Lecture Notes for Geometry of Curves and Surfaces and the Riemann Uniformization Theorem

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Part I

Course Notes

Those are the course notes on geometry of curves and surfaces taken by Dr. Somnath Basu in Spring of 2022. The course mostly followed **Elementary Differential Geometry** by **Andrew Pressley**. I presonally found **Differential geometry by Spivak(vol 2)** to be particularly good, especially if one has covered vol 1 to a certain extent.

Chapter 1

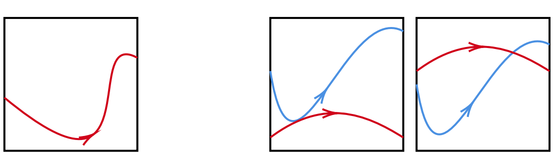
What to expect

First we shall see some theorems and examples and get to know the terms on an intuitive sense. The formal definitions of everything will be provided down the line(probably).

1.1 Four vertex theorem

Theorem 1 (Four Vertex Theorem)**.** *The curvature of a simple closed smooth curve has at least 2 local minima and two local maxima.*

[Planar] Curve: A curve is a map f from $\mathbb{R} \to \mathbb{R}^2$. Note: other types of curves exists. For example, a space curve is a map from $\mathbb{R} \to \mathbb{R}^3$.
Smooth: Smooth implies that the k^{th} derivative of f defined in the previous section exists and is continuous for all $k \in \mathbb{N}$.
Closed: Intuitively, a closed curve has the same starting and ending point. We define $f : [0, c] \to \mathbb{R}^2$ as closed if $f(0) = f(c)$. If we want to define it according to the definition we gave earlier, we say f is periodic. This implies for every $p \in \mathbb{R}$, $f(p + c) = f(p)$. The smallest such c is known as the period of the function. We can also view this in an geometric way: A closed curve is a mapping of S^1 to $\mathbb{R}^2$(S^1 is the unit circle in $\mathbb{R}^2$ plane). Let $e^{i\theta}$ be a point on S^1. Then a closed curve is a mapping $\tilde{f} : S^1 \to \mathbb{R}^2$. Now we can of course relate $\tilde{f}$ and f in a very natural way: $\tilde{f}(e^{i\theta}) = f(\frac{c}{2\pi}\theta)$(This result comes from the fact that the parameters which define S^1, i.e $\sin\theta$ and $\cos\theta$ are periodic in nature).
Simple : Intuitively, a simple curve is a curve which doesn't cross or touch itself. Therefore, if you zoom close enough at any point, it looks like a regular arc. mathematically we need $\tilde{f}$ to be injective or one-one.



$Simple$ $\qquad$ $Not\ Simple$

Curvature : Curvature is basically a function from the points of the curve to $\mathbb{R}$. Specifically $\mathcal{K} : \mathbb{R}^2 \to \mathbb{R}$, where $\mathcal{K}(\tilde{f}(e^{i\theta}))$ gives the inverse of the radius of the circle with the best fit at the point $\tilde{f}(e^{i\theta})$. Now this circle of best fit needs to be defined with some care. for one the the point and the circle drawn share the same normal and tangent. One way to drive this concept home

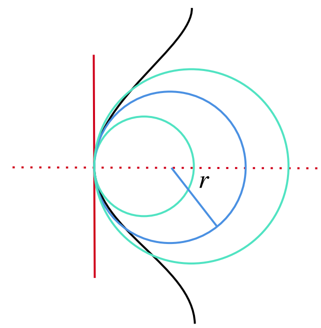
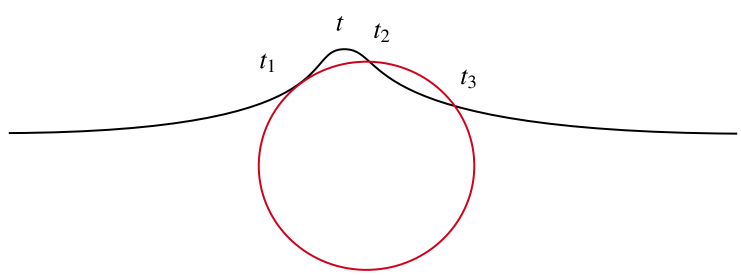


Figure 1.1: The red lines are common normal and common tangent. Note, the circle of medium size somehow seems to be a better fit than the others. So the radius of curvature at that point is $\frac{1}{r}$

for planar curve is to consider the following process: take any three points t_1, t_2, t_3 around t. Draw the circumcircle of those points. the radius of curvature will be the radius of the circle formed when $\lim t_1, t_2, t_3 \to t$



The figure above shows the process. While this is nice way to view things, it is has it's own set of problems: How do we know that a limit exists? More over in higher dimensional spaces, the plane in which the circle lies will also change with the points that are being considered. Nevertheless it's good enough for a naive mental picture. The rest of the theorem now makes sense.

1.2 Fary-Milnor theorem

Theorem 2 (Fary-Milnor theorem)**.** *If the total absolute curvature of a knot K is at most 4π, then K is an unknot.*

Knot :A knot is a simple curve in 3-space. **Total absolute curvature :**What it says. We can't of course take an infinite sum for all the points so we integrate.

$$S = \int_S |\mathcal{K}(s)|\, ds \tag{1.1}$$

Unknot :A unknot is a knot which can be deformed to a circle.

