MΔT1100110

of course in base 2

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Tutorial 4

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Example 1

1.55. (+) **Let F be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for F.**

- It's a good idea to always do multiplication table first since we might need it to fill in the addition table
- 1st 0 times every other element is 0.

.	0	1	X
0	0	0	0
1	0		
X	0		

Example 1 cont'd

1.55. (+) **Let F be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for F.**

- 2^{nd} 1 times every element is that element since 1 is the multiplicative identity.
- $X.1 = X$
- $1.1 = 1$

.	0	1	X
0	0	0	0
1	0	1	X
X	0	X	

Example 1 cont'd

1.55. (+) Let $\mathbf{F}$ be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for $\mathbf{F}$.

- 3rd use elimination process to see what u get for X.X
- If $X.X \ = \ 0 \rightarrow X = 0 \; False$
- If $X.X \ = X \rightarrow X = 1 \; False$
- If $X.X \ = 1 \; True$

.	0	1	X
0	0	0	0
1	0	1	X
X	0	X	1

Example 1 cont'd

1.55. (+) **Let F be a field consisting of exactly three elements** $0, 1, x$. **Prove that** $x + x = 1$ **and that** $x \cdot x = 1$. **Obtain the addition and multiplication tables for F.**

- Now lets do the addition table.
- 1st since 0 is the additive identity every element plus 0 is it self.
- $0 + 0 = 0$
- $0 + 1 = 1$
- $0 + X = X$

+	0	1	X
0	0	1	X
1	1		
X	X		

Example 1 cont'd

1.55. (+) Let $\mathbf{F}$ be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for $\mathbf{F}$.

- Consider the 2nd column and use process of elimination to find $\mathbf{1} + \boldsymbol{X}$.
- If $1 + X = 1 \to \boldsymbol{X} = \mathbf{0}\ False$
- If $1 + X = X \to \mathbf{1} = \mathbf{0}\ False$
- If $1 + X = 0 \to True$

+	0	1	X
0	0	1	X
1	1		0
X	X	0	

Example 1 cont'd

1.55. (+) **Let F be a field consisting of exactly three elements** $0, 1, x$. **Prove that** $x + x = 1$ **and that** $x \cdot x = 1$. **Obtain the addition and multiplication tables for F.**

- Consider the 2nd column and use process of elimination to find $\mathbf{1+1.}$
- If $1 + 1 = 1 \rightarrow \mathbf{1 = 0}\ False$
- If $1 + 1 = 0$, we also know $1 + X = 0 \rightarrow 1 + 1 = 1 + X \rightarrow \mathbf{1 = X}\ false$
- *By process of elimination:*
- $\mathbf{1 + 1 = X}$

+	0	1	X
0	0	1	X
1	1	X	0
X	X	0	

Example 1 cont'd

1.55. (+) **Let F be a field consisting of exactly three elements $0, 1, x$. Prove that $x + x = 1$ and that $x \cdot x = 1$. Obtain the addition and multiplication tables for F.**

- Consider the 3rd column and use process of elimination to find $\boldsymbol{X + X}$.
- If $X + X = X \rightarrow \boldsymbol{X = 0}\ False$
- If $X + X = 0 \wedge X + 1 = 0 \rightarrow$ $X + X = X + 1 \rightarrow$ $X = 1\ false$
- If $X + X = 1 \rightarrow True$ by elimination process

+	0	1	X
0	0	1	X
1	1	X	0
X	X	0	1

Example 2

2.37. Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?

a) $A \Rightarrow C$.
b) $B \Rightarrow C$.
c) $(A \wedge B) \Rightarrow C$.
d) $(A \wedge B) \Rightarrow (C \vee D)$.
e) $C \Rightarrow (A \wedge B)$.
f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.
g) $(A \vee C) \Rightarrow B$.

