Nearest Neighbor Classification

Machine Learning

THE UNIVERSITY OF UTAH

This lecture

- K-nearest neighbor classification
  - The basic algorithm
  - Different distance measures
  - Some practical aspects
- Voronoi Diagrams and Decision Boundaries
  - What is the hypothesis space?
- The Curse of Dimensionality

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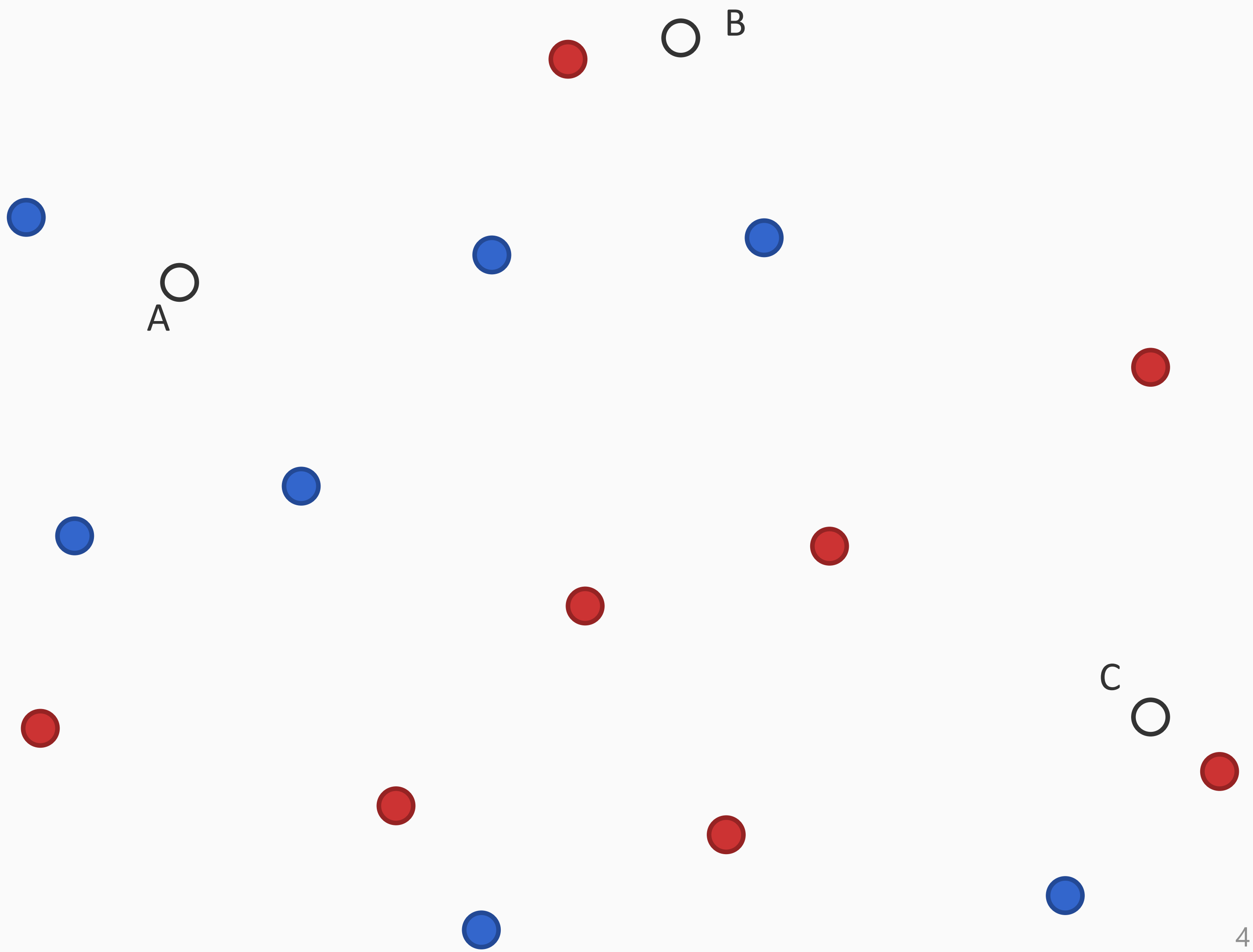
How would you color the blank circles?



