Multivariable Calculus – Cheat Sheet

1. Partial Derivatives

Definition:

- If $z = f(x, y)$, the **partial derivatives** are:
  - **With respect to x:** $f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$
  - **With respect to y:** $f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$

Higher-Order Partial Derivatives:

- **Second-order partial derivatives:** $f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$
- **Clairaut's Theorem:**
  If f_{xy} and f_{yx} are continuous, then: $f_{xy} = f_{yx}$

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2. Total Derivatives

Definition:

- If $z = f(x, y)$ and x and y are functions of another variable t, then the **total derivative** of z is: $\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

For a function of three variables:

$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$

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3. Maxima and Minima of Two Variables

Critical Points:

- **Find critical points by solving:** $f_x = 0, \quad f_y = 0$
- **Second Partial Derivative Test:**
  Compute: $D = f_{xx} f_{yy} - (f_{xy})^2$
  - **If $D > 0$ and $f_{xx} > 0$:** Local **minimum**.
  - **If $D > 0$ and $f_{xx} < 0$:** Local **maximum**.
  - **If $D < 0$: Saddle point**.
  - **If $D = 0$:** Test is **inconclusive**.

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4. Lagrange's Method of Multipliers

Used to find extrema (max/min) of a function subject to a constraint.

Problem:

Find extrema of f(x,y)f(x, y)f(x,y) subject to the constraint g(x,y)=0g(x, y) = 0g(x,y)=0.

Steps:

1st Compute **Lagrange function:** L(x,y,λ)=f(x,y)+λg(x,y)L(x, y, \lambda) = f(x, y) + \lambda g(x, y)L(x,y,λ)=f(x,y)+λg(x,y)

2nd Solve the system of equations: ∂L∂x=0,∂L∂y=0,∂L∂λ=0\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial \lambda} = 0∂x∂L=0,∂y∂L=0,∂λ∂L=0

3rd Solve for x,y,λx, y, \lambdax,y,λ to get extrema.

Quick Summary Table

Topic	Key Formula/Steps
Partial Derivatives	fx=∂f∂x,fy=∂f∂yf_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}fx=∂x∂f,fy=∂y∂f
Total Derivative	dz/dt=fxdx/dt+fydy/dtdz/dt = f_x dx/dt + f_y dy/dtdz/dt=fxdx/dt+fydy/dt
Maxima & Minima (Two Variables)	Solve fx=0,fy=0f_x = 0, f_y = 0fx=0,fy=0, use second derivative test.
Lagrange's Method	Solve f=λ g\nabla f = \lambda \nabla g f=λ g.