MΔT1100110

of course in base 2

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Tutorial 3

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Example 1

1.41. (−) Let A, B, C be sets. Explain the relationships below. Use the definitions of set operations and containment, with Venn diagrams to guide the argument.

a) $A \subseteq A \cup B$, and $A \cap B \subseteq A$.
b) $A - B \subseteq A$.
c) $A \cap B = B \cap A$, and $A \cup B = B \cup A$.
d) $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$.
e) $A \cap (B \cap C) = (A \cap B) \cap C$.
f) $A \cup (B \cup C) = (A \cup B) \cup C$.

$\boldsymbol{a})\ let's\ consider\ \boldsymbol{A \subseteq A \cup B}$

$\boldsymbol{A \cup B}\ means\ the\ elements\ that\ are\ either\ in\ A\ or\ in\ B$

Example 1 cont'd

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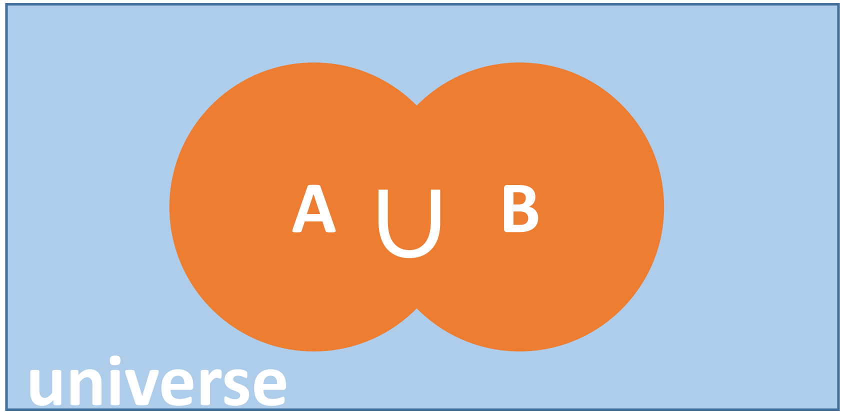
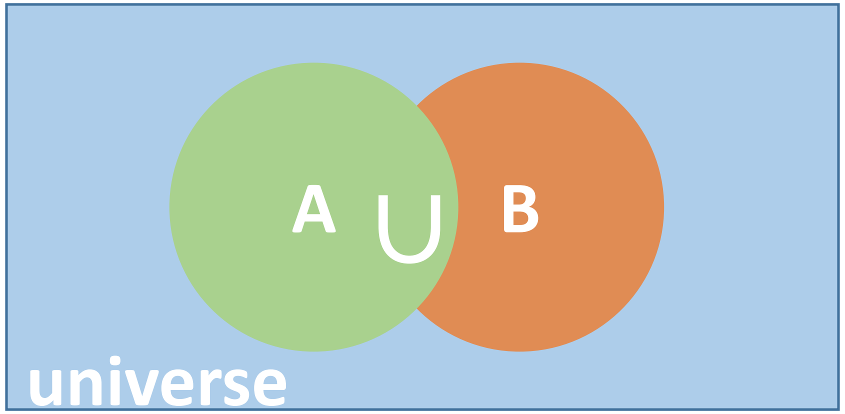


- $\boldsymbol{A \subseteq A \cup B\ \ means}$:
- $A\ is\ included\ in\ the\ \boldsymbol{A \cup B}$,



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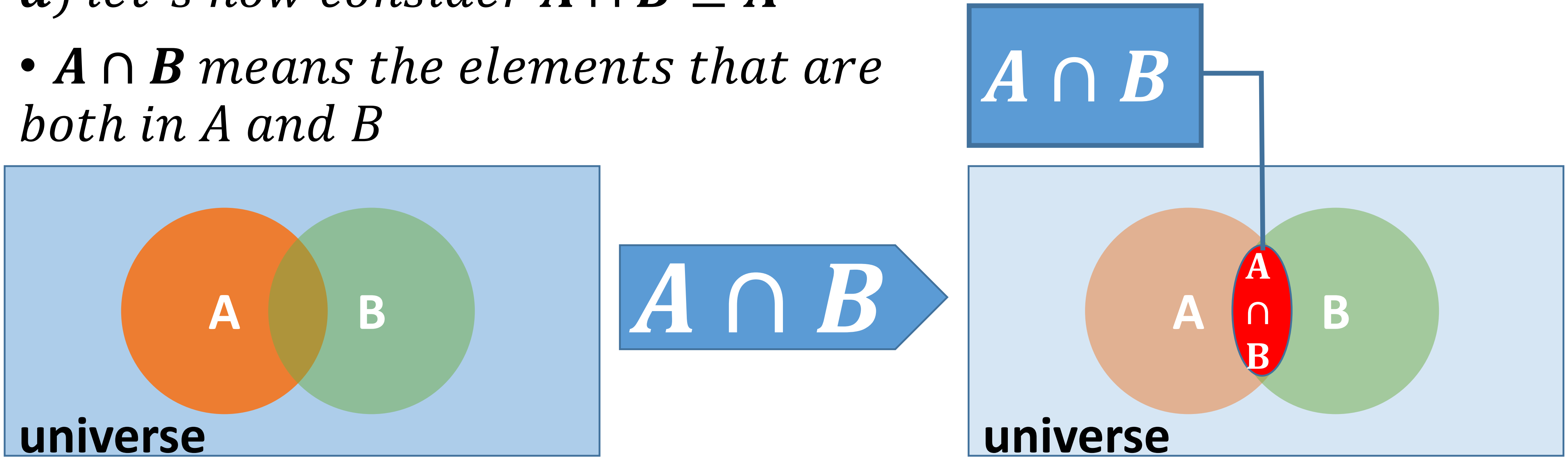
$\boldsymbol{a}$) *let's now consider* $\boldsymbol{A \cap B \subseteq A}$