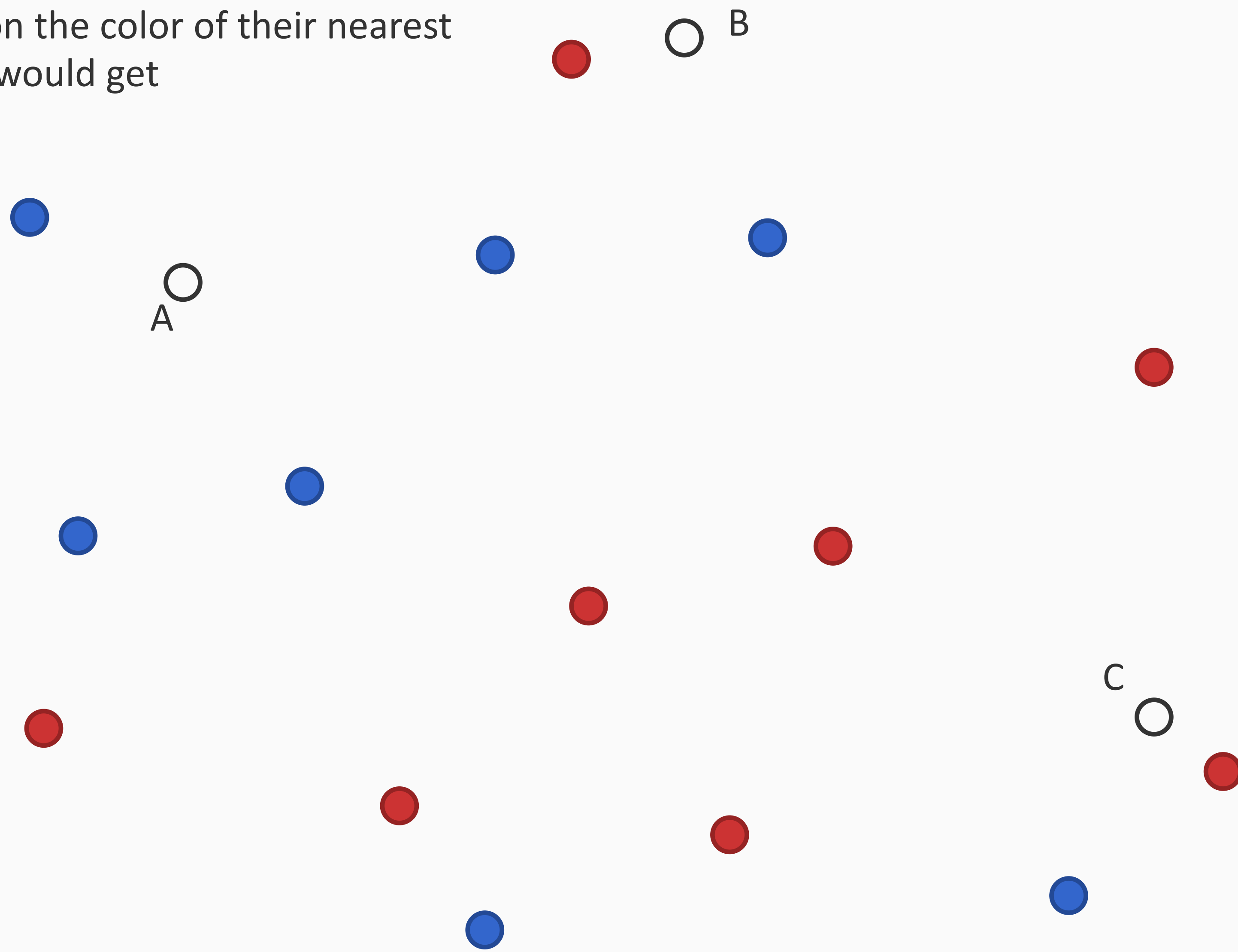
How would you color the blank circles?

