MΔT0x66

of course in base 16

Arash Gholami

Email: arash.gholami@mail.utoronto.ca

Tutorial 6

LinkedIn

Credits to Larry Zhang

Principle of Simple Induction

Let P(n) be the statment for some number n

If

(i) If $P(b)$ is True,

(ii) And $P(n) \Rightarrow P(n+1)$ is True for all $n \geq b$,

$P(n)$ is True for **all** integers $n \geq b$.

then

Recipe for writing a proof using simple induction

- Step 1: Define the predicate P(n), i.e. the statment you want to prove.
- Step 2: Base Case: show that P(b) is True, e.g., b = 0
- Step 3: Induction Step: show P(k) => P(k+1),
- for k >= b, i.e.,
  - a. Assume P(k) is True (Inductive Hypothesis)
  - b. Show that P(k+1) is True
- Done.

Example 1

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

- Step 1:
  - Define the predicate, i.e. statement you want to prove.

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Example 1

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

- Step 2:
  - Base case:
    - When n = 0, need to show...

$$P(0) : \sum_{i=0}^{0} i = \frac{0(0+1)}{2} = 0$$

  - P(0) is True
  - This is True, so based case done.

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Example 1

- Step 3: Induction Step
  - Need to show:
    - P(k) => P(k+1)

$$P(n) : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

- I.H.: Assume P(k): $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

- Want to show P(k+1):

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

Prove that for every natural number $n \geq 0$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Example 1

The statment we want to prove:

$$P(n) : \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

- Calculations

Induction Hypothesis

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2}.$$

Induction step

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

$$\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$ **By I.H.**

$$= (k+1)\left(\frac{k}{2} + 1\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

Example 2

Prove by simple induction that, n is a natural number:

$$\forall n \geq 1, \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 2

- Step 1:
  - Define the statment you want to prove:
    - For all n more than or equal to 1

$$\forall n \geq 1, \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 2

$$\forall n \geq 1, \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

- Step 2:
  - Verify that the statment holds for the base case:
    - Let n=1:
      - then $if n = 1, LeftSide = \sum_{i=1}^{1} i^3 = 1^3$

$$if n = 1, RightSide = \frac{n^2(n+1)^2}{4} = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

since LS=RS => the statment holds for the base case

Example 2

- Induction step:
  - Need to show:
    - If the statment holds for k => it also holds for k+1
  - Induction Hypothesis:
    - Assume that the statment holds for n=k, then we have,

$$I.H.: assume for n = k, \sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

Example 2

- Induction step:
  - Show that for n = k+1 the statment still holds,

$$show:$$

$$for\ n = k+1, \sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2\big((k+1)+1\big)^2}{4}$$

Example 2

$$show: for n = k+1, \sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2\big((k+1)+1\big)^2}{4}$$

- From I.H. we know:

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$$

- Now let n = k+1, then we have:

$$\sum_{i=1}^{k+1} i^3 = \boldsymbol{\sum_{i=1}^{k} i^3} + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

By I.H.

$$= \frac{(k+1)^2\big((k+1)+1\big)^2}{4}$$

Q.E.D

Exercise 3

- Prove by induction that a set with n elements has exactly 2^n subsets.
- Base Case: Since the empty set has 1 subset (itself), and $2^0 = 1$, then a set with 0 elements has 2^0 subsets.

Example 3

- Induction Step:
- Induction Hypothesis:
  - Assume every k-element set has 2^k subsets.
- Show that every k+1 element set has $2^{(k+1)}$ subsets.
- Now let $A = \{ a_1, a_2, a_3, \ldots, a_k, a_{k+1} \}$, so that A has k+1 elements. We partition the number of subsets into 2 cases:
  - The ones that exclude a_{k+1}
  - The ones that include a_{k+1}

Example 3

- So we have 2 colection of subsets:
- Subsets without a_{k+1}, which by I.H. we have 2^n number of them

$$\{\ \ \}$$

$$\{a_1\}$$

$$\{a_1, a_2\}$$

$$\{a_1, a_2, \ldots, a_k\}$$

Example 3

- Subsets with a_{k+1} , are exactly subsets without that element that now include it so there are 2^n such subsets. Again by I.H.

$$\{a_{k+1}\}$$
$$\{a_1, a_{k+1}\}$$
$$\{a_1, a_2, a_{k+1}\}$$
$$\{a_1, a_2, \ldots, a_k, a_{k+1}\}$$

Example 3

- Since the collection of all subsets of A has been partitioned into these two sub-collections, we see that A must have $2^k + 2^k = 2(2^k) = 2^{(k+1)}$ subsets.

Q.E.D.

Citations:

Book

Mathematical Thinking: Problem-solving and Proofs

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