Binary Search Trees

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Maps

Definition

A **record** is a key-value pair: (k, v)

A **map** is an abstract data type for maintaining a set of records

- No two records can have the same key
- However, two records can have same values though
- Association of keys to values define a **mapping**: $f(\texttt{key}) = \texttt{value}$

Examples of maps

- UNF maintains a map of `(N#, student information)` records
- A social media company maintains a map of `(email address, user account information)` records
- An assembler maintains a symbol table (a map) of `(opcode, hex)` records
- A text-editor maintains a map of `(color, RGB representation)` records

How to implement a map?

Common map operations

- **Insert** a record (k, v)
- **Retrieve** a record having key k
- **Delete** a record having key k

Approach 1: maintain a sorted list of records

- **Insertion.** will take $O(\log n)$ time for figuring out the correct spot for the incoming record using binary search; then $O(n)$ time for shifting items to the right to accommodate the new record; total time taken is $O(\log n) + O(n) = O(n)$
- **Retrieval.** will take $O(\log n)$ time using binary search
- **Deletion.** will take $O(\log n)$ time to locate it using a binary search; then then $O(n)$ time for left shifting items to kill the empty spot; total time taken $O(\log n) + O(n) = O(n)$

How to implement a map?

Common map operations

- **Insert** a record (k, v)
- **Retrieve** a record having key k
- **Delete** a record having key k

Approach 2: maintain an unsorted list of records

- **Insertion.** will take $O(1)$ time (add the new record at the end)
- **Retrieval.** will take $O(n)$ time using a linear search; may need to search the whole list in the worst-case
- **Deletion.** will take $O(n)$ time to locate it using a linear search; then $O(n)$ time for left shifting items to kill the empty spot; total time taken $O(n)+O(n) = O(n)$

Our aim

Common map operations

- **Insert** a record (k, v)
- **Retrieve** a record having key k
- **Delete** a record having key k

☞ To accomplish the above three tasks in $O(\log n)$ time each

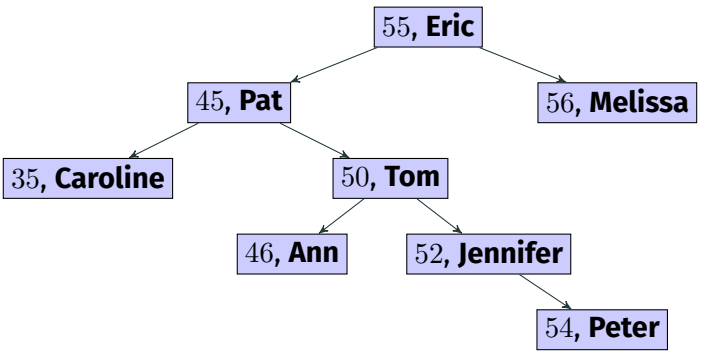
☞ **Balanced** binary search trees is the solution; stay tuned ...

The map ADT

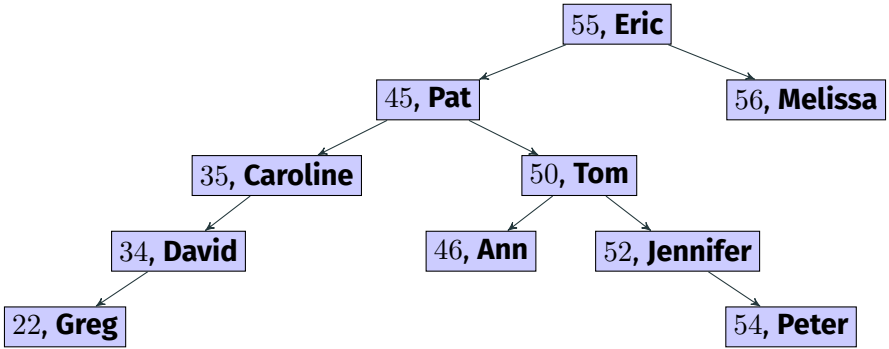
```
public interface MapADT<K,V> {
   boolean put(K k, V v); // adds a new record with key k and value v
   V remove(K k); // removes the record having key k
   V get(K k); // return the value part of the record whose key is k
   V updateValue(K k, V v); // updates the value part of the record whose key is k with a new value v
   int size(); // returns the number of records stored in the map
   void clear(); //Removes all records from the map
}
```

What is a Binary Search Tree?

- It is a **binary tree** where every node contains a **<key, value>** pair (a **record**); keys must be **comparable** but the values don't need to be
- Moreover, for every node p in the tree, the following 2 properties hold
  1. Keys stored in the left subtree of p are $<$ the key stored at p
  2. Keys stored in the right subtree of p are $>$ the key stored at p



What is a Binary Search Tree?

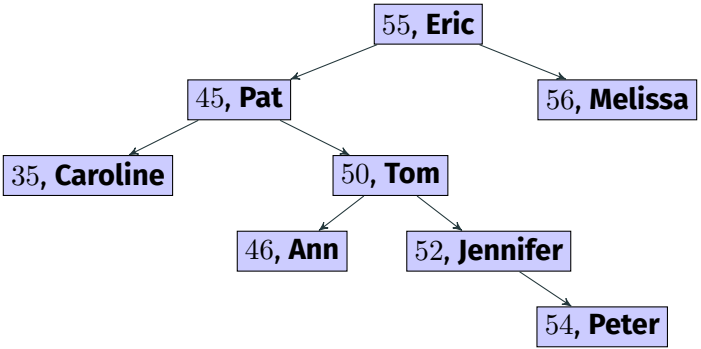


Use

BSTs can be used to implement **maps** and are commonly used for fast searching (typically need far less comparisons than lists)