If we based it on the color of their nearest neighbors, we would get

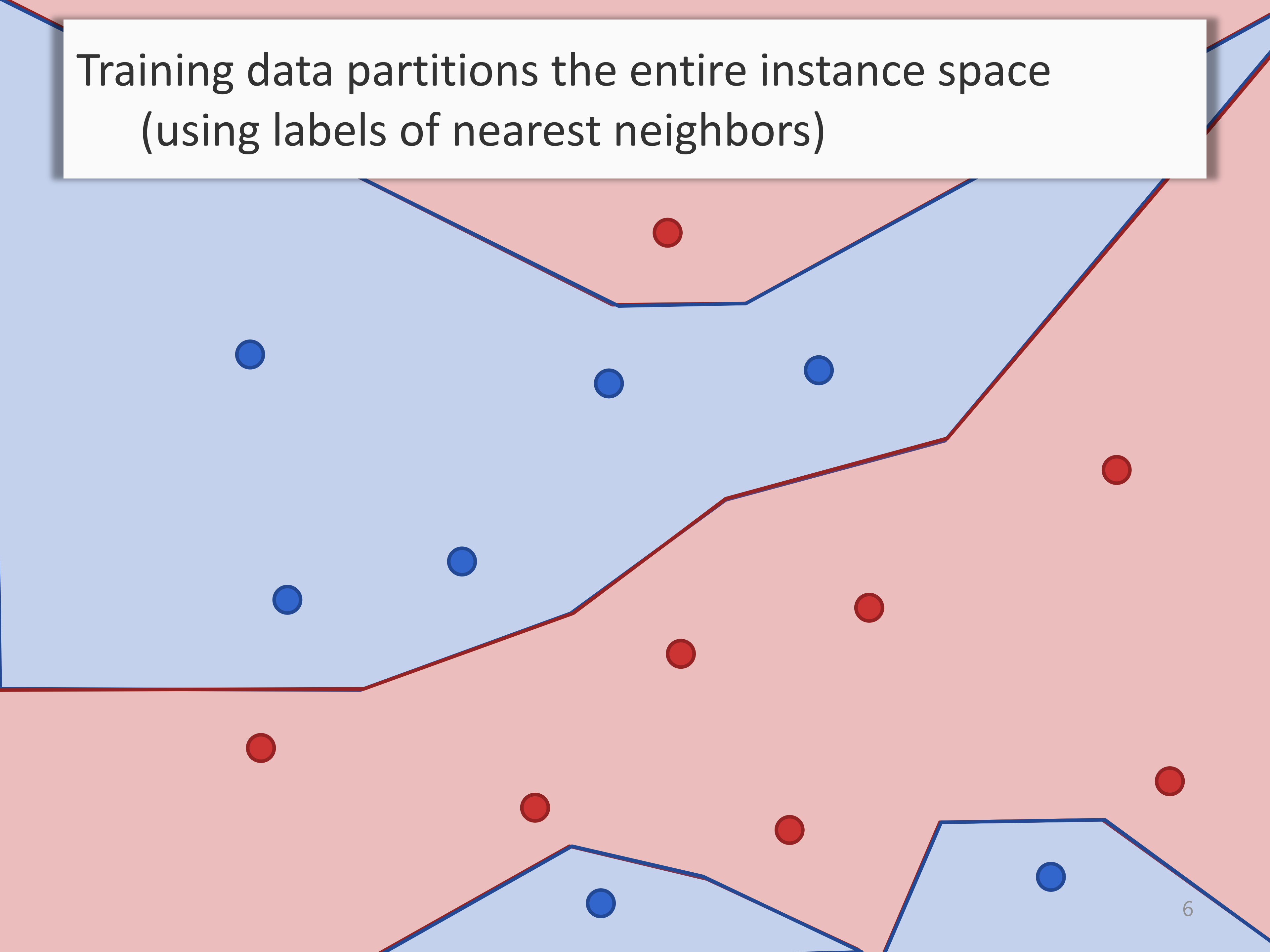
A: Blue

B: Red

C: Red



Training data partitions the entire instance space (using labels of nearest neighbors)



Nearest Neighbors: The basic version

- Training examples are vectors $\mathbf{x}_i$ associated with a label y_i

- Learning: Just store all the training examples

- Prediction for a new example $\mathbf{x}$
  - Find the training example $\mathbf{x}_i$ that is *closest* to $\mathbf{x}$
  - Predict the label of $\mathbf{x}$ to the label y_i associated with $\mathbf{x}_i$

K-Nearest Neighbors

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  - Construct the label of $\mathbf{x}$ using these k points. *How?*
  - For classification: ?

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  - For classification: Every neighbor votes on the label. Predict the most frequent label among the neighbors.
  - For regression: ?

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  - For classification: Every neighbor votes on the label. Predict the most frequent label among the neighbors.
  - For regression: Predict the mean value

Instance based learning

- A class of learning methods
  - Learning: Storing examples with labels
  - Prediction: When presented a new example, classify the labels using *similar* stored examples

- k-nearest neighbors is an example of this class of methods

- Also called *lazy* learning, because most of the computation (in the simplest case, *all* computation) is performed only at prediction time

Questions?

Distance between instances

- In general, a good place to inject knowledge about the domain

- Behavior of this approach can depend on this

- How do we measure distances between instances?

Distance between instances

Numeric features, represented as n dimensional vectors

Distance between instances

Numeric features, represented as n dimensional vectors

- Euclidean distance

$$||\mathbf{x}_1 - \mathbf{x}_2||_2 = \sqrt{\sum_{i=1}^{n} (\mathbf{x}_{1,i} - \mathbf{x}_{2,i})^2}$$

Distance between instances

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- L_p-norm
  - Euclidean = L_2
  - Manhattan = L_1
  - Exercise: What is L_∞?

$$||\mathbf{x}_1 - \mathbf{x}_2||_p = \left(\sum_{i=1}^{n} |\mathbf{x}_{1,i} - \mathbf{x}_{2,i}|^p\right)^{\frac{1}{p}}$$

Distance between instances

What about symbolic/categorical features?