- $\boldsymbol{A \cap B}$ *means the elements that are both in A and B*



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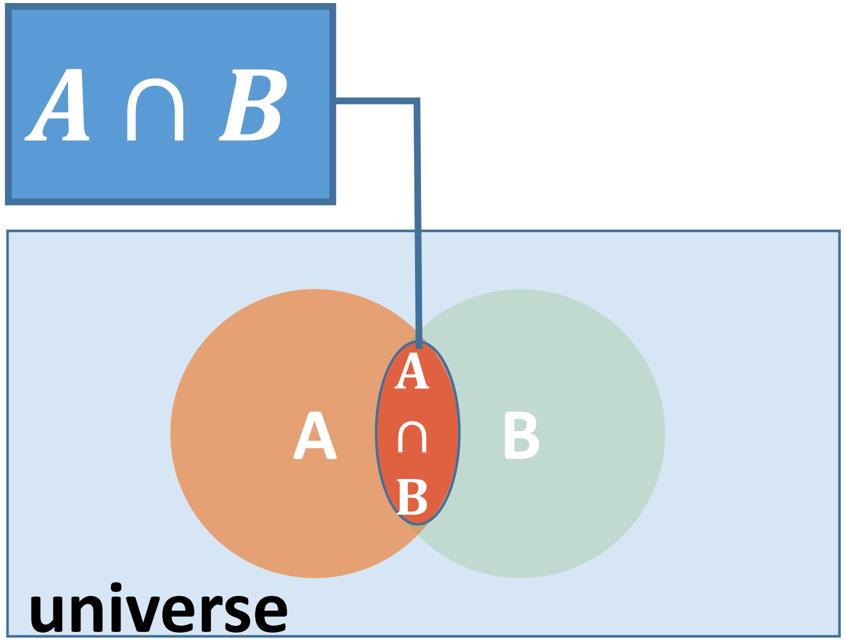
$\boldsymbol{a}$) $let's\ now\ consider\ \boldsymbol{A \cap B \subseteq A}$

- $\boldsymbol{A \cap B}$ $means\ the\ elements\ that\ are\ both\ in\ A\ and\ B$
- $As\ you\ can\ see$ $\boldsymbol{A \cap B}$ $is\ included\ in\ A$
- $\boldsymbol{therefore}$: $\boldsymbol{A \cap B \subseteq A}$ **is True**



Example 2

1.49. (!) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. For the sum and product of f and g (see Definition 1.25), determine which statements below are true. If true, provide a proof; if false, provide a counterexample.

a) If f and g are bounded, then $f+g$ is bounded.
b) If f and g are bounded, then fg is bounded.
c) If $f+g$ is bounded, then f and g are bounded.
d) If fg is bounded, then f and g are bounded.
e) If both $f+g$ and fg are bounded, then f and g are bounded.

- $\boldsymbol{a})\ TRUE.$
- $If\ f\ and\ g\ are\ bounded, then\ there\ exist\ M_1\ and\ M_2\ such\ that$ $|f(x)| \leq M_1\ and\ |g(x)| \leq\ M_2\ for\ all\ x \in \mathbb{R}.$
- $Using\ the\ triangle\ inequality(|x+y| \leq |x| + |y|)\ we\ conclude\ that$
- $|(f+g)(x)| = |f(x) +\ g(x)| \leq\ |f(x)| + |g(x)| \leq\ M_1\ + M_2 = M$
- $for\ all\ x \in \mathbb{R}, and\ hence\ f\ +\ g\ is\ a\ bounded\ function.$

