ADM formalism and its geometry

Tin Nguyen
Department of Mathematics and Department of Physics,
The University of Texas at Austin

A thesis submitted in partial fulfillment for the Bachelor of Science in Mathematics: Honors Option and the Bachelor of Science in Physics: Honors Option.

January 12, 2014

Lorenzo Sadun Supervising Professor in Mathematics	Date
David Rusin Honors Advisor in Mathematics	Date
Richard Matzner Supervising Professor in Physics	Date
Greg Sitz Honors Advisor in Physics	Date

ABSTRACT

Since its conception in 1959 by Richard Arnowitt, Stanley Deser and Charles W. Misner, the ADM formalism have played an important role in both quantum gravity and numerical relativity. The ADM formalism is a Hamiltonian formulation of general relativity and in essence, it is a geometric approach to recast the 4 dimensional space-time into a time-evolving 3 dimensional spatial manifold. Thus ADM formalism was able to rewrite the Einstein tensor of the 4 dimensional space-time in terms of time-evolving geometric objects on the 3 dimensional spatial manifold, and ultimately, its time-dependent components in a local coordinates. Despite the importance of Einstein's tensor components in local coordinates, it should be clear that the geometry of the ADM formalism is equally interesting. Thus a detail and rigorous work need to be done in trying to understand the formulation in terms of geometric objects that give rise to the physical world we are living in. In this work, we carefully foliate a 4 dimensional manifold and write the components of Riemann tensor in terms of both geometric objects and local coordinates. In the process, we show many important and essential properties and relationships between those geometric terms that are crucial in order to produce the well known results of Einstein tensor.

One geometry cannot be more true than another; it can only be more convenient.
- Henri Poincaré

ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisors: Prof. Lorenzo Sadun - Department of Mathematics, and Prof. Richard Matzner - Department of Physics, for their invaluable guidance and mentorship throughout the entirety of this work.

Secondly, I also would like to thank my mentor and friend, Luis Suazo - Department of Physics, for his continuous encouragement and indispensable help in this work. I'm extremely grateful for his countless hours spent explaining every question that I had and guiding me every step of the way, without which this work would not be possible.

Finally, I would like to thank my family and friends for their unwavering support throughout this endeavor.

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1 Background and notations

1.1 The ADM formulation of Einstein's equation

While the Einstein tensor, which encompasses all the physics of space-time in one single statement, is very beautiful, it is a statement on the geometry of a 4 dimensional manifold. Thus, despite its geometric significant, the Einstein tensor is almost impossible to use for any calculation without putting it in some local observable coordinates that we are familiar with. The ADM formulation of Einstein's equation is the process to foliate the 4 dimensional manifold to a more familiar time-evolving 3 dimensional space, by which one can rewrite the Einstein tensor into ten component equations, where 4 are *constraint* equations that the spatial metric and extrinsic curvature must satisfy, while 6 are *evolutionary* equations that describe how the spatial metric changes with time. [1]

1.2 Notation convention

Throughout this paper, we will use $\mathcal{E}(TM)$ to denote the space of smooth sections on the tangent bundle of manifold M. Also, when we are dealing with local coordinates, we will use the Einstein summation convention and the use of Roman letters, i.e. {i,j,k,...} , will denote the sum over spatial coordinates, that is from 1 to 3, while the use of Greek letters, i.e. $\{\mu, \nu, \rho, \dots\}$, will denote the sum over all coordinates, that is from 0 to 4. We will also write π_i to denote the standard canonical projection onto the i^{th} coordinate. Furthermore, in order to simplify the notation, we will write ∂_i in place of $\frac{\partial}{\partial x^i}$, which will make some equations significantly easier to follow.

2 Foliation of space-time

Given a smooth 4 dimensional space-time manifold M, we will assume the existence of a diffeomorphism

$$\Phi : \mathbb{R} \times \Sigma \to M \tag{1}$$

where Σ is a smooth 3 dimensional manifold.

Let $g \in \mathcal{E}(T^*M \otimes T^*M)$ be the metric on M and $\nabla \in \mathcal{E}(T^*M \otimes TM)$ be the Levi-Civita connection associated with the metric g. Then we know ∇ is unique and fully specified by g with the following properties [1]:

- Metric compatibility: $\nabla_U \langle V, W \rangle = \langle \nabla_U V, W \rangle + \langle V, \nabla_U W \rangle$
- Torsion free: $\nabla_U V - \nabla_V U = [U, V]$

where $U, V, W \in \mathcal{E}(TM)$ and $\langle U, V \rangle = g(U, V)$.

Employed with the diffeomorphism Φ, we can define a natural *global time function* $T : M \to \mathbb{R}$ by composing the inverse map Φ^{-1} with the projection π_1 on the first coordinate $\mathbb{R}$ as follow

$$T(p) = \pi_1 \circ \Phi^{-1}(p) \tag{2}$$

where p is a point in M. Let Σ_t be a *spatial slice* at time t, then clearly $\Sigma_t = T^{-1}(t) \subset M$ is a sub-manifold of M.

We will now define the *unit normal vector field* to be

$$n := \frac{\pm dT^{\#}}{\sqrt{(sgn)\langle dT^{\#}, dT^{\#}\rangle}} \tag{3}$$

and

$$(sgn) := \frac{\langle dT^{\#}, dT^{\#}\rangle}{|\langle dT^{\#}, dT^{\#}\rangle|} = \langle n, n\rangle \tag{4}$$

where $dT^{\#}$ is the vector field defined by $\langle dT^{\#}, U\rangle = dT(U) = U(T)$ and the ± 1 denotes the choice of directions for the normal vector field. Since Φ is a diffeomorphism, $dT^{\#}$ is a nowhere vanishing vector field. If g is a Riemannian metric then $\langle dT^{\#}, dT^{\#}\rangle$ is positive and thus, our (sgn) is $+1$. If g is a Lorentzian metric, we will assume that the diffeomorphism gives us an everywhere time-like vector field $dT^{\#}$ and thus, our sgn will be -1. For the sake of generality, we will keep the (sgn) throughout in all of our derivations instead of picking a choice of signature.

Let us verify that our definition of n is actually the unit normal vector field. It is clear that $\langle n, n\rangle = 1$. Now let $X \in \mathcal{E}(TM)$ be a vector field such that $X|_p \in T\Sigma_t$ for every point $\Phi(t, s) = p \in M$, then we have:

$$\langle n, X\rangle = \frac{\pm 1}{\sqrt{(sgn)\langle dT^{\#}, dT^{\#}\rangle}}\langle dT^{\#}, X\rangle = \frac{\pm 1}{\sqrt{(sgn)\langle dT^{\#}, dT^{\#}\rangle}}X(T)$$

and since T is a constant function on each spatial slice Σ_t, $X|_p(T) = 0$ for all point $p \in M$, thus, X is perpendicular to n which confirms that n indeed is the unit normal vector field.

Using the diffeomorphism, we can will define an *at-rest curve* γ_σ for each point $\sigma \in \Sigma$ as follow

$$\gamma_\sigma : \mathbb{R} \to M \mid t \to \Phi(t, \sigma)$$

It is intuitively clear that any point that is at rest in the spatial slice perspective will follow one of at-rest curves in M. We then can define the at-rest vector field $\gamma' \in \mathcal{E}(TM)$ as follow

$$\gamma'|_{\Phi(t,\sigma)} := \gamma'_\sigma(t)$$

where $\gamma'_\sigma(t)$ is just the tangent vector to the at-rest curve γ_σ at the point $\gamma_\sigma(t)$. Here, the at-rest vector field is nothing but the push forward of the unit vector in $\mathbb{R}$ using the push forward maps $\gamma_{\sigma *}$.

One can simply think of the at-rest vector field γ' as the time-directed vector field. Thus, it is reasonable to expect γ' to never be fully tangential to any spatial slice at any point $p \in M$. Since we have

$$\langle n, \gamma'\rangle|_{\Phi(t,\sigma)} \propto \langle dT^{\#}, \gamma'\rangle|_{\Phi(t,\sigma)} \tag{5}$$

$$= \gamma'_\sigma(t)(T) = \gamma_\sigma(t)_*(\frac{d}{dt})(T) \tag{6}$$

$$= \frac{d}{dt}(T \circ \gamma_\sigma)|_t = \frac{d}{dt}(t) = 1 \tag{7}$$

it is clear that the at-rest vector field γ' always have a normal component and thus, behaves as we expected.

In general, we can not expect the at-rest vector field γ' to be the same as our normal unit vector field n. It should be obvious that there are foliation diffeomorphisms for which the at-rest vector field that have both the normal and spatial components. Thus, we will write

$$\gamma' = \alpha n + \beta$$

where $\alpha := (sgn)\langle\gamma', n\rangle \in C^\infty(M)$ is called the *lapse function* and $\beta := \gamma' - \alpha n \in \mathcal{E}(TM)$ is called the *shift vector*. Since γ' is never full tangential to any spatial slice at any point, α is nowhere vanishing.

3 Time-dependent geometric objects

Since the essence of the ADM formalism is a reformulation of Einstein tensor and other global geometric objects on M into time-evolution objects on spatial slice, we will now rigorously define each geometric objects on Σ and its time-dependent counter parts on M.

