# Lecture Notes For Analysis

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$25^{th}$ October,2022

## References:


1. Fourier Series: Texts and Readings in Mathematics, Rajendra Bhatia.
2. An Introduction to Harmonic Analysis, Yitzhak Katznelson


## 1 Motivation

**Definition 1** ($2\pi$ periodic functions)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$ is said to be $2\pi$ periodic if $f(x+2\pi) = f(x) \forall x \in \mathbb{R}$*

1. **Periodicity:** A $2\pi$ periodic function can also be thought to have $2n\pi$ Periodicity where $n \in \mathbb{Z}\{0\}$.
2. **Bijection of $2\pi$ periodic function and continuous function on circle:** A $2\pi$ periodic function $f$ is fully described by
$$f : [0, 2\pi] \to \mathbb{R}, f(0) = f(2\pi) \quad \text{or} \quad f : [-\pi, \pi] \to \mathbb{R}, f(-\pi) = f(\pi)$$
We define the $F : \mathbb{T} \to \mathbb{R}$ where $\mathbb{T} = \{z | z \in \mathbb{C}, |z| = 1\}$ such that:
$$F(e^{i\theta}) = f(\theta)$$
This establishes a bijection between $2\pi$ periodic function and continuous function on $\mathbb{T}$. Properties like continuity and differentiability are carried over.
3. **Change of variable:** Define a map $\phi$ from $X$ to $Y$. Let $\nu$ be a measure defined on $X$. Then we can define a corresponding measure $\mu$ on $Y$ defined as
$$\mu(E) = \nu(\phi^{-1}(E))$$
Then we have the following formula for changing the variable:
$$\int_E f d\mu = \int_{\phi^{-1}(E)} f \circ \phi d\nu$$
4. **Integration of functions on a circle:** We define a normalized/probabilistic Lebesgue measure defined as
$$\tilde{m}(E) = \frac{1}{2\pi} m(E)$$
For a map $F : \mathbb{T} \to \mathbb{R}$ we have:
$$\int_{\mathbb{T}} F d\tilde{m} = \frac{1}{2\pi} \int_{[-\pi,\pi]} f dm = \frac{1}{2\pi} \int_{-\pi}^{\pi} f d\theta$$

5.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta} = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

### 1.1 Some notions from complex analysis

**Definition 2** (Unit Disc, $\mathbb{D}$)**.** *We define the unit disc as the set* $\mathbb{D} = \{z : |z| < 1\}$

**Definition 3** (Closure of Unit Disc, $\overline{\mathbb{D}}$)**.** *We define the closure unit disc as the set* $\mathbb{D} = \{z : |z| \leq 1\}$

**Definition 4** (Unit Circle, $\mathbb{T}$)**.** *We define the boundary of a unit disc as* $\mathbb{T} = \{z : |z| = 1\}$

**Remark.** *Note that* $\mathbb{T}$ *is written as* $S^1$ *in topological contexts.*

**Definition 5** (Harmonic Functions)**.** *A function* $f$ *is said to be Harmonic in a domain* $\Omega$ *if* $\frac{\partial^2}{\partial x^2}f$ *and* $\frac{\partial^2}{\partial y^2}f$ *are well-defined in* $\Omega$ *and* $\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0$

### 1.2 The Dirichlet Problem and Fourier's big idea

This is a type of PDE. Assume we are given a function $f \in C(\mathbb{T})$. We are interested in knowing if there exists a harmonic function $u$ such that $u|_{C(\mathbb{T})} = f$. Fourier noticed that $e^{in\theta}$ is harmonic. So if we can represent $f$ is the following form

$$f = \sum_{i=\mathbb{Z}} a_n e^{in\theta}$$

then we are done. But the problem with this idea is that this doesn't happen for all functions.(It happens for a large class of functions though.)

## 2 Preliminary Ideas

**Lemma 1.** *If* $f$ *is continuous function on* $\mathbb{R}$ *then the following holds:*

1. $\int_a^b f(x)dx = \int_{a+2\pi}^{b+2\pi} f(x)dx$
2. $\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} f(x+a)dx$
3. $\int_a^{a+2\pi} f(x)dx$ *is independent of* $a$.

*Those holds even if* $f \in L^1[-\pi, \pi]$

We define the map $\Gamma_{\theta_o}$ as below:

$$\Gamma_{\theta_o} F(e^{i\theta}) = F(e^{i(\theta-\theta_o)})$$

We note that

$$\int_{\mathbb{T}} \Gamma_{\theta_o} F d\tilde{m} = \int_{\mathbb{T}} F d\tilde{m}$$

Therefore, $\Gamma_{\theta_o}$ is an isometry on $L^1(\mathbb{T})$. As $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ for $1 \leq p \leq \infty$, we can extend the previous idea and conclude $\Gamma_{\theta_o}$ is an isometry on $L^p(\mathbb{T})$ for $1 \leq p \leq \infty$.

**Lemma 2.** *If* $f \in L^p(\mathbb{T})$ *then the following hold:*

1. $\Gamma_\theta \Gamma_{\theta_o} f = \Gamma_{\theta+\theta_o} f$
2. $\lim_{\theta\to\theta_0} ||\Gamma_\theta f - \Gamma_{\theta_o} f||_p = 0$

# 3 Convolution

**Definition 6** (Convolution)**.** *We define the convolution of two functions $f, g$ as*

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)g(y)dy$$

**Lemma 3.**

1. *If $f, g \in L^1(\mathbb{T})$ then $f * g \in L^1(\mathbb{T})$ and*

$$||f * g||_1 \leq ||f||_1 ||g||_1$$

2. *If $f \in L^p(\mathbb{T}), g \in L^1(\mathbb{T})$ then:*

$$||f * g||_p \leq ||f||_p ||g||_1$$

3. *If $f \in L^p(\mathbb{T}), g \in L^{p*}(\mathbb{T})$ such that $1/p + 1/p* = 1$ then:*

$$||f * g||_\infty \leq ||f||_p ||g||_{p*}$$

**Theorem 4** (Young's inequality)**.** *If $f \in L^p(\mathbb{T}), g \in L^q(\mathbb{T})$, then*

$$||f * g||_r \leq ||f||_p ||g||_q$$

*where $1 + 1/r = 1/p + 1/q$*