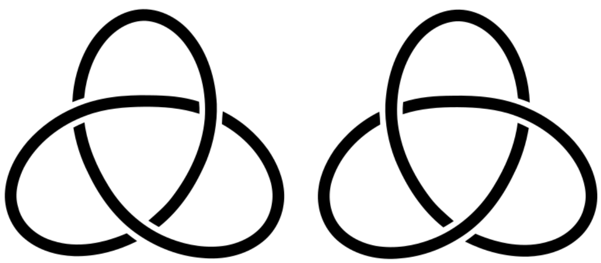


Figure 1.2: The two trefoil knots. They are not unknots(Copied from Wikipedia)

1.3 Hirsch-Smale Theory

Theorem 3 (Hirsch-Smale Theory)**.** *Any two immersed loop in $\mathbb{R}^2$ are isotopic if and only if the winding numbers match.*

Loop: A loop is a closed curve
Immersed: $f : \mathbb{R} \to \mathbb{R}^k$ is said to be immersed if $f'(t)$ is never zero. Note, the drawing of any curve is the trace of a curve: It does not represent the curve itself.
Isotopic: This means they are the same. It is what isomorphism is for groups and homeomorphism is for topologies. But when are two loops isotopic? But how to check this? We construct a

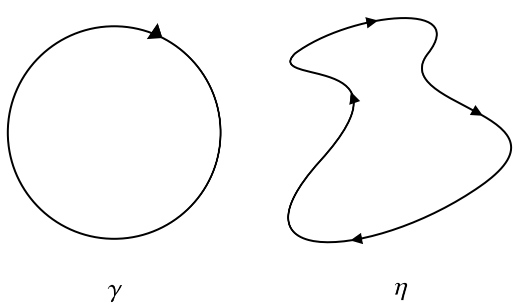


Figure 1.3: Two isotopic curves: The second one can be "straightened out" into the first one

f which deforms η to γ. But note: merely the presence of such a function is not enough. This would imply any two loops are isotopic. We furthermore require at any instant of the transformation, the resulting curve is immersed. Now we shall talk a bit about the function. Let $H : S^1 \times I \to \mathbb{R}^2$, where $I = [0,1]$. Now note that the set $S^1 \times I \to \mathbb{R}^2$ looks like a cylinder. S^1 represents the base, I represents the height. We think of moving along I as moving along time. The base of the cylinder is the curve when we start(i.e. $H(S^1 \times \{0\})$) and the top of the cylinder is the curve we end up with(i.e. $H(S^1 \times \{1\})$). We also need $H(S^1 \times t)$ to be immersed.

Winding number: It is the total number of loops we make

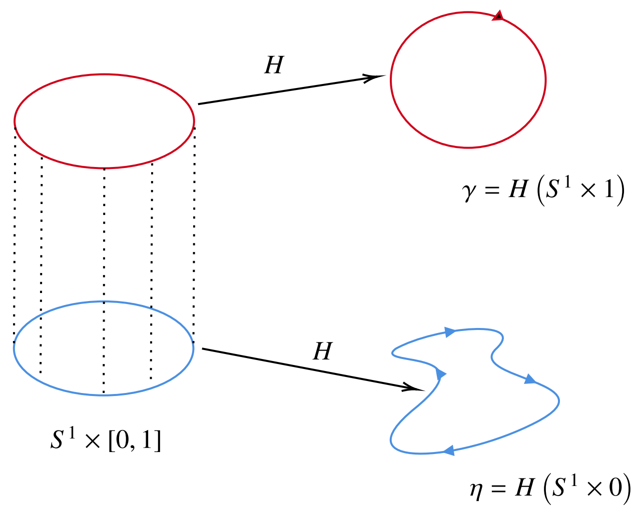


Figure 1.4: Representation of H

when we complete traversing closed curve. Alternatively, imagine holding out a compass needle. Then the total number of complete turns the needle makes will be the winding number.

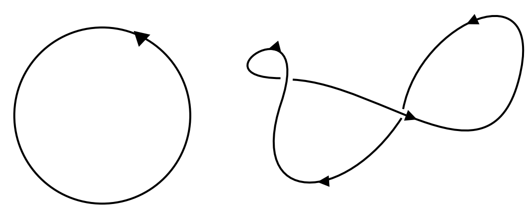


Figure 1.5: Winding number 1

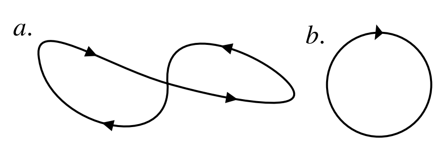


Figure 1.6: a. Winding number 0 b. Winding number -1

But what if we consider curves in S^2? Then the theorem changes as follows:

Theorem 4 (Hirsch-Smale Theory, 2^{nd} part)**.** *Any two immersed loop in S^2 are isotopic if and only if the winding numbers match modulo 2.*

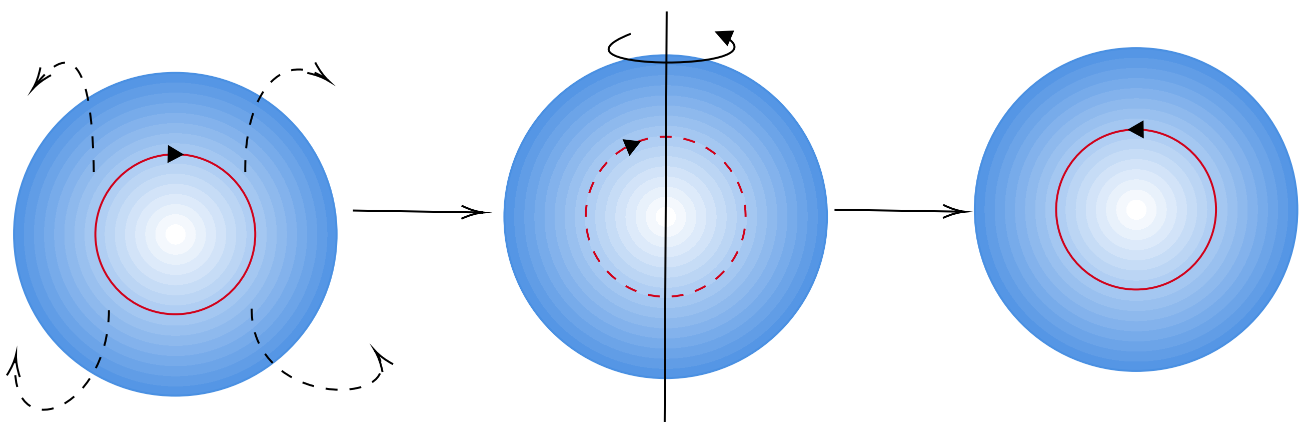


Figure 1.7: Example of part two of theorem. The winding number changes from -1 to +1. The dotted lines represent the curve on the rear side of the sphere.

