Third Internal Exam Solutions

Name: Aakash Ghosh **Course Name:** MA3205
Roll No: 19MS129 **Date:** 14^{th} April, 2022

Problem 1. *Let $\mathscr{S}$ be the pseudo-sphere.*

1. *Compute the Gauss map for this surface and show that it is differentiable.*
2. *For a regular chart (U, σ) and a region $R \subseteq U$, define the area of R to be:*

$$A_\sigma(R) := \int\int_R ||\sigma_u \times \sigma_v|| dudv$$

Fix a regular chart (U, σ) around $p \in \mathscr{S}$ so that $\sigma(0) = p$. Using the restricted Gauss Map $\mathscr{G} : U \to S^2$, we define

$$A_N(R) = \int\int_R ||N_u^\sigma \times N_v^\sigma|| dudv$$

Compute the limit

$$\lim_{r \to 0} \frac{A_N(B_r(0))}{A_\sigma(B_r(0))}$$

Solution :

1. Define $S = \{(x, y) | 0 < x^2 + y^2 < 1\}$. We parmeterize $\mathscr{S}$ by the map:

$$\sigma : S \to \mathscr{S}, \sigma(u, v) = (u, v, \cosh^{-1}\left(\frac{1}{\sqrt{u^2 + v^2}}\right) - \sqrt{1 - (u^2 + v^2)})$$

For the sake of convinience, we set $\sqrt{u^2 + v^2} = r$. Then we can rewrite the map as:

$$\sigma(u, v) = (u, v, \cosh^{-1}\left(\frac{1}{r}\right) - \sqrt{1 - r^2})$$

We now calculate σ_u and σ_v to get:

$$\sigma_u(u, v) = \left(1, 0, u\left(\frac{1}{\sqrt{1 - u^2 - v^2}} - \frac{1}{(u^2 + v^2)\sqrt{1 - u^2 - v^2}}\right)\right) = \left(1, 0, -u\frac{\sqrt{1 - r^2}}{r^2}\right)$$
$$\sigma_v(u, v) = \left(0, 1, v\left(\frac{1}{\sqrt{1 - u^2 - v^2}} - \frac{1}{(u^2 + v^2)\sqrt{1 - u^2 - v^2}}\right)\right) = \left(0, 1, -v\frac{\sqrt{1 - r^2}}{r^2}\right)$$

We now compute the cross product:

$$\sigma_u \times \sigma_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -u\frac{\sqrt{1-r^2}}{r^2} \\ 0 & 1 & -v\frac{\sqrt{1-r^2}}{r^2} \end{vmatrix} = \left(u\frac{\sqrt{1 - r^2}}{r^2}, v\frac{\sqrt{1 - r^2}}{r^2}, 1\right)$$

$$||\sigma_u \times \sigma_v|| = \sqrt{(u^2 + v^2)\frac{1 - r^2}{r^4} + 1} = \sqrt{\frac{1 - r^2}{r^2} + 1} = \frac{1}{r}$$

We therefore get the Gauss Map N^σ

$$N^\sigma = \frac{\sigma_u \times \sigma_v}{||\sigma_u \times \sigma_v||} = \left(u\frac{\sqrt{1-r^2}}{r}, v\frac{\sqrt{1-r^2}}{r}, r\right)$$

As $||N^\sigma||^2 = (u^2+v^2)\frac{1-r^2}{r^2} + r^2 = 1$, $N^\sigma(u,v) \in S^2$
As $r = \sqrt{u^2+v^2} \neq 0$, N^σ is differentiable in S.

2. Let $\gamma = \sigma^{-1}$. Consider the chart $\tilde{\sigma}(u,v) = \sigma\left((u,v) + \gamma(p)\right)$, $\tilde{\sigma}(0) = p$. Then the following holds true:

- $\tilde{\sigma}_u(u,v) = \sigma_u((u,v)+\gamma(p)), \tilde{\sigma}_v(u,v) = \sigma_v((u,v)+\gamma(p))$
- $N^{\tilde{\sigma}}(u,v) = N^\sigma((u,v)+\gamma(p))$
- $N^{\tilde{\sigma}}_u(u,v) = N^\sigma_u((u,v)+\gamma(p))$,$N^{\tilde{\sigma}}_v(u,v) = N^\sigma_v((u,v)+\gamma(p))$

