MAT102
Welcome

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Tutorial 1
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Example 0.1

1.2. (–) Fill in the blanks. The equation $x^2 + bx + c = 0$ has exactly one solution when ________________, and it has no solutions when ________________.

- Note:
- We know that for a quadratic equation of form $ax^2 + bx + c = 0$
  - The discriminant is Δ = $b^2 - 4ac$, where,
    - Δ<0 means no real solution,
    - Δ = 0 means exactly one real solution,
    - and Δ>0 means exactly 2 distinct real solutions exist for the corresponding quadratic equation.
- The solutions to the equation are x_1, x_2 = $\frac{-b \pm \sqrt{\Delta}}{2a}$

Example 0.2

1.7. (–) The statement below is not always true for $x, y \in \mathbb{R}$. Give an example where it is false, and add a hypothesis on y that makes it a true statement.

"If x and y are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."

- $initialy, if\ x > 0 > y,$
- $then\ -\frac{1}{x} < 0,$
- $whilst\ -\frac{1}{y} > 0$
- $therefore\ -\frac{1}{x} > \frac{1}{y}\ is\ \boldsymbol{false}$

Example 0.2 cont'd

1.7. **(−) The statement below is not always true for $x, y \in \mathbb{R}$. Give an example where it is false, and add a hypothesis on y that makes it a true statement.**

"If x and y are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."

- $how\ to\ make\ it\ always\ true?$
- $the\ problem\ happened\ since\ with\ only\ x > y\ is?$
- $y\ can\ be\ negative\ while\ x\ is\ positive\ which\ results\ in:$
- $-\frac{1}{y} > 0\ and\ \ -\frac{1}{x} < 0 \rightarrow -\frac{1}{x} \ngtr -\frac{1}{y}$
- $to\ stop\ this\ from\ happening, don't\ let\ y\ change\ sign:$
- $so\ x > y > 0\ works,$
- $but\ note: 0 > x > y\ does\ not\ try\ it\ ur\ self. let\ x = -2, y = -3$

Example 0.3

1.11. (−) A store offers a 15% promotional discount for its grand opening. The clerk believes that the law requires the discount to be applied first and then the tax computed on the resulting amount. A customer argues that the discount should be applied to the total after the 5% sales tax is added, expecting to save more money that way. Does it matter? Explain.

- $does\ it\ matter?$
- NO

1.11. *Promotional discount.* When a 15% discount is applied to an amount x, the actual cost is $.85x$. When 5% tax is computed on an amount y, the tax is $.05y$, and the paid total is $1.05y$. If the price of the item is z, then applying the discount before the tax yields a total cost of $1.05(.85z)$. Applying the tax first yields a total cost of $.85(1.05z)$. By the commutativity of multiplication, these are equal.

Example 0.4

1.20. **Suppose that r and s are distinct real solutions of the equation $ax^2+bx+c=0$. In terms of a, b, c, obtain formulas for $r+s$ and rs.**

- $since\ \delta > 0, \qquad r, s = \frac{-b \pm \sqrt{\delta}}{2a}$
- $r.s = \frac{-b-\sqrt{\delta}}{2a} \cdot \frac{-b+\sqrt{\delta}}{2a}$ $\quad (x-y).(x+y) = x^2 - y^2$
- $r.s = \frac{b^2-\delta}{4a^2}$ $\quad \delta = b^2 - 4ac$
- $r.s = \frac{b^2-b^2+4ac}{4a^2}$
- $r.s = \frac{4ac}{4a^2} = \frac{c}{a}$

Example 0.4

1.20. Suppose that r and s are distinct real solutions of the equation $ax^2+bx+c=0$. In terms of a, b, c, obtain formulas for $r+s$ and rs.