Chapter 2

Curves

2.1 Introduction

We shall first consider some probable definitions of curves and see why they are wrong.

1. **A curve is a set with empty interior**. This on the first glance looks pretty good: after all our by our geometric intuition, a curve has nothing inside it. But while it does take into account all things we intuitively understand as curve, there are things included in this which are not very curve like. For example, consider:

$$f(x, y) = \begin{cases} 1 & x \in \mathbb{Q} \times \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \times \mathbb{Q} \end{cases}$$

   As $\mathbb{Q}$ is dense in $\mathbb{R}$, $S = \{(x, y)|f(x, y) = 0\}$ has a empty interior. But it's obvious that S which has some weird appearance is definitely not a curve.

2. **A curve is the graph of a function.** Therefore, every curve can be represented as $(x, f(x))$ or $(x, f_1(x), f_2(x))$. This definition suffers from the opposite problem of the previous definition. Note, structures like a vertical line($y = k$) or a parabola($x = y^2$) can not be graphs of functions because they fail the vertical line test and are yet curves.

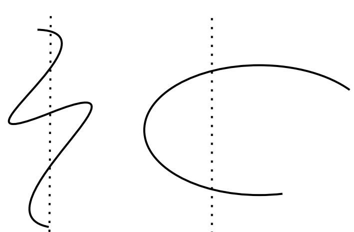


Figure 2.1: Two curves which are not graphs of a function

3. **A curve is the set of points where a function vanishes.** We might think this works: after all, we can now circumnavigate around our previous problem by making an suitable function like $f(x, y) := y - k$ or $f(x, y) := x - y^2$. But this again makes things which are intuitively not a curve, a curve.

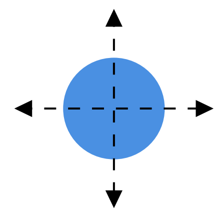


Figure 2.2: Set $f(x, y) = \max\{x^2 + y^2 - 1, 0\}$. The resulting figure, clearly, doesn't form something we would like to recognise as a curve

2.2 Whitney's Teorem

Theorem 5 (Whitney's Theorem)**.** *Let $\Omega \in \mathbb{R}^n$ be any open subset. Given a closed $C \subseteq \Omega$, there exists a smooth function $f : \Omega \to \mathbb{R}$ such that $C = f^{-1}(0)$*

Proof. **Step 1(Covering):** Note, $V = \Omega \setminus C$ is open in Ω(and in fact, it is also open in $\mathbb{R}^n$). We want to cover this by a collection of open balls. We do this by taking every $q_n(\in \mathbb{Q} \times \mathbb{Q} \dots)$ and an appropriate radius for each q_n. Therefore:

$$V = \bigcup_k B(\overrightarrow{p_k}, r_k) \quad [\overrightarrow{p_k} \in \mathbb{Q}^n \cap V, r_k \in \mathbb{Q}]$$

Note, this is a covering of V because Q is dense in $\mathbb{R}$.
Step 2(Choosing Bump Functions): Choose $f_k : \mathbb{R}^n \to \mathbb{R}$(those are smooth functions) such that

$$f_k^{-1}(0) = \mathbb{R}^n \setminus B(\overrightarrow{p_k}, r_k)$$

$$f_k^{-1}(1) = \overline{B(\overrightarrow{p_k}, r_k/2)}$$

This has been done as solution to problem 2 in exercise 1. Essentially, this looks like a smooth pudding(I feel sand heap is a better example) with a flat top. Moreover, all derivative of f_k vanishes on $\mathbb{R}^n \setminus \overline{B(\overrightarrow{p_k}, r_k)}$
Step 3(Combining those functions): We want to combine(by adding) those bump functions appropriately with weights such that the partial sums converge uniformly and their derivatives converge uniformly too. Note, for every point outside the closed set, it is present in some open ball with the bump. Therefore, if we assign a positive weight to every bump function, then , if the

sum converges, $f^{-1}(0) = C$ as all the other points lie in some bump. We define

$$c_k = \max_{|\alpha|\leq k, \overrightarrow{y}\in B(\overrightarrow{p_k}, r_k)} \left| \frac{\partial^\alpha}{\partial x^m} f_k(\overrightarrow{y}) \right|$$

Note on notation: α is a *multi-index* and each x^m is differentiation w.r.t a particular index α times. For example, if $\alpha = (1,0,2)$, it would mean $\frac{\partial^3}{\partial x_1 \partial x_3^2} f_k(x)$. As f_k is assumed to be smooth, order of derivatives don't matter. In particular, for $\alpha = 1$, the max will be calculated between $f(\overline{y})$, $\frac{\partial}{\partial x_1}(\overline{y})$, $\frac{\partial}{\partial x_2}(\overline{y})$, $\frac{\partial}{\partial x_3}(\overline{y})$ on the whole set.
Now as f and it's derivative are continuous and we are considering the maximum on a compact set, c_k is well defined and exists. Those c_k will become our weights. We define f as

