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L_p-cohomologies of Riemannian horns.

Praca licencjacka
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dra hab. Andrzeja Webera
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Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data | Podpis kierującego pracą

Oświadczenie autora (autorów) pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

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Abstract

In this thesis L_p-cohomologies of Riemannian f-horn are calculated.

Słowa kluczowe

kohomologie de Rhama, topologia różniczkowa

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.0 Matematyka, Informatyka:
11.1 Matematyka

Klasyfikacja tematyczna

14 Algebraic Geometry
14F (Co)homology theory
14F40 de Rham cohomology

Tytuł pracy w języku angielskim

An implementation of a difference blabalizer based on the theory of $\sigma - \rho$ phetors

Contents

Chapter 1

Introduction

In [?] the author considers a cone over Riemannian pseudomanifold. The cone is given a metric of the form $dt \otimes dt \oplus t^2 g$ and g is the metric on the nonsingular part of the original pseudomanifold. In the mentioned work, L_p cohomology of this space is presented.

This line of research can be traced back to Cheeger [?]. A similar approach was presented by Youssin [?], where f-horns were considered.

We present a slight modification of this notions, by considering manifolds where the Riemannian metric is of the for

Chapter 2

Preliminaries

2.1. Vector spaces and tensors

Let us recall some basic facts about behaviour of norm when scaling tensors. If we consider a finite-dimensional vector space V with a given metric $||\cdot||$ and define a new metric $||x||_r = r||x||$. Then in the space $(V, ||\cdot||_r)^*$ dual to $(V, |||\cdot|||)$, the normed is scaled by the factor $\frac{1}{r}$, that is for any $\varphi \in V^*$ we get $||\varphi||_r = \frac{1}{r}||\varphi||$.

We will make a simple observation that we will later use in the computations. We consider a Riemannian manifold M and tagent $T_x\mathrm{M}$ and cotangent $T_x^*\mathrm{M}$ spaces in the point x. Let the bases of these spaces be $e_1, e_2, ..., e_n$ and dual $e_1^*, e_2^*, ..., e_n^*$. The volume form of this manifold is $d\mathrm{vol} = \pm e_1^* \wedge e_2^* \wedge ... \wedge e_n^*$.

We now want to compute how forms from $\Lambda(\mathbb{M})$ are scaled with respect to such a change in the norm. Suppose we are considering space of k-forms on $\mathbb{M}$ at some arbitrary point. Then every k-form can be locally expressed in a basis consisting of products of covectors belonging to basis dual to the standard basis. That is every k-form in the point x using some local coordinates $(x_1, x_2, ...x_n)$ can be written $\sum_{\mathrm{I}\in I} a_{\mathrm{I}} dx_{\mathrm{i}}$, where I is the set of k-indices of form $\underbrace{(i_1, i_2, ..., i_k)}_{k \text{ times}}$, with $i_1, i_2, ... \in \{1, 2, ..., n\}$. (following Einstein convention). Let us see how basis vector is scaled:

$$||dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}||_t = ||dx_{i_1}||_t \cdot ||dx_{i_2}||_t \cdot ... \cdot ||dx_{i_k}||_t = \frac{1}{t}||dx_{i_1}|| \cdot \frac{1}{t}||dx_{i_2}||_t \cdot ... \cdot \frac{1}{t}||dx_{i_k}||_t = \frac{1}{t^k}||dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}||$$

This means that any k form is scaled by $1/t^k$ when metric is scaled by a factor of t. This applies also to the volume form, so we obtain:

$$d\mathrm{vol}_t = \frac{1}{t^n} d\mathrm{vol}$$

2.2. Differetial forms

Riemannian metric is a smooth symmetric covariant 2-tensor field on manifold $\mathcal{M}$ that is positive definite at each point.(attaching a field of linear functions that takes two variables to every point of the manifold).

