Chapter 4
Image Enhancement in the Frequency Domain

第四章：频率域图像增强

Notice

- **作业和project见课程网站**

致谢: 本课件部分页面和图片来自台湾彭明辉教授的教学课件，特此表示感谢。

第4章 频率域图像增强

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目的和内容

- 建立对傅里叶变换和频率域的基本理解
- 学习频率域图像增强技术
- 了解空间域和频率域图像增强工具之间的关系
- 更灵活的构造图像增强算法

变换是信号处理和图像处理的重要工具.

Preview

4.1 Background

4.2 Fourier Transformation and the Frequency Domain

4.3 Smoothing Frequency-Domain Filters

4.4 Sharpening Frequency-Domain Filters

4.5 Homomorphic Filtering (同态滤波)

4.6 Implementation

（第三版：4.4、4.5、4.6、4.7、4.8、4.9、4.10）

Jean Baptiste Fourier (1768-1830)

Had crazy idea (1807):

> ***Any** periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

Don't believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!

But it's true!

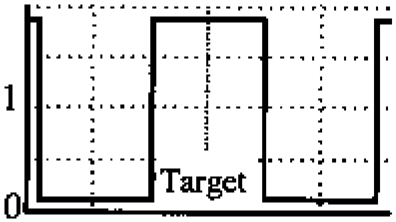
- called Fourier Series

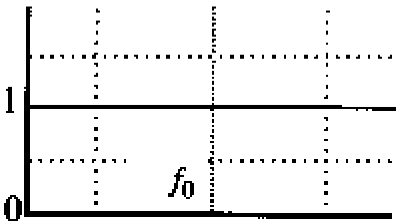
A Sum of Sine Waves

Our building block:

$$A\sin(\omega x + \phi)$$

Add enough of them to get any signal *f(x)* you want!





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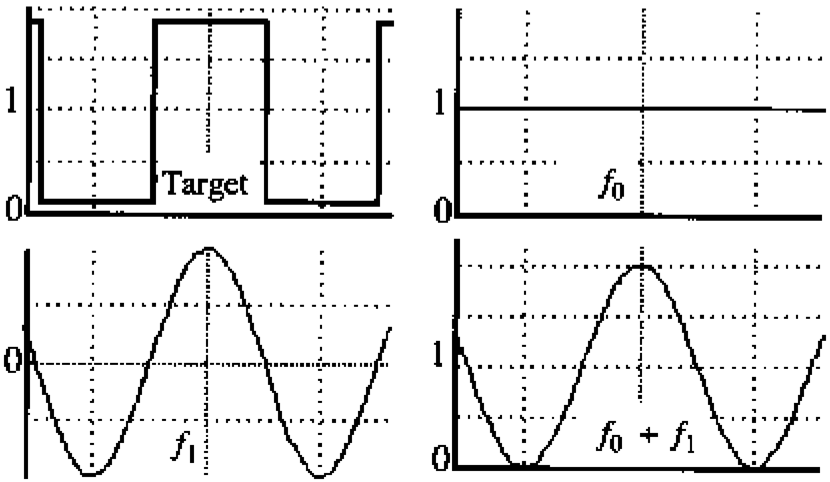

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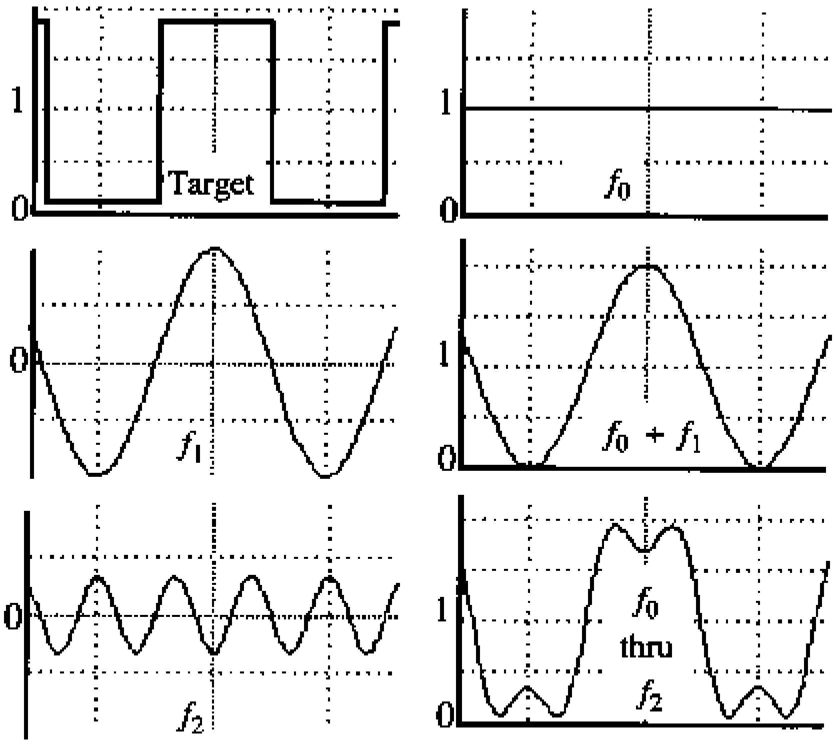

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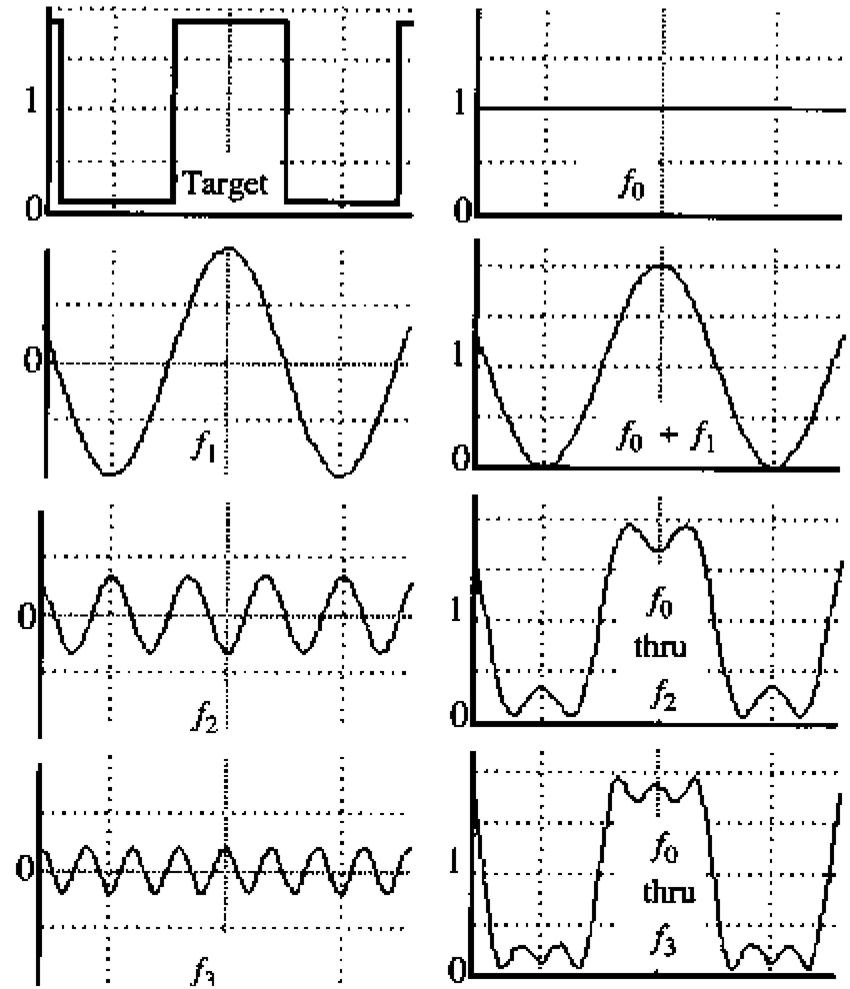

A Sum of Sine Waves

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Our building block:

