# Recursion

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# What is recursion?

**Recursion** is a technique for solving a computational problem where the final solution to the problem is constructed using the solutions of smaller subproblems, obtained recursively.

# Factorial

For a non-negative integer $n$, we define $n!$ (read as $n$ factorial) as:

$$n! = 1 \times 2 \times \ldots \times n$$

Factorial can also be defined recursively as:

$$n! = \begin{cases} 1 & \text{if } n = 0, 1 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

Expressing using functions we obtain:

$$f(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ n \cdot f(n-1) & \text{otherwise} \end{cases}$$

# Recursive code

$$f(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ n \cdot f(n-1) & \text{otherwise} \end{cases}$$

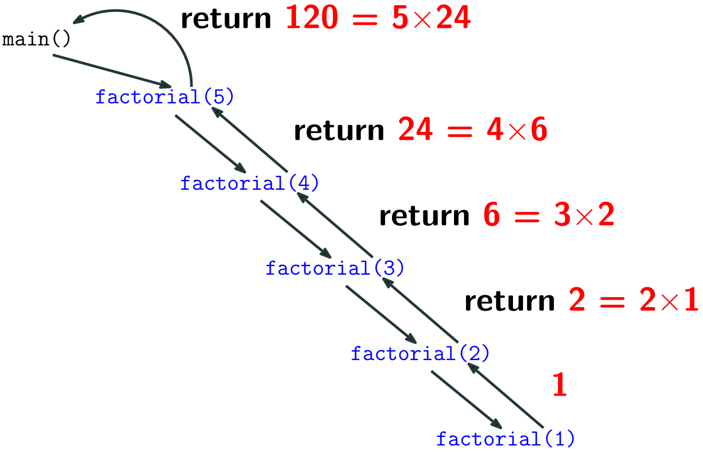
```java
public class Factorial {
   public static long factorial(int n) {
      if( n < 0 )
         throw new IllegalArgumentException("n must non-negative!");
      else if( n == 0 || n == 1 ) // base cases
         return 1;
      else
         return n * factorial(n-1); // recursive call
   }

   public static void main(String[] args) {
      System.out.println( factorial(5) );
   }
}
```

# Recursive code

```java
public class Factorial {
   public static long factorial(int n) {
      if( n < 0 )
         throw new IllegalArgumentException("n must non-negative!");
      else if( n == 0 || n == 1 ) // base cases
         return 1;
      else
         return n * factorial(n-1); // recursive call
   }

   public static void main(String[] args) {
      System.out.println( factorial(5) );
   }
}
```

Every recursive method contains the following two things:

1. **Base case(s)**. the case(s) for which we know how to calculate the answer without recursion; at least one base case is always required; every possible chain of recursive calls must eventually reach a base case.
2. **Recursive call(s)**. these are the calls to the current method. Each recursive call should be defined so that it makes progress towards a base case.




☞ The system uses a **stack** in the background to run recursive code

How to recursively reverse the subarray that starts at index $i$ and ends at index $j$?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 99 | • • • | 23 | 15 | • • • | 5 | 67 | • • • | 49 |
| $0$ | | $i$ | $i+1$ | | $j-1$ | $j$ | | $n-1$ |

Swap $A[i]$ and $A[j]$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 99 | • • • | 67 | 15 | • • • | 5 | 23 | • • • | 49 |
| $0$ | | $i$ | $i+1$ | | $j-1$ | $j$ | | $n-1$ |

Reverse the subarray $A[i+1 \ldots j-1]$

# Reversing an array

```java
import java.util.Arrays;

public class ReverseArray{
   public static void reverseArray(int[] A, int i, int j) {
        if (i > j)
          throw new IllegalArgumentException("i <= j is required.");

        int hold = A[i];
        A[i] = A[j];
        A[j] = hold;

        if( i + 1 < j - 1 )
           reverseArray(A, i + 1, j - 1); // recursive call
   }

   public static void main(String[] args) {
      int[] arr = {10, 20, 30, 40, 50};
      reverseArray(arr,0,arr.length-1);
      System.out.print(Arrays.toString(arr));
   }
}
```