BSTs are sorted maps

An inorder traversal of a BST always gives the sorted sequence based on the keys



Inorder traversal

35, **Caroline**; 45, **Pat**; 46, **Ann**; 50, **Tom**; 52, **Jennifer**; 54, **Peter**; 55, **Eric**; 56, **Melissa**

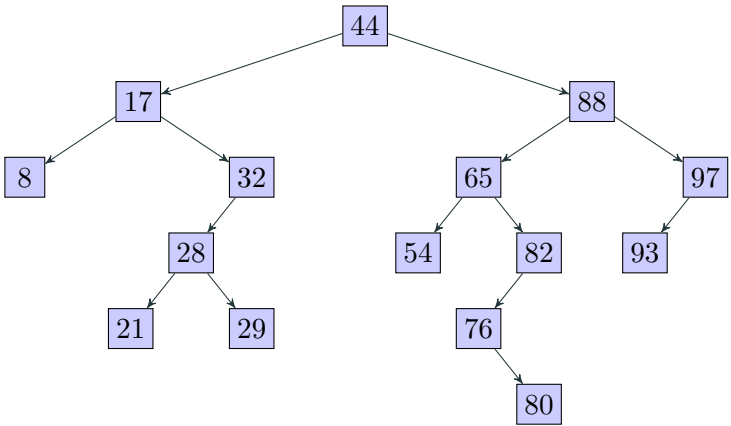
Let's say you need to look for the record that has the key k; how will you do this?

Algorithm

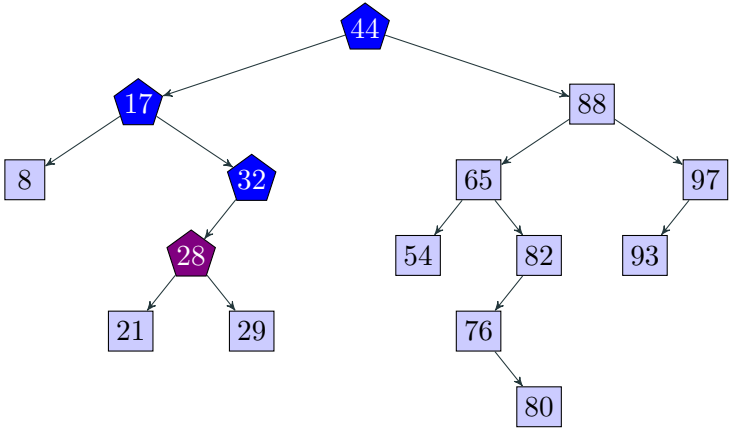
- Start at the root
- If the root's key is k, then search is successful
- If $k <$ root's key, search recursively (or iteratively) in the left subtree of the root
- Otherwise, search recursively (or iteratively) in the right subtree of the root
- If we have reached a `null` link, no record exists in the tree with key k

To save space in the figures, we will write only the record keys inside the nodes and avoid the corresponding values

Example

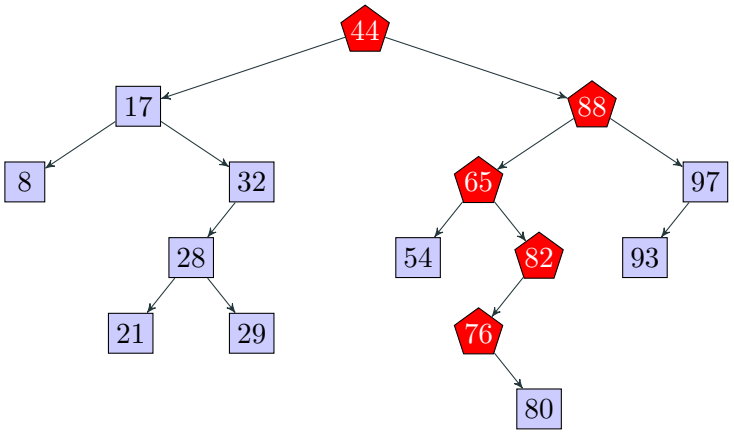


Example: search for 28



Found!

Example: search for 68



Not found; a null link is reached (the left link of 76)

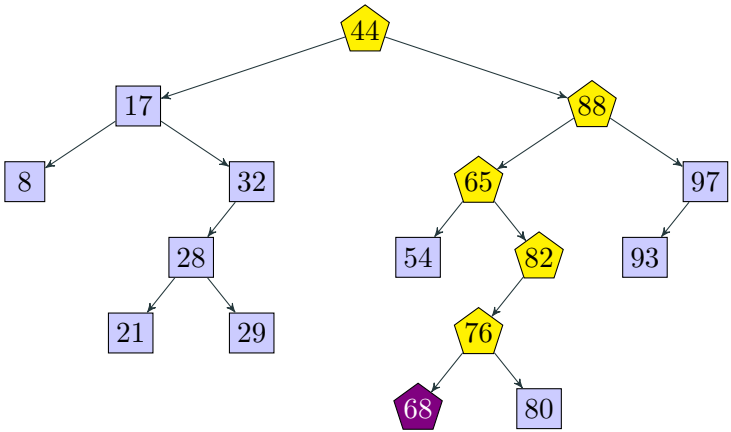
$O(h)$, where h is the height of the BST under consideration, since, for searching, we need to traverse a path whose length is h in the worst-case

How to insert a record into a BST having key k?

