

AAKASH GHOSH (19MS129)

# ANALYSIS 5



# *Contents*

<b>1 Signed Measure</b>	<b>5</b>
1.1 Introduction . . . . .	5
1.2 Upper and lower continuity . . . . .	5
1.3 Positive, Negative and Null Set . . . . .	5
1.4 Hahn Decomposition . . . . .	6
1.5 Jordan Decomposition . . . . .	6
1.6 Total Variation Measure . . . . .	7
1.7 Absolute Continuity . . . . .	8
1.8 Radon-Nikodym theorem . . . . .	8
1.9 Solutions to Real Analysis By Folland, Section 3.1,3.2 . .	9
1.9.1 Solutions to problems in section 3.1 . . . . .	9
1.9.2 Solutions to problems in section 3.2 . . . . .	15
<b>Bibliography</b>	<b>17</b>



# 1 Signed Measure

## 1.1 Introduction

**Definition 1** (Signed Measure). *Given a measurable space  $(X, \mathcal{M})$ , a signed measure is a function  $\nu : \mathcal{M} \rightarrow [-\infty, \infty]$  with the following properties:*

1.  $\nu(\emptyset) = 0$
2.  $\nu$  can assume either  $\infty$  or  $-\infty$  but not both
3. If  $\{E_j\}$  is a sequence of disjoint sets in  $\mathcal{M}$ , then  $\sum_{i=1}^{\infty} \nu(E_i) = \nu(\cup_{i=1}^{\infty} E_i)$

One should note that normal measures are also signed measure, the only difference is the extension of the range of the measure function to cover almost all of  $\mathbb{R}$ .

## 1.2 Upper and lower continuity

**Theorem 1** (Uppercontinuity). *Let  $\{E_i\}$  be a countable collection of measurable set with  $E_i \subseteq E_{i+1}$ . Then:*

$$\lim_{i \rightarrow \infty} \nu(E_i) = \nu\left(\bigcup_{i=1}^{\infty} E_i\right) \quad (1.1)$$

**Theorem 2** (Lowercontinuity). *Let  $\{E_i\}$  be a countable collection of measurable set with  $E_{i+1} \subseteq E_i$ . Then:*

$$\lim_{i \rightarrow \infty} \nu(E_i) = \nu\left(\bigcap_{i=1}^{\infty} E_i\right) \quad (1.2)$$

*Proof.* Same as what we do for unsigned measure □

## 1.3 Positive, Negative and Null Set

**Definition 2** (Positive set). *A set whose every measurable subset  $E$  satisfies  $\nu(E) \geq 0$  is called a positive set.*

In a similar fashion we define :

**Definition 3** (Negative set). *A set whose every measurable subset  $E$  satisfies  $\nu(E) \leq 0$  is called a negative set.*

**Definition 4** (Null set). *A set whose every measurable subset  $E$  satisfies  $\nu(E) = 0$  is called a positive set.*

We consider an example. Let  $\mu$  be an unsigned measure and let  $f$  be a measurable  $L^1$  function. Let us define a measure  $\nu$  as:

$$\nu(E) = \int_E f d\mu \quad (1.3)$$

Then  $\nu$  is a signed measure. If  $E$  is a set such that  $f \geq 0$   $\mu$ -a.e on  $E$  then  $E$  is a positive set. Similarly we can find negative and null sets.

**Lemma 3.** 1. *Subsets of positive sets are positive*

2. *Countable<sup>1</sup> union of positive sets are positive*  
*Similar results are also valid for null and negative sets.*

<sup>1</sup> A countable union is needed as in case of uncountable union, there will be a chance that the union will not belong to the sigma algebra; a sigma algebra is closed in countable union and not under arbitrary union

The next lemma will be required for the proof of **Hahn Decomposition Theorem** in the next section.