# Summing up an array

## Recursive idea

To add the numbers inside the subarray $A[0]$ to $A[i]$, first **recursively** add the numbers inside the subarray $A[0]$ to $A[i-1]$ and then add the number $A[i]$ to the result.

```java
public class ArraySummer {

   public static int add(int[] A, int i) {
      if ( i < 0 )
         throw new IllegalArgumentException("i should be non-negative.");
      else if( i == 0 )
         return A[0];
      else
         return add(A, i-1) + A[i]; // recursive call
   }

   public static void main(String[] args) {
      int[] arr = {10, 20, 30, 40, 50};
      System.out.print( add(arr,arr.length-1) );
   }
}
```

# Binary search

- Given a **sorted** array $A$ of $n$ items, how fast can you search a given element?
- One can search by scanning $A$ from left to right (**linear search**), but this takes $O(n)$ time
- Can we do it faster? Use the fact that the array is already sorted
- Yes, we can using binary search; runs in $O(\log n)$ time

### Recursive algorithm (asssumption: $A$ is sorted)

- If the target equals $A[\text{mid}]$, then we have found the target!
- If the target is less than $A[\text{mid}]$, search recursively in the left half
- Othewise, search recursively in the right half

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 5 | 7 | 8 | 9 | 12 | **14** | 17 | 19 | 22 | 25 | 27 | 28 | 33 | 37 |
| low | | | | | | | mid | | | | | | | | high |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 5 | 7 | 8 | 9 | 12 | 14 | 17 | 19 | 22 | **25** | 27 | 28 | 33 | 37 |
| | | | | | | | | low | | | mid | | | | high |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 5 | 7 | 8 | 9 | 12 | 14 | 17 | **19** | 22 | 25 | 27 | 28 | 33 | 37 |
| | | | | | | | | low | mid | high | | | | | |

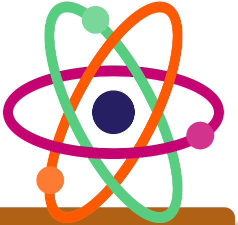
| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 5 | 7 | 8 | 9 | 12 | 14 | 17 | 19 | **22** | 25 | 27 | 28 | 33 | 37 |
| | | | | | | | | | | low = mid = high | | | | | |

Searching for **22** in the array

# Code

```java
public class BinarySearch {
   public static boolean binarySearchRec(int[] A, int target, int low, int high) {
      if( low > high )
         return false;
      else {
         int mid = (low + high) / 2; // mid takes the floor of (low + high) / 2
         if( target == A[mid] )        return true;
         else if( target < A[mid] )    return binarySearchRec(A, target, low, mid - 1 );  // recursive call
         else                          return binarySearchRec(A, target, mid + 1, high ); // recursive call
      }
   }

   public static boolean binarySearch(int[] A, int target) {
      return binarySearchRec(A, target, 0, A.length-1);
   }

   public static void main(String[] args) {
      int[] A = {2,4,5,7,8,9,12,14,17,19,22,25,27,28,33,37};
      System.out.println(binarySearch(A,22)); // prints true; search successful
      System.out.println(binarySearch(A,21)); // prints false; search unsuccessful
   }
}
```

- At every recursive call, approximately half of the array is discarded
- Also, at every recursive call, it does a constant amount of work – $O(1)$
- Let $m$ be the number of recursive calls made
- At every recursive call, array size gets halved
- After $m$ recursive calls, array size equals $n/2^m$
- In the worst case, we stop when $n/2^m = 1 \implies 2^m = n$
- Taking log of both sides we obtain, $m = \log_2 n = O(\log n)$
- **Time complexity.** $O(\log n) \times O(1) = O(\log n)$



**Fun fact**

Number of atoms in this universe: $10^{80} \approx 2^{266}$

Even if we have a dataset as large as this, binary search will make just $\log(2^{266}) = 266 \cdot \log_2 2 = 266 \cdot 1 = 266$ recursive calls in the worst case!