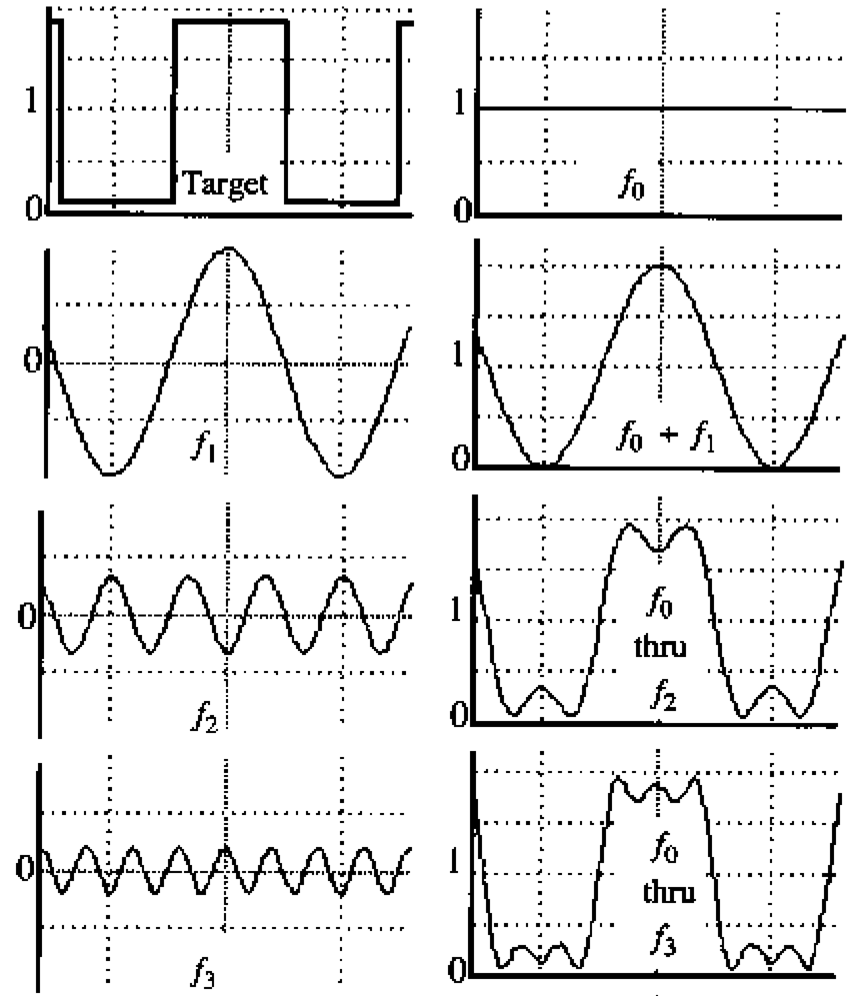
$$A\sin(\omega x + \phi)$$

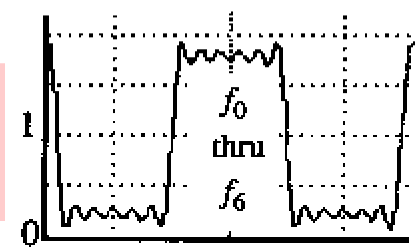
Add enough of them to get any signal *f(x)* you want!

How many degrees of freedom?

What does each control?

Which one encodes the coarse vs. fine structure of the signal?



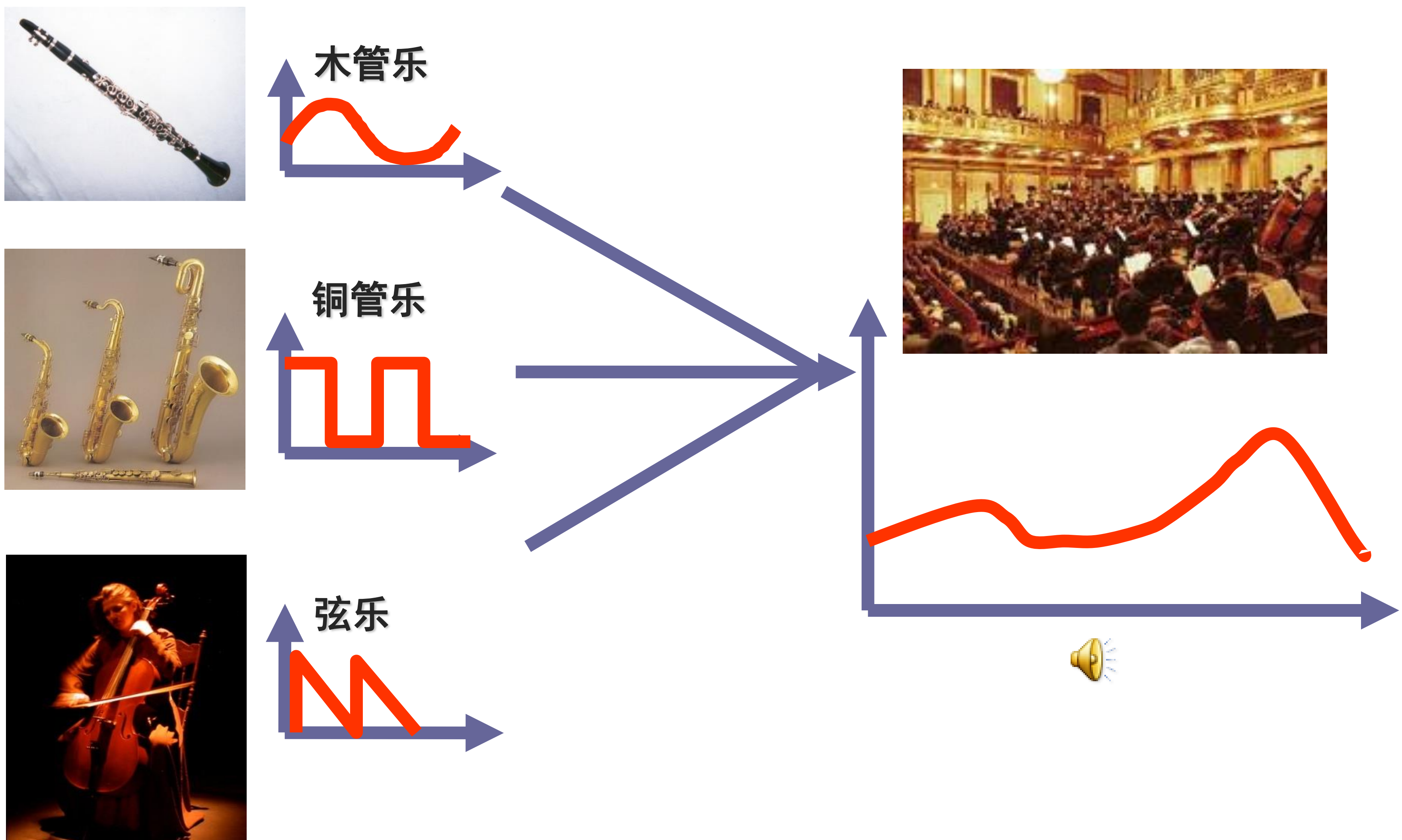


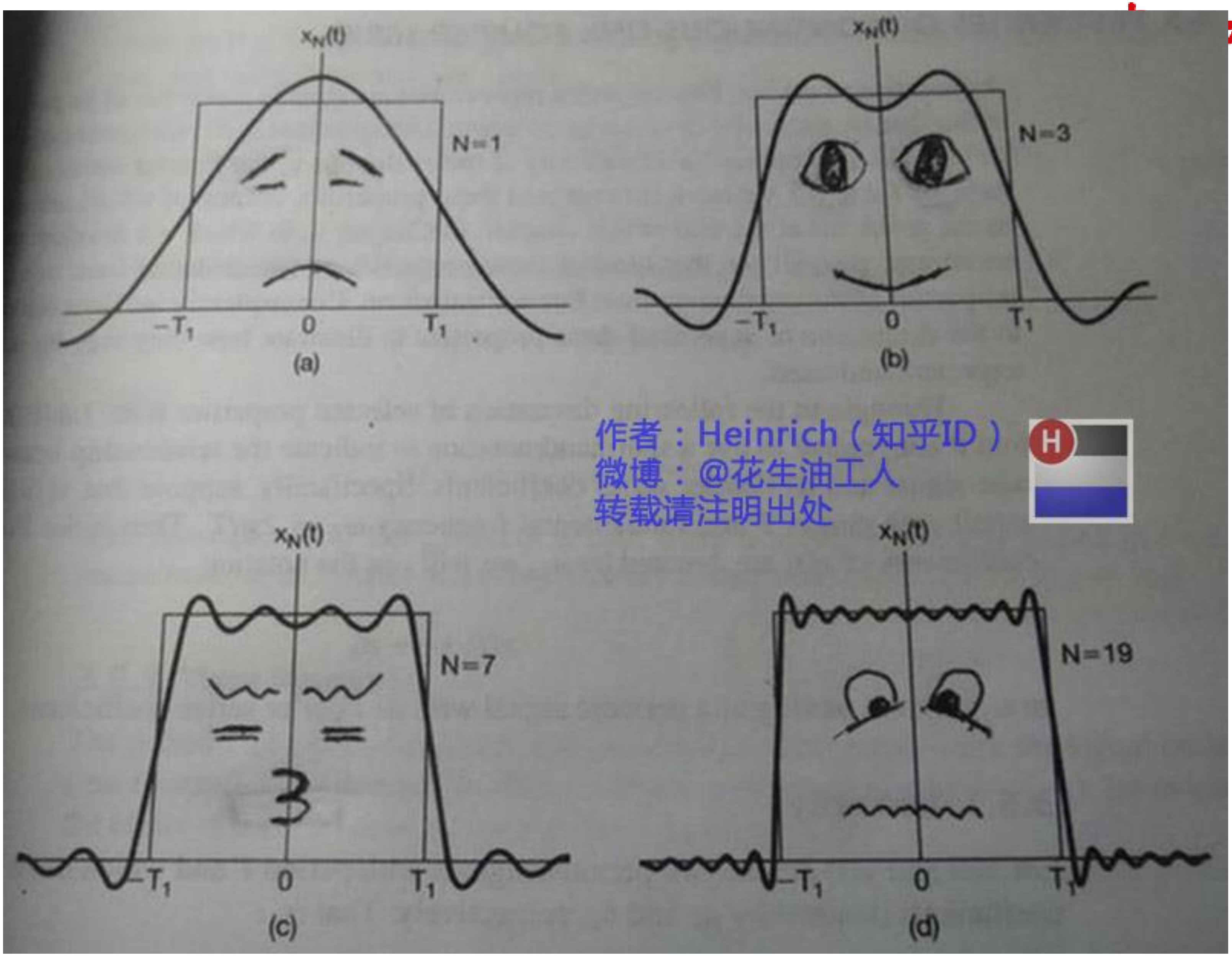
[An Interactive Guide To The Fourier Transform]

