Aakash Ghosh
Email: ag19ms129@iiserkol.ac.in | ID: 19MS129
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1 Hilbert Space

Problem 7.1
Let V be a real Banach space and assume that the parallelogram identity holds in V. Define

$$< u, v >= 1/4(||u+v||^2 - ||u-v||^2)$$

Show that this defines an inner product which induces the given norm and hence that V is a Hilbert space.

Solution: This is the Von Neumman-Fréchet theorem.

1. $\langle x, y\rangle = \langle y, x\rangle$ follows from properties of norm.
2. Let $x, y, z \in V$. Then we have:
$$||x+y+z|| + ||x-y+z|| = 2||x+z||^2 + 2||y||^2$$
$$||x+y+z|| + ||-x+y+z|| = 2||y+z||^2 + 2||x||^2$$

   Sum and half it to get:
$$||x+y+z|| = ||x+z|| + ||y+z|| + ||x|| + ||y|| - (1/2)||x-y+z|| - (1/2)||-x+y+z||$$
   Replace z by $-z$ to get:
$$||x+y-z|| = ||x+z|| + ||y+z|| + ||x|| + ||y|| - (1/2)||x-y-z|| - (1/2)||-x+y-z||$$
   Note that $||x-y-z|| = ||-x+y+z||, ||-x+y-z|| = ||x-y+z||$. Subtract the last two equations to get:
$$\langle x+y, z\rangle = ||x+y+z|| - ||x+y-z|| = ||x+z|| + ||y+z|| - ||x+z|| - ||y+z|| = \langle x, z\rangle + \langle y, z\rangle$$
3. Set $y = x$ to get $\langle 2x, z\rangle = 2\langle x, z\rangle$. Note that $\langle 0, z\rangle = 0$. Set $y = -x$ to get $\langle -x, z\rangle = -\langle x, z\rangle$. Extend by induction to get $\langle nx, z\rangle = n\langle x, z\rangle$ for $n \in \mathbb{Z}$.
4. For some rational p/q where $p, q \in \mathbb{Z}, q \neq 0$, we note $\langle (p/q)x, z\rangle = \langle (p/q)x, (q/q)z\rangle = pq\langle x/q, z/q\rangle = (pq)/q^2\langle x, z\rangle = (p/q)\langle x, z\rangle$.[The fact that $\langle x/q, z/q\rangle = 1/q^2\langle x, z\rangle$ follows from direct computation.]
5. Note that $||\cdot||$ is continuous function. So for any real number $r \in \mathbb{R}$ we have a sequence $\{q_n\}, q_n \in \mathbb{N}$ such that $q_n \to r$ and so $\langle rx, z\rangle = \lim_{n\to\infty}\langle q_n x, z\rangle = \lim_{n\to\infty} q_n\langle x, z\rangle = r\langle x, z\rangle$
6. It follows trivially that $\langle x, x\rangle = (1/4)||2x||^2 = ||x||^2 \Rightarrow ||x|| = \sqrt{\langle x, x\rangle}$.

Problem 7.4
Let H be a Hilbert space and let M be a non-zero and proper closed subspace of H. Let $P : H \to M$ be the orthogonal projection of H onto M. Show that $||P|| = 1$.

Solution: For $m \in M$, we have: $P(m) = m$. Set $\tilde{m} = \frac{1}{||m||}m$ to get $P(\tilde{m}) = \tilde{m}$. As $||\tilde{m}|| = 1$ it follows that $||P|| \geq 1$. For any $v \in H$, we have: $v = v_M + v_{M^\perp}$. Therefore,

$$\langle P(v), P(v)\rangle = \langle v_M, v_M\rangle \leq \langle v_M, v_M\rangle + \langle v_{M^\perp}, v_M\rangle + \langle v_M, v_{M^\perp}\rangle + \langle v_{M^\perp}, v_{M^\perp}\rangle$$
$$\Rightarrow ||P(v)||^2 \leq ||v||^2$$
$$\Rightarrow ||P|| \leq 1$$

Therefore, $||P|| = 1$.