Write a non-recursive (iterative) binary search

For a given value of $n$, we need to print a string made up of $n-1$ **comp**s, **computing**, and $n-1$ **ting**s; here are few examples for you...

| $n$ | **Output** |
|---|---|
| $1$ | `computing` |
| $2$ | `compcomputingting` |
| $3$ | `compcompcomputingtingting` |
| $4$ | `compcompcompcomputingtingtingting` |
| $5$ | `compcompcompcompcomputingtingtingtingting` |

# Code

```java
public class RecursiveStringPrinter {

   public static String printer(int n) {
      if( n <= 0)      return null;
      else if( n == 1) return "computing";
      else             return "comp" + printer(n-1) + "ting";
   }

   public static void main(String[] args) {
      System.out.print(printer(5));
   }
}
```

# Self-referential classes

```
private static class Node<E> {
   private E element;
   private Node<E> prev, next; // defined recursively

   // ...
}
```

A **self-referential class** contains an instance variable that refers to another object of the same class type

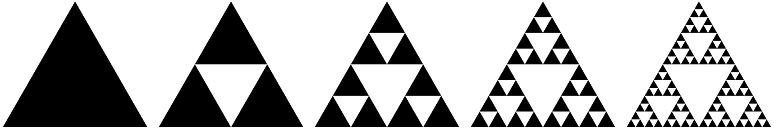
# Using recursion for linked-lists

```java
public class DoublyLinkedList<E> implements Iterable<E>{
   // other methods, variables, classes
   public String print() {
      return (printRecursive(head)).toString();
   }

   private StringBuilder printRecursive(Node<E> n) {
      if( n == null )
         return new StringBuilder();

      StringBuilder s = new StringBuilder(n.element.toString() + " ");
      s.append(printRecursive(n.next));
      return s;
   }
   // other methods, variables, classes
}
```

# Fractals

**What are fractals?** `https://en.wikipedia.org/wiki/Fractal`

Fascinating geometric figures that can be drawn recursively



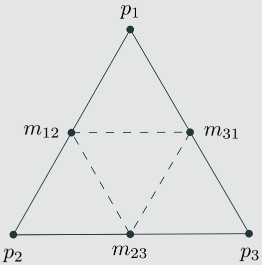
Sierpiński triangle (source: Wikipedia)

Wacław Sierpiński (source: Wikipedia)

# Pseudo-code

Sierpiński triangle (source: Wikipedia)

```
private static void drawTriangles(Graphics g, int d, Point p1, Point p2, Point p3) {
   if (d == 0) {  // depth is 0, draw the triangle; base case
      Polygon P = new Polygon();
      P.addPoint(p1.x,p1.y); P.addPoint(p2.x,p2.y); P.addPoint(p3.x,p3.y);
      g.fillPolygon(P); // draws a filled triangle
      return;
   }

   Point m12 = midpoint(p1,p2);
   Point m23 = midpoint(p2,p3);
   Point m31 = midpoint(p3,p1);

   // Draw 3 Sierpinski triangles recursively of depth d-1
   drawTriangles(g, d - 1, p1, m12, m31); // recursive call 1
   drawTriangles(g, d - 1, m12, p2, m23); // recursive call 2
   drawTriangles(g, d - 1, m31, m23, p3); // recursive call 3
}
```



# The number of triangles drawn

**Fact**

A Sierpiński triangle of depth $n$ is made up of $3^n$ triangles. So, the drawing algorithm we just looked at has a time complexity of $O(3^n)$ (exponential) and it cannot be made faster since we must draw $3^n$ triangles.

| $n$ | $3^n$ |
|---|---|
| $0$ | $1$ |
| $1$ | $3$ |
| $10$ | $59,049$ |
| $100$ | $515377520732011331036461129765621272702107522001$ |
| $2000$ | $\approx 1.75 \times 10^{954}$ ($955$ decimal digits) |

# Merge sort

- Merge sort runs in $O(n \log n)$ time
- It uses a linear-time algorithm known as **merging** for sorting the input
- Let us begin by understanding what is meant by merging two sequences …

