Energy Systems
Technical University of Berlin
Department of Digital Transformation in Energy Systems
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SOLUTIONS IV: ELECTRICITY MARKETS

Will be worked on in the exercise session on Friday, 25 June 2020.

SOLUTION IV.1 (SHADOW PRICES OF LIMITS ON CONSUMPTION).

Suppose that the utility for the electricity consumption of an industrial company is given by

$$U(d) = 70d - 3d^2 [€/\text{h}] \quad , \quad d_{min} = 2 \leq d \leq d_{max} = 10,$$

where d is the demand in MW and d_{min}, d_{max} are the minimum and maximum demand.

Assume that the company is maximising its net surplus for a given electricity price π, i.e. it maximises $\max_d [U(d) - \pi d]$.

(a) If the price is $\pi = 5$ €/MWh, what is the optimal demand d^*? What is the value of the KKT multiplier μ_{max} for the constraint $d \leq d_{max} = 10$ at this optimal solution? What is the value of μ_{min} for $d \geq d_{min} = 2$?

We convert the exercise to an optimisation problem with objective

$$f(d) = U(d) - \pi d = (70 - \pi)d - 3d^2 \tag{1}$$

$$\max_{d \in \mathbb{R}} f(d) \tag{2}$$

with constraints

$$d \leq d_{max} = 10 \qquad \leftrightarrow \qquad \mu_{max} \tag{3}$$

$$-d \leq -d_{min} = -2 \qquad \leftrightarrow \qquad \mu_{min} \tag{4}$$

From stationarity for the optimal point we get:

$$0 = \frac{\partial \mathcal{L}}{\partial d} = \frac{\partial f}{\partial d} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial d} - \sum_j \mu_j^* \frac{\partial h_j}{\partial d} \tag{5}$$

$$0 = \frac{\partial}{\partial d} \left((70 - \pi)d - 3d^2 \right) - \mu_{max} + \mu_{min} \tag{6}$$

$$= (70 - \pi) - 6d - \mu_{max} + \mu_{min} \tag{7}$$

Note that it does not matter whether you pull in the constant of the right-hand side of the respective constraint. The marginal utility curve is $U'(d) = 70 - 6d$ [€/MWh]. At $\pi = 5$ and if the demand were unconstrained, the demand would be determined by

