Lecture Notes for Graphs and Surfaces

Aakash Ghosh

Instructor: Prof. Somnath Basu

Contents

Chapter 1

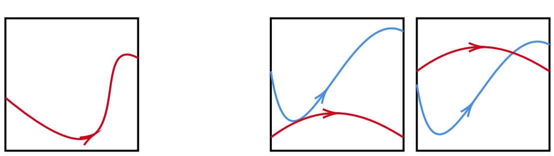
What to expect

First we shall see some theorems and examples and get to know the terms on an intuitive sense. The formal definitions of everything will be provided down the line(probably).

1.1 Four vertex theorem

Theorem 1 (Four Vertex Theorem)**.** *The curvature of a simple closed smooth curve has at least 2 local minima and two local maxima.*

[Planar] Curve: A curve is a map f from $\mathbb{R} \to \mathbb{R}^2$. Note: other types of curves exists. For example, a space curve is a map from $\mathbb{R} \to \mathbb{R}^3$.
Smooth: Smooth implies that the k^{th} derivative of f defined in the previous section exists and is continuous for all $k \in \mathbb{N}$.
Closed: Intuitively, a closed curve has the same starting and ending point. We define $f : [0, c] \to \mathbb{R}^2$ as closed if $f(0) = f(c)$. If we want to define it according to the definition we gave earlier, we say f is periodic. This implies for every $p \in \mathbb{R}$, $f(p + c) = f(p)$. The smallest such c is known as the period of the function. We can also view this in an geometric way: A closed curve is a mapping of S^1 to $\mathbb{R}^2$(S^1 is the unit circle in $\mathbb{R}^2$ plane). Let $e^{i\theta}$ be a point on S^1. Then a closed curve is a mapping $\tilde{f} : S^1 \to \mathbb{R}^2$. Now we can of course relate $\tilde{f}$ and f in a very natural way: $\tilde{f}(e^{i\theta}) = f(\frac{c}{2\pi}\theta)$(This result comes from the fact that the parameters which define S^1, i.e $\sin\theta$ and $\cos\theta$ are periodic in nature).
Simple : Intuitively, a simple curve is a curve which doesn't cross or touch itself. Therefore, if you zoom close enough at any point, it looks like a regular arc. mathematically we need $\tilde{f}$ to be injective or one-one.



Simple *Not Simple*

Curvature : Curvature is basically a function from the points of the curve to $\mathbb{R}$. Specifically $\mathcal{K} : \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{K}(\tilde{f}(e^{i\theta}))$ gives the inverse of the radius of the circle with the best fit at the point $\tilde{f}(e^{i\theta})$. Now this circle of best fit needs to be defined with some care. for one the the point and the circle drawn share the same normal and tangent. One way to drive this concept home for planar

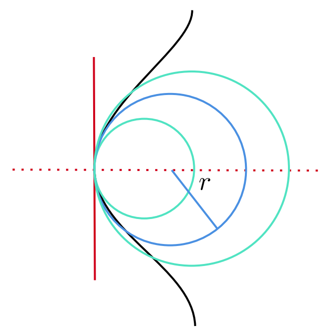
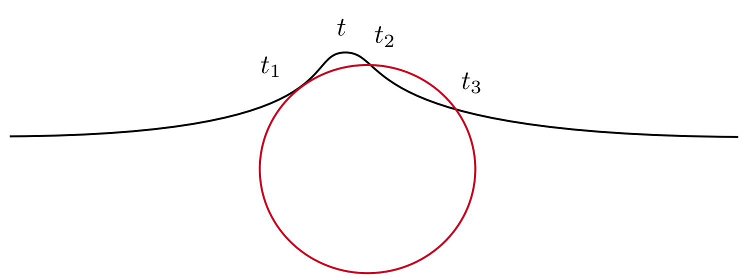


Figure 1.1: The red lines are common normal and common tangent. Note, the circle of medium size somehow seems to be a better fit than the others. So the radius of curvature at that point is $\frac{1}{r}$

curve is to consider the following process: take any three points t_1, t_2, t_3 around t. Draw the circumcircle of those points. the radius of curvature will be the radius of the circle formed when $\lim t_1, t_2, t_3 \to t$



The figure above shows the process. While this is nice way to view things, it is has it's own set of problems: How do we know that a limit exists? More over in higher dimensional spaces, the plane in which the circle lies will also change with the points that are being considered. Nevertheless it's good enough for a naive mental picture. The rest of the theorem now makes sense.