We will first need to introduce the embedding map e_t for each $t \in \mathbb{R}$ defined as follow

$$e_t : \Sigma \to M \mid e_t(\sigma) = \Phi(t, \sigma)$$

Then instead of dealing with vector fields on the 4-dimensional manifold M that are tangent to the spatial slices, we can work with a family of time-dependent vector fields on Σ. We will define a time-dependent family of vector fields the natural way

$$U : \mathbb{R} \to \mathcal{E}(T\Sigma) | t \to U(t)$$

where $U(t)$ is a *smooth* vector field on Σ for all $t \in \mathbb{R}$. Now, we can use the embedding maps to push forward the time-dependent family of vector field on Σ to a vector field on M. We will define the push-forward of U as vector field $\tilde{U}$ on M where

$$\tilde{U}|_{\Phi(t,\sigma)} := e_{t*}|_\sigma U(t)|_\sigma$$

Since each $U(t)$ is a vector field on Σ, it is reasonable for us to expect that the pushed-forward of U will be a vector field that is tangent to each spatial slice. We can check this fact by showing that $\tilde{U}$ is perpendicular with the normal vector field. We have

$$\begin{aligned} \langle\tilde{U}, n\rangle|_{\Phi(t,\sigma)} &= \frac{1}{\sqrt{(sgn)\langle dT^\#, dT^\#\rangle}} \langle\tilde{U}, dT^\#\rangle|_{\Phi(t,\sigma)} & (8) \\ &\propto \tilde{U}(T) = e_{t*}|_\sigma(U(t))(T)|_{\Phi(t,\sigma)} = U(t)|_\sigma(T \circ e_t)|_\sigma & (9) \end{aligned}$$

Since $T \circ e_t$ take all $\sigma \in \Sigma$ to $t \in \mathbb{R}$, $T \circ e_t$ is a constant function and thus $U(t)|_\sigma(T \circ e_t)|_\sigma = 0$, which confirms that $\tilde{U}$ is indeed perpendicular to the normal vector field.

It is important to notice here that the push-forward of a family of *smooth* vector fields U does not need to be a smooth vector field in M. In general, we can have a family of *smooth* vector fields that does not vary smoothly from one time slice to the next. For now, we implicitly assume here that the time-dependent family of vector fields on Σ is infinitely differential with respect to the time direction. We will be more precise about how we define the time-derivative of a time-dependent vector field in later section. From now on, we will also imply smoothness when we write vector fields, unless otherwise specified.

Now that we can relate a time-dependent family of vector fields on Σ to vector fields on M, it is easy to see that given a spatial vector field in M, one can relate such vector field to a push-forward of a time-dependent family of vector fields on Σ. Here, by spatial vector field in M, we just mean a vector field that is every where perpendicular to the normal vector field. The proof for this relation is in Appendix (A.1).

3.1 The induced spatial metric

Our next step would be to relate the given global metric g on M to a time-dependent family of *spatial metrics* on Σ. Clearly, the only way we can relate this two objects is when were are considering their actions on two spatial vector fields and their induced time-dependent vector fields on Σ. We will define the **induced family of spatial metrics** ${}^3g : \mathbb{R} \to \mathcal{E}(T^*\Sigma \otimes T^*\Sigma)$ the only way we can

$$ {}^3g(t)(U(t), V(t))|_\sigma = g(\tilde{U}, \tilde{V})|_{\Phi(t,\sigma)} \tag{10}$$

where $\tilde{U}$, $\tilde{V}$ are spatial vector fields on M and U, V are their induced time-dependent vector fields on Σ respectively. It is worthwhile to note here that for each $t \in \mathbb{R}$, we should get a metric ${}^3g(t) \in \mathcal{E}(T^*\Sigma \otimes T^*\Sigma)$ and its actions on vector fields on Σ should only depends on the vector fields $U(t)$, $V(t)$ and not the entire family U, V. One can easily see that because the value of $g(\tilde{U}, \tilde{V})|_{\Phi(t,\sigma)}$ only depend on the vectors $\tilde{U}|_{\Phi(t,\sigma)}, \tilde{V}_{\Phi(t,\sigma)}$ which in turn are just the push-forward of vectors $U(t)|_\sigma, V(t)|_\sigma$ by the map e_t.

Once again, it is important for us to consider the smoothness of the induced family of spatial metrics. It is clear that the smoothness of the induces family of spatial metrics entirely depends on the smoothness of global metric g in the "time-direction". Here, we will assume such smoothness is given. Since the metric g is assumed to be a smooth section of $T^*M \otimes T^*M$, it is a reasonable assumption for us to expect.

3.2 The Levi-Civita spatial connection

Now that we have define a metric ${}^3g$ on each spatial slice Σ_t, it is time for us to try and define a connection, especially the Levi-Civita connection, that associated with the spatial metric. Starting out with two vector fields $U(t), V(t)$ on Σ at time $t \in \mathbb{R}$, we want to define ${}^3\nabla(t)_{U(t)}V(t)$ where ${}^3\nabla(t)$ is the Levi-Civita connection associated with the metric ${}^3g(t)$. First, we can arbitrary expand $U(t), V(t)$ into two time-dependent family of vector fields U, V. [1] We then can push-forward these family to get the vector fields $\tilde{U}$ and $\tilde{V}$ on M, which then can be acted on by the Levi-Civita connection ∇ on M. Since $\nabla_{\tilde{U}}\tilde{V} \in \mathcal{E}(TM)$, in general, it needs not to be a spatial vector field. There are two different way of converting $\nabla_{\tilde{U}}\tilde{V}$ into a vector field on Σ. One way would be remove the normal component of $\nabla_{\tilde{U}}\tilde{V}$ and then push-forward to Σ using the projection map. We then would have

$$ ({}^3\nabla(t)_{U(t)}V(t))|_\sigma := (\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(\nabla_{\tilde{U}}\tilde{V})|_{\Phi(t,\sigma)} $$

The second way would be to just push-forward the vector field $\nabla_{\tilde{U}}\tilde{V}$ to Σ using the projection map as follow

$$ ({}^3\nabla(t)_{U(t)}V(t))|_\sigma := (\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(\nabla_{\tilde{U}}\tilde{V} - (sgn)\langle\nabla_{\tilde{U}}\tilde{V}, n\rangle n)|_{\Phi(t,\sigma)} \tag{11}$$

[1] The results that we get will only depend on $U(t)$ and $V(t)$ on the slice Σ_t and not on the expansion. [3]

Since

$$(\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}((sgn)\langle \nabla_{\tilde{U}} \tilde{V}, n\rangle n)|_{\Phi(t,\sigma)} \propto (\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(n)|_{\Phi(t,\sigma)}$$

and for any function f on Σ, we have

$$(\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(n)|_{\Phi(t,\sigma)}(f) = n(f \circ \pi_\Sigma \circ \Phi^{-1})|_{\Phi(t,\sigma)} \tag{12}$$

which clearly needs not to be zero. Here, $n(f \circ \pi_\Sigma \circ \Phi^{-1})|_{\Phi(t,\sigma)} = 0$ only in the case that the at-rest vector field γ' is proportional to the normal vector field n. Thus, these two definitions result in different vector fields in general. One can easily check that both definition will satisfy all the properties for a connection, however, it turns out that only the first definition satisfies both metric compatibility and torsion free and thus it is the Levi-Civita connection associated with ${}^3g$ [1]. The second definition in general will not satisfy the metric compatibility condition.

Here, it is important to discuss the significance of the two different perspectives in defining a connection above. It may seem more natural for some to use the second definition for several reasons including its simplicity and natural push-forward map. However, there are a few things that we should consider. First, the second definition converting a vector field on M to a vector field on Σ by projecting it along the "time-direction", in other words, along γ', and hence, by equation (12) above, the projection will not kill the normal component completely in general, but instead converts it partly into spatial vector fields. Again, in this case, the projection along the "time-direction" will only kill the normal component completely if γ' is proportional to the normal vector field n. Second, the push-forward map is actually not very natural to the space-time M at all. As the matter fact, the push-forward map here is completely depends on the foliation diffeomorphism and thus, just a perspective on the geometry of M. Therefore, it is reasonable to see that using the second definition by projecting along the "time-direction", we are choosing a definition that depends more on the foliation than the topology itself. On the other hand, it is much more natural for us to project a vector fields on M along the normal vector field since it is clearly canonical to the topology of each slices and how they are fitted into M. One may argue that we still use the foliation diffeomorphism map to get each spatial slice and hence the normal vector field itself, so it still depends on the foliation perspective. But it should be clear that even though the foliation diffeomorphism give us the spatial slice, the normal vector field to those slice is something canonical to the topology of the space-time and it does not depend on how one parametrizes the time-evolution of each slice, as opposed to γ'.

3.3 The Riemann curvature and extrinsic curvature tensors

Next, we will define the **spatial Riemann curvature tensor** ${}^3R(t) \in \mathcal{E}(T^*\Sigma \otimes T^*\Sigma \otimes T\Sigma \otimes T^*\Sigma)$ for each $t \in \mathbb{R}$ as follows

$${}^3R(t)(U(t), V(t))W(t) = ({}^3\nabla(t)_{U(t)}{}^3\nabla(t)_{V(t)} - {}^3\nabla(t)_{V(t)}{}^3\nabla(t)_{U(t)} - {}^3\nabla(t)_{[U(t),V(t)]})W(t) \tag{13}$$

A few things should be mentioned about this map. Clearly, this map is defined using a time-dependent family of connections and vector fields, however, as we points out above, the value of each connection and vector field only depend on their behavior on each slices Σ_t and thus, this definition is well defined. One can also check that our ${}^3R(t)$ above also satisfies all the properties of a tensor.