**Lemma 4.** *Let  $\nu$  be a signed measure which doesn't attain  $\infty$ . A set with a positive measure has a positive subset.*

## 1.4 Hahn Decomposition

**Theorem 5** (Hahn Decomposition Theorem). *If  $\nu$  is a signed measure on  $(X, \mathcal{M})$ , there exist a positive set  $P$  and a negative set  $N$  for  $\nu$  such that  $P \cup N = X$  and  $P \cap N = \emptyset$ . Moreover if  $P', N'$  is another such pair, then  $P \Delta P' (= N \Delta N')$  is null in  $\nu$ .*

**Proof Outline :**

1. Define  $m = \sup_{\text{positive sets}} \nu(P)$
2. Take a sequence  $\{p_i\}$  such that  $\lim_{i \rightarrow \infty} \nu(p_i) = m$
3. Show if  $P = \bigcup p_i$  then  $\nu(P) = m$
4. Show if  $N = P^c$  and if  $N$  has a set with positive measure, then by lemma 4, there is contradiction.
5. If  $E \subseteq P \Delta P'$  and  $\nu(E) \neq 0$ . Without loss of generality assume  $E' = E \cap P$  is not null. Then  $E' \subseteq P'^c = N'$  which contradicts negativity of  $N'$

## 1.5 Jordan Decomposition

**Definition 5** (Mutually singular measures). *Two measures  $\nu$  and  $\mu$  are said to be mutually singular if there exists a partition of  $X$  in  $E$  and  $F$  such that  $X = E \sqcup F$  and  $E$  is null in  $\mu$  and  $F$  is null in  $\nu$ <sup>2</sup>*

<sup>2</sup> That is to say that the measures  $\nu$  and  $\mu$  "lives" on different sets.

**Notation:** If  $\nu$  and  $\mu$  are mutually singular, then we denote it as:

$$\nu \perp \mu$$

**Theorem 6** (Jordan Decomposition Theorem). *Given a (signed) measure  $\nu$  there exists **unique** positive measures  $\nu^+, \nu^-$  such that:*

$$\nu = \nu^+ - \nu^- \quad \nu^+ \perp \nu^- \quad (1.4)$$

**Proof Outline:** <sup>3</sup>

1. Existence follows by Hahn decomposition.
2. Start by assuming the decomposition is not unique and there exists two such decomposition  $\nu = \nu^+ - \nu^- = \mu^+ - \mu^-$ .
3. There exists partition of  $X$  in  $E, F$  due to  $\mu^+, \mu^-$  and in  $P, N$  due to  $\nu^+, \nu^-$ . If  $A$  is measurable, show that

$$\mu^+(A) = \nu(A \cap E) = \nu(A \cap E \cap P) + \nu(A \cap E \cap N)$$

4. As  $E$  is positive and  $N$  is negative, show that  $A \cap E \cap N$  is a null set. Repeat for  $\nu^+$  and get similar results
5. Show  $\nu^+ = \mu^+$  and in a similar way  $\mu^- = \nu^-$

<sup>3</sup> As I understand it, the main idea is if there is two decomposition as outlined in step 2 and 3, then we have 4 sets to deal with:

- $P \cap F$  and  $E \cap N$ : which are null as they are intersection of positive and negative sets
- $P \cap E$  where  $\mu^+$  and  $\nu^+$  agree and  $\mu^-, \nu^- = 0$
- $N \cap F$  where  $\mu^-$  and  $\nu^-$  agree and  $\mu^+, \nu^+ = 0$

Make this nice and you get the proof outlined.

## 1.6 Total Variation Measure

**Definition 6** (Total Variation Measure). *If a measure  $\nu$  decomposes in singular  $\nu^+$  and  $\nu^-$  then we define the total variation measure  $|\nu|$  as*

$$|\nu| = \nu^+ + \nu^- \quad (1.5)$$

**Lemma 7.** *The following statements are equivalent:*

1.  $E$  is null in  $\nu$
2.  $\nu^+(E) = 0$  and  $\nu^-(E) = 0$ <sup>4</sup>
3.  $|\nu|(E) = 0$

**Lemma 8.** *The following statements are equivalent:*

1.  $\nu \perp \mu$
2.  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$
3.  $|\nu| \perp \mu$

Proof for lemma 7 and 8 is at the end, they are given as exercise in Folland, ch3. Other properties which gets reflected are finiteness and  $\sigma$ -finiteness.