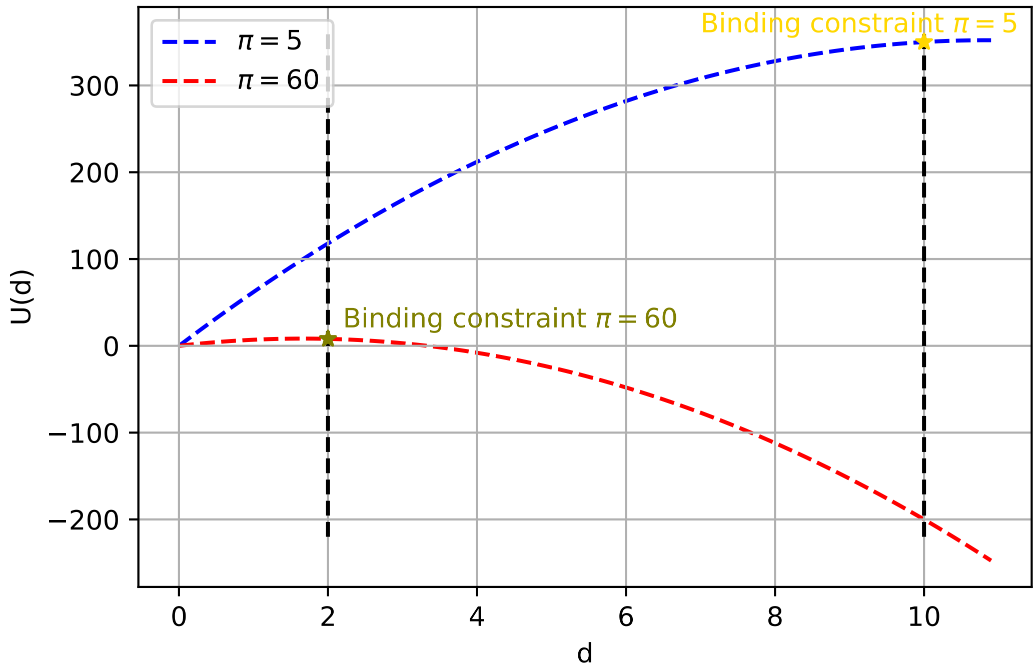

Figure 1: utility for the electricity consumption with demand d

$5 = 70 - 6d$, i.e. $d = 65/6 = 10.8333$, which is above the consumption limit $d_{max} = 10$. Therefore the optimal demand is
$d^* = 10$,
the upper limit is binding such that $\mu_{max} \geq 0$ and the lower limit is non-binding such that
$\mu_{min} = 0$.
To determine the value of μ_{max} we use (7) to get

$$\mu_{max} = (70 - \pi) - 6d^* - \mu_{min} = 70 - 5 - 60 = 5$$

(b) Suppose now the electricity price is $\pi = 60$ €/MWh. What are the optimal demand d^*, μ_{max} and μ_{min} now?

At $\pi = 60$, the demand would be determined by $60 = 70 - 6d$, i.e. $d = 10/6 = 1.667$, which is below the consumption limit $d_{min} = 2$. Therefore the optimal demand is
$d^* = 2$,
the upper limit is non-binding such that
$\mu_{max} = 0$
and the lower limit is binding such that $\mu_{min} \geq 0$.

To determine the value of μ_{min} we use (7) to get

$$\mu_{min} = -(70 - \pi) + 6d^* + \mu_{max} = -(70 - 60) + 12 + 0 = 2.$$

SOLUTION IV.2 (ECONOMIC DISPATCH IN A SINGLE BIDDING ZONE).

Consider an electricity market with two generator types, one with the cost function $C_1(g_1) = c_1 g_1$ with variable cost $c_1 = 20$ €/MWh, capacity $G_1 = 300$ MW and a dispatch rate of g_1 [MW], and another with the cost function $C_2(g_2) = c_2 g_2$ with variable cost $c_2 = 50$ €/MWh, capacity $G_2 = 400$ MW and a dispatch rate of g_2 [MW]. The demand has utility function $U(d) = 8000d - 5d^2$ [€/h] for a consumption rate of d [MW].

(a) What are the objective function and constraints required for an optimisation problem to maximise short-run social welfare in this market?

The optimisation problem has the objective function:

$$f(d, g_1, g_2) = U(d) - C_1(g_1) - C_2(g_2) = 8000d - 5d^2 - c_1 g_1 - c_2 g_2$$
$$\max_{d, g_1, g_2} f(d, g_1, g_2)$$

with constraints:

$$\begin{aligned} d - g_1 - g_2 = 0 &\leftrightarrow \lambda \\ g_1 \leq G_1 &\leftrightarrow \bar{\mu}_1 \\ g_2 \leq G_2 &\leftrightarrow \bar{\mu}_2 \\ -g_1 \leq 0 &\leftrightarrow \underline{\mu}_1 \\ -g_2 \leq 0 &\leftrightarrow \underline{\mu}_2 \end{aligned}$$

(b) Write down the Karush-Kuhn-Tucker (KKT) conditions for this problem.

Stationarity gives for d:

$$\begin{aligned} 0 = \frac{\partial \mathcal{L}}{\partial d} &= \frac{\partial f}{\partial d} - \sum_i \lambda_i^* \frac{\partial g_i}{\partial d} - \sum_j \mu_j^* \frac{\partial h_j}{\partial d} \\ &= \frac{\partial U}{\partial d} - \lambda \\ &= 8000 - 10d - \lambda \end{aligned}$$

Stationarity for g_1 gives:

$$-\frac{\partial C_1}{\partial g_1} + \lambda - \bar{\mu}_1 + \underline{\mu}_1 = -c_1 + \lambda - \bar{\mu}_1 + \underline{\mu}_1 = 0$$

Stationarity for g_2 gives:

$$-\frac{\partial C_2}{\partial g_2} + \lambda - \bar{\mu}_2 + \underline{\mu}_2 = -c_2 + \lambda - \bar{\mu}_2 + \underline{\mu}_2 = 0$$

Primal feasibility is just the generator limits above in (a).
Dual feasibility is $\bar{\mu}_i^*, \underline{\mu}_i^* \geq 0$ and
complementary slackness is $\bar{\mu}_i^*(g_i^* - G_i) = 0$ and $\underline{\mu}_i^* g_i^* = 0$ for $i = 1, 2$.

