Topology, Munkres
Chapter 3

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1 Exercise set-1

Problem 1.1. *Let τ and τ' be two topologies on X. If $\tau \subset \tau'$, what does connectedness of X in one topology imply about connectedness in the other?*

Solution : Let X be connected in τ'. Then we claim that X is connected in τ. For if U, V be a separation in τ such that $X = U \cup V$ with $U \cap V = \phi$ then U, V is a separation of X in τ' too, leading to a contradiction.
We note the converse is not true. Let $X = \{a, b, c\}$ and $\tau = \{\{a, b\}, \{b, c\}, \{b\}, \{a, b, c\}, \{a, b, c\}\}$. It is easy to check X is not separable in τ. Set τ' to be the discrete topology.It follows that $\tau \subset \tau'$ and a separation of X exists in τ'.

Problem 1.2. *Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} = \phi$ for all n. Show that $\bigcup A_n$ is connected.*

Solution : Let U, V be non-empty open sets such that $\bigcup A_n = U \cup V$ and $U \cap V = \phi$. Note that each A_i lies entirely in U or V (i.e $A_i \subset U$ or $A_i \subset V$) else $A_i \cap U$ and $A_i \cap V$ will be a separation of A_i leading to contradiction. Without loss of generality we can assume A_1 lies in U. As V is non empty, by the well ordering principle, there exist a minimum $n > 1$ such that $A_n \subset V$. But as $p = A_n \cap A_{n-1} \in V$, $A_{n-1} \subset V$ which contradicts the minimality of n.

Problem 1.3. *Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cup A_\alpha \neq \phi$ for all α, then $A \cap (\bigcup A_\alpha)$ is connected.*

Solution : Let U, V be non-empty open sets such that $A \cup (\bigcup A_\alpha) = U \cup V$ and $U \cap V = \phi$. As we discussed in $P2$, each A_α and A is entirely in U or V. Without loss of generality assume $A \subset U$. For each A_α, there exist $p_\alpha \in A \cap A_\alpha$. As $p_\alpha \in U$, $A_\alpha \subset U$ for all α. Therefore, $A \cup (\bigcup A_\alpha) \subset U$. As $U \subset A \cup (\bigcup A_\alpha)$ too, $A \cup (\bigcup A_\alpha) = U$. Therefore, V is empty, leading to contradiction.

Problem 1.4. *Show that if X is an infinite set, it is connected in the finite complement topology.*

Solution : Let U, V be non-empty open sets such that $X = U \cup V$ and $U \cap V = \phi$. As X is infinite, either U or V is infinite (else $U \cup V$ will be finite, leading to contradiction). Without loss of generality assume U is infinite. As V is open $V^c = U$ is finite, leading to contradiction.

Problem 1.5. *A space is* ***totally disconnected*** *if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?*

Let $A \subset X$, have more than 1 point. Let $a \in A$. Then $\{a\}$ and $A \setminus \{a\}$ is a separation of A. Therefore, X is totally disconnected in discrete topology.
We define a topology on $\mathbb{Z}^+$ given by $\tau = \{A_n | A_n = \{n, n+1, n+2 \ldots\}\} \cup \{\phi\}$. It is easy to check this is indeed a topology:

1. $\mathbb{Z}^+ = A_1 \in \tau, \phi \in \tau$

2. Note, $A_i \cup A_j = A_j$ if $i \geq j$. Let $\{A_{\alpha_i}\}$ be a collection of open subsets where each $\alpha^+_{\mathbb{Z}}$. Then by the well ordering principle, there exist a minimum $\alpha_i = \alpha$. Then: $\bigcup A_{\alpha_i} = A_\alpha \in \tau$.

3. Note, $A_i \cap A_j = A_i$ if $i \geq j$. Therefore for a finite subcollection of sets of τ, $A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3} \dots A_{\alpha_n}$ we have: $\bigcap A_{\alpha_i} = A_\alpha \in \tau$ where $\alpha = \max(\alpha_i)$.