$$f = \sum_{k=0}^{\infty} \frac{f_k}{2^k c_k}$$

Step 4(Analytical verification):
Check $f^{-1}(0) = C$. If $\overrightarrow{p} \neq C$ then $\overrightarrow{p} \in V$, whence $\overrightarrow{p} \in B(\overrightarrow{p_k}, r_k)$, then $f(p) > \frac{f_k(p)}{2^k c_k} > 0$. If $\overrightarrow{p} \in C$, then $p \notin B(\overrightarrow{p_k}, r_k)$ for any k so $f(p) = 0$. Now we show f is C^0, by showing convergence of the sequence of f. Given $\epsilon > 0$ choose N such that $\frac{1}{2^N} < \epsilon$. Compare $S_m(f)$ and $S_n(f)$ with $m > n \geq N$. $S_n(f)$ is the partial sum $\sum_{k=1}^{n} \frac{f_k}{2^k c_k}$. Note:

$$\sup_{x\in\mathbb{R}^n} |f_m(x) - f_n(x)| = \sum_{i=m}^{n} \frac{f_k}{2^k c_k} \leq \sum_{i=m}^{n} \frac{1}{2^k} \leq \sum_{i=m}^{\infty} \frac{1}{2^k} \leq \frac{1}{2^m}$$

The thing to note is we have $\frac{f_k}{c_k} < 1$ due to how c_k is defined. For higher derivatives it still holds(again, due to how c_k is defined) albeit with slight modifications. The $1/2^k$ is present to ensure convergence.

Note: This theorem is not very important in this course but the proof shows a classic old trick: finding covers, attaching weights and then performing analysis to get the desired result. □

2.3 Parameterized Curve

Definition 6 (Parameterized Curve)**.** *A paameterized smooth curve is a smooth function $\gamma : (a,b) \to \mathbb{R}^n$ where $-\infty \leq a \leq b \leq \infty$.(we include $\pm\infty$ as the function may map the whole of $\mathbb{R}$ to $\mathbb{R}^n$).*

Generally, we use lower Greek letters (like η, φ, etc.) to denote curves.

2.3.1 Reparameterization of curve

Definition 7 (Reparameterization of a curve)**.** *Let $\varphi(\alpha', \beta') \to (\alpha, \beta)$ be a homeomorphism. If γ is a parameterised curve as defined above, then $\eta : (\alpha', \beta') \to \mathbb{R}^n, \eta = \gamma \circ \varphi$ is called the reparameterization of γ.*

If φ is a diffeomorphism and γ is smooth then η is smooth. (see appendix, section 4). In particular, if γ is C^k then η is C^k if φ is diffeomorphism.

$$\eta'(t) = \underbrace{\gamma'(\varphi(t))}_{=Vector} \underbrace{\varphi'(t)}_{=Scalar}$$

If φ is C^0 then φ is monotonic. This is ascertained by the intermediate value theorem. If φ is atleast C^1 then we can further confirm $\varphi \neq 0$. Let ψ be the inverse of φ. Then:

$$\psi \circ \varphi(x) = x \Rightarrow \psi'(\varphi(x))\varphi'(x) = 1$$

Therefore, $\varphi(x) \neq 0$ and φ is strictly increasing or strictly decreasing. If $\varphi(\alpha') = \alpha', \varphi(\beta') = \beta$ then η is orientation preserving and if $\varphi(\alpha') = \beta', \varphi(\beta) = \alpha'$ then η is orientation reversing.

2.3.2 Arc length parameter

Definition 8 (Arc-Length)**.** *The arc length of a differentiable curve $\gamma(\alpha, \beta) \to \mathbb{R}^n$ starting at t_0 is defined as:*

$$S(t) = \int_{t_o}^{t} ||\gamma'(u)||du$$

This formula is quite intutive. We can consider the curve to be path of a particle, $\gamma(u)$ to be the position at time u. Then $\gamma'(u)$ gives the velocity at u and taking it's magnitude gives the speed. Integrating speed at various points gives the distance covered, or in this case, length of the arc traversed.
As γ' is atleast C^0 and the interval is closed and bounded, the integral is well defined and finite. Moreover:

$$\frac{d}{dt} S(t) = ||\gamma'(t)||$$

Definition 9 (Regular Curve)**.** *A curve γ is called regular if $\gamma' \neq 0$*

Example: A curve with unit speed(i.e $||\gamma'(u)|| = 1$ everywhere).

Lemma 10.

1. *Let $\gamma(\alpha, \beta) \to \mathbb{R}^n$ be a regular smooth curve. Then the arc length parameter S is smooth function of t.*
2. *The arc length parameter is a diffeomorphism onto it's image.*
3. *Let φ denote tha map: $S^{-1}(\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$. Then $\gamma \circ \varphi$ is a unit speed curve reaparameterization.*
4. *Any other unit speed reparameterization is a shift or reflection of the above.*

Proof.

1. As γ' is smooth and $<,>$ is smooth, we know $< \gamma'(t), \gamma'(t) >$ is smooth. Therefore: $\frac{ds}{dt} = ||\gamma'(t)|| = \sqrt{< \gamma'(t), \gamma'(t) >}$ is a composition of smooth function and is smooth.(*Note:* the distance function is not smooth around zero, but this poses no problem as the curve is regular).
2. Inverse of smooth function is smooth and as $||\gamma'(t)|| > 0$, S is monotonic. So S is smooth,invertible and has a smooth inverse.