(c) Determine the optimal rate of production of the generators and the value of all KKT multipliers. What is the interpretation of the respective KKT multipliers?

The marginal utility at the full output of the generators,

$$G_1 + G_2 = 300 \text{ MW} + 400 \text{ MW} = 700 \text{ MW}$$
$$U'(700) = 8000 - 10 \cdot 700 = 1000 \text{ €/MWh}$$

which is higher than the costs c_i, so we'll find optimal rates

$$g_1^* = G_1 = 300 \text{ MW}$$
$$g_2^* = G_2 = 400 \text{ MW}$$
$$d^* = G_1 + G_2 = 700 \text{ MW}$$

This means (from stationarity)

$$\lambda^* = \left.\frac{\partial U}{\partial d}\right|_{d=d^*} = 8000 - 10d^* = 1000\text{€/MWh},$$

which is the market price. Because the lower constraints on the generator output are not binding, from complementary slackness we have $\underline{\mu}_i = 0$. The upper constraints are binding, so $\bar{\mu}_i \geq 0$.
From stationarity $\bar{\mu}_i = \lambda - c_i + \underline{\mu}_i$, which is the increase in social welfare if Generator i could increase its capacity by a marginal amount.

$$\bar{\mu}_1 = 1000 - 20 = 980\text{€/MWh}$$
$$\bar{\mu}_2 = 1000 - 50 = 950\text{€/MWh}$$

Solution IV.3 (Efficient dispatch in a two-bus power system).

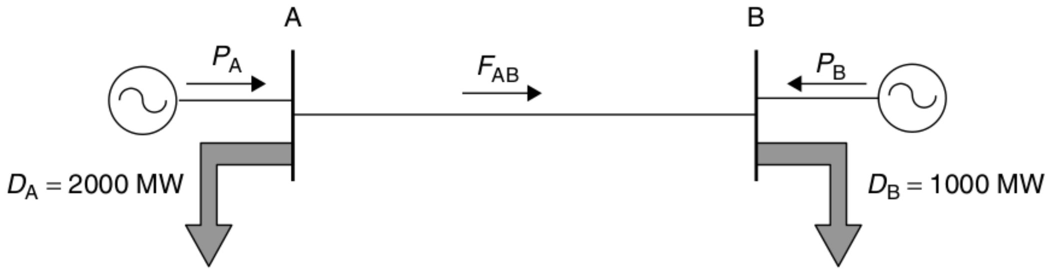

Figure 2: A simple two-bus power system.

Consider the two-bus power system shown in Figure 2, where the two nodes represent two markets, each with different total demand D_i, and one generator at each node producing G_i. At node A the demand is $D_A = 2000$MW, whereas at node B the demand is $D_B = 1000$MW. Furthermore, there is a transmission line with a capacity denoted by F_{AB}. The marginal cost of production of the generators connected to buses A and B are given respectively by the following expressions:

$$\begin{aligned} MC_A &= 20 + 0.03P_A \qquad &\text{€ /MW h} \\ MC_B &= 15 + 0.02P_B \qquad &\text{€ /MW h} \end{aligned}$$

Assume that the demands D_A and D_B are constant and insensitive to price, that energy is sold at its marginal cost of production and that there are no limits on the output of the generators.

(a) Calculate the price of electricity at each bus, the production of each generator, and the flow on the line for the following cases. You may also calculate the values of any KKT multiplier as a bonus.

The price of electricity is the value of the dual variable at the nodal balance equation.