We will now define the last geometric object that we will need which is the **extrinsic curvature tensor** [2] $K(t) \in \mathcal{E}(T^*\Sigma \otimes T^*\Sigma)$ for each $t \in \mathbb{R}$ as follows

$$K(t)(U(t), V(t))|_\sigma := (sgn)\langle \nabla_{\tilde{U}} \tilde{V}, n \rangle|_{\Phi(t,\sigma)} \tag{14}$$

We can check that $K(t)$ is symmetric and $C^\infty(\Sigma)$-linear in both of its arguments. To see the symmetry, first consider the following:

$$\begin{aligned} \langle [\tilde{U}, \tilde{V}], n \rangle &\propto dT([\tilde{U}, \tilde{V}]) = [\tilde{U}, \tilde{V}](T) \\ &= \tilde{U}(\tilde{V}(T)) - \tilde{V}(\tilde{U}(T)) \end{aligned}$$

Since both $\tilde{U}$ and $\tilde{V}$ are spatial vector fields and T is constant on each Σ slice, both $\tilde{V}(T)$ and $\tilde{U}(T)$ will be zero, then

$$\langle [\tilde{U}, \tilde{V}], n \rangle = 0 \tag{15}$$

Now using the torsion free property of ∇, we have

$$\begin{aligned} K(t)(U(t), V(t))|_\sigma &:= (sgn)\langle \nabla_{\tilde{U}} \tilde{V}, n \rangle|_{\Phi(t,\sigma)} \\ &= (sgn)\langle \nabla_{\tilde{V}} \tilde{U} + [\tilde{U}, \tilde{V}], n \rangle|_{\Phi(t,\sigma)} \\ &= (sgn)\langle \nabla_{\tilde{V}} \tilde{U}, n \rangle|_{\Phi(t,\sigma)} \\ &= K(t)(V(t), U(t))|_\sigma \end{aligned}$$

Since, $K(t)$ is symmetric with both of its arguments, we will only need to show linearity for one of them. Let $U(t), V(t), W(t) \in \mathcal{E}(T\Sigma)$ and $f \in C^\infty(\Sigma)$, we have

$$\begin{aligned} K(t)(U(t), V(t) + fW(t))|\sigma &= (sgn)\langle \nabla_{\tilde{U}}(\tilde{V} + (f \circ \pi_\Sigma \circ \Phi^{-1})\tilde{W}, n \rangle|_{\Phi(t,\sigma)} \\ &= (sgn)\langle \nabla_{\tilde{U}} \tilde{V} + \tilde{U}(f \circ \pi_\Sigma \circ \Phi^{-1})\tilde{W} + (f \circ \pi_\Sigma \circ \Phi^{-1})\nabla_{\tilde{U}} \tilde{W}, n \rangle|_{\Phi(t,\sigma)} \end{aligned}$$

Since $\tilde{W}$ is perpendicular to n, we then have

$$\begin{aligned} K(t)(U(t), V(t) + fW(t))|\sigma &= (sgn)\langle \nabla_{\tilde{U}} \tilde{V}, n \rangle|_{\Phi(t,\sigma)} + (sgn)f(\sigma)\langle \nabla_{\tilde{U}} \tilde{W}, n \rangle|_{\Phi(t,\sigma)} \\ &= K(t)(U(t), V(t))|_\sigma + f(\sigma)K(t)(U(t), W(t))|_\sigma \end{aligned}$$

The definition of the *extrinsic curvature* is precisely the perpendicular component when we converting from $\nabla_{\tilde{U}} \tilde{V}$ to ${}^3\nabla_{U(t)} V(t)$ and thus, it will help us recover the lost information when we rewrite the Riemann and Einstein tensors in terms of time-dependent objects on Σ. Again, since $K(t)$ only depends on the induced connection and metric, it is completely specified by the information on a particular slice Σ_t and is independent of any other slices.

4 Differentiability in time direction

As we mention in the last section, it is important for us to assume that our time-dependent geometric objects are "varying smoothly" in the time direction even though their counterpart in M do not

[2]In other references and text books, this object will also be refer to as the **second fundamental form**.

have a notion of time. In this section, we will try to understand the time derivatives of time-dependent tensor families in Σ and define the notion of time-differentiability in a more precise language.

Let U be a time-dependent family of vector field as defined before, we will define its derivative at time $t \in \mathbb{R}$ and $\sigma \in \Sigma$ as follows:

$$\left.\frac{dU}{dt}\right|_{(t,\sigma)} := \lim_{h\in\mathbb{R}, h\to 0} \frac{U(t+h)|_\sigma - U(t)|_\sigma}{h} \tag{16}$$

Here, we are defining a time derivative for time-dependent family of vector fields, however, one can define the time derivative of any rank tensor family in a similar manner. Now, for any given time $t \in \mathbb{R}$, if the time derivative of U exists for all points $\sigma \in \Sigma$, we can define a time-derivative vector field $\frac{dU}{dt}(t)$ as

$$\left.\frac{dU}{dt}(t)\right|_{\sigma} := \left.\frac{dU}{dt}\right|_{(t,\sigma)} \tag{17}$$

It is important to point out that the result vector field defined as above need not to be a smooth vector field on Σ_t in general. We will say that a vector field is *n-times differentiable in the time direction* if we can take n-consecutive time derivatives of the result vector fields and still get a smooth vector field each time. Hence, we will call a vector field *infinitely differentiable in the time direction* or **time-differentiable** if we can always take time derivative of the results and get a smooth vector field.

In the last section, we mention that for any time-differentiable family of vector fields on Σ, we can embed it into M and get a smooth vector fields. It is also true that given any family of vector fields on M, its projection on the spatial slices will also give you a time-differentiable family of vector fields on Σ. It should be reasonable to see that by requiring time-differentiability, we make sure that the time-dependent family of vector fields will "vary smoothly" with time and hence, its embedded image in M will not only be smooth in spatial directions but also in the complement time direction, and thus, results in a smooth vector fields in M.

Now, given a family of vector fields $U : \mathbb{R} \to \mathcal{E}(T\Sigma)$, we can define its time derivative family of vector fields $U' = \frac{dU}{dt} : \mathbb{R} \to \mathcal{E}(T\Sigma)$. We then can embed both of these into M and get two vector fields $\tilde{U}$ and $\tilde{U}'$. In terms of family of vector fields, we understand the relationship between these two vector fields as U' is the time derivative of U as defined in this section. However, it is crucial for us to understand the relationship between $\tilde{U}$ and $\tilde{U}'$ as geometric objects in M. It turns out that the time-derivative operation on the time-dependent family of vector fields corresponds precisely to the Lie-derivative along the at-rest vector field $\gamma' \in \mathcal{E}(TM)$ of its counterpart vector field in M. We will now try to prove the relationship claim above.

The one parameter family of diffeomorphism induced by γ' is

$$\Psi : \mathbb{R} \times M \to M | (h, \Phi(t,\sigma)) \to \gamma_\sigma(t+h) = \Phi(t+h, \sigma)$$

For each choice of parameter $h \in \mathbb{R}$, we will denote the result diffeomorphism as Ψ_h. Now note that since $\Psi_{-h} \circ e_{t+h}(\sigma) = \Psi_{-h}(\Phi(t+h,\sigma)) = \Phi(t,\sigma)$, we have

$$e_t = \Psi_{-h} \circ e_{t+h} : \Sigma \to M$$

and

$$e_{t*}|_\sigma = \Psi_{(-h)*}|_{e_{t+h}(\sigma)} \circ e_{(t+h)*}|_\sigma = \Psi_{(-h)*}|_{\Phi(t+h,\sigma)} \circ e_{(t+h)*}|_\sigma$$

Then for a family of vector fields $U : \mathbb{R} \to \mathcal{E}(T\Sigma)$, we have

$$\begin{aligned}
e_{t*}|_\sigma \left(\frac{dU}{dt}(t)\bigg|_\sigma \right) &= e_{t*}|_\sigma \left(\lim_{h\to 0} \frac{U(t+h)|_\sigma - U(t)|_\sigma}{h} \right) \\
&= \lim_{h\to 0} \frac{e_{t*}|_\sigma U(t+h)|_\sigma - e_{t*}|_\sigma U(t)|_\sigma}{h} \\
&= \lim_{h\to 0} \frac{\Psi_{(-h)*}|_{\Phi(t+h,\sigma)} \circ e_{(t+h)*}|_\sigma U(t+h)|_\sigma - e_{t*}|_\sigma U(t)|_\sigma}{h} \\
&= \lim_{h\to 0} \frac{\Psi_{(-h)*}|_{\Phi(t+h,\sigma)} \tilde{U}|_{\Phi(t+h,\sigma)} - e_{t*}|_\sigma U(t)|_\sigma}{h} \\
&= (\mathcal{L}_{\gamma'}\tilde{U})|_{\Phi(t,\sigma)} \\
&= [\gamma', \tilde{U}]|_{\Phi(t,\sigma)}
\end{aligned}$$

where $\mathcal{L}_{\gamma'}\tilde{U}$ denotes the Lie-derivative of $\tilde{U}$ along the at-rest vector field γ' and $[\gamma', \tilde{U}]$ denotes the commutator of the two vector fields. Note it that by assuming *time-differentiable*, the vector $\mathcal{L}_{\gamma'}\tilde{U}$ is then a spatial vector field on M.

5 Foliation perspective of Riemann tensor

In this section, we will try to rewrite the Riemann tensor, whose components make up the Einstein tensor, in terms of time-dependent geometric objects on Σ. Since the Riemann tensor include several terms with covariant derivative ∇ on M, it is then our first job to express ∇ in terms on time-dependent objects on Σ. By definition, we have

$$({}^3\nabla(t)_{U(t)}V(t))|_\sigma := (\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(\nabla_{\tilde{U}}\tilde{V} - (sgn)\langle \nabla_{\tilde{U}}\tilde{V}, n\rangle n)|_{\Phi(t,\sigma)}$$

By using the embedding map, we then have

$$\begin{aligned}
e_{t*}|_\sigma({}^3\nabla(t)_{U(t)}V(t))|_\sigma &= e_{t*}|_\sigma \circ (\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(\nabla_{\tilde{U}}\tilde{V} - (sgn)\langle \nabla_{\tilde{U}}\tilde{V}, n\rangle n)|_{\Phi(t,\sigma)} \\
&= (e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(\nabla_{\tilde{U}}\tilde{V} - (sgn)\langle \nabla_{\tilde{U}}\tilde{V}, n\rangle n)|_{\Phi(t,\sigma)} \qquad (18)
\end{aligned}$$

Now, we will take a closer look at the map $(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}$. It turns out that the map $(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)} : T|_{\Phi(t,\sigma)}M \to T|_{\Phi(t,\sigma)}M$ takes $\gamma'_{\Phi(t,\sigma)}$ to 0. One can easily see this by consider the following:

$$\begin{aligned}
(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(\gamma'|_{\Phi(t,\sigma)})(f) &= \gamma'|_{\Phi(t,\sigma)}(f \circ e_t \circ \pi_\Sigma \circ \Phi^{-1}) \\
&= \frac{d}{dt}(f \circ e_t \circ \pi_\Sigma \circ \Phi^{-1} \circ \gamma_\sigma)|_t
\end{aligned}$$

where $f \in C^\infty(M)$. Notice that the map $(f \circ e_t \circ \pi_\Sigma \circ \Phi^{-1} \circ \gamma_\sigma)$ take $t' \in \mathbb{R}$ and return $f(\Phi(t,\sigma))$, thus it is a constant function with respect to t. Then we have

$$(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(\gamma'|_{\Phi(t,\sigma)}) = 0$$

