First-Order Logic, 2019 Fall

Week 1-2

- Formal Language
- Meta & Obeject Language
- Truth Value
- Semantics of Propositional Logic

> Lewis实质蕴涵，与自然语言中“如果...那么”的区别

Semantics:

- 赋值：$V(p)=1$
- 赋值满足：$V\vDash\phi$
- 语义后承：$\Sigma\Vdash\phi$

Theorem 1 对任意公式ϕ: $\varnothing\Vdash\phi$ iff ϕ是有效的。

Theorem 2 $\{\sigma_1,\cdots,\sigma_n\}\Vdash\phi$ iff $(\sigma_1\to(\sigma_2\to\cdots\to(\sigma_n\to\phi)\cdots))$是有效的。

> Formal Proof
> Math Proof Math Objects

Week 3

- Truth Assignment Function Theory - Semantics
- Calculus - Syntax

- *semantically equivalent*: 若$\phi \vDash \alpha$且$\alpha \vDash \phi$，记为$\phi ⫤\vDash \alpha$.
- 二元逻辑n元真值函数个数：2^{2^n}，以此类推，m元逻辑n元真值函数个数：m^{m^n}.

Literal(字节):

- prop & neg prop;
- $\perp, \neg\perp$

CNF (Conjunctive Normal Form) & **DNF** (Disjunctive Normal Form)

Notation
$\bigvee\Sigma, \bigwedge\Phi$
$\bigvee\{\alpha\} := \alpha$
$\bigwedge\varnothing := \neg\perp$
$\bigvee\varnothing := \perp$

α realizes g:

- g: n-ary truth function with arguments $x_1, \cdots, x_n$;
- $A = \{P_1, \cdots, P_n\}$;
- wff α with $\Phi(\alpha) = A$.

If for every $(z_1, \cdots, z_n) \in \{0, 1\}^n$, we have $g(z_1, \cdots, z_n) = 1$ iff $\{P_i \in A \mid z_i = 1\} \vDash \alpha$.

<u>**Note**:</u> 也可用λ表达式来写真值函数

> Some conclusions about *realization*:

> 1. If α, β are realized by the same g, then $\alpha \dashv\vdash \beta$;
> 2. Every truth function g can be realized by a wff in DNF.

Proof of 1:
Suppose a truth valuation U, note:

$$y_i^u = \begin{cases} 1, & P_i \in U \\ 0, & P_i \notin U \end{cases}$$

$V \models \alpha$ iff $V \cap A \models \alpha$ iff $\{P_i \in A \mid y_i^v = 1\} \models \alpha$ iff $g(y_1^v, \cdots, y_n^v) = 1$.

Proof of 2:

1. $g = 0$, let α be $\perp \wedge P_1 \wedge \cdots \wedge P_n$;
2. $g \neq 0$, then there exists a non-empty set $S \subseteq \{0,1\}^n$ s.t. for every $y_1, \cdots, y_n \in \{0,1\}$, $(y_1, \cdots, y_n) \in S$ iff $g(y_1, \cdots, y_n) = 1$. Obviously S is finite.
   Enumerate S as $\{(x_{11}, \cdots, x_{1n}), \cdots, (x_{k1}, \cdots x_{kn})\}$, let $\alpha := \gamma_1 \vee \cdots \vee \gamma_k$, $\gamma_k := \beta_{i1} \wedge \cdots \wedge \beta_{in}$.

   $$\beta_{ij} = \begin{cases} P_j, & x_{ij} = 1 \\ \neg P_j, & x_{ij} = 0 \end{cases}$$

   Then it is sufficient to check α realizes g.
   Note $A = \{P_1, \cdots P_n\}$, $s = (y_1, \cdots, y_n) \in \{0,1\}^n$, let $V_s = \{P_h \in A \mid y_h = 1\}$.
   $(\Rightarrow)$: Suppose $g(s) = 1$, then $s \in S$, $s = (y_1, \cdots, y_n) = (x_{i1}, \cdots, x_{in})$.
   For any $j \in \{1, \cdots, n\}$:
   Case 1. $x_{ij} = 1$, then $y_j = 1$, so $P_j \in V_s$, then $V_s \models \beta_{ij}$;
   Case 2. $x_{ij} = 0$, then also $V_s \models \beta_{ij}$.
   In both cases, we have $V_s \models \beta_{ij}$. For j is coincident, then $V_s \models \gamma_j$, which makes $V_s \models \alpha$.
   $(\Leftarrow)$: Suppose $g(s) = 0$, then $s \notin S$.
   For any $i \in \{1, \cdots, k\}$, there is a $j \in \{1, \cdots, n\}$ s.t. $x_{ij} \neq y_j$.
   Case 1. $x_{ij} = 0, y_j = 1$, then $P_j \in V_s$, so $V_s \not\models \beta_{ij}$;
   Case 2. $x_{ij} = 1, y_j = 0$,then $P_j \notin V_s$, so $V_s \not\models \beta_{ij}$.
   In both cases, we have $V_s \not\models \beta_{ij}$, then $V_s \not\models \gamma_i$. For i is coincident, then $V_s \not\models \alpha$.

