$$\frac{d}{dt}\tilde{\rho}_{AB}(t) = -i\left[\tilde{V}_I(t), \tilde{\rho}_{AB}(t)\right] \longrightarrow \tilde{\rho}_{AB}(t) = \rho_{AB}(0) - i\int_0^t dt_1 \left[V_I(t_1), \tilde{\rho}_{AB}(t_1)\right]$$

$$\downarrow$$

$$\frac{d}{dt}\tilde{\rho}_{AB}(t) = (-i)\left[\tilde{V}_I(t), \tilde{\rho}_{AB}(0)\right] + (-i)^2\int_0^t dt_1 \left[\tilde{V}_I(t), \left[\tilde{V}_I(t_1), \tilde{\rho}_{AB}(t_1)\right]\right]$$

Born Approximation: $\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t) + \rho_{corr} \approx \rho_A(t) \otimes \rho_B(t)$

If $\rho_B(0) = \rho_B^G$, then $\rho_B(t) = \rho_B^G$. Thus, we approximate

$$\rho_{AB}(t) \approx \rho_A(t) \otimes \rho_B^G$$

We further assume that $V_I = \sum_\alpha A_\alpha \otimes X_\alpha$. Then,

$$\frac{d}{dt}\tilde{\rho}_{AB}(t) = (-i)\left[\tilde{V}_I(t), \tilde{\rho}_{AB}(0)\right] + (-i)^2\int_0^t dt_1 \left[\tilde{V}_I(t), \left[\tilde{V}_I^\dagger(t_1), \tilde{\rho}_{AB}(t_1)\right]\right]$$

$$\downarrow$$

$$\frac{d}{dt}\left\{\tilde{\rho}_A(t) \otimes \rho_B^G\right\} = (-i)\sum_\alpha \left[\tilde{A}_\alpha(t) \otimes \tilde{X}_\alpha(t), \tilde{\rho}_A(0) \otimes \rho_B^G\right] \qquad \tilde{\rho}_B^G = \rho_B^G \text{ because } [H_B, \rho_B^G] = 0$$

$$+ (-i)^2 \sum_{\alpha,\beta} \int_0^t dt_1 \left[\tilde{A}_\alpha(t) \otimes \tilde{X}_\alpha(t), \left[\tilde{A}_\beta^\dagger(t_1) \otimes \tilde{X}_\beta^\dagger(t_1), \tilde{\rho}_A(t_1) \otimes \rho_B^G\right]\right]$$

We now take partial trace to obtain the dynamics of $\tilde{\rho}_A$:

$$\frac{d}{dt}\mathrm{Tr}_B\left\{\tilde{\rho}_A(t) \otimes \rho_B^G\right\} = (-i)\sum_\alpha \mathrm{Tr}_B\left\{\left[\tilde{A}_\alpha(t) \otimes \tilde{X}_\alpha(t), \tilde{\rho}_A(0) \otimes \rho_B^G\right]\right\}$$

$$+ (-i)^2 \sum_{\alpha,\beta} \int_0^t dt_1\, \mathrm{Tr}_B\left\{\left[\tilde{A}_\alpha(t) \otimes \tilde{X}_\alpha(t), \left[\tilde{A}_\beta^\dagger(t_1) \otimes \tilde{X}_\beta^\dagger(t_1), \tilde{\rho}_A(t_1) \otimes \rho_B^G\right]\right]\right\}$$

