# MΔT0x66

of course in base 16


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Tutorial 7

Credits to Larry Zhang

# Simple and complete induction

# Todays topics

1. **Review of simple induction**
2. **2 examples of simple induction**
3. **Principle of complete induction**
4. **1 example for complete induction**
5. **Review example for proving by contradiction**

# Review of Principle of Simple Induction

Let P(n) be the statement for some number n,

If

(i) If $P(b)$ is True,

(ii) And $P(n) \Rightarrow P(n+1)$ is True for all $n \geq b$,

then

$P(n)$ is True for **all** integers $n \geq b$.

# Recipe for writing a proof using simple induction

- Step 1: Define the predicate P(n), i.e. the statement you want to prove.
- Step 2: Base Case: show that P(b) is True, e.g., b = 0
- Step 3: Induction Step: show P(k) => P(k+1),
- for k >= b, i.e.,
  - a. Assume P(k) is True (Inductive Hypothesis)
  - b. Show that P(k+1) is True

- Done.

# Example 4

**3.26.** Let $\langle a \rangle$ be a sequence such that $a_1 = 1$ and $a_{n+1} = a_n + 3n(n+1)$ for $n \in \mathbb{N}$. Prove that $a_n = n^3 - n + 1$ for $n \in \mathbb{N}$.

- Step 1: Define the Statement you want to prove. (i.e. the predicate P(n))

$$a_n = n^3 - n + 1$$

- Step 2: check for the base case: since n is an element of Natural numbers, base case is n = 1;
  - $a_1 = 1$
  - $a_1 = 1^3 - 1 + 1 = 1$
  - Since the formula yields the same result for $a_1$ compared to what we have defined $a_1$ to be, **Base case <u>holds</u>**.

# Example 4 cont'd **Prove that $a_n = n^3 - n + 1$ for $n \in \mathbb{N}$.**

- Step 3: Induction Step: show P(k) => P(k+1),
- Assume that for n=k, **$a_k = k^3-k+1$**, **Induction Hypothesis**
- Show that for n=k+1 ; **$a_{k+1}=(k+1)^3-(k+1)+1$**
- By the recursive definition $a_{n+1} = a_n +3n(n+1)$
- Then; $a_{k+1}=\mathbf{a_k}+3k(k+1)$

$a_k = k^3-k+1$ by I.H.

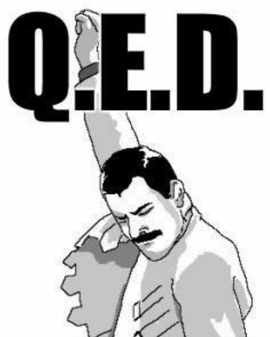
$=\mathbf{k^3-k+1}+3k(k+1)$

$=k^3 +3k^2 +3k+1 -k$

Add -1+1 = 0

$=(k+1)^3 -k \mathbf{-1 +1}$

$\mathbf{= (k+1)^3-(k+1)+1}$



# Example 5

- Prove that for all natural n ∈ N:
  - $$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1$$
- Step 1: define the predicate:
  - $$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1$$
- Note: understand how the summation of this series is formed:
- $$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}+\frac{1}{n+n+1}+\cdots+\frac{1}{n+n+n}+\frac{1}{3n+1}>1$$

+1 +n-1 +1 +n +1

+2n

# Example 5 cont'd

**P(n):** $\mathbf{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1}$

- Step 2: check for the base case, n=1:
  - $\frac{1}{1+1} + \frac{1}{1+2} + \frac{1}{1+3} > 1$

    Since for n=1, $\frac{1}{3n+1} = \frac{1}{4}$ we stop at $\frac{1}{4}$
  - $\Rightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$
  - $\Rightarrow \frac{6+4+3}{12} > 1$
  - $\Rightarrow \frac{13}{12} > 1$ True, Therefore it <u>**holds**</u> for the **base case**

# Example 5 cont'd

**P(n):** $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1$

- Step 3: Induction step: show P(k) => P(k+1),

- Let the induction hypothesis be; for n=k,

Induction hypothesis $\Rightarrow$ **P(k):** $\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{3k+1}>1$ ***holds***

- Want to show that; for n=k+1,

**P(k+1):** $\frac{1}{(k+1)+1}+\frac{1}{(k+1)+2}+\cdots+\frac{1}{3(k+1)+1}>1$ ***holds***

# Example 5 cont'd

**P(n):** $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1$

- Step 3: Induction step: show P(k) => P(k+1),

$$\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{3k+1}>1 \quad \textit{I.H.}$$

