Linear Algebra – Cheat Sheet

1. Rank of a Matrix

- The **rank** of a matrix is the maximum number of **linearly independent rows or columns**.
- **Methods to find rank:**
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  - **Row Echelon Form (REF):** Reduce the matrix to echelon form and count nonzero rows.
  - **Determinant Method:** The rank is the order of the largest nonzero determinant of a square submatrix.
  - **Properties:**
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    - Rank $\leq$ min(rows, columns).
    - Full rank: If rank = number of rows (row rank) or columns (column rank).
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2. Systems of Linear Equations

General Form:

A system of equations:
$Ax = b$
where **A is a coefficient matrix**, **x is a variable vector**, and **b is a constant vector**.

Types of Solutions:

- **Consistent System** → At least one solution.
- **Inconsistent System** → No solution.
- **Unique Solution** → If determinant $|A| \neq 0$.
- **Infinite Solutions** → If rank of A = rank of augmented matrix but determinant = 0.

Solution Methods:

- **Gaussian Elimination** – Converts to row echelon form.
- **Cramer's Rule** – Uses determinants (only for square matrices).
- **Matrix Inversion Method** – $x = A^{-1}b$, if A is invertible.

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3. Characteristic Equation

- The **characteristic equation** of a square matrix A is given by: $|A - \lambda I| = 0$ where **λ** are the **eigenvalues** of A.
- **Steps to Find:**

A
B Compute **$|A - \lambda I|$** (determinant).
C Solve for **λ**.
D

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4. Cayley-Hamilton Theorem

- **Statement:** Every square matrix satisfies its own characteristic equation.
- If the characteristic equation is: $f(A) = |A - \lambda I| = 0$ then replacing A into $f(A)$ results in **zero matrix**.

- **Applications:**
  - Used to compute powers of matrices.
  - Simplifies matrix exponentiation problems.

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5. Eigenvalues & Eigenvectors

Definitions:

- **Eigenvalues (λ\lambdaλ)**: Roots of the characteristic equation A−λI =0|A - \lambda I| = 0 A−λI =0.
- **Eigenvectors (xxx)**: Nonzero vectors that satisfy: (A−λI)x=0(A - \lambda I)x = 0(A−λI)x=0

Properties:

- Sum of eigenvalues = **Trace(A)** (sum of diagonal elements).
- Product of eigenvalues = **Det(A)**.
- If AAA is **triangular or diagonal**, eigenvalues are diagonal elements.

Steps to Find Eigenvalues and Eigenvectors:

1st Compute **A−λI =0|A - \lambda I| = 0 A−λI =0** to get **eigenvalues**.
2ndSubstitute each eigenvalue into (A−λI)x=0(A - \lambda I)x = 0(A−λI)x=0 to get **eigenvectors**.

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6. Diagonalization of Matrices

- A matrix AAA is **diagonalizable** if it has **nnn linearly independent eigenvectors**.
- **Diagonalization Formula:** A=PDP−1A = P D P^{-1}A=PDP−1 where:
  - PPP is the matrix of **eigenvectors**.
  - DDD is a **diagonal matrix** with eigenvalues on the diagonal.

Steps to Diagonalize a Matrix:

1st Find **eigenvalues** by solving A−λI =0|A - \lambda I| = 0 A−λI =0.
2ndFind **eigenvectors** for each eigenvalue.
3rdForm **PPP** using eigenvectors as columns.
4th Compute **D=P−1APD = P^{-1} A PD=P−1AP** (should be diagonal).

Quick Summary Table

Topic	Key Points
Rank of a Matrix	Max number of linearly independent rows/columns.
Systems of Linear Equations	Solve using Gaussian elimination, Cramer's rule, or matrix inversion.
Characteristic Equation	(
Cayley-Hamilton Theorem	A matrix satisfies its own characteristic equation.
Eigenvalues & Eigenvectors	Eigenvalues: roots of characteristic equation, Eigenvectors: solve (A−λI)x=0(A - \lambda I)x = 0(A−λI)x=0.

Topic	Key Points
Diagonalization	A=PDP−1A = P D P^{-1}A=PDP−1, where PPP is eigenvectors, and DDD is diagonal.