Consulting page 328 of Lee gives us that in any smooth local coordinates (x^i), Riemannian metric can be written as:

$$g = g_{ij} dx^i \otimes dx^j = g_{ij} dx^i dx^j$$

where g_{ij} is a positive definite matrix of smooth functions.

The simplest example of Riemannian metric is *Euclidean metric* on $\mathbb{R}^n$ given in standard coordinates by

$$g = \delta_{ij} dx^i dx^j.$$

Citing prof. Lee, it is common to abbreviate the symmetric product of a tensor α with itself by α^2, so the Euclidean metric can also be written as

$$g = (dx^1)^2 + ... + (dx^n)^2,$$

so now it is way easier to understand what exactly is meant by $dt \otimes dt + f^2 g$, which shoudl be the same as $dt^2 + f^2 g$.

Chapter 3

Computation

The purpose of this chapter is to present the computation of L_p-cohomologies of Riemannian f-horns.

3.1. Setting

Here we introduce definitions and make the first observations. The setting is largely similar to the setting presented in [?], [?], [?].

Definition 3.1.1 (f-horn)**.** Let $\mathcal{M}$ be a Riemannian manifold. Consider a space $\mathbb{R}_{\geq 0} \times \mathcal{M}$. Define a Riemannian tensor on this product by $dt^2 \oplus f^2(t)g$, where g is the metric on $\mathcal{M}$. Such a space will be called an f**-horn**. We will denote it by $\mathrm{c}^f\mathcal{M}$.

Remark 3.1.1. This terminology is present in works of Cheeger.

At first, we will focus our attention on scaling functions from family $f_\alpha(x) = \mathrm{e}^{\alpha x}$, parametrized by $\alpha \in \mathbb{R}$. The intuition behind such manifolds is best presented graphically, as in the Figure **??**.

We can make a simple observation here about differential forms associated with f-horn. The tangent space in the point (t, m) is:

$$T_{(t,m)}(\mathrm{c}^f\mathcal{M} = \mathbb{R} \times T_m\mathcal{M}$$

In terms of differential forms associated with f-horn, it means that $\Lambda^k(\mathbb{R} \times T_m\mathcal{M}) = \Lambda^k(\mathbb{R}) \oplus \Lambda^k(\mathcal{M})$. It can be rephrased in friendlier terms in the following way.

Remark 3.1.2. Every k-form $\omega \in \Lambda^k T(\mathrm{c}^f\mathcal{M})$ can be written as $\omega = \eta + \xi \wedge dt$, where both η and ξ do not contain dt. Please note that η is k-form and ξ is $k-1$ form.

Remark 3.1.3 (norms of forms). If we have the standard inclusion:

$$i_r : \mathcal{M} \to \mathrm{c}^f\mathcal{M},$$

$$i_r(x) = (x, r),$$

then we can define $||\omega||_r := ||\omega|_{\mathcal{M}\times\{r\}}|| = f(r)^{n/p-k}||\omega_r||$

Let

$$\pi : \mathrm{c}^f\mathcal{M} \to \mathcal{M}$$

denote the projection. We can now establish how to calculate a norm for $\eta \in L_p^*()$, namely $||\eta||_r := ||\pi^*\eta||_r = f(r)^{n/p-k}||\eta_r||$.

Our goal is to give the homotopy operator (see in Lee why ??). We do so by defining I_r:

$$I_r : \Omega^*(\mathrm{c}^f\mathcal{M}) \to \Omega^{*-1}(\mathrm{c}^f\mathcal{I})$$

$$M_r(\omega)(x,t) = \int_r^t \xi(x,s)ds$$

We now have to estimate $\int \xi$.

