TD 4: Büchi automata

Exercise 1 (Succintness of LTL+P)**.** Let $\mathrm{AP}_{n+1} = \{p_0, \dots, p_n\} = \mathrm{AP}_n \cup \{p_n\}$ be a set of atomic propositions, defining the alphabet $\Sigma_{n+1} = 2^{\mathrm{AP}_{n+1}}$. We want to show the existence of an $O(n)$-sized formula with past such that any equivalent pure future LTL formula is of size $\Omega(2^n)$.

First consider the following LTL formula of exponential size:

$$\phi_n = \bigwedge_{S \subseteq \mathrm{AP}_n} \Big(\big(\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \wedge p_n \big) \Rightarrow \mathsf{G}\big(\big(\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \big) \Rightarrow p_n \big)$$
$$\wedge \big(\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \wedge \neg p_n \big) \Rightarrow \mathsf{G}\big(\big(\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \big) \Rightarrow \neg p_n \big) \Big)$$

Done in TD2:

1. Describe which words of Σ_{n+1}^ω are the models of ϕ_n.
2. Give an LTL formula with past operators which checks whether the current position is the initial position of the word. Give an LTL formula with past operators ψ_n of size $O(n)$ initial equivalent to ϕ_n.

The goal now is to show that any pure future LTL formula satisfying ϕ_n must have size $\Omega(2^n)$.

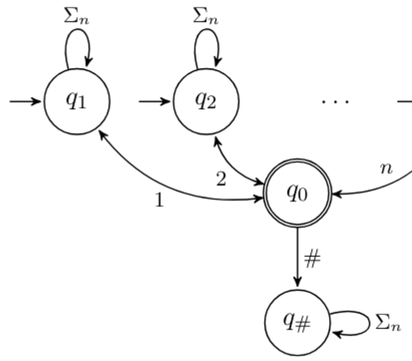
1. Consider the language $L_n = \{\sigma \in \Sigma_{n+1}^\omega \mid \sigma \models \mathsf{G}\,\phi_n\}$. We want to show that any generalized Büchi automaton that recognises L_n has at least 2^{2^n} states.

   Fix a permutation $a_0 \dots a_{2^n-1}$ of the symbols of Σ_n and consider all possible subsets K of $\{0, \dots, 2^n - 1\}$. For each K, define the word $w_K = b_0 \dots b_{2^n-1}$ such that for $0 \le i \le 2^n - 1$,
   $$b_i = \begin{cases} a_i & \text{if } i \in K, \\ a_i \cup \{p_n\} & \text{otherwise.} \end{cases}$$
   Using the words w_K for different subsets K, show that all generalized Büchi automata for $\mathsf{G}\,\phi_n$ require at least 2^{2^n} states.
2. Deduce the lower bound on the size of any LTL formula equivalent to ϕ_n.

Exercise 2. Build the Büchi automaton for LTL formula $a\,\mathsf{U}\,(\mathsf{X}\,b \wedge \neg a)$ using the method seen during the course.

Exercise 3 (Complementation is expensive)**.** Consider the alphabet $\Sigma_n = \{1, 2, \dots, n, \#\}$ and the language $L_n \subseteq \Sigma_n^\omega$ defined by the Büchi automaton (note the bi-directional transitions) in the given figure.

Our goal is to prove that the complement of L_n in Σ_n^ω cannot be recognized by a Büchi automaton having less than $n!$ states.



1. Let $(i_1, \ldots, i_n)$ be a permutation of the elements of $\{1, \ldots, n\}$. Show that the infinite word $(i_1 \ldots i_n \#)^\omega$ is *not* in L_n.
2. Let $a_1 \ldots a_k$ be a finite word in $\{1, \ldots, n\}^*$. Show that all infinite words described by $(\Sigma_n^* a_1 a_2 \Sigma_n^* a_2 a_3 \Sigma_n^* \ldots \Sigma_n^* a_{k-1} a_k \Sigma_n^* a_k a_1)^\omega$ is in L_n.
3. Consider two different permutations $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ of $\{1, \ldots, n\}$. The words $\rho = (i_1 \ldots i_n \#)^\omega$ and $\sigma = (j_1 \ldots j_n \#)^\omega$ are recognized by all Büchi automata $\mathcal{B}$ that recognize the complement of L_n. Show that if ρ loops infinitely in a subset R and σ loops infinitely in a subset S of the states of $\mathcal{B}$, then R and S must be disjoint. (Note: ρ visits each state of R infinitely often. Same for σ.)
4. Conclude.

Exercise 4. For the LTL formula $\phi_n = (p \wedge X^n p) \vee (\neg p \wedge X^n \neg p)$,

1. Give a Büchi automaton for ϕ_n with $O(n)$ states.
2. Show that all Büchi automata for $\mathsf{G}\,\phi_n$ must have at least 2^n states.
3. Deduce that the Büchi automaton of the complement of a language of a Büchi automaton with $O(n)$ states can have an exponentially more number of states.

Homework

To hand in on October 24 at the beginning of the exercise session or anytime sooner by email at `nicolas.dumange@ens-paris-saclay.fr`, with file name of the form "FirstnameLastname.pdf". Answers can be written in French or in English.

Homework 1. Build the Büchi automaton for the following formulae using the method seen during the course: (1) $\mathsf{G}(a \to (b \mathbin{\mathsf{U}} c))$, and (2) $\mathsf{G}\,\mathsf{F}\,b \to \mathsf{F}\,\mathsf{G}\,a$.

Homework 2. Give an LTL formula φ_n of size $O(n)$ over $\mathrm{AP} = \{p_0, \ldots, p_{n-1}\}$ that simulates an n-bit counter. To be more precise, the models of φ_n are exactly the infinite words in $(2^{AP})^\omega$ where, at each position i, the valuation of $(p_0, \ldots, p_{n-1})$ encodes an n-bit binary number v_i, and for all $i \geq 0$, $v_{i+1} = (v_i + 1) \pmod{2^n}$.

Also provide a tight lower bound on the size of any equivalent Büchi automaton.