Problem 7.5
Let $H = l_2^n$. Let J be the $n \times n$ matrix all of whose elements are $1/n$. Show that

$$||J||_{2,n} = ||I - J||_{2,n} = 1$$

Solution: Consider a vector given by $v = (v_1, v_2 \dots v_n)$. Set $s = \sum_{i=1}^n v_n$. Then $J(v) = (s/n, s/n \dots s/n)$. We note that:

$$\begin{aligned} ||J(v)||^2 &= \sum_{i=1}^n s^2/n^2 \\ &= s^2/n \\ &= \left(\sum_{i=1}^n v_i\right)^2 /n \leq \left(\sum_{i=1}^n v_i^2\right) = ||v||^2 \end{aligned}$$

Therefore, $||J|| \leq 1$. But we note that for $v = (1, 1, \dots 1)$, $J(v) = v$. Therefore, $||J|| \geq 1$. So $||J|| = 1$. Note that $(I - J)(v) = (v_1 - s, v_2 - s \dots v_i - s \dots v_n - s)$. Therefore,

$$\begin{aligned} ||J(v)||^2 &= \sum_{i=1}^n (v_i - s/n)^2 \\ &= \sum_{i=1}^n v_i^2 - 2(s/n) \sum_{i=1}^n v_i + s^2/n \\ &= ||v|| - 2(s/n)(s) + s^2/n \\ &= ||v|| - s^2/n \leq ||v|| \end{aligned}$$

Therefore, $||J|| \leq 1$. Set $v = (1, -1, 0, 0 \dots)$. Note that $s - 0$ in this case and so $||J(v)|| = ||v||$. So $||J|| \geq 1$. It follows that $||J|| = 1$.

Problem 7.6
Show that the following matrix describes an orthogonal projection in l_2^3. Find the range of the projection.

$$T = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Solution: We note that $e_1 = \frac{1}{\sqrt{2}}(1, -1, 0)$ and $e_2 = \frac{1}{\sqrt{2}}(1, 0, -1)$ are eigen vectors with eigen values 1. Moreover, we note that $T(1, 1, 1) = 0$. Set $e_3 = (1, 1, 1)$. We note that e_3 is orthogonal to $span\langle e_1, e_2\rangle$ and $v = \sum_{i=1}^3 c_i e_i = \sum_{i=1}^2 c_i e_i$. So $T^2 = T$ and thus T is projection on $span\langle e_1, e_2\rangle$. It follows that range of T is $span\langle e_1, e_2\rangle$ as well.

Problem 7.7
Let

$$K = \{(a, b) \in \mathbb{R}^2 | a, b \geq 0\}$$

If $a \leq 0$ and $b \leq 0$, calculate $P_K(z)$ where $z = (a, b)$.

Solution: Let $\alpha = (x, y) \in K$. Then $\langle z, \alpha\rangle = \langle (a, b), (x, y)\rangle = ax + by$. Now as $a, b \leq 0$ and $x, y \geq 0$, we have $\langle z, \alpha\rangle \leq 0$. This can be written as $\langle z - 0, \alpha - 0\rangle \leq 0$. As K is closed and convex, it follows that $P_K(z) = 0$

Problem 7.8
Let H be a real Hilbert space and let K be a closed convex cone in H with vertex at the origin.

1. If $x, y \in K$ and if α and β are non-negative scalars, show that $\alpha x + \beta y \in K$.

2. If $x \in H$, show that $P_K(x)$ is characterized by the following relations:

$$(x, P_K(x)) = ||P_K(x)||^2 \text{ and } (x - P_K(x), y) = 0 \forall y \in K$$

Solution:

Definition (Closed convex cone with tip at origin)**.** *A cone is a subset of a vector space with the property that if $x \in V$ then $cx \in V$ for all $c \in \mathbb{R}^+$.*

1. If $x, y \in K$ then $(\alpha+\beta)x, (\alpha+\beta)y \in K$. It follows that $\frac{\alpha}{\alpha+\beta}(\alpha+\beta)x + \frac{\beta}{\alpha+\beta}(\alpha+\beta)y = \alpha x + \beta y \in K$.

2. The statement is false. Ref: 7.7. Note that K is closed convex cone and the second condition is not fullfilled.

Problem 7.9
Let $H = L^2(-\pi, \pi)$. Write down explicitly the orthogonal projection of each of the following closed subspaces

1. $M = \{f \in H | f(t) = f(-t) \text{ for every } t \in (-\pi, \pi)\}$
2. $M = \{f \in H | \int_{-\pi}^{\pi} f(t)dt = 0\}$
3. $M = \{f \in H | f \equiv 0 \text{ on } (-\pi, 0)\}$

Solution:

1. We claim $P_M(f) = \frac{f(x)+f(-x)}{2}$. Let $g \in M$. Then note that $g - P_M(f)$ is even. Note that $(f - P_M(f))(x) = \frac{f(x)-f(-x)}{2}$ is odd. So $(g - P_M(f))(f - P_M(f))$ is odd and thus

$$\langle f - P_M(f), g - P_M(f) \rangle = \int_{-\pi}^{\pi} (g - P_M(f))(f - P_M(f)) = 0$$

So the claimed form of $P_M(f)$ is correct.

