MΔT1100110

of course in base 2

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Tutorial 2

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Example 0.1

Prove that for any $a, b, c > 0$ we have that:

$(a + b)(b + c)(a + c) \geq 8abc.$

Hint: Use $t + \frac{1}{t} \geq 2 \; for \; any \; t > 0.$

- $t + \frac{1}{t} \geq 2 \; for \; any \; t > 0. \Rightarrow multiply \; by \; t$
- $t^2 + 1 \geq 2t \; now \; try \; find/make \; this \; in \; the \; inequality.$
- $want \; to \; prove$: $(a + b)(b + c)(a + c) \geq 8abc.$
- $since \; a, b, c > 0, devide \; both \; sides \; by \; abc.$
- $\Rightarrow \frac{(a+b)(b+c)(a+c)}{abc} \geq \frac{8abc}{abc}.$
- $\Rightarrow \frac{a+b}{a} . \frac{b+c}{b} . \frac{a+c}{c} \geq 8$

Example 0.1

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$(a + b)(b + c)(a + c) \geq 8abc.$
Hint: Use $t + \frac{1}{t} \geq 2 \; for \; any \; t > 0.$

- $\Rightarrow \frac{a+b}{a} . \frac{b+c}{b} . \frac{a+c}{c} \geq 8$
- $(1 + \frac{b}{a}).(1 + \frac{c}{b}).\left(1 + \frac{a}{c}\right) \geq 8$
- $t^2 + 1 \geq 2t$
- Now let:
- $t_1^2 = \frac{b}{a} \Rightarrow t_1 = \sqrt{\frac{b}{a}}, t_2^2 = \frac{c}{b} \Rightarrow t_2 = \sqrt{\frac{c}{b}}, t_3^2 = \frac{a}{c} \Rightarrow t_3 = \sqrt{\frac{a}{c}}$

Example 1

1.31. (+) ***Extensions of the AGM Inequality.***

a) Prove that $4xyzw \le x^4 + y^4 + z^4 + w^4$ for real numbers x, y, z, w. (Hint: Use the inequality $2tu \le t^2 + u^2$ repeatedly.)

- Note : to prove $4xyzw \le x^4 + y^4 + z^4 + w^4$, it is enough to start from one side and end in the other. Here we want to :
- $start\ from\ x^4 + y^4 + z^4 + w^4 \ge \cdots \ge \cdots \ge 4xyzw$ end here
- Recall that $\frac{a+b}{2} \ge \sqrt{ab} \to a + b \ge 2\sqrt{ab}$ where $a, b > 0$.
- Let's set $a = x^4 + y^4,\ b = z^4 + w^4$
- 1)$\boldsymbol{\left(x^4 + y^4\right) + \left(z^4 + w^4\right) \ge 2\sqrt{(x^4 + y^4)(z^4 + w^4)}}$
- On the other hand Let's set $a = x^4, b = y^4$
- 2) $\boldsymbol{x^4 + y^4 \ge 2x^2y^2}$
- On the other hand Let's set $a = z^4, b = w^4$
- 3) $\boldsymbol{z^4 + w^4 \ge 2z^2w^2}$

Example 1 cont'd

1.31. (+) ***Extensions of the AGM Inequality.***

a) Prove that $4xyzw \le x^4 + y^4 + z^4 + w^4$ for real numbers x, y, z, w. (Hint: Use the inequality $2tu \le t^2 + u^2$ repeatedly.)

- *start from* $x^4 + y^4 + z^4 + w^4 \ge \cdots \ge \cdots \ge 4xyzw$ end here
- Recall that $\frac{a+b}{2} \ge \sqrt{ab} \to a + b \ge 2\sqrt{ab}$ where $a, b > 0$.

Make this part out of inequality number 2 and 3

- 1) $(x^4 + y^4) + (z^4 + w^4) \ge 2\sqrt{(x^4 + y^4)(z^4 + w^4)}$
- 2) $x^4 + y^4 \ge 2x^2y^2$ Since both sides are positive we can square root both sides
- 4) $\sqrt{x^4 + y^4} \ge \sqrt{2x^2y^2}$
- 3) $z^4 + w^4 \ge 2z^2w^2$ Since both sides are positive we can square root both sides
- 5) $\sqrt{z^4 + w^4} \ge \sqrt{2z^2w^2}$

Example 1 cont'd

1.31. (+) ***Extensions of the AGM Inequality.***

a) Prove that $4xyzw \le x^4 + y^4 + z^4 + w^4$ for real numbers x, y, z, w. (Hint: Use the inequality $2tu \le t^2 + u^2$ repeatedly.)

- $start\ from\ x^4 + y^4 + z^4 + w^4 \ge \cdots \ge \cdots \ge 4xyzw$ end here

Make this part out of inequality number 2 and 3

- $1)(x^4 + y^4) + (z^4 + w^4) \ge \underbrace{2\sqrt{(x^4 + y^4)(z^4 + w^4)}}$
- $4)\ \sqrt{x^4 + y^4} \ge \sqrt{2x^2y^2}$
- $5)\ \sqrt{z^4 + w^4} \ge \sqrt{2z^2w^2}$

$\boldsymbol{if\ a > b > 0\ and\ c > d > 0 \Rightarrow ac > bd > 0}$

- $6)\sqrt{x^4 + y^4}\sqrt{z^4 + w^4} \ge \sqrt{2x^2y^2}\ \sqrt{2z^2w^2}$
- $6)\sqrt{(x^4+y^4)(z^4 + w^4)} \ge \sqrt{4x^2y^2z^2w^2} = 2|x||y||z||w|$

$\boldsymbol{|x||y| \ge xy}$

- $7)\sqrt{(x^4+y^4)(z^4 + w^4)} \ge 2|x||y||z||w| \ge 2xyzw$
- $7)\sqrt{(x^4+y^4)(z^4 + w^4)} \ge 2xyzw$

Example 1 cont'd

1.31. (+) ***Extensions of the AGM Inequality.***

a) Prove that $4xyzw \le x^4 + y^4 + z^4 + w^4$ for real numbers x, y, z, w. (Hint: Use the inequality $2tu \le t^2 + u^2$ repeatedly.)

- $start\ from\ x^4 + y^4 + z^4 + w^4 \ge \cdots \ge \cdots \ge 4xyzw$ end here
- $1)(x^4 + y^4) + (z^4 + w^4) \ge 2\sqrt{(x^4 + y^4)(z^4 + w^4)}$
- $7)\sqrt{(x^4+y^4)(z^4 + w^4)} \ge 2xyzw$ ***just multiply both sides by 2***
- $7)2\sqrt{(x^4+y^4)(z^4 + w^4)} \ge 4xyzw$
- $by\ inequality\ 1\ and\ 7:$
- $(x^4 + y^4) + (z^4 + w^4) \ge 2\sqrt{(x^4 + y^4)(z^4 + w^4)} \ge 4xyzw$
- $(x^4 + y^4) + (z^4 + w^4) \ge 4xyzw\ \boldsymbol{Q.E.D.}$