This works before as by Hahn-Jordan, the decomposition is unique. The definition is important as by Lemma 7 and Lemma 8, we see that properties of  $\nu$  is reflected in  $|\nu|$

<sup>4</sup> For unsigned measures, being null and having a measure 0 is same.

## 1.7 Absolute Continuity

**Definition 7** (Absolute Continuity). Let  $\mu$  be an unsigned measure. We say  $\nu$  is absolutely continuous with respect to  $\mu$  if for any measurable set  $E$ ,  $\mu(E) = 0 \implies \nu(E) = 0$

**Notation:**  $\nu$  is absolutely continuous with respect to  $\mu$  is denoted by:

$$\nu \ll \mu$$

Unlike mutual singularity,  $\nu \ll \mu$  doesn't imply  $\mu \ll \nu$ . In a sense, being mutually singular and being absolutely continuous are exclusive concepts. If  $\nu \perp \mu$  and  $\nu \ll \mu$  then  $\nu = 0$

**Lemma 9.** The following statements are equivalent:

1.  $\nu \ll \mu$
2.  $\nu^+ \ll \mu$  and  $\nu^- \ll \mu$
3.  $|\nu| \ll \mu$

**Lemma 10.** If  $\nu$  and  $\mu$  are finite measures,  $\nu \ll \mu$  if and only if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|\nu(E)| < \epsilon$  whenever  $\mu(E) < \delta$

<sup>5</sup> This lemma gives some motivation for the nomenclature of absolute continuity

**Proof Outline:**

1. By Lemma 9, we need to show this is true for  $|\nu|$  and we will be done. This is why, without loss of generality, we can assume  $\nu$  is unsigned.
2. **Don't understand why this is trivial**
3. Make a decreasing sequence of measurable sets
4. Show if there exists  $\epsilon$  with no such  $\delta$  then  $\mu$  of intersection goes to 0 but  $\nu$  of intersection stays above  $\epsilon$ . This contradicts absolute continuity.

## 1.8 Radon-Nikodym theorem

**Theorem 11** (Radon-Nikodym theorem). The theorem has two parts:

1. For a measure space, with  $\sigma$ -finite measure signed measure  $\nu$  and  $\sigma$ -finite measure unsigned measure  $\mu$ , there is a unique decomposition of  $\nu$  in  $\nu_1$  and  $\nu_2$  such that  $\nu_1 \ll \mu$  and  $\nu_2 \perp \mu$
2. There exists a function  $f$  which is integrable in the extended sense such that  $\nu_1(E) = \int_E f d\mu$ . Moreover, if there are two such functions  $f_1, f_2$  then  $f_1 = f_2$   $\mu$ -a.e.



**Proof Outline:****1. Step 1:  $\nu, \mu$  are finite**

- (a) Note that  $\nu(E) = \int_E f d\mu + \nu_2(E) \Rightarrow \nu(E) \geq \int_E f d\mu$
- (b) Make a family of function  $\mathcal{F}$  which satisfy this.
- (c) Let  $\alpha$  be suprema of the integral of  $f$  in family. Find  $f_n$  whose integral approach  $\alpha$ . Set  $g_n(x) = \max\{f_1(x), f_2(x) \dots f_n(x)\}$ . Show  $g_n$  is increasing and is in  $\mathcal{F}$ . Find limit of  $g_n$  as  $g$ . Use MCT to show that  $\alpha$  is attained by  $g$ .<sup>6</sup>
- (d) Set  $\nu_2 = \nu - \nu_1$ . Show  $\nu_2 \perp \mu$ .

**2. Step 2: Assume  $\sigma, \mu$  are  $\sigma$ -finite.**

- (a) Divide  $X$  in disjoint countable  $B_i$  each with finite measure.
- (b) Restrict  $\mu$  and  $\nu$  in  $B_i$  to get  $\mu_i, \nu_i$ . Repeat step 1 to get  $f_i, \nu_i^1, \nu_i^2$  in  $B_i$ . Set  $f = \sum f_i, \nu^1 = \sum \nu_i^1, \nu^2 = \sum \nu_i^2$