Therefore, τ is a topology. Let $X \subset \mathbb{Z}^+$ with more than one point. Let U, V be a separation of X. Then we can write $U = A_u \cap X$, $V = A_v \cap X$ for some A_u, A_v. Without loss of generality, assume $u > v$. Then $A_v \subset A_u$ and $U \subset V$ leading to contradiction. Therefore, $(\mathbb{Z}^+, \tau)$ is totally disconnected where τ is not the discrete topology. Therefore, the converse doesn't hold.

Problem 1.6. *Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects $Bd(A)$.*

Solution : Assume C doesn't intersects $\text{Bd}(A)$. Note we can interpret $\text{Bd}(A){=}\overline{A}\setminus\text{Int}\,A$. It follows $A\setminus\text{Bd}\,A = \text{Int}\,A$ is open and $\text{Int}(X\setminus A) = X\setminus\overline{A} = (X\setminus A)\setminus\text{Bd}(A)$ is open. Therefore $C\cap X = C\cap\text{Int}\,X$ and $C\cap(X\setminus A) = C\cap\text{Int}(X\setminus A)$ is a separation of C, leading to contradiction.

Problem 1.7. *Is the space $\mathbb{R}_l$ connected? Justify your answer.*

Solution : No. $(-\infty, 0) = \bigcup_{n\in\mathbb{N}}[-n, 0)$ and $[0, \infty) = \bigcup_{n\in\mathbb{N}}[0, n)$ is a separation.

Problem 1.8. *Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.*

Problem 1.9. *Let A be a proper subset of X, and let B be a proper subset of Y . If X and Y are connected, show that*

$$(X \times Y) - (A \times B)$$

is connected.

Solution : First note that:

$$(X \times Y) - (A \times B) = (X \times B^c) \cup (A^c \times Y)$$

Note, $X \times \{y\}$,$y \in B^c$ is connected(Else if U_1, V_1 is a separation of $X \times \{y\}$ then $\pi_1(U_1)$ and $\pi_1(V_1)$ is a separation of X). Similarly, $\{x\} \times Y, x \in A^c$ is connected. Let U, V is a separation. Let $(x, y) \in U$ where $(x, y) \in X \times B^c$. Then $X \times \{y\} \subset U$.

- **Case 1:** $(x_1, y_1) \in A^c \times Y$. As $X \times \{y\} \subset U$, $(x_1, y) \in U$. Therefore, $\{x_1\} \times Y \subset U$ and $(x_1, y_1) \in U$.
- **Case 2:** $(x_2, y_2) \in X \times B^c$. Choose some $x_1 \in A^c$. Then $(x_1, y_2) \in A^c \times Y$. So $(x_1, y_2) \in U$ and $X \times \{y_2\} \subset U$. Therefore, $(x_1, y_2) \in U$.

Therefore, V is empty leading to a contradiction. Therefore, the set is connected.

Note: The set looks like inter crossed lines. We first show each line is connected and then show any two line lie on the same set, U.

Problem 1.10. *Let $\{X_\alpha\}_\alpha \in J$ be an indexed family of connected spaces; let X be the product space*

$$X = \prod_{\alpha\in J} X_\alpha$$

Let $a = (a_0)$ be a fixed point of X.

1. *Given any finite subset K of J , let X_K denote the subspace of X consisting of all points $x = (x_\alpha)$ such that $x_\alpha = a_\alpha$ for $\alpha \notin K$. Show that X_K is connected*
2. *Show that the union Y of the spaces X_K is connected.*
3. *Show that X equals the closure of Y ; conclude that X is connected.*

Solutions :

1.

> **Problem 1.11.** *Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(y)$ is connected, and if Y is connected, then X is connected.*

Solution : Let be the equivalence relation such that $p(x) = p(y)$ and let $[x]$ be the equivalence class of x under . Assume U, V is a separation of X. Let $x \in U$.