??? Why ? cited: The form $I_r\omega$ is smooth for $r \in (0,1)$, but will also consider $r = 0$ in certain cases. If $r > 0$ then the homotopy formula holds:

$$\omega - \pi^*(\omega_r) = dI_r\omega + I_r d\omega.$$

In reference to Lemma 10.1 from prof's Weber's Let $k < (n+1)/p$ Then the form π^* is p-integrable for each p-integrable form $\eta \in L_p^k(\mathcal{M})$. TODO: Think what the real difference between your idea and this below is:

which would make my proof be:

$$||\pi^*\eta||^p = \int_{c\mathcal{M}} |\pi^*\eta(x,t)|^p d\mathrm{vol}(c\mathcal{M}) = \int_0^\infty ||\eta||_t^p dt = ||\eta||^p \int_0^\infty f(t)^{n-pk}dt$$

so, if f is given by $f(x) = e^\alpha$, the last integral is $||\eta||^p \int_0^\infty e^{\alpha(n-pk)}dt$. This integral is convergent if $\alpha(n-pk) < 0$.

??? It seems now, that we want to say whether and when (depending on k, n, p, r) for a p-integrable form $\omega \in \Omega^k(c^f\mathcal{M})$ the form $I_r\omega$ is integrable. (Lemma 10.3 from prof Weber's).

So due to reasons obove, we now estimate $\int \xi$. Let $\xi \in L_p^i(c^f\mathcal{M}$ and $r \in \mathbb{R}, r > 0$. Then

$$||\int_r^t \xi_s ds|| \leq \int_r^t ||\xi_s||ds =$$

??? is that what I am supposed to do??

$$= \int_s^t f^{i-n/p}(s) f^{n/p-i}(s)||\xi_s||ds$$

The updated plan is to really understand this one here throughly: What and where are we integrating. Why should e^t be integrable on $\mathrm{R}_\geq$. Not looking too good..

New understanding note: what prof Weber is doing here, is just calculating the cohomologies of this pseudoriemannian horn, by showing ... ? That's what should be understood now.

Mike's note: It seems that what is happening here is that prof Weber is saying here, is that $r > 0$ standard homotopy formula holds, and now he is trying to compute whether analogous formula in the L_p space holds.

What's the general plan? Try to dig through prof's Weber's paper and make sense of the whole estimation section, and later apply same ideas to make your research.

What's the point of all these operators? Will try to explain here, using Lee, Weber, Cheeger, Hatcher. One clue is that we have to compute/prove something like Lee, page 444. Say we have $F, G : M \to N$ which are smooth maps. We want to prove that induced maps at the homotopies are equal, $F^* = g^*$.

Digression in digression: **Induced map** For any smooth map $F : M \to N$ between two smooth manifolds with or without boundary, the pullback $F^* : \Omega^p N \to \Omega^p M$ carries closed forms to closed forms and exact forms to exact forms. It thus decsends to a linear map, denoted by $F^* : H^p N \to H^p M$, too.

Digression in digression in digression: **Pullback** of F^* is

$$(F^*\omega)_p(v_1, ..., v_n) = \omega_{F(p)}(dF_p(v_1), ..., dF_p(v_k)).$$

Back to the main thread of thought: If we have two smooth maps $F, G : M \to N$ and we want to prove that the induced maps are equal $F^* = G^*$. Given a closed $p-$form ω on N, we need to produce a $(p-1)$-form η no M such that

$$G^*\omega - F^*\omega = d\eta$$

from this, it will follow that $G^*[\omega] - F^*[\omega] = [d\eta] = 0$, where [] is just taking homotopy equivalence class of given form. The author suggests a way to make it more systematic, by finding an operator h, which transforms closed p-forms on N to $(p-1)$-forms on M and satisfies

$$d(h\omega) = G^*\omega - F^*\omega.$$

Instead of defining $h\omega$ only when ω is close, it turns out to be far easier to define a map h from the space of *all* smooth p-forms on N to the space of smoooth $(p-1)$-forms on M, which satisfies:

$$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega,$$

which implies the above equality when ω is closed. (To be completly precise, we define a family of maps, one for each p, which satisfy said equalities on adequate levels.

$$H(\mathcal{M} \times \mathbb{R}_{\geq})^*_{dR} = H(\mathcal{M})^*_{dR}$$

Bibliography

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