**3. Step 3: Uniqueness of decomposition:** If  $\nu_1, \nu_2$  and  $\hat{\nu}_1, \hat{\nu}_2$  are two decomposition then  $\nu_1 - \hat{\nu}_1 = \hat{\nu}_2 - \nu_2$ . Now  $(\nu_1 - \hat{\nu}_1) \ll \mu$  and  $\nu_2 - \hat{\nu}_2 \perp \mu$ . So  $\nu_1 - \hat{\nu}_1 = 0$ **4. Step 4: Uniqueness of  $f$  :** If  $f, g$  are two such functions then  $\int_E (f - g) = 0$  or  $f = g$   $\mu.a.e.$ 

The decomposition of  $\nu$  is called **Leabesgue decomposition**. The  $f$  obtained in part 2 is called the **Radon-Nikodym derivative**.

If  $\nu \ll \mu$  then the existence of Radon Derivative doesn't require  $\sigma$  finiteness of  $\nu$ <sup>7</sup>

<sup>6</sup> The reason to take a in first place is because we don't know if  $f_n$  converges. The reason we take such a  $g_n$  is so that we can interchange the limit and integral by applying MCT.

<sup>7</sup> This is a really nasty theorem- Check for all conditions before applying. Both  $\nu$  and  $\mu$  should be sigma finite for decomposition to exist.  $\nu$  should be absolutely cts and  $\mu$  should be  $\sigma$  finite for the derivative to exist.

**1.9 Solutions to Real Analysis By Folland, Section 3.1, 3.2****1.9.1 Solutions to problems in section 3.1****Problem 3.1**

Prove Proposition 3.1.

Propotion 3.1 is Theorem 1&2 mentioned here.

**Solution 1 outline:** We do it as instructed in the book, by copying Theorem 1.8 from the book.

**Solution 2 outline:** Decompose  $\mu$  by Hahn Decomposition and apply upper/lower continuity on each of them individually.<sup>8</sup>

**Proof 1:**

By the second condition of definition 1, we can assume  $\mu > -\infty$ .

<sup>8</sup> Proof of Hahn decomposition doesn't assume upper/lower continuity

**Proving Uppercontinuity:** If some  $E_i = \emptyset$  we are done. Else, set  $E_0 = \emptyset$ . Define  $F_i = E_i \setminus E_{i-1}$ . Note that any two  $F_i, F_j$  is disjoint. Then  $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} F_i$ . It follows that:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) = \lim_{i \rightarrow \infty} \mu(E_i)$$

The last step follows by countable additivity.

**Proving Lowercontinuity:** Set  $F_j = E_1 \setminus E_j$ . Then  $F_i \subseteq F_{i+1}$  and  $\mu(E_1) = \mu(F_j) + \mu(E_j)$ . Also,  $\cup_{i=1}^{\infty} F_j = E_1 \setminus (\cap_{i=1}^{\infty} E_j)$ . Apply uppercontinuity to get:

$$\begin{aligned} \mu(E_1) &= \mu\left(\bigcap_{i=1}^{\infty} E_j\right) + \lim_{j \rightarrow \infty} \mu(F_j) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right) + \mu(E_1) - \lim_{j \rightarrow \infty} \mu(E_j) \\ &\Rightarrow \lim_{j \rightarrow \infty} \mu(E_j) = \mu\left(\bigcap_{i=1}^{\infty} E_j\right) \end{aligned}$$

### Problem 3.2

If  $\nu$  is a signed measure,  $E$  is  $\nu$ -null iff  $|\nu|(E) = 0$ . Also, if  $\nu$  and  $\mu$  are signed measures,  $\nu \perp \mu$  iff  $\nu^+ \perp \mu$  and  $\nu^- \perp \mu$

**Solution outline:** This is the problem corresponding to lemma 7 and 8. In both cases we shall show  $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ .

#### Solution part-1 (Proof of lemma 7):

The Steps are based on lemma 7.

- **Step 1:**  $1 \Rightarrow 2$

Let  $P, N$  be the decomposition of  $N$  in positive and negative sets using HJD [Hahn Jordan Decomposition]. Let  $E$  be a null set. Then:

$$\begin{aligned} \nu^+(E) &= \nu^+(E \cap P) - \nu^-(E \cap N) \\ &= \nu^+(E \cap P) \\ &= \nu(E \cap P) - \nu^-(E \cap P) \\ &= \nu(E \cap P) = 0 \end{aligned}$$

Similar result is obtained for  $\nu^-$ . For unsigned measures, a measure zero set is null set, so we are done.