2. We claim $P_M(f)(x) = f(x) - \frac{\int_{-\pi}^{\pi} f(t)dt}{2\pi}$. Let $g \in M$. Set $h = g - P_M(f)$. Note that $h \in M$. Then note that:

$$\langle f - P_M(f), h \rangle = \int_{-\pi}^{\pi} \left[\frac{\int_{-\pi}^{\pi} f(t)dt}{2\pi} \right] h(x)dx = \left[\frac{\int_{-\pi}^{\pi} f(t)dt}{2\pi} \right] \int_{-\pi}^{\pi} h(x)dx = 0$$

So the claimed form of $P_M(f)$ is correct.

3. Define:

$$f_1(t) = \begin{cases} f(t) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad f_2(t) = \begin{cases} 0 & \text{if } t < 0 \\ f(t) & \text{if } t \geq 0 \end{cases}$$

It follows trivially that $f = f_1 + f_2$. We claim $P_M(f)(x) = f_2(x)$. Let $g \in M$. Set $h = g - P_M(f)$. Note that $h \in M$. It follows that:

$$\langle f - P_M(f), h \rangle = \int_{-\pi}^{\pi} f_1(t)h(t) = \int_{-\pi}^{0} f_1(t) \times 0 + \int_{0}^{\pi} 0 \times h(t) = 0$$

So the claimed form of $P_M(f)$ is correct.

Problem 7.19
Consider the space $L^2(0,1)$. Define $r_0(t) \equiv 1$ and

$$r_n(t) = \sum_{i=1}^{2^n} (-1)^{i-1} \chi_{[\frac{i-1}{2^n}, \frac{i}{2^n}]}(t)$$

1. Show that $r_n(t) = sgn(\sin 2^n \pi t), 0 \leq t \leq 1$
2. Show that $\{r_n(t)\}_{n=0}^{\infty}$ is orthonormal in $L^2(0,1)$, but it is not complete.

Solution:

1. Follows from the fact that for $x \in \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ we have $2^n \pi x \in [(i-1)\pi, i\pi]$ and so $sgn(\sin(2^n \pi x)) = (-1)^{i-1}$.

2. Consider $\langle r_n, r_m \rangle$ where $n > m$. Note that for $1 \leq i \leq 2^m$:

$$\begin{aligned}
\int_{\frac{i-1}{2^m}}^{\frac{i}{2^m}} r_n(t) r_m(t) dt &= (-1)^i \int_{\frac{i-1}{2^m}}^{\frac{i}{2^m}} r_n(t) dt \\
&= (-1)^i \int_{\frac{i-1}{2^m}}^{\frac{i}{2^m}} \sum_{j=2^{n-m} i}^{2^{n-m} i} (-1)^{j-1} \chi_{[\frac{j-1}{2^n}, \frac{j}{2^n}]}(t) dt \\
&= (-1)^i \sum_{j=2^{n-m} i}^{2^{n-m} i} \int_{\frac{i-1}{2^m}}^{\frac{i}{2^m}} (-1)^{j-1} \chi_{[\frac{j-1}{2^n}, \frac{j}{2^n}]}(t) dt \\
&= (-1)^i \sum_{j=2^{n-m} i}^{2^{n-m} i} \frac{(-1)^{j-1}}{2^n} = 0
\end{aligned}$$

Therefore, $\langle r_n, r_m \rangle = \int_0^1 r_n(t) r_m(t) dt = \sum_{i=1}^{2^m} \int_{\frac{i-1}{2^m}}^{\frac{i}{2^m}} r_n(t) r_m(t) dt = 0$. It is easy to see $\langle r_n, r_n \rangle = \int_0^1 |r_n(t)| dt = 1$. This proves orthonormality.
Assume the given basis is complete. Define f as below:

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} r_n(x) & \text{if } x \leq \frac{1}{2} \\ 0 & \text{if } x > \frac{1}{2} \end{cases} = \sum_{i=1}^{\infty} \frac{1}{(n+1)^2} r_n(x) \chi_{(0,\frac{1}{2}]}$$

Note that $|f(x)| \leq \sum_{i=1}^{\infty} \frac{1}{(n+1)^2} < \infty$. As f is bounded on bounded set, it is in $L^2(0,1)$. By our assumption, we can write $f(t) = \sum_{i=0}^{\infty} c_i r_i(t)$ where

$$\begin{aligned}
c_n = \langle f, r_n \rangle &= \left\langle \sum_{i=1}^{\infty} \frac{1}{(i+1)^2} r_i(x) \chi_{(0,\frac{1}{2}]}, r_n \right\rangle \\
&= \sum_{i=1}^{\infty} \frac{1}{(i+1)^2} \int_0^1 r_i(x) \chi_{(0,\frac{1}{2}]} r_n(x) dx \\
&= \sum_{i=1}^{\infty} \frac{1}{(i+1)^2} \int_0^{1/2} r_i(x) r_n(x) dx
\end{aligned}$$

Define $t = 2x$. Note that $r_n(x) = sgn(\sin 2^n \pi x) = sgn(\sin 2^{n-1} \pi 2x) = sgn(\sin 2^{n-1} \pi t) = r_{n-1}(t)$.

Substituting above we get:

$$\begin{aligned}
c_n &= \sum_{i=1}^{\infty} \frac{1}{(i+1)^2} \int_0^{1/2} r_i(x) r_n(x) dx \\
&= \sum_{i=1}^{\infty} \frac{1}{2(i+1)^2} \int_0^{1} r_{i-1}(t) r_{n-1}(t) dt \\
&= \sum_{i=0}^{\infty} \frac{1}{2(i+2)^2} \int_0^{1} r_{i}(t) r_{n-1}(t) dt \\
&= \sum_{i=0}^{\infty} \frac{1}{2(i+2)^2} \langle r_i, r_{n-1} \rangle = \frac{1}{2\left[(n-1)+2\right]^2} = \frac{1}{2(n+1)^2}
\end{aligned}$$

But note that for $x \leq 1/2$, we have $f(x) - \sum_{i=0}^{\infty} c_i r_i(x) = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} r_n(x) - \sum_{i=0}^{\infty} \frac{1}{2(i+1)^2} r_i(x) = f(x)/2 - \frac{1}{2} \neq 0$ which contradicts the assumption of completion. So the given basis is not complete.

Problem 7.20
Let $(a, b) \in \mathbb{R}$ be a finite interval and let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2(a, b)$. Define

$$\Phi_{i,j}(x, y) = \phi_(x) \times \phi_j(y)$$

for $(x, y) \in (a, b) \times (a, b)$. Show that $\{\Phi_{i,j}\}_{i,j\in\mathbb{R}}$ is a basis for $L^2(a, b) \times (a, b)$.

Solution: Orthonormality follows from the following:

$$\langle \Phi_{i,j}, \Phi_{i',j'} \rangle = \int\int_{(a,b)\times(a,b)} \phi_i(x)\phi_j(y)\phi_{i'}(x)\phi_{j'}(y) dx dy = \int_{(a,b)} \phi_i(x)\phi_{i'}(x) dx \int_{(a,b)} \phi_j(y)\phi_{j'}(y) dy = \delta_{i,i'}\delta_{j,j'}$$

Let there exist f such that $\langle f, \Phi_{i,j} \rangle = 0$ for all $i, j \in \mathbb{N}$. Then we note that:

$$\begin{aligned}
&\int_a^b \int_a^b f \Phi_{i,j} dx dy = \int_a^b \int_a^b f(x, y)\phi_i(x)\phi_j(y) dx dy = 0 \\
\Rightarrow &\int_a^b \left(\int_a^b f(x, y)\phi_i(x) dx \right) \phi_j(y) dy = 0
\end{aligned}$$

Define $f_1(y) = \int_a^b f(x, y)\phi_i(x) dx$. Substituting we have for all $j \in \mathbb{N}$:

$$\begin{aligned}
&\int_a^b \left(\int_a^b f(x, y)\phi_i(x) dx \right) \phi_j(y) dy = 0 \\
\Rightarrow &\int_a^b f_1(y)\phi_j(y) dy = 0 \\
\Rightarrow &\langle f_1, \phi_j \rangle = 0
\end{aligned}$$