- $since\ \delta > 0, \qquad r, s = \dfrac{-b \pm \sqrt{\delta}}{2a}$
- $r + s = \dfrac{-b-\sqrt{\delta}}{2a} + \dfrac{-b+\sqrt{\delta}}{2a}$
- $= \dfrac{-b-\sqrt{\delta}-b+\sqrt{\delta}}{2a} = \dfrac{-2b}{2a}$
- $= -\dfrac{b}{a}$

Example 0.5

Determine the set of real solutions to $\frac{x}{x+1} \leq 1$:

- $\frac{x}{x+1} \leq 1$ Subtract 1 from both sides
- $\frac{x}{x+1} - 1 \leq 0$ Bring -1 in the fraction
- $\frac{x-x-1}{x+1} \leq 0$
- $\frac{-1}{x+1} \leq 0$ Multiply by -1
- $\frac{1}{x+1} \geq 0$
- $this\ can\ never\ equal\ 0, and\ is\ only\ positive\ when\ x+1\ is\ positive$
- $However\ x+1 \neq 0, so\ x+1 > 0, then\ x > -1$

Example 1

Given that $x + y = 100$, what is the maximum value of xy?

- AGM inequality: $xy \leq \left(\frac{x+y}{2}\right)^2$
- If x = y, then $xy = \left(\frac{x+y}{2}\right)^2$
- Can help us to calculate maximum value for x.y, as well as minimum value for x + y.

Example 1 cont'd

Given that $x + y = 100$, what is the maximum value of xy?

- AGM inequality: $xy \leq \left(\frac{x+y}{2}\right)^2$
- Substitute for $x + y$ in the AGM $xy \leq \left(\frac{100}{2}\right)^2$ $=(50)^2$ = 2500,
- So $xy \leq$ 2500, (i.e. x.y maximally can be 2500)
- Moreover the equality holds when x = y,
- In other words, In order to reach the maximum value, x and y have to be equal.
- If x = y $\Rightarrow$ xy = 2500

Example 2

Given that xy = 3600, calculate the minimum value for x + y?

- AGM inequality: $xy \leq \left(\frac{x+y}{2}\right)^2$
- Substitute for xy in the AGM,
- $3600 \leq \left(\frac{x+y}{2}\right)^2$
- $\Rightarrow \sqrt{3600} \leq \frac{x+y}{2}$
- $\Rightarrow 60 \leq \frac{x+y}{2}$
- $\Rightarrow 120 \leq x + y$

Example 2 cont'd

Given that xy = 3600, calculate the minimum value for x + y?

- So $120 \leq x + y$
- (i.e. $x + y$ minimally is at least 120)
- Moreover the equality holds when x = y,
- In other words, In order to reach the minimum value, x and y have to be equal
- If x = y $\Rightarrow$ $x + y$ = 120

Example 3

1.19. What are the dimensions of a rectangular carpet with perimeter 48 feet and area 108 square feet? Given positive numbers p and a, under what conditions does there exist a rectangular carpet with perimeter p and area a?

- Perimeter = $2(x+y) = 48 \Rightarrow x + y = 24$
- Area = xy = 108 $\Rightarrow$ y = $\frac{108}{x}$
- Substitute y = $\frac{108}{x}$ in $x + y = 24$
- $\Rightarrow x + \frac{108}{x} = 24$
- Since x > 0, then multiply both sides of the equation by x,
- $x^2 + 108 = 24x \Rightarrow x^2 - 24x + 108 = 0$

Example 3 cont'd

1.19. What are the dimensions of a rectangular carpet with perimeter 48 feet and area 108 square feet? Given positive numbers p and a, under what conditions does there exist a rectangular carpet with perimeter p and area a?

- So we let $y = \frac{108}{x}$,
- And need to solve $x^2 - 24x + 108 = 0$
- Note:
- We know that for a quadratic equation in the form $ax^2 + bx + c = 0$
- The discriminant is $\Delta = b^2 - 4ac$, where,
- $\Delta<0$ means no real solution,
- $\Delta = 0$ means exactly one real solution,
- and $\Delta>0$ means exactly 2 distinct real solutions exist for the corresponding quadratic equation.
- The solutions to the equation are $x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$

Example 3 cont'd

1.19. What are the dimensions of a rectangular carpet with perimeter 48 feet and area 108 square feet? Given positive numbers p and a, under what conditions does there exist a rectangular carpet with perimeter p and area a?