1.2 Fary-Milnor theorem

Theorem 2 (Fary-Milnor theorem)**.** *If the total absolute curvature of a knot K is at most 4π, then K is an unknot.*

Knot :A knot is a simple curve in 3-space. **Total absolute curvature :**What it says. We can't of course take an infinite sum for all the points so we integrate.

$$S = \int_{\mathcal{S}} |\mathcal{K}(s)|\, ds \tag{1.1}$$

Unknot :A unknot is a knot which can be deformed to a circle.

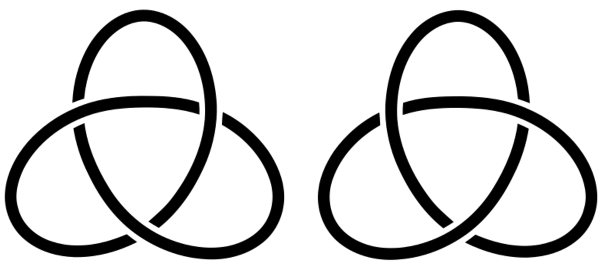


Figure 1.2: The two trefoil knots. They are not unknots(Copied from Wikipedia)

1.3 Hirsch-Smale Theory

Theorem 3 (Hirsch-Smale Theory)**.** *Any two immersed loop in $\mathbb{R}^2$ are isotopic if and only if the winding numbers match.*

Loop: A loop is a closed curve
Immersed: $f : \mathbb{R} \to \mathbb{R}^k$ is said to be immersed if $f'(t)$ is never zero. Note, the drawing of any curve is the trace of a curve: It does not represent the curve itself.
Isotopic: This means they are the same. It is what isomorphism is for groups and homeomorphism is for topologies. But when are two loops isotopic? But how to check

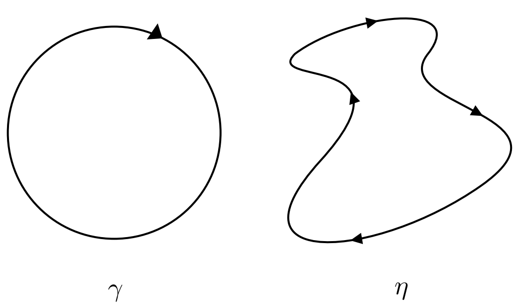


Figure 1.3: Two isotopic curves: The second one can be "straightened out" into the first one

this? We construct a f which deforms η to γ. But note: merely the presence of such a function is not enough. This would imply any two loops are isotopic. We furthermore require at any instant of the transformation, the resulting curve is immersed. Now we shall talk a bit about the function. Let $H : S^1 \times I \to \mathbb{R}^2$, where $I = [0, 1]$. Now note that the set $S^1 \times I \to \mathbb{R}^2$ looks like a cylinder. S^1 represents the base, I represents the height. We think of moving along I as moving along time. The base of the cylinder is the curve when we start(i.e. $H(S^1 \times \{0\})$) and the top of the cylinder is the curve we end up with(i.e. $H(S^1 \times \{1\})$). We also need $H(S^1 \times t)$ to be immersed.
Winding number: It is the total number of loops we