- **Step 2:**  $2 \Rightarrow 3$

This is the easy step.

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$$

• **Step 3:**  $3 \Rightarrow 1$

Note that for any measurable subset  $A$  of  $E$  we have  $|\nu|(A) = 0$ .

We also have:

$$|\nu(A)| = |\nu^+(A) - \nu^-(A)| \leq \nu^+(A) + \nu^-(A) = |\nu|(A) = 0$$

$$\Rightarrow \nu(A) = 0$$

Therefore,  $E$  is null in  $\nu$

**Solution part-2(Proof of lemma 8):**

The Steps are based on lemma 8.

• **Step 1:**  $1 \Rightarrow 2$

Let  $P, N$  be the decomposition of  $X$  in positive and negative sets using HJD[Hahn Jordan Decomposition]. Let  $A, B$  be the disjoint decomposition of  $X$  for  $\nu$  and  $\mu$ . Then it is easy to check every element lies in one of the four sets:  $A \cap P, A \cap N, B \cap P, B \cap N$ .

Now note,

- Decomposition of  $X$  for  $\nu^+$  and  $\mu$  is achieved by  $A \cap P$  and  $(A \cap N) \cup (B \cap P) \cup (B \cap N)$
- Decomposition of  $X$  for  $\nu^-$  and  $\mu$  is achieved by  $A \cap N$  and  $(A \cap P) \cup (B \cap P) \cup (B \cap N)$

• **Step 2:**  $2 \Rightarrow 3$

Let decomposition of  $X$  for  $\nu^+$  and  $\mu$  be  $E_1, F_1$  and for  $\nu^-$  and  $\mu$  be  $E_2, F_2$ . We claim the decomposition of  $X$  for  $|\nu|$  and  $\mu$  is given by  $E_1 \cup E_2$  and  $F_1 \cap F_2$ .<sup>9</sup> Note that  $|\nu|$  is null in  $F_1 \cap F_2$  both  $\nu^+$  and  $\nu^-$  is null in  $F_1, F_2$ .  $\mu$  is null in both  $E_1$  and  $E_2$ . By lemma 3, part 2,  $\mu$  is null in  $E_1 \cup E_2$ .

<sup>9</sup> This is low-key motivated by the decomposition in step 1.

• **Step 3:**  $3 \Rightarrow 1$

Let  $A, B$  be the disjoint decomposition of  $X$  for  $|\nu|$  and  $\mu$ . We claim this is the appropriate decomposition for  $\nu$  and  $\mu$  as well. It is already known  $\mu$  is null in  $A$ . As  $|\nu|$  is null in  $B$ ,  $\nu$  is null in  $B$  follows from lemma 7( $3 \Rightarrow 1$ ).

**Problem 3.3**

Let  $\nu$  be a signed measure on  $(X, \mathcal{M})$ . Prove

1.  $\mathcal{L}^1(\nu) = \mathcal{L}^1(|\nu|)$
2. If  $f \in \mathcal{L}^1(\nu)$ ,  $|\int f d\nu| \leq \int |f| d|\nu|$
3. If  $e \in \mathcal{M}$ ,  $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$

**Solution outline:** Our main goal will be to study  $f$  on the decomposition of  $X$  made by  $HJD$  due to  $\nu$

**Solution part-1:**

Let  $X$  be decomposed into positive set  $P$  and negative set  $N$ . We assume  $\nu > -\infty$ . Let  $f \in \mathcal{L}^1(\nu)$ . Let  $\chi_E$  denote the characteristic function on  $E$ . Then we have:

$$\int |f|d|\nu| = \int |f|(\chi_P + \chi_N)d(\nu^+ + \nu^-) = \int |f|d\nu^+ + \int |f|d\nu^- < \infty \quad (1.6)$$

Therefore,  $f \in \mathcal{L}^1(|\nu|)$ .