Given two **sorted** sequences $S_1, S_2$, how fast can you **merge** them into one final sorted sequence $S$?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 244 | 311 | 478 | | | | | |
| $S_2$ | 324 | 415 | 499 | 505 | 666 | | | |
| $S$ | 244 | 311 | 324 | 415 | 478 | 499 | 505 | 666 |

Assume that $S_1$ has $k_1$ elements and $S_2$ has $k_2$ elements
Clearly, $S$ has $k_1 + k_2$ elements

We need to do it in $O(k_1 + k_2)$ time

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 244 | 311 | 478 | | | | | |
| $S_2$ | 324 | 415 | 499 | 505 | 666 | | | |
| $S$ | | | | | | | | |

# Merging two sorted sequences

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 244 | 311 | 478 | | | | | |
| $S_2$ | 324 | 415 | 499 | 505 | 666 | | | |
| $S$ | | | | | | | | |

# Merging two sorted sequences

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 244 | $\underline{311}$ | 478 | | | | | |
| $S_2$ | $\underline{324}$ | 415 | 499 | 505 | 666 | | | |
| $S$ | 244 | | | | | | | |

## Merging two sorted sequences

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 244 | 311 | <u>478</u> | | | | | |
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| $S$ | 244 | 311 | | | | | | |

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| $S$ | 244 | 311 | 324 | 415 | 478 | 499 | | |

# Merging two sorted sequences

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|---|---|---|---|---|---|---|---|---|
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| $S$ | 244 | 311 | 324 | 415 | 478 | 499 | 505 | |

# Merging two sorted sequences

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 244 | 311 | 478 | | | | | |
| $S_2$ | 324 | 415 | 499 | 505 | 666 | | | |
| $S$ | 244 | 311 | 324 | 415 | 478 | 499 | 505 | 666 |

Merging takes time proportional to the total number of blue cursors movements, and they move $k_1 + k_2$ times to the right in total. Further, at each cursor movement, we spend $O(1)$ time for comparing two elements, sending an element to $S$, and incrementing a blue pointer.

So, the time complexity amounts to
$(k_1 + k_2) \times O(1) = O(k_1 + k_2)$

- It is a recursive **divide and conquer** sorting algorithm
- Runs in $O(n \log n)$ time (faster than Insertion, Bubble, and Selection sorts)

**The algorithm**

Let the input be denoted by $S$

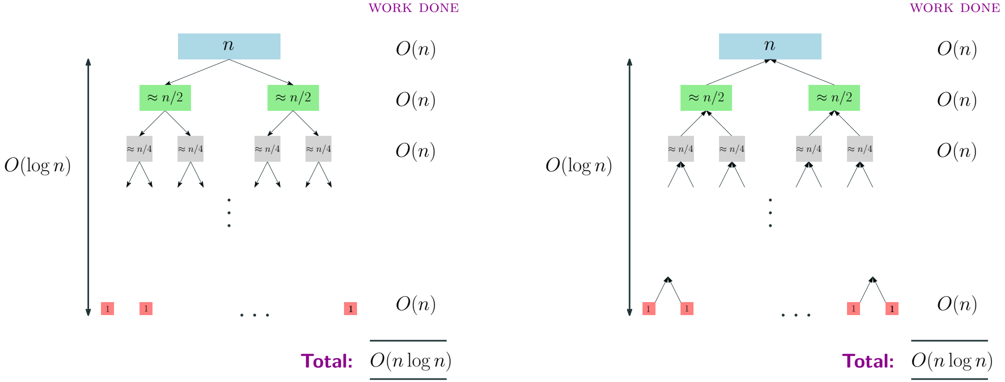
1. **Divide**. **Split** the array into two halves $S_1, S_2$
2. **Conquer**.
   1. **Recursively** sort the left half $S_1$
   2. **Recursively** sort the right half $S_2$
3. **Combine**. **Merge** the two sorted halves $S_1, S_2$ into $S$

# Visualization

`https://opendsa-server.cs.vt.edu/embed/mergesortAV`

Try: **`85 24 63 45 17 31 96 50 67 88 11`**

# Time complexity of merge sort




Left: total time spent to create the two halves $S_1, S_2$ at every recursive call;
Right: total time spent for merging.

