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L_p-cohomologies of Riemannian horns.

Praca licencjacka
na kierunku MATEMATYKA

Praca wykonana pod kierunkiem
dra hab. Andrzeja Webera
Instytut Matematyki

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Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data | Podpis kierującego pracą

Oświadczenie autora (autorów) pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

Data | Podpis autora (autorów) pracy

Streszczenie

ąęźćżźżżżżż
W pracy przedstawiono prototypową implementację blabalizatora różnicowego bazującą na teorii fetorów σ-ρ profesora Fifaka. Wykorzystanie teorii Fifaka daje wreszcie możliwość efektywnego wykonania blabalizy numerycznej. Fakt ten stanowi przełom technologiczny, którego konsekwencje trudno z góry przewidzieć.

Słowa kluczowe

blabaliza różnicowa, fetory σ-ρ, fooizm, blarbarucja, blaba, fetoryka, baleronik

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.0 Matematyka, Informatyka:
11.1 Matematyka

Klasyfikacja tematyczna

14 Algebraic Geometry
14F (Co)homology theory
14F40 de Rham cohomology

Tytuł pracy w języku angielskim

An implementation of a difference blabalizer based on the theory of $\sigma - \rho$ phetors

Spis treści

Introduction

In [**?**] the author considers a cone over Riemannian pseudomanifold. The cone is given a linear metric and a computation of L_p cohomology of this space is presented. We present a slight extension of this by considering manifolds where the metric is blabla. This can be

Computation

The purpose of this paper is to compute L_p-cohomologies of Riemannian horns.

0.1. Setting

In this section we introduce basic definitions and make the most straightforward observations.

Let us consider a space $\mathbb{R}_{\geqslant 0} \times \mathcal{M}$, where $\mathcal{M}$ is Riemannian mainfold. We will define a Riemannian tensor on this product by $dt^2 + f^2(t)g$, where g is the metric on $\mathcal{M}$. Such a space is called by Cheeger an **f-horn**. We will denote it by $\mathrm{c}^f\mathcal{M}$

At first, we will focus our attention on functions $f_1(x) = \mathrm{e}^x$ and $f_2(x) = \mathrm{e}^{-x}$. The intuition behind such manifolds is best presented graphically, as in the Figure **??**.

If we consider a finite-dimensional vector space V with a given metric $||\cdot||$ and define a new metric $|||x||| = r||x||$. Then in the space $(V, ||\cdot||)^*$ dual to $(V, |||\cdot|||)$, the normed is scaled by the factor $\frac{1}{r}$. The bases in these spaces are $e_1, e_2, ..., e_n$ and dual $e_1^*, e_2^*, ..., e_n^*$. Please note that $d\mathrm{vol} = \pm e_1^*, e_2^*, ..., e_n^*$. This siplifies greatly the computation of L_p cohomology of the manifold in consideration.

Also, make a writeup here from lee about the whole volume form deal.

Let us now

$$T_{(t,m)} = \mathbb{R}_+ \times T_m\mathcal{M}$$

Let us take some $\omega \in \Lambda^k(\mathbb{R} \oplus T_m\mathcal{M}) = \Lambda^k(\mathbb{R}) \oplus \Lambda^k(\mathcal{M})$. This equality lets us state that every k-form can be written as $\omega = \eta + \xi \wedge dt$, where both η and ξ do not contain dt. Please note that η is k-form and ξ is $k-1$ form.

Mike's note: Why there is this squared thing? Lee, page 328. **Riemannian metric** is a smooth symmetric covariant 2-tensor field on manifold $\mathcal{M}$ that is positive definite at each point.(attaching a field of linear functions that takes two variables to every point of the manifold).

Consulting page 328 of Lee gives us that in any smooth local coordinates (x^i), Riemannian metric can be written as:

$$g = g_{ij}dx^i \otimes dx^j = g_{ij}dx^i dx^j$$

where g_{ij} is a positive definite matrix of smooth functions.

The simplest example of Riemannian metric is *Euclidean metric* on $\mathbb{R}^n$ given in standard coordinates by

$$g = \delta_{ij}dx^i dx^j.$$

Citing prof. Lee, it is common to abbreviate the symmetric product of a tensor α with itself by α^2, so the Euclidean metric can also be written as

$$g = (dx^1)^2 + ... + (dx^n)^2,$$

so now it is way easier to understand what exactly is meant by $dt \otimes dt + f^2 g$, which shoudl be the same as $dt^2 + f^2 g$.