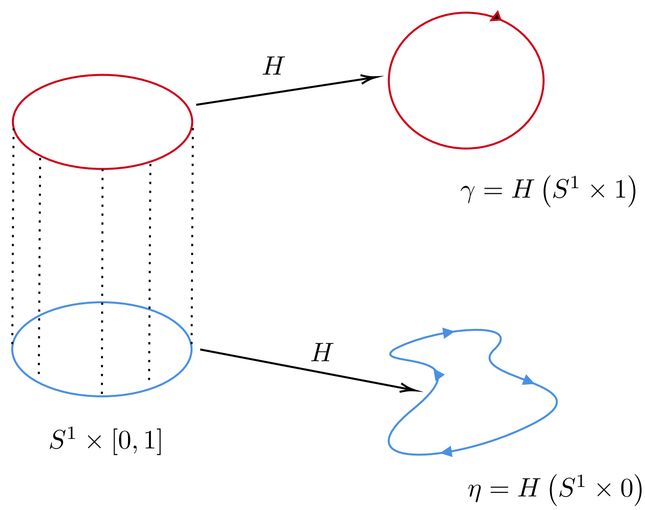


Figure 1.4: Representation of H

make when we complete traversing closed curve. Alternatively, imagine holding out a compass needle. Then the total number of complete turns the needle makes will be the winding number.

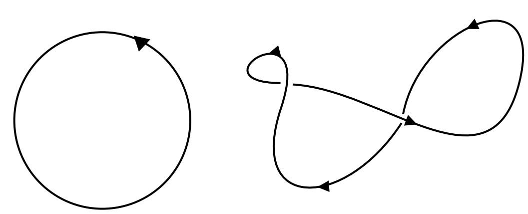


Figure 1.5: Winding number 1

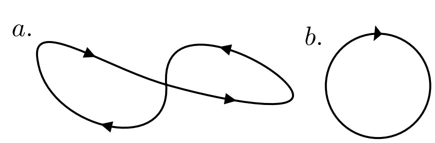


Figure 1.6: a. Winding number 0 b. Winding number -1

But what if we consider curves in S^2? Then the theorem changes as follows:

Theorem 4 (Hirsch-Smale Theory, 2^{nd} part)**.** *Any two immersed loop in S^2 are isotopic if and only if the winding numbers match modulo 2.*

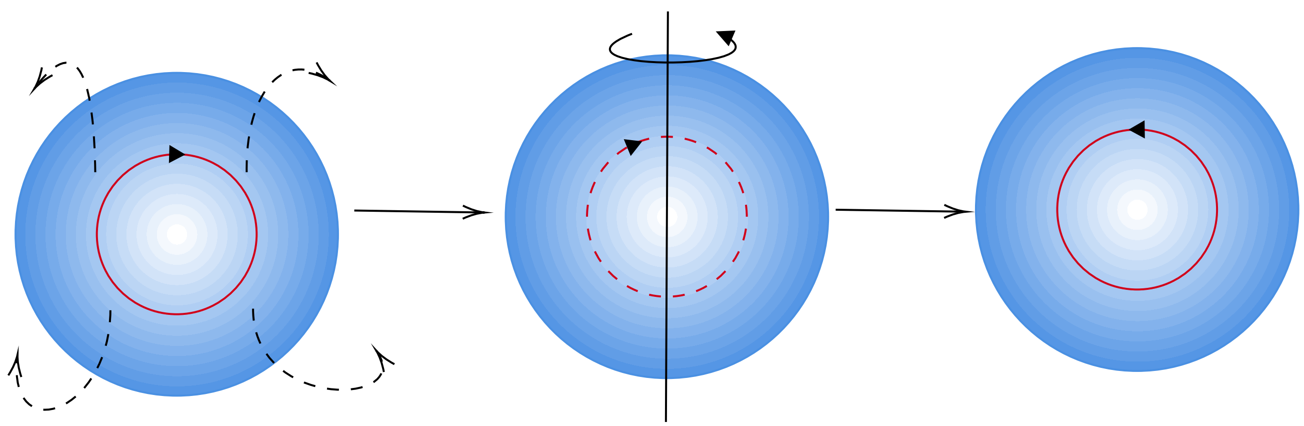


Figure 1.7: Example of part two of theorem. The winding number changes from -1 to +1. The dotted lines represent the curve on the rear side of the sphere.

Chapter 2

Curves

2.1 Introduction

We shall first consider some probable definitions of curves and see why they are wrong.