Algorithm

- Start at the root
- If the root is `null`, replace empty tree with a new tree with the new record as the root, and signal **SUCCESS**
- If k equals root's key, signal **FAILURE** since a record with key k already exists
- If $k <$ root's key, insert recursively (or, iteratively) in the left subtree of the root
- Otherwise, insert recursively (or, iteratively) in the right subtree of the root

Example: insert 68



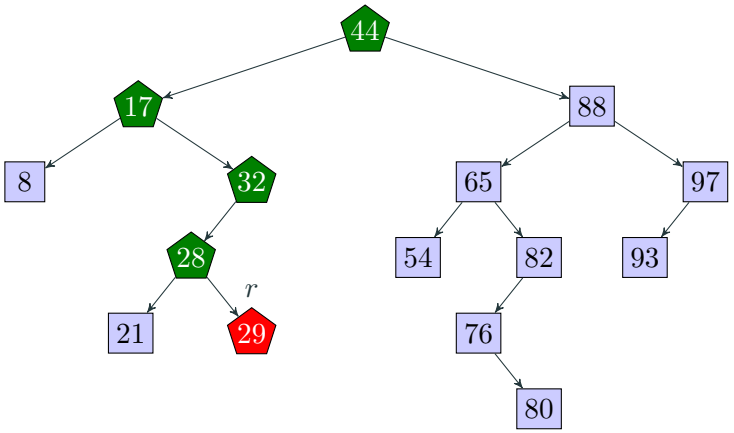
$O(h)$, where h is the height of the BST under consideration, since, for inserting a new record, we need to traverse a path whose length is h in the worst-case

How to delete the record from a BST having key k?

Idea

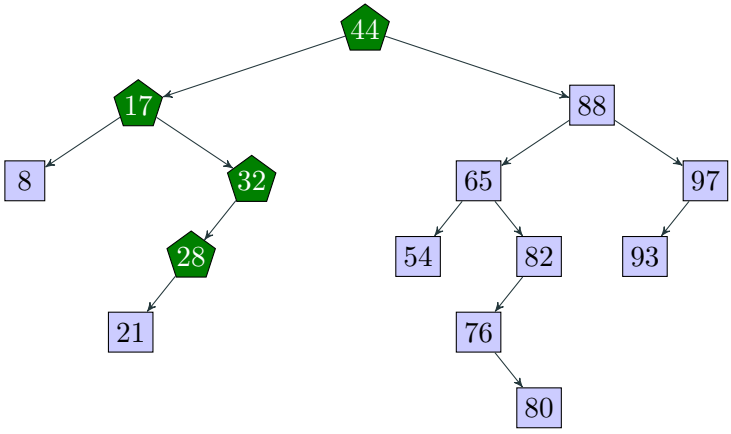
- Traverse through the tree to find the record having key k
- If the record cannot be found, no action is needed
- Now assume that we have found the record that has key k at node r
  1. r is a leaf node (no child)
  2. r has exactly one child (either left or right)
  3. r has two children (both left and right)

