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L_p-cohomologies of Riemannian f-horns.

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Oświadczenie kierującego pracą

Potwierdzam, że niniejsza praca została przygotowana pod moim kierunkiem i kwalifikuje się do przedstawienia jej w postępowaniu o nadanie tytułu zawodowego.

Data | Podpis kierującego pracą

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Świadom odpowiedzialności prawnej oświadczam, że niniejsza praca dyplomowa została napisana przeze mnie samodzielnie i nie zawiera treści uzyskanych w sposób niezgodny z obowiązującymi przepisami.

Oświadczam również, że przedstawiona praca nie była wcześniej przedmiotem procedur związanych z uzyskaniem tytułu zawodowego w wyższej uczelni.

Oświadczam ponadto, że niniejsza wersja pracy jest identyczna z załączoną wersją elektroniczną.

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Abstract

In this thesis L_p-cohomologies of Riemannian f-horn are calculated.

Słowa kluczowe

kohomologie de Rhama, topologia różniczkowa

Dziedzina pracy (kody wg programu Socrates-Erasmus)

11.0 Matematyka, Informatyka:
11.1 Matematyka

Klasyfikacja tematyczna

14 Algebraic Geometry
14F (Co)homology theory
14F40 de Rham cohomology

Tytuł pracy w języku angielskim

An implementation of a difference blabalizer based on the theory of $\sigma - \rho$ phetors

Contents

Chapter 1

Introduction

In [Weber] the author considers a cone over Riemannian pseudomanifold. The cone is given a metric of the form $dt \otimes dt \oplus t^2 g$ and g is the metric on the nonsingular part of the original pseudomanifold. In the mentioned work, L_p cohomology of this space is presented.

This line of research can be traced back to Cheeger [Cheeger]. A similar approach was presented by Youssin [Youssin], where f-horns were considered.

We present a slight modification of this notions, by considering manifolds where the Riemannian metric is of the for

Chapter 2

Preliminaries

2.1. Vector spaces and tensors

Let us recall some basic facts about behaviour of norm when scaling tensors. If we consider a finite-dimensional vector space V with a given metric $||\cdot||$ and define a new metric $||x||_r = r||x||$. Then in the space $(V, ||\cdot||_r)^*$ dual to $(V, |||\cdot|||)$, the normed is scaled by the factor $\frac{1}{r}$, that is for any $\varphi \in V^*$ we get $||\varphi||_r = \frac{1}{r}||\varphi||$.

We will make a simple observation that we will later use in the computations. We consider a Riemannian manifold M and tagent T_xM and cotangent T_x^*M spaces in the point x. Let the bases of these spaces be $e_1, e_2, ..., e_n$ and dual $e_1^*, e_2^*, ..., e_n^*$. The volume form of this manifold is $d\text{vol} = \pm e_1^* \wedge e_2^* \wedge ... \wedge e_n^*$.

We now want to compute how forms from $\Lambda(\mathcal{M})$ are scaled with respect to such a change in the norm. Suppose we are considering space of k-forms on $\mathbb{M}$ at some arbitrary point. Then every k-form can be locally expressed in a basis consisting of products of covectors belonging to basis dual to the standard basis. That is every k-form in the point x using some local coordinates $(x_1, x_2, ...x_n)$ can be written $\sum_{\mathrm{I} \in I} a_\mathrm{I} dx_\mathrm{i}$, where I is the set of k-indices of form $\underbrace{(i_1, i_2, ..., i_k)}_{k \text{ times}}$, with $i_1, i_2, ... \in \{1, 2, ..., n\}$. (following Einstein convention). Let us see how basis vector is scaled:

$$||dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}||_t = ||dx_{i_1}||_t \cdot ||dx_{i_2}||_t \cdot ... \cdot ||dx_{i_k}||_t = \\ \frac{1}{t}||dx_{i_1}|| \cdot \frac{1}{t}||dx_{i_2}||_t \cdot ... \cdot \frac{1}{t}||dx_{i_k}||_t = \frac{1}{t^k}||dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}||$$

This means that any k form is scaled by $1/t^k$ when metric is scaled by a factor of t. This applies also to the volume form, so we obtain:

$$d\text{vol}_t = \frac{1}{t^n} d\text{vol}$$

2.2. Differetial forms

Riemannian metric is a smooth symmetric covariant 2-tensor field on manifold $\mathcal{M}$ that is positive definite at each point.(attaching a field of linear functions that takes two variables to every point of the manifold).

