Counter example to show composition of two covers is not a cover

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1 Introduction

We shall try to prove the following :

Theorem 1 *Let $\phi : Y \to X$ be a covering of X by Y and let $\psi : Z \to Y$ be a covering of Y by Z. Then $\phi \circ \psi : Z \to X$ need not be a covering of X by Z.*

2 Construction of X: The Hawaiian Earring

We set X to be the Hawaiian Earring. We define

$$C_n = \left\{ z : \left| z - \frac{i}{2n} \right| = 1/2n, x \in \mathbb{C} \right\}$$

i.e. C_n is the circle of radius $1/2n$ centered at $i/2n$. Note that $0 \in C_n \forall n \in \mathbb{N}$. The Hawaiian Earring is defined as $H = \bigcup_{n \in \mathbb{N}} C_n$. We give it the subspace topology.

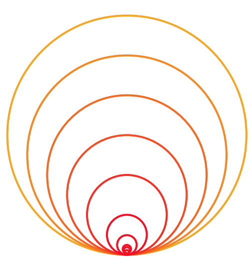

Figure 1: The Hawaiian Earring(X)

Let $x \in X$. It is easy to see that for $x \neq 0$, there exist an open set U_x such that $x \in U_x$ and U_x is homeomorphic to a line segment. We note that this

Figure 2: a. A neighborhood of $x \in X$, $x \neq 0$ which is homeomorphic to a line segment in $\mathbb{R}$. b. An arbitrary neighbourhood of $x = 0$

is not true for $x = 0$. This is because any open set containing 0 contains infinitely many C_n. Therefore, there is no simply connected neighborhood of 0 and there exists no U_0 of the form mentioned above.

3 Construction of Y

Define $\tilde{H} = \bigcup_{i=2}^{n} C_i$. $\tilde{H}$ is like the Hawaiian earring with the outermost shell removed. We define $Y = \left(\bigcup_{z \in \mathbb{Z}} (z + \tilde{H}) \right) \cup \mathbb{R}$. We give Y the subspace topology.

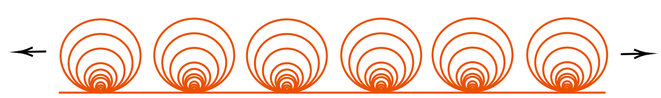

Figure 3: Visual represntation of Y.

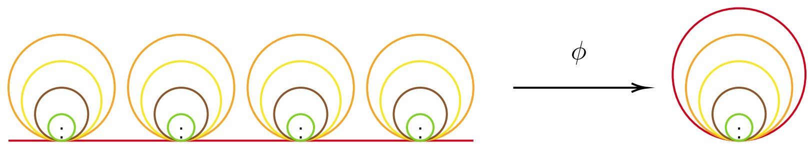

Figure 4: Visual representation of ϕ. A point and it's image has the same colour.

3.1 Y as a cover of X

We define $\phi : Y \to X$ as below:

$$\phi(\alpha) = \begin{cases} i + e^{2\pi i\alpha - i\pi/2} & \text{if } \alpha \in \mathbb{R} \\ h & \text{if } \alpha = z + h \text{ and} \\ & h \in \tilde{H}, z \in \mathbb{Z} \end{cases}$$

We note that ϕ is a covering map. Points in the outer shell have a neighbourhood which is evenly covered by $\mathbb{R}$. All the other points have a neighborhood which is evenly covered by their copies in $\tilde{H}$ at each $z \in \mathbb{Z}$.

4 Construction of Z

Set $S_n = \{z : |z - 1/2n| = 1/2n, arg(z - 1/2) \in [-\pi/2, \pi/2]\}$ We define Z as follows:

$$\begin{aligned} Z = \bigcup_{z\in\mathbb{Z}} (&\{x : Re(x) \in [z - 1/2, z + 1/2], Im(x) = 0\} \\ &\cup \{x : Re(x) \in [z - 1/2, z + 1/2], Im(x) = 2\} \\ &\cup \left(\bigcup_{n>|z|} (C_n + z) \right) \cup \left(\bigcup_{n>|z|} (2i - C_n + z) \right) \\ &\cup \left(\bigcup_{n=1}^{|z|} (S_n + z) \right) \cup \left(\bigcup_{n=1}^{|z|} (2i - S_n + z) \right) \\ &\cup \left(\bigcup_{n=1}^{|z|} \{x : Re(x) = z - 1/2n, \right. \\ &\left. Im(x) \in [1/2n, 2 - 1/2n]\} \right) \\ &\cup \left(\bigcup_{n=1}^{|z|} \{x : Re(x) = z + 1/2n, \right. \\ &\left. Im(x) \in [1/2n, 2 - 1/2n]\} \right)) \end{aligned}$$