Show **P(k+1):** $\frac{1}{(k+1)+1}+\frac{1}{(k+1)+2}+\cdots+\frac{1}{3(k+1)+1}>1$ ***holds***

$$\Rightarrow \frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{3k+4}>1$$

$$\Rightarrow \frac{1}{k+2}+\frac{1}{k+3}+\cdots\ \frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3K+3}+\frac{1}{3k+4}>1$$

-1 -1 -1

# Example 5 cont'd

**P(n):** $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$

- Step 3: Induction step: show P(k) => P(k+1),
- $\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{3k+1} > 1$ ***I.H.***
- $\boldsymbol{Show} : P(k+1)$: $\frac{1}{k+2} + \frac{1}{k+3} + \cdots \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3K+3} + \frac{1}{3k+4} > 1$
- **Need to change this somehow to use the I.H., since it does not have the $\frac{1}{k+1}$ term??????**

Solution: add $\frac{1}{k+1}$ to both sides

$$\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3K+3} + \frac{1}{3k+4} > 1 + \frac{1}{k+1}$$

# Example 5 cont'd

$\boldsymbol{Show}: P(k+1)$: $\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{3k+4}>1$

- Step 3: Induction step: show P(k) => P(k+1),

$$\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{3k+1}>1 \quad \textit{I.H.}$$

**P(k+1):** $\underbrace{\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots\frac{1}{3k+1}}_{\text{I.H.}}+\frac{1}{3k+2}+\frac{1}{3K+3}+\frac{1}{3k+4}>1+\frac{1}{k+1}$

- **Find the I.H.**

- Since we know by I.H. that $\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{3k+1}>1$, now for the inequality to be true, $\frac{1}{3k+2}+\frac{1}{3K+3}+\frac{1}{3k+4}>\frac{1}{k+1}$, in other words we can subtract I.H. from Our inequality, since we assume it holds, and then verify that the remaining inequality holds.

# Example 5 cont'd

- Step 3: Induction step: Cont'd

- **verify** $\frac{1}{3k+2} + \frac{1}{3K+3} + \frac{1}{3k+4} > \frac{1}{k+1}$ for k be a natural number.

$$\left(\frac{1}{3k+2} + \frac{1}{3K+3} + \frac{1}{3k+4}\right) > \left(\frac{1}{k+1}\right)$$

**Multiply both sides by (k+1)**

$\Rightarrow (k+1).\left(\frac{1}{3k+2} + \frac{1}{3K+3} + \frac{1}{3k+4}\right) > \left(\frac{1}{k+1}\right).(k+1)$ **; since $(k+1) > 1$.**

$$\Rightarrow \frac{k+1}{3k+2} + \frac{1}{3} + \frac{k+1}{3k+4} > 1$$

**Subtract $\frac{1}{3}$ from both sides**

# Example 5 cont'd

- Step 3: Induction step: Cont'd

$$\Rightarrow \quad \frac{k+1}{3k+2} + \frac{k+1}{3k+4} > \frac{2}{3}$$

Take the common denominator

$$\Rightarrow \frac{(k+1)(\mathbf{3k+4})+(k+1)(\mathbf{3k+2})}{(3k+2)(3k+4)} > \frac{2}{3}$$

Multiply by $(\mathbf{3k+2})(\mathbf{3k+4})$

$$\Rightarrow (k+1)(3k+4) + (k+1)(3k+2) > \frac{\mathbf{2}}{\mathbf{3}}((\mathbf{3k+2})(\mathbf{3k+4}))$$