As ϕ_j is a complete orthonormal basis it follows that $f_1(y)$ is 0 almost everywhere. Therefore, for all $i \in \mathbb{N}$:

$$\int_a^b f(x, y)\phi_i(x) dx = 0 \Rightarrow \langle f(x, y), \phi_x \rangle = 0$$

Again, as ϕ_j is a complete orthonormal basis it follows that $f(x, y)$ is 0 almost everywhere. As $\langle f, \Phi_{x,y} \rangle = 0 \forall x, y \in \mathbb{N} \Rightarrow f = 0$, we claim $\{\Phi_{i,j}\}_{i,j\in\mathbb{N}}$ is a complete orthonormal basis.

Problem 7.21
Show that the sets

$$\left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}}\cos nt | n \in \mathbb{N}\right\}$$

is a complete orthonormal set in $L^2(0,\pi)$.

Solution: Orthonormality of the first set with respect to the rest follows trivially. Write $\cos nt = \frac{1}{2}(e^{int} + e^{-int})$. Then we have:

$$\left\langle \sqrt{\frac{2}{\pi}}\cos nt, \sqrt{\frac{2}{\pi}}\cos mt \right\rangle = \frac{1}{2\pi}\int_0^\pi \left(e^{i(n+m)t} + e^{-i(n+m)t} + e^{i(n-m)t} + e^{i(m-n)t}\right) dt$$

The above expression is 0 if $n \neq m$ and 1 if $n = m$. This proves orthonormality. Extend f evenly by defining

$$f_0(x) = \begin{cases} f(x) & \text{if } x > 0 \\ f(-x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Consider the fourier expansion of f_0. As f_0 is even and sin is odd function,

$$\int_{-\pi}^{\pi} f_0 \sin nt dt = 0$$

We get a fourier expansion of $f_0(x) = a_0 + \sum_{i=1}^\infty a_n \cos nt$. But $f = f_0|_{(0,\pi)}$, Therefore $f(x) = a_0 + \sum_{i=1}^\infty a_n \cos nt$ for any $f \in L^2(0,\pi)$ which proves completeness.

Problem 7.22
Let $f, g \in L^2(-\pi,\pi)$ and let their fourier series be given by

$$f(t) = \frac{a_0}{2} + \sum_{i=1}^\infty (a_n \cos nt + b_n \sin nt)$$
$$g(t) = \frac{c_0}{2} + \sum_{i=1}^\infty (c_n \cos nt + d_n \sin nt)$$

Show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t) = \frac{a_0 c_0}{2} + \sum_{n=1}^\infty (a_n c_n + b_n d_n)$$

Solution: We know $\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \bigcup_{n\in\mathbb{N}} \left\{\frac{1}{\sqrt{\pi}}\cos nt\right\} \cup \bigcup_{n\in\mathbb{N}} \left\{\frac{1}{\sqrt{\pi}}\sin nt\right\}$ forms a complete orthonormal basis for $L^2(-\pi,\pi)$. Rewrite f, g as follows:

$$f(t) = \frac{a_0\sqrt{2\pi}}{2} \times \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^\infty (a_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\cos nt + b_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\sin nt)$$
$$g(t) = \frac{c_0\sqrt{2\pi}}{2} \times \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^\infty (c_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\cos nt + d_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\sin nt)$$

Define:

$$f_N(t) = \frac{a_0\sqrt{2\pi}}{2} \times \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^N (a_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\cos nt + b_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\sin nt)$$
$$g_N(t) = \frac{c_0\sqrt{2\pi}}{2} \times \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^N (c_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\cos nt + d_n\sqrt{\pi} \times \frac{1}{\sqrt{\pi}}\sin nt)$$

We know that $f_N \underset{L^2}{\longrightarrow} f, g_N \underset{L^2}{\longrightarrow} g$. As the inner product is continuous, it follows that $\langle f, g\rangle = \lim_{n\to\infty}\langle f_N, g_N\rangle$. Note that

$$\langle f, g\rangle = \lim_{n\to\infty}\langle f_N, g_N\rangle = \lim_{n\to\infty}\left(\pi a_0c_0 + \sum_{i=1}^{N}(\pi a_nc_n + \pi b_nd_n)\right) = \pi a_0c_0 + \sum_{n\in\mathbb{N}}(\pi a_nc_n + \pi b_nd_n)$$

The statement follows by re-arrangement.