☞ Merge sort runs in $O(n \log n) + O(n \log n) = 2 \times O(n \log n) = O(n \log n)$ time

# Space complexity of merge sort

## Merge sort

- For creating the two subsequences $S_1, S_2$ we need $O(n/2) + O(n/2) = O(n)$ extra space.
- So, the total amount of extra space needed by a series of recursive call from the root to a leaf of the recursion tree also amounts to $O(n)$ since

$$O(n)+O\left(\frac{n}{2}\right)+O\left(\frac{n}{4}\right)+\ldots+O(1) = O\left(n+\frac{n}{2}+\frac{n}{4}+\ldots+1\right) = O(2n) = O(n)$$

- For recursion, a stack is needed of size $O(\log n)$
- Total space complexity: $O(n) + O(\log n) = O(n)$

# The `Comparable` interface in Java

https://docs.oracle.com/en/java/javase/17/docs/api/java.base/java/lang/Comparable.html

## Why you should use the `Comparable` interface?

If we ever need to compare two objects of a class, there must be a comparison method for the class. This interface forces the class to define such a method if it is not already defined inside it. For the wrapper classes such as **`Integer`**, **`Double`**, **`Character`** etc. comparison methods are already defined. Generic sorting methods in Java use the comparison method for sorting by comparisons.

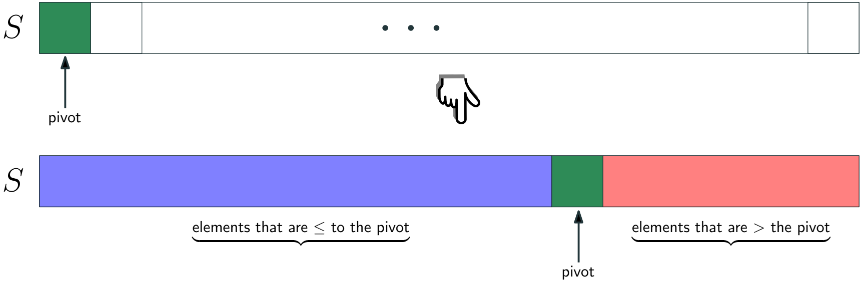
☞ The comparison method must be named **`compareTo`**, as declared inside the **`Comparable`** interface

☞ **`obj1.compareTo(obj2) < 0`** if **`obj1`** is **less than** **`obj2`**;
**`obj1.compareTo(obj2) == 0`** if **`obj1`** is **equals** **`obj2`**;
**`obj1.compareTo(obj2) > 0`** if **`obj1`** is **greater than** **`obj2`**;

See the class `MergeSort`

# Quick sort

# Partitioning an array




## The algorithm

Denoted the input by $S$

1. Select the first element in $S$; call it **pivot**
2. Find the elements in $S$ that are less than equal to pivot and send them to the left part of $S$ and the ones that are greater than the pivot to the right part of $S$
3. Put the pivot at the appropriate location in $S$, meaning put it at the location where it would appear if $S$ is sorted

# An example

| first | | | | | | | | last |
|---|---|---|---|---|---|---|---|---|
| 44 | 75 | 23 | 43 | 55 | 12 | 64 | 77 | 33 |
| up | | | | | | | | down |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 44 | 75 | 23 | 43 | 55 | 12 | 64 | 77 | 33 |
| | up | | | | | | | down |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 44 | 33 | 23 | 43 | 55 | 12 | 64 | 77 | 75 |
| | | | | up | down | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 44 | 33 | 23 | 43 | 12 | 55 | 64 | 77 | 75 |
| | | | | down | up | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 12 | 33 | 23 | 43 | 44 | 55 | 64 | 77 | 75 |
| | | | | down | | | | |

# Pseudocode

1. pivot $= S[$**first**$]$, up $=$ **first**, down $=$ **last**
2. **do**
   2.1 Increment up until up selects the first element greater than the pivot value or up has reached **last**
   2.2 Decrement down untill down selects the first element less than or equal to the pivot value or down has reached **first**
   2.3 if up $<$ down, exchange $S[$up$]$ and $S[$down$]$
3. **while** up is to the left of down
4. Exchange $S[$first$]$ and $S[$down$]$

