Probability and Wiener process

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Timeline

We are going to look at:

1. Probability from a measure theoretic viewpoint

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Probability from a measure theoretic viewpoint

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3. We define the distribution of X to be $F(t) = P(X < t)$
4. For random variables $\{X_i\}_{i=1}^n$, we can define a joint distribution by considering a map from $\Omega \to \mathbb{R}^n$ and the measure defined by the distribution function given by $F(t_1, t_2 \ldots) = P(\cap\{X_i \leq t_i\})$

Independence of events

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A set of events $\{E_i\}_{i=1}^{n}$ is called A set of Independent Events if for any subset $\{E_{i_k}\}$ of those events we have:

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We can extend this definition to an infinite collection $\{E_i\}_{i=1}^{\infty}$ by requiring $\{E_i\}_{i=1}^{k}$ to be an independent set of event for all $k \in \mathbb{N}$.

Extension of independence to sub sigma algebras

Let $\{\mathcal{A}_i\}_{i=1}^n$ be a collection of sub sigma algebras. We call this set independent if for any choice of $A_i \in \mathcal{A}_i$ we have $\{A_i\}_{i=1}^n$ to be an independent set of events.

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A thing to note is if we define $\mathcal{A}_i$ to be the sigma algebra defined by E_i, then we get the definition given before this.

Independence of Random variables

We call a set of Random variables $\{X_i\}_{i=1}^{n}$ to be an Indpendent set of random variables if for any collection of measurable sets $B_i \in \mathcal{B}_{\mathbb{R}}$ we have $\{X_i^{-1}(B_i)\}$ to be a set of independent events.

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Such a collection is uniquely characterized by the fact that the joint distribution is the product of the individual distributions, i.e.

$$\boxed{P_{(X_1,X_2,X_3\ldots)} = \prod P_{(X_i)}}$$

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Law of large numbers(LLN)

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There are some restriction on random variables for which this law can be applied, and we look at three variations.

The Weak Law of large numbers.

The Weak Law of Large Numbers that if $\{X_j\}_1^\infty$ is a sequence of independent L^2 random variables with means $\{\mu_j\}$ and variances $\{\sigma_j^2\}$, and if $n^{-2}\sum_1^n \sigma_j^2 \to 0$ as $n \to \infty$, then $n^{-1}\sum_1^n (X_j - \mu_j) \to 0$ in probability as $n \to \infty$.

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✗As the name of the theorem suggests, the conditions can be relaxed.

The strong law of natural numbers

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✓Relaxes the bounds on $\lim_{n\to\infty} n^{-2}\sum_{i=1}^n \sigma_i^2$

✗Requires some work for a full prove. In particular, the Borel-Cantelli lemma and Kolmogorov’s Inequality identity is required.

The strong law of natural numbers for IIDs

If $\{X_n\}_1^\infty$ is a sequence of independent identically distributed L^1 random variables with mean μ, then $n^{-1}\sum_1^n X_j \to \mu$ almost surely as $n \to \infty$.

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- For IIDs relaxation we get a relaxation with respect to the presence of the second moment.

The Central Limit Theorem (CLT)

Weak/Vague convergence of measure

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If $X = \mathbb{R}$, then by Riesz Representation theorem, we have some function f_n such that $\mu_n(E) = \int_E f_n dm$. Then it is sufficient that f_n converges to f on the points of continuity of f.

The Central Limit Theorem (CLT)

Let $\{X_j\}$ be a sequence of independent identically distributed L^2 random variables with mean μ and variance σ^2. As $n \to \infty$, the distribution of $(\sigma\sqrt{n})^{-1} \sum_1^n (X_j - \mu)$ converges vaguely to the standard normal distribution ν_0^1, and for all $a \in \mathbb{R}$,

$$\lim_{n\to\infty} P\left(\frac{1}{\sigma\sqrt{n}} \sum_1^n (X_n - \mu) \le a\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-t^2/2} dt$$

Wiener Processes

A physical background

Wiener Processes occur naturally in nature in the form of Brownian motion. We start with the following assumptions:

1. For an increasing sequence $\{t_i\}_{i=1}^{\infty}$, the set of random variables $\{X(t_i + 1) - X(t_i)\}_{i=1}^{\infty}$ are independent.

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3. Divide n in small sections of length δ. Then we note $X(t+n) - X(n) = \sum_{i=1}^{t/\delta} X(t + (k+1)\delta) - X(t + k\delta)$. We can therefore consider the distribution to be the limiting sum of and infinite series of itself.

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4. Moreover, it is easy to see from the properties of IID's that $\sigma^2(X(t+k) - X(t)) = k\sigma^2(X(t+1) - X(t))$

The distribution function

From the above heuristic considerations, owe guess that X has the following distribution :

$$\boxed{X(t+k) - X(k) \sim \nu_0^{C(k)}}$$

$\ldots which indeed supports our assumptions.$