Therefore:

$$\begin{aligned}
A_{\tilde{\sigma}}(R) &= \int\int_R ||\tilde{\sigma}_u(u,v) \times \tilde{\sigma}(u,v)||dudv \\
&= \int\int_R ||\sigma_u((u,v)+\gamma(p)) \times \sigma((u,v)+\gamma(p)||dudv \\
&= \int\int_{R+p} ||\sigma_u(u,v) \times \sigma(u,v)||dudv = A_\sigma(R+p)
\end{aligned}$$

$$\begin{aligned}
A_{N^{\tilde{\sigma}}}(R) &= \int\int_R ||N^{\tilde{\sigma}}_u(u,v) \times N^{\tilde{\sigma}}(u,v)||dudv \\
&= \int\int_R ||N^\sigma_u((u,v)+\gamma(p)) \times N^\sigma((u,v)+\gamma(p)||dudv \\
&= \int\int_{R+p} ||N^\sigma_u(u,v) \times N^\sigma(u,v)||dudv = A_{N^\sigma}(R+p)
\end{aligned}$$

We first calculate N^σ_u and N^σ_v

$$\begin{aligned}
N^\sigma_u &= \left(-\frac{u^4+2v^2u^2+v^4-v^2}{\sqrt{1-u^2-v^2}\,(u^2+v^2)^{\frac{3}{2}}}, -\frac{vu}{\sqrt{1-u^2-v^2}\,(u^2+v^2)^{\frac{3}{2}}}, \frac{u}{\sqrt{u^2+v^2}}\right) \\
&= \left(-\frac{r^4-v^2}{r^3\sqrt{1-r^2}}, -\frac{uv}{r^3\sqrt{1-r^2}}, \frac{u}{r}\right)
\end{aligned}$$

$$\begin{aligned}
N^\sigma_v &= \left(-\frac{vu}{\sqrt{1-u^2-v^2}\,(u^2+v^2)^{\frac{3}{2}}}, -\frac{u^4+2v^2u^2+v^4-u^2}{\sqrt{1-u^2-v^2}\,(u^2+v^2)^{\frac{3}{2}}}, \frac{v}{\sqrt{u^2+v^2}}\right) \\
&= \left(-\frac{uv}{r^3\sqrt{1-r^2}}, -\frac{r^4-u^2}{r^3\sqrt{1-r^2}}, \frac{v}{r}\right)
\end{aligned}$$

We now calculate the cross product $N^\sigma_u \times N^\sigma_v$

$$N_u^\sigma \times N_v^\sigma = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{r^4-v^2}{r^3\sqrt{1-r^2}} & -\frac{uv}{r^3\sqrt{1-r^2}} & \frac{u}{r} \\ -\frac{uv}{r^3\sqrt{1-r^2}} & -\frac{r^4-u^2}{r^3\sqrt{1-r^2}} & \frac{v}{r} \end{vmatrix}$$

$$= \left(\frac{u(r^4-u^2-v^2)}{r^4\sqrt{1-r^2}}, \frac{v(r^4-u^2-v^2)}{r^4\sqrt{1-r^2}}, \frac{(r^4-u^2)(r^4-v^2)}{r^6(1-r^2)} - \frac{u^2v^2}{r^6(1-r^2)} \right)$$

$$= \left(-u\frac{\sqrt{1-r^2}}{r^2}, -v\frac{\sqrt{1-r^2}}{r^2}, -1 \right)$$

$$||N_u^\sigma \times N_v^\sigma|| = \sqrt{(u^2+v^2)\frac{1-r^2}{r^4}+1} = \sqrt{\frac{1-r^2}{r^2}+1} = \frac{1}{r} = ||\sigma_u \times \sigma_v||$$

Therefore:

$$\lim_{r\to 0} \frac{A_{N^{\tilde{\sigma}}}(B_r(0))}{A_{\tilde{\sigma}}(B_r(0))} = \lim_{r\to 0} \frac{A_{N^\sigma}(B_r(p))}{A_\sigma(B_r(p))} = \lim_{r\to 0} \frac{\int\int_{B_r(p)} ||\sigma_u \times \sigma_v|| dudv}{\int\int_{B_r(p)} ||N_u^\sigma \times N_v^\sigma|| dudv} = 1$$