Problem 7.25
Compute the fourier series of the function:

$$f(x) = \begin{cases} -1 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$$

Solution: We write

$$f(t) = a_0 \times \frac{1}{\sqrt{2\pi}} + \sum_{i=1}^{\infty}\left(a_n \times \frac{1}{\sqrt{\pi}}\cos nt + b_n \times \frac{1}{\sqrt{\pi}}\sin nt\right)$$

We know $\left\{\frac{1}{\sqrt{2\pi}}\right\}\cup\bigcup_{n\in\mathbb{N}}\left\{\frac{1}{\sqrt{\pi}}\cos nt\right\}\cup\bigcup_{n\in\mathbb{N}}\left\{\frac{1}{\sqrt{\pi}}\sin nt\right\}$ forms a complete orthonormal basis for $L^2(-\pi,\pi)$. So we have:

$$a_0 = \int_{-\pi}^{\pi} f(t) \times \frac{1}{\sqrt{2\pi}}dt = 0$$
$$a_n = \int_{-\pi}^{\pi} f(t) \times \frac{1}{\sqrt{\pi}}\cos ntdt = 0$$
$$b_n = \int_{-\pi}^{\pi} f(t) \times \frac{1}{\sqrt{\pi}}\sin ntdt = 2\int_{0}^{\pi} f(t) \times \frac{1}{\sqrt{\pi}}\sin ntdt = 2\int_{0}^{\pi} \frac{1}{\sqrt{\pi}}\sin ntdt = \frac{2}{\sqrt{\pi}}\frac{1-\cos n\pi}{n}$$

Therefore, the required expansion is:

$$f(x) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t\dots$$

Problem 7.26
Compute the Fourier cosine series of the function $f(t) = \sin t$ On $[0,\pi]$.

Solution: By 7.21, we know that

$$\left\{\frac{1}{\sqrt{\pi}}\right\}\cup\left\{\sqrt{\frac{2}{\pi}}\cos nt | n\in\mathbb{N}\right\}$$

forms a complete orthonormal basis. Write $f = \frac{a_0}{\sqrt{\pi}} + \sum_{i=1}^{\infty} a_n\sqrt{\frac{2}{\pi}}\cos nt$. Then we have:

$$a_0 = \int_0^{\pi}\sin t\frac{1}{\sqrt{\pi}}dt = \frac{2}{\sqrt{\pi}}$$
$$a_i = \int_0^{\pi}\sin t\frac{\sqrt{2}}{\sqrt{\pi}}\cos ntdt = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}(1-n^2)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Problem 7.27

1. Compute the Fourier sine series and the Fourier cosine series of the function $f(t) = t$ on $[0, \pi]$.
2. Evaluate:
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}$$
using Parseval's identity.

Solution:

1. The basis of the cosine series is given by
$$\left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}} \cos nt | n \in \mathbb{N}\right\}$$
This is an orthogonal basis. The coefficients are given by:
$$a_0 = \int_0^{\pi} t \frac{1}{\sqrt{\pi}} dt = \frac{\pi^{3/2}}{2}$$
$$a_n = \int_0^{\pi} t \frac{\sqrt{2}}{\sqrt{\pi}} \cos ntdt = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\cos(n\pi - 1)}{n^2} = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{-2}{n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$
The basis of sine series is given by:
$$\left\{\sqrt{\frac{2}{\pi}} \sin nt | n \in \mathbb{N}\right\}$$
This is an orthogonal basis. The coefficients are given by:
$$a_n = \int_0^{\pi} t \frac{\sqrt{2}}{\sqrt{\pi}} \sin ntdt = -\frac{\sqrt{2}\pi \cos{(n\pi)}}{\sqrt{\pi}\, n}$$

2. By Perseval's identity we have
$$||f||_{L^2} = \sum_{i=0}^{\infty} {a_i}^2$$
Note that $||f||_{L^2} = \int_0^{\pi} f^2(t)dt = \frac{\pi^3}{3}$. By using $\left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}} \cos nt | n \in \mathbb{N}\right\}$ as our orthonormal basis we get:
$$\frac{\pi^3}{3} = \frac{\pi^3}{4} + \sum_{n=\text{odd}} \frac{2}{\pi} \frac{4}{n^4}$$
$$\Rightarrow \sum_{n=\text{odd}} \frac{1}{n^4} = \frac{\pi^4}{96}$$
Set $S = \sum_{n\in\mathbb{N}} \frac{1}{n^4}$. We have:
$$S = \sum_{n\in\mathbb{N}} \frac{1}{n^4} = \sum_{n=\text{even}} \frac{1}{n^4} + \sum_{n=\text{odd}} \frac{1}{n^4} = \sum_{n\in\mathbb{N}} \frac{1}{(2n)^4} + \sum_{n=\text{odd}} \frac{1}{n^4}$$
$$\Rightarrow S = \frac{S}{16} + \frac{\pi^4}{96}$$
$$\Rightarrow S = \frac{\pi^4}{90}$$
Doing the same with the sine series gives $\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$

