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ANALYSIS 5

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1 Signed Measure

1.1 Introduction

Definition 1 (Signed Measure)**.** *Given a measurable space* $(X, \mathcal{M})$*, a signed measure is a function* $\nu : \mathcal{M} \to [-\infty, \infty]$ *with the following properties:*

1. $\nu(\phi) = 0$
2. ν *can assume either* ∞ *or* $-\infty$ *but not both*
3. *If* $\{E_j\}$ *is a sequence of disjoint sets in* $\mathcal{M}$*, then* $\sum_{i=1}^{\infty} \nu(E_i) = \nu\left(\cup_{i=1}^{\infty} E_i\right)$

One should note that normal measures are also signed measure, the only difference is the extension of the range of the measure function to cover almost all of $\mathbb{R}$.

1.2 Upper and lower continuity

Theorem 1 (Uppercontinuity)**.** *Let* $\{E_i\}$ *be a countable collection of measurable set with* $E_i \subseteq E_{i+1}$*. Then:*

$$\lim_{i\to\infty} \nu(E_i) = \nu\left(\bigcup_{i=1}^{\infty} E_i\right) \tag{1.1}$$

Theorem 2 (Lowercontinuity)**.** *Let* $\{E_i\}$ *be a countable collection of measurable set with* $E_{i+1} \subseteq E_i$*. Then:*

$$\lim_{i\to\infty} \nu(E_i) = \nu\left(\bigcap_{i=1}^{\infty} E_i\right) \tag{1.2}$$

Proof. Same as what we do for unsigned measure □

1.3 Positive,Negetive and Null Set

Definition 2 (Positive set)**.** *A set whose every mesurable subset* E *satisfies* $\nu(E) \geq 0$ *is called a positive set.*

In a similar fashion we define :

Definition 3 (Negetive set)**.** *A set whose every mesurable subset* E *satisfies* $\nu(E) \leq 0$ *is called a positive set.*

Definition 4 (Null set). *A set whose every mesurable subset E satisfies $\nu(E) = 0$ is called a positive set.*

We consider an example. Let μ be an unsigned measure and let f be a measurable L^1 function . Let us define a measure nu as:

$$\nu(E) = \int_E f d\mu \tag{1.3}$$

Then ν is a signed measure. If E is a set such that $f \geq 0$ $\mu.a.e$ on E then E is a positive set. Similarly we can find negetive and null sets.

Lemma 3. *1. Subsets of positive sets are positive*

2. Countable[1] union of positive sets are positive
Similar results are also valid for null and negetive sets.

[1] A countable union is needed as in case of uncountable union, there will be a chance that the union will not belong to the sigma algebra; a sigma algebra is closed in countable union and not under arbitary union

The next lemma will be required for the proof of **Hahn Decomposition Theorem** in the next section.

Lemma 4. *Let ν be a signed measure which doesn't attain ∞. A set with a positive measure has a positive subset.*

1.4 Hahn Decomposition

Theorem 5 (Hahn Decompositiopn Theorem). *lf ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \phi$ Moreover if P', N' is another such pair, then $P \Delta P' (= N \Delta N')$ is null in ν.*

Proof Outline :

1. Define $m = \sup_{\text{positive sets}} \nu(P)$
2. Take a sequence $\{p_i\}$ such that $\lim_{i\to\infty} \nu(p_i) = m$
3. Show if $P = \bigcup p_i$ then $\nu(P) = m$
4. Show if $N = P^c$ and if N has a set with positive measure, then by lemma 4, there is contradiction.
5. If $E \subseteq P \Delta P'$ and $\nu(E) \neq 0$. Without loss of generality assume $E' = E \cap P$ is not null. Then $E' \subseteq P'^c = N'$ which contradicts negetivity of N'

1.5 Jordan Decomposition

Definition 5 (Mutually singular measures). *Two measures ν and μ are said to be mutually singular if there exists a partition of X in E and F such that $X = E \sqcup F$ and E is null in μ and F is null in ν*[2]

[2] That is to say that the measures ν and μ "lives" on different sets.