Consulting page 328 of Lee gives us that in any smooth local coordinates (x^i), Riemannian metric can be written as:

$$g = g_{ij} dx^i \otimes dx^j = g_{ij} dx^i dx^j$$

where g_{ij} is a positive definite matrix of smooth functions.

The simplest example of Riemannian metric is *Euclidean metric* on $\mathbb{R}^n$ given in standard coordinates by

$$g = \delta_{ij} dx^i dx^j.$$

Citing prof. Lee, it is common to abbreviate the symmetric product of a tensor α with itself by α^2, so the Euclidean metric can also be written as

$$g = (dx^1)^2 + ... + (dx^n)^2,$$

so now it is way easier to understand what exactly is meant by $dt \otimes dt + f^2 g$, which shoudl be the same as $dt^2 + f^2 g$.

Induced map For any smooth map $F : M \to N$ between two smooth manifolds with or without boundary, the pullback $F^* : \Omega^p N \to \Omega^p M$ carries closed forms to closed forms and exact forms to exact forms. It thus decsends to a linear map, denoted by $F^* : H^p N \to H^p M$, too.

Digression in digression in digression: **Pullback** of F^* is

$$(F^*\omega)_p(v_1, ..., v_n) = \omega_{F(p)}(dF_p(v_1), ..., dF_p(v_k)).$$

2.3. Explaination about induced maps etc.

If we have two smooth maps $F, G : M \to N$ and we want to prove that the induced maps are equal $F^* = G^*$. Given a closed $p-$form ω on N, we need to produce a $(p-1)$-form η no M such that

$$G^*\omega - F^*\omega = d\eta$$

from this, it will follow that $G^*[\omega] - F^*[\omega] = [d\eta] = 0$, where [] is just taking homotopy equivalence class of given form. The author suggests a way to make it more systematic, by finding an operator h, which transforms closed p-forms on N to $(p-1)$-forms on M and satisfies

$$d(h\omega) = G^*\omega - F^*\omega.$$

Instead of defining $h\omega$ only when ω is close, it turns out to be far easier to define a map h from the space of *all* smooth p-forms on N to the space of smoooth $(p-1)$-forms on M, which satisfies:

$$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega,$$

which implies the above equality when ω is closed. (To be completly precise, we define a family of maps, one for each p, which satisfy said equalities on adequate levels.

$$H(\mathcal{M} \times \mathbb{R}_{\geq})^*_{dR} = H(\mathcal{M})^*_{dR}$$

Chapter 3

Computation

The purpose of this chapter is to present the computation of L_p-cohomologies of Riemannian f-horns.

3.1. Setting

Here we introduce definitions and make the first observations. The setting is largely similar to the setting presented in [Weber], [Youssin], [Cheeger].

Definition 3.1.1 (f-horn)**.** Let $\mathcal{M}$ be a Riemannian manifold. Consider a space $\mathbb{R}_{\geq 0} \times \mathcal{M}$. Define a Riemannian tensor on this product by $dt^2 \oplus f^2(t)g$, where g is the metric on $\mathcal{M}$. Such a space will be called an **f-horn**. We will denote it by $\mathrm{c}^f\mathcal{M}$.

Remark 3.1.1. This terminology is present in works of Cheeger.

At first, we will focus our attention on scaling functions from family $f_\alpha(x) = \mathrm{e}^{\alpha x}$, parametrized by $\alpha \in \mathbb{R}$. The intuition behind such manifolds is best presented graphically, as in the Figure **??**.

We can make a simple observation here about differential forms associated with f-horn. The tangent space in the point (t, m) is:

$$T_{(t,m)}(\mathrm{c}^f\mathcal{M}) = \mathbb{R} \times T_m\mathcal{M}$$

In terms of differential forms associated with f-horn, it means that $\Lambda^k(\mathbb{R} \times T_m\mathcal{M}) = \Lambda^k(\mathbb{R}) \oplus \Lambda^k(\mathcal{M})$. It can be rephrased in friendlier terms in the following way.

Remark 3.1.2. Every k-form $\omega \in \Lambda^k T(\mathrm{c}^f\mathcal{M})$, and consequently every form in the space of p integrable forms $L^p_k(\mathrm{c}^f\mathcal{M}$ can be written as $\omega = \eta + \xi \wedge dt$, where both η and ξ do not contain dt. Please note that η is k-form and ξ is $k-1$ form.