Problem 7.30
Let H be an infinite dimensional separable Hilbert space. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a bounded sequence of scalars. For $x \in H$, define

$$A(x) = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k$$

Show that A is well-defined for each $x \in H$ and that $A \in \mathcal{L}(H)$

Solution: For $x, y \in H$ and $c_1, c_2 \in \mathbb{R}$. Then we note

$$A(c_1 x + c_2 y) = \sum_{k=1}^{\infty} \lambda_k \langle c_1 x + c_2 y, e_k \rangle = \sum_{k=1}^{\infty} \lambda_k (c_1 \langle x, e_k \rangle + c_2 \langle y, e_k \rangle) = c_1 A(x) + c_2 A(y)$$

Let $\lambda_i \leq M \forall i \in \mathbb{N}$. Then:

$$||A(x)||^2 = \sum_{n \in \mathbb{N}} \lambda_k^2 \langle x, e_k \rangle^2 \leq \sum_{n \in \mathbb{N}} M^2 \langle x, e_k \rangle^2 = \sum_{n \in \mathbb{N}} \langle Mx, e_k \rangle^2 = ||Mx||^2$$

So $||A(x)|| \leq ||Mx|| = M||x|| \Rightarrow ||A|| \leq M$. As A is linear and bounded, it is continuous.
As $||A||$ is well-defined, the series converges and the operator is well-defined.

2 CT-2, 2021

Almost every question is of the form " Justify true or false". If it is true prove it otherwise give counter example or proper reason.

Problem 1
Let $B = \{e_n, n \in \mathbb{N}\}$ be an orthonormal basis of Hilbert space H over $\mathbb{R}$. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$ and set a new sequence

$$y_n = \frac{x_1 e_1 + x_2 e_2 + ..x_n e_n}{n}, n \in \mathbb{N}$$

(i) Find (with justification)

$$\lim_{n \longrightarrow \infty} \|y_n\|_H$$

(ii) The sequence $\sqrt{n} y_n \to 0$ in $\sigma(H, H^*)$ i.e. weakly converges to zero. Justify your answer.

*Solution:*Let M be an upper bound of $|x_n|$.
(i) We note that:

$$||y_n||^2 = \sum_{i=1}^{n} \frac{x_i^2}{n^2} \leq \sum_{i=1}^{n} \frac{M^2}{n^2} = \frac{M^2}{n}$$

It follows trivially that $\lim_{n \to \infty} ||y_n||_H = 0$
(ii) Let T be linear operator in H. By Riez Representation, we have $T(\cdot) = \langle \cdot, w \rangle$. Let $w = \sum_{n \in \mathbb{N}} w_n e_n$ where $\lim_{n \to \infty} |w_n| < M'/\sqrt{n}$ for any $M' \in \mathbb{R}^+$. We know this happens because $\sum_{n \in \mathbb{N}} w_n^2$ converges. Then:

$$\lim_{n \to \infty} T(\sqrt{n} y_n) = \lim_{n \to \infty} \sqrt{n} \langle y_n, w \rangle = \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i w_i < \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{MM'}{\sqrt{i}}$$

We approximate by integrals:

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} = 1 + \sum_{i=2}^{n} \frac{1}{\sqrt{i}} \leq 1 + \sum_{i=2}^{n} \int_{i-1}^{i} \frac{1}{\sqrt{t}} dt = \alpha + \beta \sqrt{n}$$

Putting above we get:

$$\lim_{n\to\infty} T(\sqrt{n}y_n) \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}}(\alpha+\beta\sqrt{n})MM'$$

Now $\frac{1}{\sqrt{n}}(\alpha+\beta\sqrt{n})$ is bounded and M' can be taken to be as close to 0 as we want. So $\lim_{n\to\infty} T(\sqrt{n}y_n) = 0$. As this holds for any T, $\sqrt{n}y_n \rightharpoonup 0$.