Now we will also show that the map $(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}$ is an identity map on spatial vectors that are tangent to each slice Σ_t. Let $v \in T_{\Phi(t,\sigma)}M$ be perpendicular to the normal n. Then we have

$$(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(v)(f) = v(f \circ e_t \circ \pi_\Sigma \circ \Phi^{-1})|_{\Phi(t,\sigma)}$$

Note that the map $(e_t \circ \pi_\Sigma \circ \Phi^{-1})$ take $\Phi(t', \sigma)$ to $\Phi(t, \sigma)$ and thus is the identity function on each slice Σ_t. Since v is a spatial vector at each point $\Phi(t,\sigma) \in M$, the value $v(f)|_{\Phi(t,\sigma)}$ depends only on the changes of f on the slice Σ_t. Therefore, we have

$$(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}(v)(f) = v(f)|_{\Phi(t,\sigma)} \tag{19}$$

Then Eq. 18 becomes

$$e_{t*}|_\sigma({}^3\nabla(t)_{U(t)}V(t))|_\sigma = (\nabla_{\tilde{U}}\tilde{V} - (sgn)\langle\nabla_{\tilde{U}}\tilde{V}, n\rangle n)|_{\Phi(t,\sigma)}$$

or

$$\boxed{\nabla_{\tilde{U}}\tilde{V} = e_{t*}|_\sigma({}^3\nabla(t)_{U(t)}V(t))|_\sigma + K(t)(U(t), V(t))|_\sigma n|_{\Phi(t,\sigma)}} \tag{20}$$

Similarly, for any two time-dependent families of vector fields $U, V : \mathbb{R} \to \mathcal{E}(T\Sigma)$, we have

$$\langle\tilde{U}, \tilde{V}\rangle|_{\Phi(t,\sigma)} = {}^3g(t)(U(t), V(t))|_\sigma$$

We will now show the detailed proofs of all the components for the Riemann tensor by decomposing it into time-dependent geometric objects in the foliation perspective. But first, to simplify the equations, we will start by introduce some short-hand notations that will be used throughout all of the proofs from now on. First, we will implicitly assume the time-dependent for geometric objects in Σ, i.e. we will write U, ${}^3\nabla$, etc instead of $U(t)$ and ${}^3\nabla(t)$, etc. The time dependent should be understood by contex and we will also assume the use of the push-forward map e_{t*} to embed time-dependent family of vector fields on Σ to vector field on M when necessary. Thus we will write

$$({}^3\nabla_U V)|_{\Phi(t,\sigma)} = e_{t*}|\sigma({}^3\nabla(t)_{U(t)}V(t))|_\sigma$$

and

$$K(U, V)|_{\Phi(t,\sigma)} = K(t)(U(t), V(t))|_\sigma$$

These will then result in a much nicer expression for relating covariant derivative on M and Σ

$$\nabla_{\tilde{U}}\tilde{V} = {}^3\nabla_U V + K(U, V)n$$

Again, it is important to understand by contex here that the embedding of the vector field ${}^3\nabla_U V$ into M by the family e_{t*} is implicitly assumed for the equation to be well-defined. Such simplification will result in much easier proofs to follow but at the price of glossing over some important and subtle details. Therefore, throughout the proofs, we will pause and discuss the significance of any crucial but minute detail as necessary. Needless to say, since most of our expression are evaluated at a particular point $\Phi(t, \sigma)$, the embedding map is more of a formality than any geometric importance. Note that here $K(U, V)$ is a smooth function on M. We will also assume that all vector fields such as $\tilde{U}, \tilde{V}, \tilde{W}$ are spatial vector fields unless otherwise specified. [3]

[3] Notice that the reason we can denote all spatial global vector fields on M as the push-forward of time-dependent families of vector fields on Σ, i.e. $\tilde{U}$, is discussed in Appendix (A.1).

5.1 Gauss-Codazzi Equation

We will start with the definition of the Riemann tensor. Let $\tilde{U}, \tilde{V}, \tilde{W}$ be global spatial vector fields on M, then we have [4]

$$\begin{aligned}
(R(\tilde{U},\tilde{V})\tilde{W})|_{\Phi(t,\sigma)} &= (\nabla_{\tilde{U}}\nabla_{\tilde{V}}\tilde{W} - \nabla_{\tilde{V}}\nabla_{\tilde{U}}\tilde{W} - \nabla_{[\tilde{U},\tilde{V}]}\tilde{W})|_{\Phi(t,\sigma)} \\
&= \nabla_{\tilde{U}}{}^3\nabla_V W + \nabla_{\tilde{U}}(K(V,W)n) - \nabla_{\tilde{V}}{}^3\nabla_U W + \nabla_{\tilde{V}}(K(U,W)n) \\
&\quad -{}^3\nabla_{[U,V]}W - K([U,V],W)n \\
&= {}^3\nabla_U{}^3\nabla_V W + K(U,{}^3\nabla_V W)n + U(K(V,W))n + K(V,W)\nabla_{\tilde{U}}n \\
&\quad -{}^3\nabla_V{}^3\nabla_U W - K(V,{}^3\nabla_U W)n - V(K(U,W))n - K(U,W)\nabla_{\tilde{V}}n \\
&\quad -{}^3\nabla_{[U,V]}W - K([U,V],W)n
\end{aligned}$$

Note that the reason we can write $[\tilde{U},\tilde{V}]$ as $[U,V]$ is discussed in Appendix (A.2). Rearrange the equation a little bit and group all the terms that are proportional to n together, we get

$$\begin{aligned}
R(\tilde{U},\tilde{V})\tilde{W} &= {}^3\nabla_U{}^3\nabla_V W - {}^3\nabla_V{}^3\nabla_U W - {}^3\nabla_{[U,V]}W \\
&\quad +(U(K(V,W)) - K(V,{}^3\nabla_U W) - V(K(U,W)) + K(U,{}^3\nabla_V W) - K([U,V],W))n \\
&\quad +K(V,W)\nabla_{\tilde{U}}n - K(U,W)\nabla_{\tilde{V}}n
\end{aligned}$$

Here, note that ${}^3\nabla_U{}^3\nabla_V W - {}^3\nabla_V{}^3\nabla_U W - {}^3\nabla_{[U,V]}W$ is exactly ${}^3R(U,V)W$. Also since

$$({}^3\nabla_U K)(V,W) = U(K(V,W)) - K({}^3\nabla_U V,W) - K(V,{}^3\nabla_U W)$$

we have

$$U(K(V,W)) - K(V,{}^3\nabla_U W) = ({}^3\nabla_U K)(V,W) + K({}^3\nabla_U V,W)$$

and similarly

$$-V(K(U,W)) + K(U,{}^3\nabla_V W) = -({}^3\nabla_V K)(U,W) - K({}^3\nabla_V U,W)$$

However, since ${}^3\nabla$ is torsion free, we also know that

$$[U,V] = {}^3\nabla_U V - {}^3\nabla_V U$$

or

$$K([U,V],W) = K({}^3\nabla_U V,W) - K({}^3\nabla_V U,W)$$

Using all the above relations, we obtain the **Gauss-Codazzi Equation**:

$$R(\tilde{U},\tilde{V})\tilde{W} = {}^3R(U,V)W + K(V,W)\nabla_{\tilde{U}}n - K(U,W)\nabla_{\tilde{V}}n + (({}^3\nabla_U K)(V,W) - ({}^3\nabla_V K)(U,W))n$$

To get the **Gauss equation** we take the inner product of the *Gauss-Codazzi equation* with a spatial vector field $\tilde{X}$. Using the metric compatibility condition, we have the following relation

$$\langle\nabla_{\tilde{U}}n,\tilde{X}\rangle = \tilde{U}\langle n,\tilde{X}\rangle - \langle n,\nabla_{\tilde{U}}\tilde{X}\rangle = -(sgn)K(U,X)$$

[4] For simplicity, we will drop the evaluations after the first step.

Therefore, we have

$$\boxed{\langle R(\tilde{U},\tilde{V})\tilde{W},\tilde{X}\rangle = {}^3g({}^3R(U,V)W,X) - (sgn)K(V,W)K(U,X) + (sgn)K(U,W)K(V,X)} \tag{21}$$

On the other hand, to obtain the **Codazzi Equation** we only need to take the inner product of the *Gauss-Codazzi equation* with the unit normal n

$$\boxed{\langle R(\tilde{U},\tilde{V})\tilde{W},n\rangle = (sgn)(({}^3\nabla_U K)(V,W) - ({}^3\nabla_V K)(U,W))} \tag{22}$$

5.2 Ricci-Mainardi Equation

The last component of the Riemann tensor that we need is $\langle R(n,\tilde{V})\tilde{W},n\rangle$. First, since $\langle n,\nabla_n n\rangle = 0$, we have

$$\begin{aligned}
\langle \nabla_n \nabla_{\tilde{V}} \tilde{W}, n\rangle &= n\langle \nabla_{\tilde{V}}\tilde{W},n\rangle - \langle \nabla_{\tilde{V}}\tilde{W},\nabla_n n\rangle \\
&= (sgn)n(K(V,W)) - \langle {}^3\nabla_V W + K(V,W)n, \nabla_n n\rangle \\
&= (sgn)n(K(V,W)) - \langle {}^3\nabla_V W, \nabla_n n\rangle
\end{aligned} \tag{23}$$

We will now spend sometime proving the following lemma, which will become useful throughout the rest of the proof.