$\downarrow$

$$\frac{d}{dt}\tilde{\rho}_A(t) = (-i)\sum_{\alpha}\left[\tilde{A}_\alpha(t), \tilde{\rho}_A(0)\right]\langle\tilde{X}_\alpha(t)\rangle_{\rho_B^G}$$

$$+(-i)^2\sum_{\alpha,\beta}\int_0^t dt_1\left\{\left[\tilde{A}_\alpha(t), \tilde{A}_\beta^+(t_1)\tilde{\rho}_A(t_1)\right]G_{\alpha,\beta}(t,t_1)\right.$$

$$\left.-\left[\tilde{A}_\alpha(t), \tilde{\rho}_A(t_1)\tilde{A}_\beta^+(t_1)\right]G_{\alpha,\beta}^*(t,t_1)\right\}$$

where $G_{\alpha,\beta}(t,t_1) \equiv \langle\tilde{X}_\alpha(t)\tilde{X}_\beta^+(t_1)\rangle_{\rho_B^G}$. WLOG, we can set $\langle\tilde{X}_\alpha(t)\rangle_{\rho_B^G} = \langle X_\alpha\rangle_{\rho_B^G} = 0$. We now introduce the Markov approximation,

$$\tilde{\rho}_A(t_1) = \tilde{\rho}_A(t) + (t_1 - t)\dot{\tilde{\rho}}_A(t) + \ldots \approx \tilde{\rho}_A(t)$$

With these, we can write

$$\frac{d}{dt}\tilde{\rho}_A(t) = -\sum_{\alpha,\beta}\int_0^t dt_1\left\{\left[\tilde{A}_\alpha(t), \tilde{A}_\beta^+(t_1)\tilde{\rho}_A(t)\right]G_{\alpha,\beta}(t,t_1)\right.$$

$$\left.-\left[\tilde{A}_\alpha(t), \tilde{\rho}_A(t)\tilde{A}_\beta^+(t_1)\right]G_{\alpha,\beta}^*(t,t_1)\right\}$$

$$= -\sum_{\alpha,\beta}\left[\tilde{A}_\alpha(t), \tilde{\Lambda}_{\alpha,\beta}(t)\tilde{\rho}_A(t) - \tilde{\rho}_A(t)\tilde{\Lambda}_{\alpha,\beta}^+(t)\right]$$

where

$$\tilde{\Lambda}_{\alpha,\beta}(t) = \int_0^t dt_1\,\tilde{A}_\beta^+(t_1)G_{\alpha,\beta}(t,t_1)$$

Now we will convert back to the Schrödinger picture:

$$\frac{d}{dt}\tilde{\rho}_A(t) = \frac{d}{dt}\left(e^{iH_A t}\rho_A(t)e^{-iH_A t}\right)$$

$$= iH_A e^{iH_A t}\rho_A(t)e^{-iH_A t} + e^{iH_A t}\frac{d\rho_A}{dt}e^{-iH_A t} - ie^{iH_A t}\rho_A(t)H_A e^{-iH_A t}$$

$$= e^{iH_A t}\left\{i[H_A, \rho_A(t)] + \frac{d\rho_A}{dt}\right\}e^{-iH_A t}$$

$$[\tilde{A}_\alpha(t), \tilde{\Lambda}_{\alpha\beta}(t)\tilde{\rho}_A(t)] = \tilde{A}_\alpha(t)\tilde{\Lambda}_{\alpha\beta}(t)\tilde{\rho}_A(t) - \tilde{\Lambda}_{\alpha\beta}(t)\tilde{\rho}_A(t)\tilde{A}_\alpha(t)$$

$$= e^{iH_A t}A_\alpha e^{-iH_A t}\left\{\int_0^t dt_1 e^{iH_A t_1}A_\beta^+ e^{-iH_A t_1}G_{\alpha\beta}(t,t_1)\right\}e^{iH_A t}\rho_A(t)e^{-iH_A t}$$