f(target)= $f_1 + f_2 + f_3 ... + f_n + ...$

空域图像

Listen……




$x_N(t)$
N=1
$-T_1$
0
T_1
(a)
$x_N(t)$
N=3
$-T_1$
0
T_1
(b)
$x_N(t)$
N=7
3
$-T_1$
0
T_1
(c)
$x_N(t)$
N=19
$-T_1$
0
T_1
(d)
作者：Heinrich（知乎ID）
微博：@花生油工人
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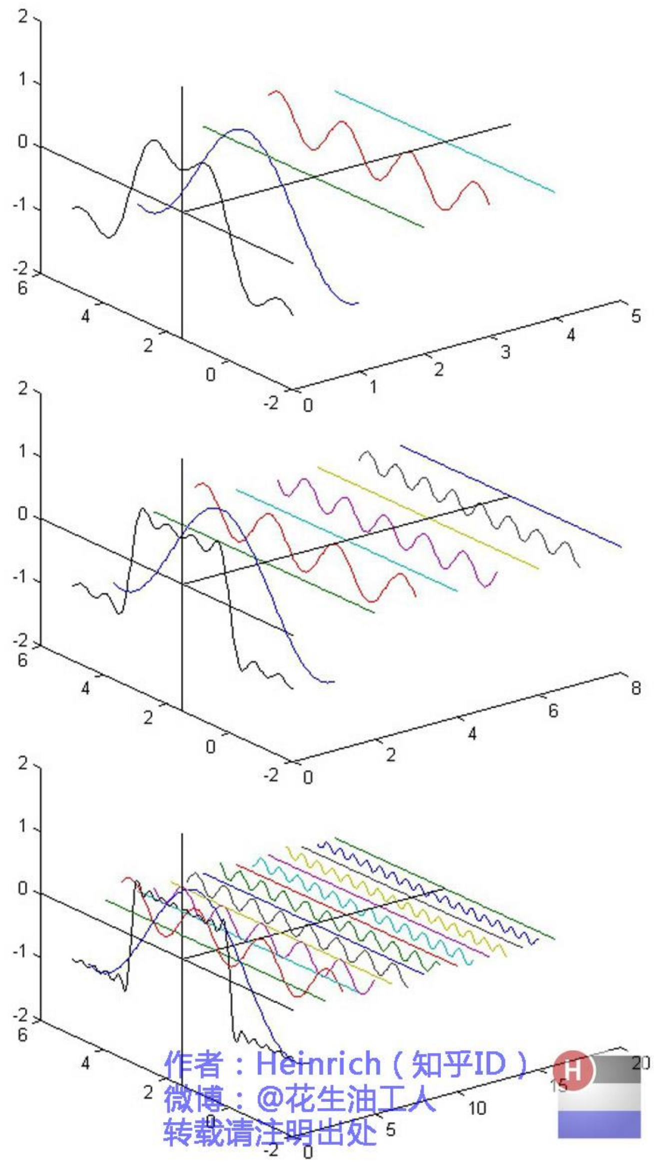

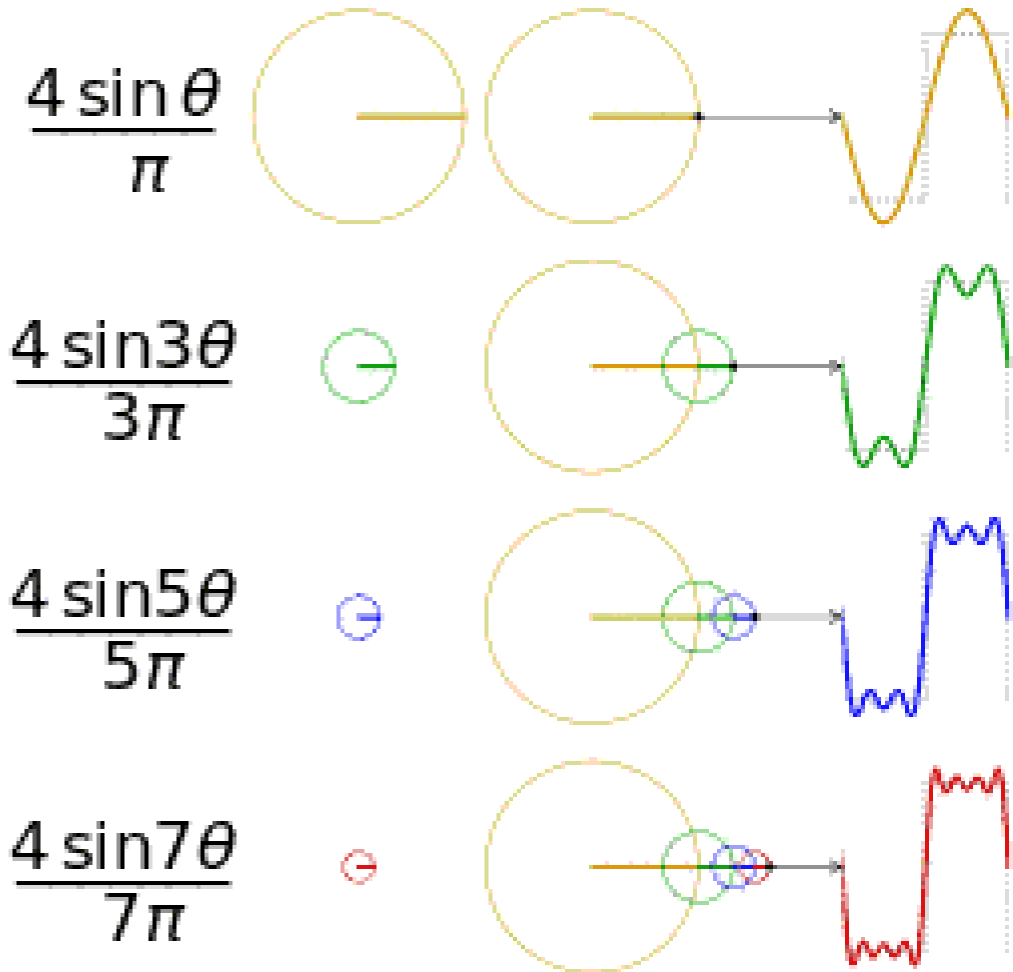
$\frac{4\sin\theta}{\pi}$
$\frac{4\sin 3\theta}{3\pi}$
$\frac{4\sin 5\theta}{5\pi}$
$\frac{4\sin 7\theta}{7\pi}$

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关于向量空间基底和信息的表示：

N维实数空间R^N中的任一向量$\boldsymbol{x}$可以表示为$\boldsymbol{x} = (x_1, x_2, \ldots, x_N)^T$, 或者

$$\boldsymbol{x} = x_1 \cdot \mathbf{e}_1 + x_2 \cdot \mathbf{e}_2 + \ldots + x_N \cdot \mathbf{e}_N$$

其中$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$是$R^N$的单位向量，构成$R^N$的一组标准基底，称为标准基底。$(x_1, x_2, \ldots, x_N)^T$是向量$\boldsymbol{x}$在这组基底下的表示（“坐标”、系数或“幅度”）。表示了用$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$合成（叠加成）向量$\boldsymbol{x}$时，各个分量的“大小”。

如果给另外一组基底$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$, 则会有另外一种表示方法

$$\boldsymbol{x} = y_1 \cdot \mathbf{u}_1 + y_2 \cdot \mathbf{u}_2 + \ldots + y_N \cdot \mathbf{u}_N$$

同样，$(y_1, y_2, \ldots, y_N)^T$表示用$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N\}$,合成（叠加成）向量$\boldsymbol{x}$时，各个分量的“大小”。

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关于向量空间基底：

$$\boldsymbol{x} = x_1 \cdot \mathbf{e}_1 + x_2 \cdot \mathbf{e}_2 + \ldots + x_N \cdot \mathbf{e}_N$$

$$\boldsymbol{x} = y_1 \cdot \mathbf{u}_1 + y_2 \cdot \mathbf{u}_2 + \ldots + y_N \cdot \mathbf{u}_N$$

向量在不同基底下，表现形式不同，但代表的都是同一个向量。它们可以从不同角度展示向量本身的性质，也可以用于对向量在不同的“域”（domain）做不同的处理。例如压缩。正确的选择基底是数据处理中的一个重要问题。把一个向量从一组基底下的表示转换为另外一组基底下的表示，称为“变换”。在线性空间的线性变换可以用矩阵乘法表示。

例如：向量$\boldsymbol{t} = (x_1, x_2, \ldots, x_N)^T$和 $\boldsymbol{s} = (y_1, y_2, \ldots, y_N)^T$之间的关系可以用矩阵乘法表示：$\boldsymbol{s} = \mathbf{A}\boldsymbol{t}$，其中 矩阵$\mathbf{A}$是矩阵$[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N]$的逆矩阵

