ADS Homework 3

Problem 3.1

(a) $f(n) = 9n$, $g(n) = 5n^3$

- $f \overset{?}{\in} \Theta(g) \iff 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{9n}{5n^3} = \lim_{n\to\infty} \frac{9}{5n^2} = 0$

$0 < 0 < \infty$ (F) $\iff f \notin \Theta(g)$

- $f \overset{?}{\in} O(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$;

$0 < \infty$ (T)

- $f \overset{?}{\in} o(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$\lim_{n\to\infty} \frac{9n}{5n^3} = 0$, $0 = 0 \implies f \in o(g)$

- $f \overset{?}{\in} \Omega(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$

$\lim_{n\to\infty} \frac{9n}{5n^3} = 0$, $0 > 0$ F $\implies f \notin \Omega(g)$

- $f \overset{?}{\in} \omega(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

$\lim_{n\to\infty} \frac{9n}{5n^3} = 0$; $0 = \infty$ (F) $\not\Rightarrow f \in \omega(g)$

- $g \in \Theta(f) \iff 0 < \lim_{n\to\infty} \frac{g(n)}{f(n)} < \infty$

$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{5n^3}{9n} = \lim_{n\to\infty} \frac{5n^2}{9} = \infty$

$0 < \infty < \infty$ F $\implies g \notin \Theta(f)$

- $g \in O(f) \iff \lim_{n\to\infty} \frac{g(n)}{f(n)} < \infty$

$$\lim_{n\to\infty} \frac{5n^3}{9n} = \infty$$

$\infty < \infty$ F $\Rightarrow g \notin O(f)$

- $g \in o(f) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

$$\lim_{n\to\infty} \frac{5n^3}{9n} = \lim_{n\to\infty} \frac{5n^2}{9} = \infty$$

$\infty = 0$ F $\Rightarrow g \notin o(f)$

- $g \in \Omega(f) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} > 0$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{5n^3}{9n} = \infty$$

$\infty > 0 \Rightarrow$ T $\Rightarrow g \in \Omega(f)$

- $g \in \omega(f) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$

$$\lim_{n\to\infty} \frac{5n^3}{9n} = \infty$$

$\infty = \infty \Rightarrow g \in \omega(f)$

(e) $f(n) = 9n^{0.8} + 2n^{0.3} + 14 \log n$ and $g(n) = \sqrt{n}$

$g(n) = \sqrt{n} = n^{\frac{1}{2}} = n^{0,5}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{9n^{0,8} + 2n^{0.3} + 14 \log n}{n^{0.5}} = \infty$$

(we always ~~check~~ compare the power of the leading term in that case : $0,8 > 0.5 \Rightarrow \lim_{n\to\infty} = \infty$

- $f \in \Theta(g) \Leftrightarrow 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

$$\lim_{n\to\infty} \frac{9n^{0.8} + 2n^{0.3} + 14 \log n}{n^{0.5}} = \infty$$

$\infty < \infty$ F $\Rightarrow f \notin \Theta(g)$

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- $f \in O(g) \Leftrightarrow \lim\limits_{n \to \infty} \frac{f(n)}{g(n)} < \infty$

$$\lim_{n \to \infty} \frac{9n^{0.8} + 2n^{0.3} + 14 \log n}{n^{0.5}} = \infty$$

- $\infty < \infty$ F $\Rightarrow \lim\limits_{n \to \infty}$ $f \notin O(g)$

- $f \in \Theta(g) \Leftrightarrow \lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = c$

$$\lim_{n \to \infty} \frac{9n^{0.8} + 2n^{0.3} + 14 \log n}{n^{0.5}} = \infty$$

$\infty = c$ F $\Rightarrow f \notin \Theta(g)$

- $f \in \Omega(g) \Leftrightarrow \lim\limits_{n \to \infty} \frac{f(n)}{g(n)} > 0$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{9n^{0.8} + 2n^{0.3} + 14 \log n}{n^{0.5}} = \infty$$

$\infty > 0 \Rightarrow f \in \Omega(g)$

- $f \in \omega(g) \Leftrightarrow \lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

$$\lim_{n \to \infty} \frac{9n^{0.8} + 2n^{0.3} + 14 \log n}{n^{0.5}} = \infty$$

$\infty = \infty \Rightarrow f \in \omega(g)$

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{n^{0.5}}{9n^{0.8} + 2n^{0.3} + 14 \log n} = 0$$

The power of the leading term from the numerator is less than the power from the leading term of the denominator