Example 2 cont'd

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- $\boldsymbol{b})TRUE.$
- $If\ f\ and\ g\ are\ bounded, then\ there\ exist\ M_1\ and\ M_2\ such\ that$ $|f(x)| \leq M_1\ and\ |g(x)| \leq\ \ M_2\ for\ all\ x \in \mathbb{R}.$
- $we\ know\ that\ (|x.y| = |x|.|y|)\ we\ conclude\ that$
- $|(fg)(x)| = |f(x).g(x)| =\ \ |f(x)|.|g(x)| \leq\ \ M_1\ .M_2 = M$
- $for\ all\ x \in \mathbb{R}, and\ hence\ f\ .g\ is\ a\ bounded\ function.$

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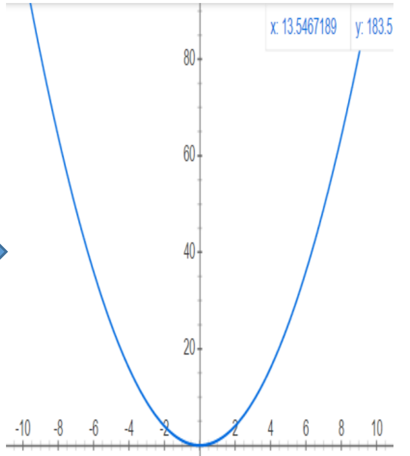

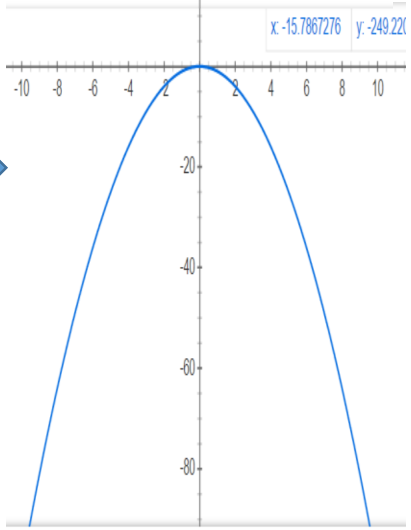

- $\mathbf{c)}\boldsymbol{False}.$ *come up with a counter example.*
- $let\ f(x) = x^2\ and\ g(x) = -x^2$
- $\Rightarrow (f+g)(x) = f(x) + g(x) = x^2 + (-x^2) = 0$
- $As\ you\ can\ see\ f(x)\ and\ g(x)\ are\ both\ unbounded.$
- $however, (f+g)(x) = 0\ which\ is\ bounded\ by\ any\ real\ number \geq 0, eg. 0. |(f+g)(x)| \leq 0$

Example 2 cont’d

1.49. (!) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. For the sum and product of f and g (see Definition 1.25), determine which statements below are true. If true, provide a proof; if false, provide a counterexample.

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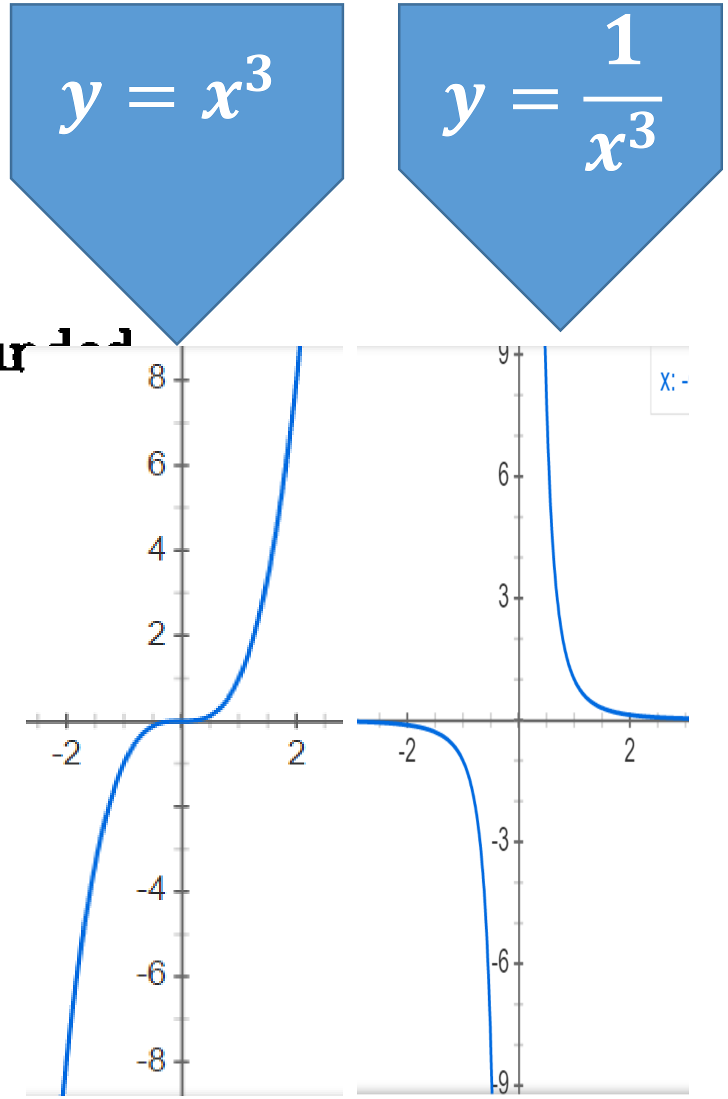
- $\boldsymbol{d)False}.$ *come up with a counter example.*
- $let\ f(x) = x^3\ and\ g(x) = \frac{1}{x^3}$
- $\Rightarrow (f.g)(x) = f(x).g(x) = x^3.\frac{1}{x^3} = 1$
- $As\ you\ can\ see\ f(x)\ and\ g(x)\ are\ both\ unbounded.$
- $however, (f.g)(x) = 1\ which\ is\ bounded\ by\ any$
$real\ number \geq 1, eg.\ 2., |(f.g)(x)| \leq 2$



