A Report On Structural Systems Theory

Gabriel Herman, Member, IEEE

May 5, 2024

1 Introduction

We investigate Structural Systems Theory as a recently developed sub-field of Dynamical Systems and Control, seeking to understand its situation and use within the broader field, and to explain a few recent advances. Started by C.T. Lin's *Structural Controllability* [3], Structural Systems Theory seeks to answer questions about the typical system properties of interest—stability, controllability, observability—with respect to a generic system structure, as opposed to a specific realization of a system with specified parameter values. We consider the standard state-space model of a linear and time-invariant (LTI) dynamical system with control input u:

$$\dot{x} = Ax + Bu \tag{1}$$

where x is a vector of state variables and $\dot{x}$ is its time derivative. In practice, we could determine the desired properties for any specific system using standard methods, i.e. stability is determined by the sign of the real part of the eigenvalues of the system matrix, and controllability is determined by the Kalman rank condition. In structural systems theory, we wish to derive these properties for a system irrespective of the specific parameter values, and so we represent the generic system structure in a sparsity pattern of the system matrices (A, B). We'll continue with a specific example for illustrative purposes.

Consider the classic inverted-pendulum-on-a-cart setup. Let the state vector be $x = (x_1\ x_2\ x_3\ x_4)^T = (x\ v\ \theta\ \omega)^T$, where x is the cart position, v is the cart velocity, θ is the pendulum angle, and ω is the pendulum angular velocity. If we linearize this system about the pendulum up position ($\theta = \pi$), we obtain the following state-space realization [6]:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\delta}{M} & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\delta}{ML} & -\frac{(m+M)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} u \tag{2}$$

where m is the pendulum mass, M is the cart mass, L is the pendulum length, g is the acceleration of gravity, δ is the friction damping on the cart, and u is the control force pushing the cart. If we instead apply a structural systems theoretic approach, we ignore the specific formulas and replace all nonzero entries with a $\star$, and represent the system by the following sparse pair:

$$A = \begin{bmatrix} 0 & \star & 0 & 0 \\ 0 & \star & \star & 0 \\ 0 & 0 & 0 & \star \\ 0 & \star & \star & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \star \\ 0 \\ \star \end{bmatrix} \tag{3}$$

The motivation for this representation is simple. We may be concerned not simply with the controllability of a particular pendulum-on-a-cart, but with the controllability of the general structure. One could imagine that certain parameters like the mass of the pendulum may be open to manipulation, or even that the structure of the pendulum could vary in some trivial way; say that the pendulum arm is instead made up of multiple physical parts, perhaps with differing moments of inertia, in such a way that would complicate the particular formulas in the model, but not change the location of the zero elements. The pendulum-cart model is simply one example; in general, it often behooves us to be able to speak to system properties with respect to the underlying structure of the system, whether it be a physical, cyber, or cyber-physical system (CPS).

Formally, we say an LTI system, defined by the sparsity pattern of its system matrices, satisfies a given system property if there exists a particular realization of the system which satisfies said property in the traditional sense, e.g. sparse pair (A, B) is structurally controllable if there exists a realization of (A, B) which satisfies the Kalman rank condition. In that case we say the sparsity pattern can *sustain* controllability.

The central insight of structural systems theory is that we can express the sparsity pattern for given system matrices in terms of a directed graph, and then reduce the structural system properties to simpler graph properties. The notion of representing a system using a directed graph is very intuitive if we consider the flow of information in the system, or analogously the direction of causal relationships in the system. Going back to the pendulum on a cart example, we build a graph by adding four state nodes which we denote α_i, and one control node which we label β. We then draw an edge from α_i to α_j if $a_{ji} = \star$, and an edge from β to α_i if $b_i = \star$. The result is the digraph in Figure 1. We can see that there are edges going from β to α_2 and α_4, cart velocity and pendulum velocity respectively, and this makes sense, knowing that the actuation affects only the acceleration of the cart and pendulum in the first instance. On the other hand, we can see that α_1 has no outgoing edges, and this also makes sense. Intuitively, none of the state variables depend on the cart position at any given time (unless the cart happens to be sitting in a nonuniform force field), and the ability to balance the pendulum should not depend on where the cart is. In a slight abuse of terminology, we can say that information flows from the control force to the cart and pendulum velocity, and that no information flows from the cart position, and so on for the various other edges. **Remark:** This digraph representation becomes even more intuitive in the context of a multi-