Example 2

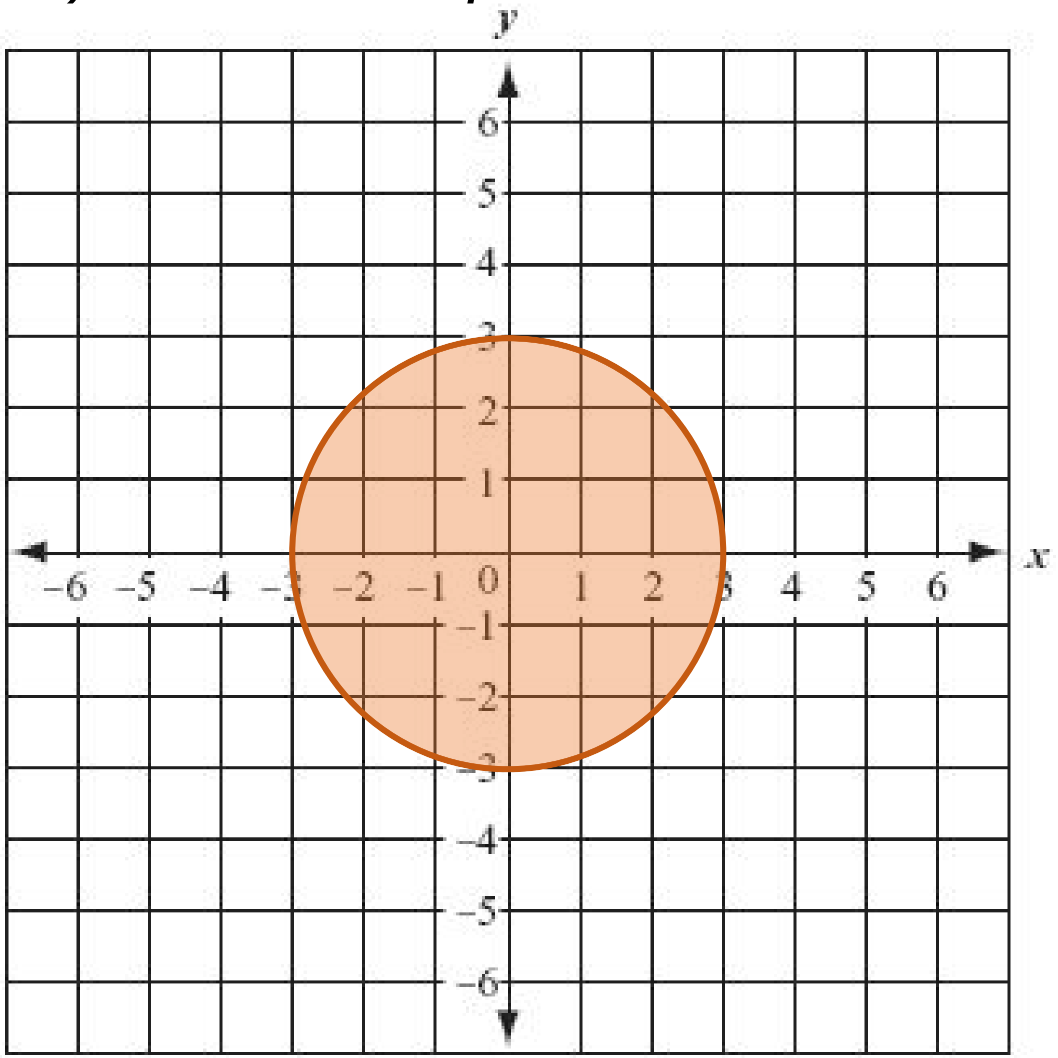
1. (a) Find all real solutions of the equation $x^2 + 2x + 4 = 0$ if there are any.

- The discriminant $\Delta = b^2 - 4ac \; where \; b = 2, a = 1, c = 4$.
- $\Delta = 4 \; - 4(1)(4) = 4 - 16 = -12$
- $Note \; when$:
- $\Delta < 0 \Rightarrow no \; real \; root$
- $\Delta = 0 \Rightarrow exactly \; one \; real \; root$
- $\Delta > 0 \Rightarrow 2 \; distinct \; real \; roots$
- $Since \; \Delta < 0 \; there \; exist \; no \; real \; root \; for \; the \; equation$.

Example 3

$Let\ S = \{(x,y): x^2 + y^2 \leq 9, (x,y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$
$and\ F = \{(x,y): x, y \geq 0\}. Graph\ (\boldsymbol{T} - \boldsymbol{S}) \cap \boldsymbol{F}\ on\ the\ plane.$

- $Let's\ consider\ S\ first$:
- $x^2 + y^2 \leq 9\ is\ the\ area\ within\ a\ circle\ with\ radius\ 3\ centered\ at\ the\ origin.$
- $and\ since\ it\ includes\ x^2 + y^2 = 9, the\ border\ of\ the\ circle\ is\ also\ included (solid).$



Example 3 cont'd

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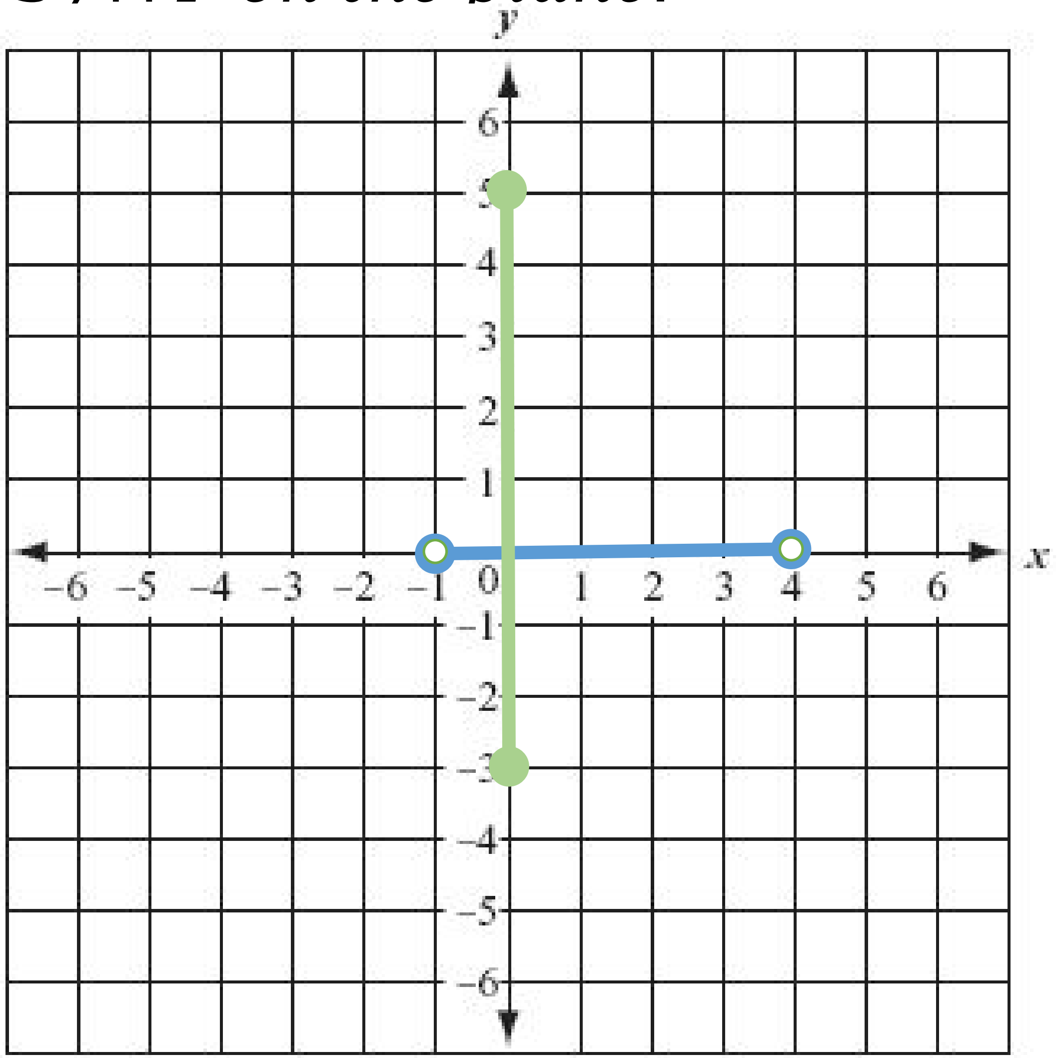
- $Now\ let's\ consider\ T$:
- $T = (-1,4) \times [-3,5],$
- Note: in a Cartesian product: the first interval or set of elements belong to the domain(i.e. x axis), and the interval after X symbol, belong to codomain(i.e. y axis).
- Therefore T is the cross product of the intervals shown on the graph.



