Calibration of SABR Stochastic Volatility Model

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TABLE OF CONTENTS

1. Introduction

SABR model is a CEV model augmented by stochastic volatility that assumes the forward rate evolves under the associated forward (terminal) measure $\mathbb{Q}^T$ according to:

$$\begin{aligned} &d\varphi = \alpha\varphi^\beta dW_1, \qquad d\alpha = \nu\alpha dW_2, \qquad dW_1 dW_2 = \rho dt \\ &\text{initial condition:} \qquad \varphi_0 = f, \qquad \alpha_0 = a \end{aligned} \tag{1}$$

where the factors φ and α are the stochastic forward and volatility process. The parameters β, ρ and ν are deterministic and remain constant. The constant ν is known as the volatility of volatility, a model feature which acknowledges that volatility obeys well-known clustering in time. The parameter $\beta \in [0,1]$ controls the relationship between the forward price and the at-the-money volatility. A $\beta < 1$ ("non-lognormal" case) leads to skews in the implied volatilities. In the case of $\beta \approx 1$, if the market were to move up or down, the level of the at-the-money implied volatility σ_f would not be significantly affected, whereas when $\beta < 1$ the volatility increases as price falls (i.e. volatility and price move in opposite direction). The closer to 0 the more pronounced would be this effect. The correlation parameter ρ plays a similar role as the β does. It defines how the market moves in sync with the volatility dynamics. There are two more latent (i.e. invisible) parameters, the initial value of φ and α at time $t = 0$

$$\varphi_{t=0} = f \qquad \text{and} \qquad \alpha_{t=0} = a \tag{2}$$

The f is the forward price quoted in market at $t = 0$. The a is a "volatility-like" parameter that cannot be observed from the market. However it can be derived analytically from the at-the-money implied volatility σ_f, as we shall see this in due course.

2. Asymptotic Solution by Hagan et al.

Using singular perturbation techniques, Hagan et al. [1] provide a closed form asymptotic solution (up to the accuracy of a series expansion) for prices of vanilla instruments. The price of a vanilla option under the SABR model is given by the appropriate Black formula provided that the correct Black implied volatility is used. Given the initial foward f and expiry time τ, the Black implied volatility can be derived as a function of strike price K from a given set of SABR parameters α, β, ρ, ν

$$\sigma_K = \frac{\zeta}{\chi_\zeta} \cdot \frac{\alpha\left[1 + \left(\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\beta\nu\alpha}{4(fK)^{(1-\beta)/2}} + \frac{(2-3\rho^2)\nu^2}{24}\right)\tau\right]}{(fK)^{\frac{(1-\beta)}{2}}\left[1 + \frac{(1-\beta)^2 q^2}{24} + \frac{(1-\beta)^4 q^4}{1920}\right]} \tag{3}$$

$$\text{where} \qquad q = \log\frac{f}{K}, \qquad \zeta = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}} q, \qquad \chi_\zeta = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right)$$

When $K = f$, equation (3) reduces to give the at-the-money implied volatility

$$\sigma_f = \frac{\alpha\left[1 + \left(\frac{(1-\beta)^2\alpha^2}{24 f^{2(1-\beta)}} + \frac{\rho\beta\nu\alpha}{4 f^{1-\beta}} + \frac{(2-3\rho^2)\nu^2}{24}\right)\tau\right]}{f^{1-\beta}} \tag{4}$$

The (4) shows that there exists a relationship

$$\ln\sigma_f = \ln\alpha - (1-\beta)\ln f + \cdots \tag{5}$$

It indicates that the value of β can be estimated from a log-log regression of σ_f and f with historical data, ignoring terms involving τ. Alternatively, since the parameters β and ρ in SABR model control the distribution function in similar ways (i.e. both control the skewness of the distribution), the redundancy between the two parameters allows one to calibrate the model by fixing β to an assumption (e.g. $\beta = 0.5$). The decision is often made on the basis of market experience. The remaining parameters α, ρ and ν have different effects on the volatility curve. The parameter α mainly controls the overall magnitude of the curve, the ρ controls the curve's skew, and the ν controls how much smile (i.e. convexity) the curve exhibits.

As shown in (4) the parameter α has a functional form with the at-the-money volatility σ_f. Inverting the equation gives the value of α as a root of a cubic equation (in general, the smallest positive root would be taken if there were three real roots) when the ρ and ν are given

$$\frac{(1-\beta)^2\tau}{24f^{2(1-\beta)}}a^3+\frac{\rho\beta\nu\tau}{4f^{1-\beta}}a^2+\left[1+\frac{(2-3\rho^2)\nu^2\tau}{24}\right]a-\sigma_f f^{1-\beta}=0 \tag{6}$$

This indicates that in SABR model we only need to calibrate ρ and ν to an existing implied volatility surface, providing that the value of β is prescribed and the at-the-money implied volatility σ_f is given.

The calibration is performed for each slice of the implied volatility smiles defined by different expiry times. For simplicity, the objective function of the calibration can defined as a sum of squared residuals (or sum of vega weighted squared residuals)

$$\underset{\rho,\nu}{\operatorname{argmin}}\sum_{i=1}^{N}\left(\sigma_{K_i}^{mkt}-\sigma_{K_i|f,a,\beta,\rho,\nu}^{SABR}\right)^2 \tag{7}$$

This is a nonlinear constrained minimization problem. Many nonlinear optimization routines can be used to carry out the calibration, for example, Levenberg-Marquardt Method or Nelder-Mead Simplex Method.