$I = e^{iH_A t}e^{-iH_A t}$

$$-\left\{\int_0^t dt_1 e^{iH_A t_1}A_\beta^+ e^{-iH_A t_1}G_{\alpha\beta}(t,t_1)\right\}e^{iH_A t}\rho_A(t)\cancel{e^{-iH_A t}e^{iH_A t}}A_\alpha e^{-iH_A t}$$

$$= e^{iH_A t}\left\{A_\alpha\left[\int_0^t dt_1 e^{-iH_A(t-t_1)}A_\beta^+ e^{iH_A(t-t_1)}G_{\alpha,\beta}(t,t_1)\right]\rho_A(t)\right.$$

$$\left. - \left[\int_0^t dt_1 e^{-iH_A(t-t_1)}A_\beta^+ e^{iH_A(t-t_1)}G_{\alpha,\beta}(t,t_1)\right]\rho_A(t)A_\alpha\right\}e^{-iH_A t}$$

$$= e^{iH_A t}[A_\alpha, \Lambda_{\alpha,\beta}(t)\rho_A(t)]e^{-iH_A t}, \quad \Lambda_{\alpha\beta}(t) = \int_0^t dt_1 e^{-iH_A(t-t_1)}A_\beta^+ e^{iH_A(t-t_1)}G_{\alpha\beta}(t,t_1)$$

$$[\tilde{A}_\alpha(t), \tilde{\rho}_A(t)\tilde{\Lambda}^+_{\alpha\beta}(t)] = \tilde{A}_\alpha(t)\tilde{\rho}_A(t)\tilde{\Lambda}^+_{\alpha\beta}(t) - \tilde{\rho}_A(t)\tilde{\Lambda}^+_{\alpha\beta}(t)\tilde{A}_\alpha(t)$$

$$= e^{iH_At} A_\alpha \cancel{e^{-iH_At} e^{iH_At}} \rho_A(t) e^{-iH_At}\left\{\int_0^t dt_1\, e^{iH_At_1} A_\beta e^{-iH_At_1} G^*_{\alpha\beta}(t,t_1)\right\} e^{iH_At}e^{-iH_At}$$

$$- e^{iH_At}\rho_A(t)e^{-iH_At}\left\{\int_0^t dt_1\, e^{iH_At_1}A_\beta e^{-iH_At_1}G^*_{\alpha\beta}(t,t_1)\right\}e^{iH_At}A_\alpha e^{-iH_At}$$

$$= e^{iH_At}\left\{A_\alpha\rho_A(t)\left[\int_0^t dt_1\, e^{-iH_A(t-t_1)}A_\beta e^{iH_A(t-t_1)}G^*_{\alpha\beta}(t,t_1)\right]\right.$$

$$\left. - \rho_A(t)\left[\int_0^t dt_1\, e^{-iH_A(t-t_1)}A_\beta e^{iH_A(t-t_1)}G^*_{\alpha\beta}(t,t_1)\right]A_\alpha\right\}e^{-iH_At}$$

$$= e^{iH_At}[A_\alpha, \rho_A(t)\Lambda^+_{\alpha,\beta}(t)]e^{-iH_At}$$

Thus, the dynamics in the Schrödinger picture are given by

$$\frac{d}{dt}\tilde{\rho}_A(t) = -\sum_{\alpha,\beta}\left[\tilde{A}_\alpha(t), \tilde{\Lambda}_{\alpha\beta}(t)\tilde{\rho}_A(t) - \tilde{\rho}_A(t)\tilde{\Lambda}^+_{\alpha\beta}(t)\right]$$

$\downarrow$

$$\cancel{e^{iH_At}}\left\{i[H_A,\rho_A(t)] + \frac{d\rho_A(t)}{dt}\right\}\cancel{e^{-iH_At}} = -\cancel{e^{iH_At}}\left\{\sum_{\alpha,\beta}\left([A_\alpha,\Lambda_{\alpha\beta}(t)\rho_A(t)] - [A_\alpha,\rho_A(t)\Lambda^+_{\alpha\beta}(t)]\right)\right\}\cancel{e^{-iH_At}}$$

$\downarrow$

$$\boxed{\frac{d\rho_A(t)}{dt} = -i[H_A,\rho_A(t)] - \sum_{\alpha,\beta}\left[A_\alpha, \Lambda_{\alpha\beta}(t)\rho_A(t) - \rho_A(t)\Lambda^+_{\alpha\beta}(t)\right]}$$