Example 3 cont'd

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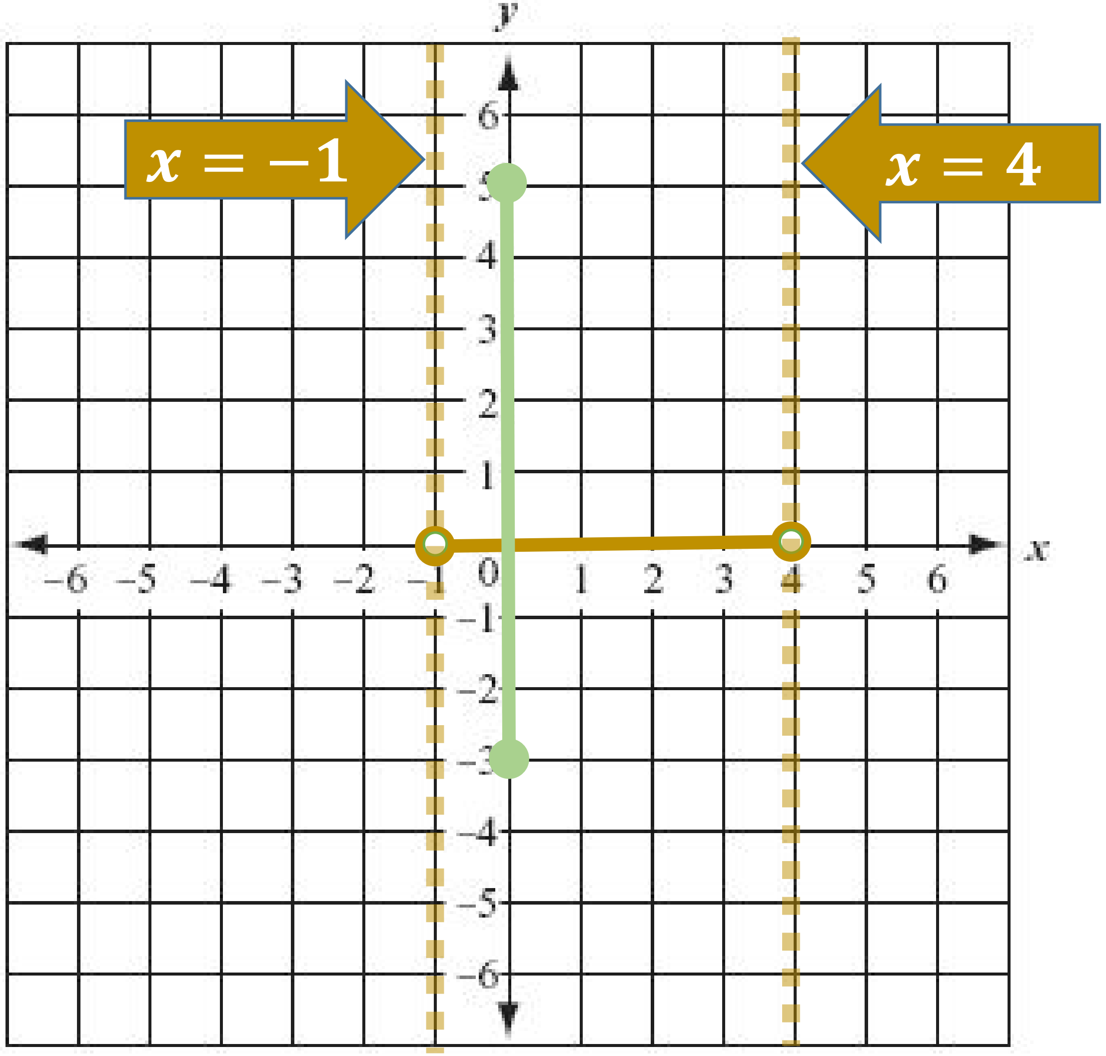
$and\ F = \{(x, y): x, y \geq 0\}. Graph\ (\boldsymbol{T} - \boldsymbol{S}) \cap \boldsymbol{F}\ on\ the\ plane.$

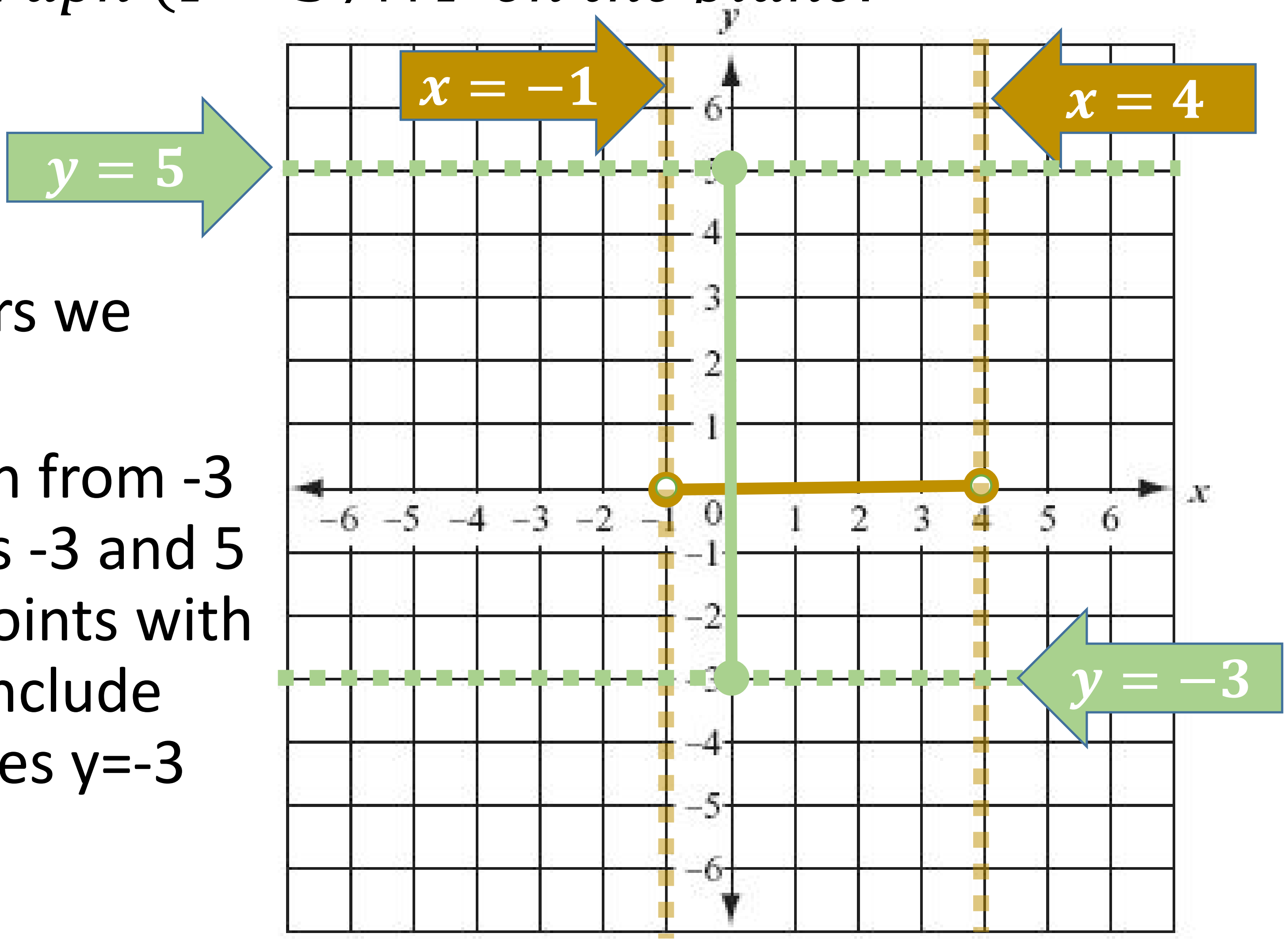
- $Now\ let's\ consider\ T$:
- $T = (-1,4) \times [-3,5],$
- Let's now see what (x,y) pairs we need to include.
- Since the domain does contain from -1 to 4 but excluding -1 and 4, all the point with x=-1, and x=4 are excluded. i.e. all the points that lie on those lines are excluded.