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Example：

$\boldsymbol{x}$是$\mathbf{R}^8$中的一个向量（信号），$\boldsymbol{x}\in\mathbf{R}^8$。在标准基底{$\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_8$}下的坐标为

$$\boldsymbol{t} = (16, 9, 4, 0, 4, 9, 16, 25)^T$$

$\boldsymbol{t}$ 即是向量$\boldsymbol{x}$在标准基底下的表示。如果选另外一组基底{$\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_N$}如下：则向量$\boldsymbol{x}$在这组基底下表示为

$$\boldsymbol{s} = (12.5, 2, 6.5, 15.5, -3.5, -2, 2.5, 4.5)^T$$

$$\begin{array}{cccccccc}
\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 & \mathbf{u}_6 & \mathbf{u}_7 & \mathbf{u}_8 \\
\begin{bmatrix}1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, &
\begin{bmatrix}0\\0\\1\\1\\0\\0\\0\\0\end{bmatrix}, &
\begin{bmatrix}0\\0\\0\\0\\1\\1\\0\\0\end{bmatrix}, &
\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\1\end{bmatrix}, &
\begin{bmatrix}-1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, &
\begin{bmatrix}0\\0\\-1\\1\\0\\0\\0\\0\end{bmatrix}, &
\begin{bmatrix}0\\0\\0\\0\\-1\\1\\0\\0\end{bmatrix}, &
\begin{bmatrix}0\\0\\0\\0\\0\\0\\-1\\1\end{bmatrix}
\end{array}$$

如果令后四个分量（坐标）为零（相当于“滤波”处理），则恢复后的向量（信号）为：

$$\mathbf{(12.5, 12.5, 2, 2, 6.5, 6.5, 15.5, 15.5)^T}$$

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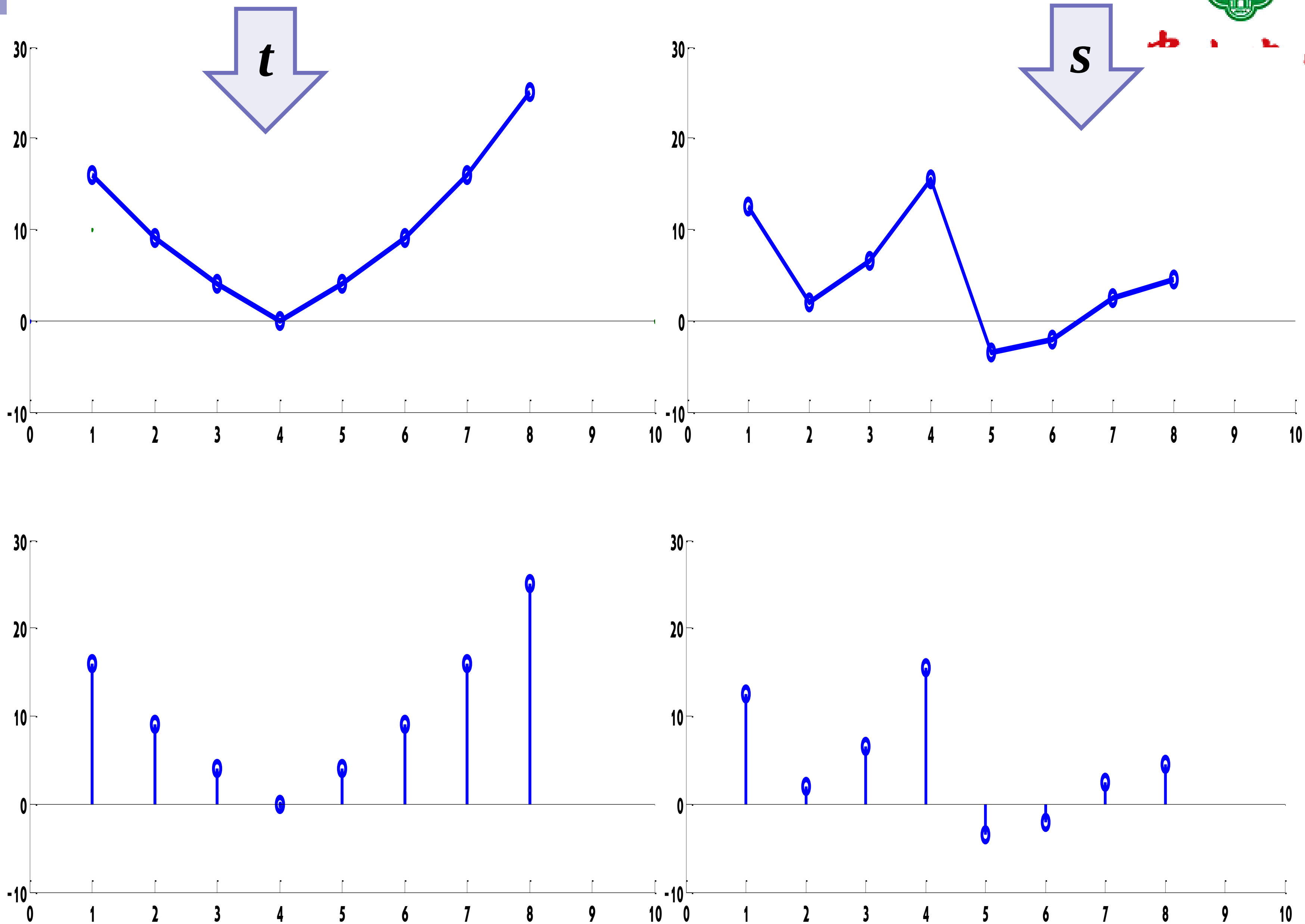
Example:

写成矩阵形式： $\boldsymbol{s} = \mathbf{A}\boldsymbol{t}$

$$
\begin{bmatrix} 12.5 \\ 2 \\ 6.5 \\ 15.5 \\ -3.5 \\ -2 \\ 2.5 \\ 4.5 \end{bmatrix}
=
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix} 16 \\ 9 \\ 4 \\ 0 \\ 4 \\ 9 \\ 16 \\ 25 \end{bmatrix}
$$


t
s

2D Example:

Aerial image of Fort Hood, TX (used as input for the CDF wavelet)

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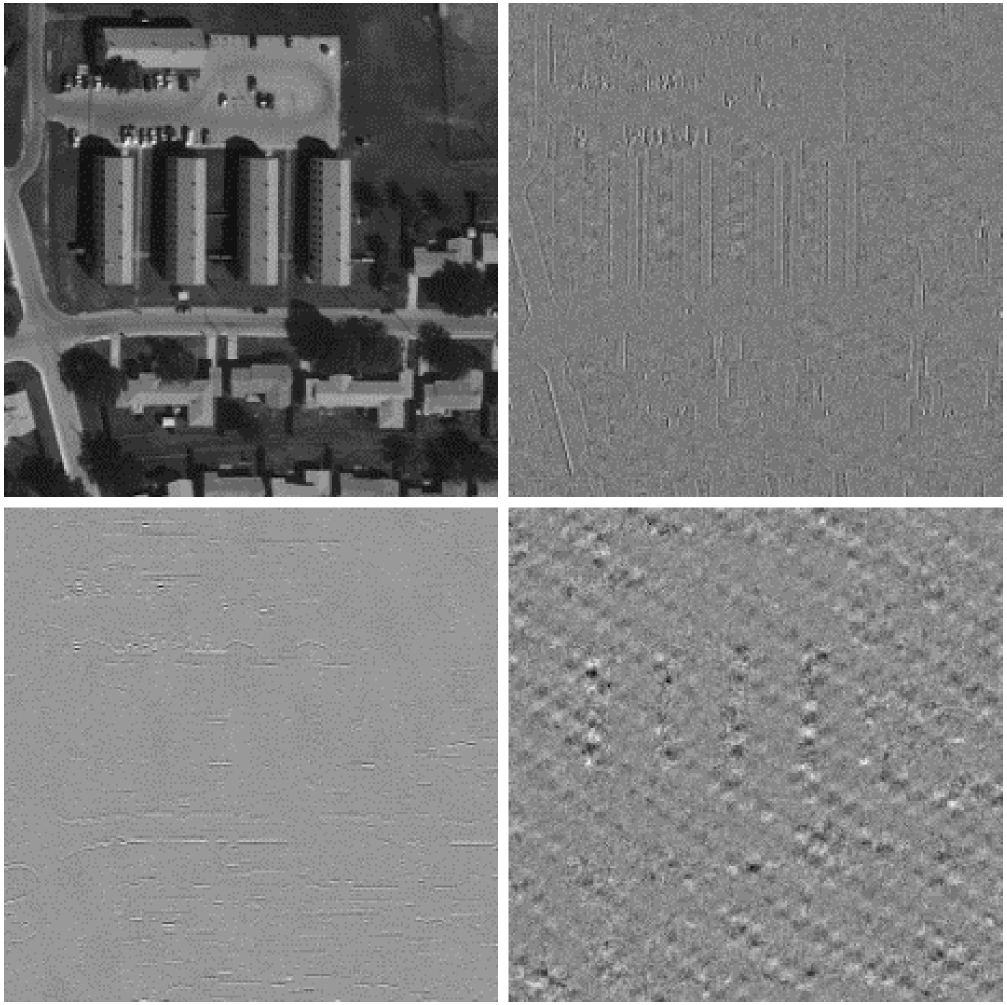
Example:



Aerial image of Fort Hood, TX (used as input for the CDF wavelet)

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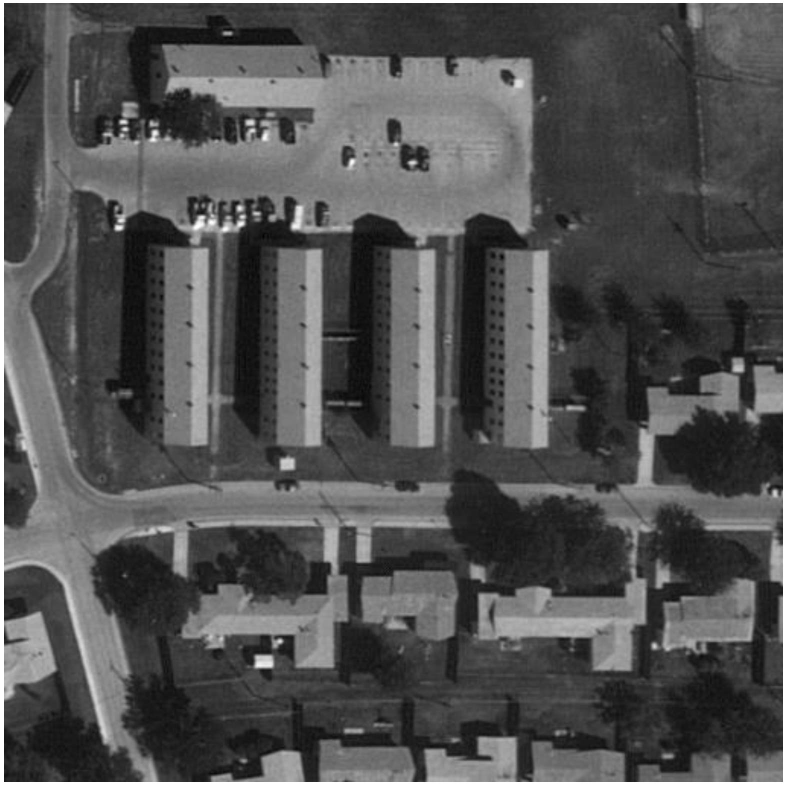
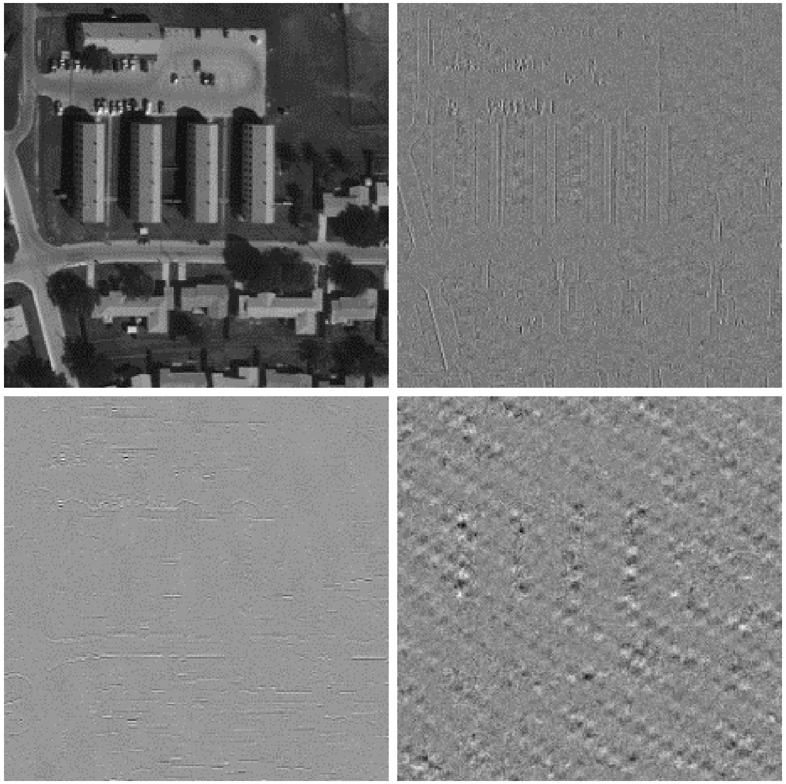
Example:





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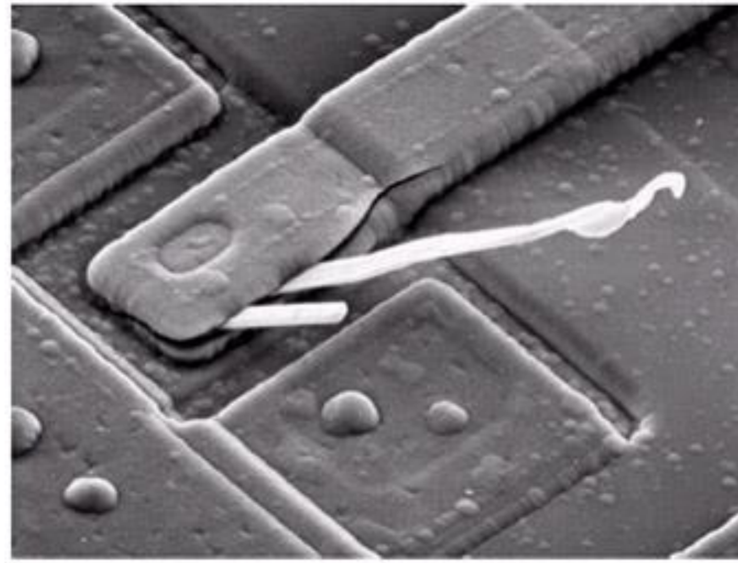
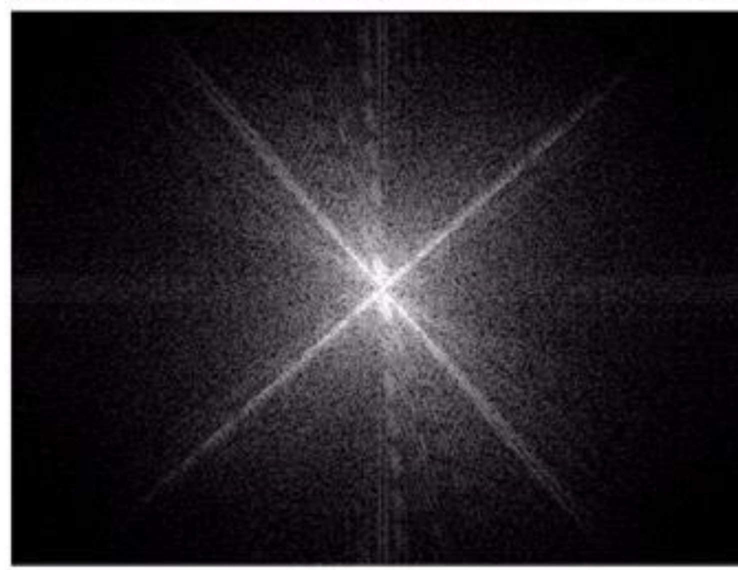
关于向量空间基底：

对复数向量空间C^N有类似的基底选择问题。而任一个实数向量，可以看成是一个特殊的复向量。**Fourier变换（分析）则是选择了一组特殊的向量作为基底。这组基底恰好是不同频率的谐波（正弦波或余弦波）**

关于图像变换：

图像可以用矩阵表示。因此，图像可以看作是一个高维向量，同样在不同的基底下有不同的表示。Fourier 基底是使用最多的基底，在傅立叶基底下的表示通常被称为频谱表示。这是因为Fourier的基底恰恰是不同频率的“波”。





motivation

Figure 8. The original image and the descreen image using regular SUSAN filter , the screen patterns is not removed cleanly.