Lemma 1. *For $x \in U$, $[x] \subset U$*

Proof. If this is not so then $U \cap [x]$ and $V \cap [x]$ will be a separation of $p^{-1}(x)$ which leads to a contradiction ☐

Lemma 2. $U = p^{-1}(p(U))$,

Proof. Let this be not true. Then there exist $x \notin U$ such that $x \in p^{-1}(p(U))$. Therefore, $p(x) \in U$. Let $p(x) = p(y)$ where $y \in U$. Then $[y] \notin U$ leading to a contradiction. ☐

Similarly we get $V = p^{-1}(p(V))$ Therefore we note:

1. $p(U)$ and $p(v)$ are non empty and open as U, V is non-empty and open.
2. $p(U)$ and $p(V)$ are disjoint. Else if $r \in p(U) \cap p(V)$ then there exists $u \in U$ and $v \in V$ such that $p(u) = p(v)$ and thus $[u] \not\subset U$ leading to contradiction.

But this implies $p(U), p(V)$ is a separation of Y leading to contradiction.

> **Problem 1.12.** *Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X \setminus Y$, then $Y \cup A$ and $Y \cup B$ are connected.*

Solution :

2 Exercise Set-2

> **Problem 2.1.** 1. *Show that no two of the spaces $(0, 1), (0, 1]$ and $[0, 1]$ are homeomorphic. [Hint: What happens if you remove a point from each of these spaces?)]*
>
> 2. *Suppose that there exist imbeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.*
>
> 3. *Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.*

Solution :

1. Let $[0, 1)$ and $(0, 1)$ be homeomorphic with φ as the homeomorphism. Note $[0, 1) - \{0\} = (0, 1)$ is connected. Therefore $(0, 1) - \{\varphi(0)\}$ is connected (as continuous image of connected spaces are connected). But it is not true as $(0, \varphi(0))$ and $(\varphi(0), 1)$ is a separation which is a contradiction. The same logic can be used to show $[0, 1]$ and $(0, 1)$ is not homeomorphic.
   Let $[0, 1)$ and $[0, 1]$ be homeomorphic with ψ as the homeomorphism. As before, we note $[0, 1) - \{\psi(0), \psi(1)\}$ is connected as $[0, 1] - \{0, 1\}$ is connected. We consider two cases:

   - Either $\psi(0)$ or $\psi(1)$ is 0: Without loss of generality, assume that $\psi(0) = 0$. Then $(0 = \psi(0), \psi(0))$ and $(\psi(1), 1)$ is a separation of $[0, 1] - \{\psi(0), \psi(1)\} = (0, 1] - \{\psi(1)\}$
   - $\psi(0), \psi(1) \neq 0$: Without loss of generality, assume that $\psi(0) < \psi(1)$[As ψ is homeomorphism, bijection is not possible]. Then $[0, \psi(0)) \cup (\psi(0), \psi(1))$ and $(\psi(1), 1)$ is a a separation of $[0, 1] - \{\psi(0), \psi(1)\}$

   In either case as the image set is disconnected, ψ can't be homeomorphism, leading to a contradiction.

2. Set $X = (0,1)$ and $Y = (0,1]$. Set $f : X \to Y, f(x) = x$ and $g : Y \to X, g(y) = \frac{y}{2}$. It is easy to check they are embedding and non-homeomrphism follows from above.

3. Let φ be the homeomorhism. Note $\mathbb{R}^n - \{0\}$ is path connected, Therefore, $\varphi(\mathbb{R}^n - \{0\}) = \mathbb{R} - \{\varphi(0)\}$ is path connected which is a contradiction.

Problem 2.2. *Let $f : S^1 \to \mathbb{R}$ be a continuous map. Show there exists a point x of S^1 such that $f(x) = f(-x)$.*

Solution : Set $h : S^1 \to \mathbb{R}, h(x) = f(x) - f(-x)$. Take any $x \in S^1$. If $f(x) = f(-x)$ then we are done. If $f(x) > f(-x)$ set $x' = x$, if $f(-x) > f(x)$ set $x' = -x$. Note $f(x') > f(-x')$. Then $h(x') > 0$ and $h(-x') < 0$. As S^1 is (path) connected and $\mathbb{R}$ is connected, by the intermediate value theorem, there exists x_0 such that $h(x_0) = 0 \Rightarrow f(x_0) = f(-x_0)$.