Case 1: r is a leaf node, delete 29

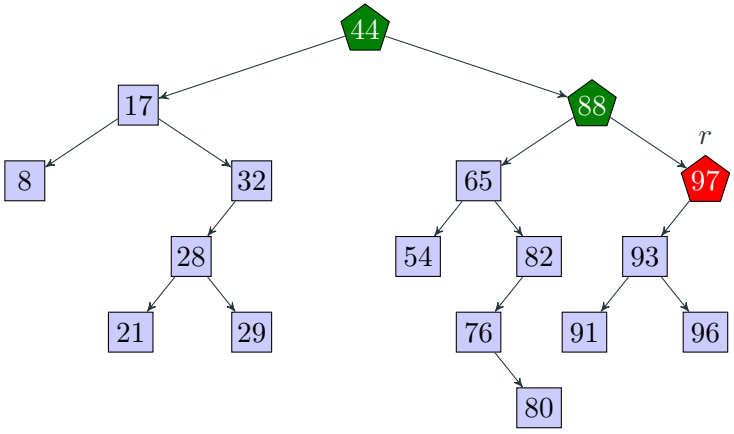


Easy! just delete it right away

Case 1: r is a leaf node, delete 29

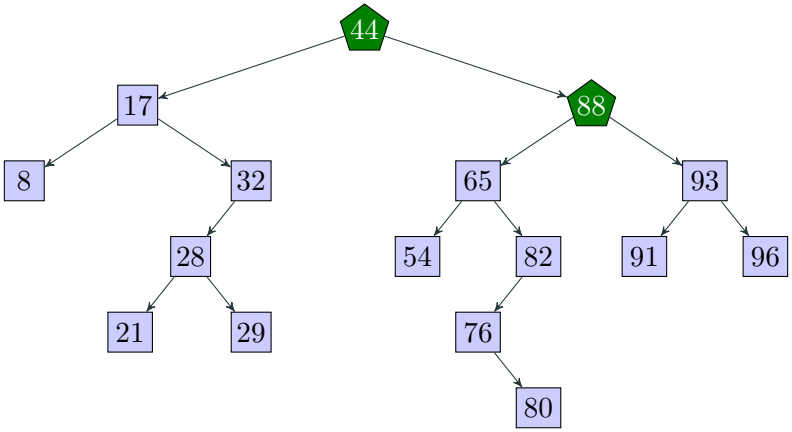


Case 2: r has one child, delete 97

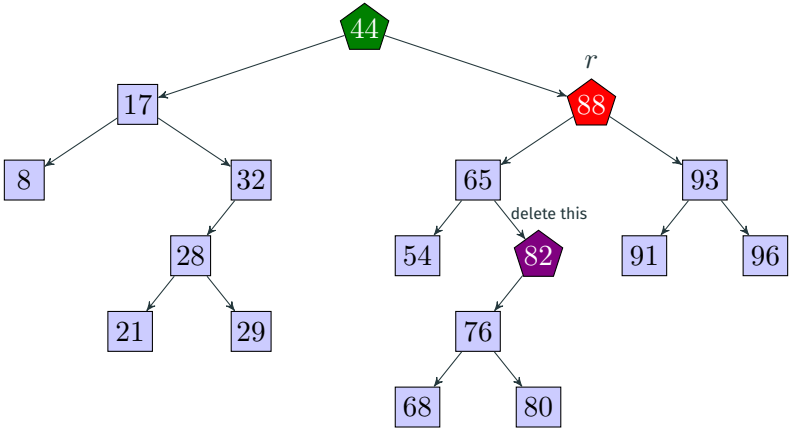


Make the parent of r point to the only child of r instead of r

Case 2: r has one child, delete 97

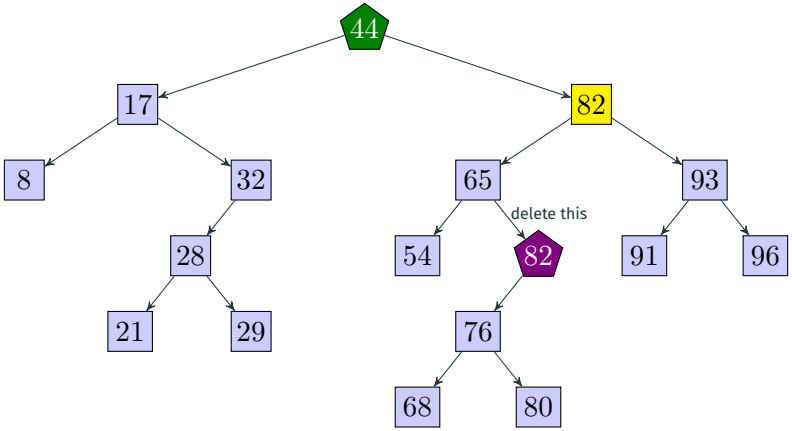


Case 3: r has two children, delete 88



Replace the record at r with the record at the inorder predecessor p of n; then delete p using Case 1 or 2, depending on the number of children of p. Note that the inorder predecessor is either a leaf node or has a left child only (no right child).

Case 3: r has two children, delete 88



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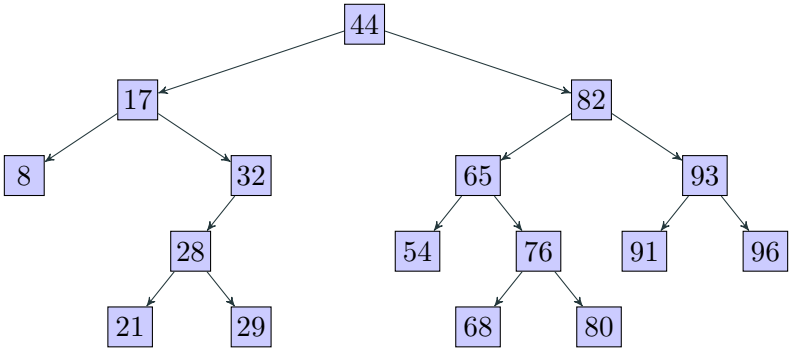
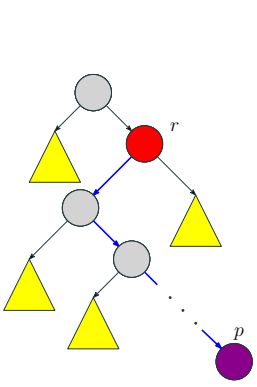
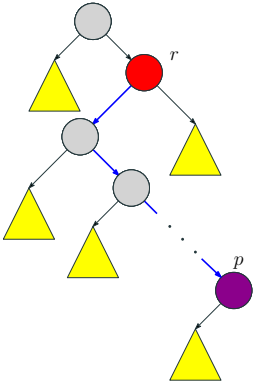


Replace the record at r with the record at the inorder predecessor p of n; then delete p using Case 1 or 2, depending on the number of children of p. Note that the inorder predecessor is either a leaf node or has a left child only (no right child).

Finding the inorder predecessor in Case 3





The inorder predecessor p of node r is either a leaf node (left figure) or has a left child but no right child (right figure). So, to find p, go left of r and then keep on moving right as long as possible. It takes $O(h)$ time to find p, where h is the height of the tree. The yellow triangles represent subtrees (some of them could be empty).

Time complexity of deletion

- It takes $O(h)$ time for locating the record to be deleted, where h is the height of the BST
- Case 1 takes $O(1)$ time to delete a record
- Case 2 takes $O(1)$ time to delete a record
- Case 3 takes $O(h)$ time to delete a record since we need to locate the inorder predecessor p of node r in $O(h)$ time and then delete p in $O(1)$ time using Case 1 or 2
- So, a single deletion takes $O(h)$ time in the worst-case

Code

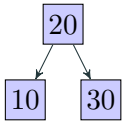
A built-in record class in Java

`java.util.AbstractMap.SimpleEntry<K,V>`

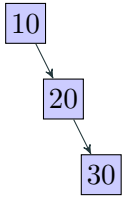
Reference. https://docs.oracle.com/en/java/javase/17/docs//api/java.base/java/util/AbstractMap.SimpleEntry.html

See the class `TreeMapBST`

BSTs are not unique



Insertion sequence: $20, 10, 30$



Insertion sequence: $10, 20, 30$

Their structures really depend on the insertion sequence of records

An application

Problem

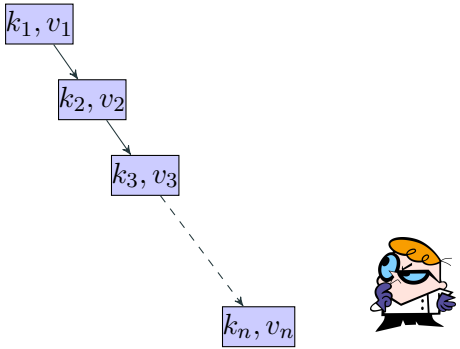
Given a text, find out the unique words in it along with their counts. We also need to output the distinct words with their count.