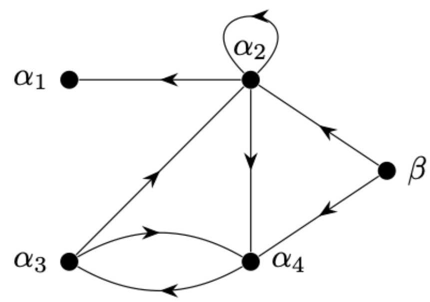

Figure 1: Graph G corresponding to (A,B)

agent system (MAS) in which an edge from α_i to α_j represents the ability of agent i to communicate with (send information to) agent j.

Now that we have a graph structure, we can determine various system properties by considering the properties of our graph. In the Results section, we'll explore a few significant contributions to the structural systems theory, starting with structural controllability, the subject of the field's seminal paper [1].

2 Results

2.1 Structural Controllability

We recall that a sparsity pattern is structurally controllable if and only if it sustains a controllable pair (A, B). In the following section, we will introduce a necessary and sufficient condition on the digraph G corresponding to (A, B). In order to explain this result, we need to utilize some basic graph theory notation. We'll denote the set of state nodes by V_α and the set of control nodes V_β. Then, the set of all nodes is $V = V_\alpha \cup V_\beta$, the set of all edges is E, and we denote the entire system graph by $G = (V, E)$. We say that α_i is an *in-neighbor* of α_j if there exists an edge from α_i to α_j, and we denote by $N_{in}(\alpha_j) = \{v_i \in V : v_i\alpha_j \in E\}$ the set of all in-neighbors of α_j. Finally, for any set A, $|A|$ is its cardinality, and $N_{in}(A) = \bigcup_{a_j \in A} N_{in}(a_j)$ denotes the set of in-neighbors of A. **Theorem:** the result is as follows: G is structurally controllable if and only if for any $V' \subseteq V_\alpha,\ |N_{in}(V')| \geq |V'|$. In words, for any subset of the state nodes, the number of in-neighbors of the subset must be greater than or equal to the cardinality of the subset.

This is a surprisingly simple condition, but to test it would in general require checking all $2^n - 1$ possible subsets of V_α, which could be computationally intractable for large systems. We can simplify the process by deriving an analogous condition. **Theorem:** Hall's Marriage Theorem states that for a bipartite graph with node sets (X, Y), $|X| \leq |Y|$, there exists an X-perfect matching if and only if, for any subset $W \subseteq X$, $|W| \leq |N(W)|$, where $N(W)$ is the collection of all nodes in Y which are adjacent to nodes in X [2]. If we construct

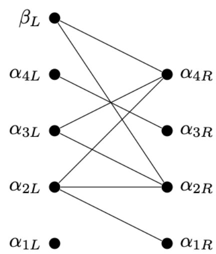

Figure 2: figure
Right-partite graph for G

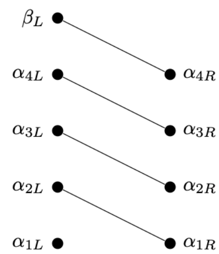

Figure 3: figure
Corresponding right-perfect matching

a bipartite graph with node sets $X = V_\alpha$ and $Y = V$, it becomes clear that $N(W \subseteq X) = N_{in}(V' \subseteq V_\alpha)$, and thus the condition of Hall's Marriage Theorem is equivalent to the graph condition for structural controllability. This is particularly useful because, while the first condition requires checking $2^n - 1$ subsets, there are existing algorithms that can check for the existence of a perfect matching with complexity $\mathcal{O}(n^3)$ by computing the *max flow*.

We now have an equivalent condition for structural controllability in the existence of of a V_α-perfect matching in the bipartite graph $G = (V_\alpha, V, E)$. Figure 2 shows the relevant bipartite graph for our pendulum-cart example, and Figure 3 shows one possible right-perfect matching, the existence of which proves that G is structurally controllable.

