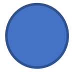
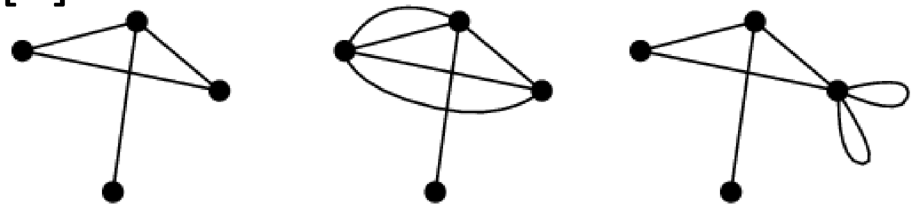
MAT344 Combinatorics
Graph Theory

Session 1

Definitions

1. **Graph**: A diagram made up from Vertices and Edges that connect some pairs of Vertices.[1]
2. **Vertices** 
3. An **Edge**: A line that connects a pair of vertices.
4. **Degree** of a Vertex: The number of Edges connected to that vertex
5. **Loops**: Edges that start and end in the same vertex.[1]
6. **Simple Graph**: If a graph G has no **loops**, and has no **multiple** edges between the same pair of vertices, then we will say that G is a simple graph.[1]



simple graph *nonsimple graph with multiple edges* *nonsimple graph with loops*

Definitions

1. A **Trail**: A sequence of **distinct** edges $e_1, e_2, \dots, e_i$ is called a trail if we can take a continuous walk in our graph. [1,2]
2. **Eulerian trail**: If a trail uses all edges of G, then we call it an Eulerian trail.[2]
3. **Path:** If a trail does not touch any vertex twice, then we call it a path.[2]
4. A **subgraph** H of a graph G: is a graph whose set of vertices is a subset of the set of vertices of G, and whose set of edges is a subset of the set of edges of G.[2]
5. **Connected graph**: for any two vertices x and y in a graph, one can find a path from x to y.[2]
6. **Degree sum formula**: In any undirected graph, the number of vertices with odd degree is **even**.[3]

$$d_1 + d_2 + d_3 + \cdots + d_n = 2|E|$$

1. Prove that in any simple graph there are two vertices with the same degree.

- Proof by contradiction:
- Assume, that given a simple graph of **n** vertices, there does not exist any 2 vertices with the same degrees, i.e. degrees are **distinct** for distinct vertices.
- The set of possible distinct degrees for **n** vertices is **{0, 1, 2, …, n-1},** which is of size n.
- Since we have n vertices and want them to have distinct degrees, we have to use all of the possible distinct degrees.
- Considering our graph; now we have a vertex with degree **0** and a vertex with degree **n-1**, which is impossible as the first one is connected to **no other vertex** and the latter is connected to **every other vertex**. **Contradiction**.

2. Ten players participate in a chess tournament. Eleven games have been played so far. Prove that at least one player has played at least three games.

- Loops are **not** allowed, as no player plays against him/her self.
- Proof by contradiction.
- Assume, that it is possible that **11 games** have been played and each player has played **at most 2** games.
- Each player is a vertex, and the number of games they have played is the degree of that vertex.
- Players who have played a **game** against each other are connected by an edge.
- By our assumption the degree of each player is **at most 2**:
    - **d1 <=2**
    - **d2<=2**
    - **d3<=2**
    - **d10<=2**

2. Ten players participate in a chess tournament. Eleven games have been played so far. Prove that at least one player has played at least three games.

$$d_1 \le 2$$
$$d_2 \le 2$$
$$d_3 \le 2$$
$$\vdots$$
$$d_{10} \le 2$$

Then: $d_1+d_2+d_3+...+d_{10} \le 20 = 2\times 10$, then $|E| \le 10$

- By the degree sum formula,
- **the sum of the total number of degrees = 2 x total number of edges.**
- This implies that the **maximum** number of edges, i.e. games, is **10**, which is a **contradiction**.
- Therefore; there has to be at least one player who has played at least 3 games.

1. Miklós Bóna. A Walk Through Combinatorics:An Introduction to Enumeration and Graph Theory (p. 206). World Scientific Publishing Company. Kindle Edition.
2. Miklós Bóna. A Walk Through Combinatorics:An Introduction to Enumeration and Graph Theory (p. 207). World Scientific Publishing Company. Kindle Edition.
3. Miklós Bóna. A Walk Through Combinatorics:An Introduction to Enumeration and Graph Theory (p. 209). World Scientific Publishing Company. Kindle Edition.