Calculus:

- 归结演算

- Hilbert 演算

Define a type. **(Backus-Naur Form)**
$\phi := \bot \mid p \mid \phi \to \phi$.

Classical Propositional Calculus(3 Axioms Schemes and MP Rule):

1. $\alpha \to (\beta \to \alpha)$;
2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$;
3. $((\beta \to \bot) \to (\alpha \to \bot)) \to (\alpha \to \beta)$;
4. If $\alpha \to \beta, \alpha$, then β.

Notation. $\vdash \alpha$ means α is *provable* or *have a proof*.
Call α an *inner theorem*.

Example.
Abbrev. write $\phi \to \phi$ as $(\phi\phi)$
Prove: $\vdash \phi \to \phi$

1. $\vdash (\phi(\phi\phi)\phi)((\phi(\phi\phi))(\phi\phi))$
2. $\vdash (\phi(\phi\phi)\phi)$
3. $\vdash (\phi(\phi\phi))(\phi\phi)$
4. $\vdash \phi(\phi\phi)$
5. $\vdash \phi\phi$

Week 5

- Derivation
- Deduction Theorem
- Lindenbaum Lemma

Derivation: a finite tree of formulas from Γ to ϕ, each leaf is either an axiom or a wff in Γ; each node is from the earlier nodes by the application of MP.

- $\Sigma \vdash \phi, \vdash \phi$;
- If $\Gamma \vdash \phi$, then ϕ is called a *syntactical consequence* of Γ

> **Deduction Theorem**
> Suppose α, β, Σ, then $\Sigma \vdash \alpha \to \beta$ iff $\Sigma \cup \{\alpha\} \vdash \beta$.

Proof: Omitted.

Consistent Set:
Σ is said to be consistent, if $\Sigma \nvdash \bot$.

Two Consequences:

- for any α, $\{\alpha, \neg\alpha\}$ is not consistent;
- Σ is consistent iff $\Sigma \nvdash \alpha$ for some α.

Proof:
($\Rightarrow$): By definition;
($\Leftarrow$): Suppose $\Sigma \vdash \bot$.
For any arbitary α, $\vdash ((\alpha \to \bot) \to (\bot \to \bot)) \to (\bot \to \alpha)$, since $\vdash (\bot \to \bot)$ and $\vdash (\bot \to \bot) \to ((\alpha \to \bot) \to (\bot \to \bot))$.
Then by an application of MP, we have $\vdash \bot \to \alpha$.
Since $\Sigma \vdash \bot \to \alpha$ and $\Sigma \vdash \bot$, by MP, we have $\Sigma \vdash \alpha$.

Maximal Consistent Set (MCS):
If $\Sigma \nvdash \bot$, and for every $\Delta \supsetneq \Sigma$, we have $\Delta \vdash \bot$, then Σ is said to be a MCS.

