AAKASH GHOSH (19MS129)

ANALYSIS 5

Contents

1 Signed Measure

1.1 Introduction

Definition 1 (Signed Measure)**.** *Given a measurable space* $(X, \mathcal{M})$*, a signed measure is a function* $\nu : \mathcal{M} \to [-\infty, \infty]$ *with the following properties:*

1. $\nu(\phi) = 0$

2. ν *can assume either* ∞ *or* $-\infty$ *but not both*

3. *If* $\{E_j\}$ *is a sequence of disjoint sets in* $\mathcal{M}$*, then* $\sum_{i=1}^{\infty} \nu(E_i) = \nu\left(\cup_{i=1}^{\infty} E_i\right)$

One should note that normal measures are also signed measure, the only difference is the extension of the range of the measure function to cover almost all of $\mathbb{R}$.

1.2 Upper and lower continuity

Theorem 1 (Uppercontinuity)**.** *Let* $\{E_i\}$ *be a countable collection of measurable set with* $E_i \subseteq E_{i+1}$*. Then:*

$$\lim_{i\to\infty} \nu(E_i) = \nu\left(\bigcup_{i=1}^{\infty} E_i\right) \tag{1.1}$$

Theorem 2 (Lowercontinuity)**.** *Let* $\{E_i\}$ *be a countable collection of measurable set with* $E_{i+1} \subseteq E_i$*. Then:*

$$\lim_{i\to\infty} \nu(E_i) = \nu\left(\bigcap_{i=1}^{\infty} E_i\right) \tag{1.2}$$

Proof. Same as what we do for unsigned measure □

1.3 Positive,Negetive and Null Set

Definition 2 (Positive set)**.** *A set whose every mesurable subset* E *satisfies* $\nu(E) \geq 0$ *is called a positive set.*

In a similar fashion we define :

Definition 3 (Negetive set)**.** *A set whose every mesurable subset* E *satisfies* $\nu(E) \leq 0$ *is called a positive set.*

Definition 4 (Null set). *A set whose every mesurable subset E satisfies $\nu(E) = 0$ is called a positive set.*

We consider an example. Let μ be an unsigned measure and let f be a measurable L^1 function . Let us define a measure nu as:

$$\nu(E) = \int_E f d\mu \tag{1.3}$$

Then ν is a signed measure. If E is a set such that $f \geq 0$ $\mu.a.e$ on E then E is a positive set. Similarly we can find negetive and null sets.

Lemma 3. *1. Subsets of positive sets are positive*

2. Countable[1] *union of positive sets are positive*
Similar results are also valid for null and negetive sets.

[1] A countable union is needed as in case of uncountable union, there will be a chance that the union will not belong to the sigma algebra; a sigma algebra is closed in countable union and not under arbitary union

The next lemma will be required for the proof of **Hahn Decomposition Theorem** in the next section.

Lemma 4. *Let ν be a signed measure which doesn't attain ∞. A set with a positive measure has a positive subset.*

1.4 *Hahn Decomposition*

Theorem 5 (Hahn Decompositiopn Theorem). *If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \phi$ Moreover if P', N' is another such pair, then $P \Delta P' (= N \Delta N')$ is null in ν.*

Proof Outline :

1. Define $m = \sup_{\text{positive sets}} \nu(P)$
2. Take a sequence $\{p_i\}$ such that $\lim_{i \to \infty} \nu(p_i) = m$
3. Show if $P = \bigcup p_i$ then $\nu(P) = m$
4. Show if $N = P^c$ and if N has a set with positive measure, then by lemma 4, there is contradiction.
5. If $E \subseteq P \Delta P'$ and $\nu(E) \neq 0$. Without loss of generality assume $E' = E \cap P$ is not null. Then $E' \subseteq P'^c = N'$ which contradicts negetivity of N'

1.5 *Jordan Decomposition*

Definition 5 (Mutually singular measures). *Two measures ν and μ are said to be mutually singular if there exists a partition of X in E and F such that $X = E \sqcup F$ and E is null in μ and F is null in ν*[2]

[2] That is to say that the measures ν and μ "lives" on different sets.