3. Correction to Hagan et al. Solution

The ζ in (3) was defined in equation (A.57c) in [1]. When assuming CEV model (i.e. $C_\varphi=\varphi^\beta$) for the forward process, it has the form

$$\zeta=\frac{\nu}{a}\int_K^f\frac{du}{u^\beta}=\frac{\nu}{a}\cdot\frac{f^{1-\beta}-K^{1-\beta}}{1-\beta} \tag{8}$$

By expanding, we have

$$\begin{aligned}&f^{1-\beta}-K^{1-\beta}=(1-\beta)(fK)^{\frac{1-\beta}{2}}q\left[1+\frac{(1-\beta)^2q^2}{24}+\frac{(1-\beta)^4q^4}{1920}+\cdots\right]\qquad\text{and}\\&f-K=(fK)^{\frac{1}{2}}q\left(1+\frac{q^2}{24}+\frac{q^4}{1920}+\cdots\right)\qquad\text{When}\qquad\beta=0\end{aligned} \tag{9}$$

Thus ζ can be written as

$$\zeta=\frac{\nu}{a}(fK)^{\frac{1-\beta}{2}}q\left[1+\frac{(1-\beta)^2q^2}{24}+\frac{(1-\beta)^4q^4}{1920}+\cdots\right] \tag{10}$$

Clearly the value of ζ adapted in (3) is just an approximation of (10) truncating all higher order terms of q. This leads to a correction to the original Hagan et al. solution proposed by Obloj [2] in 2008, where he uses (8) to calculate ζ.

4. Summary

In the following, we are going to summarize the Hagan et al. solution and the correction. For simplicity, let's define

$$\theta = \frac{(1-\beta)^2 a^2}{24(fK)^{1-\beta}} + \frac{\rho\beta\nu a}{4(fK)^{(1-\beta)/2}} + \frac{(2-3\rho^2)\nu^2}{24} \tag{11}$$

The original Hagan et al. solution in (3) becomes

$$\sigma_K = \frac{\zeta}{\chi_\zeta}\frac{a}{\eta}(1+\theta\tau) \tag{12}$$

where

$$\begin{aligned} &\eta = \frac{f^{1-\beta} - K^{1-\beta}}{(1-\beta)q}, \qquad q = \log\frac{f}{K}, \qquad \zeta = \frac{\nu q}{a}\cdot (fK)^{\frac{1-\beta}{2}} \qquad \text{and} \\ &\chi_\zeta = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right) \end{aligned} \tag{13}$$

Since we know from (9) that

$$\eta = (fK)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2 q^2}{24} + \frac{(1-\beta)^4 q^4}{1920} + \cdots\right] \tag{14}$$

Hence when $K = f$ (i.e. at-the-money strike), we have

$$\lim_{K\to f}\eta = (fK)^{\frac{1-\beta}{2}} = f^{1-\beta} \qquad \text{and} \qquad \lim_{K\to f}\frac{\zeta}{\chi_\zeta} = 1 \tag{15}$$

therefore (12) reduces to

$$\sigma_K = \frac{a}{f^{1-\beta}}(1+\theta\tau) \tag{16}$$

When $\beta = 1$ (i.e. log-normal forward process), we have

$$\lim_{\beta \to 1} \eta = 1 \quad \text{and} \quad \lim_{\beta \to 1} \zeta = \frac{\nu}{a} (fK)^{\frac{1-\beta}{2}} q = \frac{\nu q}{a} \tag{17}$$

therefore (12) reduces to

$$\sigma_K = \frac{\nu q}{\chi_\zeta} (1 + \theta\tau) \tag{18}$$

The correction made by Obloj [2] provides the Black implied volatility as

$$\sigma_K = \frac{\nu q}{\chi_\zeta} (1 + \theta\tau) \tag{19}$$

where, as we have mentioned, it has a different definition for ζ

$$\zeta = \frac{\nu}{a} \cdot \frac{f^{1-\beta} - K^{1-\beta}}{(1-\beta)} \tag{20}$$

Following the previous definition of η in (13), the ζ can be written as

$$\zeta = \frac{\nu q \eta}{a} \tag{21}$$

Therefore, the (19) converts to

$$\sigma_K = \frac{\zeta}{\chi_\zeta} \frac{a}{\eta} (1 + \theta\tau) \tag{22}$$

which is the same form as the original Hagan et al. solution in (12), except for different definition of ζ (i.e. Hagan et al. take an approximation for ζ). Hence when $K = f$ and when $\beta = 1$, the (22) reduces to the same formulas as in (16) and (18) respectively.

REFERENCES

1. Hagan, P.; Kumar, D.; Lesniewski, A. and Woodward, D., "Managing smile risk", *Wilmott Magazine*, pages 84–108, July 2002.
2. Obloj, J., "Fine-tune your smile: Correction to Hagan et al.", *Wilmott Magazine*, 35, 102–104, 2008.