☞ Note that the index down is a correct place for the pivot since in that case, everything to its left will be less than pivot and everything to its right will be greater than pivot

# Partition

```java
private static <K extends Comparable<K>> void swapTheItemsAt(K[] S, int i, int j) {
   K hold = S[i];
   S[i] = S[j];
   S[j] = hold;
}

private static <K extends Comparable<K>> int partition(K[] S, int first, int last) {
   K pivot = S[first];
   int up = first, down  = last;

   do {
      while( (up < last) && (pivot.compareTo(S[up]) >= 0))
         up++;

      while( pivot.compareTo(S[down]) < 0)
         down--;

      if( up < down )
         swapTheItemsAt(S,up,down);

   }while(up < down);

   swapTheItemsAt(S,first,down);
   return down;
}
```

# Quick sort

- It is another divide and conquer sorting algorithm
- Runs in $O(n^2)$ time (explained next)

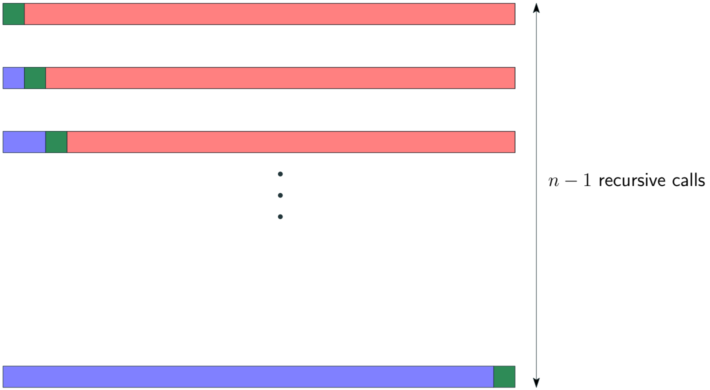
**The algorithm**

Let the input be denoted by $S[\texttt{first}, \ldots, \texttt{last}]$

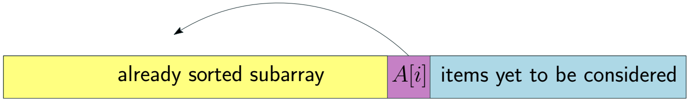
1. **Divide**. **Partition** the array so that the pivot item reaches its correct place in the array (its index is `pivIndex`)
2. **Conquer**.
   1. **Recursively** sort the subarray $\texttt{first}, \ldots, \texttt{pivIndex-1}$ (the subarray to the left of pivot)
   2. **Recursively** sort the subarray $\texttt{pivIndex+1}, \ldots, \texttt{last}$ (the subarray to the right of pivot)

See the class `QuickSort`

# Time and space complexities of quick sort




- When the array is sorted, at every recursive call, we find that all the other elements are bigger than the pivot! This is the worst case in fact
- So, we make $n - 1 = O(n)$ recursive calls; we spend $O(n)$ time for paritioning at every level
- Total time taken $O(n^2)$
- Space complexity: $O(n)$ since recursion depth can be at most $n - 1$




**Insertion sort, needed for comparison; runs in $O(n^2)$ time**

**Input**: An array $A$ of $n$ comparable elements

for $i = 1$ to $n - 1$ do

  Insert $A[i]$ at the proper spot within the sorted subarray $A[0], A[1], \ldots, A[i]$;

☞ If $A$ is already sorted then every $A[i]$ is already in its correct position. As a result, every iteration of the for-loop runs in $O(1)$ time. Consequently, insertion sort takes $O(n)$ time when $A$ is already sorted.