2.2 Structural Ensemble Controllability

A related field within Systems and Control which also lends itself to a structural approach is that of Ensemble Systems theory, and in particular ensemble control. We will again concern ourselves with the state-space model of an LTI system with control input u, given by (1). An ensemble system is, in general, a collection of individual systems, interpreted as realizations of a single parameterization. It is expressed as follows:

$$\dot{x}(t, \sigma) := A(\sigma)x(t, \sigma) + B(\sigma)u(t) \tag{4}$$

where $\sigma \in \Sigma$, and Σ, the *parameterization space*, can be finite, countably infinite, or in the most general case, a continuum set. A and B are then matrices with elements being functions of σ, and each possible value of σ is associated with its own LTI system. **Remark:** Importantly, each individual system takes the same common control input $u(t)$. This is crucial for most applications, in which we typically want to control a complex system with many agents or components, yet lack the resources or the precision required to apply a unique control input to each system. Examples include quantum systems, in which we may apply a uniform magnetic field to a space of many particles, or entire national economies, in which monetary policy is employed to affect spending and borrowing activities by adjusting interest rates.

We also need to define a *profile*, which is a function that maps the parameterization space Σ to $\mathbb{R}^n$; or rather, maps every possible value of σ to a specific state vector: $x(t, \bullet) : \sigma \Longrightarrow x(t, \sigma)$. We can then imagine a *profile space* (in general an infinite dimensional function space) as the ensemble analog to state-space, in which we define an initial profile $x(0, \bullet)$, the set of initial conditions of each individual system, and a target profile $x^*(\bullet)$, allowing us to pose a control problem. **Definition 1:** An ensemble system of the form (4) is *uniformly controllable* if given any initial and target profiles, and error tolerance $\epsilon \geq 0$, there exists a time T and control input $u(t)$ such that $||x(T, \sigma) - x^*(\sigma)|| \leq \epsilon$. A generalization of the Kalman rank condition says that the parameterization (A, B) is uniformly controllable if and only if the controllable subspace, denoted $\mathcal{L}(A, B)$, is equal to the set of all continuous functions mapping Σ to $\mathbb{R}^n$. **Definition 2:** The *controllable subspace* is defined as the L^∞-closure of the vector space spanned by the columns of $A^k B$, $\forall k \geq 0$.

The *structural ensemble controllability* problem is posed analogously to the single-system structural controllability problem. Formally, it is the question of existence of a uniformly controllable pair (A, B) conforming to a given sparsity pattern, where the sparsity pattern is fully determined and represented by its corresponding digraph G. As in Section 2.1, we can answer this question using properties of the graph G. The result is provided by Chen [4].

We define two relevant graph properties. **Definition 3:** A digraph G is *accessible* to control nodes if for each state node α_i, there is a path from a control node to α_i. **Definition 4:** A digraph G admits a *Hamiltonian decomposition* if a there is a subset of its edge set which forms a disjoint union of cycles. **Theorem:** We then say G is structurally controllable if it is accessible to control nodes and the subgraph of its state nodes admits a Hamiltonian decomposition. The first condition is quite intuitive. If there exists a state node(s) that is not accessible by a control node, then that state variable(s) is effectively isolated from the control. It may (or may not) affect other state variables, but the dynamics affecting its own state value are decoupled and uncontrollable. The second condition is not so straightforward and the proof of its necessity is beyond the scope of this report.

The pendulum-on-a-cart system is not a typical application for ensemble controllability, but for sake of demonstration we can still check the structural controllability conditions (perhaps we imagine we have a bunch of cart-pendulum systems each with a different value for one of the physical parameters). The accessibility condition can be easily checked by inspection of Figure 1, and it turns out to be satisfied. However, we can see that there is no Hamiltonian decomposition that includes the α_1 node, and therefore the second condition is not satisfied. This should makes sense; it would be quite surprising if would could balance a collection of cart-pendulums with a common control input when each pendulum has say a unique mass or friction constant. **Remark:** We didn't need it in this situation, but there is a useful result that Hamiltonian decompositions correspond one-to-one to perfect matchings, allowing for an algorithmic implementation nearly identical to that for single-system structural controllability. There is also a no-go result which says that if the dimension of Σ is greater than

1, then there is no real analytic controllable pair (A,B).

2.3 Structural Averaged Controllability

If we consider the typical applications of ensemble controllability, as in quantum systems, biological systems, and social and economic networks, it is often the case that we need not control the state of each individual system in the ensemble, rather it is sufficient to control the average of the states of all the systems.