• $g \in \Theta(f) \Leftrightarrow \lim_{n \to \infty} \quad 0 < \lim_{n \to \infty} \frac{g(n)}{f(n)} < \infty$

$$\lim_{n \to \infty} \frac{n^{0.5}}{9n^{0.8} + 2n^{0.3} + 14 \log n} = 0$$

$0 < 0 < \infty$ F $\Rightarrow g \notin \Theta(f(g))$

• $g \in O(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} < \infty$

$$\lim_{n \to \infty} \frac{n^{0.5}}{9n^{0.8} + 2n^{0.3} + 14 \log n} = 0$$

$0 < \infty \Rightarrow g \in O(f)$

• $g \in o(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$

$$\lim_{n \to \infty} \frac{n^{0.5}}{9n^{0.8} + 2n^{0.3} + 14 \log n} = 0$$

• $g \in \Omega(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} > 0$

$$\lim_{n \to \infty} \frac{n^{0.5}}{9n^{0.8} + 2n^{0.3} + 14 \log n} = 0$$

$0 > 0$ F $\Rightarrow g \notin \Omega(f)$

• $g \in \omega(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty$

$$\lim_{n \to \infty} \frac{n^{0.5}}{9n^{0.8} + 2n^{0.3} + 14 \log n} = c, \quad c \neq \infty$$

$\Rightarrow g \notin \omega(f)$

c) $f(n) = \frac{n^2}{\log n}$, $g(n) = n \log n$

$$\lim_{n \to \infty} \frac{\frac{n^2}{\log n}}{n \log n} = \lim_{n \to \infty} \frac{n^2}{\log n} : n \log n = \lim_{n \to \infty} \frac{n^2}{\log n} \cdot \frac{1}{n \log n}$$

$$= \lim_{n \to \infty} \frac{n}{\log^2 n} = \infty$$

$(\log^2 n << n)$

- $f \in \Theta(g) \Leftrightarrow 0 < \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$

$\lim_{n \to \infty} \frac{n}{\log^2 n} = \infty$, $0 < \infty < \infty$ $\neq$ $\Rightarrow$

$\Rightarrow f \notin \Theta(g)$

- $f \in O(g) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$

$\lim_{n \to \infty} \frac{n}{\log^2 n} = \infty$

$\infty < \infty$ F $\Rightarrow f \notin O(g)$

- $f \in o(g) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n}{\log^2 n} = \infty$

$\infty = 0$ F $\Rightarrow f \notin o(g)$

- $f \in \Omega(g) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$

$\lim_{n \to \infty} \frac{n}{\log^2 n} = \infty$, $\infty > 0$ T $\Rightarrow f \in \Omega(g)$

- $f \in \omega(g) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

$\lim_{n \to \infty} \frac{n}{\log^2 n} = \infty$, $\infty = \infty$ (T) $\Rightarrow f \in \omega(g)$

$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \frac{n \cdot \log n}{\frac{n^2}{\log n}} = \frac{1}{\cancel{n}} \log n \cdot \frac{\log n}{n} = \frac{\log^2 n}{n} = 0$

$g \in \Theta(f) \Leftrightarrow 0 < \lim_{n \to \infty} \frac{g(n)}{f(n)} < \infty$

$\lim_{n \to \infty} \frac{g(n)}{f(n)} \Leftrightarrow 0 < \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$

$\lim_{n \to \infty} \frac{\log^2 n}{n} = 0 \Rightarrow 0 < 0 < \infty \text{ (F)}$

$\Rightarrow g \notin \Theta(f)$

- $g \in O(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} < \infty$

$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{\log^2 n}{n} = 0$

$0 < \infty \Rightarrow g \in O(f)$

- $g \in o(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$

$\lim_{n \to \infty} \frac{\log^2 n}{n} = 0$

$0 = 0 \text{ (T)} \Rightarrow g \in o(f)$

- $g \in \Omega(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} > 0$

$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{\log^2 n}{n} = 0$

$0 > 0 \text{ F} \Rightarrow g \notin \Omega(f)$

- $g \in \omega(f) \Leftrightarrow \lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty$

$\lim_{n \to \infty} \frac{g(n)}{f(n)} = \lim_{n \to \infty} \frac{\log^2 n}{n} = 0$

$0 = \infty \text{ F} \Rightarrow g \notin \omega(f)$

(d) $f(n) = (\log(3n))^3$, $g(n) = 9\log n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \left(\frac{(\log(3n))^3}{9\log n}\right)$$