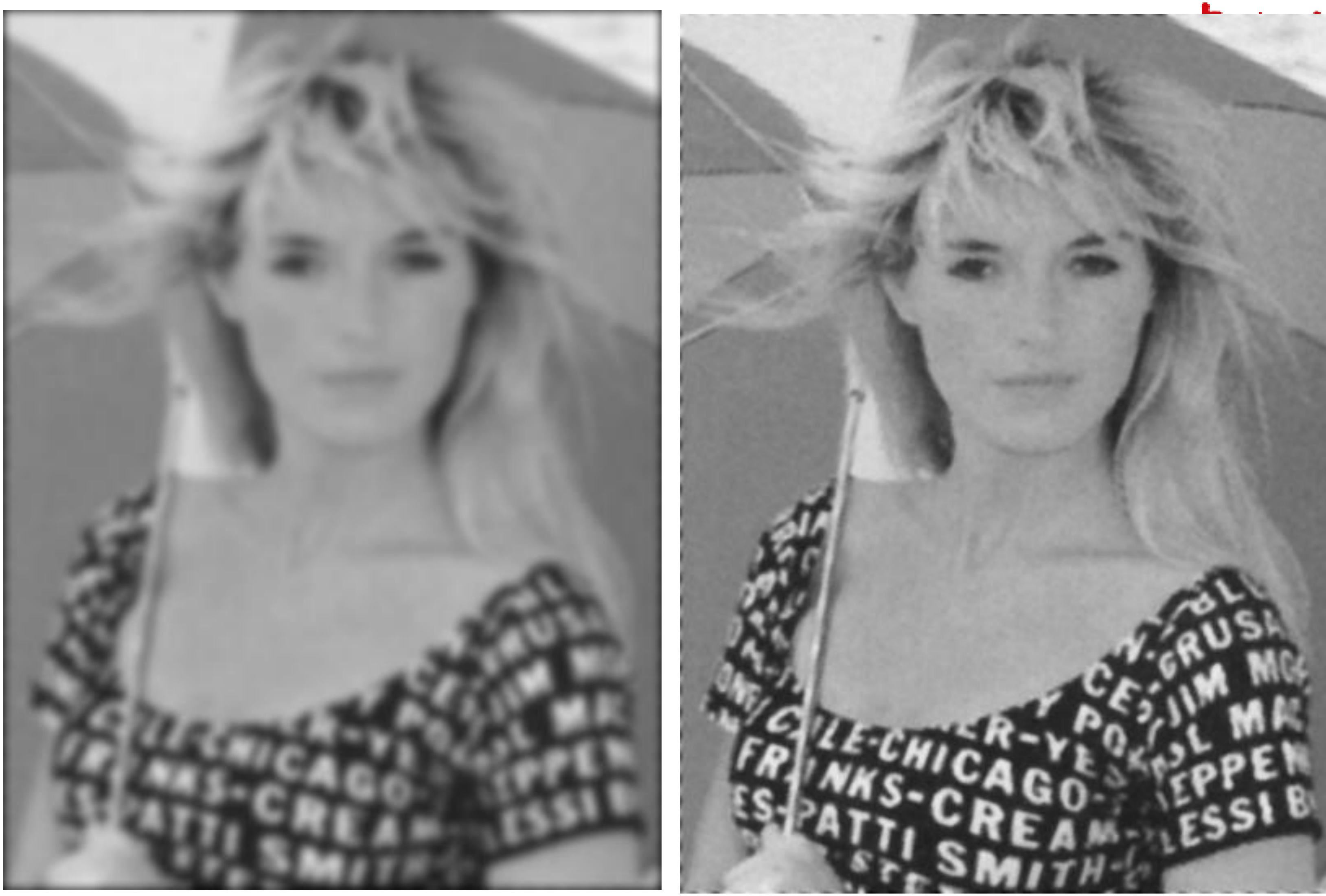


Figure 9. The left image is processed with Gaussian filter (σ=3.2, filter size 15x15), and the right image is processed with our method.

- Because of the halftone screen have periodicity, so we can seen some halftone peaks in the Fourier spectrum of the scanned halftone image (Figure 6). Then we can calculate the screen rule from the halftone peak's position.

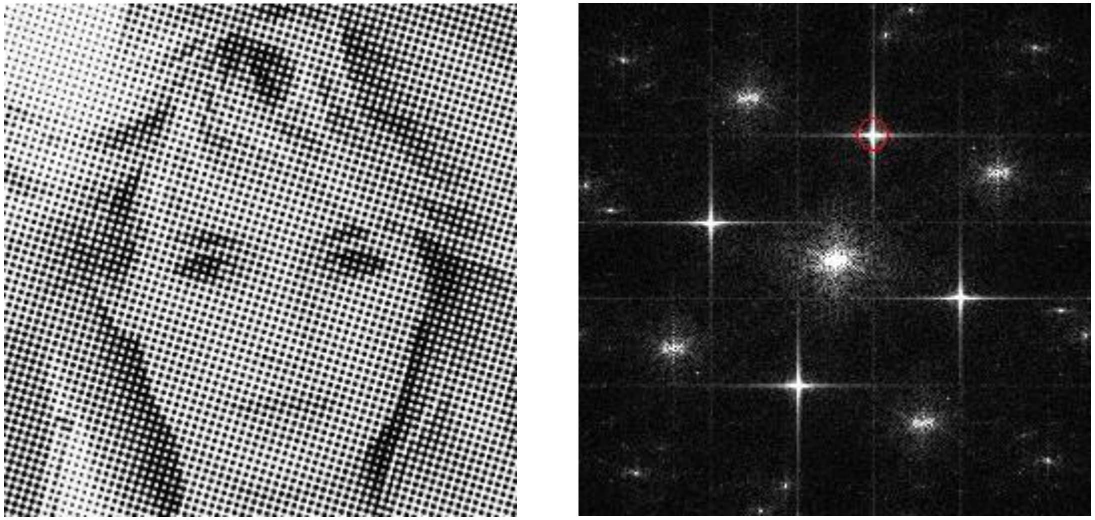


Figure 6. A scanned halftone image and it's Fourier spectrum, one of halftone peaks is marked.

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关于函数空间基底：

如果F是区间[a, b]上所有函数组成的空间，则当这些函数满足一定条件（如连续、可导等）， F中的函数可以在某组函数基底下展开，也就是说，函数可以写成级数的形式，如多项式级数展开，三角级数展开。多项式函数序列、三角函数序列就是这个函数空间的基底。这时，一个函数就和一个序列等价。

注：如果不关注细节，一元函数可以看成是一个无穷（不可数）维数的向量。一个n维向量，亦可看成是一元函数的离散化表示。

投影的概念

4.1-3 背景—Background

- Fourier 分析理论的出现是学术界和工业界的一次革命；快速傅里叶变换算法的出现, 使信号处理领域出现了重大变革.
- Fourier分析目前是很多应用学科的基础.上世纪末出现的小波分析本质上也是建立在傅氏分析的基础之上的.
- 傅里叶分析中最重要的结论就是几乎“所有”的函数(信号)都可以表示为(分解成)简单的(加权)正弦波和余弦波（或者简谐波）之和。从而提供了一种具有物理意义的函数表达方式。

设：$f(x)$是以T为周期的函数, 满足一定的条件, 例如平方可积, 则有

$$f(x)=\sum_{k=-\infty}^{+\infty} c_k \cdot e^{jk\left(\frac{2\pi}{T}\right)x} \qquad c_k=\frac{1}{T}\int_0^T f(x)\cdot e^{-jk\left(\frac{2\pi}{T}\right)x}dx$$

怎么来的？

A classical conclusion:

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设：$f(x)$是以T为周期的函数，满足一定的条件，例如平方可积，则有

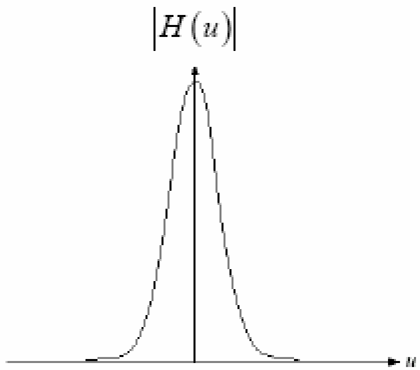
$$f(x)=a_0+\sum_{n=1}^{+\infty}\left[a_n\cdot\cos\left(\frac{2\pi nx}{T}\right)+b_n\cdot\sin\left(\frac{2\pi nx}{T}\right)\right]$$

$$a_0=\frac{1}{T}\int_0^T f(x)dx \qquad a_n=\frac{2}{T}\int_0^T f(x)\cdot\cos\left(\frac{2\pi nx}{T}\right)dx$$

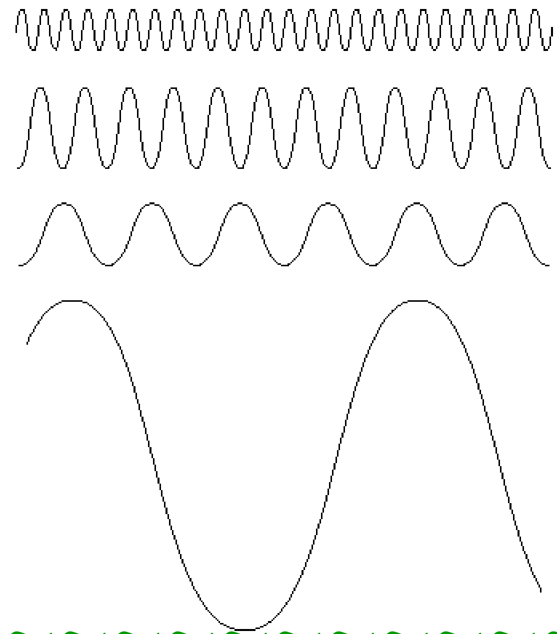
$$b_n=\frac{2}{T}\int_0^T f(x)\cdot\sin\left(\frac{2\pi nx}{T}\right)dx$$

Furthermore, we have an equivalent expression:

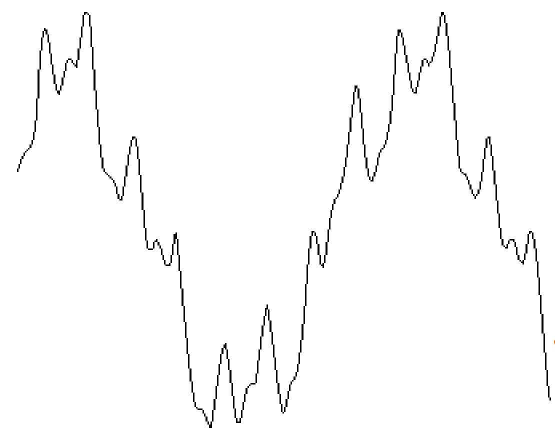
$$f(x)=\sum_{k=-\infty}^{+\infty}c_k\cdot e^{jk\left(\frac{2\pi}{T}\right)x} \qquad c_k=\frac{1}{T}\int_0^T f(x)\cdot e^{-jk\left(\frac{2\pi}{T}\right)x}dx$$



Frequency domain

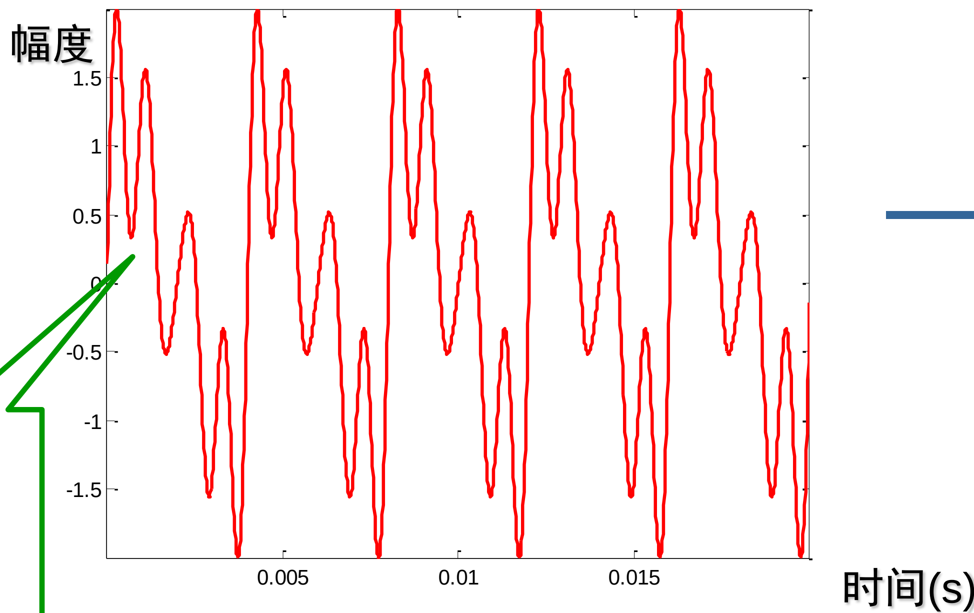


- An image can be treated as a two-dimensional signal generalized from 1-D signal discussed in *Control Theory or Signal Processing*.
- A continuous signal can be decomposed into the sum of a series of simple harmonic functions with phase differences.



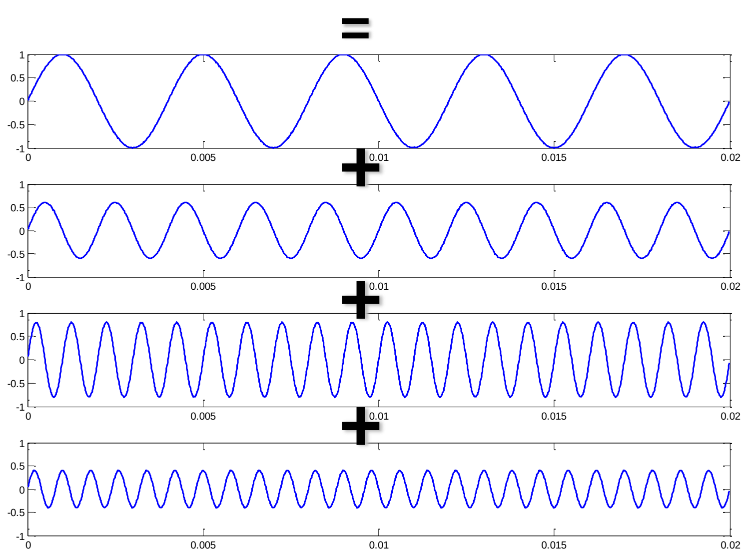
Spatial domain

是四个波形的线性叠加



是四个波形的线性叠加

$$x(t) = \sin(2\pi f t) + 0.6\sin(2\pi 2 f t) + 0.8\sin(2\pi 4 f t) + 0.4\sin(2\pi 5 f t)$$



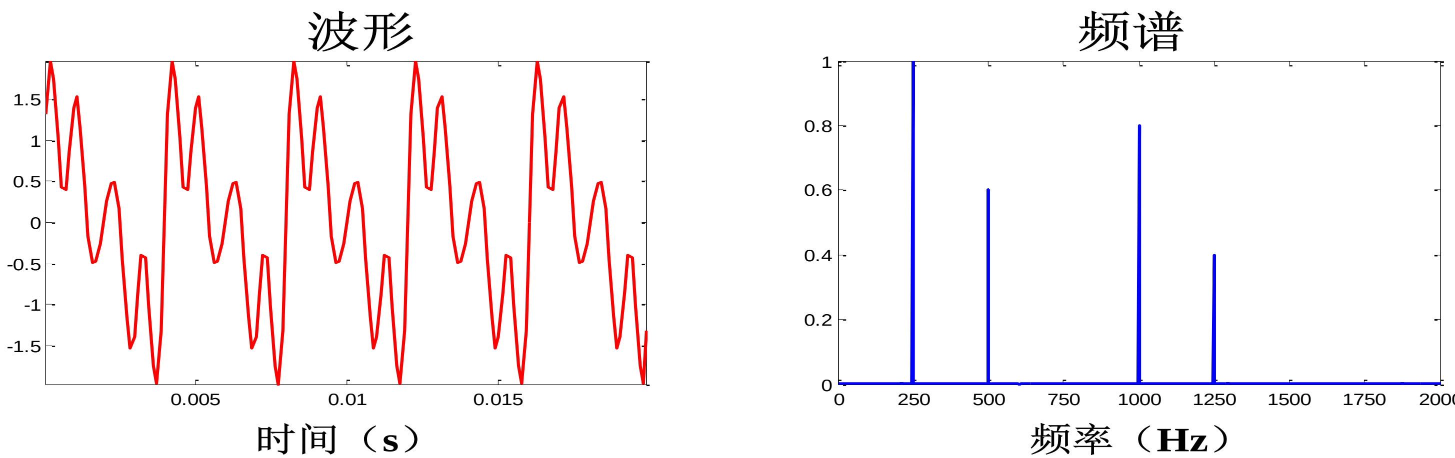
$\sin(2\pi f t)$

$0.6\sin(2\pi 2 f t)$

$0.8\sin(2\pi 4 f t)$

$0.4\sin(2\pi 5 f t)$

时（空）域与频域表示



信号的时域与频域表示

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如何来看待$f(x)$的傅里叶级数展开公式?

- 傅里叶系数$\{ c_k \}_{(-\infty, +\infty)}$可等价地表示原函数$f(x)$, 包含了原函数$f(x)$的所有信息. 对傅里叶系数$\{ c_k \}_{(-\infty, +\infty)}$的处理也等价于处理原函数$f(x)$. 可看成是频谱域图像处理的基础;
- 本质上可以看成是一种变换, 把函数变成了数列。而$f(x)$的级数表达公式相当于反变换，把数列还原成函数.

变换就是把数据(或者函数)在一组基底下的表示变换到另一组基底下的表示. 因为两种表示是等价的, 所以本质上上所有可逆的变换都可以考虑在图像处理(信号处理)中发挥作用, 只要这个变换(对应的基底)具有某种我们需要的性质.

变换通常会破坏图像的可视性.

关键还是变换的特性.

