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SOLUTIONS TO A COURSE IN COMBINATORICS

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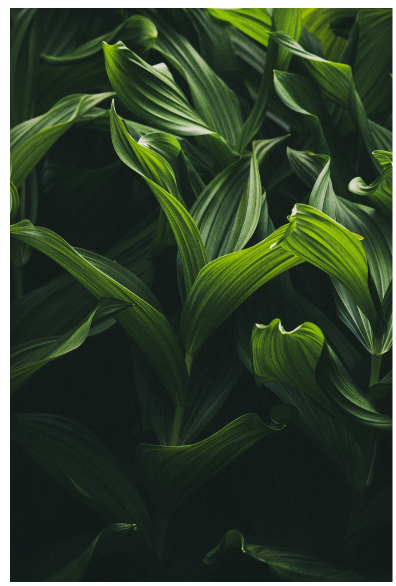
Elementary Counting and Stirling Numbers

Problems on Counting are coloured Red. Problems on Stirling numbers are coloured Blue. I would suggest to read generating functionology by Wilf before reading the second part of this chapter. All in all, it was painful.

Problem 13A
On a circular array with n positions, we wish to place the integers $1, 2, \ldots, r$ in order, clockwise, such that consecutive integers, including the pair $(r, 1)$, are not in adjacent positions on the array. Arrangements obtained by rotation are considered the same. In how many ways can this be done?



Solution 13A
Since arrangements we get by rotations are considered to be the same, we shall first rotate the array in such a way that 1 is placed in the first position and the rest of the $r-1$ elements wll be plced in the remaining $n-1$ slots. Let i is placed in position x_i. Define $y_i = x_{i+1} - x_i - 1$ for $i < r$ and $y_r = n - x_r$ [1]. It follows trivially that each $y_i > 0$. Note that:

[1] y_i is supposed to be the number of available positions between i and $i+1$ for $i < r$. For $i = r$ it is the number of available positions between r and 1

$$\sum_{i=1}^{r} y_i = n - x_1 - (r-1) = n - 1 - r + 1 = n - r$$

The total number of solutions is $\binom{n-r-1}{r-1}$ with each solution describing one arrangement.

Problem 13B
Show that the following formula for binomial coefficients is a direct consequence of (10.6)

$$\binom{n+1}{a+b+1} = \sum_{k=0}^{n} \binom{k}{a}\binom{n-k}{b}$$

Formula 10.6:

$$\sum_{i=0}^{\infty} \binom{a+i}{i} x^i = \frac{1}{(1+x)^{a+1}}$$

Solution 13B

$$\begin{aligned}\sum_{k=0}^{n} \binom{k}{a}\binom{n-k}{b} &= \sum_{k=0}^{n} \binom{k}{k-a}\binom{n-k}{n-k-b} \\ &= \sum_{k=0}^{n} \binom{a+(k-a)}{k-a}\binom{b+(n-k-b)}{n-k-b}\end{aligned}$$

Now, $\binom{a+(k-a)}{k-a}$ is coefficient of $k-a$ in $(1-x)^{-a-1}$ and $\binom{b+(n-k-b)}{n-k-b}$ is coefficient of $n-k-b$ in $(1-x)^{-b-1}$. As we sum over all possible values of k, the sum becomes equal to the coefficient of x^{n-a-b} in $(1-x)^{-(a+b+1)-1}$, which is equal to $\binom{a++b+1+n-a-b}{n-a-b} = \binom{n+1}{n-a-b} = \binom{n+1}{a+b+1}$

For such sums where the sum is taken over all values of a parameter where the function is non-zero, we can consider the sum to be from 0 to ∞. This is because in cases where the sum is not defined, the terms become 0 on their own and it saves us quite a bit of book-keeping. In this case, it would mean considering the sum to be taken from $k = 0$ to ∞. For ore details check out the first chapter of Generatingfunctionology by Herbert Wilf

Problem 13B.5
Give a combinatorial proof of above by considering $(a+b+1)$-subsets of the set $\{0, 1, \ldots, n\}$, ordering them in increasing order, and then looking at the value of the integer in position $a+1$.

Solution 13B.5
Let the $a+1^{th}$ element be $k+1$. Then there are a elements choosen from $1 \ldots k$ which can be done in $\binom{k}{a}$ ways and b elements choosen from $k+2, k+3 \ldots n+1$ which can be done in $\binom{n-k}{b}$ ways. We sum over all possible values of $a+1^{th}$ element to get all $a+b+1$ subset of $n+1$ to get the required relation.

Problem 13C
Give a solution involving binomial coefficients and a combinatorial solution to the following question. How many pairs (A_1, A_2) of subsets of $\{1, 2, ..., n\}$ are there such that $A_1 \cap A_2 = \phi$?

Solution 13C
Each element has three choices: it either belongs to A_1 or A_2 or nither in A_1 or A_2. Therefore, the total number of pairs of

A_1, A_2 is 3^n.