Notation: If ν and μ are mutuallysingular, then we denote it as:

$$\nu \perp \mu$$

Theorem 6 (Jordan Decomposition Theorem)**.** *Given a (signed) measure ν there exists* ***unique*** *positive measures ν^+, ν^- such that:*

$$\nu = \nu^+ - \nu^- \quad \nu^+ \perp \nu^- \tag{1.4}$$

Proof Outline: [3]

1. Existance follows by Hahn decomposition.
2. Start by assuming the decomposition is not unique and theere exists two such decomposition $\nu = \nu^+ - \nu^- = \mu^+ - \mu^-$.
3. There exists partition of X in E, F due to μ^+, μ^- and in P, N due to ν^+, ν^-. If A is measurable , show that
$$\mu^+(A) = \nu(A \cap E) = \nu(A \cap E \cap P) + \nu(A \cap E \cap N)$$
4. As E is positive and N is negetive, show that $A \cap E \cap N$ is a null set. Repeat or ν^+ and get similar results
5. Show $\nu^+ = \mu^+$ and in a similar way $\mu^- = \nu^-$

[3] As I understand it, the main idea is if there is two decomposition as outlined in step 2 and 3, then we have 4 sets to deal with:

- $P \cap F$ and $E \cap N$: which are null as they are intersection of positive and negetive sets
- $P \cap E$ where μ^+ and ν^+ agree and $\mu^-, \nu^- = 0$
- $N \cap F$ where μ^- and ν^- agree and $\mu^+, \nu^+ = 0$

Make this nice and you get the proof outlined.

1.6 Total Variation Measure

Definition 6 (Total Variation Measure)**.** *If a measure ν decomposes in singular ν^+ and ν^- then we define the total variation measure $|\nu|$ as*

$$|\nu| = \nu^+ + \nu^- \tag{1.5}$$

This works before as by Hahn-Jordan, the decomposition is unique. The definition is important as by Lemma 7 and Lemma 8, we see that properties of ν is reflected in $|\nu|$

Lemma 7. *The following statements are equivalent:*

1. *E is null in ν*
2. *$\nu^+(E) = 0$ and $\nu^-(E) = 0$*[4]
3. *$|\nu|(E) = 0$*

[4] For unsigned measures, being null and having a measure 0 is same.

Lemma 8. *The following statemwents are equivalent:*

1. *$\nu \perp \mu$*
2. *$\nu^+ \perp \mu$ and $\nu^- \perp \mu$*
3. *$|\nu| \perp \mu$*

Proof for lemma 7 and 8 is at the end, they are given as exercise in Folland, ch3. Other properties which gets reflected are finiteness and $\sigma-$finiteness.

1.7 Absolute Continuity

Definition 7 (Absolute Continuity)**.** *Let μ be an unsigned measure. We say ν is absolutely continious with respect to μ if for any measurable set E, $\mu(E) = 0 \implies \nu(E) = 0$*

Notation: ν is absolutely continious with respect to μ is denoted by:

$$\nu \ll \mu$$

Unlike mutual singulaity, $\nu \ll \mu$ doesn't imply $\mu \ll \nu$. In a sense, being mutually singular and being absolutely continious are exclusive concepts. If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$

Lemma 9. *The following statements are equivalent:*

1. $\nu \ll \mu$
2. $\nu^+ \ll \mu$ *and* $\nu^- \ll \mu$
3. $|\nu| \ll \mu$

Lemma 10. *If ν and μ are finite measures, $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ usch that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$* [5]

[5] This lemma gives some motivation for the nomenclature of absolute continuity

Proof Outline:

1. By Lemma 9, we need to show this is true for $|\nu|$ and we will be done. This is why, without loss of generality, we can assume ν is unsigned.
2. ***Don't understand why this is trivial***
3. Make a decresing sequence of mesurable sets
4. Show if there exists ϵ with no such δ then μ of intersection goes to 0 but ν of intersection stays above ϵ. This contradicts absolute continuity.