$\log(3n) = \log 3 + \log n = \log n + \log 3$

$(\log n + \log 3)^3 = (\log n)^3 + 3\log^2 n \log 3 +$

$+ 3\log n \log^2 3 + (\log 3)^3$

$$\Rightarrow \lim_{n\to\infty} \frac{(\log n)^{\cancel{3}}}{9\log n} + \frac{\cancel{3}\log^{\cancel{2}} n \log 3}{\underset{3}{\cancel{9}}\log n} + \frac{\cancel{3}\log n \log^2 3}{\underset{3}{\cancel{9}}\log n}$$

$$+ \frac{(\log 3)^3}{9\log n} = \lim_{n\to\infty} \underbrace{\frac{\log n^2}{9}}_{\to\infty} + \underbrace{\frac{\log n \cdot \log 3}{3}}_{\to\infty} +$$

$$+ \underbrace{\frac{\log^2 3}{3}}_{\to\frac{\log^2 3}{3}} + \underbrace{\frac{(\log 3)^3}{9\log n}}_{\to 0} = \infty$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{\log(3n)^3}{9\log n}\right) = \infty$$

1) $f \in \Theta(g) \Leftrightarrow 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

$\lim_{n\to\infty} \left(\frac{\log(3n)}{9\log n}\right)^3 = \infty$, $0 < \infty < \infty$ F

$\Rightarrow f \notin \Theta(g)$

2) $f \in O(g) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$

$\lim_{n\to\infty} \left(\frac{\log(3n)}{9\log n}\right)^3 = \infty$, $\infty < \infty$ F $\Rightarrow$

$\Rightarrow f \notin O(g)$

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3) $f \in o(g) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$\lim_{n\to\infty} \frac{(\log(3n))^3}{9\log n} = \infty$, $\infty \neq 0$

$\Rightarrow f \notin o(g)$

4) $f \in \Omega(g)$ $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$

$\lim_{n\to\infty} \frac{(\log(3n))^3}{3\log n} = \infty$, $\infty > 0$ T $\Rightarrow f \in \Omega(g)$

5) $f \in \omega(g) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

$\lim_{n\to\infty} \frac{(\log(3n))^3}{3\log n} = \infty$, $\infty = \infty$

$\Rightarrow f \in \omega_g$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{9\log n}{(\log(3n))^3}$$

$$\frac{9\log n}{(\log(3n))^3} < \frac{9\cancel{\log n}}{(\log(3n))^{\cancel{3}2}}$$

$$\frac{9\log n}{(\log(3n))^3} < \frac{9}{(\log(n))^2}$$

$$\lim_{n\to\infty} \frac{9}{(\log(n))^2} = 0$$

$\Rightarrow \lim_{n\to\infty} \frac{9\log n}{(\log(3n))^3} = 0$

(comparison test)

6) $g \in \Theta(f)$

$\Leftrightarrow 0 < \lim_{n\to\infty} \frac{g(n)}{f(n)} < \infty$

$0 < 0 < \infty$ F $\Rightarrow g \notin \Theta(f)$

7) $g \in O(f) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} < \infty$

$0 < \infty \Rightarrow g \in O(f)$

8) $g \in o(f) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

$\Rightarrow 0 = 0 \Rightarrow g \in o(g)$

9) $g \in \Omega(f) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} > 0$

$0 > 0$ F $\Rightarrow g \notin \Omega(f)$

10) $g \in \omega(f) \Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$

$0 = \infty$ F $\Rightarrow g \in \omega(f)$

a) We implemented the code in VS code

a) Selection sort correctness
We can make use of the loop invariant to prove this

Definition (Loop Invariant) :
Let a ($A[0 \dots n-1]$) be an array of size n => before each iteration j in the outer loop, the subarray $A[0 \dots j-1]$ consists of the smallest elements in ascending order

Proof by induction :

I Base case : $j=0$
The sub array $A[0]$ consists of only one element => we can safely assume it is in ascending order

II Inductive step
We can assume that in the ($j-1$)th iteration ~~iteration~~ / the subarray $A[0, \dots, j-1]$ is in ascending order.
In the $j-1+1=j$ th iteration, the algorithm finds the smallest element in the unsorted part ($A[j, \dots, n-1]$, and we store the index of the smallest index in indmin, we then swap $A[j]$ with $A[indmin]$.
After this swap, the new sorted subarray is $A[0, \dots, j]$ will contain the ($j+1$)

smallest elements in ascending order $\Rightarrow$ the loop invariant holds for every iteration.

we know that the outer loop runs until $j = n-1$ (for ($j=0$; $j<$ dim-1, j++)) $\Rightarrow$