- So we let $y = \frac{108}{x}$,
- And need to solve $x^2 - 24x + 108 = 0$
- $\Delta = b^2 - 4ac \Rightarrow \Delta = 24^2 - 4(1)(108) = 576 - 432 = 144$
- The solutions to the equation are $x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$
- $x_1, x_2 = \frac{-(-24) \pm \sqrt{144}}{2(1)} = \frac{24 \pm 12}{2(1)} \Rightarrow x_1 = 6, x_2 = 18$
- Now if $x_1 = 6 \Rightarrow y_1 = \frac{108}{6} = 18$, if $x_2 = 18 \Rightarrow y_2 = \frac{108}{18} = 6$, therefore the dimensions are 6 by 18

Example 3 cont'd

1.19. What are the dimensions of a rectangular carpet with perimeter 48 feet and area 108 square feet? Given positive numbers p and a, under what conditions does there exist a rectangular carpet with perimeter p and area a?

- Generally for Perimeter p = $2(x+y)$ and Area a = xy , we can make a general quadratic equation for the dimensions as follows:
  - p = $2(x+y) \Rightarrow x+y = \frac{p}{2}$
  - a = $xy \Rightarrow y = \frac{a}{x}$
- Now substitute for y : $x + \frac{a}{x} = \frac{p}{2}$ since x is positive, multiply both sides by x
- $\Rightarrow x^2 + \text{a} = \frac{p}{2}x \Rightarrow x^2 - \frac{p}{2}x + \text{a} = 0$
- Now $\Delta = b^2 - 4ac \Rightarrow \Delta = (-\frac{p}{2})^2 - 4(1)(a) = \frac{p^2}{4} - 4\text{a}$
- The solutions to the equation are $x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a} = = \frac{\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - 4\text{a}}}{2(1)}$, which only has solutions if $\Delta = \frac{p^2}{4} - 4\text{a} \geq 0 \Rightarrow \frac{p^2}{4} \geq 4\text{a} \Rightarrow p^2 \geq 16\text{a}$

Example 3 cont'd

1.19. What are the dimensions of a rectangular carpet with perimeter 48 feet and area 108 square feet? Given positive numbers p and a, under what conditions does there exist a rectangular carpet with perimeter p and area a?

- Home exercise :
- Given the carpet with perimeter 48 feet, what is the maximum area that the carpet can have?
- Given the carpet with area 108 feet, what is the minimum perimeter that the carpet has?

Example 4

1.28. (!) *Application of the AGM Inequality.*

a) Use Proposition 1.4 to prove that $x(c - x)$ is maximized when $x = c/2$.

b) For $a > 0$, use part (a) to find the value of y maximizing $y(c - ay)$.

- AGM inequality: $xy \leq \left(\frac{x+y}{2}\right)^2$
- Let $x = x$, and let $y = c - x$
- $x(c-x) \leq \left(\frac{x+c-x}{2}\right)^2 = \left(\frac{c}{2}\right)^2 = \frac{c^2}{4}$
- The left side reaches the maximum value of $\frac{c^2}{4}$ if and only if $x = y = (c - x)$
- $\Rightarrow x = (c - x) \ \Rightarrow 2x = c \Rightarrow x = \frac{c}{2}$

Example 4 cont'd

1.28. (!) *Application of the AGM Inequality.*

a) Use Proposition 1.4 to prove that $x(c-x)$ is maximized when $x = c/2$.

b) For $a > 0$, use part (a) to find the value of y maximizing $y(c - ay)$.

- B)
- As a function of y, $y(c - ay)$ is maximized at the same value of y where $ay(c - ay)$ is maximized, since the ratio between these is the constant a.
- Now let $z = ay$,
- we know that $z(c - z)$ is maximized when $z = \frac{c}{2}$.
- At this value of z, we have $y = \frac{c}{2a}$.

Example 5

1.30. (!) Let x, y, u, v be real numbers.

a) Prove that $(xu + yv)^2 \le (x^2 + y^2)(u^2 + v^2)$.

b) Determine precisely when equality holds in part (a).

Rough work:

- $(xu + yv)^2 \le (x^2 + y^2)(u^2 + v^2)$
- $(xu)^2 + 2xuyv + (yv)^2 \le x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2$
- $(xu)^2 + 2xuyv + (yv)^2 \le (xu)^2 + x^2v^2 + y^2u^2 + (yv)^2$ $\boldsymbol{-((xu)^2 + (yv)^2)}$
- $2xuyv \le x^2v^2 + y^2u^2$
- $0 \le x^2v^2 - 2xuyv + y^2u^2$
- $0 \le (xv)^2 - 2xuyv + (yu)^2$
- $0 \le (xv - yu)^2$ **Which is True**

Example 5 cont'd

1.30. (!) Let x, y, u, v be real numbers.

a) Prove that $(xu + yv)^2 \le (x^2 + y^2)(u^2 + v^2)$.

b) Determine precisely when equality holds in part (a).