Distance between instances

Symbolic/categorical features

Most common distance is the *Hamming distance*

- Number of bits that are different
- Or: Number of features that have a different value
- Also called the *overlap*
- Example:

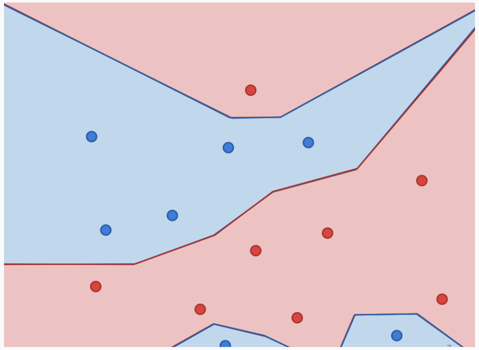
  $\mathbf{X_1}$: {Shape=Triangle, Color=Red, Location=Left, Orientation=Up}

  $\mathbf{X_2}$: {Shape=Triangle, Color=Blue, Location=Left, Orientation=Down}

  Hamming distance = 2

Advantages

- Training is *very fast*
  - Just adding labeled instances to a list
  - More complex indexing methods can be used, which slow down learning slightly to make prediction faster



- Can learn very complex functions
- We always have the training data
  - For other learning algorithms, after training, we don't store the data anymore. What if we want to do something with it later…

Disadvantages

- Needs a lot of storage
  - Is this really a problem now?
- Prediction can be slow!
  - Naïvely: $O(dN)$ for N training examples in d dimensions
  - More data will make it slower
  - Compare to other classifiers, where prediction is very fast
- Nearest neighbors are fooled by irrelevant attributes
  - Important and subtle

Questions?

Summary: k-Nearest Neighbors

- Probably the first “machine learning” algorithm
  - Guarantee: If there are enough training examples, the error of the nearest neighbor classifier will converge to the error of the optimal (i.e. best possible) predictor
- In practice, use an odd k. Why?

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- Feature normalization could be important
  - Often, good idea to center the features to make them zero mean and unit standard deviation. Why?

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- Feature normalization could be important
  - Often, good idea to center the features to make them zero mean and unit standard deviation. Why?
  - Because different features could have different scales (weight, height, etc); but the distance weights them equally

- Variants exist
  - Neighbors’ labels could be weighted by their distance

Where are we?

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  - What is the hypothesis space?
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The decision boundary for kNN

Is the k nearest neighbors algorithm explicitly building a function?

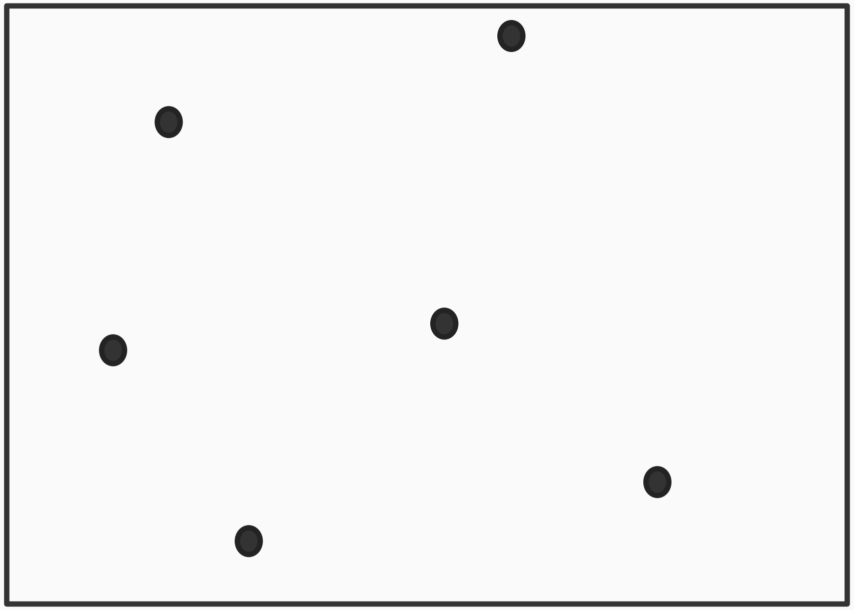
The decision boundary for kNN

Is the k nearest neighbors algorithm explicitly building a function?