- $a)\ A \Rightarrow B, means\ if\ \ A\ holds\ then\ B\ is\ true$.
- For such an statement to be false, A needs to hold and B needs to not be satisfied
- i.e. the condition holds but the expected result doesn't appear
- Which results in the conditional statement to be false

Example 2 cont'd

2.37. Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?

a) $A \Rightarrow C$.

b) $B \Rightarrow C$.

c) $(A \wedge B) \Rightarrow C$.

d) $(A \wedge B) \Rightarrow (C \vee D)$.

e) $C \Rightarrow (A \wedge B)$.

f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.

g) $(A \vee C) \Rightarrow B$.

- $a)\ A \Rightarrow C, means\ if\ A\ holds\ then\ C\ is\ true.$
- $here\ it\ means: if\ \frac{1}{2} < x < \frac{5}{2}, then\ x^2 = 1$
- $This\ is\ false, since\ x\ is\ a\ real\ number, pick\ x = \frac{3}{2}$
- $Then\ by\ C, x^2\ =\ 1, whilst\ x^2\ =\ \left(\frac{3}{2}\right)^2\ = \frac{9}{4}$
- $So\ False$

Example 2 cont'd

2.37. Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?

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e) $C \Rightarrow (A \wedge B)$.

f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.

g) $(A \vee C) \Rightarrow B$.

- $b)B \Rightarrow C\ means, if\ x \in \mathbb{Z}, then\ x^2 = 1$
- $False\ just\ pick\ x\ to\ be\ some\ integer\ other\ than\ 1\ and\ -1$
- $If\ x = 2,\ then\ x^2 = 4\ not\ 1, a\ contradiction$

Example 2 cont'd

2.37. Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?

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d) $(A \wedge B) \Rightarrow (C \vee D)$.

e) $C \Rightarrow (A \wedge B)$.

f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.

g) $(A \vee C) \Rightarrow B$.

- $c)(A \wedge B) \Rightarrow C.$
- $The\ condition\ is\ A \wedge B, which\ means:$
- $if\ \frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z}$
- $The\ expected\ result\ is\ C, i.e. X^2 = 1,$
- $To\ contradict\ it, condition\ must\ hold\ and\ the\ result\ doesn't.$
- $if\ \frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z}\ implies\ x \in \{1, 2\}$
- $pick\ x = 2, then\ x^2 \neq 1, a\ contradiction$

Example 2 cont'd

2.37. Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?

a) $A \Rightarrow C$.
b) $B \Rightarrow C$.
c) $(A \wedge B) \Rightarrow C$.
d) $(A \wedge B) \Rightarrow (C \vee D)$.
e) $C \Rightarrow (A \wedge B)$.
f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.
g) $(A \vee C) \Rightarrow B$.

- $d)(A \wedge B) \Rightarrow (C \vee D).$
- $The\ condition\ is\ A \wedge B, which\ means: if\ \frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z}$
- $The\ expected\ result\ is\ (C \vee D). i.e. X^2 = 1\ or\ x = 2\ .$
- $if\ \frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z}\ implies\ x \in \{1, 2\},$
- $Then\ the\ result\ holds, since\ if\ x = 1, then\ C\ holds.$
- $if\ x = 2, then\ D\ holds.$
- $Therefore\ this\ statement\ d\ is\ True$

Example 2 cont'd

2.37. **Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?**

a) $A \Rightarrow C$.
b) $B \Rightarrow C$.
c) $(A \wedge B) \Rightarrow C$.
d) $(A \wedge B) \Rightarrow (C \vee D)$.
e) $C \Rightarrow (A \wedge B)$.
f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.
g) $(A \vee C) \Rightarrow B$.

- $d) C \Rightarrow (A \wedge B)$
- $The\ condition\ is\ C, which\ means: if\ x^2 = 1, which\ means\ x \in \{-1, 1\}.$
- $The\ expected\ result\ is (A \wedge B). i.e. \frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z}.$
- $By\ the\ condition\ x\ is\ either\ -1\ or\ 1.$
- $For\ the\ expected\ result\ to\ hold\ both\ A\ and\ B\ need\ to\ hold$
- $In\ other\ words\ x\ must\ be\ an\ integer\ and\ be\ in\ (\frac{1}{2}, \frac{5}{2})$
- $If\ x = -1, then\ it\ is\ an\ integer, so\ B\ holds, but\ it\ is\ not\ in (\frac{1}{2}, \frac{5}{2}), so\ A\ doesn't$
- $Therefore\ statement\ e\ is\ false.$

Example 2 cont'd

2.37. Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?

a) $A \Rightarrow C$.
b) $B \Rightarrow C$.
c) $(A \wedge B) \Rightarrow C$.
d) $(A \wedge B) \Rightarrow (C \vee D)$.
e) $C \Rightarrow (A \wedge B)$.
f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.
g) $(A \vee C) \Rightarrow B$.