Example 3 cont'd

$Let\ S = \{(x,y): x^2 + y^2 \leq 9, (x,y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$
$and\ F = \{(x,y): x,y \geq 0\}. Graph\ (\boldsymbol{T} - \boldsymbol{S}) \cap \boldsymbol{F}\ on\ the\ plane.$

- $Now\ let's\ consider\ T$:
- $T = (-1,4) \times [-3,5]$,
- Let's now see what (x,y) pairs we need to include.
- Since the range does contain from -3 to 5, and includes the points -3 and 5 as well, we include all the points with ys in that interval and also include the points that lie on the lines y=-3 and y=5



Example 3 cont'd

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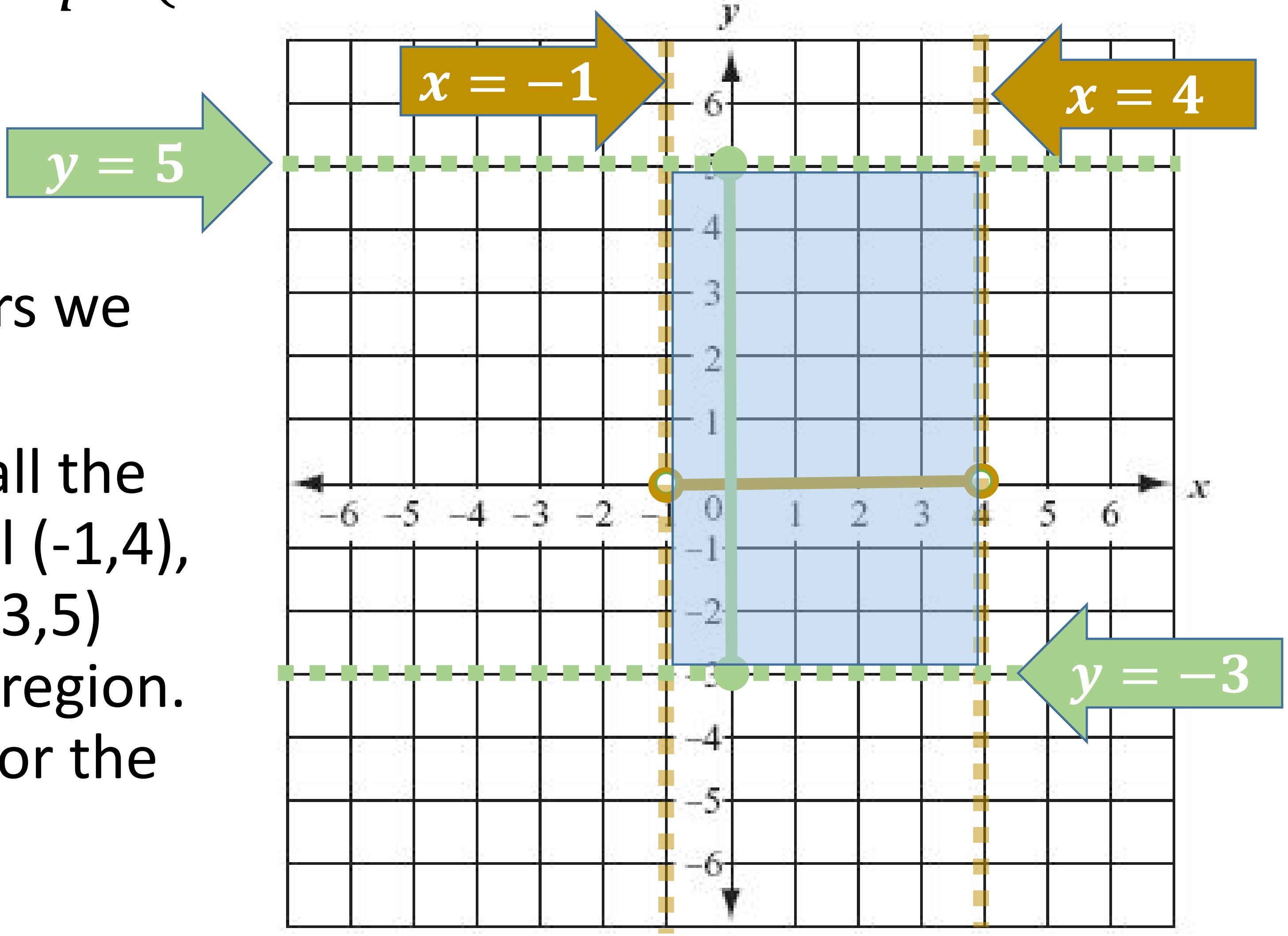
- $Now\ let's\ consider\ T$:
- $T = (-1,4) \times [-3,5],$
- Let's now see what (x,y) pairs we need to include.
- So in summary, we include all the points with Xs in the interval (-1,4), and with Ys in the interval (-3,5) therefore, the square is the region. But we still need to decide for the borders and the sides.