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c) If $f+g$ is bounded, then f and g are bounded.
d) If fg is bounded, then f and g are bounded.
e) If both $f+g$ and fg are bounded, then f and g are bounded.

- $\boldsymbol{e)True}$.
- We want to show that $|f(x)| \leq M_1 \; and \; |g(x)| \leq \; M_2$ $for \; all \; x \in \mathbb{R}$.
- $Note$: $showing \; that \sqrt{f^2(x) + g^2(x)} \leq M, is \; enough \; to \; conclude$ $|f(x)| \leq M_1 \; and \; |g(x)| \leq \; M_2, since$
- $|f(x)| = \sqrt{f^2(x)} \leq \sqrt{f^2(x) + g^2(x)} \leq M,$
- $and \; the \; same \; holds \; for \; g(x)$

Example 2 cont'd

- $\boldsymbol{e)we\ are\ given\ that\ f+g\ and\ fg\ are\ bounded}$.
- $we\ want\ to\ show\ that\ \sqrt{f^2(x)+g^2(x)} \leq M$
- $we\ know\ |(f+g)(x)| \leq M_1\ and\ |fg(x)| \leq M_2$
- $|(f+g)(x)| = |f(x)+g(x)| \leq M_1$ **Square both sides**
- $|f(x)+g(x)|^2 = \left(f(x)+g(x)\right)^2 \leq M_1^2$ **expand**
- $\left(f(x)+g(x)\right)^2 = f^2(x)+2f(x).g(x)+g^2(x) \leq M_1^2$ $\boldsymbol{-2f(x).g(x)}$
- $f^2(x)+g^2(x) \leq M_1^2 - 2f(x).g(x)$ $\boldsymbol{a \leq |a|}$
- $f^2(x)+g^2(x) \leq M_1^2 - 2f(x).g(x) \leq M_1^2 + |2f(x).g(x)|$ $\boldsymbol{|fg(x)| \leq M_2}$
- $f^2(x)+g^2(x) \leq M_1^2 + |2f(x).g(x)| \leq M_1^2 + M_2$
- $So\ f^2(x)+g^2(x) \leq M_1^2 + M_2$
- $but\ we\ wanted \sqrt{f^2(x)+g^2(x)} \leq M$

Example 2 cont'd

- $\boldsymbol{e)we\ are\ given\ that\ f+g\ and\ fg\ are\ bounded}$.
- $we\ want\ to\ show\ that\ \sqrt{f^2(x)+g^2(x)} \leq M$

- $So\ \ f^2(x)+g^2(x) \leq M_1^2 + M_2$ **square root**
- $\Rightarrow \sqrt{f^2(x)+g^2(x)} \leq \sqrt{M_1^2+M_2}$
- *Now lets bound f(x) and g(x)*
- $|f(x)| = \sqrt{f^2(x)} \leq \sqrt{f^2(x)+g^2(x)} \leq \sqrt{M_1^2+M_2} = M$
- $|g(x)| = \sqrt{g^2(x)} \leq \sqrt{f^2(x)+g^2(x)} \leq \sqrt{M_1^2+M_2} = M$
- *Therefore if* $f+g$ *and* fg *are bounded , then* $f(x)$ *and* $g(x)$ *are also bounded.*

Example 3

Show that for any three sets A, B, C we have: $\boldsymbol{A-(B-C)=(A-B)\cup(A\cap C)}$.
Write a formal solution, and draw the appropriate Venn diagram.