**Applet.** `https://visualgo.net/en/sorting`

Now see the class `SortingSpeedComparison`

# Quick sort vs Merge sort, output in some run, $n = 1M$

```
n = 1,000,000
QuickSort (O(n^2)): 624 ms
MergeSort (O(n*log n)): 1264 ms
```


*Quick sort* ***beats*** *merge sort in practice on randomly ordered arrays despite having worse time complexity!*

# Merge sort vs. Quick sort vs. Insertion sort

```
n  = 100,000
QuickSort (O(n^2)): 85 ms
MergeSort (O(n log n)): 124 ms
InsertionSort (O(n^2)): 19002 ms
```

*Insertion sort is unusable in general when $n$ is large. However, if the input was already sorted, it could finish in just $10$ ms when $n = 100K$. In such cases, every iteration of the for-loop takes $O(1)$ time since $A[i]$ is already at its correct spot! As result, insertion sort takes $O(n)$ time on sorted inputs.*

# Quick sort performs terribly when the input is already sorted!

**Output,** $n = 10K$

```
n = 10,000
QuickSort (O(n^2)) on a random array: 6 ms
QuickSort on a sorted array: 200 ms
```

☞ *When the input is sorted, quick sort runs in quadratic time*

# What happens when the input size is 50,000?

```
n = 50,000
Exception in thread "main" java.lang.StackOverflowError
  at recursion.QuickSort.recurseAndSort(QuickSort.java:12)
  at recursion.QuickSort.recurseAndSort(QuickSort.java:13)
  at recursion.QuickSort.recurseAndSort(QuickSort.java:13)
  at recursion.QuickSort.recurseAndSort(QuickSort.java:13)
  at recursion.QuickSort.recurseAndSort(QuickSort.java:13)
  at recursion.QuickSort.recurseAndSort(QuickSort.java:13)
  at recursion.QuickSort.recurseAndSort(QuickSort.java:13)
  .
  .
  .
```

☞ *The call stack runs out of space since quick sort strives to make around $50K$ calls in this case!*

## How to avoid StackOverflowError exception?

- Make quick sort non-recursive by using a stack explicitly
- ```
  LinkedStack<Pair> stack = new LinkedStack<>();
  stack.push( new Pair(0,S.length-1) ); // push the left and right indices of the source array

  while( !stack.isEmpty() ) {
     var currentPair = stack.pop(); // pop a pair from the stack

     if( currentPair.leftIndex >= currentPair.rightIndex ) continue;

     int pivotIndex = partition(S, currentPair.leftIndex, currentPair.rightIndex);
     stack.push( new Pair(currentPair.leftIndex, pivotIndex - 1) ); // push the left and right indices of the left chunk
     stack.push( new Pair(pivotIndex+1, currentPair.rightIndex) );  // push the left and right indices of the right chunk
  }
  ```
- See the class `NonRecursiveQuickSort`
- The non-recursive version is still **slow** on **sorted** inputs but unlike the recursive version, it doesn't crash by throwing a `StackOverflowError` exception
- Now, `NonRecursiveQuickSort` can sort a $50K$-sized **sorted** input in $2608$ ms
  ```

# How to 'almost' avoid quadratic runtime in practice?

## Speeding up quick sort in practice

- Quick sort slows down on sorted inputs because of the bad pivots which generate empty left chunks (all the other elements goes to the right chunks)

☞ **Idea.** choose pivots **randomly** instead of sticking to the first element every time

- It will be then **unlikely** a bad pivot is chosen every time a partition is executed
- It can be shown theoretically that this small change will result in $O(n \log n)$ behavior in practice (proof is out of scope)
- See the class `RandomizedQuicksort` (**recursive**)
- 
```java
Random generator = new Random();
int pivotIndex =  generator.nextInt(first, last + 1); // generate a random index for pivot selection
K pivot = S[pivotIndex];
```
- The randomized **recursive** version can sort a $50K$-sized sorted input in well under $30$ ms (previously it took $2608$ ms!)

# Avoiding `StackOverflowError` exceptions

- Although very unlikely, bad pivots can still be chosen resulting in `StackOverflowError` exceptions since `RandomizedQuicksort` is recursive
- **Solution.** make it non-recursive
- See the class `NonRecursiveRandomizedQuickSort`
- No more stack overflows and painful slowdowns on sorted datasets are unlikely!

# Making quick sort faster in practice

## Median of three heuristic

Use the median of the three items `S[first]`, `S[(first+last)/2]`, `S[last]` as the pivot. In this case, median is the second item of the sorted sequence of the above three items.

- In the randomized version, we choose pivots randomly
- Finding a pivot using random number generator is **slower** than computing the median of the above three items since only comparison operators and swapping can be used to select the median