– **No,** it never forms an explicit hypothesis

But we can still ask: Given a training set what is the implicit function that is being computed

The Voronoi Diagram

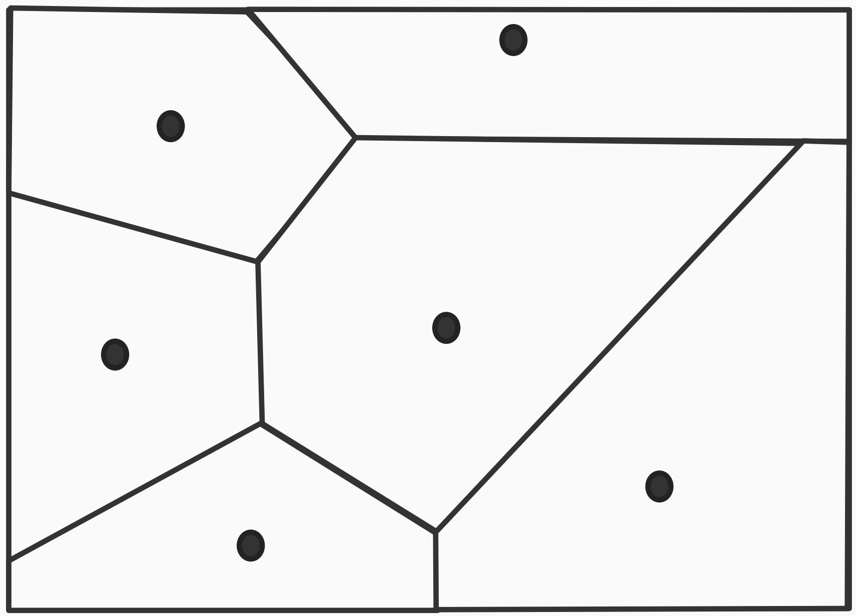


For any point **x** in a training set S, the Voronoi Cell of **x** is a polyhedron consisting of all points closer to **x** than any other points in S

The Voronoi diagram is the union of all Voronoi cells

- Covers the entire space

The Voronoi Diagram



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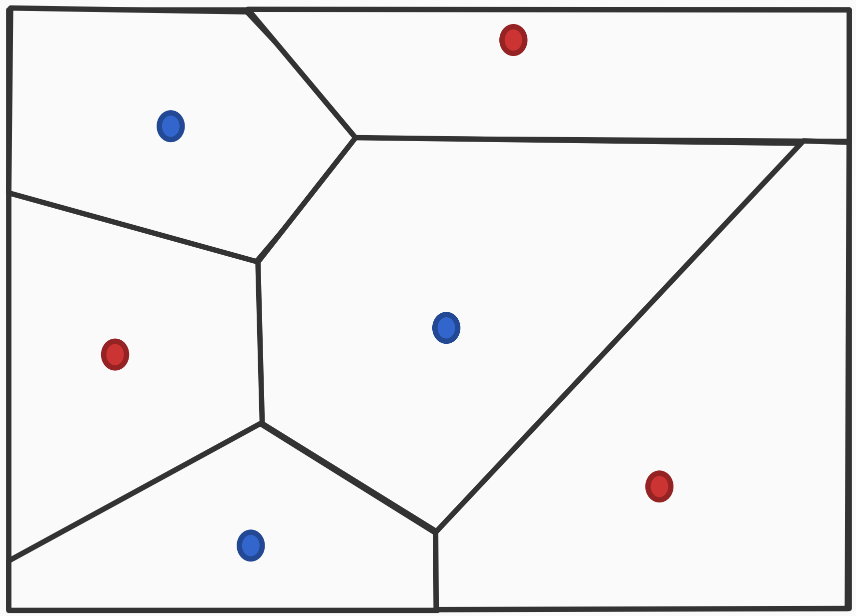
Voronoi diagrams of training examples



Points in the Voronoi cell of a training example are closer to it than any others

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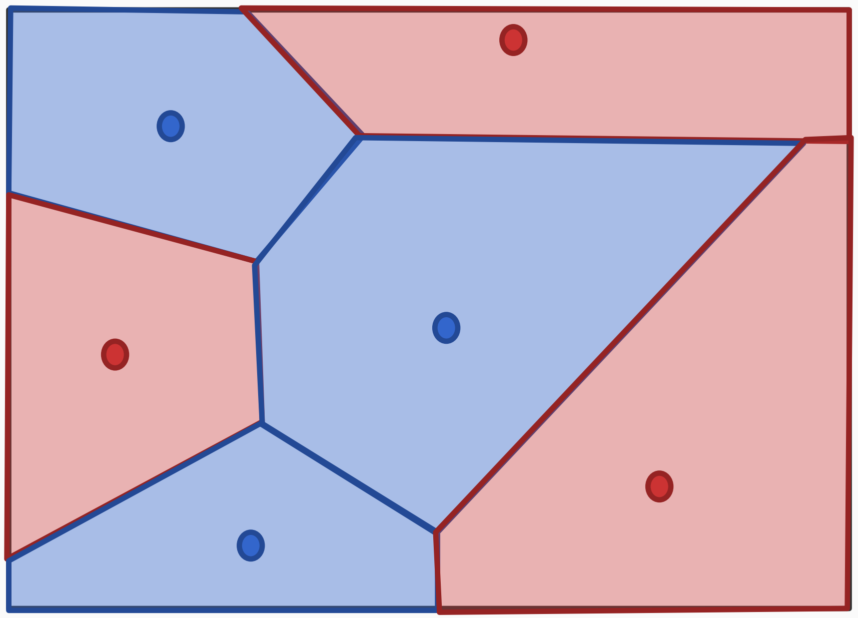