1.8 Radon-Nikodym theorem

Theorem 11 (Radon-Nikodym theorem)**.** *The theorem has two parts:*

1. *For a measure space, with $\sigma-$finite measure signed measure ν and $\sigma-$finite measure unsigned measureμp, there is a unique decomposition of ν in ν_1 and ν_2 such that $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$*
2. *There exists a function f which is integrable in the etended sense such that $\nu_1(E) = \int_E f d\mu$. Moreover, if there are two such functions f_1, f_2 then $f_1 = f_2$ μ.a.e.*

Proof Outline:

1. **Step 1: ν, μ are finite**
   (a) Note that $\nu(E) = \int_E f d\mu + \nu_2(E) \Rightarrow \nu(E) \geq \int_E f d\mu$
   (b) Make a family of function $\mathcal{F}$ which satisfy this.
   (c) Let α be suprema of the integral of f in family. Find f_n whose integral approach α. Set $g_n(x) = \max\{f_1(x), f_2(x) \dots f_n(x)\}$. Show g_n is increasing and is in $\mathcal{F}$. Find limit of g_n as g. Use MCT to show that α is attained by g.[6]
   (d) Set $\nu_2 = \nu - \nu_1$. Show $\nu_2 \perp \mu$.

2. **Step 2: Assume σ, μ are $\sigma-$finite.**
   (a) Divide X in disjoint countable B_i each with finite measure.
   (b) Restrict μ and ν in B_i to get μ_i, ν_i. Repeat step 1 to get f_i, ν_i^1, ν_i^2 in B_i. Set $f = \sum f_i, \nu^1 = \sum \nu_i^1, \nu^2 = \sum \nu_i^2$

3. **Step 3: Uniqueness of decomposition**: If ν_1, ν_2 and $\hat{\nu}_1, \hat{\nu}_2$ are two decomposition then $\nu_1 - \hat{\nu}_1 = \hat{\nu}_2 - \nu_2$. Now $(\nu_1 - \hat{\nu}_1) \ll \mu$ and $\nu_2 - \hat{\nu}_2 \perp \mu$. So $\nu_1 - \hat{\nu}_1 = 0$

4. **Step 4: Uniqueness of f :** If f, g are two such functions then $\int_E (f - g) = 0$ or $f = g$ $\mu.a.e.$

The decomposition of ν is called **Leabesgue decomposition**. The f obtained in part 2 is called the **Radon-Nikodym derivative**.
If $\nu \ll \mu$ then the existance of Radon Derivative doesn't require σ finiteness of ν[7]

[6] The reason to take a in first place is because we don't know if f_n converges. The reason we take such a g_n is sothat we can intechange the limit and integral by applying MCT.

[7] This is a really nasty tyeorem- Check for all conditions before applying. Both ν and μ should be sigma finite for decomposition to exist. ν should be absolutely cts and μ should be σ finite for the derivative to exist.

1.9 *Solutions to Real Analysis By Folland, Section 3.1,3.2*

1.9.1 *Solutions to problems in section 3.1*

Problem 3.1

Prove Proposition 3.1.
Propotion 3.1 is Theorem 1&2 mentioned here.

Solution 1 outline: We do it as instructed in the book, by copying Theorem 1.8 from the book.
Solution 2 outline: Decompose μ by Hahn Decomposition and apply upper/lower continuity on each of them individually.[8]
Proof 1:
By the second condition of definition 1, we can assume $\mu > -\infty$.