Lemma: For a spatial vector field $\tilde{X}$, we have

$$\langle \tilde{X},\nabla_n n\rangle = -\frac{(sgn)}{\alpha}\tilde{X}(\alpha)$$

Proof:

$$\begin{aligned}
\langle \tilde{X},\nabla_n n\rangle &= -\langle \nabla_n \tilde{X}, n\rangle \\
&= -\langle [n,\tilde{X}],n\rangle \\
&= \langle [\tilde{X},\frac{1}{\alpha}(\gamma'-\beta)],n\rangle \\
&= \tilde{X}\left(\frac{1}{\alpha}\right)\langle \gamma'-\beta,n\rangle + \frac{1}{\alpha}\langle[\tilde{X},\gamma'-\beta],n\rangle \\
&= -\frac{1}{\alpha^2}\tilde{X}(\alpha)\langle \alpha n,n\rangle + \frac{1}{\alpha}(\langle[\beta,\tilde{X}],n\rangle + \langle \mathcal{L}_{\gamma'}\tilde{X},n\rangle)
\end{aligned}$$

Since both $[\beta,\tilde{X}]$ and $\mathcal{L}_{\gamma'}\tilde{X}$ are spatial vector fields, we have

$$\langle \tilde{X},\nabla_n n\rangle = -\frac{(sgn)}{\alpha}\tilde{X}(\alpha) \tag{24}$$

Using this lemma and replace n with $\frac{1}{\alpha}(\gamma'-\beta)$ we have

$$\langle \nabla_n \nabla_{\tilde{V}}\tilde{W},n\rangle = \frac{(sgn)}{\alpha}((\gamma'-\beta)(K(V,W)) + ({}^3\nabla_V W)(\alpha))$$

We will now consider the second term

$$\begin{aligned}
\langle \nabla_{\tilde{V}}\nabla_n \tilde{W},n\rangle &= \tilde{V}\langle \nabla_n \tilde{W},n\rangle - \langle \nabla_n \tilde{W},\nabla_{\tilde{V}} n\rangle \\
&= -\tilde{V}\langle \tilde{W},\nabla_n n\rangle - \langle \nabla_{\tilde{W}} n,\nabla_{\tilde{V}} n\rangle - \langle [n,\tilde{W}],\nabla_{\tilde{V}} n\rangle
\end{aligned}$$

Since we have

$$\begin{aligned}
-\langle [n,\tilde{W}], \nabla_{\tilde{V}} n\rangle &= \langle [\tilde{W}, \frac{1}{\alpha}(\gamma' - \beta), \nabla_{\tilde{V}} n\rangle \\
&= \tilde{W}\left(\frac{1}{\alpha}\right)\langle \gamma' - \beta, \nabla_{\tilde{V}} n\rangle + \frac{1}{\alpha}\langle [\tilde{W}, \gamma' - \beta], \nabla_{\tilde{V}} n\rangle \\
&= \frac{1}{\alpha}\langle \mathcal{L}_{\beta}\tilde{W} - \mathcal{L}_{\gamma'}\tilde{W}, \nabla_{\tilde{V}} n\rangle \\
&= \frac{(sgn)}{\alpha}(K(V, \mathcal{L}_{\gamma'}\tilde{W}) - K(V, \mathcal{L}_{\beta}\tilde{W}))
\end{aligned}$$

and using the lemma we have

$$-\tilde{V}\langle \tilde{W}, \nabla_n n\rangle = (sgn)\tilde{V}\left(\frac{1}{\alpha}\tilde{W}(\alpha)\right) = \frac{(sgn)}{\alpha}\tilde{V}(\tilde{W}(\alpha)) - \frac{(sgn)}{\alpha^2}\tilde{V}(\alpha)\tilde{W}(\alpha)$$

Apply both of the relations above, we then have

$$\langle \nabla_{\tilde{V}}\nabla_n \tilde{W}, n\rangle = \frac{(sgn)}{\alpha}(K(V, \mathcal{L}_{\gamma'}\tilde{W}) - K(V, \mathcal{L}_{\beta}\tilde{W}) + \tilde{V}(\tilde{W}(\alpha)) - \frac{1}{\alpha}\tilde{V}(\alpha)\tilde{W}(\alpha)) - \langle \nabla_{\tilde{W}} n, \nabla_{\tilde{V}} n\rangle \quad (25)$$

Now, let's take a look at the final term $\langle \nabla_{[n,\tilde{V}]}\tilde{W}, n\rangle$. First, we have

$$[n,\tilde{V}] = -\tilde{V}\left(\frac{1}{\alpha}\right)\alpha n + \frac{1}{\alpha}[\gamma' - \beta, \tilde{V}] = \frac{1}{\alpha}\tilde{V}(\alpha)n + \frac{1}{\alpha}(\mathcal{L}_{\gamma'}\tilde{V} - \mathcal{L}_{\beta}\tilde{V})$$

Then

$$\begin{aligned}
\nabla_{[n,\tilde{V}]}\tilde{W} &= \nabla_{\frac{1}{\alpha}\tilde{V}(\alpha)n + \frac{1}{\alpha}(\mathcal{L}_{\gamma'}\tilde{V} - \mathcal{L}_{\beta}\tilde{V})}\tilde{W} \\
&= \frac{1}{\alpha}\tilde{V}(\alpha)\nabla_n \tilde{W} + \frac{1}{\alpha}(\nabla_{\mathcal{L}_{\gamma'}\tilde{V}}\tilde{W} - \nabla_{\mathcal{L}_{\beta}\tilde{V}}\tilde{W})
\end{aligned}$$

Taking inner product with the unit normal n we have

$$\begin{aligned}
\langle \nabla_{[n,\tilde{V}]}\tilde{W}, n\rangle &= \frac{1}{\alpha}\tilde{V}(\alpha)\langle \nabla_n \tilde{W}, n\rangle + \frac{1}{\alpha}\langle (\nabla_{\mathcal{L}_{\gamma'}\tilde{V}}\tilde{W} - \nabla_{\mathcal{L}_{\beta}\tilde{V}}\tilde{W}), n\rangle \\
&= -\frac{1}{\alpha}\tilde{V}(\alpha)\langle \tilde{W}, \nabla_n n\rangle + \frac{(sgn)}{\alpha}(K(\mathcal{L}_{\gamma'}\tilde{V}, W) - K(\mathcal{L}_{\beta}\tilde{V}, W)) \\
&= \frac{(sgn)}{\alpha}\left(\frac{1}{\alpha}\tilde{V}(\alpha)\tilde{W}(\alpha) + K(\mathcal{L}_{\gamma'}\tilde{V}, W) - K(\mathcal{L}_{\beta}\tilde{V}, W)\right)
\end{aligned}$$

Putting all the terms together, we get

$$
\begin{aligned}
\langle R(n,\tilde{V})\tilde{W},n\rangle &= \frac{(sgn)}{\alpha}((\gamma'-\beta)(K(V,W))+({}^3\nabla_V W)(\alpha)) \\
&\quad -\left(\frac{(sgn)}{\alpha}(K(V,\mathcal{L}_{\gamma'}\tilde{W})-K(V,\mathcal{L}_\beta\tilde{W})+\tilde{V}(\tilde{W}(\alpha))-\frac{1}{\alpha}\tilde{V}(\alpha)\tilde{W}(\alpha))-\langle\nabla_{\tilde{W}}n,\nabla_{\tilde{V}}n\rangle\right) \\
&\quad -\frac{(sgn)}{\alpha}\left(\frac{1}{\alpha}\tilde{V}(\alpha)\tilde{W}(\alpha)+K(\mathcal{L}_{\gamma'}\tilde{V},W)-K(\mathcal{L}_\beta\tilde{V},W)\right) \\
&= \frac{(sgn)}{\alpha}\left(\gamma'(K(V,W))-K(\mathcal{L}_{\gamma'}\tilde{V},W)-K(V,\mathcal{L}_{\gamma'}\tilde{W})\right) \\
&\quad -\frac{(sgn)}{\alpha}\left(\beta(K(V,W))-K(\mathcal{L}_\beta\tilde{V},W)-K(V,\mathcal{L}_\beta\tilde{W})\right) \\
&\quad +\frac{(sgn)}{\alpha}\left(({}^3\nabla_V W)(\alpha)-\tilde{V}(\tilde{W}(\alpha))\right)+\langle\nabla_{\tilde{W}}n,\nabla_{\tilde{V}}n\rangle
\end{aligned}
$$

Since we know that (see Appendix (A.2))

$$\mathcal{L}_\beta\tilde{V}|_{\Phi(t,\sigma)}=[\beta.\tilde{V}]|_{\Phi(t,\sigma)}=[\bar{\beta},V]|_\sigma=(\mathcal{L}_{\bar{\beta}}V)|_\sigma$$

where $\bar{\beta}:\mathbb{R}\to\mathcal{E}(T\Sigma)$ is the projection of β onto Σ. Again, note that the time-dependent and the embedding map are implicitly assumed. Since $K(t)$ is a tensor, we have

$$\left(\bar{\beta}(K(V,W))-K(\mathcal{L}_{\bar{\beta}}\tilde{V},W)-K(V,\mathcal{L}_{\bar{\beta}}\tilde{W})\right)=(\mathcal{L}_{\bar{\beta}}K)(V,W)$$

Similarly we have

$$\left(\gamma'(K(V,W))-K(\mathcal{L}_{\gamma'}\tilde{V},W)-K(V,\mathcal{L}_{\gamma'}\tilde{W})\right)=(\mathcal{L}_{\gamma'}K)(V,W)=\left(\frac{d}{dt}K\right)(V,W)$$

Finally, we also have

$$
\begin{aligned}
({}^3\nabla_V W)(\alpha)-\tilde{V}(\tilde{W}(\alpha)) &= ({}^3\nabla_V W)(\alpha)-V(W(\bar{\alpha})) \\
&= d\bar{\alpha}({}^3\nabla_V W)-V(d\bar{\alpha}(W)) \\
&= -({}^3\nabla_V d\bar{\alpha}(W))
\end{aligned}
$$

where $\bar{\alpha}(t)=\alpha\circ e_t:\Sigma\to\mathbb{R}$. Putting everything together we obtain the **Ricci-Mainardi equation**:

$$\boxed{\langle R(n,\tilde{V})\tilde{W},n\rangle=\frac{(sgn)}{\bar{\alpha}}\left(\left(\frac{d}{dt}K\right)(V,W)-(\mathcal{L}_{\bar{\beta}}K)(V,W)-{}^3\nabla_V d\bar{\alpha}(W)\right)+\langle\nabla_{\tilde{W}}n,\nabla_{\tilde{V}}n\rangle} \quad (26)$$

