TD 8: Binary Decision Diagrams

Exercise 1 (Some BDDs)**.** Draw the BDDs for the following functions, using the order of your choice on the variables $\{x_1, x_2, x_3\}$:

1. the majority function $m(x_1, x_2, x_3)$: its value is 1 iff the majority of the input bits are 1's,
2. the hidden weighted bit function $h(x_1, x_2, x_3)$: its value is that of variable x_s, where $s = \sum_{i=1}^{3} x_i$ and x_0 is defined as 0.

Exercise 2 (Symmetric Functions)**.** A *symmetric function* of n variables has the same value for all permutations of the same n tuple of arguments.

Show that a BDD for a symmetric function has at most $\frac{n(n+1)}{2} + 1$ nodes (when omitting the 0-node).

Exercise 3 (An Upper Bound on the Size of BDDs)**.** The size $B(f)$ of a BDD for a function f is defined as the number of its nodes. Consider an arbitrary boolean function f on the ordered set $x_1 \cdots x_n$, and consider a variable x_k.

1. Show that we can bound the number of nodes labeled by $\{x_1, \ldots, x_k\}$ by $2^k - 1$.
2. How many different subfunctions on the ordered set of variables $x_{k+1} \cdots x_n$ exist? Deduce another bound for the number of nodes labeled by $\{x_{k+1}, \ldots, x_n\}$.
3. What global bound do you obtain for $k = n - \log_2(n - \log_2 n)$?

Exercise 4 (Finding the Optimal Order)**.** Reduce the problem of finding the optimal order for the variables in a BDD (one that minimizes the number of vertices in the BDD) to the problem of finding a path of minimal weight in a weighted graph with subsets of $\{x_1, \ldots, x_n\}$ as vertices.

Exercise 5 (Implication operator)**.** Let us consider variables with the order $u < v < w < x < y$. Let F_1, F_2 be propositional formulae with

$$F_1 := u \rightarrow (\neg w \wedge (x \leftrightarrow y)) \qquad F_2 := v \wedge (x \rightarrow y)$$

(1) Draw BDDs for F_1 and F_2, using the given order.

Let F, G be any two formulae and x a variable. Using the Shannon partitioning $F = ite(x, F[x/1], F[x/0])$ we derived the following equation for the conjunction of two formulae:

$$F \wedge G \equiv \begin{cases} F & \text{if } F \equiv G \\ 0 & \text{if } F = 0 \text{ or } G = 0 \\ G & \text{if } F = 1 \\ F & \text{if } G = 1 \\ ite(x, F[x/1] \wedge G[x/1], F[x/0] \wedge G[x/0]) & \text{otherwise.} \end{cases}$$

(2) What is the corresponding equation for *implication* $F \rightarrow G$?

(3) Using the above, construct a BDD for $F_1 \rightarrow F_2$.

Exercise 6 (Algorithms)**.** Describe linear-time algorithms for the following problems:

1. Counting the number of solutions of a boolean function f represented by a BDD, i.e. of the number of valuations ν s.t. $\nu \models f$.
2. Given two BDDs for boolean functions f and g, finding a valuation in the symmetric difference of f and g (i.e., it can be a satisfying assignment for f and not g or vice versa)