be updated using measurements. First, we should take care to consider how well the proposed measurement model actually matches the application.

Note: The measurement equation $y_{t_k} = h(P_{t_k}) + \nu_{t_k}$ assumes each sample includes an independent zero-mean Gaussian random variable. Not only is population measurement noisy and intermittent, but there are many methods used to estimate populations in real life, each with their own sources of error. The measurement method attempting to be modeled here is one in which the population is assumed to exist in approximately uniform density over a large but constrained region. The number of individuals in a small subregion is recorded exactly, and the total population number is extrapolated by proportion. We can analyze this by making the somewhat gross assumption that each individual is equally likely to be anywhere in the region, and therefore the probability that it is found within the subregion is equal to the fraction of the greater region occupied by the subregion, and its binary location is represented by a Bernoulli random variable with that probability. The random variable describing the number of individuals found in the subregion is then the sum of P_{t_k} Bernoulli random variables. This is exactly a Binomial random variable with parameter P_{t_k} and probability the same as the Bernoulli variables, and it is known that Binomial distributions approximate Normal distributions for sufficiently large n, and p not too close to 0 or 1, which makes sense given that a sum of Bernoulli random variables is a Binomial random variable, but is also government by the central limit theorem, just like any other sum of random variables. Thus, we are safe to assume that the measurement noise is gaussian.

With a solution for the probability density $\rho(x,t)$ in hand, the measurement update rule is straightforwardly given by Bayes' Rule. Consider a measurement at time t_k. The density just before the measurement $\rho(x, t_{k^-})$ is called the *prior*. The measurement is denoted by y_{t_k}, and the probability of observing that measurement when the state is some x_{t_k} is referred to as the *likelihood*, denoted

$L(y_{t_k}|x_{t_k})$. Bayes' rules states that the (unnormalized) posterior distribution, i.e. the conditional density, is given by the product of the prior distribution and the likelihood function:

$$\rho(x_{t_{k+}}|y_{t_{k+}}) = L(y_{t_k}|x_{t_k}) \times \rho(x_{t_{k-}}, t_{k^-}) \tag{14}$$

Because the measurement equation (6) includes additive gaussian noise, the likelihood is given by:

$$L(y_{t_k}|x_{t_k}) = \frac{1}{\sqrt{2\pi\sigma_\nu^2}} \exp(-\frac{1}{2\sigma_\nu^2}(y_{t_k} - h(x_{t_k}))^2) \tag{15}$$

The complete conditional density update after a measurement is then:

$$\rho(x_{t_{k+}}|y_{t_{k+}}) = \frac{1}{\sqrt{2\pi\sigma_\nu^2}} \exp(-\frac{1}{2\sigma_\nu^2}(y_{t_k} - h(x_{t_k}))^2) \times \rho(x_{t_{k-}}, t_{k^-}) \tag{16}$$

From here, the conditional mean and variance could easily be derived.

3 Control

The nonlinearity that makes estimation difficult also makes finding the optimal control quite challenging. One option is to linearize the system to make use of standard linear optimal control algorithms. To do so we can make use of the unstable equilibrium introduced by the Allee effect. In the presence of harvesting that equilibrium may be non-trivial, but with zero harvesting it occurs simply at the population level S. Consider the deterministic logistic model with the Allee effect and no harvesting:

$$\dot{P} = rP(1 - \frac{P}{K})(\frac{P}{S} - 1) = -\frac{rP^3}{KS} + (\frac{r}{K} + \frac{r}{S})P^2 - rP \tag{17}$$

The linearized system is given by:

$$\dot{P} = \left.\frac{\partial f}{\partial P}\right|_{P=S} P \tag{18}$$

Plug in the right hand side:

$$\dot{P} = -\frac{3rP^2}{KS} + 2(\frac{r}{K} + \frac{r}{S})P - r\Big|_{P=S} = -\frac{3rS}{K} + \frac{2rS}{K} + r \tag{19}$$

For a final result:

$$\dot{P} = r(1 - \frac{S}{K})P \tag{20}$$

Converting back to a stochastic model yields:

$$dP_t = r(1 - \frac{S}{K})P_t dt + \sigma P_t dW_t \tag{21}$$

We now have a completely linear system with dynamics given by (21) and output given by (6), and both gaussian process and measurement noise, so we can apply the linear quadratic gaussian (LQG). If we add a control term into the equation as shown,

$$dP_t = r(1 - \frac{S}{K})P_t dt + \lambda u + \sigma P_t dW_t \tag{22}$$

the optimal control is $u^*(t) = -\lambda K_0 \mu_t$, from the LQR, where K_0 is given by the algebraic Riccati equation (ARE) for this system:

$$2r(1 - \frac{S}{K})K_0 - K_0^2\lambda^2 + L = 0 \tag{23}$$

L expresses the weight placed on the running cost relative to the control penalty, and μ_t is the mean of the conditional density. If we consider a period $[0, T]$ with a single measurement at time $\frac{T}{2}$, and assume that the measurement function $h(P_{t_k})$ is in fact a linear function hP_{t_k}, then we can express the conditional mean in the following way derived from a single Bayes' update with two gaussian random variables

$$\mu(t) = e^{r(1-\frac{S}{K})t}P_0 + \int_0^t e^{r(1-\frac{S}{K})(t-s)}\lambda u^*(s)ds \ , \quad 0 \leq t \leq \frac{T}{2} \tag{24}$$

$$\mu(t) = e^{r(1-\frac{S}{K})(t-\frac{T}{2})}\mu(\frac{T}{2}^+) + \int_{\frac{T}{2}}^t e^{r(1-\frac{S}{K})(t-\frac{T}{2}-s)}\lambda u^*(s)ds \ , \quad \frac{T}{2} \leq t \leq T \tag{25}$$