Notation: If ν and μ are mutuallysingular, then we denote it as:

$$\nu \perp \mu$$

Theorem 6 (Jordan Decomposition Theorem)**.** *Given a (signed) measure ν there exists* ***unique*** *positive measures ν^+, ν^- such that:*

$$\nu = \nu^+ - \nu^- \quad \nu^+ \perp \nu^- \tag{1.4}$$

Proof Outline: [3]

1. Existance follows by Hahn decomposition.
2. Start by assuming the decomposition is not unique and theere exists two such decomposition $\nu = \nu^+ - \nu^- = \mu^+ - \mu^-$.
3. There exists partition of X in E, F due to μ^+, μ^- and in P, N due to ν^+, ν^-. If A is measurable , show that
$$\mu^+(A) = \nu(A \cap E) = \nu(A \cap E \cap P) + \nu(A \cap E \cap N)$$
4. As E is positive and N is negetive, show that $A \cap E \cap N$ is a null set. Repeat or ν^+ and get similar results
5. Show $\nu^+ = \mu^+$ and in a similar way $\mu^- = \nu^-$

[3] As I understand it, the main idea is if there is two decomposition as outlined in step 2 and 3, then we have 4 sets to deal with:

- $P \cap F$ and $E \cap N$: which are null as they are intersection of positive and negetive sets
- $P \cap E$ where μ^+ and ν^+ agree and $\mu^-, \nu^- = 0$
- $N \cap F$ where μ^- and ν^- agree and $\mu^+, \nu^+ = 0$

Make this nice and you get the proof outlined.

1.6 *Total Variation Measure*

Definition 6 (Total Variation Measure)**.** *If a measure ν decomposes in singular ν^+ and ν^- then we define the total variation measure $|\nu|$ as*

$$|\nu| = \nu^+ + \nu^- \tag{1.5}$$

This works before as by Hahn-Jordan, the decomposition is unique. The definition is important as by Lemma 7 and Lemma 8, we see that properties of ν is reflected in $|\nu|$

Lemma 7. *The following statements are equivalent:*

1. *E is null in ν*
2. *$\nu^+(E) = 0$ and $\nu^-(E) = 0$*[4]
3. *$|\nu|(E) = 0$*

[4] For unsigned measures, being null and having a measure 0 is same.

Lemma 8. *The following statemwents are equivalent:*

1. *$\nu \perp \mu$*
2. *$\nu^+ \perp \mu$ and $\nu^- \perp \mu$*
3. *$|\nu| \perp \mu$*

Proof for lemma 7 and 8 is at the end, they are given as exercise in Folland, ch3. Other properties which gets reflected are finiteness and $\sigma-$finiteness.

1.7 Absolute Continuity

Definition 7 (Absolute Continuity)**.** *Let μ be an unsigned measure. We say ν is absolutely continious with respect to μ if for any measurable set E, $\mu(E) = 0 \implies \nu(E) = 0$*

Notation: ν is absolutely continious with respect to μ is denoted by:

$$\nu \ll \mu$$

Unlike mutual singulaity, $\nu \ll \mu$ doesn't imply $\mu \ll \nu$. In a sense, being mutually singular and being absolutely continious are exclusive concepts. If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$

Lemma 9. *The following statements are equivalent:*

1. $\nu \ll \mu$
2. $\nu^+ \ll \mu$ *and* $\nu^- \ll \mu$
3. $|\nu| \ll \mu$

Lemma 10. *If ν and μ are finite measures, $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ usch that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$* [5]

[5] This lemma gives some motivation for the nomenclature of absolute continuity

Proof Outline:

1. By Lemma 9, we need to show this is true for $|\nu|$ and we will be done. This is why, without loss of generality, we can assume ν is unsigned.
2. ***Don't understand why this is trivial***
3. Make a decresing sequence of mesurable sets
4. Show if there exists ϵ with no such δ then μ of intersection goes to 0 but ν of intersection stays above ϵ. This contradicts absolute continuity.

1.8 Radon-Nikodym theorem

Theorem 11 (Radon-Nikodym theorem)**.** *The theorem has two parts:*

1. *For a measure space, with $\sigma-$finite measures ν(unsigned) and μ(unsigned), there is a unique decomposition of ν in ν_1 and ν_2 such that $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$*
2. *There exists a function f which is integrable in the etended sense such that $\nu_1(E) = \int_E f d\mu$. Moreover, if there are two such functions f_1, f_2 then $f_1 = f_2$ μ.a.e.*

Proof Outline:

1. **Step 1: ν, μ are finite**
   (a) Note that $\nu(E) = \int_E f d\mu + \nu_2(E) \Rightarrow \nu(E) \geq \int_E f d\mu$
   (b) Make a family of function $\mathcal{F}$ which satisfy this.
   (c) Let α be suprema of the integral of f in family. Find f_n whose integral approach α. Set $g_n(x) = \max\{f_1(x), f_2(x) \dots f_n(x)\}$. Show g_n is increasing and is in $\mathcal{F}$. Find limit of g_n as g. Use MCT to show that α is attained by g.[6]
   (d) Set $\nu_2 = \nu - \nu_1$. Show $\nu_2 \perp \mu$.

