Outer Measure and Related Things

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1 Outer Measure

For a given X, a outer measure $\mu* : \mathcal{P}(X) \to (0, \infty]$ is defined to be function with the following properties:

1. $\mu^*(\phi) = 0$
2. If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$
3. $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)\ \forall A_i \in \mathcal{P}(X)$.

Example: Let $\epsilon \subseteq \mathcal{P}(X)$ and $f : \epsilon : [0, \infty)$ such that:

1. $\phi, X \in \epsilon$
2. $f(\phi) = 0$

Then for a subset $A \subseteq X$ define:

$$\mu^*(A) = \inf \left\{ \left(\sum_{i=0}^{\infty} \mu^*(A_i) \right) | A \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$$

We show that μ^* defined above is an outer measure.

1. $\mu^*(\phi) = 0$ as $\phi \subseteq \phi$
2. Let $A \subseteq B$. As all covers of B are also covers of A, $\mu^*(A) < \mu^*(B)$.
3. Let $\cup_{j=1}^{\infty} E_i^j$ be a cover of A_i in ϵ. Then, $\bigcup_{i,j} E_i^j$ covers A. Therefore, $\mu * \left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)\ \forall A_i \in \mathcal{P}(X)$

In particular, we can set $X = \mathbb{R}$, $\epsilon = \{(a, b) | a, b \in [-\infty, \infty]\}$, and $f(I) =$ length if interval I.

Definition 1 (μ^* measurable sets)**.** *A set $A \subseteq X$ is called μ^* measurable if for all $E \in \mathcal{P}(X)$ we have:*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Note, as $E \cap A$ and $E \cap A^c$ covers A, then it is often enought to prove $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The other way follows trivially from the definition of μ^*

2 Caratheodory's Extension Theorem

THis theorem is used to generates mesure from outer measures.

Definition 2 (Complete Measure)**.** *Let μ be a measure. Let A have size zero and $B \subseteq A$. μ is a complete measure if B is measurable and $\mu(B) = 0$.*

Theorem 3. *Let X be a non empty set with outer measure μ^*. Let $\mathcal{M}$ be the set of all μ^* measurable subset of X. Let $\mu = \mu^*|_{\mathcal{M}}$. Then $(X, \mathcal{M}, \mu)$ forms a complete measure.*

Proof. Put $A = \phi$ and $A = \phi$ in place of A in the statement of μ^* measurable function to get $X, \phi \in \mathcal{M}$. If $A \in \mathcal{M}$ then $\forall E \in \mathcal{P}(X)$, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) + \mu^*(E \cap (A^c)^c)$. Therefore $A^c \in \mathcal{M}$.
For showing $\mathcal{M}$ is closed under taking countable union, we shall first prove it for the finite case and then take a limit under apropiate conditions.

Showing $\mathcal{M}$ is closed under finite union: We just need to show that it holds for union of two sets and then the proof for the union of any arbitary $n < \infty$ set follows from induction.

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \tag{1}$$
$$= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \tag{2}$$

We go from (1) to (2) by using the μ^* measurability of B for $E \cap A$ and $E \cap A^c$. We know: $A \cup B = (A \cap B) \cup (A^c \cap B) \cup (A \cap B^c)$. Therefore:

$$\mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \tag{3}$$
$$=\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap A^c \cap B^c) \tag{4}$$
$$=\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \tag{5}$$

As we have mentioned before, this is enough to conclude $A \cup B$ is μ^* measurable. Note:

$$\mu^*(A \cup B) = \mu^*(A \cup B \cap A) + \mu^*(A \cup B \cap A^c) = \mu^*(A) + \mu^*(B) \tag{6}$$

As μ^* and μ agree on measurable sets, the result for finite unions is proved.
Now we sahll attempt the case for the union of infinite number of sets. Define:

$$B_n = \bigcup_{i=1}^{n} A_i \quad B = \bigcup_{i=1}^{\infty} A_i$$

where each $A_i \in \mathcal{M}$. Therefore:

$$\mu^*(E \cap B_n) = \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \tag{7}$$

As this is a recurrence relation we get:

$$\mu^*(E \cap B_n) = \sum_{i=1}^{n} \mu^*(E \cap A_n) \tag{8}$$

Therefore:

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \tag{9}$$
$$\geq \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap B_n^c) \tag{10}$$

Now we take the limit $n \to \infty$ to get:

$$\mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B_n^c) \tag{11}$$
$$\geq \mu^*(E \cap \bigcup_{i=1}^{\infty} (A_i) + \mu^*(E \cap B_n^c) \tag{12}$$
$$\geq \mu^*(E \cap B) + \mu^*(E \cap B^c) \tag{13}$$

Therefore, $\mathcal{M}$ is closed under countable unions. Take $B = E$ in (11) to get $\mu^*(B) \geq \sum_{i=1}^{\infty} \mu^*(B \cap A_i) = \sum_{i=1}^{\infty} \mu^*(A_i)$. As by definition, $\mu^*(B) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$, equality holds. Therefore, μ is a measure. Now we need to show this measure is complete. Let $\mu^*(A) = 0$ and $B \subseteq A$. Then:

$$\mu^*(E) \geq \mu^*(A \cap E) + \mu^*(A^c \cap E) \geq \mu^*(B \cap E) + \mu^*(B^c \cap E) \geq \mu^*(E)$$

Therefore, equality must hold and $\mu^*(B)$ is measurable. This completes the proof. $\square$

3 Leabesgue Measure

We perform Caratheodory extension on the outer measure on $\mathbb{R}$ defined in the example. The measure we get is known as the leabesgue measure and is written as $(\mathbb{R}, \mathcal{M}_1, m_i)$. The notion can be generalised for higher dimensions where we take $(a_1, b_1) \times (a_2, b_2) \ldots$ to be elements of ϵ and $f((a_1, b_1) \times (a_2, b_2))$ =Volume/Area of the figure.