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Probability-II
Assignment-1

Problem 3
Let $\{X_n\}$ be a sequence of identically distributed random variables and let $M_n = \max\{|X_1|, |X_2|, \ldots, |X_n|\}$.
(a) If $\mathbb{E}[|X_1|] < \infty$, then show that M_n/n converges to zero almost surely.
(b) Generalization of part (a): If $\mathbb{E}[|X_1|^\alpha] < \infty$ for some $\alpha \in (0, \infty)$, then show that $M_n/n^{1/\alpha}$ converges to zero almost surely.
(c) Show that if the sequence $\{X_n\}$ is also independent, then $M_n/n^{1/\alpha}$ converges to zero almost surely implies that $\mathbb{E}[|X_1|^\alpha] < \infty$.
[Hint : $M_n \geq |X_n| \cdot$]

Solution: We shall use the following lemma:

Lemma 1. *If* $a_n/n \to 0$ *then* $\max_{1\leq i\leq n} a_i/n \to 0$

Proof. Take any $\epsilon > 0$. There exists N such that for $n > N$ $|an/N| < \epsilon$. Let $M = max_{1\leq i\leq n} a_i$. If $M \geq \sup_{n>N} a_n$ then $\max_{1\leq i\leq n} a_i/n \leq M/n$ and we are done. Else, there exists N' such that for $n > N'$, $\max_{1\leq i\leq n} a_i = a_j$ where $j > N$. But then we have $\max_{1\leq i\leq n} a_i/n = a_j/n = (a_n/j)(j/n) \leq \epsilon(j/n) \leq \epsilon$. Therefore, we conclude that $\max_{1\leq i\leq n} a_i/n$ converges to 0. ☐

(a) By problem 2(a)[Note that in 2a we don't need independence assumptions], $E[|X_1] < \infty \Rightarrow X_n/n \to 0$ almost surely. Consider the set:

$$\mathcal{X} = \left\{\omega | \frac{|X_n(\alpha)|}{n} \to 0\right\}$$

We know that $P(\mathcal{X}) = 1$. For $\alpha \in \mathcal{X}$, $X_n(\alpha)/n \to 0$. We use the lemma above to conclude that $M_n(\alpha)/n \to 0$. Therefore, $\{\omega | M_n(\omega)/n \to 0\} \subset \mathcal{X}$ and thus $P(M_n/n \to 0) \geq P(|X_n|/n \to 0) = 1$ or $M_n \to 0$ almost surely.
(b) Note that:

$$\max\{|X_1|^\alpha, |X_2|^\alpha, |X_3|^\alpha ... |X_n|^\alpha\} = M_n^\alpha$$

By (a) we conclude that $M_n^\alpha/n \to 0$ almost surely. Note $f(x) = x^{\frac{1}{\alpha}}$ is a continuous map. Therefore, by the continuous mapping theorem, $M_n/n^{1/\alpha} \to 0$ almost surely.
(c) If $M_n/n^{1/\alpha} \to 0$ almost surely, then by continuous mapping theorem $M_n^\alpha/m \to 0$ almost surely. As $X_n^\alpha(\omega) > \epsilon n \Rightarrow M_n(\omega)^\alpha > \epsilon n$, $P(M_n^\alpha > \epsilon n) \geq P(|X_n|^\alpha > \epsilon n)$. Therefore, $\sum_{n=1}^\infty P(X_n/n^\alpha < \epsilon) \leq \sum_{n=1}^\infty P(M_n/n^\alpha < \epsilon) < \infty$, and so we can conclude