6 Foliation Coordinates

We will start out by define a coordinate chart $(\mathbb{R}\times U,(id_{\mathbb{R}},X))$ around the point (t,σ) for the smooth product manifold $\mathbb{R}\times\Sigma$ [4] as follows

$$(id_{\mathbb{R}},X):\mathbb{R}\times U\to\mathbb{R}\times V|(t,\sigma)\to(t,X(\sigma))$$

where U is an open set in Σ, V is an open set in $\mathbb{R}^3$ and $X : U \to V$ is a diffeomorphism. Composing the above map with the diffeomorphism $\Phi^{-1} : M \to \mathbb{R} \times \Sigma$ we get a diffeomorphism

$$(id_{\mathbb{R}}, X) \circ \Phi^{-1} : \Phi(\mathbb{R} \times U) \to \mathbb{R} \times V | p \to (T(p), X \circ \pi_\Sigma \circ \Phi^{-1}(p))$$

where $T : M \to \mathbb{R}$ is the global time function as defined before. Then the set $\phi(\mathbb{R} \times U$ and the diffeomorphism $(id_{\mathbb{R}}, X) \circ \Phi^{-1}$ is a coordinate chart on M. For simplicity, we will denote the map $(id_{\mathbb{R}}, X) \circ \Phi^{-1}$ as $\tilde{X}$. Then $\tilde{X}$ is a **foliation coordinate chart**. Then the **foliation coordinate functions** associated with $\tilde{X}$ are

$$t := \pi_0 \circ (id_{\mathbb{R}}, X) \circ \Phi^{-1} : M \to \mathbb{R}$$
$$\tilde{x}^i := \pi_i \circ (id_{\mathbb{R}}, X) \circ \Phi^{-1} : M \to \mathbb{R} \qquad \text{for i = 1,2,3}$$

where π_0, π_i are the canonical projections on to the corresponding coordinates. Note that the map t is the same as the global time function T.

It turns out that the time coordinate vector field is the same as the at-rest vector field. To see this, consider the following

$$\begin{aligned}
\left.\frac{\partial}{\partial t}\right|_p f &= \left.\frac{\partial}{\partial x^0}\left(f \circ \tilde{X}^{-1}\right)\right|_{\tilde{X}(p)} \\
&= \left.\frac{\partial}{\partial x^0}\left(f \circ \Phi \circ (id_{\mathbb{R}}, X^{-1})\right)\right|_{\tilde{X}(p)} \\
&= \lim_{h \to 0} \frac{f \circ \Phi \circ (id_{\mathbb{R}}, X^{-1})(t_p + h, x_1, x_2, x_3) - f \circ \Phi \circ (id_{\mathbb{R}}, X^{-1})(t_p, x_1, x_2, x_3)}{h} \\
&= \lim_{h \to 0} \frac{f(\Phi(t_p + h, \sigma_p)) - f(\Phi(t_p, \sigma_p))}{h} \\
&= \lim_{h \to 0} \frac{f(\gamma_{\sigma_p}(t_p + h) - f(\gamma_{\sigma_p}(t_p))}{h} \\
&= \gamma'|_p f
\end{aligned}$$

where $f \in C^\infty(M)$, $p = \Phi(t_p, \sigma_p)$ and $X(\sigma_p) = (x_1, x_2, x_3)$.

Now, let $\{x^1, x^2, x^3\}$ and $\{\partial x^1, \partial x^2, \partial x^3\}$ be the coordinate functions and coordinate vectors respectively on an open set in Σ. Then denote the induced coordinate functions and induced coordinate vectors on the corresponding open set in M as $\{t, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3\}$ and $\{\partial\tilde{x}^0, \partial\tilde{x}^1, \partial\tilde{x}^2, \partial\tilde{x}^3\}$.

In this coordinate system, we have

$$^3g(t)\left.\left(\frac{\partial}{\partial x^i}(t), \frac{\partial}{\partial x^j}(t)\right)\right|_\sigma = g\left.\left(\frac{\partial \tilde{x}^i}{\partial \tilde{x}^j}\right)\right|_{\Phi(t,\sigma)}$$

Then

$$\boxed{^3g_{ij}(t, \sigma) = g_{ij} \circ \Phi(t, \sigma) \qquad \text{for i,j = 1,2,3}}$$

We also have

$$\begin{aligned}
g\left.\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \tilde{x}^i}\right)\right|_{\Phi(t,\sigma)} &= \left.\left\langle \gamma', \frac{\partial}{\partial \tilde{x}^i}\right\rangle\right|_{\Phi(t,\sigma)} \\
&= \left.\left\langle \beta, \frac{\partial}{\partial \tilde{x}^i}\right\rangle\right|_{\Phi(t,\sigma)} \\
&:= (\beta^j g_{ij})|_{\Phi(t,\sigma)} = \beta_i(\Phi(t, \sigma))
\end{aligned}$$

and

$$g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)\Big|_{\Phi(t,\sigma)} = (sgn)\alpha^2(\Phi(t,\sigma)) + \beta^i\beta_i(\Phi(t,\sigma))$$

Then the metric $g \in \mathcal{E}(T^*M \otimes T^*M)$ in foliation coordinates is

$$g_{\mu\nu} = \left[\begin{array}{c|ccc} (sgn)\alpha^2 + {}^3g_{km}\beta^k\beta^m & {}^3g_{1k}\beta^k & {}^3g_{2k}\beta^k & {}^3g_{3k}\beta^k \\ \hline {}^3g_{1k}\beta^k & & & \\ {}^3g_{2k}\beta^k & & [{}^3g_{ij}] & \\ {}^3g_{3k}\beta^k & & & \end{array}\right]$$

and the inverse metric is

$$g^{\mu\nu} = \left[\begin{array}{c|ccc} \frac{(sgn)}{\alpha^2} & -(sgn)\frac{\beta^1}{\alpha^2} & -(sgn)\frac{\beta^2}{\alpha^2} & -(sgn)\frac{\beta^3}{\alpha^2} \\ \hline -(sgn)\frac{\beta^1}{\alpha^2} & & & \\ -(sgn)\frac{\beta^2}{\alpha^2} & & [{}^3g_{ij} + \frac{(sgn)}{\alpha^2}\beta^i\beta^j] & \\ -(sgn)\frac{\beta^3}{\alpha^2} & & & \end{array}\right]$$

or in tensorial forms

$$g = (sgn)\alpha^2 dt \otimes dt + {}^3g_{ij}(dx^i + \beta^i dt) \otimes (dx^j + \beta^j dt)$$

and

$$g^{-1} = \frac{(sgn)}{\alpha^2}(\beta^i\partial_i - \partial_t) \otimes (\beta^j\partial_j - \partial_t) + {}^3g^{ij}\partial_i \otimes \partial_j$$

Now by replacing $\tilde{U} = \partial\tilde{x}^i$, $\tilde{V} = \partial\tilde{x}^j$, $\tilde{X} = \partial\tilde{x}^k$, and $\tilde{W} = \partial\tilde{x}^l$, we can rewrite the **Gauss-Codazzi equation** in foliation coordinates as follows

$$(R_{ij}{}^{\mu}{}_l\tilde{\partial}_\mu) = ({}^3R_{ij}{}^k{}_l)\tilde{\partial}_k + K_{jl}(\nabla_{\tilde{\partial}_i n}) - K_{il}(\nabla_{\tilde{\partial}_j n}) + ({}^3\nabla_i K_{jl} - {}^3\nabla_j K_{il})n$$

where the notation ${}^3\nabla_i K_{jl}$ stands for the jl component of the (0,2) tensor ${}^3\nabla_{\partial_i}K$ and not the covariant derivative of the function K_{jl}.

Now using metric compatibility condition, we can see that

$$\langle\nabla_{\tilde{\partial}_i} n, n\rangle = 0$$

Then we can write $\nabla_{\tilde{\partial}_i} n$ as a combination of spatial coordinate vectors. We have

$$\langle\nabla_{\tilde{\partial}_i} n, \tilde{\partial}_l\rangle = -\langle n, \nabla_{\tilde{\partial}_l}\rangle = -(sgn)K(\partial_i, \partial_l) = -(sgn)K_{il}$$

Then

$$\nabla_{\tilde{\partial}_i} n = -(sgn)K_i{}^l\tilde{\partial}_l$$

Using this relation, the **Gauss-Codazzi equation** becomes

$$(R_{ij}{}^{\mu}{}_l\tilde{\partial}_\mu) = ({}^3R_{ij}{}^m{}_l + (sgn)(K_{il}K_j^m - K_{jl}K_i^m)\tilde{\partial}_m + ({}^3\nabla_i K_{jl} - {}^3\nabla_j K_{il})n$$

Now, by taking the inner product with a spatial coordinate vector $\tilde{\partial}_k$, we have the **Gauss equation**

$$R_{ijkl} = {}^3R_{ijkl} + (sgn)(K_{il}K_{jk} - K_{jl}K_{ik})$$

Alternately, we can take the inner product with the normal vector field n and obtain the **Codazzi equation**

$$R_{ij}{}^{\nu}{}_{l}n_{\nu} = (sgn)({}^{3}\nabla_{i}K_{jl} - {}^{3}\nabla_{j}K_{il})$$

And finally, we can write the **Ricci-Mainardi equation** in components as follows

$$R_{\mu j}{}^{\nu}{}_{l}n^{\mu}n_{\nu} = \frac{(sgn)}{\bar{\alpha}}\left(\frac{d}{dt}K_{jl} - \mathcal{L}_{\bar{\beta}}K_{jl} - {}^{3}\nabla_{j}d\bar{\alpha}_{l}\right) + K_{jm}K^{m}_{l}$$

where again, $\frac{d}{dt}K_{jl}$ is the jl component of $\frac{d}{dt}K$, $\mathcal{L}_{\bar{\beta}}K_{jl}$ stands for the jl component of $\mathcal{L}_{\bar{\beta}}K$ and ${}^{3}\nabla_{j}d\bar{\alpha}_{l}$ is the l component of the one form ${}^{3}\nabla_{j}d\bar{\alpha}$.