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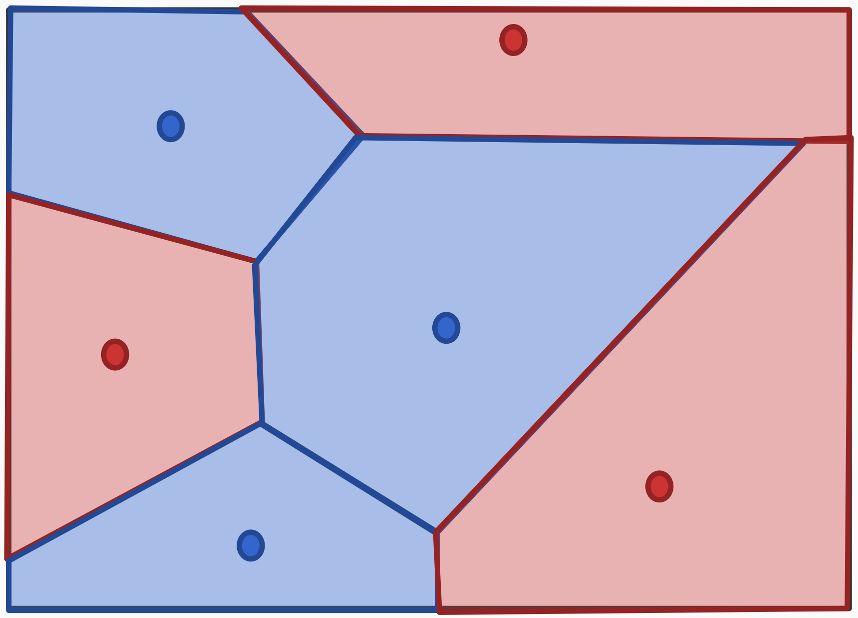

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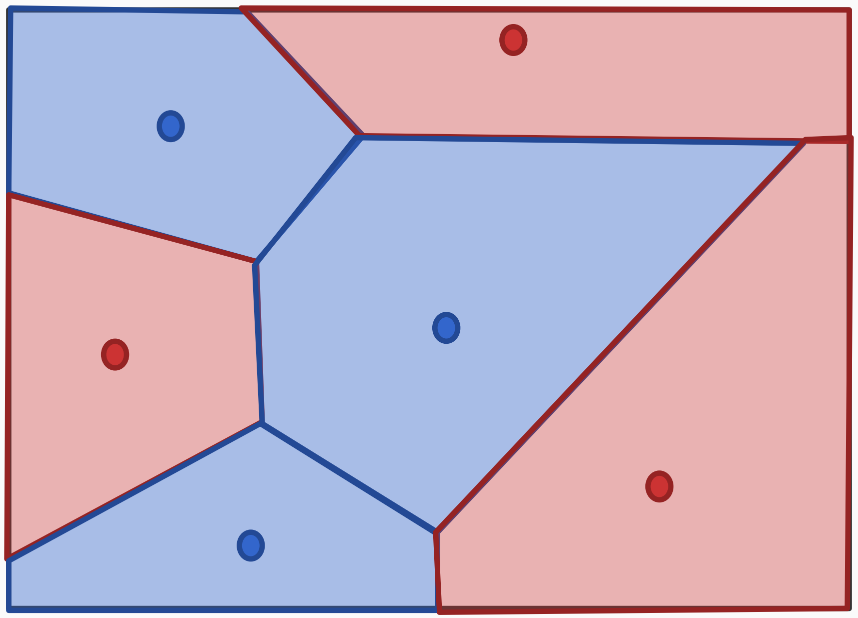
Picture uses Euclidean distance with 1-nearest neighbors.

What about K-nearest neighbors?

Also partitions the space, but much more complex decision boundary

Voronoi diagrams of training examples

What about points on the boundary? What label will they get?