1. **A curve is a set with empty interior**. This on the first glance looks pretty good: after all our by our geometric intuition, a curve has nothing inside it. But while it does take into account all things we intuitively understand as curve, there are things included in this which are not very curve like. For example, consider:

$$f(x,y) = \begin{cases} 1 & x \in \mathbb{Q} \times \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \times \mathbb{Q} \end{cases}$$

   As $\mathbb{Q}$ is dense in $\mathbb{R}$, $S = \{(x,y) | f(x,y) = 0\}$ has a empty interior. But it's obvious that S which has some weird appearance is definitely not a curve.

2. **A curve is the graph of a function.** Therefore, every curve can be represented as $(x, f(x))$ or $(x, f_1(x), f_2(x))$. This definition suffers from the opposite problem of the previous definition. Note, structures like a vertical line($y = k$) or a parabola($x = y^2$) can not be graphs of functions because they fail the vertical line test and are yet curves.

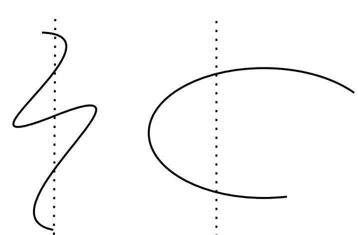


Figure 2.1: Two curves which are not graphs of a function

3. **A curve is the set of points where a function vanishes.** We might think this works: after all, we can now circumnavigate around our previous problem by making an suitable function like $f(x,y) := y - k$ or $f(x,y) := x - y^2$. But this again makes things which are intuitively not a curve, a curve.

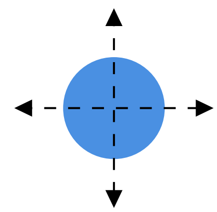


Figure 2.2: Set $f(x,y) = \max\{x^2+y^2-1, 0\}$. The resulting figure, clearly, doesn't form something we would like to recognise as a curve

2.2 Whitney's Teorem

Theorem 5 (Whitney's Theorem)**.** *Let $\Omega \in \mathbb{R}^n$ be any open subset. Given a closed $C \subseteq \Omega$, there exists a smooth function $f : \Omega \to \mathbb{R}$ such that $C = f^{-1}(0)$*

Proof. **Step 1(Covering):** Note, $V = \Omega \setminus C$ is open in Ω(and in fact, it is also open in $\mathbb{R}^n$). We want to cover this by a collection of open balls. We do this by taking every $q_n (\in \mathbb{Q} \times \mathbb{Q} \ldots)$ and an appropriate radius for each q_n. Therefore:

$$V = \bigcup_k B(\overrightarrow{p_k}, r_k) \quad [\overrightarrow{p_k} \in \mathbb{Q}^n \cap V, r_k \in \mathbb{Q}]$$

Note, this is a covering of V because Q is dense in $\mathbb{R}$.
Step 2(Choosing Bump Functions): Choose $f_k : \mathbb{R}^n \to \mathbb{R}$(those are smooth functions) such that

$$f_k^{-1}(0) = \mathbb{R}^n \setminus B(\overrightarrow{p_k}, r_k)$$

$$f_k^{-1}(1) = \overline{B(\overrightarrow{p_k}, r_k/2)}$$

This has been done as solution to problem 2 in exercise 1. Essentially, this looks like a smooth pudding(I feel sand heap is a better example) with a flat top. Moreover, all derivative of f_k vanishes on $\mathbb{R}^n \setminus \overline{B(\overrightarrow{p_k}, r_k)}$
Step 3(Combining those functions): We want to combine(by adding) those bump functions appropriately with weights such that the partial sums converge uniformly and their derivatives converge uniformly too. Note, for every point outside the closed set, it is present in some open ball with the bump. Therefore, if we assign a positive

weight to every bump function, then , if the sum converges, $f^{-1}(0) = C$ as all the other points lie in some bump. We define

$$c_k = \max_{|\alpha|\le k, \overrightarrow{y} \in B(\overrightarrow{p_k}, r_k)} \left| \frac{\partial^\alpha}{\partial x^m} f_k(\overrightarrow{y}) \right|$$