3. Note that:

$$S \circ \varphi(t) = t \Rightarrow S'(\varphi(t))\varphi'(t) = 1$$

Therefore:

$$\|(\gamma \circ \varphi)'(t)\| = |\varphi'(t)| \times \|\gamma'(\varphi(t))\| = |\varphi'(t)|S'(\varphi(t)) = 1$$

The key step involves noting $\|\gamma'(\varphi(t))\| = \frac{d}{dt}S(\varphi(t))$

4. This was supposed to be proved in a homework. I will give the outline of the proof here.

   (a) Prove any diffeomorphism of (a, b) is a open interval

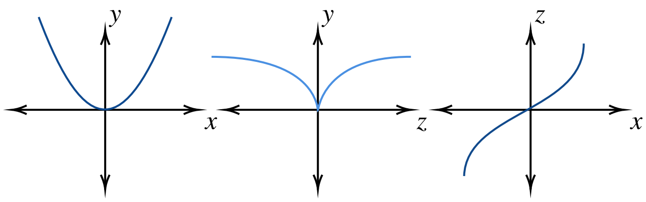
   (b) Prove that if γ is unit speed then the length of domain for all unit speed parameterizations are same

   (c) Any two open interval of same lenght are either a shift or a shit+reflection.

$\square$

2.3.3 Example - Twisted Cubic

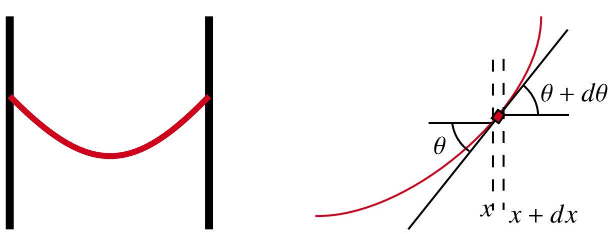
The Twisted Cubic is a space curve defined by $\gamma(t) = (t, t^2, t^3)$. Given below are the projections on $x - y$, $y - z$ and $x - z$ planes.



We set the starting point to be $(0, 0, 0)$. $\gamma'(t) = (1, 2t, 3t^2)$. $S(t) = \int_0^t \sqrt{1 + 4u^2 + 9u^4}du$

2.3.4 Example - Catenary

A catenary is basically the curved formed by hanging uniform cable.



a) The general shape. The thick red line denotes the hanging cable. b) Analysis using a unit element(shown as thick red box)

Let the equation be given by $y = f(x)$. Then we note the following:

1. $f'(x) = \tan(\theta)$
2. $T(x)\cos(\theta) = T(x + \Delta x)\cos(\theta + \Delta\theta)$
3. $T(x)\sin(\theta) + g\delta\Delta S = T(x + \Delta x)\sin(\theta + \Delta\theta)$

In the equations above, $T(x)$ is the tension at $(x, f(x))$, δ is the mass per unit length and ΔS is the length of the unit element. From equation 1 and 2 we get:

$$\frac{T(x)}{\sqrt{1 + [f'(x)]^2}} = T_0(constant)$$

Put this in 3 to get:

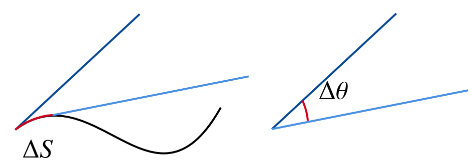
$$\begin{aligned}
&T(x)\frac{\sin(\theta + \Delta\theta)}{\cos(\theta + \Delta\theta)}\cos(\theta) - T(x)\sin(\theta) = g\delta\Delta S \\
&\Rightarrow T(x)[\tan(\theta + \Delta\theta) - \tan(\theta)] = g\delta\Delta S\ \sec(\theta) \\
&\Rightarrow T(x)\lim_{\Delta x \to 0}\frac{\tan(\theta + \Delta\theta) - \tan(\theta)}{\Delta x} = g\delta\sec(\theta)\lim_{\Delta x \to 0}\frac{\Delta S}{\Delta x} \\
&\Rightarrow T(x)f''(x) = g\delta\sec(\theta)\sqrt{1 + [f'(x)]^2} \\
&\Rightarrow f''(x) = \frac{g\delta}{T(x)\cos(\theta)}\sqrt{1 + [f'(x)]^2} \\
&\Rightarrow f''(x) = c\sqrt{1 + [f'(x)]^2}
\end{aligned}$$

Where $c = \frac{g\delta}{T_0}$ is a constant. Now we solve this equation(preferebly with the help of wolfram alpha) to get:

$$f(x) = \frac{\cosh cx}{c}$$

2.4 Bending of a curve: Curvature

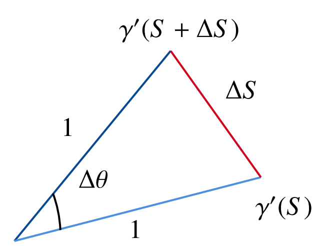
Definition 11 (Curvature)**.** *Let γ be a regular curve in $\mathbb{R}^n$. Let $\Delta\theta$ and ΔS be as shown below.*



Then the curvature is defined to be rate of change of the tangent vector with curve length or

$$\kappa(t) = \lim_{\Delta t \to 0}\frac{\Delta\theta}{\Delta s}$$

For a unit speed curve, $\|\gamma'(S)\| = 1$. We have the following diagram:



We note:

$$\|\gamma'(S + \Delta S) - \gamma'(S)\| = 2\sin\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{\Delta\theta}{\Delta S} = \frac{\Delta\theta/2}{\sin\left(\frac{\theta}{2}\right)}\frac{\|\gamma'(S + \Delta S) - \gamma'(S)\|}{\Delta S} = |\gamma''(S)|$$

We immidietly notice for curvature to be defined we eed:

1. γ to be Regular
2. γ'' to be defined

*Notation:*If γ is unit speed then we use s as the parameter.

