Generalized eigenvalue problems: find generalized eigenvalues of (L, D), where we assume $D > 0$. (If a Hermitian matrix A is spd, we write $A > 0$.) Let's also assume that L is symmetric. Solve for (λ, v):

$$Lv = \lambda Dv.$$

Note some things:

- Let's premultiply by D^{-1}: Solve for (λ, v):

  $$D^{-1}Lv = \lambda v.$$

  Why don't we just solve *standard* eigenvalue problems? This is *not* a symmetric eigenvalue problem. The matrix $D^{-1}L$ could be *defective*, meaning that it could have an *incomplete* set of eigenvectors. I.e., the algebraic multiplicity of an eigenvalue could be strictly larger than it's geometric multiplicity. E.g., the matrix

  $$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

  is defective. $\lambda = 1$ is an eigenvalue of algebraic multiplicity 2. But it's of geometric multiplicity 1. (Because the invariant subpsace associated to $\lambda = 1$ has dimension 1.) Another problem: mathematically speaking, the *stability* ("condition number") of the map $D^{-1}L \mapsto V$, where V is a matrix of eigenvectors "can be large". But the stability of the problem of finding an eigenvector matrix from a *Hermitian* matrix is unity. (This notion of conditioning is not (really) the condition number of a matrix.)
- Alternative: if $D > 0$, then Spectral Theorem:

  $$D = UTU^T,$$

  where T is diagonal with strictly positive (diagonal) entries, and U is unitary. I can define

  $$D^{1/2} = UT^{1/2}U^T, \qquad T^{1/2} = \operatorname{diag}\left(\sqrt{t_1}, \ldots, \sqrt{t_N}\right)$$

  Can verify: $D^{1/2}D^{1/2} = D$, and that $D^{1/2}$ is symmetric. So $D^{1/2}$ is **a** symmetric square root of D. (In fact, this is the only symmetrtic sqrt of D, but it's not the only sqrt of D.) So not only does D have a unitary diagonalization, but so does $D^{1/2}$, and in particular $D^{1/2}$ is invertible.

  Let's look again at

  $$\begin{aligned} Lv &= \lambda Dv \\ &= \lambda D^{1/2}D^{1/2}v \\ LD^{-1/2}D^{1/2}v &= \lambda D^{1/2}D^{1/2}v \\ \left(D^{-1/2}LD^{-1/2}\right)D^{1/2}v &= \lambda D^{1/2}v \\ Aw &= \lambda w, \end{aligned}$$

  where $A \coloneqq D^{-1/2}LD^{-1/2}$ and $w \coloneqq D^{1/2}v$. Since A is symmetric: there exist a unitary diagonalization,

  $$A = W\Lambda W^T,$$

  where W is unitary. Note: Λ contains our generalized eigenvalues. And each column of W is "sort of" a generalized eigenvector. The "sort of" is resolved by $w = D^{1/2}v$. I.e.,

  $$V \coloneqq D^{-1/2}W,$$

  is a matrix with our generalized eigenvectors.