A solution

- Use a BST T where the key-type is `String` and the value-type is `Integer`. This means every node will store a word from the text and its count in the same text.
- For every word w in the text, first check if a node exists in T, where the stored key is w
  - If such a node does not exist, insert a new node in T with key w and value 1
  - If such a node exists in T, increment the stored value (essentially a counter) by 1

See the class `WordCounter`

Worst case scenario: skewed binary trees



In this case, $k_1 < k_2 < k_3 < \ldots < k_n$ and the tree is right-skewed

Note that when $k_n < k_{n-1} < \ldots < k_1$, the tree will be left-skewed (try!)

So, in the worst-case, $h = n - 1 = O(n)$

Therefore, searching, insertion, deletion take $O(h) = O(n)$ time each!

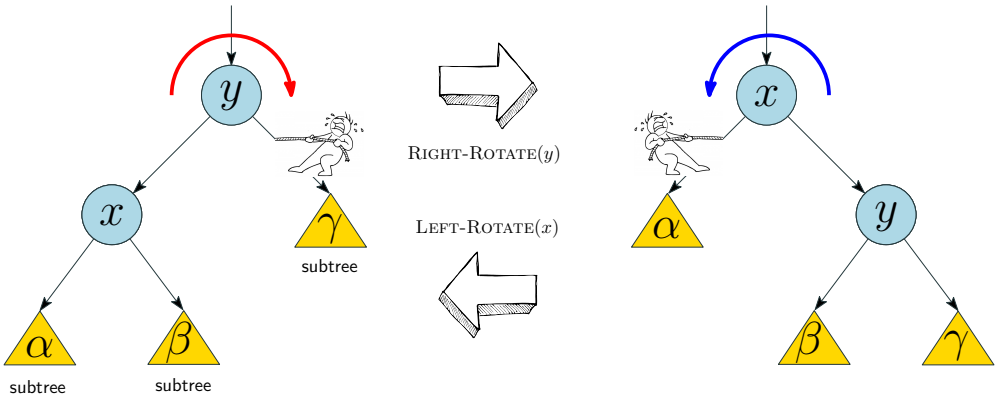
This is as bad as using singly linked-lists!

Do something so that h remains under control (logarithmic)
We aim for $h = O(\log n)$

A solution. use **Red-Black** trees

Red-Black trees are never skewed or close to being skewed unlike the plain binary search trees we just talked about 👍

Rotation on BSTs

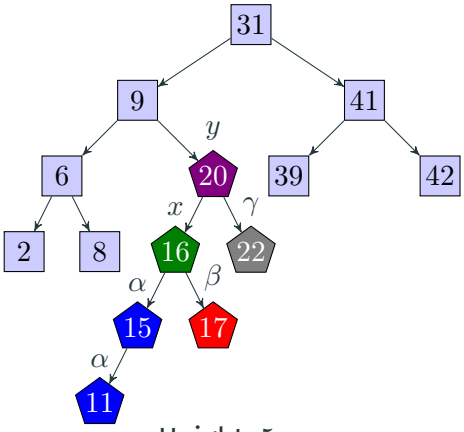


Rotations help in **reducing** height of BSTs; this means faster operations on BSTs

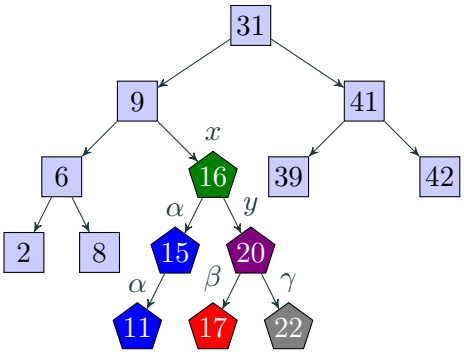
A single rotation can be done in $O(1)$ time

☞ Some of the three subtrees (α, β, γ) can be empty (devoid of nodes).

An example of rotation



Height: 5

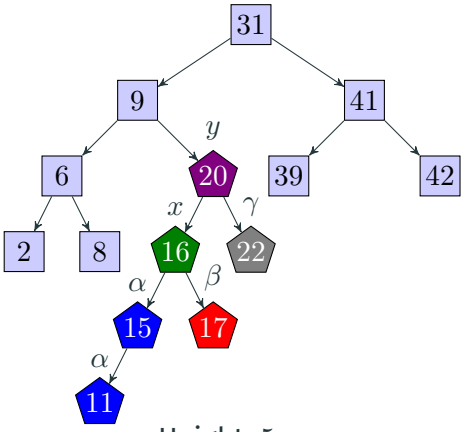


Height: 4

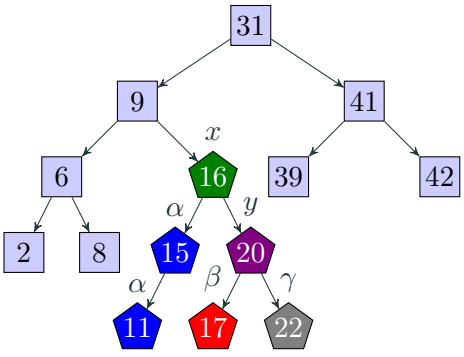
☞ If a **right** rotation is performed at node y in the left tree, we get the tree on the right.

☞ If a **left** rotation is performed at node x in the right tree, we get back the tree on the left.

An example of rotation



Height: 5



Height: 4

Note that rotations never alter the inorder traversal sequences. It also means that after a rotation, the resulting binary tree is still a binary search tree.