$\Rightarrow A[0, \ldots, n-1]$ contains all n elements in ascending order $\Rightarrow$ the entire array is sorted correctly

c) see code file

d) see code file

e) Looking at the figure based on our inputs in Matlab, we can see that, regardless of the case, the values seem to be mostly overlapping, with some variations. This happens because Selection Sort always has an $O(n^2)$ time complexity since the algorithm performs the same number of comparisons in every case, the overall execution time remains similar, as comparisons dominate the runtime of Selection Sort. Even though the best case requires fewer swaps, the time of execution difference is minimal

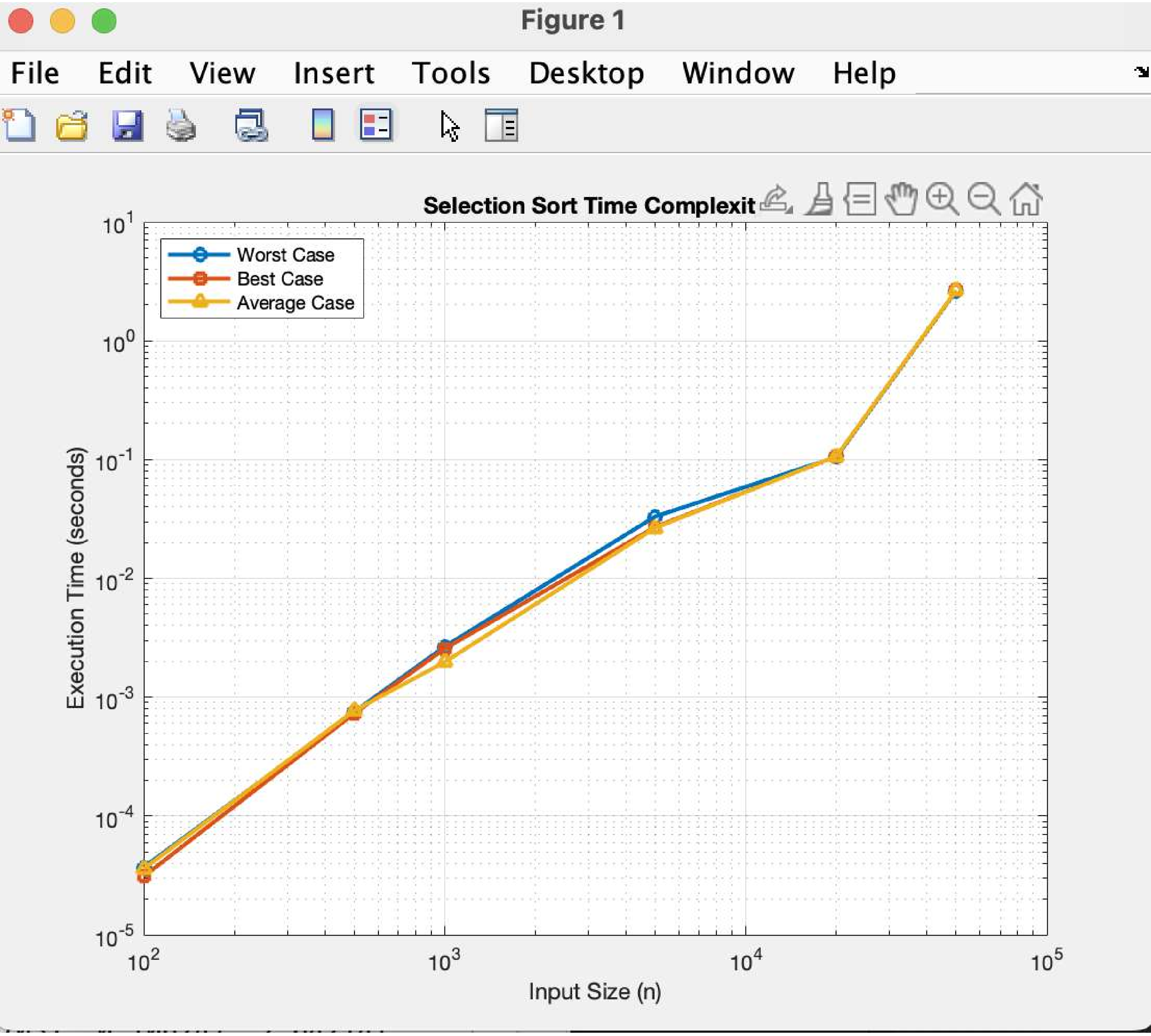
Figure 1
File
Edit
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Tools
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Best Case
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Execution Time (seconds)
10^{1}
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10^{4}
10^{5}
Input Size (n)