Now assume  $f \in \mathcal{L}^1(|\nu|)$ . Then as before,

$$\infty > \int |f|d|\nu| = \int |f|(\chi_P + \chi_N)d(\nu^+ + \nu^-) = \int |f|d\nu^+ + \int |f|d\nu^- \quad (1.7)$$

But as  $\nu^+, \nu^-$  are both unsigned we can conclude that  $\int |f|d\nu^+, \int |f|d\nu^- < \infty$ . Therefore,  $f \in \mathcal{L}^1(\nu)$

**Solution part-2:**

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f(\chi_P + \chi_N)d(\nu^+ - \nu^-) \right| \\ &= \left| \int f d\nu^+ - \int f d\nu^- \right| \\ &\leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f|d\nu^+ + \int |f|d\nu^- = \int |f|d|\nu| \end{aligned}$$

**Solution part-2:**

First we show that  $|\nu|(E)$  is an upper bound and then we show that it is attained. For any measurable  $f$  with  $|f| \leq 1$  we have:

$$\left| \int_E f d\nu \right| \leq \int_E |f|d|\nu| \leq \int_E d|\nu| = |\nu|(E)$$

Therefore,  $|\nu|(E)$  is an upper bound. Now set  $f = \chi_P - \chi_N$ . For any  $x \in X$ , either  $x \in P$  or  $x \in N$ . Therefore  $f(x) \in \{1, -1\}$ .

$$\begin{aligned} \left| \int f_E d\nu \right| &= \left| \int_E (\chi_P - \chi_N)(\chi_P + \chi_N)d(\nu^+ - \nu^-) \right| \\ &= \left| \int_E (\chi_P^2 - \chi_N^2)d(\nu^+ - \nu^-) \right| \\ &= \left| \int_E \chi_P^2 d\nu^+ + \int_E \chi_N^2 d\nu^- \right| \\ &= \left| \int_E \chi_P d\nu^+ + \int_E \chi_N d\nu^- \right| \\ &= |\nu^+(E \cap P) + \nu^+(E \cap N)| \\ &= |\nu^+(E \cap P) + \nu^+(E \cap N) + \nu^-(E \cap N) + \nu^-(E \cap P)| \\ &= \nu^+(E) + \nu^-(E) = |\nu|(E) \end{aligned}$$

**Problem 3.4**

If  $\nu$  is a signed measure and  $\lambda$  and  $\mu$  are unsigned measures such that  $\nu = \lambda - \mu$  then  $\lambda \geq \nu^+, \mu \geq \nu^-$

**Solution outline:** There are two ways to do this. The easy but long way is to show that this is true for positive and negative sets respectively and then stitch things up. The hard and short way is to do the same stitching but in a single line.

**Proof :**

Let  $X = P \sqcup N$  be the Hahn Jordan decomposition of  $X$  in positive set  $P$  and negative set  $N$ . Assume  $E \subseteq P$ . Then:

$$\nu(E) = \nu^+(E) + \nu^-(E) = \nu^+(E) = \lambda(E) - \mu(E) \leq \lambda(E)$$

$$\nu^-(E) = 0 \leq \mu(E)$$

The same steps can be repeated on subsets of  $N$ . Now for any measurable set  $E$ , we have:

$$\nu^+(E) = \nu^+(E \cap P) + \nu^+(E \cap N) \leq \lambda(E \cap P) + \lambda(E \cap N) = \lambda(E)$$

$$\nu^-(E) = \nu^-(E \cap P) + \nu^-(E \cap N) \leq \mu(E \cap P) + \mu(E \cap N) = \mu(E)$$

Which completes the proof.