Alternate Solution :

Assume $w_i, x_i \geq 0$ for all i. Let $w^N = \sum_{n=1}^{N} w_i e_i$. Now $w^N \to w$ and $y_n \to 0$. Therefore

$$\lim_{n\to\infty} \langle \sqrt{n}y_n, w\rangle = \lim_{n\to\infty}\lim_{m\to\infty} \langle \sqrt{n}y_n, w^m\rangle = \lim_{m\to\infty}\lim_{n\to\infty} \langle \sqrt{n}y_n, w^m\rangle = 0$$

If all the coefficients are not in $\mathbb{R}^+$, then:

$$|\lim_{n\to\infty} \langle \sqrt{n}y_n, w^m\rangle| = \left|\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{n}\frac{x_i w_i}{n}\right| \leq \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{n}\frac{|x_i||w_i|}{n}$$

Set $y_n' = \sum_{i=1}^{n} |x_i| e_i/n$ and $w' = \sum_{i=1}^{\infty} |w_i| e_i$ and apply the above result to get:

$$|\lim_{n\to\infty} \langle \sqrt{n}y_n, w\rangle| \leq 0$$

So $\lim_{n\to\infty} T(\sqrt{n}y_n) = 0$ and $\sqrt{n}y_n \rightharpoonup 0$.

Problem 2

(i) Let $E = C[0,1]$ with sup norm. If $g_n \in E$ and $g_n \to g$ in $\sigma(E, E^*)$ i.e. weakly converges to g. Then the sequence $\{g_n\}$ converges point wise on $[0,1]$. Justify your answer.

(ii) If a sequence $\{h_n\} \in E$ is point wise convergent, then it is a weak convergent sequence. Justify your answer.

Solution:

(i)Set $J_x = f(x)$. As $g_n \rightharpoonup g$, for any $x \in \mathbb{R}$ we have $J_x(g_n) \to J_x(g)$ or $g_n(x) \to g(x)$. Therefore, $g_n \to g$ point wise.

(ii) No. Consider $T(f) = \int_0^1 f dx$. Set

$$h_n(x) = \begin{cases} nx & \text{if } x \leq 1/n \\ 2n - nx & \text{if } 1/n \leq x \leq 2/n \\ 0 & \text{if } x \geq 2/n \end{cases}$$

Note that $h_n \to 0$ point wise but $T(h_n) \not\to T(0)$. So it is not weakly convergent.

Problem 3

(i) Let H be a Hilbert space over $\mathbb{R}$. If $x_n \to x$ in $\sigma(H, H^*)$ i.e. weakly converges to x and $\|x_n\|_H$ converges to $\|x\|_H$ in $\mathbb{R}$. Then $x_n \longrightarrow x$ in H i.e. converges in normed $\|.\|_H$ topology or strong topology. Justify your answer.

(ii) Let H be a Hilbert space. If $K \subset H$ is compact in weak topology $\sigma(H, H^*)$ then K is bounded and closed in weak topology $\sigma(H, H^*)$. Justify your answer.

Solution:

(i) Note that

$$\langle x - x_n, x - x_n\rangle = \langle x, x\rangle + \langle x_n, x_n\rangle + 2\langle x_n, x\rangle$$

Now we know $\langle x_n, x_n\rangle = ||x_n||^2 \to ||x||^2$. As $T = \langle \cdot, x\rangle$ is a linear functional of H and $x_n \rightharpoonup x$, it follows that $T(x_n) \to T(x)$. Putting those substitutions above we get:

$$\langle x - x_n, x - x_n\rangle = \langle x, x\rangle + \langle x_n, x_n\rangle - 2\langle x_n, x\rangle \to 2||x||^2 - 2||x||^2 = 0$$

Therefore $||x - x_n|| \to 0$which implies strong convergence.
(ii) As K is compact and every $T \in H^*$ is continuous, $T(K)$ is compact and bounded. As image of K is bounded in any linear functional, it is bounded in normed topology. Let $x \in K^c$. As compact sets are closed we have some weakly open set U_x such that $U_x \cap K = \phi$. But as weak topology is a subset of normed topology, we can conclude that U_x is also open in normed topology and x is an exterior point. As every point in K^c is exterior in normed topology, we claim K^c is strongly open or K is strongly closed.