Remark 3.1.3 (Norms of forms)**.** We have the standard inclusion:

$$i_r : \mathcal{M} \to \mathrm{c}^f\mathcal{M},$$

$$i_r(x) = (x, r).$$

With this in mind, we will write $\omega_r = i_r^*(\omega) = i_r^*(\eta)$. Further, let us denote $||\omega|_{\mathcal{M}\times\{r\}}|| = f(r)^{n/p-k}||\omega_r||$ as $||\omega||_r$. Moreover, if

$$\pi : \mathrm{c}^f\mathcal{M} \to \mathcal{M}$$

is the projection, we establish that for $\eta \in L_p^*(\mathrm{c}^f\mathcal{M})$ we can write $||\eta||_r := ||\pi^*\eta||_r = f(r)^{n/p-k}||\eta_r||$.

As we are computing cohomologies, we should define the homology operator. Let

$$I_r : \Omega^*(\mathrm{c}^f\mathcal{M}) \to \Omega^{*-1}(\mathrm{c}^f\mathcal{I})$$

$$I_r(\omega)(x,t) = \int_r^t \xi(x,s)ds$$

???

The form $I_r\omega$ is smooth for $r \in (0,1)$, but will also consider $r = 0$ in certain cases. If $r > 0$ then the homotopy formula holds:

$$\omega - \pi^*(\omega_r) = dI_r\omega + I_r d\omega.$$

This one probably is Poincare lemma in a version suited for our situation. This is a good place to rewrite it here using Bott
??

Lemma 3.1.1. *Let $k < (n+1)/p$ Then the form π^* is p-integrable for each p-integrable form $\eta \in L_p^k(\mathcal{M})$.*

Proof.

$$||\pi^*\eta||^p = \int_{c\mathcal{M}} |\pi^*\eta(x,t)|^p d\mathrm{vol}(c\mathcal{M}) = \int_0^\infty ||\eta||_t^p dt = ||\eta||^p \int_0^\infty f(t)^{n-pk} dt$$

so, if f is given by $f(x) = e^\alpha$, the last integral is $||\eta||^p \int_0^\infty e^{\alpha(n-pk)}dt$. This integral is convergent if $\alpha(n-pk) < 0$. ☐

The following lemma will be useful in further computation.

Lemma 3.1.2 (Estimation of $\int \xi$)**.** *blabla bla*

Proof.

$$||\int_t^r \xi_x dx|| \leq \int_t^r ||\xi_x||dx =$$

$$\int_r^t f(x)^{i-n/p} f(x)^{n/p-i}||\xi_x||dx \leq$$

$$|\int_r^t f(x)^{i-n/p}dx|^{1/q} \cdot |\int_r^t f(x)^{n/p-i}||\xi_s||ds|^{1/p}$$

☐

Lemma 3.1.3. *Let $k < (n+1)/p$. Then for p-integrable $\omega \in \Omega^k(\mathrm{c}^f\mathcal{M})$ the form $I_r\omega$ is integrable.*

Proof. For the case $k < (n+1)/p = n/p - 1/q + 1$ we get from the previous lemma:

$$||I_r\omega|| = \int_0^\infty || \int_r^t \xi_s ds||dt \leq C\int_0^\infty f(t)^{p/q},$$

and for $k = (n+1)/p$, we get:

$$||I_r\omega|| \leq \int_0^1 |t(\log r - \log t)|^{p/q} dt$$

□

Remark 3.1.4. Based on the above lemma, we get that if $k < (n+1)/p$ then a p-integrable closed form is cohomologous to $\pi^*\omega_r$ for almost all r.

$$\omega = dI_0\omega + I_0 d\omega$$

Proof. We have $\omega - \pi^*(\omega_r) = dI_r\omega$ and $I_r\omega$ is p-integrable. The form ω_r is p-integrable for almost all r. So for almost all r the form $dI_r\omega$ is also p-integrable. □

Lemma 3.1.4. *Let $k > (n+1)/p$ Then $I_0\omega$ is p-integrable for $\omega \in L_p\Omega^k(\mathrm{c}^f\mathcal{M})$ and the homotopy formula holds:*

$$\omega = dI_0\omega + I_0 d\omega$$

Proof. Again, we estimate

$$||I_0\omega|| = \int_0^\infty || \int_0^t \xi_s ds||_t^p dt \leq ...,$$

so this norm is finite. The form $I_0\omega$ may not be smooth. We □

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