Code for rotation

Right rotation at node y

```
private void rightRotateAt(Node<K,V> y) {
   Node<K,V> x = y.left;
   y.left = x.right;

   if( x.right != null )
      x.right.parent = y;

   x.parent = y.parent;

   if( y.parent == null )
      root = x;
   else if( y == y.parent.right )
      y.parent.right = x;
   else
      y.parent.left = x;

   x.right = y;
   y.parent = x;
}
```

Left rotation at node x

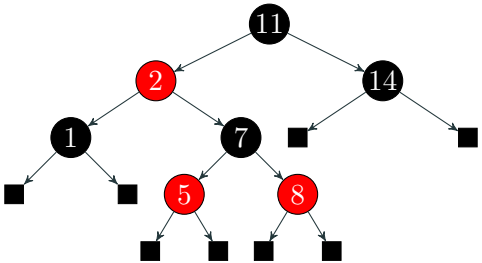
```
private void leftRotateAt(Node<K,V> x) {
   Node<K,V> y = x.right;
   x.right = y.left;

   if( y.left != null )
      y.left.parent = x;

   y.parent = x.parent;

   if( x.parent == null )
      root = y;
   else if( x == x.parent.left )
      x.parent.left = y;
   else
      x.parent.right = y;

   y.left = x;
   x.parent = y;
}
```

A single rotation (left/right) takes $O(1)$ time

Definition

A **red**-**black** tree is a self-balancing BST (height never gets too bad) that has the following properties:

1. Every node is either **red** or **black**
2. The root is **black**
3. If a node is **red**, then both its children are **black**. The `null` links (shown using black squares in the figure) of the leaves are **black**.
4. The number of **black** nodes in any path from the root to a leaf is the same



How to maintain the colors?

Since the colors are bichromatic, a boolean variable `color` is enough to specify the color of a node. Set `color` to `false`, when the node is painted **RED**, or else, set it to `true` to specify the color is **BLACK**.

☞ There is no magic behind these two colors; feel free to use any two colors

A node class implementing for RB-tree

```
public class TreeMapRBTree<K extends Comparable<K>, V> implements MapADT<K,V>, Iterable< SimpleEntry<K,V> > {
   private static final boolean RED = false, BLACK = true;

   private static class Node<K, V> {
      final private K key;
      private V val;

      private Node<K, V> left, right, parent;

      private boolean color;

      public Node(K k, V v) {
         key = k; val = v;
         left = right = parent = null;
         color = RED;
      }

      public String toString() {
         String colorString = (color == RED)? "RED" : "BLACK";
         if (val != null) return "<" + key.toString() + ", " + val.toString() + ", " + colorString + ">" ;
         else return key.toString();
      }

      // other variables and methods
}
```

Height of a red-black tree

- A RB-tree containing n records has height $\leq 2\log_2(n+1) = O(\log n)$
- **Intuition.** By constraining the node colors on any simple path from the root to a leaf (property 4), it can be ensured that no such path is more than twice as long as any other so that the tree's height is always logarithmic.

Implication

When $n = 1,000,000$, $h \leq 2\log_2(n+1) < 40$.
To search for a record, at most 41 comparisons are required in this case.

In contrast, when a plain binary search tree is used, $h \leq 999,999$.
For searching for a record, at most $1,000,000$ comparisons are required in this case.

This demonstrates the advantage of using RB-trees over plain BSTs and also shows that RB-trees can never be badly skewed (can never look like a very long linked-list).

☞ *Searching, insertion, and deletion in RB-trees take $O(h) = O(\log n)$ time each.*

The three primary operations

1. **Search**. same as the search operation for plain BSTs; takes $O(h) = O(\log n)$ time (note that RB-trees are also BSTs, so the same search algorithm works here too!)
2. **Insertion**. we **will** discuss this; takes $O(\log n)$ time
3. **Deletion**. we **won't** discuss this; takes $O(\log n)$ time

☞ The `TreeMap` class in Java implements RB-Tree

`https://docs.oracle.com/en/java/javase/17/docs/api/java.base/java/util/TreeMap.html`

Insertion

Inserting a new node z into a RB-tree

1. Color z **red** and insert it as you would in a plain BST
2. If necessary, start fixing the tree (using rotations and recoloring) as long as you see z's parent is **red** (z may change as we climb up the tree); see the cases next

```
while( z.parent != null && z.parent.color == RED ){
   // deal with the cases inside this loop
}
```

3. At the end, color the root using **black**

```
root.color = BLACK;
```

Terminologies

Uncle of a node

The uncle of a node is the sibling of its parent. In some cases, it could be a null link if there is no such sibling node.

```
private boolean isLeftChild( Node<K,V> node ) {
   return node.parent != null && node.parent.left == node;
}

private Node<K,V> uncle( Node<K,V> node ){
   if( node == null ) return null;
   return ( isLeftChild(node.parent) ) ? node.parent.parent.right : node.parent.parent.left;
}
```

Grandparent of a node

The grandparent of a node is the parent of its parent. In some cases, it could be a null link if there is no such grandparent node.

```
private Node<K,V> grandParent( Node<K,V> node ){
   return node.parent.parent;
}
```

The six cases of insertion

- **(Case 1a)** z's uncle y is **RED** and z is a right child
- **(Case 1b)** z's uncle y is **RED** and z is a left child
- **(Case 2a)** z is a right child, z's parent is a left child, and z's uncle y is **BLACK**
- **(Case 2b)** z is a left child, z's parent is a right child, and z's uncle y is **BLACK**
- **(Case 3a)** z is a left child, z's parent is left child, and z's uncle y is **BLACK**
- **(Case 3b)** z is a right child, z's parent is a right child, and z's uncle y is **BLACK**