- $f) D \Rightarrow [A \wedge B \wedge (C^c)]$
- $The\ condition\ is\ D, which\ means: if x = 2.$
- $The\ expected\ result\ is [A \wedge B \wedge (C^c)].$
- $Note\ C^c\ means\ not(x^2 = 1), since\ x^2 = 1\ means\ x \in \{-1, 1\}, \rightarrow\ C^c\ means\ x \neq\ -1\ and\ x \neq 1.$
- $i.e. \frac{1}{2} < x < \frac{5}{2} \wedge x \in \mathbb{Z} \wedge (x \neq -1 \wedge x \neq 1).$
- $If\ D\ holds, means\ x = 2,$
- $Then \frac{1}{2} < 2 = \frac{4}{2} < \frac{5}{2} \wedge\ 2 \in \mathbb{Z} \wedge (2 \neq\ -1 \wedge\ 2 \neq\ 1)\ is\ a\ true\ statement.$
- $So\ the\ whole\ statement\ is\ True$

Example 2 cont'd

2.37. Given a real number x, let A be the statement "$\frac{1}{2} < x < \frac{5}{2}$", let B be the statement "$x \in \mathbb{Z}$", let C be the statement $x^2 = 1$, and let D be the statement "$x = 2$". Which statements below are true for all $x \in \mathbb{R}$?

a) $A \Rightarrow C$.
b) $B \Rightarrow C$.
c) $(A \wedge B) \Rightarrow C$.
d) $(A \wedge B) \Rightarrow (C \vee D)$.
e) $C \Rightarrow (A \wedge B)$.
f) $D \Rightarrow [A \wedge B \wedge (\neg C)]$.
g) $(A \vee C) \Rightarrow B$.

- $f)(A \vee C) \Rightarrow B.$
- $The\ condition\ is (A \vee C), which\ means:$
- $if\ \frac{1}{2} < x < \frac{5}{2} \vee x^2 = 1.\ i.e.\ x \in \left(\frac{1}{2}, \frac{5}{2}\right) \cup \{-1, 1\}$
- $x \in \left(\frac{1}{2}, \frac{5}{2}\right) \cup \{-1\}$
- $The\ expected\ result\ is\ B.\ i.e.\ x \in \mathbb{Z}.$
- $If\ we\ pick\ x = \frac{3}{2}\ the\ condition\ holds.$
- $However\ \frac{3}{2} \notin \mathbb{Z}, so\ the\ result\ is\ false.$
- *So the whole statement is false.*

Example 3

2.47. Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "$x^2 - 1$ is divisible by 8". Determine whether the following statements are true:

a) $(\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]$.

b) $(\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]$.

- (a) This statement is TRUE. To prove that, we show that for any $x \in \mathbb{Z}$, $\boldsymbol{if\ P(x)\ is\ true, then\ Q(x)\ is\ true}$.
- For a given integer x, if x is an odd number, then it has the form $x = 2k + 1$ for some integer k.
- Therefore we get that $x^2 - 1 = (2k + 1)^2 - 1$
- $= 4k^2 + 4k + 1 - 1 = 4k(k + 1)$
- One of the numbers $k, k + 1$must be even, which means that $k(k + 1)$ is an even number.
- When we multiply by 4 we conclude that $x^2 - 1 = 4k(k + 1)$ must be divisible by 8, as needed.