[8] Proof of Hahn decomposition doesn't assume upper/lower continuity

Proving Uppercontinuity: If some $E_i = \infty$ we are done. Else , set $E_0 = \phi$. Define $F_i = E_i \setminus E_{i-1}$. Not that any two F_i, F_j is disjoint. Then $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} F_i$. It follows that:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) = \lim_{i\to\infty} \mu(E_i)$$

The last step follows by countable additivity.
Proving Lowercontinuity: Set $F_j = E_1 \setminus E_j$. Then $F_i \subseteq F_{i+1}$ and $\mu(E_1) = \mu(F_j) + \mu(E_j)$. Also, $\cup_{i=1}^{\infty} F_j = E_1 \setminus (\cap_{i=1}^{\infty} E_j)$. Apply uppercontinuity to get:

$$\mu(E_1) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right) + \lim_{j\to\infty} \mu(F_j) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right) + \mu(E_1) - \lim_{j\to\infty} \mu(E_j)$$

$$\Rightarrow \lim_{j\to\infty} \mu(E_j) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right)$$

Problem 3.2

If ν is a signed measure, E is ν-null iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$

Solution outline: This the the problem corresponding to lemma 7 and 8. In both case we shall show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.
Solution part-1(Proof of lemma 7):
The Steps are based on lemma 7.

- **Step 1:** $1 \Rightarrow 2$
  Let P, N be the decomposition of N in positive and negetive sets using HJD[Hahn Jordan Decomposition]. Let E be a null set. Then:

$$\begin{aligned}
\nu^+(E) =& \nu^+(E \cap P) - \nu^-(E \cap N) \\
=& \nu^+(E \cap P) \\
=& \nu(E \cap P) - \nu^-(E \cap P) \\
=& \nu(E \cap P) = 0
\end{aligned}$$

  Similar reult is obtained for ν^-. For unsigned measures, a measure zero set is null set, so we are done.

- **Step 2:** $2 \Rightarrow 3$
  This is the easy step.

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$$

- **Step 3:** $3 \Rightarrow 1$
  Note that for any measurable subset A of E we have $|\nu|(A) = 0$. We also have:

$$|\nu(A)| = |\nu^+(A) - \nu^-(A)| \le \nu^+(A) + \nu^-(A) = |\nu|(A) = 0$$
$$\Rightarrow \nu(A) = 0$$

  Therefore, E is null in ν

Solution part-2(Proof of lemma 8):
The Steps are based on lemma 8.

- **Step 1:** $1 \Rightarrow 2$
  Let P, N be the decomposition of N in positive and negetive sets using HJD[Hahn Jordan Decomposition]. Let A, B be the disjoint decomposition of X for ν and μ. Then it is easy to check every element lies in one of the four sets: $A \cap P, A \cap N, B \cap P, B \cap N$. Now note,
  - Decomposition of X for ν^+ and μ is achived by $A \cap P$ and $(A \cap N) \cup (B \cap P) \cup (B \cap N)$
  - Decomposition of X for ν^- and μ is achived by $A \cap N$ and $(A \cap P) \cup (B \cap P) \cup (B \cap N)$
- **Step 2:** $2 \Rightarrow 3$
  Let decomposition of X for ν^+ and μ be E_1, F_1 and for ν^- and μ be E_2, F_2. We calim the decomposition of X for $|\nu|$ and μ is given by $E_1 \cup E_2$ and $F_1 \cap F2$.[9] Note that $|\nu|$ is null in $F_1 \cap F_2$ both ν^+ and ν^- is null in F_1, F_2. μ is null in both E_1 and E_2. By lemma 3, part 2, μ is null in $E_1 \cup E_2$.
- **Step 3:** $3 \Rightarrow 1$
  Let A, B be the disjoint decomposition of X for $|\nu|$ and μ. We claim this is the appropiate decomposition for ν and μ as well. It is already known μ is null in A. As $|\nu$ is null in B, ν is null in B follows from lemma 7($3 \Rightarrow 1$).

[9] This is low-key motivated by the decomposition in step 1.

