Solutions to Questions in Ring theory and Modules for MA3102

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Contents

Chapter 1

Tutorial Sheet 1

1. Give an example of a non-commutative ring without identity.
Solution : $2\mathbb{Z} = \{x|x = 2z, z \in \mathbb{Z}\}$

2. Let R be a ring with 1 and $a \in R$. Suppose there exists a positive integer n such that $a^n = 0$. Show that $1 + a$ is a unit, and so is $1 - a$.
Solution : Note that:

$$(1-a)(1+a+a^2+a^3 \ldots a^{n-1}) = 1 - a^n = 1$$

Therefore, $1 - a$ is an unit. Set $b = -a$. Then $b^n = 0$ and by above $1 - b = 1 + s$ is a unit too.

3. Let R be a ring such that $r^2 = r$ for all $r \in R$. Show that R is commutative.
Solution : Let $x, y \in R$. Then note:

$$\begin{aligned} x + y &= (x+y)^2 = x^2 + y^2 + xy + yx = x + y + xy + yx \\ \Rightarrow 0 &= xy + yx \end{aligned}$$

Set $y = x$. Then we get $0 = x^2 + x^2 = x + x$. Therefore, $x = -x \forall x \in R$. Apply above to get:

$$xy = -yx = yx$$

Therefore, R is commutative.

4. Let R be a ring with 1 and $a, b \in R$. Prove that $1 - ab \in R^*$ if and only if $1 - ba \in R^*$
R^* is the set of units of a ring.
Solution : As $1 - ab$ is a unit, there exits $v \in R$ such that $(1 - ab)v = 1$. Then note:

$$(1-ba)(1+bva) = 1 - ba + bva - babva = 1 - ba + b(1-ba)va = 1 - ba + ba = 1$$

Therefore, $1 - ba$ is an unit too.
*Intution:*We note that $v = 1/(1 - ab) = 1 + ab + abab \ldots$. Now note: $1/(1 - ba) = 1 + ba + baba \ldots = 1 + b(1 + ab + abab \ldots)a = 1 + bva$. Now, of course, such expressions may not exist. But, if we assume the relation to still hold, we get the desired solution.

5. Does there exist a non-commutative ring with 77 elements?
Solution : No.

Theorem 1. *Rings over cyclic groups are always commutative*

Proof. Let $G =< c >$ be a cyclic group. Then for $x, y \in G$ we can write $x = mc$ and $y = nc$ where $mc = c + c + c + \ldots (m$ times). Then it is easy to see that $xy = mc \cdot nc = mn(c \cdot c) = nm(c \cdot c) = nc \cdot mc = yx$. □

Now, we just need to show if $|G| = 77$ then G is cyclic. Note: $77 = 7 * 11$. pick a element g of G. If order of g is 77 then $G \cong C_{77}$ and we are done. If order of g is 7, then $< g >$ is normal(subgroup of abelian groups are normal) and

$G/<g>\cong C_{11}$. Therefore, we can write $G\cong C_{11}\times C_7\cong C_{77}$. A similar line of argument follows if g has order 11. Therefore, G is cyclic.

6. Let R be a ring and $R[x]$ be the polynomial ring over R. Show that $R[x]$ forms a ring under the usual addition and multiplication of polynomials.
Solution : Trivial

7. Does there exist an infinite ring with finite characteristic?
Solution : Yes. $R=(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})\dots$. There are infinite elements but characteristic is 2.

8. Does there exist a finite ring with characteristic zero?
Solution : No. As G is finite, for every $g\in G$, there exists n_g such that $n_g g=0$. Set $n=\prod_{g\in G}n_g$. Then $ng=0\forall g\in G$. Therefore, a finite characteristic exists.

9. Determine the smallest subring of Q that contains 1/2.
Solution :$R=\{x|x=\sum_{i\in\mathbb{Z}}a_i2^i$ such that a_i is 0 above for some $i>I_x, a_i\in\{0,1\}\}$
Alternatively, we consider the question in boolean. Given $0.1_2\in R$. Then $1\in R$ and by extension $10_2^k\in R$ for all $k\in\mathbb{Z}$. All of their linear combination is in R which is equivalent to the set above.

10. Let R be an integral domain and $kq=0$ for some non-zero $q\in R$ and some integer $k\neq 0$. Prove that R is of finite characteristic.
Solution : Note: $(np)\cdot q=p\cdot(nq)=0$. Therefore, $q=0$ or $np=0$. As the first case can't occur, $np=$ for all $p\in R$.

11. Give an example of a non-commutative simple ring.
Solution :

Chapter 2

Tutorial Sheet 6