Priority Queues and Heaps

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What is a Priority Queue?

Definition

It is an **abstract data type** that can store a collection of **comparable** items supporting the following operations:

1. **`insert`**(e): creates a new item e in the priority queue
2. **`min`**(): returns (but does not remove) the minimal item among the ones present in the priority queue; returns `null` if empty
3. **`removeMin`**(): removes and returns the minimal item among the ones present in the priority queue; returns `null` if empty
4. **`size`**(): returns the number of items in the priority queue
5. **`isEmpty`**(): returns a boolean indicating whether the priority queue is empty

Example

Operation	Return value	PQ contents
`insert`(5)		$\{5\}$
`insert`(9)		$\{5, 9\}$
`insert`(3)		$\{3, 5, 9\}$
`min()`	3	$\{3, 5, 9\}$
`removeMin()`	3	$\{5, 9\}$
`insert`(7)		$\{5, 7, 9\}$
`removeMin()`	5	$\{7, 9\}$
`removeMin()`	7	$\{9\}$
`removeMin()`	9	$\{\ \}$
`removeMin()`	`null`	$\{\ \}$
`isEmpty()`	`true`	$\{\ \}$

Priority queue ADT

```java
public interface PriorityQueue<E extends Comparable<E>> {
   int size(); // returns the number elements currently stored
   boolean isEmpty(); // checks if the priority queue is empty
   E min(); // returns the minimum element present in the priority queue
   E removeMin(); // returns and removes the minimum element present in the priority queue
   void insert(E e); // inserts a new element into the priority queue
}
```

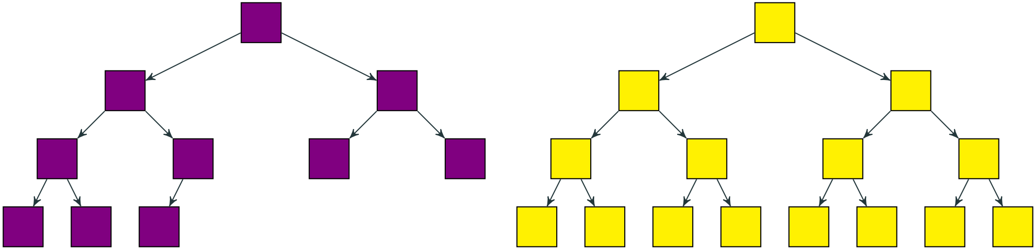
Applications

- Applications in daily life: servicing customers having higher priorities, etc.
- Job scheduling in operating systems and computer networks
- Use as an auxiliary data structure for other algorithms: graph algorithms, heap-sort, etc.
- ...

So how to implement PQs?

- Arrays and lists will result in linear runtimes
- One can use a RB-tree to implement a priority queue (logarithmic runtimes can be guaranteed)
- But a simpler and fast array-based (not space-wasting) data structure exists
- That data structure is **Binary Heap**
- Let us define it …

Complete binary tree



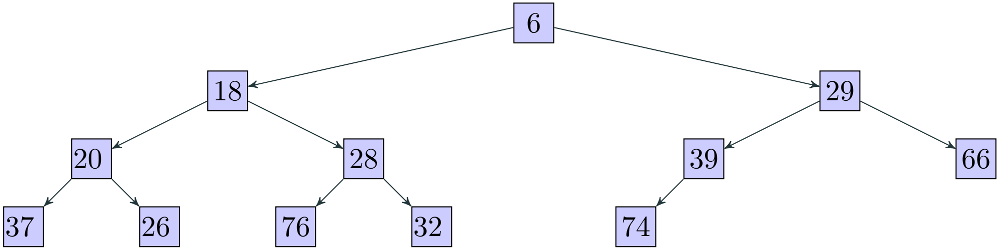
Both are complete binary trees

Definition

A binary tree T is **complete** if

- all the levels are completely filled except possibly the lowest one (the lowest level may or may not be full)
- the nodes in the last level reside in the leftmost possible positions (left-aligned)

Heap



Definition

A binary tree T is a **heap** if,

1. the root of every subtree T' of T contains the smallest item in T'; this means the root of T always contains the smallest item in T
2. T is complete

Height of a heap

A heap storing n items has height
$h = \lfloor \log_2 n \rfloor = O(\log n)$

Warning!