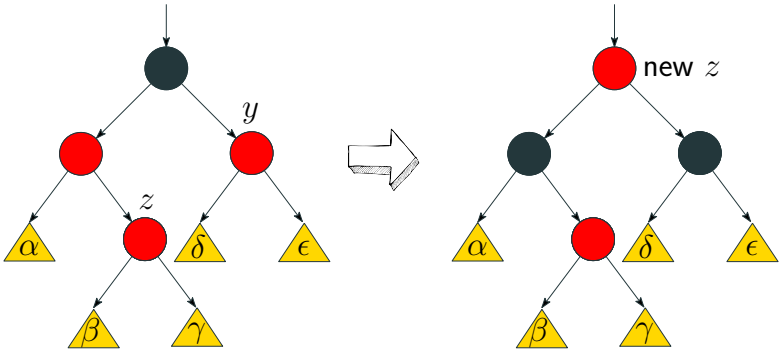
Things to note

1. The first node inserted into an empty red-black tree is always painted **BLACK**
2. Since the null links are considered to be **BLACK**, make sure that you consider their existence and color while finding the uncles of nodes
3. In each of the six cases, the parent of z is always **RED** (if it is **BLACK**, no fixing would be required)

Case 1

(Case 1a) z's uncle y is **RED** and z is a right child
(Case 1b) z's uncle y is **RED** and z is a left child

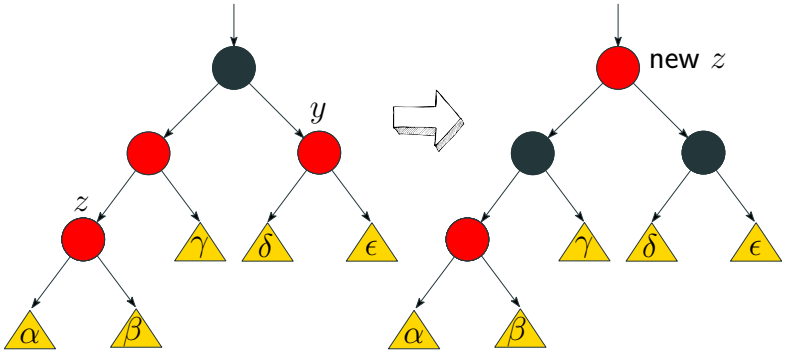
Case 1a



z's uncle y is **red** and z is a right child; recoloring is needed but no rotation; takes $O(1)$ time; **continue fixing up** using the new z node

- `z.parent.color = BLACK; y.color = BLACK; grandParent(z).color = RED;`
- `z = grandParent(z); // continue fixing up using the new 'z'`

Case 1b



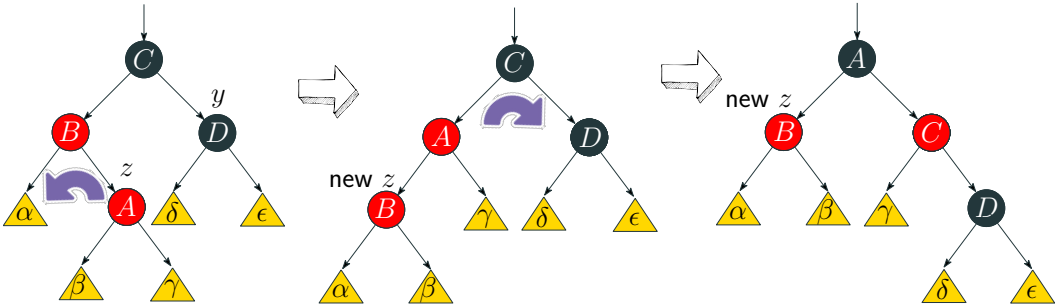
z's uncle y is **red** and z is a left child; recoloring is needed but no rotation; takes $O(1)$ time; **continue fixing up** using the new z node

- `z.parent.color = BLACK; y.color = BLACK; grandParent(z).color = RED;`
- `z = grandParent(z); // continue fixing up using the new 'z'`

Case 2

(Case 2a) z is a right child, z's parent is a left child, and z's uncle y is **BLACK**
(Case 2b) z is a left child, z's parent is a right child, and z's uncle y is **BLACK**

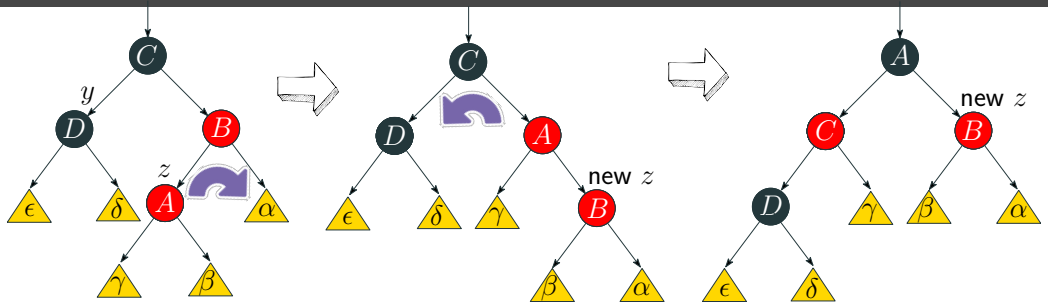
Case 2a



z is a right child, z's parent is a left child, and z's uncle y is **black**; recolorings + 2 rotations are needed; the fixing-up process **terminates** since z's parent is **black** after the two rotations; takes $O(1)$ time

- `z = z.parent; leftRotateAt(z);`
- `z.parent.color = BLACK; grandParent(z).color = RED; rightRotateAt(grandParent(z));`

Case 2b



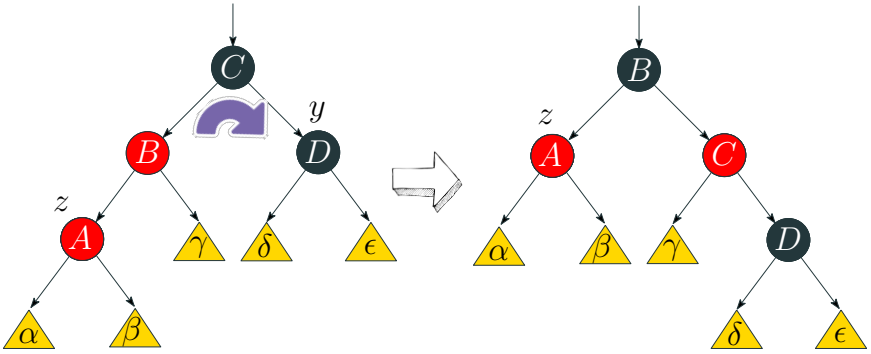
z is a left child, z's parent is a right child, and z's uncle y is **black**; recolorings + 2 rotations are needed; the fixing-up process **terminates** since z's parent is **black** after the two rotations; takes $O(1)$ time