For the general continuum case, we define the average state of a profile:

$$\bar{x}(t) = \int_{\Sigma} x(t,\sigma)\, d\sigma \tag{5}$$

We can then pose the average control problem: given an initial profile $x(0,\bullet)$, any target average $\bar{x}^* \in \mathbb{R}^n$, and any time T, find a control input $u(t) : [0,T] \to \mathbb{R}^m$ such that $\bar{x}(T) = \bar{x}^*$. There exists a straightforward extension of the Kalman rank condition for average controllability, using the simple average of the (Kalman) controllability matrix:

$$\bar{C}(A,B) \;=\; \left[\int_{\Sigma} B, \quad \int_{\Sigma} AB, \quad \int_{\Sigma} A^2B, \quad \dots\right] \tag{6}$$

And we say that (A,B) is *averaged controllable* if $\bar{C}(A,B)$ is rank n.

We can now introduce the final result of this review, provided by Gharesifard and Chen [5]. As in the single-system and ensemble controllability cases, the structural approach to averaged controllability takes as the fundamental object a sparse pair (A,B), and poses the question of existence of a controllable pair. Formally, the sparsity pattern (A,B) is structurally averaged controllable if it admits a parameterized pair (A,B) that satisfies that averaged Kalman rank condition.

The necessary and sufficient conditions for structural averaged controllability are not straightforward, so we'll present only a condition for the single input, $u \in \mathbb{R}$, case.

In general, the digraph G corresponding to (A,B) is weakly connected, but can be decomposed into strongly connected components (subgraphs), where a graph is said to be *strongly connected* if there is a path from every vertex to every other vertex [3]. If we identify the strong components within G, replace each strong component with a single vertex (this is called the condensation of G), and cut off anything with a self-loop, then we say the single input system (A,B) is structurally averaged controllable if and only if the resulting graph is a directed path.

We can now check if our original example is structurally averaged controllable. Looking at Figure 1, we can see that α_2, α_3, and α_4 for a strongly connected component, and that α_1 and β are trivial strongly connected components, leaving us with the condensation shown in Figure 4. This graph is clearly a directed path, thus, the sparsity pattern (3) is structurally averaged controllable.

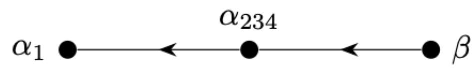

Figure 4: Condensation of G

3 Conclusion

In summary, we have reviewed three standard control problems for linear systems—single-system, ensemble, and averaged controllability—and beyond that, we have motivated a structural approach to linear control, and provided necessary and sufficient conditions for the structural formulation of each of the three control problems. To reiterate, structural systems theory takes as the object of question not a specific linear system, nor a parameterized ensemble system, but a sparsity pattern of system matrices. Concerned only with the locations of the zero and nonzero elements in the system matrices, structural systems theory attempts to answer questions of controllability and other properties by analyzing a more fundamental system structure, and therefore it can enable us to make more robust and generalizable statements about system properties.

As Lin [1] notes in his seminal paper, structural controllability is in some sense a more fundamental notion of controllability for, in physical systems especially, the system parameters are only ever approximately known. Structurally controllability therefore handles potential variation and uncertainty in system parameters, in a way the simple controllability does not. At first glance, structural controllability conditions could appear too weak to be of much use, given that they are simply conditions on the mere existence of a controllable pair; they don't say anything about the probability of any given realization being controllable. It turns out, however, that if (A, B) is a structurally controllable sparsity pattern, then almost all realizations of (A, B) are controllable pairs. (This is for much the same reason that a randomly sampled matrix is almost certain to be full rank.) Indeed, it is clear that the structural approach to linear systems theory is remarkably powerful, in contrast to the ease and simplicity of its formulations.

References

[1] Ching-Tai Lin, "Structural controllability," in IEEE Transactions on Automatic Control, vol. 19, no. 3, pp. 201-208, June 1974

[2] "Hall's marriage theorem." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 17 Apr. 2024. Web. 3 May. 2024.

[3] "Strongly connected component." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 9 Mar. 2024. Web. 5 May. 2024.

[4] X. Chen, "Sparse Linear Ensemble Systems and Structural Controllability," in IEEE Transactions on Automatic Control, vol. 67, no. 7, pp. 3337-3348, July 2022

[5] B. Gharesifard and X. Chen, "Structural Averaged Controllability of Linear Ensemble Systems," in IEEE Control Systems Letters, vol. 6, pp. 518-523, 2022..

[6] Brunton SL, Kutz JN. Data-Driven Science and Engineering: Machine Learning, Dynamical Systems, and Control. Cambridge University Press; 2019.