Proof(go backwards):

- $0 \le (xv - yu)^2$ Which is True expand
- $0 \le (xv)^2 - 2xuyv + (yu)^2$
- $0 \le x^2v^2 - 2xuyv + y^2u^2$ $+\,4xuyv$
- $2xuyv \le x^2v^2 + y^2u^2$ $+((xu)^2 + (yv)^2)$
- $(xu)^2 + 2xuyv + (yv)^2 \le (xu)^2 + x^2v^2 + y^2u^2 + (yv)^2$
- $(xu)^2 + 2xuyv + (yv)^2 \le x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2$
- $(xu + yv)^2 \le (x^2 + y^2)(u^2 + v^2)$

Example 5 cont'd

1.30. (!) Let x, y, u, v be real numbers.

a) Prove that $(xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2)$.

b) Determine precisely when equality holds in part (a).

Part b)

- $(xu + yv)^2 = (x^2 + y^2)(u^2 + v^2)$ expand
- $x^2u^2 + 2xuyv + y^2v^2 = x^2u^2 + x^2v^2 + y^2u^2 + y^2v^2$ $\boldsymbol{-(x^2u^2 + y^2v^2)}$
- $2xuyv = x^2v^2 + y^2u^2$ $\boldsymbol{-4xuyv}$
- $0 = x^2v^2 - 2xuyv + y^2u^2$
- $0 = (xv - yu)^2$
- Therefore the equality holds when $xv = yu$

Example 6

1.35. (!) Determine the set of ordered pairs (x, y) of nonzero real numbers such that $x/y + y/x \geq 2$.

- Lets check if the inequality holds for 2 cases:
- 1st when nonzero real numbers x and y have the same signs,
- 2nd when they have different signs
- 1st)Assume x and y have the same signs,
- i.e.$(x < 0 \; and \; y < 0) \; or \; (x > 0 \; and \; y > 0)$, therefore $x.y > 0$.
- $\frac{x}{y} + \frac{y}{x} \geq 2$

Multiply by xy

$\textit{Note: since } xy > 0 \rightarrow \textit{the order of inequality doesn't change}$

- $x^2 + y^2 \geq 2xy$
- $x^2 - 2xy + y^2 \geq 0$
- $(x - y)^2 \geq 0$ which is True, therefore the inequality holds for all real number pairs x and y when they have the same sign.

Example 6 cont'd

1.35. (!) Determine the set of ordered pairs (x, y) of nonzero real numbers such that $x/y + y/x \geq 2$.

- 2^{nd} when they have different signs
- Assume x and y have different signs,
- i.e.$(x > 0 \ and \ y < 0) \ or \ (x < 0 \ and \ y > 0)$, therefore $x.y < 0$.

- $\frac{x}{y} + \frac{y}{x} \geq 2$

Multiply by xy

$\boldsymbol{Note: since \ xy < 0 \rightarrow the \ order \ of \ inequality \ does \ change}$

- $x^2 + y^2 \leq 2xy$
- $x^2 - 2xy + y^2 \leq 0$
- $(x - y)^2 \leq 0$ which is False, therefore the inequality doesn't hold for all real number pairs x and y when they have different signs.

Example 7

Given $\boldsymbol{x^2 - ax + a - 1 = 0}$ where a is a real number.
Determine all values of a for which the above equation has

- exactly **1** root;
- **2 distinct** real roots.

- Note:

- We know that for a quadratic equation in the form $ax^2 + bx + c = 0$
- The discriminant is Δ = $b^2 - 4ac$,
- where, Δ<0 means no real solution,
- Δ = 0 means exactly one real solution, and
- Δ>0 means exactly 2 distinct real solutions
- exist for the corresponding quadratic equation.
- The solutions to the equation are x_1, x_2 = $\frac{-b \pm \sqrt{\Delta}}{2a}$

Example 7 cont'd

Given $\boldsymbol{x^2 - ax + a - 1 = 0}$ where a is a real number.
Determine all values of a for which the above equation has

- exactly **1** root;
- **2 distinct** real roots.