- $Note\colon to\ prove\ that\ two\ sets\ are\ equal, A=B, we\ need\ to\ show,$ $A\subseteq B\ and\ B\subseteq A.$
- So lets first show: $A\subseteq B$ I.e. starting from left side and finish at right.
- $\boldsymbol{start\ from\ A-(B-C)\ conclude\ (A-B)\cup(A\cap C)}$.
- $\boldsymbol{A-(B-C)}$ $A-B=A\cap B^c$
- $\boldsymbol{=A-(B\cap C^c)}$ $A-B=A\cap B^c$
- $\boldsymbol{=A\cap(B\cap C^c)^c}$ $\boldsymbol{by\ demorgan's\ law}$
- $\boldsymbol{=A\cap\left(B^c\cup C^{c^c}\right)=A\cap(B^c\cup C)}$ **by Distributive property**
- $\boldsymbol{=(A\cap B^c)\cup(A\cap C)}$

$\boldsymbol{demorgan's\ laws}$:
$(A\cap B)^c=A^c\cup B^c$
$(A\cup B)^c=A^c\cap B^c$

$\boldsymbol{distributive\ property}$:
$A\cup(B\cap C)\Leftrightarrow(A\cup B)\cap(A\cup C)$
$A\cap(B\cup C)\Leftrightarrow(A\cap B)\cup(A\cap C)$

Example 3 cont'd

Show that for any three sets A, B, C we have: $\boldsymbol{A - (B - C) = (A - B) \cup (A \cap C)}$.
Write a formal solution, and draw the appropriate Venn diagram.

- $Note: to\ prove\ that\ two\ sets\ are\ equal, A = B, we\ need\ to\ show,$ $A \subseteq B\ and\ B \subseteq A$.
- So lets first show: $A \subseteq B$ I.e. starting from left side and finish at right.
- $\boldsymbol{start\ from\ A - (B - C)\ conclude\ (A - B) \cup (A \cap C)}$.
- $\boldsymbol{A - (B - C)}$
- $\boldsymbol{= (A \cap B^c) \cup (A \cap C)}$ $A - B = A \cap B^c$
- $\boldsymbol{= (A - B) \cup (A \cap C)}$
- Therefore we just showed that $\boldsymbol{A - (B - C) \subseteq (A - B) \cup (A \cap C)}$
- But in order to say $\boldsymbol{A - (B - C) = (A - B) \cup (A \cap C)}$, we still need to show $\boldsymbol{A - (B - C) \supseteq (A - B) \cup (A \cap C)}$.

Example 3 cont'd

Show that for any three sets A, B, C we have: $\boldsymbol{A - (B - C) = (A - B) \cup (A \cap C)}$.
Write a formal solution, and draw the appropriate Venn diagram.

- $Note: to\ prove\ that\ two\ sets\ are\ equal, A = B, we\ need\ to\ show,$
$A \subseteq B\ and\ B \subseteq A.$
- Now lets show: $B \subseteq A$ I.e. starting from right side and finish at left.
- $\boldsymbol{start\ from\ (A - B) \cup (A \cap C)\ conclude\ A - (B - C)}$.
- $\boldsymbol{(A - B) \cup (A \cap C)}$ — $A - B = A \cap B^c$
- $\boldsymbol{= (A \cap B^c) \cup (A \cap C)}$ — by Distributive law
- $\boldsymbol{= A \cap (B^c \cup C)}$ — $by\ demorgan's\ law$
- $\boldsymbol{= A \cap (B \cap C^c)^c}$ — $A - B = A \cap B^c$
- $\boldsymbol{= A - (B \cap C^c)}$ — $A - B = A \cap B^c$
- $\boldsymbol{= A - (B - C) \Rightarrow A - (B - C) \supseteq (A - B) \cup (A \cap C)}$.

$distributive\ property$:
$A \cup (B \cap C) \Leftrightarrow (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) \Leftrightarrow (A \cap B) \cup (A \cap C)$

$demorgan's\ laws$:
$(A \cap B)^c = A^c \cup B^c$
$(A \cup B)^c = A^c \cap B^c$

Example 3 cont'd

Show that for any three sets A, B, C we have: $\boldsymbol{A-(B-C)=(A-B)\cup(A\cap C)}$.
Write a formal solution, and draw the appropriate Venn diagram.

- $Note\text{:} \, to\ prove\ that\ two\ sets\ are\ equal, A = B, we\ need\ to\ show,$
$A \subseteq B\ and\ B \subseteq A$.
- So we showed that:
- $\mathbf{1)}\ \boldsymbol{A-(B-C)\supseteq(A-B)\cup(A\cap C)}$.
- $\mathbf{2)}\ \boldsymbol{A-(B-C)\subseteq(A-B)\cup(A\cap C)}$.
- Now we can conclude that :
- $\boldsymbol{A-(B-C)=(A-B)\cup(A\cap C)}$.