Problem 2.3. *Let $f : X \to X$ be continuous. Show that if $X = [0,1]$, there is a point x such that $f(x) = x$. The point x is called a fixed point of f . What happens if X equals $[0,1)$ or $(0,1)$?*

Solution : Set $h(x) = f(x) - x$. If $f(1) = 1$ or $f(0) = 0$ then we are done. Else $1 > f(1)$, $h(1) < 0$ and $0 < f(0)$, $h(0) > 0$. Then by intermediate value theorem there exists x_0 such that $h(x_0) = 0$ and $f(x_0) = x_0$.
Let $f(x) = \frac{1+x}{2}$. Then there is no fixed point for $X = [0,1)$ and $X = (0,1)$.

Problem 2.4. *Let X be an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.*

Solution :

Lemma 3. *If $x < y$, there exists z such that $x < z < y$.*

Proof. Let no such z exist. Then $U = \{p|p \in X, p > x\}$ and $V = \{p|p \in X, p < y\}$ form a separation of X, leading to a contradiction.[If such a z exists, then the proof breaks down as $U \cap V$ at least has z in it and is therefore non-empty]. □

Lemma 4. *X has the least upper bound property.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a set of elements bounded above without a least upper bound. Let $B = \{b|b \in X, b$ is a upper bound of $\{a_n\}\}$.Then $U = \bigcup_{n\in\mathbb{N}}\{p|p \in X, p < a_n\}$ and $V = \bigcup_{b\in B}\{p|p \in X, p > b\}$ is a separation of X: They are clearly disjoint and every $x \in X$ is either an upper bound or less than a_n for some n. □

The result follows.

Problem 2.5. *Consider the following sets in the dictionary order. Which are linear continua?*

1. $Z^+ \times [0,1)$

2. $[0,1) \times Z^+$

3. $[0,1) \times [0,1]$

4. $[0,1] \times [0,1)$

Solution :

1. Yes. It is obvious that between any two elements we can get another element. Let $\{a_n\}$ be a sequence of elements bounded above. Let $z = \max(\pi_1(a_n))$ Let $S = \{a_n|\pi_1(a_n) = z\}$.
   - **Case 1:** $r = \sup\{\pi_2(a_n)|a_n \in S\} < 1$. Then (z,r) is least upper bound
   - **Case 2:** $r = 1$. Then $(z+1, 0)$ is the upper bound.

2. No. There are no element between (p,z) and $(p,z+1)$ where $p \in [0,1)$ and $z \in \mathbb{Z}^+$

3. Yes.

4. Yes.

3,4 is to be done in the same way as that of a ordered square.

Problem 2.6. *Show that if X is a well-ordered set, then $X \times [0,1)$ in the dictionary order is a linear continuum.*

A well-order (or well-ordering or well-order relation) on a set S is a total order on S with the property that every non-empty subset of S has a least element in this ordering.

Solution : Let (x_1, r_1) and (x_2, r_2) be two elements of $X \times [0,1)$. Let $(x_1, r_1) > (x_2, r_2)$. If $x_1 > x_2$, set $x = x_1, y = \frac{1+r_1}{2}$, If $x_1 = x_2$, set $x = x_1, y = \frac{r_1+r_2}{2}$. Then $(x_1, r_1) > (x, y) > (x_2, r_2)$. Let $(x_i, y_i)_{i \in \mathbb{N}}$ be a bounded sequence of elements in $X \times [0,1)$. In particular, let (a, b) be an upper bound. Let $L_x = \{\pi_1(p) | p \text{ is an upper bound of the sequence}\}$ Then by well ordering, a minimum $x \in L_x$ exists.

- **Case 1:** No element of the form (x, p) exists: Then $(x, 0)$ is the least upper bound.
- **Case 2:**Some element of the form (x, p) exists. Set $L_y = \{\pi_2(p) | p = (x, y) \text{ is in the sequence}\}$. If $\sup L_y \neq 1$ then $(x, \sup L_y)$ is the desired upper bound. Else if $\sup L_y = 1$ set $S = \{a | a > x\}$ and let s be the least element in S. Then $(s, 0)$ is the least upper bound.