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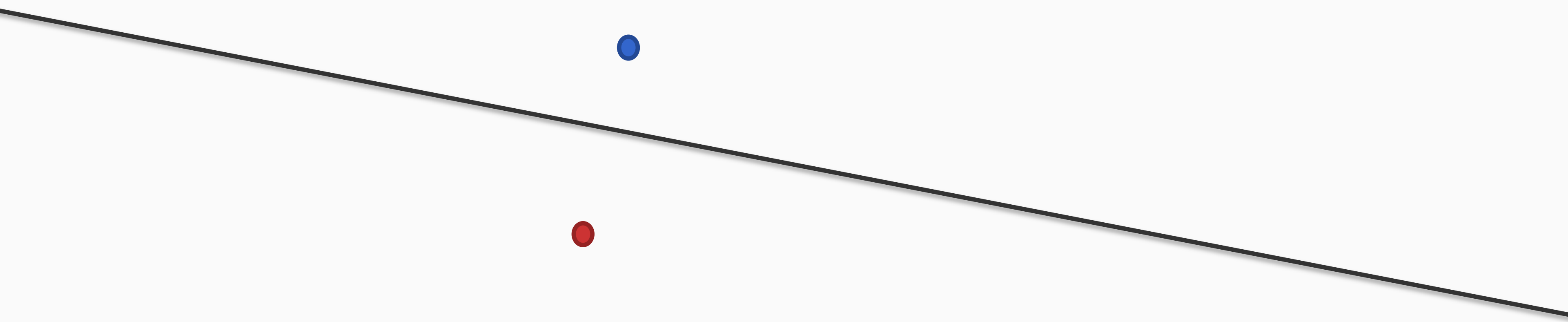

– A line bisecting the two points

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Why your classifier might go wrong

Two important considerations with learning algorithms

- Overfitting: We have already seen this
- The curse of dimensionality
  – Methods that work with low dimensional spaces may fail in high dimensions
  – What might be intuitive for 2 or 3 dimensions do not always apply to high dimensional spaces

Of course, irrelevant attributes will hurt

Suppose we have 1000 dimensional feature vectors

- But only 10 features are relevant

- Distances will be dominated by the large number of irrelevant features

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Suppose we have 1000 dimensional feature vectors

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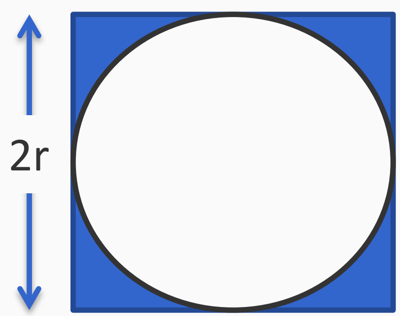
But even with only relevant attributes, high dimensional spaces behave in odd ways

The Curse of Dimensionality

Intuitions that are based on 2 or 3 dimensional spaces do not always carry over to high dimensional spaces

Example 1: What fraction of the points in a cube lie outside the sphere inscribed in it?

In two dimensions



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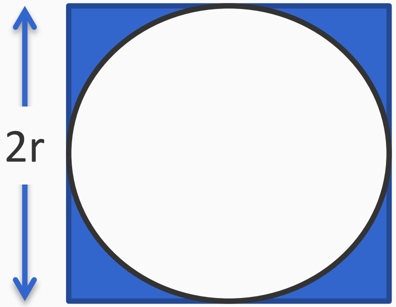

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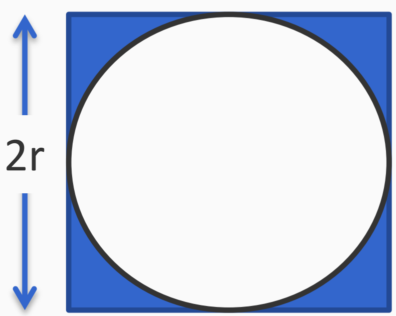

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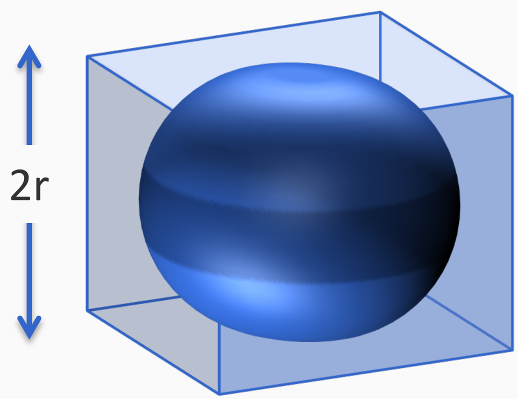

What fraction of the cube is outside the inscribed sphere in three dimensions?

