

Uniwersytet Warszawski
Wydział Matematyki, Informatyki i Mechaniki

Michał Garmulewicz

Nr albumu: 304742

**L_p -cohomologies of Riemannian
 f -horns.**

**Praca licencjacka
na kierunku MATEMATYKA**

Praca wykonana pod kierunkiem
dra hab. Andrzeja Webera
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Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data

Podpis kierującego pracą

Oświadczenie autora (autorów) pracy

Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

Data

Podpis autora (autorów) pracy

Abstract

In this thesis L_p -cohomologies of Riemannian f -horn are calculated.

Słowa kluczowe

kohomologie de Rhama, topologia różniczkowa

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.0 Matematyka, Informatyka:

11.1 Matematyka

Klasyfikacja tematyczna

14 Algebraic Geometry

14F (Co)homology theory

14F40 de Rham cohomology

Tytuł pracy w języku angielskim

An implementation of a difference blabalizer based on the theory of $\sigma - \rho$ phetors

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Chapter 1

Introduction

In [Weber] the author considers a cone over Riemannian pseudomanifold. The cone is given a metric of the form $dt \otimes dt \oplus t^2 g$ and g is the metric on the nonsingular part of the original pseudomanifold. In the mentioned work, L_p cohomology of this space is presented.

This line of research can be traced back to Cheeger [Cheeger]. A similar approach was presented by Youssin [Youssin], where f -horns were considered.

We present a slight modification of this notions, by considering manifolds where the Riemannian metric is of the for

Chapter 2

Preliminaries

2.1. Vector spaces and tensors

Let us recall some basic facts about behaviour of norm when scaling tensors. If we consider a finite-dimensional vector space V with a given metric $\|\cdot\|$ and define a new metric $\|x\|_r = r\|x\|$. Then in the space $(V, \|\cdot\|_r)^*$ dual to $(V, \|\cdot\|)$, the normed is scaled by the factor $\frac{1}{r}$, that is for any $\varphi \in V^*$ we get $\|\varphi\|_r = \frac{1}{r}\|\varphi\|$.

We will make a simple observation that we will later use in the computations. We consider a Riemannian manifold M and tangent $T_x M$ and cotangent $T_x^* M$ spaces in the point x . Let the bases of these spaces be e_1, e_2, \dots, e_n and dual $e_1^*, e_2^*, \dots, e_n^*$. The volume form of this manifold is $d\text{vol} = \pm e_1^* \wedge e_2^* \wedge \dots \wedge e_n^*$.

We now want to compute how forms from $\Lambda(\mathcal{M})$ are scaled with respect to such a change in the norm. Suppose we are considering space of k -forms on M at some arbitrary point. Then every k -form can be locally expressed in a basis consisting of products of covectors belonging to basis dual to the standard basis. That is every k -form in the point x using some local coordinates (x_1, x_2, \dots, x_n) can be written $\sum_{I \in I} a_I dx_i$, where I is the set of k -indices of form $\underbrace{(i_1, i_2, \dots, i_k)}_{k \text{ times}}$, with $i_1, i_2, \dots \in \{1, 2, \dots, n\}$. (following Einstein convention). Let us see how basis vector is scaled:

$$\begin{aligned} \|dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}\|_t &= \|dx_{i_1}\|_t \cdot \|dx_{i_2}\|_t \cdot \dots \cdot \|dx_{i_k}\|_t = \\ &= \frac{1}{t} \|dx_{i_1}\| \cdot \frac{1}{t} \|dx_{i_2}\| \cdot \dots \cdot \frac{1}{t} \|dx_{i_k}\| = \frac{1}{t^k} \|dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}\| \end{aligned}$$

This means that any k form is scaled by $1/t^k$ when metric is scaled by a factor of t . This applies also to the volume form, so we obtain:

$$d\text{vol}_t = \frac{1}{t^n} d\text{vol}$$

2.2. Differential forms

Riemannian metric is a smooth symmetric covariant 2-tensor field on manifold \mathcal{M} that is positive definite at each point. (attaching a field of linear functions that takes two variables to every point of the manifold).

Consulting page 328 of Lee gives us that in any smooth local coordinates (x^i) , Riemannian metric can be written as:

$$g = g_{ij}dx^i \otimes dx^j = g_{ij}dx^i dx^j$$

where g_{ij} is a positive definite matrix of smooth functions.