We can form A_1 of size $r \leq n$ in $\binom{n}{r}$ ways. We can form A_2 from the remaining elements by taking a subset in 2^{n-r} ways. Therefore, we get the total number of ways to choose A_1, A_2 by summing over r

$$\sum_{r=0}^{n} \binom{n}{r} 2^{n-r} = \sum_{r=0}^{n} \binom{n}{n-r} 2^{n-r} = (1+2)^n = 3^n$$

Problem 13D
Consider the set S of all ordered k-tuples $\mathcal{A} = (A_1, ..., A_k)$ of subsets of $\{1, 2, ..., n\}$. Determine

$$\sum_{\mathcal{A} \subseteq S} |A_1 \cup A_2 \dots A_k|$$

Solution 13D We shall consider the number of ways of forming $\mathcal{A}$ so that $|A_1 \cup A_2 \dots A_k| = r$. We shall use the method outlined in the book. Consider a $k \times n$ matrix M with $M[ij] = 1$ if $j \in A_i$ and 0 otherwise (So that the j^{th} entry of i^{th} row is 1 iff j is present in A_i). Now if there are r elements in $|A_1 \cup A_2 \dots A_k|$ then there are exactly $n - r$ columns which are zero. The zero columns can be choosen in $\binom{n}{n-r}$ ways. Each of the remaining r non zero columns can be filled in $2^k - 1$ way.[2] Therefore total ways of forming $\mathcal{A}$ is $\binom{n}{n-r}(2^k - 1)^r$. It follows that

[2] There are k entries on each column where each entry is either 0 or 1 and only 1 way to fill all of them with 0.

$$\begin{aligned}
\sum_{\mathcal{A} \subseteq S} |A_1 \cup A_2 \dots A_k| &= \sum_{r=0}^{n} r \times \text{\# of } \mathcal{A} \text{ with size r} \\
&= \sum_{r=0}^{n} r \times \binom{n}{n-r} (2^k - 1)^r \\
&= \sum_{r=0}^{n} r \times \binom{n}{r} (2^k - 1)^r
\end{aligned}$$

Note that:

$$\begin{aligned}(1+x)^n &= \sum_{r=0}^{n} \binom{n}{r} x^r \\ \frac{d}{dx}(1+x)^n &= \sum_{r=0}^{n} \binom{n}{r} \frac{d}{dx} x^r \\ n(1+x)^{n-1} &= \sum_{r=0}^{n} \binom{n}{r} r x^{r-1} \\ nx(1+x)^{n-1} &= \sum_{r=0}^{n} \binom{n}{r} r x^r\end{aligned}$$

Now set $x = 2^k - 1$ above to get:

$$\sum_{\mathcal{A} \subseteq S} |A_1 \cup A_2 \ldots A_k| = n(2^k - 1)2^{k(n-1)}$$

Taken from the hints & answer section: The nice closed form of the answer suggests that there is some better way to do this problem. The alternate solution is also given. I have tweaked it a bit according to my preference but the idea is the same

Alternate Solution:

Define a function $f : S \times \mathbb{N} \to \{0,1\}$ by

$$f(\mathcal{A}, i) = \begin{cases} 1 & i \in \bigcup A_j \\ 0 & \text{Otherwise} \end{cases}$$

Note that for some $\mathcal{A}$, $|A_1 \cup A_2 \ldots A_k| = \sum_{i=1}^{n} f(\mathcal{A}, i)$. Therefore,

$$\sum_{\mathcal{A} \subseteq S} |A_1 \cup A_2 \ldots A_k| = \sum_{\mathcal{A} \subseteq S} \sum_{i=1}^{n} f(\mathcal{A}, i) = \sum_{i=1}^{n} \sum_{\mathcal{A} \subseteq S} f(\mathcal{A}, i)$$

$\sum_{\mathcal{A} \subseteq S} f(\mathcal{A}, i)$ means number of $\mathcal{A}$ such that i is contained in the union. We now use M as defined above. As i is contained in the union, the i^{th} column filled with 0's and 1's in $2^k - 1$ ways. The remaining $k \times (n-1)$ entries can be filled in $2^{k(n-1)}$ ways. Therefore, $\sum_{\mathcal{A} \subseteq S} f(\mathcal{A}, i) = (2^k - 1)2^{k(n-1)}$ and

$$\sum_{\mathcal{A} \subseteq S} |A_1 \cup A_2 \ldots A_k| = n(2^k - 1)2^{k(n-1)}$$

Problem 13E
The familiar relation

$$\sum_{m=k}^{l} \binom{m}{k} = \binom{l+1}{k+1}$$

Find a combinatorial proof by counting paths from $(0,0)$ to $(l + 1, k + 1)$ in the X-Y plane where each step is of type $(x, y) \rightarrow (x+1, y)$ or $(x, y) \rightarrow (x+1, y+1)$. Then use the formula to show that the number of solutions of

$$x_1 + x_2 + x_3 \ldots + x_k \leq n$$

in nonnegative integers is $\binom{n+k}{k}$. Can you prove this result combinatorially?

Problem 13F
Show directly that the number of permutations of the integers 1 to n with an even number of cycles is equal to the number of permutations with an odd number of cycles $(n > 1)$. Also show that this is a consequence of Theorem 13.7.

Solution 13F
This is true for $n = 2$. Assume it is true for $N = n$. Let number of permutation with odd cycles for integers 1 to n be $O(n)$ and number of permutation with even integers be $E(n)$. Consider permutations of integers from 1 to $N = n+1$ with an even number of cycles. There are 2 cases possible:

1. $n+1$ forms an cycle by itself(i.e $\sigma(n+1) = n+1$): Then the remaining n elements form an permutation anong themselves with an odd number of cycles.

2. $n+1$ is not in a cycle by itself: Write down the pemutation in cyclic notation[3]. Note that removing $n+1$ gives a permutation of n elements with an even number of cycles. Conversely, given a permutation of n elements with an even number of cycles, there are n places to Insert a new element $n+1$ to form a permutation of $n+1$ elements.

[3] Check Artin's Algebra for details

Therefore we have:

$$E(n+1) = O(n) + nE(n)$$

and similarly:

$$O(n+1) = E(n) + nO(n)$$

Therefore, we get: $E(n+1) - O(n+1) = n\left(E(n) - O(n)\right) = 0$.
Therefore, the statement is true by induction.

Bibliography