Note on notation: α is a *multi-index* and each x^m is differentiation w.r.t a particular index α times. For example, if $\alpha = (1, 0, 2)$, it would mean $\frac{\partial^3}{\partial x_1 \partial x_3^2} f_k(x)$. As f_k is assumed to be smooth, order of derivatives don't matter. In particular, for $\alpha = 1$, the max will be calculated between $f(\overline{y})$, $\frac{\partial}{\partial x_1}(\overline{y})$, $\frac{\partial}{\partial x_2}(\overline{y})$, $\frac{\partial}{\partial x_3}(\overline{y})$ on the whole set.
Now as f and it's derivative are continuous and we are considering the maximum on a compact set, c_k is well defined and exists. Those c_k will become our weights. We define f as

$$f = \sum_{k=0}^{\infty} \frac{f_k}{2^k c_k}$$

Step 4(Analytical verification):
Check $f^{-1}(0) = C$. If $\overrightarrow{p} \neq C$ then $\overrightarrow{p} \in V$, whence $\overrightarrow{p} \in B(\overrightarrow{p_k}, r_k)$, then $f(p) > \frac{f_k(p)}{2^k c_k} > 0$. If $\overrightarrow{p} \in C$, then $p \notin B(\overrightarrow{p_k}, r_k)$ for any k so $f(p) = 0$. Now we show f is C^0, by showing convergence of the sequence of f. Given $\epsilon > 0$ choose N such that $\frac{1}{2^N} < \epsilon$. Compare $S_m(f)$ and $S_n(f)$ with $m > n \geq N$. $S_n(f)$ is the partial sum $\sum_{k=1}^{n} \frac{f_k}{2^k c_k}$.
Note:

$$\sup_{x\in\mathbb{R}^n} |f_m(x) - f_n(x)| = \sum_{i=m}^{n} \frac{f_k}{2^k c_k} \leq \sum_{i=m}^{n} \frac{1}{2^k} \leq \sum_{i=m}^{\infty} \frac{1}{2^k} \leq \frac{1}{2^m}$$

The thing to note is we have $\frac{f_k}{c_k} < 1$ due to how c_k is defined. For higher derivatives it still holds(again, due to how c_k is defined) albeit with slight modifications. The $1/2^k$ is present to ensure convergence.

Note: This theorem is not very important in this course but the proof shows a classic old trick: finding covers, attaching weights and then performing analysis to get the desired result. □

2.3 Parameterized Curve

Definition 6 (Parameterized Curve)**.** *A paameterized smooth curve is a smooth function $\gamma : (a, b) \to \mathbb{R}^n$ where $-\infty \leq a \leq b \leq \infty$.(we include $\pm\infty$ as the function may map the whole of $\mathbb{R}$ to $\mathbb{R}^n$).*

Generally, we use lower Greek letters (like η, φ, etc.) to denote curves.

2.3.1 Reparameterization of curve

Definition 7 (Reparameterization of a curve)**.** *Let $\varphi(\alpha', \beta') \to (\alpha, \beta)$ be a homeomorphism. If γ is a parameterised curve as defined above, then $\eta : (\alpha', \beta') \to \mathbb{R}^n, \eta = \gamma \circ \varphi$ is called the reparameterization of γ.*

If φ is a diffeomorphism and γ is smooth then η is smooth. (see appendix, section 4). In particular, if γ is C^k then η is C^k if φ is diffeomorphism.

$$\eta'(t) = \underbrace{\gamma'(\varphi(t))}_{=Vector} \underbrace{\varphi'(t)}_{=Scalar}$$

If φ is C^0 then φ is monotonic. This is ascertained by the intermediate value theorem. If φ is atleast C^1 then we can further confirm $\varphi \neq 0$. Let ψ be the inverse of φ. Then:

$$\psi \circ \varphi(x) = x \Rightarrow \psi'(\varphi(x))\varphi'(x) = 1$$

Therefore, $\varphi(x) \neq 0$ and φ is strictly increasing or strictly decreasing. If $\varphi(\alpha') = \alpha', \varphi(\beta') = \beta$ then η is orientation preserving and if $\varphi(\alpha') = \beta', \varphi(\beta) = \alpha'$ then η is orientation reversing.