- `z = z.parent; rightRotateAt(z);`
- `z.parent.color = BLACK; grandParent(z).color = RED; leftRotateAt(grandParent(z));`

Case 3

(Case 3a) z is a left child, z's parent is left child, and z's uncle y is **BLACK**
(Case 3b) z is a right child, z's parent is a right child, and z's uncle y is **BLACK**

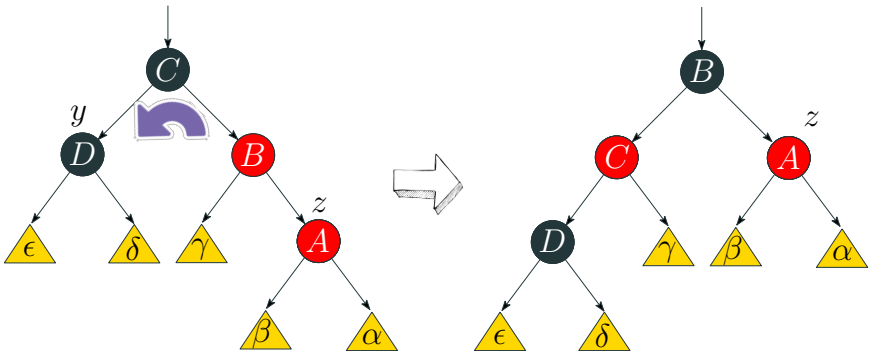
Case 3a



z is a left child, z's parent is left child, and z's uncle y is **black**; recolorings + 1 right rotation are needed; the fixing-up process **terminates** since z's parent is **black** after the rotation; takes $O(1)$ time

- `parent(z).color = BLACK; grandParent(z).color = RED; rightRotateAt(grandParent(z));`

Case 3b



z is a right child, z's parent is a right child, and z's uncle y is **black**; recolorings + 1 left rotation are needed; the fixing-up process **terminates** since z's parent is **black** after the rotation; takes $O(1)$ time

- `parent(z).color = BLACK; grandParent(z).color = RED; leftRotateAt(grandParent(z));`

Code

The second rotation of Case 2a is exactly an execution of Case 3a
Similarly, the second rotation of Case 2b is exactly an execution of Case 3b

See the class `TreeMapRBTree`

In-browser visualization

https://www.cs.csubak.edu/~msarr/visualizations/RedBlack.html

Insert: 41, 38, 31, 12, 19, 8, 7, 6

The value parts of the above records are ignored in this example

Item	Case Used	Action
41	none	The only node in the tree, color it **black**
38	none	Parent is **black**; no action is needed
31	3a	31's uncle (a null reference) is **black**; right rotate at 41
12	1b	Recoloring is needed
19	2a	Two rotations are needed
8	1b	Recoloring is needed
7	3a	Right rotate at 31
6	1b, 3a	Two cases are needed as we climb up to fix the tree

Observations

- Time taken for inserting a new node is $O(h) = O(\log n)$; after this fix-up may be needed
- During fix-up, we climb up the tree using Case 1, which only recolors but never rotates
- If we ever use Case 2 (uses 2 rotations) or 3 (uses 1 rotation), the fix-up process terminates!
- This means at every insertion of a new item, at most 2 rotations are needed
- At most h executions of Case 1 are needed plus 1 execution of Case 2/3, each taking $O(1)$ time
- So, the total time taken for fix-up is $(h+1) \times O(1) = O(\log n) \times O(1) = O(\log n)$, since, for RB-trees, $h = O(\log n)$
- Total time taken for one insertion equals time taken for inserting a new node plus total time taken for fix-up $= O(\log n) + O(\log n) = 2 \times O(\log n) = O(\log n)$
- Time complexity of searching is $O(h) = O(\log n)$, since, for RB-trees, $h = O(\log n)$
- Deletion also takes $O(h) = O(\log n)$ time but we are not going to discuss it

Plain BSTs vs RB-Trees: tree creation time (using n insertions)

n	RB-tree	Plain BST
10	1	1
100	1	1
1000	3	13
10000	7	214
100000	20	17249

When the input records were already **sorted** in ascending order, RB-trees could easily beat plain BSTs since the heights of the plain BSTs were exactly $n - 1$ everywhere (much worse than the logarithmic heights of RB-trees).

n	RB-tree	Plain BST
10	1	1
100	1	1
1000	3	2
10000	7	8
100000	72	49

When the input records were in **random** order, plain BSTs performed quite as fast as RB-trees since the heights of plain BSTs were not $n - 1$ or even close (in fact, they were almost logarithmic like RB-trees).

☞ Times are reported in milliseconds

Are plain BSTs completely useless?

- Plain BSTs perform **terribly** when the inputs are sorted (or, almost sorted) in ascending or descending order; the reason is in such cases, we get to see performance-degrading long paths in the tree
- But, BSTs are found to perform great on randomly ordered inputs
- In those cases, h is found to be much less than $n - 1$ and is almost logarithmic
- Consequently, we get to see fast searching, insertion, and deletion times
- **Example.** when $n = 5000$, heights of plain BSTs are around 30 if the input is randomly ordered. Note that this is much less than 4999 (worst case height)

Reading

https://opendsa-server.cs.vt.edu/ODSA/Books/Everything/html/BST.html