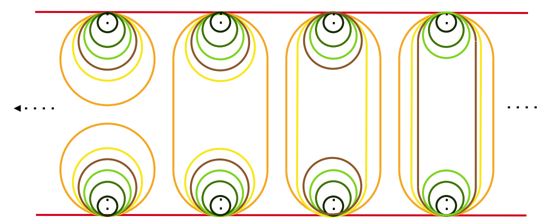

Figure 5: Visual representation of Z

Refer to figure 5. Informally speaking, the first two lines of the description are the two lines above and below. The third line describes the loops. The fourth line describes the semicircles. The fifth to eighth line describes the stright lines.

4.1 Z as a cover of Y

We define $\psi : Z \to Y$ as below

$$\psi(\alpha) = \begin{cases} i + e^{2\pi i\alpha - i\pi/2} & \text{if } Im(\alpha) = 0 \\ i + e^{2\pi i(\alpha - 2i) - i\pi/2} & \text{if } Im(\alpha) = 2 \\ \alpha - z & \text{if } \alpha = z + C_n \text{ or } \alpha = z + S_n \\ -(\alpha - z - 2i) & \text{if } \alpha = z + 2i - C_n \\ & \text{or } \alpha = z + 2i - S_n \\ e^{\pi i\left(\frac{Im(\alpha) - 1/2n}{(2-2/n)}\right)}/2n + i/2n + z & \text{if } \alpha \text{ is in the vertical} \\ & \text{line between } z \text{ and } z + 1/2 \\ e^{\pi i\left(1 - \left(\frac{Im(\alpha) - 1/2n}{(2-2/n)}\right)\right)}/2n + i/2n + z & \text{if } \alpha \text{ is in the vertical} \\ & \text{line between } z \text{ and } z + 1/2 \end{cases}$$

We can check on a case by case basis that ψ evenly covers a neighborhood around each point in Y and is therefore a covering map.

5 Proving $\phi \circ \psi$ is not a covering map

Take any small neighbourhood U around 0. Note that we can always find some C_n such that $C_n \subseteq U$.

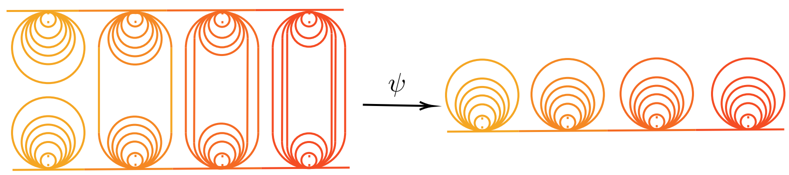

Figure 6: Visual representation of ψ. A point and it's image has the same colour.

We look at preimage of U in Z around $Z_n = \{z : z \in Z \text{ and } Re(z) \in [n - 1/2, n + 1/2]\}$. Assume the pre-image is homeomorphic to U under the map $f : U \to Z_n, \phi \circ \psi \circ f = id$. As we discussed before, only 0 does not have a neighbourhood homeomorphic to a line segment. But the pre-image of U in Z_n contains two points$(z, z+2i)$ which doesn't have a neighbourhood homeomorphic to a line segment contradicting our assumption that f is a homeomorphism. Moreover, we can't split $Z_n \cap f(U)$ in two open disjoint open sets as it is pathconnected. Therefore, no neighbourhood of 0 is evenly covered and $\phi \circ \psi$ is not a covering map.

Figure 7: $Z_n \cap f^{-1}(U)$. Note it is path connected and has two points $(z, z+2i)$ where there is no neighbourhood which is homeomorphic to a line segment in $\mathbb{R}$. b. How U might have looked like.