2.4.1 Examples

1. For a straightline given by $\gamma(t) = \vec{a} + \vec{p}t$, Curvature is 0
2. For a circle of radius r given by $\gamma(t) = (x_0, y_0) + \left(r\cos\left(\frac{t}{r}\right), r\sin\left(\frac{t}{r}\right)\right)$, curvature=$\|\gamma''\| = \frac{1}{r}$

2.4.2 Calculation of curvature for space curves

Let γ be a regular C^3 curve. The curvature is given by:

$$\kappa(t) := \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

2.4.3 Remarks

1. If γ lies on an unit sphere, with $\|\gamma''\| = 1$ Then $<\gamma(t), \gamma(t)> = 1 \Rightarrow <\gamma(t), \gamma'(t)> = 0$. Therefore, $\kappa(t) = \frac{\|\gamma''(t)\|}{\|\gamma'(t)\|^2} = \|\gamma''(t)\|$. In this case, $\gamma''(t)$ or acceleration essentially measures the rate of change of dirction of the curve.
2. Let $\eta(t) = \gamma \circ u(t)$ be a reparameterization of γ. Then:
   - $\eta'(t) = \gamma'(u(t))\frac{du}{dt}$
   - $\eta''(t) = \gamma''(u(t))\left(\frac{du}{dt}\right)^2 + \gamma'(u(t)).\frac{d^2u}{dt^2}$

   It follows $\eta'(t) \times \eta''(t) = \left(\frac{du}{dt}\right)^3 \gamma'(u(t)) \times \gamma''(u(t))$ and:

   $$\kappa_\gamma(u(t)) = \kappa_\eta(t)$$

   Therefore, we conclude curvatuere is invariant under reparameterization.

Lemma 12. *If $\gamma'' \neq 0$ for a plane curve, then γ turns left if $\det(\gamma'(0), \gamma''(0))$ is posative and right if the determinant is negetive.*

Proof. We may assume $\gamma(0) = 0$ and $\gamma'(0) = (1, 0)$(this can be achived by an isometry+ suitable reprameterizatio). We therefore have $\gamma'' \perp \gamma'$. Therefore $\gamma'' = (0, c)$ and the value of the dterminant is c. For $\Delta t > 0$ note:

$$\gamma(\Delta t) = \gamma(0) + \Delta t\gamma'(0) + \frac{1}{2}(\Delta t)^2\gamma''(0)\ldots = \left(\Delta t, \frac{1}{2}\Delta t^2 c\right) + \epsilon$$

The curve turns right if $c < 0$ and left $c > 0$ which is what we needed to prove. □

Remark. *Our expression of curvature requires a suitable definition of cross product which is not available in higher dimensions. So we have this alternate formulation for curvature:*

$$\kappa(t) = \frac{\|\gamma''\langle\gamma', \gamma'\rangle - \gamma'\langle\gamma', \gamma''\rangle\|}{\|\gamma'\|^4}$$

One can check (I haven't) that in $\mathbb{R}^3$ this reduces to the usual definition.

2.5 Osculating Circles and Planes

Let γ be a regular C^2 curve with $\||\gamma'| = 1$. Let $\gamma''(s) \neq 0$ at some $s \in \mathbb{R}$. The osculating plane is defined as span$\{\gamma', \gamma''\}$

Theorem 13. *Let γ be parameterized with $\gamma'' \neq 0$.*

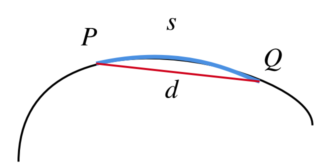
1. *For s_1, s_2, s_3 sufficently close to s, the points $\gamma(s_1), \gamma(s_2) \& \gamma(s_3)$ are not co-linear.*
2. *As s_1, s_2, s_3 tend to s, the plane spanned by s_1, s_2, s_3 tend to the oscullating plane.*
3. *The circumcircle $C(s_1, s_2, s_3)$ associated to $\gamma(s_1), \gamma(s_2), \gamma(s_3)$ tend to be a circle lying on the osculating plane & passes through $\gamma(s)$ with radius*

   $$1/\gamma''(s)$$

4. *For sufficently close to s, there is a plane with $\gamma(s_1)$ and tangent to γ at s.*

Proof. The proof was supposed to be here, but unfortunately I dont have it. I have to revisit this place later and fill in the balnks or ask sir about it. □

2.6 Relation between $d = d(P, Q)$ and s = arc length from P to Q



We can fix P as $\gamma(0)$, $\gamma'(s) = 1$ by a suitable reparameterization. Then

$$d^2 = \|\gamma(s) - \gamma(0)\|^2$$
$$s = \int_0^s \|\gamma'(u)\|du$$

Theorem 14. *If γ is C^3 then:*

1. $\lim_{s\to 0} \frac{d}{s} = 1$
2. $\lim_{s\to 0} \frac{d-s}{s^3} = \frac{-1}{24}\|\gamma''(0)\|^2$

Proof. □

2.7 Twisting of a curve: Torsion

Torsion is a measure of how much the osculating plane turn as we move along the curve.

Definition 15 (Torsion)**.** *Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a regular curve such that the osculating plane is defined at all times. We define the torsion of γ at s as*

$$\tau(s) = \lim_{\Delta s\to 0} \frac{\Delta\theta}{\Delta S}$$

where $\Delta\theta$ is the angle between the osculating planes at s and $s + \Delta s$

We can consider the unit normal to the osculating planes as

$$\mathcal{N}(s) = \frac{\gamma'(s) \times \gamma''(s)}{||\gamma'(s) \times \gamma''(s)||}$$

Note:

1. $\mathcal{N}$ is perpendicular to the osculating plane.
2. The angle between the normals is precisely the angle between the planes

Torsion is given by

$$\tau(t) = \frac{\det \begin{vmatrix} \gamma'(t) & \gamma''(t) & \gamma'''(t) \end{vmatrix}}{||\gamma'(t) \times \gamma''(t)||^2}$$