Problem 3.3

Let ν be a signed measure on $(X, \mathcal{M})$. Prove

1. $\mathcal{L}^1(\nu) = \mathcal{L}^1(|\nu|)$
2. If $f \in \mathcal{L}^1(\nu)$, $|\int f d\nu| \le \int |f| d|\nu|$
3. If $e \in \mathcal{M}$, $|\nu|(E)| = \sup\{|\int_E f d\nu| : |f| \le 1\}$

Solution outline: Our main goal will be to study f on the decomposition of X made by HJD due to ν

Solution part-1:
Let X be decomposed into positive set P and negetive set N. We assume $\nu > -\infty$. Let $f \in \mathcal{L}^1(\nu)$. Let χ_E denote the characteristic function on E. Then we have:

$$\int |f| d|\nu| = \int |f|(\chi_P + \chi_N) d(\nu^+ + \nu^-) = \int |f| d\nu^+ + \int |f| d\nu^- < \infty \tag{1.6}$$

Therefore, $f \in \mathcal{L}^1(|\nu|)$.
Now assume $f \in \mathcal{L}^1(|\nu|)$. Then as before,

$$\infty > \int |f| d|\nu| = \int |f|(\chi_P + \chi_N) d(\nu^+ + \nu^-) = \int |f| d\nu^+ + \int |f| d\nu^- \tag{1.7}$$

But as $\nu+, \nu^-$ are both unsigned we can conclude that $\int |f| d\nu^+, \int |f| d\nu^- < \infty$. Therefore, $f \in \mathcal{L}^1(\nu)$
Solution part-2:

$$\begin{aligned}
\left|\int f d\nu\right| &= \left|\int f(\chi_P + \chi_N) d(\nu^+ - \nu^-)\right| \\
&= \left|\int f d\nu^+ - \int f d\nu^-\right| \\
&\leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\
&\leq \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu|
\end{aligned}$$

Solution part-2:
Firt we show that $|\nu|(E)$ is an upper bound and then we show that it is attained. For any measurable f with $|f| \leq 1$ we have:

$$\left|\int_E f d\nu\right| \leq \int_E |f| d|\nu| \leq \int_E d|\nu| = |\nu|(E)$$

Therefore, $|\nu|(E)$ is an upper bound. Now set $f = \chi_P - \chi_N$. For any $x \in X$,either $x \in P$ or $x \in N$. Therefore $f(x) \in \{1, -1\}$.

$$\begin{aligned}
\left|\int f_E d\nu\right| &= \left|\int_E (\chi_P - \chi_N)(\chi_P + \chi_N) d(\nu^+ - \nu^-)\right| \\
&= \left|\int_E (\chi_P^2 - \chi_N^2) d(\nu^+ - \nu^-)\right| \\
&= \left|\int_E \chi_P^2 d\nu^+ + \int_E \chi_N^2 d\nu^-\right| \\
&= \left|\int_E \chi_P d\nu^+ + \int_E \chi_N d\nu^-\right| \\
&= |\nu^+(E \cap P) + \nu^+(E \cap N)| \\
&= |\nu^+(E \cap P) + \nu^+(E \cap N) + \nu^+(E \cap N) + \nu^+(E \cap P)| \\
&= \nu^+(E) + \nu^-(E) = |\nu|(E)
\end{aligned}$$

Problem 3.4

If ν is a signed measure and λ and μ are unsigned measures such that $\nu = \lambda - \mu$ then $\lambda \geq \nu^+, \mu \geq \nu^-$

Solution outline: There are two ways to do this. The easy but long way is to show that this is true for positive and negetive sets respectively and then stich things up. The hard and short way is to do the same stiching but in a single line.

Proof :

Let $X = P \sqcup N$ be the Hahn Jordan decomposition of X in positive set P and negative set N. Assume $E \subseteq P$. Then:

$$\nu(E) = \nu^+(E) + \nu^-(E) = \nu^+(E) = \lambda(E) - \mu(e) \leq \lambda(E)$$

$$\nu^-(E) = 0 \leq \mu(E)$$

The same steps can be repeated on subsets of N. Now for any measurable set E, we have:

$$\nu^+(E) = \nu^+(E \cap P) + \nu^+(E \cap N) \leq \lambda(E \cap P) + \lambda(E \cap N) = \lambda(E)$$
$$\nu^-(E) = \nu^-(E \cap P) + \nu^-(E \cap N) \leq \mu(E \cap P) + \mu(E \cap N) = \mu(E)$$

Which completes the proof.