$$1 - \frac{\frac{4}{3}\pi r^3}{8r^3} = 1 - \frac{\pi}{6}$$

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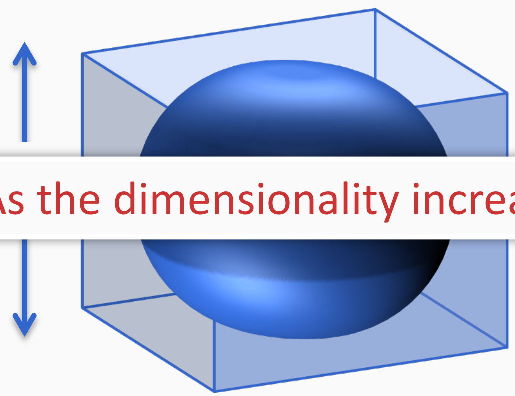

What fraction of the cube is outside the inscribed sphere in three dimensions?

As the dimensionality increases, this fraction approaches 1!!

In high dimensions, most of the volume of the cube is far away from the center!

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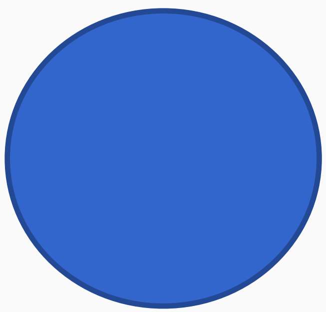

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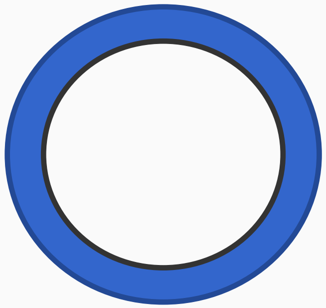

What fraction of the area of the circle is in the blue region?

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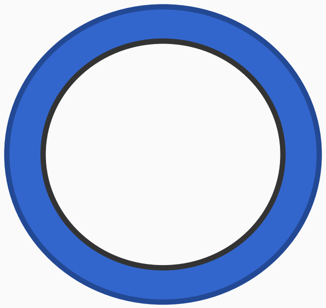

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$$\frac{\pi \cdot 1^2 - \pi(1-\epsilon)^2}{\pi \cdot 1^2} = 1 - (1-\epsilon)^2$$

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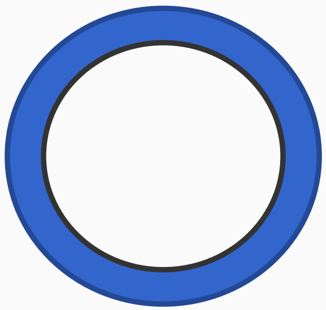

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But, distances do not behave the same way in high dimensions

In d dimensions, the fraction is $1 - (1 - \epsilon)^d$

As d increases, this fraction goes to 1!

In high dimensions, most of the volume of the sphere is far away from the center!

Questions?

The Curse of Dimensionality

- Most of the points in high dimensional spaces are far away from the origin!
  - In 2 or 3 dimensions, most points are near the center
  - Need more data to "fill up the space"

- Bad news for nearest neighbor classification in high dimensional spaces

Even if most/all features are relevant, in high dimensional spaces, most points are equally far from each other!

"Neighborhood" becomes very large

Presents computational problems too

Dealing with the curse of dimensionality

- Most "*real-world*" data is not uniformly distributed in the high dimensional space
  - Different ways of capturing the *underlying dimensionality* of the space
    - Eg: Dimensionality reduction techniques, manifold learning

- Feature selection is an art
  - Different methods exist
  - Select features, maybe by information gain
  - Try out different feature sets of different sizes and pick a good set based on a validation set

- Prior knowledge or preferences about the hypotheses can also help

Questions?

Summary: Nearest neighbors classification

- Probably the oldest and simplest learning algorithm
  - Prediction is expensive.
    - Efficient data structures help. k-D trees: the most popular, works well in low dimensions
    - Approximate nearest neighbors may be good enough some times. Hashing based algorithms exist

- Requires a distance measure between instances
  - Metric learning: Learn the “right” distance for your problem

- Partitions the space into a Voronoi Diagram

- Beware the curse of dimensionality

Questions?

Exercises

1. What happens if you choose k to the number of training examples?

2. Show that the VC dimension of 1-nearest neighbors is infinite.

3. Suppose you want to build a nearest neighbors classifier to predict whether a beverage is a coffee or a tea using two features: the volume of the liquid (in milliliters) and the caffeine content (in grams). You collect the following data:

Volume (ml)	Caffeine (g)	Label
238	0.026	Tea
100	0.011	Tea
120	0.040	Coffee
237	0.095	Coffee

What is the label for a test point with Volume = 120, Caffeine = 0.013?

Why might this be incorrect?

How would you fix the problem?