- We know that for a quadratic equation in the form $ax^2 + bx + c = 0$
- The discriminant is Δ = $b^2 - 4ac$,
- Our equation is $\boldsymbol{(1)x^2 - ax + a - 1 = 0}$ **, in which** $\boldsymbol{a = 1, b = -a, c = a - 1}$
- Now let us write the discriminant: Δ = $(\boldsymbol{-a})^2 - 4(\boldsymbol{1})(\boldsymbol{a - 1}) = a^2 - 4a + 4$
- $\rightarrow \Delta = (a-2)^2$
- Part 1) for the equation to have exactly one root, the discriminant needs to be 0,
- Therefore Δ is only zero when a = 2.
- Part 2) for the equation to have 2 distinct roots, Δ needs to be positive, therefore, since $\Delta = (a-2)^2$, Δ is positive when $a \neq 2$.

Example 8

A motorcyclist, traveling 30 miles per hour, leaves Westford heading toward Eastford. One hour later another motorcyclist, traveling 35 miles per hour, also leaves Westford heading toward Eastford.
When do the two motorcyclists meet?

- $Let\ x_1 be\ the\ distance\ that\ the\ first\ motorcyclist\ is\ traveling$
- $Let\ x_2\ be\ the\ distance\ that\ the\ second\ motorcyclist\ is\ traveling$
- $we\ know\ that\ when\ the\ second\ one\ started\ x_1 - x_2 = 30\ miles$
- $since\ the\ 2nd\ motorcyclist\ started\ traveling\ an\ hour\ later$
- $while\ the\ first\ one\ was\ traveling\ with\ the\ speed\ of\ 30\ miles\ per\ hour.$
- $x = vt + x_0 \rightarrow x_1 = 30t + 30, \quad x_2 = 35t$
- $when\ x_1 = x_2\ , they\ meet\ each\ other. i.e.\ 30t + 30 = 35t \rightarrow$
- $30 = 5t \rightarrow t = 6, therefore\ they\ meet\ eachother\ after\ 6\ hours.$

Example 9

Prove that for all real a and b we have

$$|\boldsymbol{a}-\boldsymbol{b}| \geq \big||\boldsymbol{a}|-|\boldsymbol{b}|\big|.$$

- 2 ways to prove this, 1) easy but longer, 2)more creative but shorter.
- 1) Easy but longer:

- **Rough work:**
- $|\boldsymbol{a}-\boldsymbol{b}| \geq \big||\boldsymbol{a}|-|\boldsymbol{b}|\big|$ *square both sides*
- $(|\boldsymbol{a}-\boldsymbol{b}|)^2 \geq \left(\big||\boldsymbol{a}|-|\boldsymbol{b}|\big|\right)^2 \rightarrow (\boldsymbol{a}-\boldsymbol{b})^2 \geq (|\boldsymbol{a}|-|\boldsymbol{b}|)^2$ *Note*: $a^2=|a|^2$ *expand*
- $\boldsymbol{a}^2-2\boldsymbol{ab}+\boldsymbol{b}^2 \geq |\boldsymbol{a}|^2-2|\boldsymbol{a}||\boldsymbol{b}|+|\boldsymbol{b}|^2$ *Note*: $a^2=|a|^2, b^2=|b|^2$
- *subtract* a^2 *and* b^2 $-2\boldsymbol{ab} \geq -2|\boldsymbol{a}||\boldsymbol{b}|$
- *devide by* -2 $\boldsymbol{ab} \leq |\boldsymbol{a}||\boldsymbol{b}| = |\boldsymbol{ab}|$ ***which is True,*** **since** $\mathbf{n} \leq |\mathbf{n}|$

Example 9 cont'd

Prove that for all real a and b we have

$$|a-b| \geq \big||a|-|b|\big|.$$

- 2 ways to prove this, 1) easy but longer, 2)more creative but shorter.
- 1) Easy but longer:

- **Proof:**
- $n \leq |n|$
- $ab \leq |ab| = |a||b|$ *multipy by* -2
- $-2ab \geq -2|a||b|$ *add* a^2 *and* b^2 — **Note**: $a^2 = |a|^2, b^2 = |b|^2$
- $a^2 - 2ab + b^2 \geq |a|^2 - 2|a||b| + |b|^2$ *rewrite*
- $(a-b)^2 \geq (|a|-|b|)^2$ *square root both sides* — **Note**: $\sqrt{a^2} = |a|$
- $|a-b| \geq \big||a|-|b|\big|$

