CSE512 Fall 2018 - Machine Learning - Homework 6

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ANSWER - 1

(1) To show that the co-variance of the deflated matrix,

$$\tilde{C} = \frac{1}{n}\tilde{X}\tilde{X}^T \quad \text{is given by,} \quad \text{— (i)}$$

$$\tilde{C} = \frac{1}{n}XX^T - \lambda_1 v_1 v_1^T$$

Given → $I - v_1 v_1^T$ is symmetric — (a)

$\tilde{X} = (I - v_1 v_1^T)X$ — (b)

$XX^T v_1 = n\lambda_1 v_1 \longrightarrow$ (c)

$v_1^T v_1 = 1 \longrightarrow$ (d)

Substituting (b) in (i) we get,

$$\tilde{C} = \frac{1}{n}\left((I - v_1 v_1^T)X\right)\left((I - v_1 v_1^T)X\right)^T,$$

using (a) we get.

$$= \frac{1}{n}(I - v_1 v_1^T)XX^T(I - v_1 v_1^T)$$

$$= \frac{1}{n}\left[IXX^T(I - v_1 v_1^T) - v_1 v_1^T XX^T(I - v_1 v_1^T)\right]$$

$$= \frac{1}{n}\left[XX^T - XX^T v_1 v_1^T - v_1 v_1^T XX^T + v_1 v_1^T XX^T v_1 v_1^T\right]$$

using (c) we get.

$$= \frac{1}{n}\left[XX^T - n\lambda_1 v_1 v_1^T - XX^T v_1 v_1^T + v_1 v_1^T n\lambda_1 v_1 v_1^T\right]$$

$$= \frac{1}{n}\left[XX^T - n\lambda_1 v_1 v_1^T - n\lambda_1 v_1 v_1^T + v_1 v_1^T n\lambda_1 v_1 v_1^T\right]$$

using (d) we get.

$$= \frac{1}{n}\left[XX^T - n\lambda_1 v_1 v_1^T - n\lambda_1 \cancel{v_1} v_1^T + n\lambda_1 \cancel{v_1} v_1^T\right]$$

$$= \frac{1}{n}\left[XX^T - n\lambda_1 v_1 v_1^T\right] = \frac{1}{n}XX^T - \frac{1}{\cancel{n}}(\cancel{n}\lambda_1 v_1 v_1^T)$$

$$\boxed{\tilde{C} = \frac{1}{n}XX^T - \lambda_1 v_1 v_1^T}$$ Hence Proved.

② To show that v_j is also a principal eigenvector of $\tilde{C}$. — (I) ②

Given → $Cv_j = \lambda_j v_j$ — (9)

To prove (I) we can write it has,

$$\tilde{C}v_j = \lambda_j v_j \rightarrow (i)$$

From part ① we have → $\tilde{C} = \frac{1}{n} XX^T - \lambda_1 v_1 v_1^T$ — (ii)

To reach (i) we get the same by multiplying with v_j on both sides.

$$\tilde{C}v_j = \frac{1}{n} XX^T v_j - \lambda_1 v_1 v_1^T v_j$$

If we have $v_1^T v_j$ here we know $v_1^T v_j = 0$ as $i = 1$ & $i \neq j$

$\therefore \tilde{C}v_j = \frac{1}{n} XX^T v_j - \lambda_1 v_1 (0) \Rightarrow \tilde{C}v_j = \frac{1}{n} XX^T v_j - 0$

In question we have $v_i^T v_j = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases}$

$\rightarrow \tilde{C}v_j = \frac{1}{n} X X^T v_j$ — ⓐ

Also we know that, $C = \frac{1}{n} XX^T$ from the question.

Substituting in ⓐ we get:

$\tilde{C}v_j = \frac{1}{n} XX^T v_j \Rightarrow Cv_j$; Also we have from (9)

$\tilde{C}v_j = \lambda_j v_j$

$\longrightarrow$ Hence v_j is an eigenvector of $\tilde{C}$ as well with the same eigen value.

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3 From our last example we have:

$$\tilde{C} v_j = x x^T v_j - \underbrace{\lambda_1 v_1 v_1^T v_j}_{} \quad \text{——— (i)}$$

→ this part becomes zero only when $j \neq 1$

Hence, lets assume that $j = 1$, then eqn (i) becomes

$$\tilde{C} v_1 = x x^T v_1 - \lambda_1 v_1 \underbrace{v_1^T v_1}_{}$$

→ we know $v_1^T v_1 = 1$

$$\tilde{C} v_1 = x x^T v_1 - \lambda_1 v_1$$

→ also from part 1 we can write this as.

$$\tilde{C} v_1 = C v_1 - \lambda_1 v_1 \Rightarrow \lambda_1 v_1 - \lambda_1 v_1 = 0$$

$\therefore \tilde{C} v_1 = 0$ → this is not an eigen vector of v_1.

Hence, we can write that first eigenvector is v_2 i.e. at $j = 2$.

Also, this will act as the first principal eigenvector of $\tilde{C}$ which has the property of being the greatest in value.

$\therefore$ we can write $\underline{\underline{u = v_2}}$ from the above statement.

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④ given $f \to$ function to find the leading eigen vector.

$$[\lambda, u] = f(C) \quad \text{—— ⓐ}$$

→ To find first k principal basis vector of X that uses only the special function f and simple vector arithmetic

- Also given that the input is C, k, f & output is v_j & λ_j. for $j \in \{1, \dots, k\}$

From previous examples we saw

$$\tilde{C} = \frac{1}{n} \tilde{X} \tilde{X}^T \quad \&$$

$$\tilde{C} = \frac{1}{n} X X^T - \lambda_1 v_1 v_1^T \quad \text{— using this \& above eq}^n \text{ we get}$$

$$\tilde{C} = C - \lambda_1 v_1 v_1^T \longrightarrow (i).$$

★ **ALGORITHM**:

```
function EigenVector (C, k, f):
    v_list = [ ]
    λ_list = [ ]
    for j = 1 to k do:
        [λ, v] = f(C)      —— using (a)
        v_list append (v)
        λ_list.append (λ)
        C = C - λ.v        —— using (i)
    end for
    return v_list, λ_list
end function
```