The following hard and short proof was given by Nitish:

$$\nu^+(E) = \nu(E \cap P) = \lambda(E \cap P) - \mu(E \cap P) \leq \lambda(E \cap P) \leq \lambda(E)$$

**Problem 3.5**

If  $\nu_1, \nu_2$  are signed measures that both omit  $\infty$  or  $-\infty$  then

$$|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$$

**Proof :**

We note that:

$$\nu_1 + \nu_2 = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$$

We use the result of the previous question. Treat  $\nu_1^+ + \nu_2^+$  as  $\lambda$  and  $\nu_1^- + \nu_2^-$  as  $\mu$ .

$$|\nu_1 + \nu_2| = (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \leq (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) = |\nu_1| + |\nu_2|$$

**Problem 3.6**

Suppose  $\nu(E) = \int f d\mu$ , where  $\mu$  is a positive measure and  $f$  is an extended  $\mu$ -integrable function. Describe the Hahn decompositions of  $\nu$  and the positive, negative, and total variations of  $\nu$  in terms of  $f$  and  $\mu$ .

**Solution outline :**

By the nature of Hahn Jordan we will be done once we find out any suitable decomposition: the theorem will take care of uniqueness. The proof uses the following insight:  $E$  will be a positive set in  $\nu$  if  $f \geq 0$  in  $E$ .

**Proof :**

Set:

$$P = \{x | f(x) \geq 0\}$$

$$N = \{x | f(x) < 0\}$$

It is easy to see that  $X = P \sqcup N$ . It follows that:

$$\nu(E) = \int_{E \cap P} f d\mu + \int_{E \cap N} f d\mu = \int_E |f| \chi_P d\mu - \int_E |f| \chi_N d\mu$$

Note that  $\nu^+(E) = \int_E |f| \chi_P d\mu$ ,  $\nu^-(E) = \int_E |f| \chi_N d\mu$  is a valid decomposition of  $\nu$  in positive measures. It follows that the total variation measure is:

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \int_E |f| (\chi_P + \chi_N) d\mu = \int_E |f| d\mu$$

**Problem 3.7**

Suppose  $\nu$  is signed measure on  $(X, \mathcal{M})$  and  $E \in \mathcal{M}$ . Then:

1.  $\nu^+(E) = \sup\{\nu(F) | F \subseteq E, F \in \mathcal{M}\}$  and  $\nu^-(E) = \inf\{\nu(F) | F \subseteq E, F \in \mathcal{M}\}$
2.  $|\nu|(E) = \sup\{\sum_{i=1}^n |\nu(E_j)| | n \in \mathbb{N}, E = \bigsqcup_{j=1}^n E_j\}$

**Solution outline :** For both cases we show that the relevant figures are upper bound and the upper bound is attained.

**Proof :**

1.  $\nu^+(E)$  is an upper bound as  $\nu^+(E) \geq \nu^+(F) \geq \nu(F)$ . Set  $F = E \cap P$  where  $P$  is the positive set in Hahn Jordan decomposition of  $X$ . Similar thing is to be done for  $\nu^-$ .
2. For any  $\{E_j\}, 1 \leq j \leq n, E = \bigsqcup_{j=1}^n E_j$  note that  $|\nu|(E_j) \geq |\nu(E_j)|$  and  $\sum_{i=1}^n |\nu(E_j)| \leq \sum_{i=1}^n |\nu|(E_j) = |\nu|(\bigsqcup_{j=1}^n E_j) = |\nu|(E)$ . The bound is attained by setting  $n = 2, E_1 = P, E_2 = N$  where  $P, N$  are positive and negative sets in  $X$  as found by Hahn Jordan decomposition.

## 1.9.2 Solutions to problems in section 3.2

## Problem 3.8

Show that the following statements are equivalent:

1.  $\nu \ll \mu$
2.  $\nu^+ \ll \mu, \nu^- \ll \mu$
3.  $|\nu| \ll \mu$

**Proof :**

Let  $X = P \sqcup N$  be the decomposition of  $X$  in positive set  $P$  and negative set  $N$  according to Hahn Jordan decomposition.