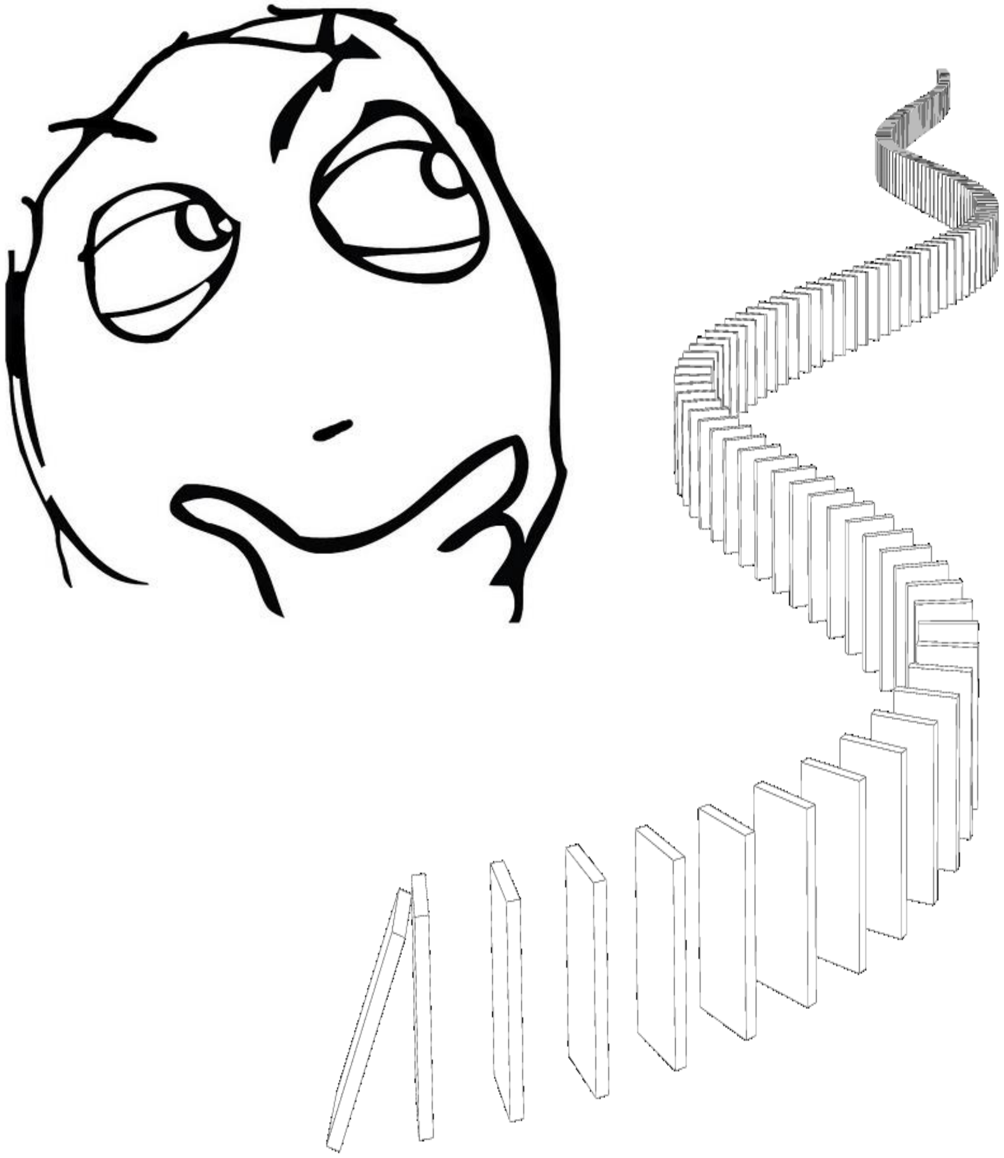
$$\Rightarrow (3k^2 + 7k + 4) + (3k^2 + 5K + 2) > \frac{2}{3}(9k^2 + 18k + 8)$$

$$\Rightarrow (3k^2 + 7k + 4) + (3k^2 + 5K + 2) > \left(6k^2 + 12k + \frac{16}{3}\right)$$

# Example 5 cont'd

- Step 3: Induction step: Cont'd
- $\Rightarrow (3k^2 + 7k + 4) + (3k^2 + 5K + 2) > \left(6k^2 + 12k + \frac{16}{3}\right)$
- $\Rightarrow \; 6K^2 + 12k + 6 > 6k^2 + 12k + \frac{16}{3}$
- $\Rightarrow \; 6 > \frac{16}{3}$
- **Since 6 can be written as** $\mathbf{\frac{18}{3}}$ $\Rightarrow \; 6 > \frac{16}{3}$ is **True,** therefore, the induction step for **n = k+1 (i.e. P(k+1)) holds**


- **Therefore for n ∈ N, P(n):** $\boldsymbol{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1.}$



# New topic: Complete (strong Induction)

- Think about the dominoes
- What simple induction says is that, **to show that d[236] falls**, all I need to know is that **d[235] falls.**
- But by knowing d[235] falls, we actually know much more…
- **We also know d[1] to d[234] all fall ○**
  - We didn't use this information because knowing that d[235] falls happened to be enough
  - But sometimes it is NOT enough and we need to use all the information we know.