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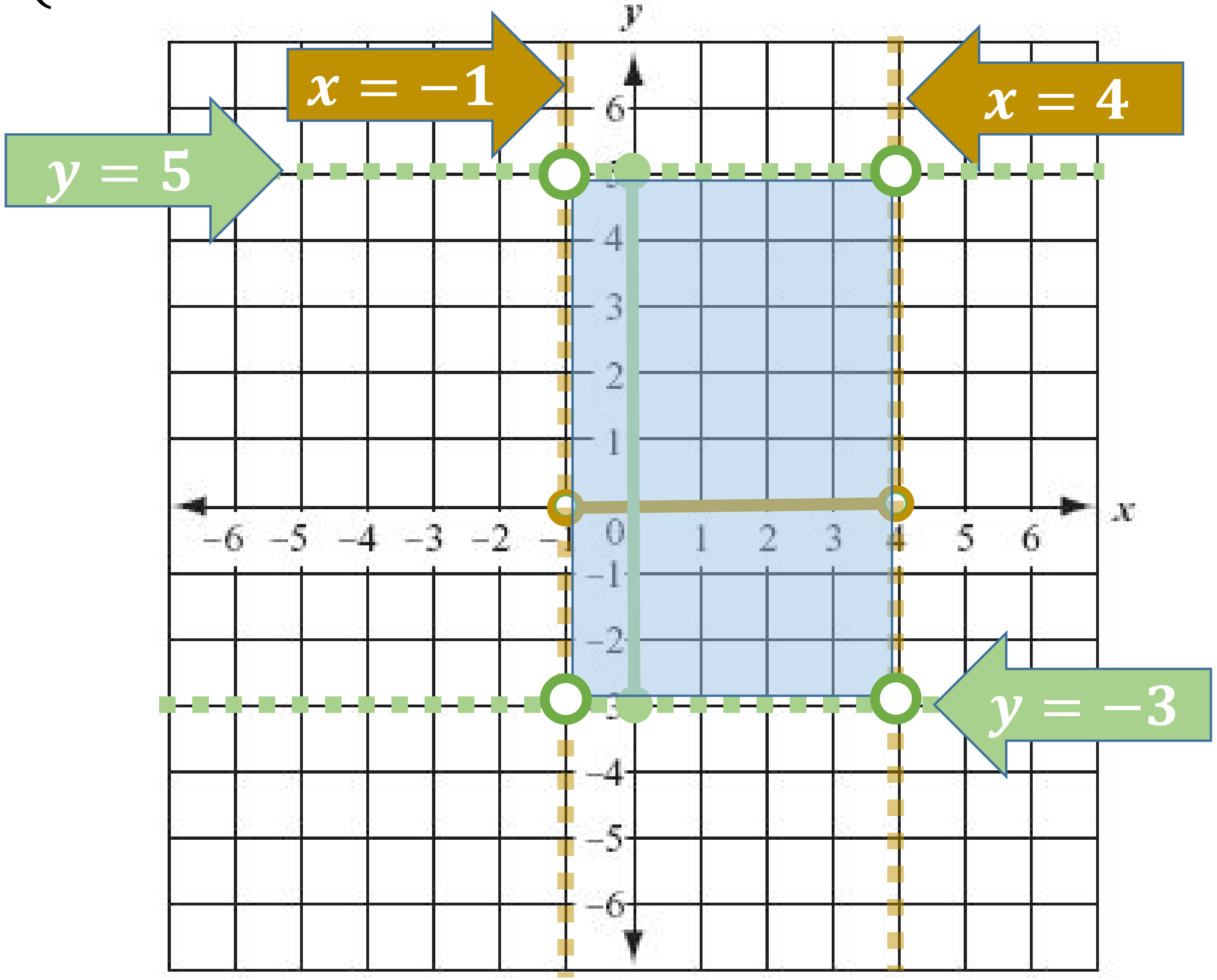
$$Let\ S = \{(x,y): x^2 + y^2 \leq 9, (x,y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$$
$$and\ F = \{(x,y): x, y \geq 0\}. Graph\ (\boldsymbol{T} - \boldsymbol{S}) \cap \boldsymbol{F}\ on\ the\ plane.$$

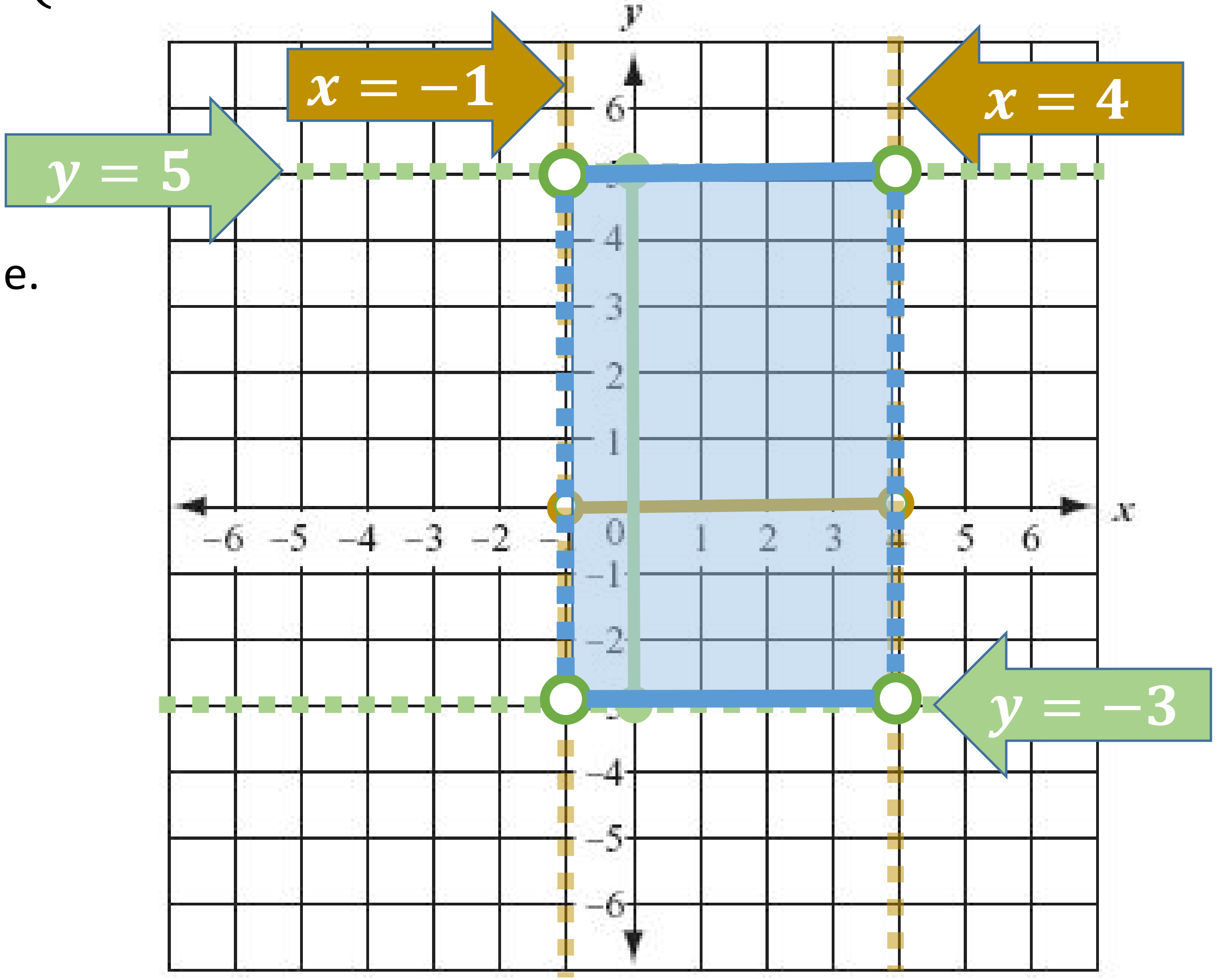
- $Now\ let's\ consider\ T$:
- $T = (-1,4) \times [-3,5]$,
- Let's now see what (x,y) pairs we need to include.
- First lets consider borders.
- (-1, 5), (-1, -3), (4, 5), (4, -3)
- Should we include (-1, 5)?
- No since x=-1 is not included
- Should we include (-1, -3)?
- No since x=-1 is not included
- Should we include (4, 5)?
- No since x=4 is not included
- Should we include (4, -3)?
- No since x=4 is not included
- As the result all those points are excluded



Example 3 cont'd

$$Let\ S = \{(x, y): x^2 + y^2 \leq 9, (x, y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$$
$$and\ F = \{(x, y): x, y \geq 0\}. Graph\ (\boldsymbol{T} - \boldsymbol{S}) \cap \boldsymbol{F}\ on\ the\ plane.$$

- $Now\ let's\ consider\ T$:
- $T = (-1,4) \times [-3,5]$,
- Let's now see what (x,y) pairs we need to include.
- At last let's consider which sides we need to include.
- We have the following sides:
- X=-1, X=4, Y=-3, Y=5
- Should we include X=-1?
- No since the domain interval is open.
- Should we include X=4?
- No since the domain Interval is open.
- Should we include Y=-3?
- Yes since the range interval is closed.
- Should we include Y=5?
- Yes since the range interval is closed.