2.3.2 Arc length parameter

Definition 8 (Arc-Length)**.** *The arc length of a differentiable curve $\gamma(\alpha, \beta) \to \mathbb{R}^n$ starting at t_0 is defined as:*

$$S(t) = \int_{t_o}^{t} ||\gamma'(u)||du$$

This formula is quite intutive. We can consider the curve to be path of a particle, $\gamma(u)$ to be the position at time u. Then $\gamma'(u)$ gives the velocity at u and taking it's magnitude gives the speed. Integrating speed at various points gives the distance covered, or in this case, length of the arc traversed.
As γ' is atleast C^0 and the interval is closed and bounded, the integral is well defined and finite. Moreover:

$$\frac{d}{dt}S(t) = ||\gamma'(t)||$$

Definition 9 (Regular Curve)**.** *A curve γ is called regular if $\gamma' \neq 0$*

Example: A curve with unit speed(i.e $||\gamma'(u)|| = 1$ everywhere).

Lemma 10.
1. *Let $\gamma(\alpha, \beta) \to \mathbb{R}^n$ be a regular smooth curve. Then the arc length parameter S is smooth function of t.*
2. *The arc length parameter is a diffeomorphism onto it's image.*
3. *Let φ denote tha map: $S^{-1}(\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$. Then $\gamma \circ \varphi$ is a unit speed curve reaparameterization.*
4. *Any other unit speed reparameterization is a shift or reflection of the above.*

Proof. 1. As γ' is smooth and $<,>$ is smooth, we know $< \gamma'(t), \gamma'(t) >$ is smooth. Therefore: $\frac{ds}{dt} = ||\gamma'(t)|| = \sqrt{< \gamma'(t), \gamma'(t) >}$ is a composition of smooth function and is smooth.(*Note:* the distance function is not smooth around zero, but this poses no problem as the curve is regular).

2. Inverse of smooth function is smooth and as $||\gamma'(t)|| > 0$, S is monotonic. So S is smooth,invertible and has a smooth inverse.

3. Note that:

   $$S \circ \varphi(t) = t \Rightarrow S'(\varphi(t))\varphi'(t) = 1$$

   Therefore:

   $$||(\gamma \circ \varphi)'(t)|| = |\varphi'(t)| \times ||\gamma'(\varphi(t))|| = |\varphi'(t)|S'(\varphi(t)) = 1$$

   The key step involves noting $||\gamma'(\varphi(t))|| = \frac{d}{dt}S(\varphi(t))$

4. This is proved in homework 3

$\square$

2.4 Bending of a curve: Curvature

Chapter 3

Appendix

3.1 Some specific curves taught in class

3.1.1 Parabola

3.2 Differentiability of curves

Definition 11 (Differentiable function)**.** *A continuous function $f : (a, b) \to \mathbb{R}$(Note: a, b might be $\pm\infty$, in which case the function is defined on whole of $\mathbb{R}$) is called differentiable or of type c^1 if*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is continuous.

In a similar way, we can define C^2, C^3 and C^k by differentiating f $2, 3, k$ times respectively. We call a function "*smooth*" if the function is C^k for any $k \in \mathbb{N}$.
Examples

1. $f(x) = |x|$: This is a C^0 function as f is not differentiable at 0.

2. $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   f' exists but is not continuous. So f' is not continuous.