Example 9 cont'd

Prove that for all real a and b we have

$$|\boldsymbol{a}-\boldsymbol{b}| \geq \big||\boldsymbol{a}|-|\boldsymbol{b}|\big|.$$

- 2 ways to prove this, 1) easy but longer, 2)more creative but shorter.
- 2) More creative but shorter:
- **By triangular inequality:** $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}|+|\boldsymbol{y}|$
- $|\boldsymbol{a}-\boldsymbol{b}| \geq \big||\boldsymbol{a}|-|\boldsymbol{b}|\big|$
- $\big||\boldsymbol{a}|-|\boldsymbol{b}|\big| \leq |\boldsymbol{a}-\boldsymbol{b}|$
- $-|\boldsymbol{a}-\boldsymbol{b}| \leq |\boldsymbol{a}|-|\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|$
- **1)** $|\boldsymbol{a}|-|\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|$ **and 2)** $-|\boldsymbol{a}-\boldsymbol{b}| \leq |\boldsymbol{a}|-|\boldsymbol{b}|$
- **Let's find each of those conjuncts separately**

Example 9 cont'd

Prove that for all real a and b we have

$$|\boldsymbol{a}-\boldsymbol{b}| \geq \big||\boldsymbol{a}|-|\boldsymbol{b}|\big|.$$

- 2 ways to prove this, 1) easy but longer, 2)more creative but shorter.
- 2) More creative but shorter:
- **By triangular inequality:** $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}|+|\boldsymbol{y}|$
- **Proof:**
- $\boldsymbol{use\ the\ triangular\ inequality\ twice}$
- $|\boldsymbol{a}| = |(\boldsymbol{a}-\boldsymbol{b})+\boldsymbol{b}|$
- $\boldsymbol{By\ triangular\ inequality}\ |(\boldsymbol{a}-\boldsymbol{b})+\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|+|\boldsymbol{b}|$
- $|\boldsymbol{a}| = |(\boldsymbol{a}-\boldsymbol{b})+\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|+|\boldsymbol{b}|$
- $|\boldsymbol{a}| \leq |\boldsymbol{a}-\boldsymbol{b}|+|\boldsymbol{b}|$
- $|\boldsymbol{a}|-|\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|$ **First conjunct**

Example 9 cont’d

Prove that for all real a and b we have

$$|\boldsymbol{a}-\boldsymbol{b}| \geq \left||\boldsymbol{a}|-|\boldsymbol{b}|\right|.$$

- 2 ways to prove this, 1) easy but longer, 2)more creative but shorter.
- 2) More creative but shorter:
- **By triangular inequality:** $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}|+|\boldsymbol{y}|$
- **Proof:**
- $\boldsymbol{use\ the\ triangular\ inequality\ twice}$
- $|\boldsymbol{b}| = |(\boldsymbol{b}-\boldsymbol{a})+\boldsymbol{a}|$
- $\boldsymbol{By\ triangular\ inequality}\ |(\boldsymbol{b}-\boldsymbol{a})+\boldsymbol{a}| \leq |\boldsymbol{b}-\boldsymbol{a}|+|\boldsymbol{a}| = |\boldsymbol{a}-\boldsymbol{b}|+|\boldsymbol{a}|$
- $|\boldsymbol{b}| = |(\boldsymbol{b}-\boldsymbol{a})+\boldsymbol{a}| \leq |\boldsymbol{a}-\boldsymbol{b}|+|\boldsymbol{a}|$
- $|\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|+|\boldsymbol{a}|$
- $|\boldsymbol{b}|-|\boldsymbol{a}| \leq |\boldsymbol{a}-\boldsymbol{b}| \rightarrow -(|\boldsymbol{a}|-|\boldsymbol{b}|) \leq |\boldsymbol{a}-\boldsymbol{b}|$ $\boldsymbol{multipy\ by\ -1}$
- $\rightarrow |\boldsymbol{a}|-|\boldsymbol{b}| \geq -|\boldsymbol{a}-\boldsymbol{b}| \rightarrow -|\boldsymbol{a}-\boldsymbol{b}| \leq |\boldsymbol{a}|-|\boldsymbol{b}|$ **second conjunct**