2. **Step 2: Assume σ, μ are $\sigma-$finite.**
   (a) Divide X in disjoint countable B_i each with finite measure.
   (b) Restrict μ and ν in B_i to get μ_i, ν_i. Repeat step 1 to get f_i, ν_i^1, ν_i^2 in B_i. Set $f = \sum f_i, \nu^1 = \sum \nu_i^1, \nu^2 = \sum \nu_i^2$

3. **Step 3: Uniqueness of decomposition**: If ν_1, ν_2 and $\hat{\nu}_1, \hat{\nu}_2$ are two decomposition then $\nu_1 - \hat{\nu}_1 = \hat{\nu}_2 - \nu_2$. Now $(\nu_1 - \hat{\nu}_1) \ll \mu$ and $\nu_2 - \hat{\nu}_2 \perp \mu$. So $\nu_1 - \hat{\nu}_1 = 0$

4. **Step 4: Uniqueness of f :** If f, g are two such functions then $\int_E (f - g) = 0$ or $f = g$ $\mu.a.e.$

[6] The reason to take a in first place is because we don't know if f_n converges. The reason we take such a g_n is sothat we can intechange the limit and integral by applying MCT.

1.9 *Solutions to Real Analysis By Folland, Section 3.1,3.2*

1.9.1 *Solutions to problems in section 3.1*

Problem 3.1

Prove Proposition 3.1.
Propotion 3.1 is Theorem 1&2 mentioned here.

Solution 1 outline: We do it as instructed in the book, by copying Theorem 1.8 from the book.
Solution 2 outline: Decompose μ by Hahn Decomposition and apply upper/lower continuity on each of them individually.[7]
Proof 1:
By the second condition of definition 1, we can assume $\mu > -\infty$.
Proving Uppercontinuity: If some $E_i = \infty$ we are done. Else , set $E_0 = \phi$. Define $F_i = E_i \setminus E_{i-1}$. Not that any two F_i, F_j is disjoint. Then $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} F_i$. It follows that:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) = \lim_{i\to\infty} \mu(E_i)$$

[7] Proof of Hahn decomposition doesn't assume upper/lower continuity

The last step follows by countable additivity.
Proving Lowercontinuity: Set $F_j = E_1 \setminus E_j$. Then $F_i \subseteq F_{i+1}$ and $\mu(E_1) = \mu(F_j) + \mu(E_j)$. Also, $\cup_{i=1}^{\infty} F_j = E_1 \setminus (\cap_{i=1}^{\infty} E_j)$. Apply uppercontinuity to get:

$$\mu(E_1) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right) + \lim_{j\to\infty} \mu(F_j) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right) + \mu(E_1) - \lim_{j\to\infty} \mu(E_j)$$

$$\Rightarrow \lim_{j\to\infty} \mu(E_j) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right)$$

Problem 3.2

If ν is a signed measure, E is ν-null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

Solution outline: This the the problem corresponding to lemma 7 and 8. In both case we shall show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.
Solution part-1(Proof of lemma 7):
The Steps are based on lemma 7.

- **Step 1:** $1 \Rightarrow 2$
  Let P, N be the decomposition of N in positive and negetive sets using HJD[Hahn Jordan Decomposition]. Let E be a null set. Then:

$$\begin{aligned}\nu^+(E) =& \nu^+(E \cap P) - \nu^-(E \cap N) \\ =& \nu^+(E \cap P) \\ =& \nu(E \cap P) - \nu^-(E \cap P) \\ =& \nu(E \cap P) = 0\end{aligned}$$

  Similar reult is obtained for ν^-. For unsigned measures, a measure zero set is null set, so we are done.

- **Step 2:** $2 \Rightarrow 3$
  This is the easy step.

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$$

- **Step 3:** $3 \Rightarrow 1$
  Note that for any measurable subset A of E we have $|\nu|(A) = 0$. We also have:

$$|\nu(A)| = |\nu^+(A) - \nu^-(A)| \leq \nu^+(A) + \nu^-(A) = |\nu|(A) = 0$$

$$\Rightarrow \nu(A) = 0$$

  Therefore, E is null in ν

Solution part-2(Proof of lemma 8):
The Steps are based on lemma 8.