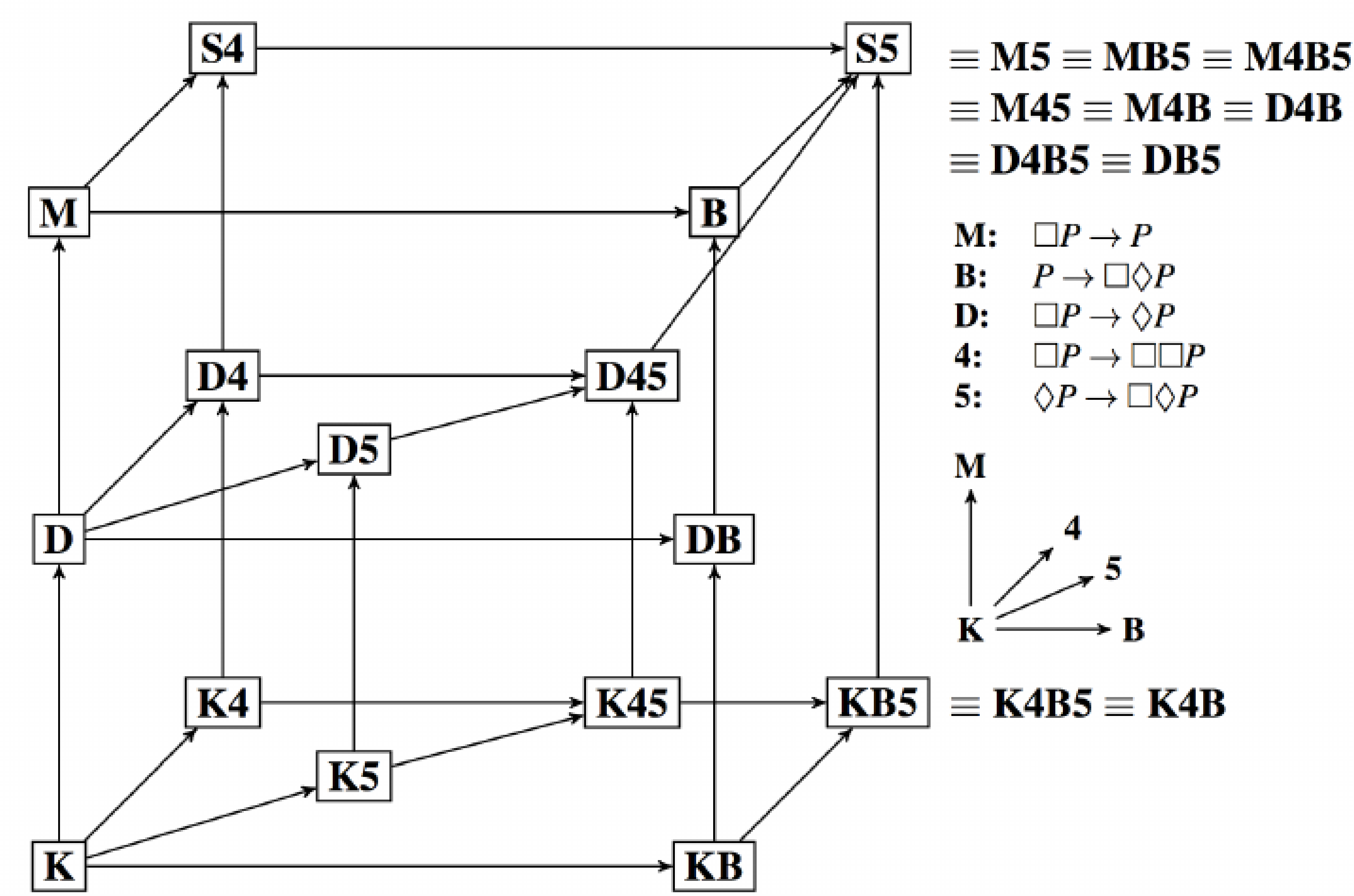
Lindenbaum Lemma(in a countable language):
If $\Sigma \nvdash \bot$, then there is a MCS $\Delta \supseteq \Sigma$.

Claims about Δ:

1. $\Sigma \subseteq \Delta$;
2. For every $i \in \mathcal{N}$, $\Gamma_i \subseteq \Gamma_{i+1}$;
3. For every $i \in \mathcal{N}$, $\Gamma_i \nvdash \bot$;
4. $\Delta \nvdash \bot$;
5. For every ϕ, either $\phi \in \Delta$ or $\phi \to \bot \in \Delta$;
6. For every ϕ, $\phi \in \Delta$ iff $\Delta \vdash \phi$;
7. Δ is a MCS.