Example 3 cont’d

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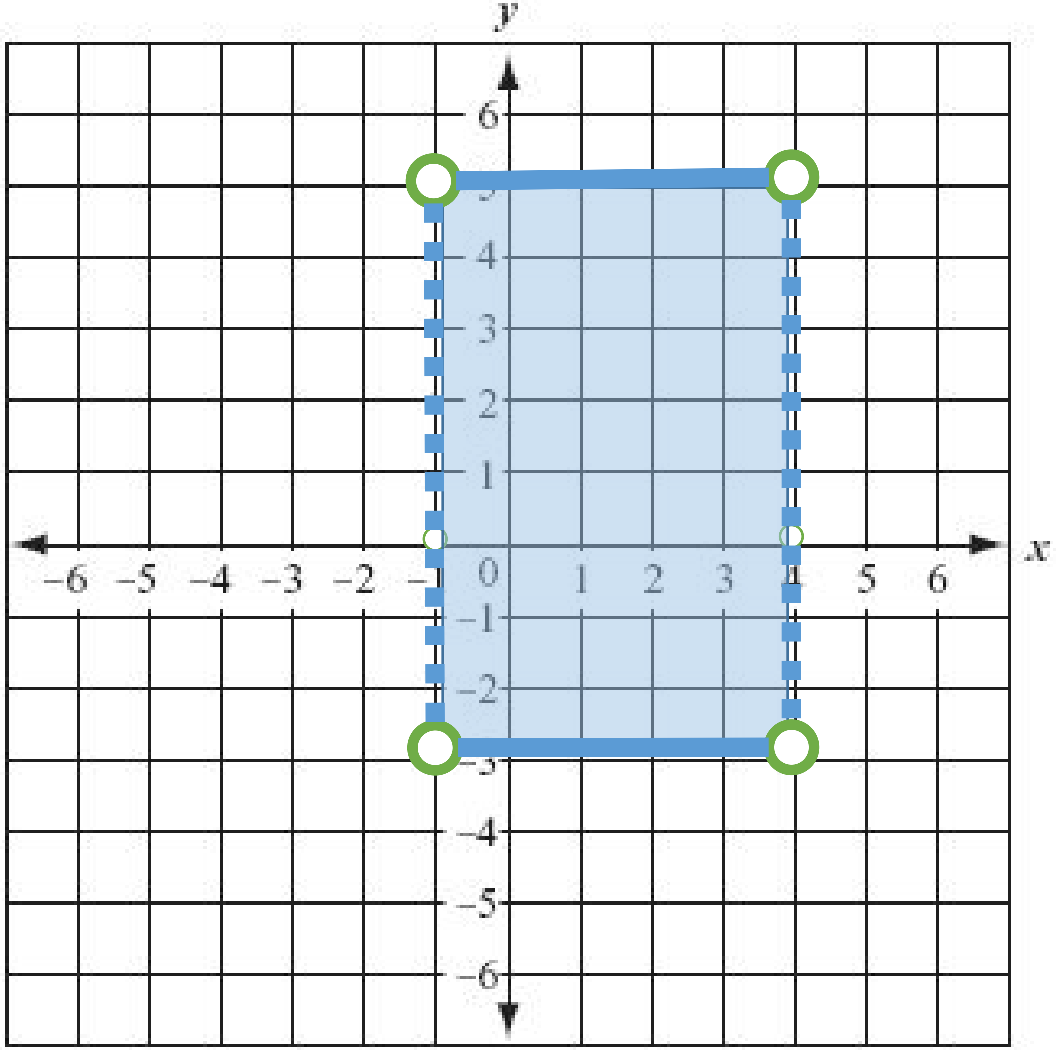
- $Now\ let's\ consider\ T$:
- $T = (-1,4) \times [-3,5]$,
- So the region for T is as follows.



Example 3 cont’d

$Let\ S = \{(x, y): x^2 + y^2 \leq 9, (x, y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$
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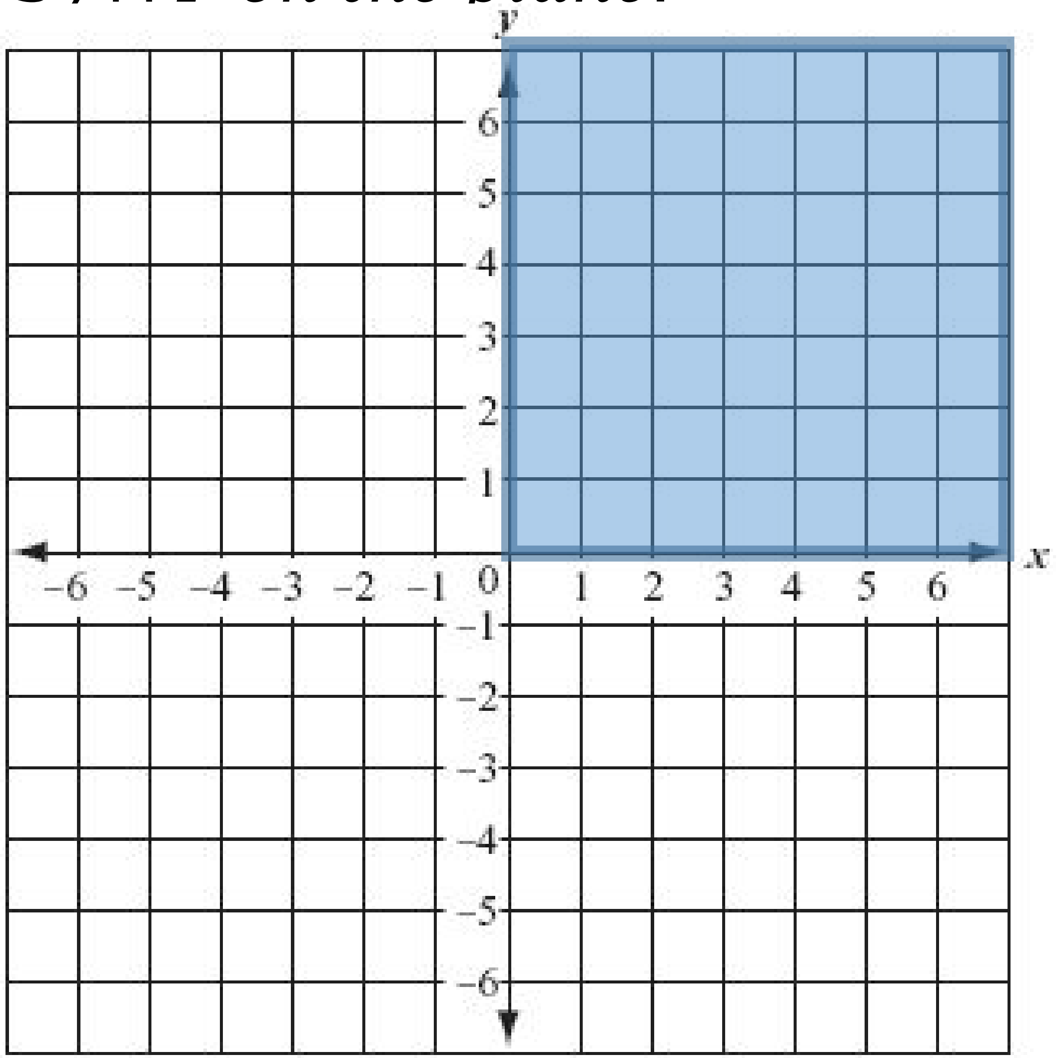
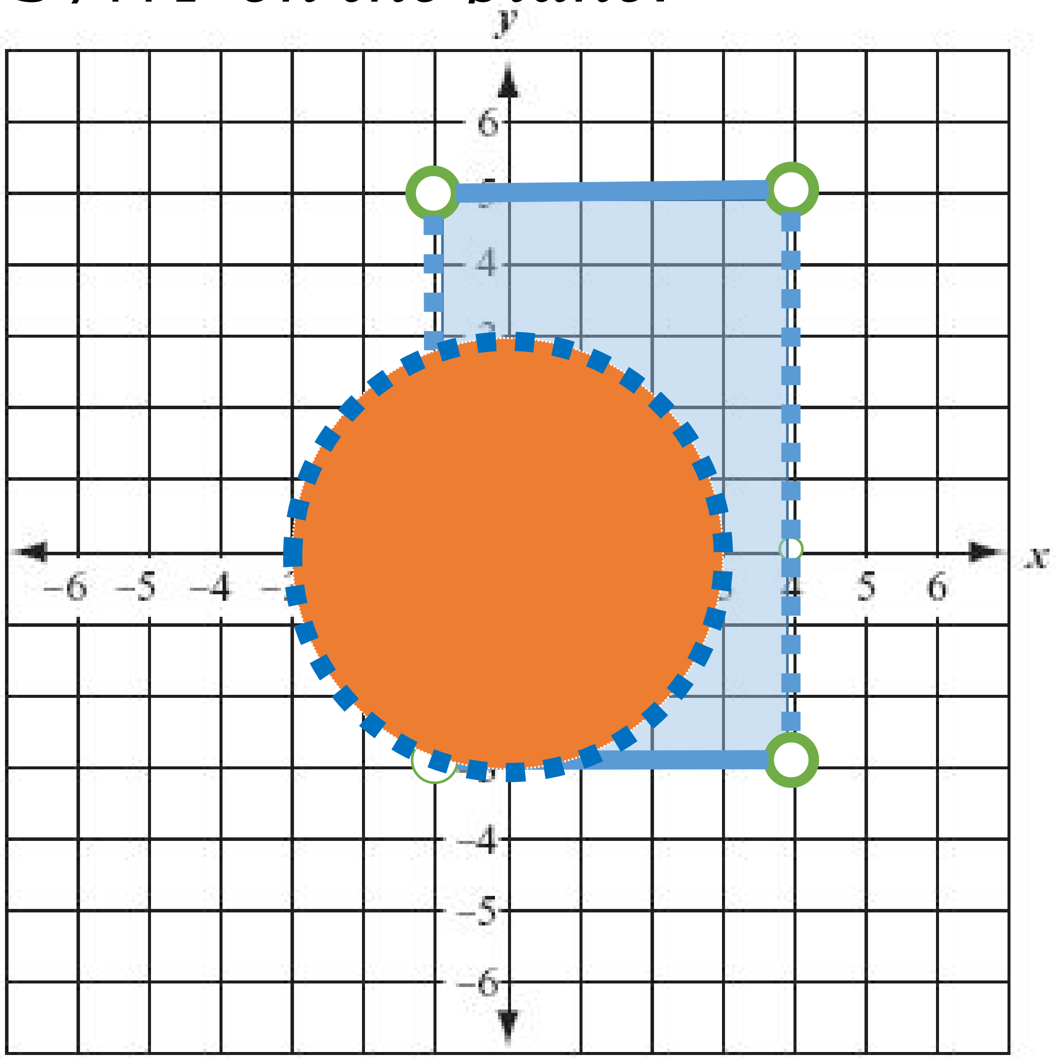
- $Now\ let's\ consider\ F$:
- $F = \{(x, y): x, y \geq 0\}$.
- It is basically all the points with nonnegative Xs and nonnegative Ys