Binary heaps are different from binary search trees!

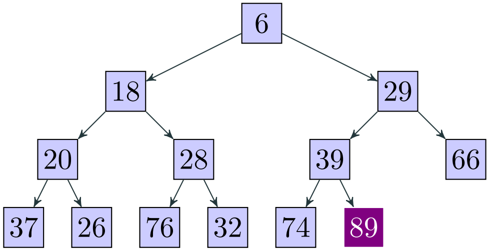
Insertion

Algorithm

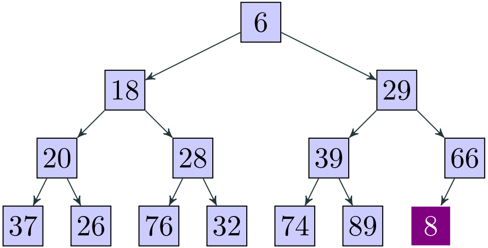
1. Insert the new item x in the next position at the last level (left-most empty position) of the heap; if the last level is full **create a new level** with x
2. **while** x is not at the root and x is smaller than its parent

    Swap x with its parent, moving the item up the heap

Inserting 89

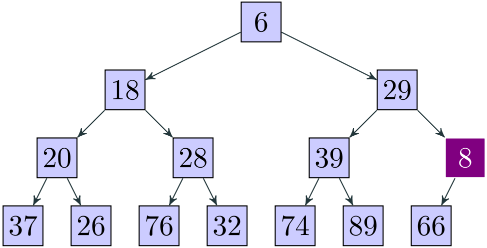


Place 89 at the first available spot at the last level; 89 is smaller than its parent 39; insertion process terminates



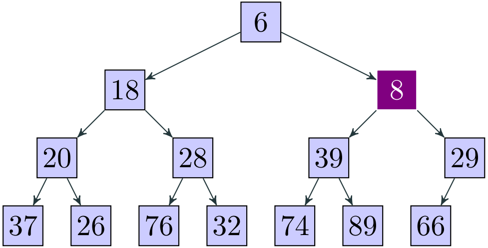
Place 8 at the first available spot at the last level; 8 is smaller than its parent 66; swap them

Inserting 8



8 is smaller than its current parent 29; swap them

Inserting 8



8 is greater than its current parent 6; stop

Try

- Insert 25
- Insert -2

Time complexity of insertion

- One insertion takes $O(h)$ time, where h is the height of a heap, since, for fixing a heap, we need to climb up its whole height in the worst-case
- Since height of a heap is $O(\log n)$; one insertion takes $O(h) = O(\log n)$ time

Time taken to create the whole heap on n elements

Since a single insertion takes $O(\log n)$ time, creation of the whole heap on n elements takes $n \times O(\log n) = O(n \log n)$

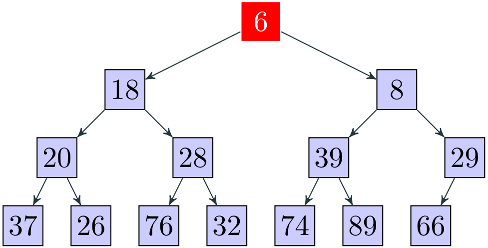
Removal

Algorithm

1. Save the root element in a variable
2. Delete and bring the rightmost element in the last level of the heap to the root
3. **while** there exists a child of the root element that is smaller than it

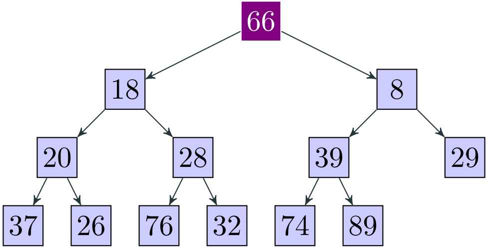
    Swap the root element with its smaller child

☞ In min-heaps, only the current minimum element can be deleted/removed



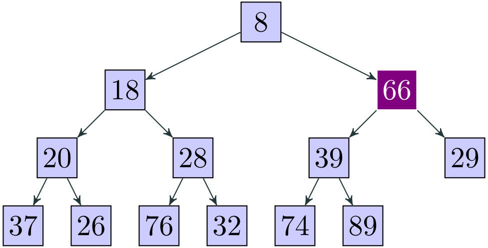
About to remove 6 (the minimum element)