- **Step 1:** $1 \Rightarrow 2$
  Let P, N be the decomposition of N in positive and negetive sets using HJD[Hahn Jordan Decomposition]. Let A, B be the disjoint decomposition of X for ν and μ. Then it is easy to check every element lies in one of the four sets: $A \cap P, A \cap N, B \cap P, B \cap N$. Now note,
  - Decomposition of X for ν^+ and μ is achived by $A \cap P$ and $(A \cap N) \cup (B \cap P) \cup (B \cap N)$
  - Decomposition of X for ν^- and μ is achived by $A \cap N$ and $(A \cap P) \cup (B \cap P) \cup (B \cap N)$

- **Step 2:** $2 \Rightarrow 3$
  Let decomposition of X for ν^+ and μ be E_1, F_1 and for ν^- and μ be E_2, F_2. We calim the decomposition of X for $|\nu|$ and μ is given by $E_1 \cup E_2$ and $F_1 \cap F2$.[8] Note that $|\nu|$ is null in $F_1 \cap F_2$ both ν^+ and ν^- is null in F_1, F_2. μ is null in both E_1 and E_2. By lemma 3, part 2, μ is null in $E_1 \cup E_2$.

[8] This is low-key motivated by the decomposition in step 1.

- **Step 3:** $3 \Rightarrow 1$
  Let A, B be the disjoint decomposition of X for $|\nu|$ and μ. We claim this is the appropiate decomposition for ν and μ as well. It is already known μ is null in A. As $|\nu$ is null in B, ν is null in B follows from lemma 7($3 \Rightarrow 1$).

Problem 3.3

Let ν be a signed measure on $(X, \mathcal{M})$. Prove

1. $\mathcal{L}^1(\nu) = \mathcal{L}^1(|\nu|)$
2. If $f \in \mathcal{L}^1(\nu)$, $|\int f d\nu| \leq \int |f| d|\nu|$
3. If $e \in \mathcal{M}$, $|\nu|(E)| = \sup\{|\int_E f d\nu| : |f| \leq 1\}$

Solution outline: Our main goal will be to study f on the decomposition of X made by HJD due to ν
Solution part-1:
Let X be decomposed into positive set P and negetive set N. We assume $\nu > -\infty$. Let $f \in \mathcal{L}^1(\nu)$. Let χ_E denote the characteristic function on E. Then we have:

$$\int |f| d|\nu| = \int |f|(\chi_P + \chi_N) d(\nu^+ + \nu^-) = \int |f| d\nu^+ + \int |f| d\nu^- < \infty \tag{1.6}$$

Therefore, $f \in \mathcal{L}^1(|\nu|)$.
Now assume $f \in \mathcal{L}^1(|\nu|)$. Then as before,

$$\infty > \int |f| d|\nu| = \int |f|(\chi_P + \chi_N) d(\nu^+ + \nu^-) = \int |f| d\nu^+ + \int |f| d\nu^- \tag{1.7}$$

But as $\nu+, \nu^-$ are both unsigned we can conclude that $\int |f| d\nu^+, \int |f| d\nu^- < \infty$. Therefore, $f \in \mathcal{L}^1(\nu)$
Solution part-2:

$$\begin{aligned}
\left|\int f d\nu\right| &= \left|\int f(\chi_P + \chi_N) d(\nu^+ - \nu^-)\right| \\
&= \left|\int f d\nu^+ - \int f d\nu^-\right| \\
&\leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\
&\leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|
\end{aligned}$$

Solution part-2:
Firt we show that $|\nu|(E)$ is an upper bound and then we show that it is attained. For any measurable f with $|f| \leq 1$ we have:

$$\left|\int_E f d\nu\right| \leq \int_E |f| d|\nu| \leq \int_E d|\nu| = |\nu|(E)$$

Therefore, $|\nu|(E)$ is an upper bound. Now set $f = \chi_P - \chi_N$. For any $x \in X$,either $x \in P$ or $x \in N$. Therefore $f(x) \in \{1, -1\}$.

$$\begin{aligned}
\left|\int f_E d\nu\right| &= \left|\int_E (\chi_P - \chi_N)(\chi_P + \chi_N) d(\nu^+ - \nu^-)\right| \\
&= \left|\int_E (\chi_P^2 - \chi_N^2) d(\nu^+ - \nu^-)\right| \\
&= \left|\int_E \chi_P^2 d\nu^+ + \int_E \chi_N^2 d\nu^-\right| \\
&= \left|\int_E \chi_P d\nu^+ + \int_E \chi_N d\nu^-\right| \\
&= |\nu^+(E \cap P) + \nu^+(E \cap N)| \\
&= |\nu^+(E \cap P) + \nu^+(E \cap N) + \nu^+(E \cap N) + \nu^+(E \cap P)| \\
&= \nu^+(E) + \nu^-(E) = |\nu|(E)
\end{aligned}$$

Bibliography