Use the following nomenclature: price $\lambda_{A/B}$, generation $G_{A/B}$, flow F_{AB}.

(i) The line between buses A and B is disconnected.

Where to start: $P_A = D_A$ and $P_B = D_B$ and substitute into MC_i.

$\lambda_A = 80$ €/MWh, $\lambda_B = 35$ €/MWh,

$G_A = 2000$ MW, $P_B = 1000$ MW, $F_{AB} = 0$

(ii) The line between buses A and B is in service and has an unlimited capacity.

Where to start: No restriction in transmission, so prices must be the same for the two nodes: $\lambda_A = \lambda_B$, therefore, $MC_A = MC_B$. Also: $P_A + P_B = D_A + D_B$.

$\lambda_A = 53$ €/MWh, $\lambda_B = 53$ €/MWh,

$G_A = 1100$ MW, $P_B = 1900$ MW, $F_{AB} = -900$ MW

(iii) The line between buses A and B is in service and has an unlimited capacity, but the maximum output of Generator B is 1500 MW.

Where to start: $P_B = 1500$ MW since it is now constrained but would have been higher in the unconstrained case (ii). Also, since there are no transmission constraints: $\lambda_A = \lambda_B$. Also: $P_A + P_B = D_A + D_B$.

$\lambda_A = 65$ €/MWh, $\lambda_B = 65$ €/MWh,

$G_A = 1500$ MW, $P_B = 1500$ MW, $F_{AB} = -500$ MW

(iv) The line between buses A and B is in service and has an unlimited capacity, but the maximum output of Generator A is 900 MW. The output of Generator B is unlimited.

Where to start: $P_A = 900$ MW since it is now constrained but would have been higher in the unconstrained case (ii). Also, since there are no transmission constraints: $\lambda_A = \lambda_B$. Also: $P_A + P_B = D_A + D_B$.

$\lambda_A = 57$ €/MWh, $\lambda_B = 57$ €/MWh,

$G_A = 900$ MW, $P_B = 2100$ MW, $F_{AB} = -1100$ MW

(v) The line between buses A and B is in service but its capacity is limited to 600 MW. The output of the generators is unlimited.

Where to start: $F_{AB} = -600$ MW since we would want even more transmission, if it were not constrained. Also, $P_A + F_{AB} = 2000$ MW.

$\lambda_A = 62$ €/MWh, $\lambda_B = 47$ €/MWh,

$G_A = 1400$ MW, $P_B = 1600$ MW, $F_{AB} = -600$ MW

(b) Calculate the generator revenues, generator profits, consumer payments and consumer net surplus for all the cases considered in the above problem. Who benefits from the line connecting these two buses?

Generator revenues R_i, generator costs C_i, generator profits P_i, consumer payments E_i. Find the generator profits by subtracting the costs from the revenue. Costs are given by integrating the marginal cost, i.e. $C_A = 20P_A + 0.015P_A^2$ and $C_B = 15P_B + 0.01P_B^2$. The generator at B and the consumers at A benefit from the line (price increases at B, decreases at B).

(c) Calculate the congestion surplus for case (v). For what values of the flow on the line between buses A and B is the congestion surplus equal to zero?

Congestion surplus is 9000 €:

$$(E_A + E_B) - (R_A + R_B) = |F_{AB}| \times (\lambda_A - \lambda_B)$$

Congestion surplus is equal to zero when the flow $F_{AB} = 0$, or when it is equal to the unconstrained value $F_{AB} = -900$ MW (then $\lambda_A = \lambda_B$).

Case	(i)	(ii)	(iii)	(iv)	(v)
E_A (€)	160,000	106,000	130,000	114,000	124,000
R_A (€)	160,000	58,300	97,500	51,300	86,800
C_A (€)	100,000	40,150	63,750	30,150	57,400
P_A (€)	60,000	18,150	33,750	21,150	29,400
E_B (€)	35,000	53,000	65,000	57,000	47,000
R_B (€)	35,000	100,700	97,500	119,700	75,200
C_B (€)	25,000	64,600	45,000	75,600	49,600
P_B (€)	10,000	36,100	52,500	44,100	25,600