Example



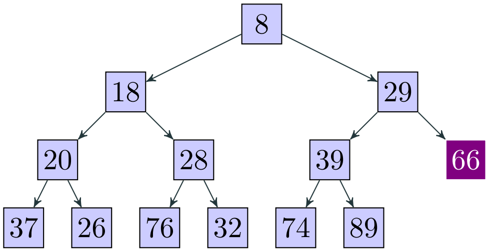
Delete and bring the rightmost element 66 from the last level to the root

Example



Choose the smaller child and swap 66 with it

Example

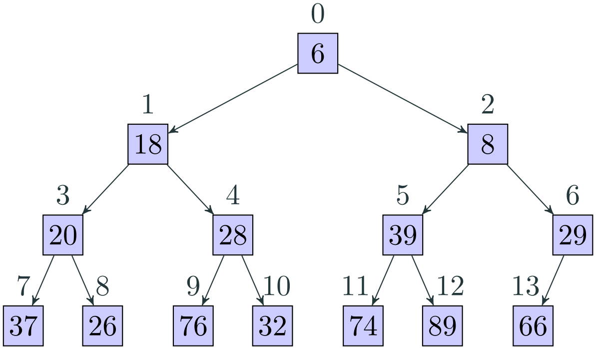


Removal process terminates

Time complexity of removal

- One deletion takes $O(h)$ time, where h is the height of a heap, since, for fixing a heap, we need to climb down its whole height in the worst-case
- Since height of a heap is $O(\log n)$; one deletion takes $O(h) = O(\log n)$ time

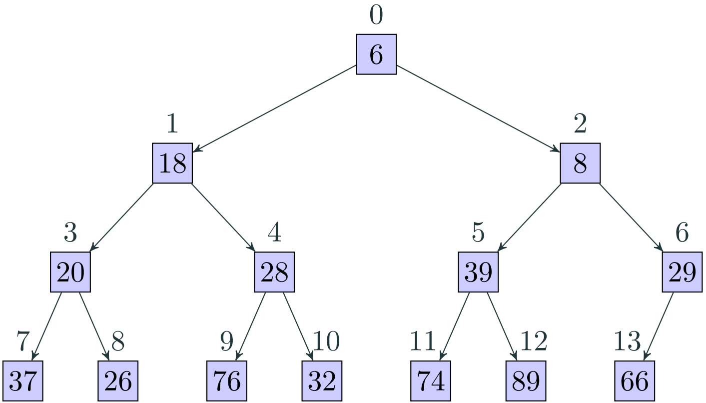
How to represent heaps using an array?



Rules

- The root is at array index index 0
- If a node has index i, then its left child (if any) is at array index $2i+1$ and its right child (if any) is at array index $2i+2$
- Parent of the node at index i can be found at array index $\lfloor (i-1)/2 \rfloor$

How to represent heaps?



ELEMENT	6	18	8	20	28	39	29	37	26	76	32	74	89	66
INDEX	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Observations

- The rightmost location of the last level (needed for the insert and delete operations) is precisely the last index of the array
- Since a heap can grow in size over time, an `ArrayList` can be used to maintain the heap
- Deleting the last element (required for heap removal operations) is easy when an `ArrayList` is used since no left shifting is needed for array compaction

See the class `MinHeap`

Algorithm

1. Insert the n items to be sorted into an empty heap H
2. Run `removeMin()` n times on H to get a sorted sequence of the input items

Time complexity

Step 1 takes $n \times O(\log n) = O(n \log n)$ time since every insertion takes $O(\log n)$ time
Step 2 takes $n \times O(\log n) = O(n \log n)$ time since every `removeMin()` takes $O(\log n)$ time

Total time taken: $O(n \log n) + O(n \log n) = 2 \times O(n \log n) = O(n \log n)$

See the class `HeapSort`

Further information

- The type of heaps we have considered so far are known as **min-heaps**
- Creation of a min-heap can be executed faster in $O(n)$ time using a different approach
- If the maximal element is always at the top, we call it a **max-heap**:
  the root of every subtree T' of T contains the largest item in T'; this means the root of T always contains the largest item in T