The following hard and short proof was given by Nitish:

$$\nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P) \leq \lambda(E \cap P) \leq \lambda(E)$$

Problem 3.5

If ν_1, ν_2 are signed measures that both omit ∞ or $-\infty$ then

$$|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$$

Proof :

We note that:

$$\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$$

We use the result of the previous question. Treat $\nu_1^+ + \nu_2^+$ as λ and $\nu_1^- + \nu_2^-$ as μ.

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) = |\nu_1| + |\nu_2|$$

Problem 3.6

Suppose $\nu(E) = \int f d\mu$, where μ is a positive measure and f is an extended $\mu-$integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ ·

Solution outline :
By the nature of Hahn Jordan we aill be done once we find out any suitable decomposition: the theorem will take care of uniqueness. The proof uses the folling insight: E will be a positive set in ν if $f >\geq 0$ in E.
Proof :
Set:

$$P = \{x|f(x) \geq 0\}$$

$$N = \{x|f(x) < 0\}$$

It is easy to see that $X = P \sqcup N$. It follows that:

$$\nu(E) = \int_{E\cap P} f d\mu + \int_{E\cap N} f d\mu = \int_E |f|\chi_P d\mu - \int_E |f|\chi_N d\mu$$

Note that $\nu^+(E) = \int_E |f|\chi_P d\mu$, $\nu^-(E) = \int_E |f|\chi_N d\mu$ is a valid decomposition of ν in positive measures. It follows that the toatal variation measure is:

$$|\nu(E)| = \nu^+(E) + \nu^-(E) = \int_E |f|(\chi_P + \chi_N)d\mu = \int_E |f|d\mu$$

Problem 3.7

Suppose ν is signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$ Then:

1. $\nu^+(E) = \sup\{\nu(F)|F \subseteq E, F \in \mathcal{M}\}$ and $\nu^-(E) = \inf\{\nu(F)|F \subseteq E, F \in \mathcal{M}\}$
2. $|\nu|(E) = \sup\{\sum_{i=1}^n |\nu(E_j)||n \in \mathbb{N}, E = \bigsqcup_{j=1}^n E_j\}$

Solution outline : For both cases we show that the relevant figures are upper bound and the upper bound is attained.
Proof :

1. $\nu^+(E)$ is an uupper bound as $\nu^+(E) \geq \nu^+(F) \geq \nu(F)$. Set $F = E \cap P$ where P is the positive set in Hahn Jordan decomposition of X. Similar thing is to be done for ν^-

2. For any $\{E_j\}, 1 \leq j \leq n, E = \bigsqcup_{j=1}^n E_j$ note that $|\nu|(E_j) \geq |\nu(E_j)|$ and $\sum_{i=1}^n |\nu(E_j)| \leq \sum_{i=1}^n |\nu|(E_j) = |\nu|\left(\cup_{j=1}^n E_j\right) = |\nu|(E)$. The bound is attained by setting $n = 2$, $E_1 = P, E_2 = N$ where P, N are positive and negetive sets in X as found by Hahn Jordan decomposition.

1.9.2 Solutions to problems in section 3.2

Those problems are harder than the ones in 3.1. This is not something I appreciate. Also I don't have much time. So I will upload hand written solutions for this part.