**Part 1: (1 $\Rightarrow$ 2)**

For  $E \in \mathcal{M}$ ,

$$\mu(E) = 0 \Rightarrow \mu(E \cap P) = 0 \Rightarrow \nu(E \cap P) = 0 \Rightarrow \nu^+(E \cap P) = 0 \Rightarrow \nu^+(E) = 0$$

$$\mu(E) = 0 \Rightarrow \mu(E \cap N) = 0 \Rightarrow \nu(E \cap N) = 0 \Rightarrow \nu^-(E \cap N) = 0 \Rightarrow \nu^-(E) = 0$$

Therefore,  $\nu^+ \ll \mu, \nu^- \ll \mu$

**Part 1: (2 $\Rightarrow$ 3)**

For  $E \in \mathcal{M}$  and  $\mu(E) = 0$  we have  $\nu^+(E) = 0, \nu^-(E) = 0$ . Therefore,  $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$  and we can conclude  $|\nu| \ll \mu$

**Part 3: (3 $\Rightarrow$ 1)**

For  $E \in \mathcal{M}$  and  $\mu(E) = 0$  we have  $|\nu|(E) = 0$ . We note that

$$0 \leq |\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E) = 0$$

It follows that  $\nu(E) = 0$ . Therefore,  $\nu \ll \mu$ .

## Problem 3.9

Suppose  $\{\nu_j\}$  is a sequence of unsigned measures.

1. If  $\nu_j \perp \mu$  for all  $j$  then  $\sum_{j=1}^{\infty} \nu_j \perp \mu$
2. If  $\nu_j \ll \mu$  for all  $j$  then  $\sum_{j=1}^{\infty} \nu_j \ll \mu$

**Solution outline :** The second part is easy. In the first part getting the appropriate decomposition of  $X$  is the crucial step. Studying the problem with  $\nu_1, \nu_2$  helped in my case. The key insight is union and intersection of null sets are null.

**Proof :**

1. For every  $j$  there exists a decomposition of  $X$  as  $F_j \sqcup E_j$  such that  $\mu(F_j) = 0, \nu(E_j) = 0$ . Set  $F = \cup_{i=1}^{\infty} F_j, E = \cap_{i=1}^{\infty} E_j$ . It follows that  $\nu_j(F) \leq \nu_j(E_j) = 0 \Rightarrow \sum_{j=1}^{\infty} \nu_j(F) = 0$  and  $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_j) = 0 \Rightarrow \mu(E) = 0$

Those problems are harder than the ones in 3.1. This is not something I appreciate. Also I don't have much time. So I will upload hand written solutions for this part.

2. Let  $\mu(E) = 0$ . Then  $\forall j \in \mathbb{N}, \nu_j(E) = 0 \Rightarrow \sum_{j=1}^{\infty} \nu_j(E) = 0$ .

**Problem 3.10**

Theorem 3.5 may fail if  $\nu$  is not finite. Show this for the following cases:

1.  $d\nu(x) = \frac{dx}{x}$  and  $d\mu(x) = dx$  on  $(0,1)$
2.  $\nu$  is the counting measure and  $\mu(E) = \sum_{n \in E} 2^{-n}$  on  $\mathbb{N}$

**Proof:**

Proving absolute continuity is easy in every case.

1. Set  $\epsilon = 1/2$ . Let there exist some  $\delta < 1$  for which the theorem holds. Consider the set  $E = (0, \min(\delta, 1))$ . Note that  $\mu(E) \leq \delta$ . But  $\int_E d\nu = \infty > \epsilon$ . We need to prove that the integral diverges. Consider the following sequence of increasing functions  $\{f_n\}$ ,  $f_n = \chi_{[\frac{1}{n}, 1]}$ . It is easy to see that  $f_n \rightarrow 1$  on  $(0, 1)$  and  $f_n$  is increasing. Note that  $\int_E f_n d\nu = \int_{1/n}^{\delta} \frac{1}{x} dx = \ln(n) + \ln(\delta)$ . By monotone convergence theorem, we have:

$$\int_E d\nu = \int_E \lim_{n \rightarrow \infty} f_n d\nu = \lim_{n \rightarrow \infty} \int_E f_n d\nu = \infty$$

2. We use the fact that  $\{\sum_{i=1}^n 1/2^i\}$  is a cauchy sequence. Set  $\epsilon = 1$ . Assume a delta exists. There exists  $N$  such that  $\sum_{i=N}^{\infty} 1/2^i \leq \delta$ . Set  $E = \{N, N+1, N+2, \dots\}$ . Therefore,  $\mu(E) < \delta$  but  $\nu(E) = \infty > \epsilon$



## *Bibliography*