```
int mid = (first + last)/2;
if (S[last].compareTo( S[first] ) < 0)   swapTheItemsAt(S, first, last);
if (S[mid].compareTo( S[first]) < 0)    swapTheItemsAt(S, mid, first);
if (S[last].compareTo( S[mid]) < 0 )    swapTheItemsAt(S, last, mid);

K pivot = S[mid]; // S[mid] is now the pivot since the median item is now present at S[mid]
```

- Also, this results in careful selection of pivots in practice
- Consequently, quick sort runs surprisingly faster!

# Further optimization

- Insertion sort works very fast for small inputs
- We leverage insertion sort in the algorithm
- When array size is at most $50$ (other small numbers may work as well), do not partition anymore, use insertion sort instead

```
 if( (currentPair.rightIndex - pivotIndex + 1) > 50)
   stack.push(new Pair(pivotIndex, currentPair.rightIndex));
else
   insertionSort(S,pivotIndex,currentPair.rightIndex);

if( (pivotIndex -  currentPair.leftIndex) > 50)
   stack.push(new Pair(currentPair.leftIndex, pivotIndex - 1));
else
   insertionSort(S,currentPair.leftIndex,pivotIndex-1);
```

See the class `MedOfThreeNonRecQuickSort`

# Demonstration

### $n = 10,000,000$, randomly generated integer array

```
n = 10,000,000
NonRecursiveRandomizedQuickSort (O(n log n) behavior expected): 6723 ms
MedOfThreeNonRecQuickSort (O(n log n) behavior expected): 4971 ms
MergeSort (O(n log n)): 5195 ms
Arrays.sort() (O(n log n)): 5435 ms
```

### $n = 10,000,000$, sorted integer array

```
n = 10,000,000
NonRecursiveRandomizedQuickSort (O(n log n) behavior expected): 3654 ms
MedOfThreeNonRecQuickSort (O(n log n) behavior expected): 1498 ms
MergeSort (O(n log n)): 4411 ms
Arrays.sort() (O(n log n)): 161 ms
```

☞ **Moral of the story.** `Arrays.sort()` is hard to beat in general

# Using multiple cores on your machine to sort faster

**$n = 10,000,000$, randomly generated integer array**

```
n = 10,000,000
Arrays.sort() (O(n log n)): 5564 ms
Arrays.parallelSort() (O(n log n)): 488 ms
```

Implement the selection sort algorithm using the `Comparable` interface

`https://en.wikipedia.org/wiki/Selection_sort`

# Timsort (optional, for algorithm lovers only)

Java uses **Timsort** for sorting an array of non-primitives

`https://en.wikipedia.org/wiki/Timsort`

# Generic binary search

```java
public class GenericBinarySearch {
   private static <K extends Comparable<K>> boolean binarySearchRec(K[] A, K target, int low, int high) {
      if( low > high )
         return false;
      else {
         int mid = (low + high) / 2; // mid takes the floor of (low + high) / 2
         if( target.equals(A[mid]) )      return true;
         else if( target.compareTo( A[mid] ) < 0 ) return binarySearchRec(A, target, low, mid - 1 ); // recursive call
         else return binarySearchRec(A, target, mid + 1, high );// recursive call
      }
   }

   public static <K extends Comparable<K>> boolean search(K[] A, K target) {
      return binarySearchRec(A, target, 0, A.length-1);
   }
}
```

The `Comparable` interface also helps us to implement a generic binary search

# Recursion tips

- Make sure every chain of recursive calls eventually reach at least one base case
- Long chains of recursive calls can throw `StackOverflowError`; be careful!
- If such long chains cannot be avoided, make your code iterative (non-recursive)

# Reading

https://opendsa-server.cs.vt.edu/ODSA/Books/Everything/html/RecIntro.html