Problem 3.8

Show that the following statements are equivalent:

1. $\nu \ll \mu$
2. $\nu^+ \ll \mu, \nu^- \ll \mu$
3. $|\nu| \ll \mu$

Proof :
Let $X = P \sqcup N$ be the decomposition of X in positive set P and negetive set N according to Hahn Jordan decompositon.
Part 1: (1⇒2)
For $E \in \mathcal{M}$,

$$\mu(E) = 0 \Rightarrow \mu(E \cap P) = 0 \Rightarrow \nu(E \cap P) = 0 \Rightarrow \nu^+(E \cap P) = 0 \Rightarrow \nu^+(E) = 0$$

$$\mu(E) = 0 \Rightarrow \mu(E \cap N) = 0 \Rightarrow \nu(E \cap N) = 0 \Rightarrow \nu^-(E \cap N) = 0 \Rightarrow \nu^-(E) = 0$$

Therefore, $\nu^+ \ll \mu, \nu^- \ll \mu$
Part 1: (2⇒3)
For $E \in \mathcal{M}$ and $\mu(E) = 0$ we have $\nu + (E) = 0, \nu^-(E) = 0$. Therefore, $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$ and we can conclude $|\nu| \ll \mu$
Part 3: (3⇒1)
For $E \in \mathcal{M}$ and $\mu(E) = 0$ we have $|\nu|(E) = 0$. We note that

$$0 \leq |\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E) = 0$$

It follows that $\nu(E) = 0$. Therefore, $\nu \ll \mu$.

Problem 3.9

Suppose $\{\nu_j\}$ is a sequence of unsigned measures.

1. If $\nu_j \perp \mu$ for all j then $\sum_{j=1}^{\infty} \nu_j \perp \mu$
2. If $\nu_j \ll \mu$ for all j then $\sum_{j=1}^{\infty} \nu_j \ll \mu$

Solution outline : The second part is easy. In the first part getting the appropiate decomposition of X is the crucial step. Studying the problem with $\nu_!, \nu_2$ helped in my case. The key insight is union and intersection of null sets are null.
Proof :

1. For every j there exists a decomposition of X as $F_j \sqcup E_j$ such that $\mu(F_j) = 0$, $\nu(E_j) = 0$. Set $F = \cup_{i=1}^{\infty} F_j, E = \cap_{i=1}^{\infty} E_j$. It follows that $\nu_j(F) \leq \nu_j(F_j) = 0 \Rightarrow \sum_{j=1}^{\infty} \nu_j(F) = 0$ and $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_j) = 0 \Rightarrow \mu(E) = 0$

2. Let $\mu(E) = 0$. Then $\forall j \in \mathbb{N}, \nu_j(E) = 0 \Rightarrow \sum_{j=1}^{\infty} \nu_j(E) = 0$.

Problem 3.10

Theorem 3.5 may fail if ν is not finite. Show this for the following cases:

1. $d\nu(x) = \frac{dx}{x}$ and $d\mu(x) = dx$ on (0,1)
2. ν is the counting measure and $\mu(E) = \sum_{n \in E} 2^{-n}$ on $\mathbb{N}$

Proof:
Proving absolute continuity is easy in every case.

1. Set $\epsilon = 1/2$. Let there exist some $\delta < 1$ for which the theorem holds. Consider the set $E = (0, \min(\delta, 1))$. Note that $\mu(E) \leq \delta$. But $\int_E d\nu = \infty > \epsilon$. We need to prove that the integral diverges. Consider the following sequence of increasing functions $\{f_n\}, f_n = \chi_{[\frac{1}{n},1]}$. It is easy to see that $f_n \to 1$ on $(0,1)$ and f_n is increasing. Note that $\int_E f_n d\nu = \int_{1/n}^{\delta} \frac{1}{x} dx = \ln(n) + \ln(\delta)$. By monotone convergence theorem, we have:

$$\int_E d\nu = \int_E \lim_{n\to\infty} f_n d\nu = \lim_{n\to\infty} \int_E f_n d\nu = \infty$$

2. We use the fact that $\{\sum_{i=1}^{n} 1/2^i\}$ is a cauchy sequence. Set $\epsilon = 1$. Assume a delta exists. There exists N such that $\sum_{i=N}^{\infty} 1/2^i \leq \delta$. Set $E = \{N, N+1, N+2 \ldots\}$. Therefore, $\mu(E) < \delta$ but $\nu(E) = \infty > \epsilon$

Bibliography