The simplest example of Riemannian metric is *Euclidean metric* on \mathbb{R}^n given in standard coordinates by

$$g = \delta_{ij}dx^i dx^j.$$

Citing prof. Lee, it is common to abbreviate the symmetric product of a tensor α with itself by α^2 , so the Euclidean metric can also be written as

$$g = (dx^1)^2 + \dots + (dx^n)^2,$$

so now it is way easier to understand what exactly is meant by $dt \otimes dt + f^2g$, which should be the same as $dt^2 + f^2g$.

Induced map For any smooth map $F : M \rightarrow N$ between two smooth manifolds with or without boundary, the pullback $F^* : \Omega^p N \rightarrow \Omega^p M$ carries closed forms to closed forms and exact forms to exact forms. It thus descends to a linear map, denoted by $F^* : H^p N \rightarrow H^p M$, too.

Digression in digression in digression: **Pullback** of F^* is

$$(F^*\omega)_p(v_1, \dots, v_n) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)).$$

2.3. Explanation about induced maps etc.

If we have two smooth maps $F, G : M \rightarrow N$ and we want to prove that the induced maps are equal $F^* = G^*$. Given a closed p -form ω on N , we need to produce a $(p-1)$ -form η on M such that

$$G^*\omega - F^*\omega = d\eta$$

from this, it will follow that $G^*[\omega] - F^*[\omega] = [d\eta] = 0$, where $[\]$ is just taking homotopy equivalence class of given form. The author suggests a way to make it more systematic, by finding an operator h , which transforms closed p -forms on N to $(p-1)$ -forms on M and satisfies

$$d(h\omega) = G^*\omega - F^*\omega.$$

Instead of defining $h\omega$ only when ω is close, it turns out to be far easier to define a map h from the space of *all* smooth p -forms on N to the space of smooth $(p-1)$ -forms on M , which satisfies:

$$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega,$$

which implies the above equality when ω is closed. (To be completely precise, we define a family of maps, one for each p , which satisfy said equalities on adequate levels.

$$H(\mathcal{M} \times \mathbb{R}_{\geq})_{dR}^* = H(\mathcal{M})_{dR}^*$$

Chapter 3

Computation

The purpose of this chapter is to present the computation of L_p -cohomologies of Riemannian f -horns.

3.1. Setting

Here we introduce definitions and make the first observations. The setting is largely similar to the setting presented in [Weber], [Youssin], [Cheeger].

Definition 3.1.1 (f -horn). Let \mathcal{M} be a Riemannian manifold. Consider a space $\mathbb{R}_{\geq 0} \times \mathcal{M}$. Define a Riemannian tensor on this product by $dt^2 \oplus f^2(t)g$, where g is the metric on \mathcal{M} . Such a space will be called an **f -horn**. We will denote it by $c^f \mathcal{M}$.

Definition 3.1.2. By $L_p^k \mathcal{M}$ we will denote the space of p -integrable k -forms with measurable coefficients.

At first, we will focus our attention on scaling functions from family $f_\alpha(x) = e^{\alpha x}$, parametrized by $\alpha \in \mathbb{R}$. The intuition behind such manifolds is best presented graphically, as in the Figure ??.

We can make a simple observation here about differential forms associated with f -horn. The tangent space in the point (t, m) is:

$$T_{(t,m)}(c^f \mathcal{M}) = \mathbb{R} \times T_m \mathcal{M}$$

In terms of differential forms associated with f -horn, it means that $\Lambda^k(\mathbb{R} \times T_m \mathcal{M}) = \Lambda^k(\mathbb{R}) \oplus \Lambda^k(\mathcal{M})$. It can be rephrased in friendlier terms in the following way.

Remark 3.1.1. Every k -form $\omega \in \Lambda^k T(c^f \mathcal{M})$, and consequently every form in the space of p -integrable forms $L_p^k(c^f \mathcal{M})$ can be written as $\omega = \eta + \xi \wedge dt$, where both η and ξ do not contain dt . Please note that η is k -form and ξ is $k - 1$ form.

3.2. Norms

Here we define norms of different forms used later in the computation.

For $\omega \in L_p^k \mathcal{M}$ we have the norm:

$$||\omega|| = (\int_{\mathcal{M}} |\omega|^p)^{1/p}$$

We have the standard inclusion:

$$\begin{aligned} i_r : \mathcal{M} &\rightarrow c^f \mathcal{M}, \\ i_r(x) &= (x, r). \end{aligned}$$

With this in mind, for $\omega = \eta + \xi \wedge dt$ we will write $\omega_r = i_r^*(\omega) = i_r^*(\eta)$.