Example 9 cont'd

Prove that for all real a and b we have

$$|\boldsymbol{a}-\boldsymbol{b}| \geq \big||\boldsymbol{a}|-|\boldsymbol{b}|\big|.$$

- 2 ways to prove this, 1) easy but longer, 2)more creative but shorter.
- 2) More creative but shorter:
- **By triangular inequality:** $|\boldsymbol{x}+\boldsymbol{y}| \leq |\boldsymbol{x}|+|\boldsymbol{y}|$
- **Proof:**
- $\boldsymbol{by\ the\ first\ and\ second\ conjunct}$:
- $|\boldsymbol{a}|-|\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|$ **First conjunct**
- $-|\boldsymbol{a}-\boldsymbol{b}| \leq |\boldsymbol{a}|-|\boldsymbol{b}|$ **second conjunct**
- **Which implies:** $-|\boldsymbol{a}-\boldsymbol{b}| \leq \ |\boldsymbol{a}|-|\boldsymbol{b}| \leq |\boldsymbol{a}-\boldsymbol{b}|$
- $\rightarrow ||\boldsymbol{a}|-|\boldsymbol{b}|| \leq |\boldsymbol{a}-\boldsymbol{b}|$

$\boldsymbol{Note}: |\boldsymbol{x}| < \mathbf{2} \Leftrightarrow -\mathbf{2} < \boldsymbol{x} < \mathbf{2}$

Example 10

Show that for any positive a and b the following inequality holds true:

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}} \geq \sqrt{a}+\sqrt{b}$$

- ***Rough work:***
- $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}} \geq \sqrt{a}+\sqrt{b}$ Square both sides $\left(\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}\right)^2 \geq \left(\sqrt{a}+\sqrt{b}\right)^2,$
- $\frac{a^2}{b}+2\sqrt{a}\sqrt{b}+\frac{b^2}{a} \geq a+2\sqrt{a}\sqrt{b}+b,$ $-2\sqrt{a}\sqrt{b}$
- $\frac{a^2}{b}+\frac{b^2}{a} \geq a+b,$ $-(a+b)$
- $\frac{a^2}{b}-a+\frac{b^2}{a}-b \geq 0,$ $multiply\ by\ ab, since\ ab>0$ $the\ sign\ of\ inequality\ doesn't\ change.$
- $a^3-a^2b+b^3-b^2a \geq 0,$ $factor\ an\ a^2 and\ a\ b^2$
- $a^2(a-b)-b^2(a-b) \geq 0,$ $factor\ (a-b)$
- $(a^2-b^2)(a-b) \geq 0$ $expand$ $(a-b)(a+b)(a-b) \geq 0,$
- $(a-b)^2(a+b) \geq 0,$ $which\ is\ True\ since\ both\ factors\ are\ positive$

Example 10

Show that for any positive a and b the following inequality holds true:

$$\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}} \geq \sqrt{a}+\sqrt{b}$$

Proof:

- $(a-b)^2(a+b) \geq 0,$ True since both factors are positive — expand
- $(a-b)(a+b)(a-b) \geq 0 \rightarrow (a^2-b^2)(a-b) \geq 0,$ expand
- $a^2(a-b)-b^2(a-b) \geq 0,$ expand
- $a^3-a^2b+b^3-b^2a \geq 0,$ devide by ab, since $ab > 0$ the sign of inequality doesn't change.
- $\frac{a^2}{b}-a+\frac{b^2}{a}-b \geq 0,$ $+(a+b)$
- $\frac{a^2}{b}+\frac{b^2}{a} \geq a+b,$ $+2\sqrt{a}\sqrt{b}$
- $\frac{a^2}{b}+2\sqrt{a}\sqrt{b}+\frac{b^2}{a} \geq a+2\sqrt{a}\sqrt{b}+b,$ factor
- $\left(\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}}\right)^2 \geq \left(\sqrt{a}+\sqrt{b}\right)^2$ **square root** $\sqrt{\frac{a^2}{b}}+\sqrt{\frac{b^2}{a}} \geq \sqrt{a}+\sqrt{b}$

Example 11

For what values of the parameter a, is the sum of squares of two distinct solutions of the equation $x^2 - 4ax + 5a = 0$ equal to 6?