Where:

$$\mu(\frac{T}{2}^+) = \mu(\frac{T}{2}^-) + \frac{h\Sigma(\frac{T}{2}^-)}{h^2\Sigma(\frac{T}{2}^-) + \sigma_v}(y_{\frac{T}{2}} - h\mu(\frac{T}{2}^-)) \tag{26}$$

And:

$$\mu(\frac{T}{2}^-) = e^{r(1-\frac{S}{K})\frac{T}{2}}P_0 + \int_0^{\frac{T}{2}} e^{r(1-\frac{S}{K})(\frac{T}{2}-s)}\lambda u^*(s)ds \tag{27}$$

$$\Sigma(\frac{T}{2}^-) = \int_0^{\frac{T}{2}} e^{2r(1-\frac{S}{K})(\frac{T}{2}-s)}\sigma_P^2 ds \tag{28}$$

Equations (24) and (25) express a piecewise definition of the conditional mean, (26) expresses its value just after the measurement, and (27) and (28) express the conditional mean and variance just before the measurement, with P_0 being the population at time 0.

We could also consider the scenario in which there is a potentially reckless harvester in the environment, and we want to know the feasibility of keeping a maintaining a population around a constant level in the presence of that harvester. As mentioned previously, increasing the harvest parameter at low levels simply corresponds to raising the unstable equilibrium up from S. The control design becomes very much the same process as zero harvest, but it is interesting to see what happens to the equilibrium as we increase the harvest parameter.

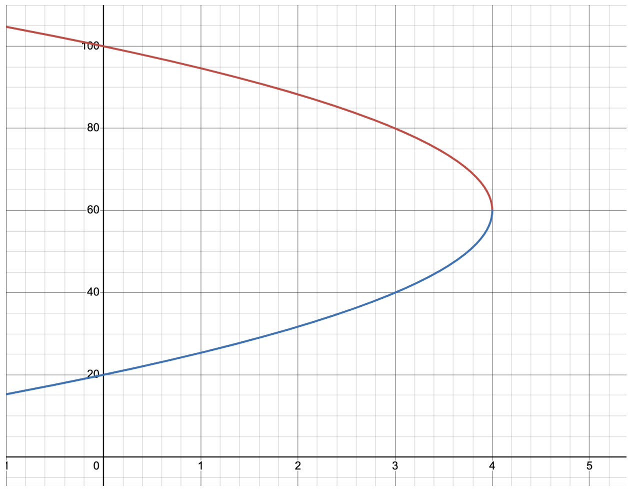

Figure 2: Equilibrium points as function of harvest parameter λ

The equilibrium is found by setting the deterministic form of the differential equation (5) equal to zero.

$$\dot{P}_t = rP_t(1 - \frac{P_t}{K})(\frac{P_t}{S} - 1) - \lambda P_t = 0 \tag{29}$$

Simplify:

$$P_t(-\frac{r}{KS}P_t^2 + (\frac{r}{K} + \frac{r}{S})P_t - (r + \lambda)) = 0 \tag{30}$$

We let $\gamma = \frac{r}{KS}$ for notational simplicity and solve the quadratic formula to find the roots:

$$P_t = 0, \frac{K+S}{2} \pm \sqrt{\frac{\gamma^2(K+S)^2 - 4\gamma(r+\lambda)}{4\gamma^2}} \tag{31}$$

In particular, we note that there is an equilibrium at 0, as expected, and that the other two roots are symmetric about the midpoint between S and K. Moreover, we would hope that for $\lambda = 0$, those roots would simplify to S and K, which we know to be equilibria. In fact this is exactly what happens. After setting $\lambda = 0$ and simplifying, the result is

$$\frac{K+S}{2} \pm \frac{K-S}{2} \tag{32}$$

which is exactly S and K. This symmetry makes complete sense and is somewhat obvious, but it is nice to see pop out in the algebra and even more enlightening in a picture.

Figure 2 shows an example with carrying capacity 100, Allee threshold 20, and growth rate $r = 5$, where the two symmetric equilibria are plotted as a function of λ. The graph showcases the symmetry we observed in the equation and the equilibria at S and K when $\lambda = 0$. It also shows what happens as λ increases. As the low equilibrium moves up from S, the high equilibrium moves down from K, and the symmetry demands that they meet at the midpoint between S and K. This is the largest λ can be before it outpaces all trajectories and sends all initial populations to zero. It makes sense that it would occur at this point because, as shown in Figure 1, that midpoint is where sample paths of the system achieve their greatest rate of growth. Note that we could also have seen this in the algebra; λ is inside a negative term under the radical (31), so it can increase only until the discriminant goes to zero, at which point the equilibrium is the midpoint, and then the roots become complex.

4 Future Directions

A natural extension of this investigation would be to test out the estimation procedure computationally. The Fokker-Planck equation could be solved numerically, and a particle filter used to propogate the measurements. In particular, a simple but useful exercise would be to measure the estimation error in simulation as a function of the measurement noise or, more literally speaking, as a function of the size of the subregion used to sample the population. This could be useful information for population estimators.

On the control side, an interesting problem would be to treat the harvest term itself as a control term, and again find the optimal regulation control, this time with a reward term representing the fact that you "like fish" and benefit from harvesting. This problem is distinguished by the fact that harvesting is a unipolar control signal— it does not provide a way to increase the number of individuals— so the population would have to be kept safely above S with some robustness guarantees given, perhaps in the form of a minimized $P[Extinction]$, representing the probability of a noise-induced phase transition into the extinction zone below $P_t = S$.