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ANALYSIS-5

Contents

1 Signed Measure

1.1 Introduction

Definition 1 (Signed Measure)**.** *Given a measurable space* $(X, \mathcal{M})$*, a signed measure is a function* $\nu : \mathcal{M} \to [-\infty, \infty]$ *with the following properties:*

1. $\nu(\phi) = 0$
2. ν *can assume either* ∞ *or* $-\infty$ *but not both*
3. *If* $\{E_j\}$ *is a sequence of disjoint sets in* $\mathcal{M}$*, then* $\sum_{i=1}^{\infty} \nu(E_i) = \nu\left(\cup_{i=1}^{\infty} E_i\right)$

One should note that normal measures are also signed measure, the only difference is the extension of the range of the measure function to cover almost all of $\mathbb{R}$.

1.2 Upper and lower continuity

Theorem 1 (Uppercontinuity)**.** *Let* $\{E_i\}$ *be a countable collection of measurable set with* $E_i \subseteq E_{i+1}$*. Then:*

$$\lim_{i \to \infty} \nu(E_i) = \nu\left(\bigcup_{i=1}^{\infty} E_i\right) \tag{1.1}$$

Theorem 2 (Lowercontinuity)**.** *Let* $\{E_i\}$ *be a countable collection of measurable set with* $E_{i+1} \subseteq E_i$*. Then:*

$$\lim_{i \to \infty} \nu(E_i) = \nu\left(\bigcap_{i=1}^{\infty} E_i\right) \tag{1.2}$$

Proof. Same as what we do for unsigned measure □

1.3 Positive,Negetive and Null Set

Definition 2 (Positive set)**.** *A set whose every mesurable subset* E *satisfies* $\nu(E) \geq 0$ *is called a positive set.*

In a similar fashion we define :

Definition 3 (Negetive set)**.** *A set whose every mesurable subset* E *satisfies* $\nu(E) \leq 0$ *is called a positive set.*

Definition 4 (Null set). *A set whose every mesurable subset E satisfies $\nu(E) = 0$ is called a positive set.*

We consider an example. Let μ be an unsigned measure and let f be a measurable L^1 function . Let us define a measure nu as:

$$\nu(E) = \int_E f d\mu \tag{1.3}$$

Then ν is a signed measure. If E is a set such that $f \geq 0$ $\mu.a.e$ on E then E is a positive set. Similarly we can find negetive and null sets.

Lemma 3. *1. Subsets of positive sets are positive*

2. *Countable[1] union of positive sets are positive*
   Similar results are also valid for null and negetive sets.

[1] A countable union is needed as in case of uncountable union, there will be a chance that the union will not belong to the sigma algebra; a sigma algebra is closed in countable union and not under arbitary union

The next lemma will be required for the proof of **Hahn Decomposition Theorem** in the next section.

Lemma 4. *Let ν be a signed measure which doesn't attain ∞. A set with a positive measure has a positive subset.*

1.4 Hanh Decomposition

Theorem 5 (Hanh Decompositiopn Theorem). *If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \phi$ Moreover if P', N' is another such pair, then $P \Delta P' (= N \Delta N')$ is null in ν.*

Proof Outline :

1. Define $m = \sup_{\text{positive sets}} \nu(P)$
2. Take a sequence $\{p_i\}$ such that $\lim_{i \to \infty} \nu(p_i) = m$
3. Show if $P = \bigcup p_i$ then $\nu(P) = m$
4. Show if $N = P^c$ and if N has a set with positive measure, then by lemma 4, there is contradiction.
5. If $E \subseteq P \Delta P'$ and $\nu(E) \neq 0$. Without loss of generality assume $E \subseteq P$. Then $E \subseteq P'^c = N'$ which contradicts negetivity of N'

Bibliography