Objectives:

A. The equation has to have 2 distinct real solutions; therefore, the discriminant $\Delta > 0$.

B. The sum of the squares of the roots must equal 6, i.e. $x_1^2 + x_2^2 = 6$

Example 11 cont'd

For what values of the parameter a, is the sum of squares of two distinct solutions of the equation $x^2 - 4ax + 5a = 0$ equal to 6?

A. The equation has to have 2 distinct real solutions; therefore, the discriminant $\Delta > 0$. $\Delta = b^2 - 4ac$

- $\Delta = (-4a)^2 - 4(5a)$
- $= 16a^2 - 20a$
- $= 16a^2 - 20a > 0$

  - $\Rightarrow 4a^2 - 5a > 0$
  - $\Rightarrow a(4a - 5) > 0$
  - $\Rightarrow a < 0 \; or \; a > 5/4$

Example 11 cont'd

For what values of the parameter a, is the sum of squares of two distinct solutions of the equation $x^2 - 4ax + 5a = 0$ equal to 6?

B. The sum of the squares of the roots must equal 6, i.e. $x_1^2 + x_2^2 = 6$

$$x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}, \Delta = b^2 - 4ac$$

$$x_1^2 + x_2^2 = 6$$

$$x_1^2 + x_2^2 = \left(\frac{-b + \sqrt{\Delta}}{2a}\right)^2 + \left(\frac{-b - \sqrt{\Delta}}{2a}\right)^2$$

$$= \frac{\left(-b + \sqrt{\Delta}\right)^2 + \left(-b + \sqrt{\Delta}\right)^2}{4a^2}$$

Example 11 cont'd

For what values of the parameter a, is the sum of squares of two distinct solutions of the equation $x^2 - 4ax + 5a = 0$ equal to 6?

B. The sum of the squares of the roots must equal 6, i.e. $x_1^2 + x_2^2 = 6$

$$x_1^2 + x_2^2 = \frac{(b^2+2b\sqrt{\Delta} + \Delta) + (b^2-2b\sqrt{\Delta} + \Delta)}{4a^2} = \frac{2b^2 + 2\Delta}{4a^2} = \frac{b^2 + \Delta}{2a^2}$$

$$= \frac{b^2 + \mathrm{b}^2 - 4\mathrm{ac}}{2a^2} = \frac{\mathrm{b}^2 - 2\mathrm{ac}}{a^2}$$

Therefore, $x_1^2 + x_2^2 = \frac{\mathrm{b}^2-2\mathrm{ac}}{a^2}$

Example 11 cont'd

For what values of the parameter a, is the sum of squares of two distinct solutions of the equation $x^2 - 4ax + 5a = 0$ equal to 6?

B. The sum of the squares of the roots must equal 6, i.e. $x_1^2 + x_2^2 = 6$

$$x_1^2 + x_2^2 = \frac{\mathrm{b}^2 - 2\mathrm{ac}}{a^2} = 6$$

Plugin the appropriate values in the equation:

$$\frac{(-4\mathrm{a})^2 - 2(1)(5\mathrm{a})}{1^2} = 6$$

$$\Rightarrow 16a^2 - 10a = 6$$

$$\Rightarrow 8a^2 - 5a - 3 = 0$$

Next find the roots for the quadratic equation

Example 11 cont'd

For what values of the parameter a, is the sum of squares of two distinct solutions of the equation $x^2 - 4ax + 5a = 0$ equal to 6?

B. The sum of the squares of the roots must equal 6, i.e. $x_1^2 + x_2^2 = 6$

$8a^2 - 5a - 3 = 0$

Now find the roots for the quadratic equation:

$\Delta = (-5)^2 - 4(8)(-3) = 25 + 96 = 121$

$$x_1, x_2 = \frac{5 \pm \sqrt{121}}{16} = \frac{5 \pm 11}{16}$$

$$\Rightarrow x_1, x_2 = 1, -\frac{3}{8}$$

Therefore, a must be either $1\ or\ -\frac{3}{8}$ for the sum of squares to equal 6.

However, since for having 2 distinct Real solutions (part a), $a < 0\ or\ a > 5/4$, the only acceptable value for a is $-\frac{3}{8}$.

Citations:

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