Example 3 cont'd

2.47. **Let $P(x)$ be the assertion "x is odd", and let $Q(x)$ be the assertion "$x^2 - 1$ is divisible by 8". Determine whether the following statements are true:**

a) $(\forall x \in \mathbb{Z})[P(x) \Rightarrow Q(x)]$.

b) $(\forall x \in \mathbb{Z})[Q(x) \Rightarrow P(x)]$.

- This statement is TRUE. To prove that:
- we show that for any $x \in \mathbb{Z}, if\ Q(x)\ is\ true, then\ P(x)\ is\ true$.
- For any integer x, if $x^2 - 1$ is divisible by 8, then in particular $x^2 - 1$ is even,
- Which implies that x^2 is odd.
- But an integer is odd if and only if its square is odd, and hence x must be an odd number, as needed.

Example 4

$$Let\ F = \{0, 1, a, b\}\ be\ a\ field\ with\ four\ elements.$$
$$Show\ that\ b^3 = 1.$$

- We need to always do multiplication table first since we need it to fill in the addition table
- 1st 0 times every other element is 0.

.	0	1	a	b
0	0	0	0	0
1	0			
a	0			
b	0			

Example 4 cont'd

$$Let\ F = \{0, 1, a, b\}\ be\ a\ field\ with\ four\ elements.$$
$$Show\ that\ b^3 = 1.$$

- 2^{nd} every element times 1 is itself, since 1 is the multiplicative identity.

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a		
b	0	b		

Example 4 cont'd

$$Let\ F = \{0, 1, a, b\}\ be\ a\ field\ with\ four\ elements.$$
$$Show\ that\ b^3 = 1.$$

- 3rd
- Consider the 3rd column and use process of elimination to find $\boldsymbol{ab}$.
- If $a.b = 0 \rightarrow \boldsymbol{a} = \mathbf{0}\ or\ \boldsymbol{b} = \mathbf{0}$
- $False$
- If $a.b = a \rightarrow \boldsymbol{b} = \mathbf{1}$
- $False$
- If $a.b = b \rightarrow \boldsymbol{a} = \mathbf{1}$
- $False$
- **So by process of elimination:**
- $\boldsymbol{a}.\boldsymbol{b} = \mathbf{1} = \mathbf{b}.\mathbf{a}$

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a		1
b	0	b	1	

Example 4 cont'd

$$Let\ F = \{0, 1, a, b\ \}\ be\ a\ field\ with\ four\ elements.$$
$$Show\ that\ b^3\ =\ 1.$$

- 4th
- Consider the 3rd column and use process of elimination to find $\boldsymbol{a.a.}$
- If $a.a = 0 \rightarrow \boldsymbol{a = 0}$
- $False$
- If $a.a = a \rightarrow \boldsymbol{a = 1}$
- $False$
- If $a.a = 1\ since\ ab = 1 \rightarrow \boldsymbol{a = b}$
- $False$
- **So by process of elimination:**
- $\boldsymbol{a.a = b}$

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	

Example 4 cont'd

$$Let\ F = \{0, 1, a, b\}\ be\ a\ field\ with\ four\ elements.$$
$$Show\ that\ b^3 = 1.$$

- 5^{th}
- Consider the 4^{th} column and use process of elimination to find $\boldsymbol{b.b}$.
- If $b.b = 0 \rightarrow \boldsymbol{b = 0}$
- $False$
- If $b.b = b \rightarrow \boldsymbol{b = 1}$
- $False$
- If $b.b = 1\ since\ ab = 1 \rightarrow \boldsymbol{a = b}$
- $False$
- **So by process of elimination:**
- $\boldsymbol{b.b = a}$

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Example 4 cont'd

$$Let\ F = \{0, 1, a, b\}\ be\ a\ field\ with\ four\ elements.$$
$$Show\ that\ b^3 = 1.$$

- Now lets calculate b^3.
- $b^3 = b.b.b$
- $= (b.b).b$ $\boldsymbol{b.b = a}$
- $= a.b$
- $= 1$

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Citations:

Book

Mathematical Thinking: Problem-solving and Proofs

D'Angelo, J.P.

West, D.B.

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