Therefore we obtain easily $||e_1^* \wedge ... \wedge e_n^*|| = \frac{1}{f^k}$ and as $d\mathrm{vol} = e_1^* \wedge ... \wedge e_n^*$.

$$\int_{\mathcal{M}} |||\omega|||^p d\mathrm{vol} = \int_{\mathcal{M}} (f^{-k}||\omega||)^p =$$

If we have the standard inclusion:

$$i_r : \mathcal{M} \to \mathrm{c}^f \mathcal{M}$$

$$i_r(x) = (x, r)$$

We define $||\omega||_r := ||\omega|_{\mathcal{M}\times\{r\}}|| = f(r)^{n/p-k}||\omega_r||$
Let

$$\pi : \mathrm{c}^f \mathcal{M} \to \mathcal{M}$$

denote the projection. We can now establish how to calculate a norm for $\eta \in L_p^*()$, namely $||\eta||_r := ||\pi^*\eta||_r = f(r)^{n/p-k}||\eta_r||$.

Mike's note: Please note we have many different norms here, which are used for things from different spaces.

Our goal is to give the homotopy operator (see in Lee why ??). We do so by defining I_r:

$$I_r : \Omega^*(\mathrm{c}^f \mathcal{M}) \to \Omega^{*-1}(\mathrm{c}^f \mathcal{M})$$

$$I_r(\omega)(x, t) = \int_r^t \xi(x, s) ds$$

We now have to estimate $\int \xi$.

??? Why ? cited: The form $I_r\omega$ is smooth for $r \in (0, 1)$, but will also consider $r = 0$ in certain cases. If $r > 0$ then the homotopy formula holds:

$$\omega - \pi^*(\omega_r) = dI_r\omega + I_r d\omega.$$

In reference to Lemma 10.1 from prof's Weber's Let $k < (n + 1)/p$ Then the form π^* is p-integrable for each p-integrable form $\eta \in L_p^k(\mathcal{M})$. TODO: Think what the real difference between your idea and this below is:

Proof from prof Weber:

$$||\pi^*\eta||^p = \int_{c\mathcal{M}} |\pi^*\eta(x, t)|^p d\mathrm{vol}(c\mathcal{M}) = \int_0^1 ||\eta||_t^p dt = ||\eta||^p \int_0^1 t^{n-pk} dt$$

which would make my proof be:

$$||\pi^*\eta||^p = \int_{c\mathcal{M}} |\pi^*\eta(x, t)|^p d\mathrm{vol}(c\mathcal{M}) = \int_0^\infty ||\eta||_t^p dt = ||\eta||^p \int_0^\infty f(t)^{n-pk} dt$$

The updated plan is to really understand this one here throughly: What and where are we integrating. Why should e^t be integrable on $\mathrm{R}_{\geqslant}$. Not looking to good..

Mike's note: It seems that what is happening here is that prof Weber is saying here, is that $r > 0$ standard homotopy formula holds, and now he is trying to compute whether analogous formula in the L_p space holds.

What's the general plan? Try to dig through prof's Weber's paper and make sense of the whole estimation section, and later apply same ideas to make your research.

What's the point of all these operators? Will try to explain here, using Lee, Weber, Cheeger, Hatcher. One clue is that we have to compute/prove something like Lee, page 444. Say we have $F, G : M \to N$ which are smooth maps. We want to prove that induced maps at the homotopies are equal, $F^* = g^*$.

Digression in digression: **Induced map** For any smooth map $F : M \to N$ between two smooth manifolds with or without boundary, the pullback $F^* : \Omega^p N \to \Omega^p M$ carries closed forms to closed forms and exact forms to exact forms. It thus decsends to a linear map, denoted by $F^* : H^p N \to H^p M$, too.

Digression in digression in digression: **Pullback** of F^* is

$$(F^*\omega)_p(v_1, ..., v_n) = \omega_{F(p)}(dF_p(v_1), ..., dF_p(v_k)).$$

Back to the main thread of thought: If we have two smooth maps $F, G : M \to N$ and we want to prove that the induced maps are equal $F^* = G^*$. Given a closed $p-$form ω on N, we need to produce a $(p-1)$-form η no M such that

$$G^*\omega - F^*\omega = d\eta$$

from this, it will follow that $G^*[\omega] - F^*[\omega] = [d\eta] = 0$, where [] is just taking homotopy equivalence class of given form. The author suggests a way to make it more systematic, by finding an operator h, which transforms closed p-forms on N to $(p-1)$-forms on M and satisfies

$$d(h\omega) = G^*\omega - F^*\omega.$$

Instead of defining $h\omega$ only when ω is close, it turns out to be far easier to define a map h from the space of *all* smooth p-forms on N to the space of smoooth $(p-1)$-forms on M, which satisfies:

$$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega,$$

which implies the above equality when ω is closed. (To be completly precise, we define a family of maps, one for each p, which satisfy said equalities on adequate levels.

$$H(\mathcal{M} \times \mathbb{R}_{\geqslant})^*_{dR} = H(\mathcal{M})^*_{dR}$$

Bibliografia

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[Leuk00] Lechoslav Leukocyt, *Oval mappings ab ovo*, Materiały Białostockiej Konferencji Hodowców Drobiu, 2000.