3.3 Differentiation (Multivariable)

Definition 12 (Differentiable multivariable function)**.** *A function $f : (a, b) \to \mathbb{R}$ is of class C^k if every $\pi_i \circ f(x)$ is C^k where π_i is projection to the i^{th} coordinate.*

Notation: We define

$$\nabla_v f(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t}$$

For C^1 functions let $v = v_1 + v_2$. Then:

$$\begin{aligned}\nabla_v f(p) - \nabla_{v_1} f(p) &= \lim_{t\to 0} \frac{f(p+tv_1+tv_2) - f(p+tv_1)}{t} \\ &= \lim_{t\to 0} \nabla_{v_2} f(p+tv_1) = \nabla_{v_2} f(p)\end{aligned}$$

$$\begin{aligned}\nabla_{kv} f(p) &= \lim_{t\to 0} \frac{f(p+tkv) - f(p)}{t} \\ &= k \lim_{t\to 0} \frac{f(p+tkv) - f(p)}{kt} \\ &= k\nabla_v f(p)\end{aligned}$$

If $v = \sum_{i=1}^n c_i e_i$, then we can apply the above formula repeatedly to get $\nabla_v f(p) = \sum_{i=1}^n c_i \nabla_{e_i} f(p)$.

Definition 13 (Differentiation of multivariable function)**.** *A function $\mathbb{R}^n \to \mathbb{R}^m$ is differntiable at a if there exists a linear map $L_a : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$f(a+v) - f(a) - L_a(v) = E_a(v)||v||$$

where $\lim_{v\to 0} E_a(v) = 0$.

A function f is called differntiable if such a linear map exists and $D_f(a) = L_a$. We can define a map $\tilde{D_f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, \tilde{D_f}(a, v) = L_a v$. This map is studied in differential geometry. A nice condition is if all the k^{th} derivative $\frac{\partial^k}{\partial x_i^{k_i}} f, \sum k_i = k$ exits and is continuous then f is C^k continuous.

3.4 Homeomorphism, Diffeomorphism and invariance of domain

Definition 14 (Diffeomorphism)**.** *Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ be open sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ repectively. A map $f : U \to V$ is a diffeomorphism if f is C^∞ and if f has a smooth inverse.*

If we consider homeomorphism between U, V then it is required that $m = n$. This proof requires techniques of algebric topology and is not discussed here.This is known as invariance of domain. But if we consider difeomorphisms, then we can prove $m = n$.

Lemma 15. *Let $f : \mathbb{R}^l \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are smooth function then $f \circ g$ is smooth.*

Notation: We shall look at a nice way to write the differentiation operator. We say

$$Df(a, v) = D_f(a)v$$

For example, we shall write the chain rule as

$$D(g \circ f)(\overrightarrow{a}, \overrightarrow{v}) = Dg(f(\overrightarrow{a}), Df(\overrightarrow{a}, \overrightarrow{v}))$$

Corollary 16. *If there exists a difeeomorphism between $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ then $m = n$.*

Proof. Without loss of generality, let $m \geq n$. Let $f : U \rightarrow V$ be the diffeomorphism and let g be it's inverse. Then $D(f \circ g)(\overrightarrow{a}, \cdot) = D(Id_{m\times m})(\overrightarrow{a}, \cdot) = Id_{m\times m}$. This follows because differentiation of a linear map T is T itself. But we also have

$$D(f \circ g)(\overrightarrow{a}, \cdot) = Dg(f(\overrightarrow{a}), Df(\overrightarrow{a}, \cdot))$$

Let $D_g(f(a)) = L^n_m$ and $D_f(a) = L^m_n$. Therefore:

$$D(f \circ g)(\overrightarrow{a}, v) = L^m_n L^n_m v$$

Now we use the rank nullity theorem.

$$\begin{aligned} dim(null(L^n_m)) &= dim\mathbb{R}^m - dim(range(L^n_m)) \\ &\geq m - dim(\mathbb{R}^n) = m - n \end{aligned}$$

Now if $dim(null(L^n_m)) \neq 0$ then we can get $v \neq 0$ such that $L^m_n L^n_m v = L^m_n 0 = 0$. But this shouldn't happen as $L^m_n L^n_m v = Id_{m\times m} v = v$. So we need $m - n = 0 \Rightarrow m = n$. □

Note: Our instructor gave a slight variation of the proof: he mentioned $L^m_n L^n_m v$ and $L^n_m v L^m_n$ can't be both invertible if $m \neq n$. I felt what he said was trivial but still need to be written out.