Remark. *1. τ, like κ is an invariant.*

2. If $\tau = 0$ everywhere, the curve lies on a plane.

2.7.1 Examples

- For a circular helix given by $\gamma(t) = (\lambda t, r\cos\omega t, r\sin\omega t)$, the torsion is given by

$$\tau(t) = \frac{\lambda\omega}{r^2\omega^2 + \lambda^2}$$

- For the twisted cubic given by $\gamma(t) = (t, t^2, t^3)$

$$\tau(t) = \frac{3}{1 + 9t^2 + 9t^4}$$

2.8 Fernet-Serret frames

This is perhaps the most important (and useful!) theorem on curves taught in this course. Define

$$\mathbb{T} = \frac{d}{dt}\gamma(t) \quad \mathbb{N} = \frac{\frac{d\mathbb{T}}{dt}}{||\frac{d\mathbb{T}}{dt}||} \quad \mathbb{B} = \mathbb{T} \times \mathbb{N}$$

Theorem 16 (Fernet-Serret)**.** *For a unit speed curve with nowhere vanishing curvature, the following holds*

$$\begin{bmatrix} \mathbb{T}' \\ \mathbb{N}' \\ \mathbb{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbb{T} \\ \mathbb{N} \\ \mathbb{B} \end{bmatrix}$$

Proof. We shall go through this step by step.

1. $\mathbb{T}, \mathbb{N}, \mathbb{B}$ forms an orthoinormal basis
   $\mathbb{N}$ and $\mathbb{T}$ are permpendicular to each other. As B is the cross product of two orthogonal vectors, $\mathbb{B} \perp \mathbb{N}, \mathbb{B} \perp \mathbb{T}$. By definition they all have length 1.

2. $\mathbb{T}' = \kappa\mathbb{N}$
   This follows rather simply as for unit speed curves we have $\kappa = ||\gamma''||$ and therefore, by definition, we have:

$$\kappa\mathbb{N} = \mathbb{T}'$$

3. $\mathbb{B}' = c\mathbb{N}$ for some scalar c.
   Note that

$$\langle \mathbb{B}, \mathbb{B} \rangle = 1 \Rightarrow \langle \mathbb{B}', \mathbb{B} \rangle + \langle \mathbb{B}, \mathbb{B}' \rangle = 0 \Rightarrow \langle \mathbb{B}', \mathbb{B} \rangle = 0$$

   As $\mathbb{T}, \mathbb{N}, \mathbb{B}$ span the space, we have: $\mathbb{B}' = a\mathbb{T} + b\mathbb{N}$ Nut we note:

$$\langle \mathbb{B}, \mathbb{T} \rangle = 0 \Rightarrow \langle \mathbb{B}', \mathbb{T} \rangle + \langle \mathbb{B}, \mathbb{T}' \rangle = 0$$

   But $\mathbb{T}' = \kappa\mathbb{N}$ and $\mathbb{N} \perp \mathbb{B}$. Therefore $\langle \mathbb{B}, \mathbb{T}' \rangle = 0$ and

$$\langle \mathbb{B}', \mathbb{T} \rangle = 0$$

   It follows that $a = 0$ and we set $c = b$.

4. $c = -\tau$
   We shall solve $\det \begin{vmatrix} \gamma'(t) & \gamma''(t) & \gamma'''(t) \end{vmatrix}$ in two ways.

   (a)

$$\begin{aligned} &\det \begin{vmatrix} \gamma'(t) & \gamma''(t) & \gamma'''(t) \end{vmatrix} = \left(\gamma'(t) \times \gamma''(t)\right) \cdot \gamma'''(t) \\ =&(\mathbb{T} \times \mathbb{T}') \cdot (\mathbb{T}'') = (\mathbb{T} \times \mathbb{T}') \cdot (\mathbb{T}'') = (\mathbb{T} \times \kappa\mathbb{N}) \cdot (\kappa'\mathbb{N} + \kappa\mathbb{N}') \\ =&\kappa^2(\mathbb{T} \times \mathbb{N}) \cdot \mathbb{N}' = \kappa^2\mathbb{B} \cdot \mathbb{N}' \end{aligned}$$

   (b) Note for unit speed curves, $\gamma' \perp \gamma''$, $||\gamma'|| = 1$ and $||\gamma''|| = \kappa$. Therefore, we have:

$$\det \begin{vmatrix} \gamma'(t) & \gamma''(t) & \gamma'''(t) \end{vmatrix} = \tau(t)||\gamma'(t) \times \gamma''(t)||^2 = \tau(t)\kappa^2(t)$$

   Therefore, we get $\mathbb{B} \cdot \mathbb{N}' = \tau\kappa$.

$$\begin{aligned} &\langle \mathbb{B}, \mathbb{N} \rangle = 0 \\ \Rightarrow &\langle \mathbb{B}', \mathbb{N} \rangle = -\langle \mathbb{B}, \mathbb{N}' \rangle = -\tau \end{aligned}$$

5. Similar to before, we have:

$$\langle \mathbb{N}, \mathbb{N} \rangle = 1 \Rightarrow \langle \mathbb{N}', \mathbb{N} \rangle = 0$$

   We write $\mathbb{N}' = \alpha\mathbb{T} + \beta\mathbb{B}$. Now note that

$$\begin{aligned} \langle \mathbb{T}, \mathbb{N} \rangle = 0 \Rightarrow \langle \mathbb{T}, \mathbb{N}' \rangle + \langle \mathbb{T}', \mathbb{N} \rangle = 0 \Rightarrow \alpha = -\kappa \\ \langle \mathbb{B}, \mathbb{N} \rangle = 0 \Rightarrow \langle \mathbb{B}, \mathbb{N}' \rangle + \langle \mathbb{B}', \mathbb{N} \rangle = 0 \Rightarrow \beta = \tau \end{aligned}$$

The proof is completed. □

Chapter 3

Surfaces-Charts,Tangent Spaces and Fundamental Forms

3.1 Surfaces

Definition 17 (Surface)**.** *A surface $\sum \subset \mathbb{R}^3$ is a subset satisfying the following property: For $p \in \sum$, there exists a open set $W \subseteq R^3$ around p and an open set $U \subseteq \mathbb{R}^2$ such that there exists a homeomorphism*

$$\varphi : U \to W \cap \sum$$

Remark.