7 Einstein Equations [6]

In this section, we will repeatedly use the following two properties (and a few others) of the Riemann tensor:

1. $R(A,B)C = -R(B,A)C$
2. $\langle R(A,B)C, D\rangle = -\langle R(A,B)D, C\rangle$

A thorough treatment of the Riemann tensor, Ricci Scalar and Einstein tensor can be found in [5].

7.1 $R_{\mu\nu}n^{\mu}n^{\nu}$ component

We have $R_{\mu\nu}n^{\mu}n^{\nu} = R_{0\mu}{}^{0}{}_{\nu}n^{\mu}n^{\nu} + R_{j\mu}{}^{j}{}_{\nu}n^{\mu}n^{\nu}$ but

$$R_{0\mu}{}^{0}{}_{\nu}n^{\mu}n^{\nu} = R_{0\mu}{}^{\rho}{}_{\nu}n^{\mu}n^{\nu}\frac{\langle\partial_{\rho}, n\rangle}{\langle\partial_{0}, n\rangle} = \frac{\langle R(\partial_{0}, n)n, n\rangle}{\langle\partial_{0}, n\rangle} = 0$$

so

$$\begin{aligned}
R_{\mu\nu}n^{\mu}n^{\nu} &= R_{ju}{}^{j}{}_{\nu}n^{\mu}n^{\nu} \\
&= R_{\mu j}{}^{\nu}{}_{\rho}n^{\mu}n_{\nu}g^{\rho j} \\
&= R_{\mu j}{}^{\nu}{}_{0}n^{\mu}n_{\nu}g^{0j} + R_{\mu j}{}^{\nu}{}_{l}n^{\mu}n_{\nu}g^{jl}
\end{aligned}$$

where $R_{\mu j}{}^{\nu}{}_{0}n^{\mu}n_{\nu} = \langle R(n,\partial_{j})\partial_{0}, n\rangle = \langle R(n,\partial_{j})(\alpha n+\beta), n\rangle = \langle R(n,\partial_{j})\beta, n\rangle$. Then

$$R_{\mu j}{}^{\nu}{}_{0}n^{\mu}n_{\nu} = \beta^{l}R_{\mu j}{}^{\nu}{}_{l}n^{\mu}n_{\nu}$$

Therefore, by using the components of the inverse metric discussed in the previous section, we have

$$\begin{aligned}
R_{\mu\nu}n^{\mu}n^{\nu} &= R_{\mu j}{}^{\nu}{}_{l}n^{\mu}n_{\nu}(g^{0j}\beta^{l} + g^{jl}) \\
&= R_{\mu j}{}^{\nu}{}_{l}n^{\mu}n_{\nu}(g^{jl} - \frac{(sgn)}{\alpha^{2}}\beta^{j}\beta^{l}) \\
&= R_{\mu j}{}^{\nu}{}_{l}n^{\mu}n_{\nu}({}^{3}g^{jl})
\end{aligned}$$

So we have

$$R_{\mu\nu}n^{\mu}n^{\nu} = \frac{(sgn)}{\bar{\alpha}}\left({}^{3}g^{jl}\left(\frac{d}{dt}K_{jl} - \mathcal{L}_{\bar{\beta}}K_{jl}\right) - {}^{3}\nabla^{l}d\bar{\alpha}_{l}\right) + K_{lm}K^{lm} \tag{27}$$

Now let us take a closer look at the term ${}^3g^{jl}\left(\frac{d}{dt}K_{jl}-\mathcal{L}_{\bar\beta}K_{jl}\right)$. We have [6]

$$\begin{aligned}
{}^3g^{jl}\left(\frac{d}{dt}K_{jl}-\mathcal{L}_{\bar\beta}K_{jl}\right) &= {}^3g^{jl}\left(\frac{d}{dt}({}^3g_{rj}\ {}^3g_{sl}K^{rs})-\bar\beta^k\partial_k({}^3g_{rj}\ {}^3g_{sl}K^{rs})-\partial_j\bar\beta^k\ {}^3g_{kr}K^r_l-\partial_l\bar\beta^k\ {}^3g_{ks}K^s_j\right)\\
&= {}^3g_{rs}\frac{d}{dt}K^{rs}+2\left(\frac{d}{dt}g_{rs}\right)K^{rs}-{}^3g_{rs}\bar\beta^k\partial_kK^{rs}-2\bar\beta^k(\partial_k\ {}^3g_{rs})K^{rs}\\
&\quad -\partial_s\bar\beta^k\ {}^3g_{kr}K^{rs}-\partial_r\bar\beta^k\ {}^3g_{kr}K^{rs}\\
&= \frac{d}{dt}({}^3g_{rs}K^{rs})+\left(\frac{d}{dt}g_{rs}\right)K^{rs}-\bar\beta^k\partial_k({}^3g_{rs}K^{rs})-\bar\beta^k(\partial_k\ {}^3g_{rs})K^{rs}\\
&\quad -\partial_s\bar\beta^k\ {}^3g_{kr}K^{rs}-\partial_r\bar\beta^k\ {}^3g_{kr}K^{rs}\\
&= \frac{d}{dt}(K)-\bar\beta^k\partial_k(K)+\left(\frac{d}{dt}g_{rs}-\bar\beta^k\partial_k\ {}^3g_{rs}-\partial_s\bar\beta^k\ {}^3g_{kr}-\partial_r\bar\beta^k\ {}^3g_{kr}\right)K^{rs}
\end{aligned}$$

We also have

$$\begin{aligned}
n\langle\partial_r,\partial_s\rangle &= \langle\nabla_n\partial_r,\partial_s\rangle+\langle\partial_r,\nabla_n\partial_s\rangle\\
&= \langle\nabla_{\partial_r}n,\partial_s\rangle+\langle[n,\partial_r],\partial_s\rangle+\langle\partial_r,\nabla_{\partial_s}n\rangle+\langle\partial_r,[n,\partial_s]\rangle\\
&= -\langle n,\nabla_{\partial_r}\partial_s\rangle+\langle[n,\partial_r],\partial_s\rangle-\langle\nabla_{\partial_s}\partial_r,n\rangle+\langle\partial_r,[n,\partial_s]\rangle\\
&= -2(sgn)K(\partial_r,\partial_s)+\langle\mathcal{L}_n\partial_r,\partial_s\rangle+\langle\partial_r,\mathcal{L}_n\partial_s\rangle
\end{aligned}$$

Since

$$\begin{aligned}
-2(sgn)K_{rs} &= n\langle\partial_r,\partial_s\rangle-\langle\mathcal{L}_n\partial_r,\partial_s\rangle-\langle\partial_r,\mathcal{L}_n\partial_s\rangle\\
&= (\mathcal{L}_ng)(\partial_r,\partial_s)\\
&= \frac{1}{\alpha}\big((\mathcal{L}_{\gamma'}g)(\partial_r,\partial_s)-(\mathcal{L}_\beta g)(\partial_r,\partial_s)\big)
\end{aligned}$$

Therefore

$$ {}^3g^{jl}\left(\frac{d}{dt}K_{jl}-\mathcal{L}_{\bar\beta}K_{jl}\right)=\frac{d}{dt}(K)-\bar\beta^k\partial_k(K)-2(sgn)\bar\alpha K_{rs}K^{rs}$$

Then we have

$$R_{\mu\nu}n^\mu n^\nu=\frac{(sgn)}{\bar\alpha}\left(\frac{d}{dt}(K)-\bar\beta^k\partial_k(K)-{}^3\nabla^l d\bar\alpha_l\right)-K_{lm}K^{lm}$$

7.2 $R_{i\mu}n^\mu$ component

We have $R_{i\mu}n^\mu=R_{\alpha i}{}^\alpha{}_\mu n^\mu=R^0_{0i\mu}n^\mu+R^j_{ji\mu}n^\mu$, but

$$R_{0i}{}^0{}_\mu n^\mu=\frac{\langle R(\partial_0,\partial_i)n,n\rangle}{\langle\partial_0,n\rangle}=0$$

Then

$$\begin{aligned}
R_{i\mu}n^\mu &= R_{ji}{}^j{}_\mu n^\mu=R_{ij}{}^\mu{}_\rho n_m u g^{\rho j}\\
&= \langle R(\partial_i,\partial_j)(\alpha n+\beta),n\rangle g^{0j}+\langle R(\partial_i,\partial_j\rangle\partial_l,n\rangle g^{jl}\\
&= \langle R(\partial_i,\partial_j)\partial_l,n\rangle(g^{0j}\beta^l+g^{jl})\\
&= \langle R(\partial_i,\partial_j)\partial_l,n\rangle({}^3g^{jl})
\end{aligned}$$