Week 6

- Completeness: for a definite propositional calculus
- Modal Logic
- Intuitionistic Logic



Week 7

- Complements on IPC
- Natural Deduction
- FOL Language

Symbols:

- quantifiers: $\forall, \exists$;
- equality symbol: $=$;
- predicate symbol: $P, R, \cdots$;

- individual variables: $v, x, y, \cdots$;
- constants: $a, b, c, \cdots$;
- function symbol: $f, g, \cdots$;
- prop connectives;
- (,)

Term:

$$t ::= c|x|f^{(n)}\underbrace{t\cdots t}_{n}$$

Wff:

$$\phi ::= \equiv tt|P^{(n)}\underbrace{t\cdots t}_{n}|\bot|\phi \to \phi|\forall x\phi$$

Polish Notation:(An Example)
$\forall u \to \exists y \in u\exists x\wedge \in xu\forall z \to\in zx\neg \in zu$

Some Concepts:

- ground term;
- binding force;
- range;
- free/bound occurrences of variables:
  - a variable is bounded/free in α if x has a bounded/free occurrence in α.
  - $\mathcal{BV}(\alpha), \mathcal{FV}(\alpha)$;
  - $\mathcal{FV}(\alpha) = \emptyset$, α is called a **sentence** or a **closed formula**.

Substitution:

1. Term: S^x_t

$$\begin{cases} c^x_t := c; \\ x^x_t := t; \\ y^x_t := y; \\ f^{(n)}t_1\cdots t_n = f^{(n)}(t_1)^x_t\cdots(t_n)^x_t. \end{cases}$$

2. Formula: ϕ^x_t

$$
\begin{cases}
(\equiv s_1 s_2)^x_t := \equiv (s_1)^x_t (s_2)^x_t; \\
P^{(n)} s_1 \cdots s_n := P^{(n)} (s_1)^x_t \cdots (s_n)^x_t \\
(\perp)^x_t := \perp; \\
(\to \alpha\beta)^x_t := \to (\alpha)^x_t (\beta)^x_t; \\
(\forall x \alpha)^x_t := \forall x \alpha; \\
(\forall y \alpha)^x_t := \forall y (\alpha)^x_t.
\end{cases}
$$

Week 8

- Model(Structure) in FOL
- Vlidity

Define **a FOL model** $\mathfrak{A} = (|\mathfrak{A}|, (\cdot)^{\mathfrak{A}})$, in which:

- $|\mathfrak{A}| \neq \emptyset$;
- $(\cdot)^{\mathfrak{A}}$

Interpretation:

- For every $c \in \mathrm{CON}$: $c^{\mathfrak{A}} \in |\mathfrak{A}|$;
- For every $f \in \mathrm{Fct}^{(n)}$: $f^{\mathfrak{A}} : |\mathfrak{A}|^n \to |\mathfrak{A}|$, which must be a total function;
- For every $P \in \mathrm{Pre}^{(n)}$: $P^{\mathfrak{A}} \subseteq |\mathfrak{A}|^n$.

Assignment(on a model):
Define an assignment $s : \mathrm{Var} \to |\mathfrak{A}|$, by which we have: $(\cdot)^{\mathfrak{A}}_s : \mathrm{Tm} \to |\mathfrak{A}|$. Specifically:

- $c^{\mathfrak{A}}_s := c^{\mathfrak{A}}$;
- $x^{\mathfrak{A}}_s := s(x)$;
- $f(t_1 \cdots t_n)^{\mathfrak{A}}_s := f^{\mathfrak{A}}((t_1)^{\mathfrak{A}}_s, \cdots, (t_1)^{\mathfrak{A}}_s)$.

Revision on an assignment:
For any assignment s on $\mathfrak{A}$, $x \in \mathrm{Var}$, $d \in |\mathfrak{A}|$, define a revision on an assignment $s(x|d)$: $\mathrm{Var} \to |\mathfrak{A}|$:

$$
s(x|d)(y) = \begin{cases} d & x = y; \\ s(y) & x \neq y. \end{cases}
$$

Some concepts:

- Valid;
- Satisfiable;
- Semantical Consequence: e.g. $\Theta \Vdash \phi$.

Say $\mathfrak{A}$ and s satisfies a formula ϕ, write $\vDash_{\mathfrak{A}} \phi[s]$, if at least one of the following holds:

1. ϕ is of the form $t_1 = t_2$, and $(t_1)_s^{\mathfrak{A}} = (t_2)_s^{\mathfrak{A}}$;
2. ϕ is of the form $Pt_1 \cdots t_n$, and $\langle (t_1)_s^{\mathfrak{A}}, \cdots, (t_n)_s^{\mathfrak{A}} \rangle \in P^{\mathfrak{A}}$;
3. ϕ is of the form $\alpha \to \beta$, and ``$\nvDash_{\mathfrak{A}} \alpha[s]$ or $\vDash_{\mathfrak{A}} \beta[s]$" holds;
4. ϕ is of the form $\forall x \alpha$, and for every $d \in |\mathfrak{A}|$, $\vDash_{\mathfrak{A}} \alpha[s(x|d)]$.