Definition 3.2.1. We define norm "at the level r " to be:

$$||\omega||_r = ||\omega|_{\mathcal{M} \times \{r\}}|| = f(r)^{n/p-k} ||\omega_r||$$

Similarly, when we consider the standard projection operator:

$$\pi : c^f \mathcal{M} \rightarrow \mathcal{M}$$

for $\eta \in L_p^*(c^f \mathcal{M})$ we can write:

$$||\eta||_r := ||\pi^* \eta||_r = f(r)^{n/p-k} ||\eta_r||.$$

As we are computing cohomologies, we should define the homology operator. Let

$$I_r : \Omega^*(c^f \mathcal{M}) \rightarrow \Omega^{*-1}(c^f \mathcal{I})$$

$$I_r(\omega)(x, t) = \int_r^t \xi(x, s) ds$$

??

The form $I_r \omega$ is smooth for $r \in (0, 1)$, but will also consider $r = 0$ in certain cases. If $r > 0$ then the homotopy formula holds:

$$\omega - \pi^*(\omega_r) = dI_r \omega + I_r d\omega.$$

This one probably is Poincare lemma in a version suited for our situation. This is a good place to rewrite it here using Bott

??

Lemma 3.2.1. Let $k < (n+1)/p$ Then the form π^* is p -integrable for each p -integrable form $\eta \in L_p^k(\mathcal{M})$.

Proof.

$$||\pi^* \eta||^p = \int_{c\mathcal{M}} |\pi^* \eta(x, t)|^p d\text{vol}(c\mathcal{M}) = \int_0^\infty ||\eta||_t^p dt = ||\eta||^p \int_0^\infty f(t)^{n-pk} dt$$

so, if f is given by $f(x) = e^\alpha$, the last integral is $||\eta||^p \int_0^\infty e^{\alpha(n-pk)} dt$. This integral is convergent if $\alpha(n-pk) < 0$. \square

The following lemma will be useful in further computation.

Lemma 3.2.2 (Estimation of $\int \xi$). *blabla bla*

Proof.

$$\begin{aligned} \left\| \int_t^r \xi_x dx \right\| &\leq \int_t^r \|\xi_x\| dx = \\ &= \int_r^t f(x)^{i-n/p} f(x)^{n/p-i} \|\xi_x\| dx \leq \\ &= \left| \int_r^t f(x)^{i-n/p} dx \right|^{1/q} \cdot \left| \int_r^t f(x)^{n/p-i} \|\xi_s\| ds \right|^{1/p} \end{aligned}$$

□

Lemma 3.2.3. *Let $k < (n+1)/p$. Then for p -integrable $\omega \in \Omega^k(\mathcal{C}^f \mathcal{M})$ the form $I_r \omega$ is integrable.*

Proof. For the case $k < (n+1)/p = n/p - 1/q + 1$ we get from the previous lemma:

$$\|I_r \omega\| = \int_0^\infty \left\| \int_r^t \xi_s ds \right\| dt \leq C \int_0^\infty f(t)^{p/q} dt,$$

and for $k = (n+1)/p$, we get:

$$\|I_r \omega\| \leq \int_0^1 |t(\log r - \log t)|^{p/q} dt$$

□

Remark 3.2.1. Based on the above lemma, we get that if $k < (n+1)/p$ then a p -integrable closed form is cohomologous to $\pi^* \omega_r$ for almost all r .

$$\omega = dI_0 \omega + I_0 d\omega$$

Proof. We have $\omega - \pi^*(\omega_r) = dI_r \omega$ and $I_r \omega$ is p -integrable. The form ω_r is p -integrable for almost all r . So for almost all r the form $dI_r \omega$ is also p -integrable. □

Lemma 3.2.4. *Let $k > (n+1)/p$. Then $I_0 \omega$ is p -integrable for $\omega \in L_p \Omega^k(\mathcal{C}^f \mathcal{M})$ and the homotopy formula holds:*

$$\omega = dI_0 \omega + I_0 d\omega$$

Proof. Again, we estimate

$$\|I_0 \omega\| = \int_0^\infty \left\| \int_0^t \xi_s ds \right\|_t^p dt \leq \dots,$$

so this norm is finite. The form $I_0 \omega$ may not be smooth. We

□

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