Therefore, we have

$$R_{i\mu}n^\mu=(sgn)({}^3\nabla_iK^l{}_l-{}^3\nabla_lK_i{}^l)$$

7.3 R_{ij} component

Now consider $R_{jl} = R_{\alpha j}{}^{\alpha}{}_{l} = R_{0j}{}^{0}{}_{l} + R_{ij}{}^{i}{}_{l}$. We have

$$R_{0j}{}^{0}{}_{l} = \frac{\langle R(\partial_0, \partial_j)\partial_l, n\rangle}{\langle \partial_0, n\rangle} = \frac{\rangle R(\alpha n + \beta, \partial_j)\partial_l, n\rangle}{\langle \partial_0, n\rangle}$$

and $R_{ij}{}^{i}{}_{l} = \langle R(\partial_i, \partial_j)\partial_l, \partial_\nu\rangle g^{\nu i}$. Then

$$\begin{aligned}
R_{jl} &= \frac{\langle R(\alpha n + \beta, \partial_j)\partial_l, n\rangle}{\langle \partial_0, n\rangle} + \langle R(\partial_i, \partial_j)\partial_l, \partial_\nu\rangle g^{\nu i} \\
&= (sgn)\langle R(n, \partial_j)\partial_l, n\rangle + \frac{(sgn)\beta^i}{\alpha}\langle R(\partial_i, \partial_j)\partial_l, n\rangle - \langle R(\partial_i, \partial_j)\partial_l, \partial_0\rangle g^{0i} + \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle g^{ik} \\
&= (sgn)\langle R(n, \partial_j)\partial_l, n\rangle + \langle R(\partial_i, \partial_j)\partial_l, n\rangle \alpha \left(\frac{(sgn)\beta^i}{\alpha^2} + g^{0i}\right) + \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle (g^{ik} + g^{0i}\beta^k) \\
&= (sgn)\langle R(n, \partial_j)\partial_l, n\rangle + \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle (g^{ik} + g^{0i}\beta^k) \\
&= (sgn)\langle R(n, \partial_j)\partial_l, n\rangle + \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle \left(g^{ik} - \frac{(sgn)}{\alpha^2}\beta^i\beta^k\right) \\
&= (sgn)\langle R(n, \partial_j)\partial_l, n\rangle + \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle \left({}^3 g^{ik}\right)
\end{aligned}$$

Therefore, we have

$$R_{jl} = \frac{1}{\bar{\alpha}}\left(\frac{d}{dt}K_{jl} - \mathcal{L}_{\bar{\beta}}K_{jl} - {}^3\nabla_j d\bar{\alpha}_l\right) + ({}^3 R_{jl} + (sgn)(2K_{jm}K_l{}^m - K_m{}^m K_{jl})$$

7.4 Ricci Scalar

Now let us consider the Ricci Scalar. We have

$$R_{\mu\nu}n^\mu n^\nu = R_{00}n^0 n^0 + R_{0j}n^0 n^j + R_{l0}n^l n^0 + R_{jl}n^j n^l$$

since $n^0 n^0 = (sgn)g^{00}$, then

$$R_{00}g^{00} = (sgn)(R_{\mu\nu}n^\mu n^\nu - R_{0j}n^0 n^j - R_{l0}n^l n^0 - R_{jl}n^j n^l)$$

We also have $n^0 n^j = -\frac{\beta^j}{\alpha^2} = (sgn)g^{0j}$, so

$$R_{00}g^{00} = (sgn)\left(R_{\mu\nu}n^\mu n^\nu - R_{jl}\frac{\beta^j\beta^l}{\alpha^2}\right) - R_{0j}g^{0j} - R_{l0}g^{l0}$$

Then the Ricci Scalar becomes

$$\begin{aligned}
R &= R_{00}g^{00} + R_{0j}g^{0j} + R_{l0}g^{l0} + R_{jl}g^{jl} \\
&= (sgn)R_{\mu\nu}n^\mu n^\nu + R_{jl}\left(g^{jl} - \frac{(sgn)}{\alpha^2}\beta^j\beta^l\right) \\
&= ((sgn)\langle R(n, \partial_j)\partial_l, n\rangle + R_{jl})\left(g^{jl} - \frac{(sgn)}{\alpha^2}\beta^j\beta^l\right) \\
&= \left((sgn)2\langle R(n, \partial_j)\partial_l, n\rangle + \langle R(\partial_i, \partial_j)\partial_l, \partial_k\rangle \left(g^{ik} - \frac{(sgn)}{\alpha^2}\beta^i\beta^k\right)\right)\left(g^{jl} - \frac{(sgn)}{\alpha^2}\beta^j\beta^l\right)
\end{aligned}$$

Then

$$R = \left({}^3R + \frac{2}{\bar{\alpha}} \left(\frac{d}{dt} K - \bar{\beta}^k \partial_k K - {}^3\nabla^m d\bar{\alpha}_m \right) - (sgn)(K_{mn}K^{mn} + (K_m{}^m)^2) \right) \tag{28}$$

7.5 Einstein Tensor

$$\begin{aligned}
G_{\mu\nu} n^\mu n^\nu &= R_{\mu\nu} n^\mu n^\nu - \frac{1}{2} g_{\mu\nu} n^\mu n^\nu R \\
&= \left(\langle R(n, \partial_j)\partial_l, n \rangle - \langle R(n, \partial_j)\partial_l, n \rangle - \frac{(sgn)}{2} \langle R(\partial_i, \partial_j)\partial_l, \partial_k \rangle (g^{ik} + g^{0i}\beta^k) \right) \left(g^{jl} - \frac{(sgn)}{\alpha^2} \beta^j \beta^l \right) \\
&= -\frac{(sgn)}{2} \langle R(\partial_i, \partial_j)\partial_l, \partial_k \rangle (g^{ik} + g^{0i}\beta^k) \left(g^{jl} - \frac{(sgn)}{\alpha^2} \beta^j \beta^l \right)
\end{aligned}$$

Then we have

$$\boxed{G_{\mu\nu} n^\mu n^\nu = -\frac{(sgn)}{2} \left({}^3R + (sgn)(K_{ml}K^{ml} - (K_m{}^m)^2 \right)}$$

8 Conclusion

Throughout this work, it is clear that there are (at least) two different perspectives that is competing in the theory. The first perspective is the covariant perspective, which is purely geometric, and the most canonical in terms of space-time structure. The second is the Newtonian perspective, where objects are living in a time evolving universe. Both perspectives revolve around the same geometry of space-time and hence, neither one is more true than the other. However, by obtaining all the components of the Riemann tensor in terms time-dependent geometric objects, we were able to better understand how our time-dependent 3 dimensional universe fits into the 4 dimensional space-time and the geometry that results in our physical world. We also were able to rewrite all of the components of Riemann tensor and Einstein tensor in local coordinates, which are in agreement with literature.

Appendices

A Some properties of global vector fields on M

A.1 Spatial vector fields on M to time-dependent family of vector fields on Σ

Given spatial global vector field $U \in \mathcal{E}(TM)$, then U is perpendicular to the normal $n \in \mathcal{E}(TM)$. We will now define a time-dependent family of vector fields $V : \mathbb{R} \to \mathcal{E}(T\Sigma)$ as follows

$$V(t)|_\sigma := (\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)} U|_{\Phi(t,\sigma)}$$

Note that in general the map $\pi_\Sigma \circ \Phi^{-1}$ is not injective, however, for a fixed $t \in \mathbb{R}$, it is a bijection. Now, we can use our procedure and embed V using the family of embedding maps $e_t : \Sigma \to M$

$$\begin{aligned}
\tilde{V}|_{\Phi(t,\sigma)} &= e_{t*}|_\sigma V(t)|_\sigma \\
&= e_{t*}|_\sigma((\pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)} U|_{\Phi(t,\sigma)}) \\
&= (e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)} U|_{\Phi(t,\sigma)}
\end{aligned}$$

But Eq. 19 on pg. 14 shows that the map $(e_t \circ \pi_\Sigma \circ \Phi^{-1})_*|_{\Phi(t,\sigma)}$ is an identity map on spatial global vector fields. Hence we have

$$\tilde{V} = U$$

Then, any spatial global vector fields on M is an image of a time-dependent family of vector fields on Σ under the family of embedding maps e_{t*}. Thus, for a given spatial global vector field on M, we can denote it as $\tilde{U}$ where U is the corresponding time-dependent family of vector fields.

A.2 Lie bracket of global vector fields on M

Let $\tilde{U}$ and $\tilde{V}$ be spatial global vector fields on M. We know that $[\tilde{U}, \tilde{V}]$ is another spatial global vector field on M, hence it is an image of time-dependent family of vector fields on Σ. As it turns out that it is the image of $[U(t), V(t)]$ under the family of embedding maps. To show this, we will consider the following

$$\begin{aligned}
[\tilde{U}, \tilde{V}]|_{\Phi(t,\sigma)} f &= \tilde{U}|_{\Phi(t,\sigma)}(\tilde{V}(f)) - \tilde{V}|_{\Phi(t,\sigma)}(\tilde{U}(f)) \\
&= e_{t*}|_\sigma(U(t)|_\sigma)(\tilde{V}(f)) - e_{t*}|_\sigma(V(t)|_\sigma)(\tilde{U}(f)) \\
&= U(t)|_\sigma(\tilde{V}(f) \circ e_t) - V(t)|_\sigma(\tilde{U}(f) \circ e_t)
\end{aligned}$$

Note that we have

$$(\tilde{V}(f) \circ e_t)|_\sigma = \tilde{V}(f)|_{\Phi(t,\sigma)} = e_{t*}|_\sigma(V(t)|_\sigma)(f)|_{\Phi(t,\sigma)} = V(t)|_\sigma(f \circ e_t)|_\sigma \tag{29}$$

Then

$$\begin{aligned}
[\tilde{U}, \tilde{V}]|_{\Phi(t,\sigma)} f &= U(t)|_\sigma(V(t)|_\sigma(f \circ e_t)|_\sigma) - V(t)|_\sigma(U(t)|_\sigma(f \circ e_t)|_\sigma) \\
&= [U(t), V(t)]|_\sigma(f \circ e_t)|_\sigma \\
&= e_{t*}|_\sigma([U(t), V(t)]|_\sigma) f
\end{aligned}$$

Therefore, if we assume the push-forward map implicitly, then we would have $[\tilde{U}, \tilde{V}]|_{\Phi(t,\sigma)} = [U(t), V(t)]|_\sigma$.

References

[1] Baez, John. *Gauge Fields, Knots and Gravity.* New Jersey: World Scientific, 1994. Print

[2] Guillemin, Victor, and Alan Pollack. *Differential Topology.* New Jersey: Prentice-Hall, 1974. Print

[3] Lee, John. *Introduction to Smooth Manifolds.* New York: Springer, 2010. Print

[4] Willard, Stephen. *General Topology.* Massachusetts: Addison-Wesley, 1970. Print

[5] Carroll, Sean. *Spacetime and Geometry: An Introduction to General Relativity.* Massachusetts: Addison-Wesley Longman, 2004. Print

[6] Suazo, Luis. *The Geometry of the ADM Formalism.* University of Texas at Austin, 2013. (In process)