Example 3 cont'd

$Let\ S = \{(x, y): x^2 + y^2 \leq 9, (x, y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$
$and\ F = \{(x, y): x, y \geq 0\}. Graph\ (\boldsymbol{T} - \boldsymbol{S}) \cap \boldsymbol{F}\ on\ the\ plane.$

- $Now\ let's\ consider\ T - S$:
- It means all the points that are within T but are not in S, i.e. if a point is within or on the circle with R=3 it is excluded.



Example 3 cont’d

$$Let\ S = \{(x,y): x^2 + y^2 \leq 9, (x,y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$$
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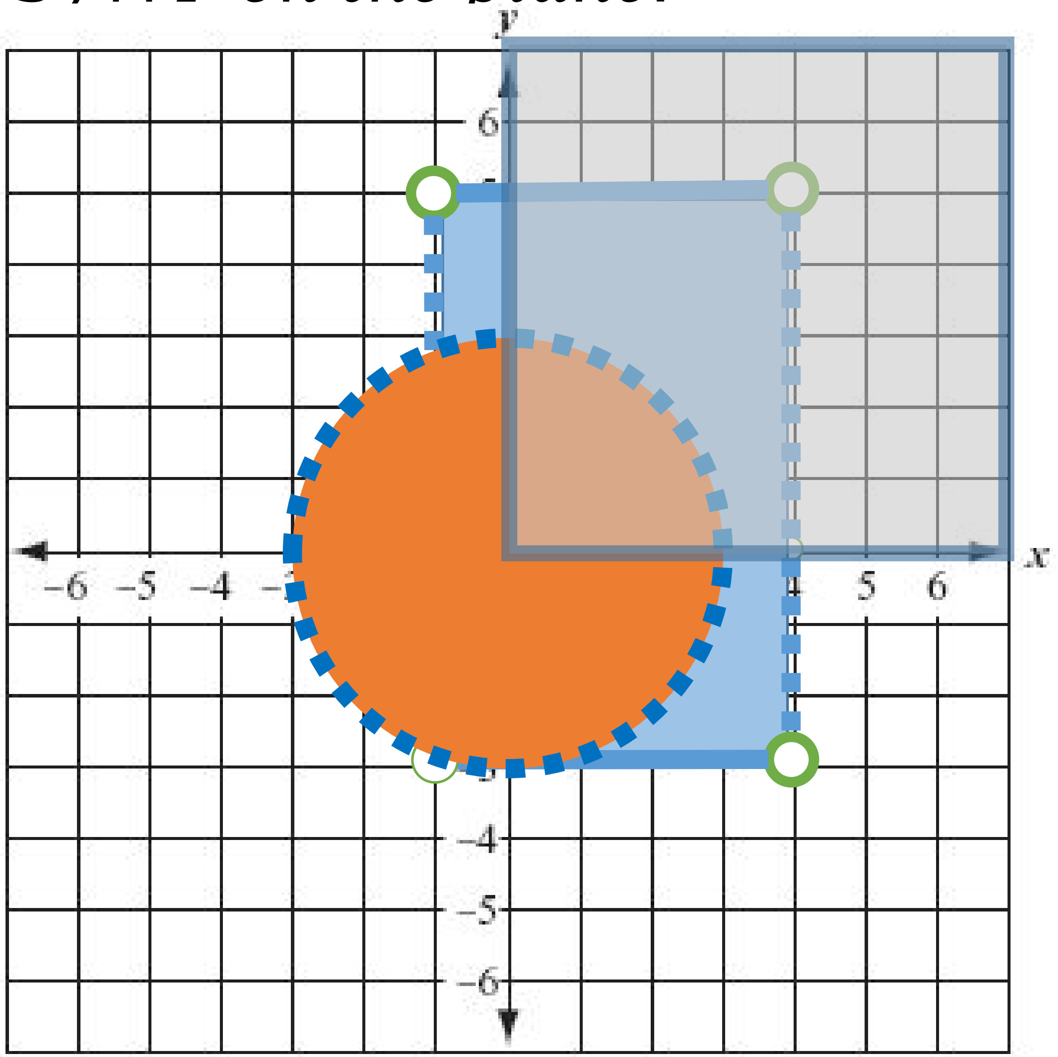
- $Now\ let's\ consider\ T - S$:
- It means all the points that are within T but are not in S, i.e. if a point is within or on the circle with R=3 it is excluded.
- $Now\ let's\ consider\ (T - S) \cap F$:
- Which basically means keep the points with nonnegative X and Y, from T-S