Example 4

Express the set $\{x \in \mathbb{R} \,|(x+3)(7-x)(x-2)^2 > 0\}$ as a union of intervals. Explain your answer briefly.

- $we\ need\ to\ simplify\ the\ condition\ of\ the\ set\ to\ see\ which\ real\ nunbers\ are\ included.$
- $(x+3)(7-x)(x-2)^2 > 0$

Use a table to see how the equation behaves on different intervals

- First find the roots of the equation
- Then see how each factor behaves before and after its root
- $x+3=0 \Rightarrow x=-3$
- $7-x=0 \Rightarrow x=7$
- $(x-2)^2=0 \Rightarrow x=2$

Example 4 cont'd

Express the set $\{x \in \mathbb{R} \,|(x+3)(7-x)(x-2)^2 > 0\}$ as a union of intervals. Explain your answer briefly.

- $we\ need\ to\ simplify\ the\ condition\ of\ the\ set\ to\ see\ which\ real\ nunbers\ are\ included.$
- E=$(x+3)(7-x)(x-2)^2 > 0$

Use a table to see how the equation behaves on different intervals

- Then see how each factor behaves before and after its root
- $x+3=0 \Rightarrow x=-3$
- $7-x=0 \Rightarrow x=7$
- $(x-2)^2=0 \Rightarrow x=2$

	$(\infty,-3)$	$(-3,2)$	$(2,7)$	$(7,\infty)$
$x+3$				
$7-x$				
$(x-2)^2$				
E				

Example 4 cont'd

Express the set $\{x \in \mathbb{R}\,|(x+3)(7-x)(x-2)^2 > 0\}$ as a union of intervals. Explain your answer briefly.

- $we\ need\ to\ simplify\ the\ condition\ of\ the\ set\ to\ see\ which\ real\ nunbers\ are\ included.$
- E=$(x+3)(7-x)(x-2)^2 > 0$

Use a table to see how the equation behaves on different intervals

- Then see how each factor behaves before and after its root
- $x+3 < 0\ when\ x < -3$
- $x+3 > 0\ when\ x > -3$
- $7-x < 0\ when\ x > 7$
- $7-x > 0\ when\ x < 7$
- $(x-2)^2 > 0\ when\ x < 2$
- $(x-2)^2 > 0\ when\ x > 2$

	$(\infty,-3)$	$(-3,2)$	$(2,7)$	$(7,\infty)$
$x+3$	−	+	+	+
$7-x$	+	+	+	−
$(x-2)^2$	+	+	+	+
E				

Example 4 cont'd

Express the set $\{x \in \mathbb{R} \,|(x+3)(7-x)(x-2)^2 > 0\}$ as a union of intervals. Explain your answer briefly.

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- E=$(x+3)(7-x)(x-2)^2 > 0$

Use a table to see how the equation behaves on different intervals

- Then see how each factor behaves before and after its root
- $Note\ now\ that$:
- $+ . + = +$
- $- . + = -$
- $+ . - = -$
- $- . - = +$

	$(\infty, -3)$	$(-3, 2)$	$(2, 7)$	$(7, \infty)$
$x+3$	$-$	$+$	$+$	$+$
$7-x$	$+$	$+$	$+$	$-$
$(x-2)^2$	$+$	$+$	$+$	$+$
E				

Example 4 cont'd

Express the set $\{x \in \mathbb{R} \,|(x+3)(7-x)(x-2)^2 > 0\}$ as a union of intervals. Explain your answer briefly.

- $we\ need\ to\ simplify\ the\ condition\ of\ the\ set\ to\ see\ which\ real\ nunbers\ are\ included.$
- E=$(x+3)(7-x)(x-2)^2 > 0$

Use a table to see how the equation behaves on different intervals

- Then see how each factor behaves before and after its root
- $Note\ now\ that$:
- $1st\ column: -.+.+= -$
- $2ndcolumn: +.+.+= +$
- $3rd\ column: +.+.+= +$
- $4th\ column: +.-.+= -$

	$(\infty, -3)$	$(-3, 2)$	$(2, 7)$	$(7, \infty)$
$x+3$	$-$	$+$	$+$	$+$
$7-x$	$+$	$+$	$+$	$-$
$(x-2)^2$	$+$	$+$	$+$	$+$
E	$-$	$+$	$+$	$-$

Example 4 cont'd

Express the set $\{x \in \mathbb{R} \,|(x+3)(7-x)(x-2)^2 > 0\}$ as a union of intervals. Explain your answer briefly.

- $we\ need\ to\ simplify\ the\ condition\ of\ the\ set\ to\ see\ which\ real\ nunbers\ are\ included.$
- E=$(x+3)(7-x)(x-2)^2 > 0$

Use a table to see how the equation behaves on different intervals

- Then see how each factor behaves before and after its root
- $Note\ now\ that$:
- $since\ we\ want\ E > 0$:
- $\boldsymbol{x} \in (\boldsymbol{-3,2}) \cup (\boldsymbol{2,7})$
- Note since at 2, E=0,

we exclude 2.

