1 The Spectral Theorem

1.1 A self-adjoint operator $T \in \mathcal{L}(V)$ is symmetric.

Proof. Consider a orthonormal basis of V: $e_1, ..., e_n$. Since we're working in an orthonormal basis,

$$Te_k =< Te_k, e_1 > e_1 + \ldots + < Te_k, e_n > e_n$$

The entry at the jth row and kth column of $[\mathcal{M}(T, (e_1, ..., e_n), (e_1, ..., e_n))]_{jk} =< Te_k, e_j >$. The entry at the kth row and the jth column of $[\mathcal{M}(T, (e_1, ..., e_n), (e_1, ..., e_n))]_{kj} =< Te_j, e_k >=< e_j, Te_k >$ Over $\mathbf{F} = \mathbf{R}$, $< Te_k, e_j >=< e_j, Te_k >$, so the corresponding matrix representation is a symmetric matrix.

1.2 For self-adjoint operator $T \in \mathcal{L}(V)$, $b, c \in \mathbf{R}$, and $b^2 - 4c < 0$, $T^2 + bT + c$ is invertible.

Proof. An operator is invertible if 0 is not an eigenvalue. For any nonzero vector $v \in V$, consider

$$\begin{aligned} < (T^2 + bT + c)v, v > &=< T^2 v, v > +b < Tv, v > +c < v, v > \\ &\geq ||Tv||^2 - b||Tv||||v|| + c||v||^2 \\ &= (||Tv|| - \frac{b}{2}||v||)^2 + (c - \frac{b^2}{4})||v||^2 \\ &> 0 \end{aligned}$$

We can conclude that $T^2 + bT + c$ is invertible because there are no 0 eigenvalues.

1.3 A self-adjoint operator $T \in \mathcal{L}(V)$ always has a real eigenvalue

Proof. For a nonzero vector v and $dim(V) = n$, consider the set of vectors $v, Tv, T^2 v, \ldots, T^n v$. Because there are $n+1$ elements, there exists constants $a_0, \ldots, a_n$ not all zero such that $a_0 v + a_1 Tv + \cdots + a_n T^n v = 0$. This can be factorized as $a_n(T^2 + b_1 T + c_1) \ldots (T^2 + b_M T + c_M)(T - \lambda_1) \ldots (T - \lambda_N)v = 0$ where $b_i^2 - 4c_i < 0$ and $\lambda, b, c \in \mathbf{R}$. From section 1.2, we know the inverses for the quadratic function exists, so $(T - \lambda_1 I) \ldots (T - \lambda_N I)v = 0$. Since we assumed v was a nonzero vector, there is some λ_i such that $(T - \lambda_i I)v = 0$, ie λ_i is an eigenvalue.

1.4 For a self-adjoint operator $T \in \mathcal{L}(V)$ and suppose U is invariant in T, then $T|_{U^\perp}$ is a self-adjoint operator.

Proof. Suppose $u \in U$ and $v \in U^\perp$, then $< u, Tv >=< Tu, v >= 0$ where the last equality is true because $Tu \in U$ and $v \in U^\perp$. This shows that $U^\perp$ is an invariant subspace as well.

Since $U^\perp$ is an invariant subspace, we know that $T|_{U^\perp} = T$ when applied on elements in $U^\perp$. Suppose $u, v \in U^\perp$, then $< T|_{U^\perp} u, v >=< Tu, v >=< u, Tv >=< u, < T|_{U^\perp} v >$, so $T|_{U^\perp}$ is a self-adjoint operator.

1.5 $T \in \mathcal{L}(V)$ is a self-adjoint operator iff it can be unitarily diagonalizable with real numbers.

Proof. We proceed by inducting on the $dim(V)$.
$\underline{BaseCase}$: $dim(V) = 1$ This is trivially true because a vector is orthogonal to itself.
$\underline{InductiveHypothesis}$: Suppose for all dimensions less than dim V ¿ 1, the self-adjoint operator can be unitarily diagonalizable with real numbers.
$\underline{InductiveStep}$: Section 1.3 tells us that T has a real eigenvalue λ. Let u be the corresponding eigenvector

and because $Tu = \lambda u$, $U = \text{span}(u)$ is an invariant subspace. Section 1.4 tells us that $T|_{U^\perp}u$ is a self-adjoint operator with dimension $dim(V) - 1$. We can apply the inductive hypothesis on $T|_{U^\perp}u$, so the orthogonal diagonalization consists of $\{u\} \cup U^\perp$.