Note that $\exists$ is defined as $\neg\forall\neg$ since:

$$\forall x \alpha \to \neg \exists x \neg \alpha$$

is valid.

Remark: Suppose σ a **closed-formula**, if $\vDash_{\mathfrak{A}} \sigma[s]$, then $\vDash_{\mathfrak{A}} \sigma$, which means the satisfiction has nothing to do with the choice of assignment on the model. We just say $\mathfrak{A}$ is a **model** of σ.

Notation: Define the set of free variables in a formula as $\mathcal{FV}$, say $\mathcal{FV}(\phi) = \{x_1, \cdots, x_n\}$, then that ϕ is satisfied under some assignment s on $\mathfrak{A}$ just means

$$\vDash_{\mathfrak{A}} \phi[\![s(x_1), \cdots, s(x_n)]\!]$$

holds.

Some Facts:

Fact 1:

For every $t \in \mathrm{Tm}$, every model $\mathfrak{M}$ and every assignment s, s' on $\mathfrak{M}$:

If $s(x) = s'(x)$ holds for every $x \in \mathcal{V}ar(t)$, then $t_s^{\mathfrak{A}} = t_s^{\mathfrak{A}}$.

Fact 2:

If $s(x) = s'(x)$ holds for every $x \in \mathcal{PV}(\phi)$, then $\vDash_{\mathfrak{A}} \phi[s]$ iff $\vDash_{\mathfrak{A}} \phi[s']$.

Fact 3:

$$(t_{t'}^{x})_s^{\mathfrak{A}} = (t)_{s(x|(t')_s^{\mathfrak{A}})}^{\mathfrak{A}}$$

Week 9

Definition: Term t is free(substituable) for variable x, if one of the following holds:

- ϕ is atomic;
- $x \notin \mathcal{FV}(\phi)$;
- ϕ is of $\alpha \to \beta$, t is free for x in both α and β;
- ϕ is $\forall y \alpha$, $y \notin \mathcal{V}ar(t)$, and t is free for x in α.

Fact 4 (Substitution Lemma):

If t is free for x in α, then:

$$\vDash_{\mathfrak{A}} \alpha[s(x|t_s^{\mathfrak{A}})] \quad \text{iff} \quad \vDash_{\mathfrak{A}} \alpha_t^x[s]$$

.

Proof:

> 1. $\vDash_{\mathfrak{A}} t_1 \equiv t_2[s(x|t_s^{\mathfrak{A}})]$
> 2. $\vDash_{\mathfrak{A}} Q\overline{r}[s(x|t_s^{\mathfrak{A}})]$
> 3. $\perp$, trivial;
> 4. $\vDash_{\mathfrak{A}} \beta \to \theta[s(x|t_s^{\mathfrak{A}})]$;
> 5. $\vDash_{\mathfrak{A}} \forall x \beta[s(x|t_s^{\mathfrak{A}})]$;
> 6. $y \notin \{x\}$, and $\vDash_{\mathfrak{A}} \forall y \beta[s(x|t_s^{\mathfrak{A}})]$.

Fact 5:

If t is free for x in a wff α, then $\forall x \alpha \to \alpha_t^x$ is valid.

Definable Set:

$\{< e_1, \cdots, e_n > \in |\mathfrak{A}|^n \mid \vDash_{\mathfrak{A}} \phi[\![e_1, \cdots, e_n]\!]\}$

Examples: $\mathfrak{A} = (\mathbb{N}, <)$

- $[x < x]^{\mathfrak{A}} = \emptyset$
- $[\exists y (y < x)]^{\mathfrak{A}} = \{(a) | a \in \mathbb{N}\}$

Note that there are uncountable relations on $\mathbb{N}$, but only countable of them are definable.

Homomorphism:(P94)

Remark:

1. Replace $\Rightarrow$ in homo definition of relations with $\Leftrightarrow$, then h is said to be a **Strong Homo**;
2. Let a strong homo h be injective, then h is said to be an **Embedding**;
3. Let an embedding h be surjective, then h is said to be an **Isomorphism**;
4. Let the range of an isomorphism h be $|\mathfrak{A}|$, then h is said to be an **Automorphism**;
5. **Self-embedding**.