The roots of the general quintic equation

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Abstract

In this paper, I describe the roots of the general quintic equation.
The paper ends with "The End"

Introduction

In a previous paper, I've described the roots to the general cubic equation. In a previous paper, I've described one of the roots of the quartic equation. In a previous paper, I've described my monic quintic identity. In a previous paper, I've described the roots of my monic quintic equation, which are expressible in radicals. In this paper, I describe the roots of the general quintic equation.

Preliminaries

The general quintic equation is

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

where a, b, c, d, e, f are constants.

If $f = 0$ then the equation reduces to

$$x(ax^4 + bx^3 + cx^2 + dx + e) = 0$$

which has the root $x = 0$ and the quartic

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

one of whose roots is known and can be factored out. The roots of the remaining cubic are known.

Similarly, if $a = 0$ then the equation reduces to the quartic equation

$$bx^4 + cx^3 + dx^2 + ex + f = 0$$

one of whose roots is known and can be factored out. The roots of the remaining cubic are known.

If $a = f = 0$ then the equation reduces to

$$x(bx^3 + cx^2 + dx + e) = 0$$

which has the root $x = 0$ and the cubic

$$bx^3 + cx^2 + dx + e = 0$$

whose roots are known.

Therefore, $a \neq 0$ and $f \neq 0$ henceforth. Moreover, we divide the general quintic equation by the leading coefficient a to transform the general quintic equation to the monic quintic equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

where $e \neq 0$ henceforth.

Comparing coefficients with my monic quintic

Recall that my monic quintic is

$$x^5+ax^4+bx^3+\left(P^3+bP+a\left(Q-P^2\right)-2PQ+\frac{e}{b-aP+P^2-Q}\right)x^2+\left(Q\left(b-aP+P^2-Q\right)+\frac{e(a-P)}{b-aP+P^2-Q}\right)x+e=0$$

Comparing coefficients we get
(1) $a=a$
(2) $b=b$
(3) $c=P^3+bP+a\left(Q-P^2\right)-2PQ+\frac{e}{b-aP+P^2-Q}$
(4) $d=Q\left(b-aP+P^2-Q\right)+\frac{e(a-P)}{b-aP+P^2-Q}$
(5) $e=e$
Thus, if suitable P and Q are obtained, then, by my monic quintic identity, we can reduce the monic quintic equation to the product of a cubic equation and a quadratic equation, whose roots are known.

Choosing P and Q

Eliminating e between equations (3) and (4) gives us the eliminant

$$(6)\ a^2P^2-a^2Q-abP+ac-2aP^3+2aPQ+bP^2+bQ-cP-d+P^4-P^2Q-Q^2=0$$

As long as $b-aP+P^2-Q\neq 0$ and we obtain corresponding P and Q, we may choose any value for either P or Q to solve the eliminant. For most quintics, $P=0$ is a valid and convenient choice. When $P=0$ is not a valid choice, other valid and convenient choices may be $P=a$, $Q=0$, $Q=P^2$ etc.

Solving the monic quintic

Once we have at least one valid value of P and one valid value of Q, by the right side of my monic quintic identity, we obtain 5 roots of the monic quintic equation.

Notes

1. Note that by following this procedure, we obtain 5 roots of the general quintic equation expressible in radicals.
2. Note that this procedure doesn't invalidate Galois' theory since our procedure is based on expressing the general quintic equation as **factored forms** but not on solving the quintic equation algebraically.

Exercises for the reader

Find the roots of the following quintic equations expressible in radicals:

1. $$x^5+x^4-266713x^3+4x^2-16244256x-31481=0$$
2. $$x^5-x-20=0$$

The End