	$(\infty, -3)$	$(-3, 2)$	$(2, 7)$	$(7, \infty)$
$x+3$	$-$	$+$	$+$	$+$
$7-x$	$+$	$+$	$+$	$-$
$(x-2)^2$	$+$	$+$	$+$	$+$
E	$-$	$+$	$+$	$-$

Example 5 cont'd

Consider the function $f: (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{4x}{x+1}$. Prove that the image of f is the interval $(0, 4)$.

- For showing that the image of a function is an interval, we basically want to show that the set of outputs of that function is equal to that interval.
- Hence we have 2 sets $A\ and\ B$ and we want to show that $A = B$.
- Therefore we need to show:
- 1st $A \subseteq B$
- 2nd $B \subseteq A$

Example 5 cont'd

Consider the function $f\colon(0,\infty)\to\mathbb{R}$ given by $f(x)=\frac{4x}{x+1}$. Prove that the image of f is the interval $(0,4)$.

- We want to show $f\big((0,\infty)\big)=(0,4)$.
- We need to show:
- 1st $f\big((0,\infty)\big)\subseteq(0,4)$.
- 2nd $f\big((0,\infty)\big)\supseteq(0,4)$.

Example 5 cont'd

Consider the function $f: (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{4x}{x+1}$. Prove that the image of f is the interval $(0, 4)$.

- Lets 1st show $f\big((0,\infty)\big) \subseteq (0,4)$.
- We want to show that every element of the output is included in the interval (0, 4).
- i.e. we want to pick an arbitrary y from the set of output and show that it is within the image.
- Take $y \in\ f((0,\infty))$ which satisfies $y\ =\ f(x)$ for some $x \in\ (0,1)$.
- This means that $y\ = \frac{4x}{x+1}$.
- We want to prove that $0\ <\ y\ <\ 4$.

Example 5 cont'd

Consider the function $f: (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{4x}{x+1}$. Prove that the image of f is the interval $(0, 4)$.

- Lets 1st show $f\big((0, \infty)\big) \subseteq (0, 4)$.
- Take $y \in f((0, \infty))$ which satisfies $y = f(x)$ for some $x \in (0, 1)$.
- $y = \frac{4x}{x+1}$.
- We want to prove that $0 < y < 4$.
- We do some rough work:
- $0 < y < 4 \to 0 < \frac{4x}{x+1} < 4$. $\boldsymbol{Since\ x + 1 > 0, multiply\ by\ it}$
- $0 < 4x < 4(x + 1) \to 0 < x < x + 1$
- Witch(lol!) is always true.

Example 5 cont'd

Consider the function $f: (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{4x}{x+1}$. Prove that the image of f is the interval $(0, 4)$.

- Lets 1st show $f\big((0, \infty)\big) \subseteq (0, 4)$.
- Take $y \in f((0, \infty))$ which satisfies $y = f(x)$ for some $x \in (0, 1)$.
- $y = \frac{4x}{x+1}$.
- We want to prove that $0 < y < 4$.
- Proof:
- Since $x \in (0, \infty)$; $0 < x < x + 1$ $\boldsymbol{Since\ x + 1 > 0, devide\ by\ it}$
- $\to$ $0 < \frac{x}{x+1} < 1$ $\boldsymbol{mul\ by\ 4}$
- $\to 0 < \frac{4x}{x+1} < 4$
- $\to 0 < y < 4.$
- $\to$ $f\big((0, \infty)\big) \subseteq (0, 4)$

Example 5 cont'd

Consider the function $f: (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{4x}{x+1}$. Prove that the image of f is the interval $(0, 4)$.

- Lets 2nd show $(0, 4) \subseteq f\big((0, \infty)\big)$.
- We want to show that every element from the interval (0, 4) is included in the set of outputs of the function.
- i.e. we want to show that every element of (0, 4) can be a valid output for our function.
- Therefore we assume an arbitrary element of (0, 4) is a valid output, then try to find if the x corresponding to it is within our domain of our function.
- In other words if there exists such an X, we surely can reach that arbitrary element that we assumed was a valid output.

Example 5 cont'd

Consider the function $f\colon (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{4x}{x+1}$. Prove that the image of f is the interval $(0, 4)$.