Example 3 cont'd

$$Let\ S = \{(x, y): x^2 + y^2 \leq 9, (x, y) \subseteq \mathbb{R}^2\}, T = (-1,4) \times [-3,5],$$
$$and\ F = \{(x, y): x, y \geq 0\}. Graph\ (\boldsymbol{T} - \boldsymbol{S}) \cap \boldsymbol{F}\ on\ the\ plane.$$

- $Now\ let's\ consider\ (T - S) \cap F$:
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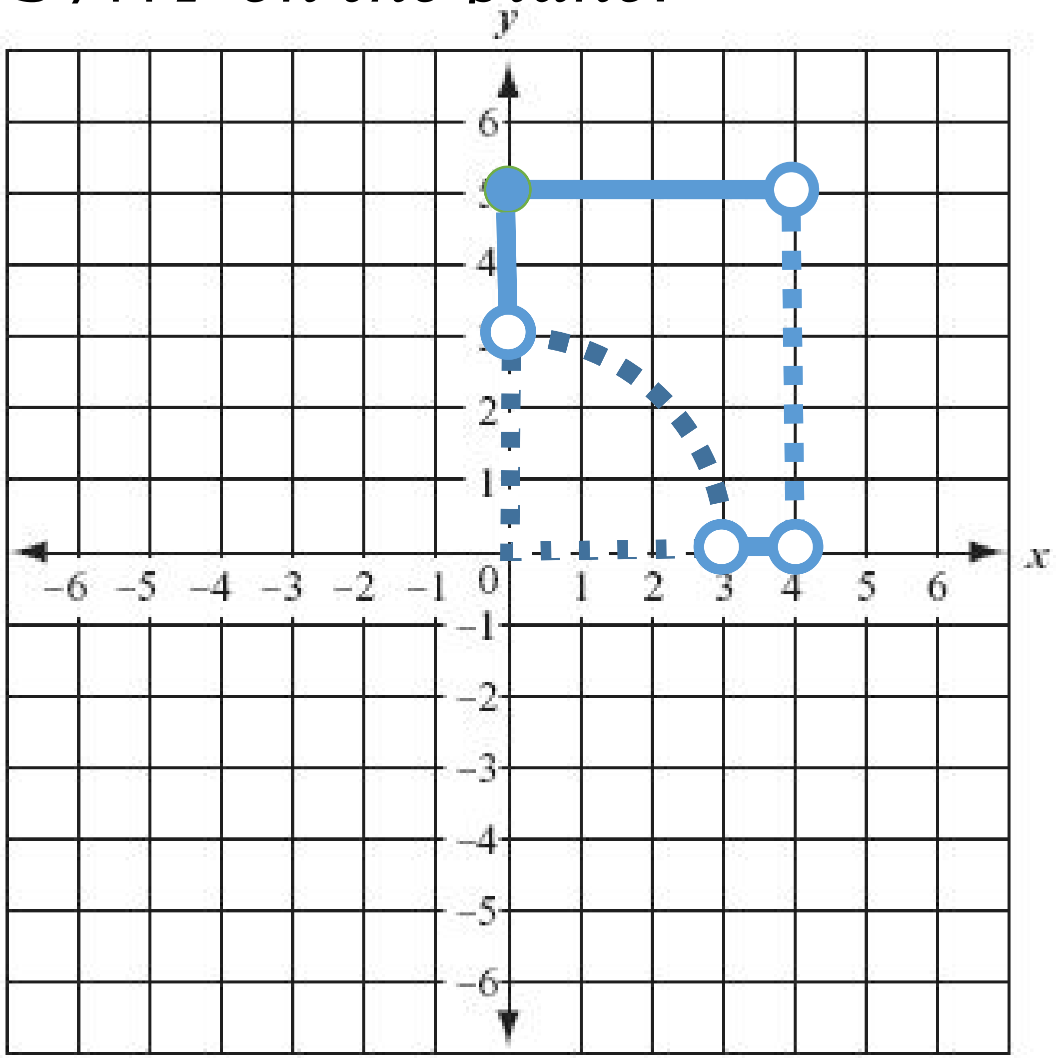

Example 4

Anna was seven cents short of the price of an alphabet book, and Alex was one cent short. They combined their money to buy one book to share, but even then they didn't have enough. How much did the book cost?

- $let\ x\ be\ the\ price\ of\ the\ book$
- $Anna\ has\ x - 7\ cents, Alex\ has\ x - 1\ cents$
- $Even\ when\ they\ combine\ their\ money, they\ cannot\ buy\ the\ book, so$
- $(x - 7) + (x - 1) < x \Rightarrow 2x - 8 < x \Rightarrow x - 8 < 0$
- $therefore\ x < 8(1),$
- $since\ Anna\ cannot\ have\ a\ negative\ amount\ of\ money\ x - 7 \geq 0$
- $i.e. x \geq 7(2)$
- $so\ x \geq 7\ and\ x < 8 \rightarrow 7 \leq x < 8, therefore\ since\ it\ needs\ to\ be\ an\ integer\ x = 7$

Example 5

For each statement, decide whether it is TRUE or FALSE. Justify your answer briefly.

- 1) $\{(x, y) : x, y \in \mathbb{R} \; and \; x - 2 = 0\} = \{(x, y): x, y \in \mathbb{R} \; and \; \frac{x-2}{x^2} = 0\}$
- **True** since both equations have the same solution set:
- $x - 2 = 0 \Leftrightarrow x = 2$
- $\frac{x-2}{x^2} = 0 \Leftrightarrow x = 2$
- 2) $\{x \in \mathbb{R}: x^2 + 2x = 1\} \subseteq \mathbb{R} - \mathbb{Q}\}$
- $solving \; for \; the \; equation: \delta = b^2 - 4ac \Rightarrow \delta = 2^2 - 4(1)(-1) = 8$
- $x_1, x_2 = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$ Note both solutions are irrational numbers
- $so \; is \; \{-1 + \sqrt{2}, -1 - \sqrt{2}\} \subseteq \mathbb{R} - \mathbb{Q}$ True?
- **True,** since $\mathbb{R} - \mathbb{Q}$ means: real numbers – the rationals = irrationals
- And the 2 solutions are included in the set of irrationals since both are irrational numbers

Example 5 cont'd

For each statement, decide whether it is TRUE or FALSE. Justify your answer briefly.

- 3)$\mathbb{R} \in \mathbb{R}$ *is this true*?
- **False** , $\mathbb{R}$ *is* ***NOT*** *an element of* $\mathbb{R}$, *however*, $\mathbb{R} \subseteq \mathbb{R}$
- 4) $\mathbb{Z} \times \mathbb{Q} \subseteq \mathbb{Q} \times \mathbb{Z}$
- **False** , first of all this means $(x_1 \in \mathbb{Z}, y_1 \in \mathbb{Q}) \subseteq (\mathbb{Q},\ \mathbb{Z})$
- For this to happen it is required that;
- x_1 which is an integer, needs to be included in $\mathbb{Q}$, which is True, satisfied.
- y_2 which is a rational, needs to be included in $\mathbb{Z}$, which is not always true, not satisfied.(i.e. $\mathbb{Z} \subseteq \mathbb{Q}$ *but* $\mathbb{Q} \nsubseteq \mathbb{Z}$)

Citations:

Book

Mathematical Thinking: Problem-solving and Proofs

D'Angelo, J.P.

West, D.B.

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