1. *The pair $\left((W \cap \sum), \varphi^{-1}\right)$ is called a chart around p.*
2. *Given a surface and a point $p \in \sum$, we can find a chart U', ψ around p such that $\psi : U \to B_1(0) \subseteq \mathbb{R}^2$*

Part II

The Riemann Uniformization Theorem

Part III

End Notes

Chapter 4

Appendix

4.1 Some specific curves taught in class

4.1.1 Parabola

4.2 Differentiability of curves

Definition 18 (Differentiable function)**.** *A continuous function $f : (a, b) \to \mathbb{R}$(Note: a, b might be $\pm\infty$, in which case the function is defined on whole of $\mathbb{R}$) is called differentiable or of type c^1 if*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is continuous.

In a similar way, we can define C^2, C^3 and C^k by differentiating f $2, 3, k$ times respectively. We call a function "*smooth*" if the function is C^k for any $k \in \mathbb{N}$.
Examples

1. $f(x) = |x|$: This is a C^0 function as f is not differentiable at 0.
2. $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   f' exists but is not continuous. So f' is not continuous.

4.3 Differentiation (Multivariable)

Definition 19 (Differentiable multivariable function)**.** *A function $f : (a, b) \to \mathbb{R}$ is of class C^k if every $\pi_i \circ f(x)$ is C^k where π_i is projection to the i^{th} coordinate.*

Notation: We define

$$\nabla_v f(p) = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t}$$

For C^1 functions let $v = v_1 + v_2$. Then:

$$\begin{aligned} \nabla_v f(p) - \nabla_{v_1} f(p) &= \lim_{t \to 0} \frac{f(p + tv_1 + tv_2) - f(p + tv_1)}{t} \\ &= \lim_{t \to 0} \nabla_{v_2} f(p + tv_1) = \nabla_{v_2} f(p) \end{aligned}$$

$$\begin{aligned} \nabla_{kv} f(p) &= \lim_{t \to 0} \frac{f(p + tkv) - f(p)}{t} \\ &= k \lim_{t \to 0} \frac{f(p + tkv) - f(p)}{kt} \\ &= k \nabla_v f(p) \end{aligned}$$

If $v = \sum_{i=1}^n c_i e_i$, then we can apply the above formula repeatedly to get $\nabla_v f(p) = \sum_{i=1}^n c_i \nabla_{e_i} f(p)$.

Definition 20 (Differentiation of multivariable function)**.** *A function $\mathbb{R}^n \to \mathbb{R}^m$ is differntiable at a if there exists a linear map $L_a : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$f(a + v) - f(a) - L_a(v) = E_a(v)||v||$$

where $\lim_{v \to 0} E_a(v) = 0$.

A function f is called differntiable if such a linear map exists and $D_f(a) = L_a$. We can define a map $\tilde{D_f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, \tilde{D_f}(a, v) = L_a v$. This map is studied in differential geometry. A nice condition is if all the k^{th} derivative $\frac{\partial^k}{\partial x_i^{k_i}} f, \sum k_i = k$ exits and is continuous then f is C^k continuous.

4.4 Homeomorphism, Diffeomorphism and invariance of domain

Definition 21 (Diffeomorphism)**.** *Let $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ be open sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ repectively. A map $f : U \to V$ is a diffeomorphism if f is C^∞ and if f has a smooth inverse.*

If we consider homeomorphism between U, V then it is required that $m = n$. This proof requires techniques of algebric topology and is not discussed here.This is known as invariance of domain. But if we consider difeomorphisms, then we can prove $m = n$.

Lemma 22. *Let $f : \mathbb{R}^l \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are smooth function then $f \circ g$ is smooth.*

Notation: We shall look at a nice way to write the differentiation operator. We say

$$Df(a, v) = D_f(a)v$$

For example, we shall write the chain rule as

$$D(g \circ f)(\overrightarrow{a}, \overrightarrow{v}) = Dg(f(\overrightarrow{a}), Df(\overrightarrow{a}, \overrightarrow{v}))$$

Corollary 23. *If there exists a difeeomorphism between $U \in \mathbb{R}^n$ and $V \in \mathbb{R}^m$ then $m = n$.*

Proof. Without loss of generality, let $m \geq n$. Let $f : U \to V$ be the diffeomorphism and let g be it's inverse. Then $D(f \circ g)(\overrightarrow{a}, \cdot) = D(Id_{m\times m})(\overrightarrow{a}, \cdot) = Id_{m\times m}$. This follows because differentiation of a linear map T is T itself. But we also have

$$D(f \circ g)(\overrightarrow{a}, \cdot) = Dg(f(\overrightarrow{a}), Df(\overrightarrow{a}, \cdot))$$

Let $D_g(f(a)) = L^n_m$ and $D_f(a) = L^m_n$. Therefore:

$$D(f \circ g)(\overrightarrow{a}, v) = L^m_n L^n_m v$$

Now we use the rank nullity theorem.

$$\begin{aligned} dim(null(L^n_m)) &= dim\mathbb{R}^m - dim(range(L^n_m)) \\ &\geq m - dim(\mathbb{R}^n) = m - n \end{aligned}$$

Now if $dim(null(L^n_m)) \neq 0$ then we can get $v \neq 0$ such that $L^m_n L^n_m v = L^m_n 0 = 0$. But this shouldn't happen as $L^m_n L^n_m v = Id_{m\times m} v = v$. So we need $m - n = 0 \Rightarrow m = n$. □

Note: Our instructor gave a slight variation of the proof: he mentioned $L^m_n L^n_m v$ and $L^n_m v L^m_n$ can't be both invertible if $m \neq n$. I felt what he said was trivial but still need to be written out.