- Lets 2nd show $(0, 4) \subseteq f\big((0, \infty)\big)$.
- Pick a $y \in (0, 4)$ and assume it is a valid output of our function, then we want to find the x, for which $f(x) = y$, and see if that x is within our domain. i.e. $x \in (0, 1)$.
- If such x exists, then it satisfies:
- $f(x) = \frac{4x}{x+1} = y$

basically solve for x and see if it is valid for the function

- $\Rightarrow y(x + 1) = 4x$
- $\Rightarrow yx + y = 4x$
- $\Rightarrow y = 4x - yx$
- $\Rightarrow \mathrm{y} = x(4 - y)$
- $\Rightarrow x = \frac{y}{4-y}$

Example 5 cont'd

Consider the function $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = \frac{4x}{x+1}$. Prove that the image of f is the interval $(0,4)$.

- Lets 2nd show $(0,4) \subseteq f\big((0,\infty)\big)$.
- $\Rightarrow x = \frac{y}{4-y}$
- Since $0 < y < 4, y > 0\ and\ 4 - y > 0 \to x > 0$
- Therefore $y \in f\big((0,\infty)\big)$.
- $\to \boldsymbol{(0,4) \subseteq f\big((0,\infty)\big)}$
- **Since** $\boldsymbol{(0,4) \subseteq f\big((0,\infty)\big)}$ **and** $\boldsymbol{(0,4) \supseteq f\big((0,\infty)\big)}$
- $\Rightarrow \boldsymbol{(0,4) = f\big((0,\infty)\big)}$ **i.e. the image is (0, 4)**

Example 6

Consider the function $f : A \to B$.

1. Prove that for any two sets $C, D \subseteq A$; we have $f(C) - f(D) \subseteq f(C - D)$.
2. Give an example of a function f, and sets C, D,
 for which $f(C) - f(D) \neq f(C - D)$.

- 1) lets first understand it:
- We know that our function maps from A to B, so A is our domain.
- We want to prove that for any 2 subsets of our domain, e.g. $C, D \subseteq A$, the set of outputs for set C, without the set of outputs for set D, is included in the set of outputs of C – D.
- One might think they are equal but here is an example that they're not.
- $f(C) - f(D)$ *and* $f(C - D)$ are not guaranteed to be equal.

Example 6 cont'd

Consider the function $f: A \to B$.

1. Prove that for any two sets $C, D \subseteq A$; we have $f(C) - f(D) \subseteq f(C - D)$.
2. Give an example of a function f, and sets C, D,
 for which $f(C) - f(D) \neq f(C - D)$.

- 1)
- One might think they are equal but here an example that they're not.
- $f(C) - f(D)\ and\ f(C - D)$ are not guaranteed to be equal.
- Let $f = x^2, A = \mathbb{Z}, C = \{-3, -2, -1, 0\}, D = \{0, 1, 2\}$
- $C - D = \{-3, -2, -1\}$
- $f(C) = \{0, 1, 4, 9\}$
- $f(D) = \{0, 1, 4\}$
- $f(C - D) = \{1, 4, 9\}$

Example 6 cont'd

Consider the function $f: A \to B$.

1. Prove that for any two sets $C, D \subseteq A$; we have $f(C) - f(D) \subseteq f(C - D)$.
2. Give an example of a function f, and sets C, D,
 for which $f(C) - f(D) \neq f(C - D)$.

- 1) Let $f = x^2$, $A = \mathbb{Z}$, $C = \{-3, -2, -1\}$, $D = \{1, 2\}$
- $C - D = \{-3, -2, -1\}$
- $f(C) = \{1, 4, 9\}$
- $f(D) = \{1, 4\}$
- $f(C - D) = \{1, 4, 9\}$
- $f(C) - f(D) = \{9\} \subseteq f(C - D) = \{1, 4, 9\}$
- The reason why they became not equal is: there are some elements in C and D that when put in the function X^2 have the same outputs ,i.e. 1, 2 and -1, -2. since they are different elements they exist in $C - D$ and their output is kept in the image of $C - D$, however when we do $f(C) - f(D)$ since those elements have the same output, those outputs are lost, therefore $f(C) - f(D) \subseteq f(C - D)$

Example 6 cont'd

Consider the function $f : A \rightarrow B$.

1. Prove that for any two sets $C, D \subseteq A$; we have $f(C) - f(D) \subseteq f(C - D)$.

2. Give an example of a function f, and sets C, D,
for which $f(C) - f(D) \neq f(C - D)$.

- 1)now let us prove: $f(C) - f(D) \subseteq f(C - D)$.
- To prove set $A \subseteq B$ we need to:
- Show that every y in A is included in B.
- Let $y \in f(C) - f(D)$. Then $y \in f(C)$ which means that $y = f(x)$ for some $x \in C$.
- We also know that $y \notin f(D)$ and hence $x \notin D$.
- Therefore, we conclude that $y = f(x)$ and $x \in C - D$.
- In other words,$y \in f(C - D)$.
- This proves that $f(C) - f(D) \subseteq f(C - D)$.

Citations:

Book

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