# In other words

- **What we did in simple induction**
- **Suppose P(0) is True**
- **Then we use P(0) to prove P(1) is True**
- **Then we use P(1) to prove P(2) is true.**
- **Then we use P(2) to prove P(3) is true**
- **……**

- **Suppose P(0) is True**
- **Then we can use P(0) to prove P(1) is True**
- **Then we can use both P(0) and P(1) to prove P(2) is true.**
- **Then we can use P(0), P(1) and P(2) to prove P(3) is true**
- **……**
- **This is called complete (strong) induction.**

$\wedge$ means "and"

# Principle of Complete Induction

(i) If $P(b)$ is True,

(ii) And $\underline{P(b) \wedge P(b+1) \wedge \ldots \wedge P(n-1)} \Rightarrow P(n)$ is True for all $n > b$,

Then $P(n)$ is True for **all** integers $n \geq b$.

Induction Hypothesis

# Example 1 Prime or Product of Primes

- Prove that every natural number greater than 1 can be written as a product of primes. Think on it for 1 minute.
- For example:

2 = 2

3 = 3

4 = 2 x 2

6 = 2 x 3

28 = 2 x 2 x 7

# Example 1 cont'd

- **Let's try simple induction ...**
- **Define predicate P(n):** n can be decomposed into a product of primes
- **Base case**: n=2
  - 2 is already a product of primes (2 is prime), so we're done.
- **Induction Step**:
  - Assume **n >= 2** and that **n** can be written as a product of primes. Need to prove that **n+1** can be written as a product of primes...
- Imagine that we know that 8 can be written as a product of primes. (2x2x2) How does this help us decompose 9 into a product of primes? (3x3) Not obvious!
- **Problem: There is no obvious relation between the decomposition of k and the decomposition of k+1. Simple induction not working!**

# Example 1 cont’d

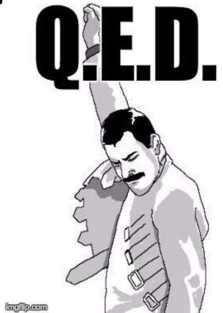
- **Use Complete Induction**

- **Define predicate P(n):** n can be decomposed into a product of primes. (same as before)

- **Base case**: n=2,
  - 2 is already a product of primes (2 is prime), so we're done. (same as before)

# Example 1 cont'd

- **Induction Step:**
- Assume $P(2) \wedge P(3) \wedge P(4) \wedge \ldots \wedge P(n-1)$, i.e., all numbers from 2 to n-1 can be written as a product of primes. (**Induction Hypothesis of Complete Induction**)
- Now need to show ***P(n)***, i.e., ***n*** can be written as a product of primes
- Case 1: n is prime …
  - then n is already a product of primes, done
- Case 2: n is composite (not prime) …
  - then n can be written as **n = a x b**, where a & b satisfies **2 <= a, b <= n-1**
    - According to **I.H.**, each of ***a*** and ***b*** can be written as a product of primes.
  - So n = **a x b** can be written as a product of primes.



# Example 2 review of contradiction proving method

- Q) Consider a group of 11 people at a party. People shake hands with each other when they meet. Prove that there should be at least 2 people with the same number of handshakes.

- Proof:
- let’s assume that the number of handshakes can be unique for each person.
- Also note that the number of handshakes a person can have is

$0 \leq Hnumber \leq 10$

Now assign a number to each person such that, every persons number is unique.

# Example 2 cont'd

- Proof cont'd:
- Now assign a number to each person such that the number every persons number is unique.
- $A_1$ ->0
- $A_2$ -> 1
- …
- $A_{11}$ ->10
- Now realize that the person $A_1$ has shaken hands with nobody, and the $A_{11}$ person has shaken hands with everyone, and **this is a contradiction**,
- Therefore there must be at least 2 people with the same number of handshakes. **Q.E.D.**

# Citations:


Book

Mathematical Thinking: Problem-solving and Proofs

D'Angelo, J.P.

West, D.B.

9780130144126

https://books.google.ca/books?id=fL6nQgAACAAJ

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Prentice Hall