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4.2 Fourier Transform and Frequency Domain

- Fourier series expansion for periodic function with period T

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}\right]$$

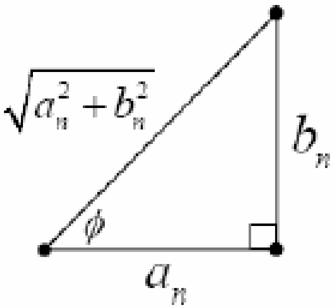
$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\frac{2n\pi x}{T}\,dx$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\frac{2n\pi x}{T}\,dx$$

and

$$\begin{aligned}
& a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T} \\
&= \sqrt{a_n^2 + b_n^2}\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}}\cos\frac{2n\pi x}{T} + \frac{b_n}{\sqrt{a_n^2 + b_n^2}}\sin\frac{2n\pi x}{T}\right) \\
&= \sqrt{a_n^2 + b_n^2}\left(\cos\phi \cdot \cos\frac{2n\pi x}{T} + \sin\phi \cdot \sin\frac{2n\pi x}{T}\right) \\
&= \sqrt{a_n^2 + b_n^2}\cos\left(\frac{2n\pi x}{T} - \phi_n\right)
\end{aligned}$$



- Fourier Series in Complex Form for periodic function with period T

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\omega_n x + b_n \sin\omega_n x\right] \quad \cdots\cdots \left(\omega_n = \frac{2n\pi}{T}\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{j\omega_n x} + e^{-j\omega_n x}}{2}\right) + b_n\left(\frac{e^{j\omega_n x} - e^{-j\omega_n x}}{2j}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}\right)e^{j\omega_n x} + \sum_{n=1}^{\infty}\left(\frac{a_n + jb_n}{2}\right)e^{-j\omega_n x}$$

两种表达形式一致

Euler's Formula

$$e^{j2\pi ux} = \cos(2\pi ux) + j\sin(2\pi ux)$$

$$e^{-j2\pi ux} = \cos(2\pi ux) - j\sin(2\pi ux)$$

$$\frac{a_n - jb_n}{2} = \frac{1}{2}\left[\frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\omega_n x \cdot dx - j\frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\omega_n x \cdot dx\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(x)\left[\cos\omega_n x - j\sin\omega_n x\right]dx$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-j\omega_n x}dx$$

$$\equiv c_n$$

$$\sum_{n=1}^{\infty}\left(\frac{a_n + jb_n}{2}\right)e^{-j\omega_n x}$$

$$= \sum_{n=-\infty}^{-1}\left(\frac{a_{-n} + jb_{-n}}{2}\right)e^{-j\omega_{-n} x}$$

$$= \sum_{n=-\infty}^{-1}\left(\frac{a_n - jb_n}{2}\right)e^{j\omega_n x}$$

$$= \sum_{n=-\infty}^{-1} c_n e^{j\omega_n x}$$

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As a result (对周期函数)

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}\right)e^{j\omega_n x} + \sum_{n=1}^{\infty}\left(\frac{a_n + jb_n}{2}\right)e^{-j\omega_n x}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{j\omega_n x} + \sum_{n=-\infty}^{-1} c_n e^{j\omega_n x} \qquad \cdots\cdots\left(a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx = c_0\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n x}$$

$$\boxed{\omega_n = \frac{2n\pi}{T} \;;\; c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,e^{-j\omega_n x}\,dx}$$

$$f(x) \xrightarrow{\mathbf{T}} \{\ldots\ldots\ c_{-n}\ \ldots\ldots,\ c_0,\ c_1,\ c_2,\ \ldots\ldots c_n,\ \ldots\ldots\}$$

特别注意

- 傅氏级数中基底的物理意义非常明确, 每一个基函数都是一个单频谐波, 而相应的系数(频谱)表明了原函数中这种频率成份的多少(原函数在这个谐波上的"投影")。
- 傅氏级数把函数(或信号)分解成不同频率成分的谐波叠加
- 从图像(信号)处理的角度, 利用谐波的物理性质可以通过对（频谱）系数的处理达到对图像的处理, 如增强、压缩等等。
- $f(x)$每一个傅氏系数c_k的计算, 需要用到函数在整个空间(或时间)上的分布情况, 缺点? 优点? 这一点和空间域情形分析有很大的不同.

$$c_k = \frac{1}{T}\int_0^T f(x)\cdot e^{-jk\left(\frac{2\pi}{T}\right)x}dx$$

大學

- One dimensional Fourier transform（非周期函数情形）

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx$$
$$= \int_{-\infty}^{\infty} f(x)[\cos 2\pi ux - j\sin 2\pi ux]dx = R(u) + jI(u)$$

对比 $$c_k = \frac{1}{T}\int_0^T f(x)\cdot e^{-jk\left(\frac{2\pi}{T}\right)x}dx$$

Euler's Formula

$$e^{j2\pi ux} = \cos(2\pi ux) + j\sin(2\pi ux)$$

$$\Im^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

- Fourier spectrum $|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$
- Phase angle $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
- Power spectrum $P(u) = |F(u)|^2 = R^2(u) + I^2(u)$

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- Two dimensional Fourier transform

$$\Im\left(f(x,y)\right)=F(u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$\Im^{-1}\left(F(u,v)\right)= f(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

- $F(u,v)=R(u,v)+jI(u,v)=\left|F(u,v)\right|e^{j\phi(u,v)}$
- Fourier spectrum $\left|F(u,v)\right|=\left[R^2(u,v)+I^2(u,v)\right]^{1/2}$
- Phase angle $\phi(u,v)=\tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
- Power spectrum $P(u,v)=\left|F(u,v)\right|^2=R^2(u,v)+I^2(u,v)$

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- One dimensional Discrete Fourier transform (DFT)

N维实数空间R^N中的任一向量$\boldsymbol{x}$可以表示为$\boldsymbol{x}=(x_1, x_2, \ldots, x_N)^T$, 或者

$$\boldsymbol{x} = x_1 \cdot \mathbf{e}_1 + x_2 \cdot \mathbf{e}_2 + \ldots + x_N \cdot \mathbf{e}_N$$

其中{$\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_N$}是R^N的单位向量，构成R^N的一组标准基底，称为标准基底。$(x_1, x_2, \ldots, x_N)^T$是向量$\boldsymbol{x}$在这组基底下的表示（“坐标”或“幅度”）。表示了用{$\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_N$}合成（叠加成）向量$\boldsymbol{x}$时，各个分量的“大小”。

如果给另外一组基底{$\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_N$}, 则会有另外一种表示方法

$$\boldsymbol{x} = y_1 \cdot \mathbf{v}_1 + y_2 \cdot \mathbf{v}_2 + \ldots + y_N \cdot \mathbf{v}_N$$

同样，$(y_1, y_2, \ldots, y_N)^T$表示用{$\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_N$},合成（叠加成）向量$\boldsymbol{x}$时，各个分量的“大小”。

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- One dimensional Discrete Fourier transform (DFT)

Let $x = 0, 1, 2, 3, \ldots, N\text{-}1$ and $f(x)$ be a discrete signal with a sampling time $\triangle x$

$$f(x) = (f(0), f(1), \ldots\ldots f(N\text{-}1))^{T}.$$

Let

$$W_N = \exp(-\frac{j2\pi}{N}) = e^{-\frac{j2\pi}{N}}$$

Fourier 基底还有不同的变种，例如DCT

Then, choose basis

$$\mathbf{v}_u = (1, W_N^{-1u}, W_N^{-2u}, \cdots, W_N^{-(N-1)u})^{T}, \quad u = 0, 1, \cdots, \mathrm{N}\text{-}1$$

In the following, we will use the similar concept to describe Fourier transform.（在新的一组基底下求向量$f(x)$的表示或者坐标。）

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- One dimensional Discrete Fourier transform (DFT)

Let $x = 0, 1, 2, 3, \ldots, N-1$ and $f(x)$ be a discrete signal with a sampling time $\triangle x$

$$f(x) = (f(0), f(1), \ldots\ldots f(N-1))^{T}.$$

New basis

$$\mathbf{v}_u = (1, W_N^{-1u}, W_N^{-2u}, \cdots, W_N^{-(N-1)u})^{T}, \quad u = 0, 1, \cdots, N-1$$

Find out F(*u*) (u=0, 1, 2, 3, …, N-1), such that

$$\left(f(0), f(1), \cdots, f(N-1)\right)^{T} = \sum_{u=0}^{N-1} F(u)\mathbf{v}_u$$

$$(f(0), f(1), \ldots\ldots f(N-1))^{T} \underset{\text{反变换}}{\overset{\text{正变换}}{\rightleftarrows}} (F(0), F(1), \ldots\ldots F(N-1))^{T}$$

- One dimensional Discrete Fourier transform (DFT)

Let x = 0, 1, 2, 3, …, N-1 and $f(x)$ be a discrete signal with a sampling time $\triangle x$, then its DFT and inverse DFT are **defined** by

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{-j2\pi ux/N}=\frac{1}{N}\sum_{x=0}^{N-1} f(x)W_N^{xu},\ u=0,1,\cdots,N-1$$

$$f(x)=\sum_{u=0}^{N-1} F(u)e^{j2\pi ux/N}=\sum_{u=0}^{N-1} F(u)W_N^{-ux},\ x=0,1,\cdots,N-1$$

$$\Longrightarrow \left(f(0),\ f(1),\cdots,\ f(N-1)\right)^{\mathrm{T}}=\sum_{u=0}^{N-1} F(u)\mathbf{v}_u$$

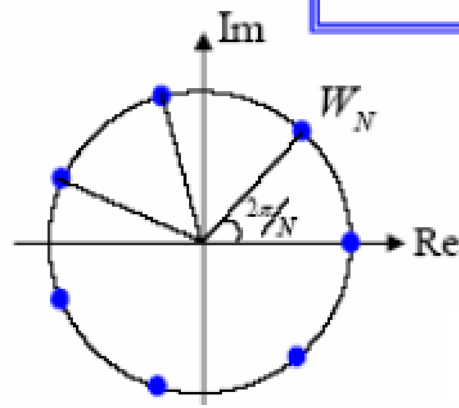
Here, $W_N=\exp(-\frac{j2\pi}{N})=e^{-\frac{j2\pi}{N}}$

$$\mathbf{v}_u=(1,W_N^{-1u},W_N^{-2u},\cdots,W_N^{-(N-1)u})^{\mathrm{T}},\quad u=0,1,\cdots,\mathrm{N}\text{-}1$$

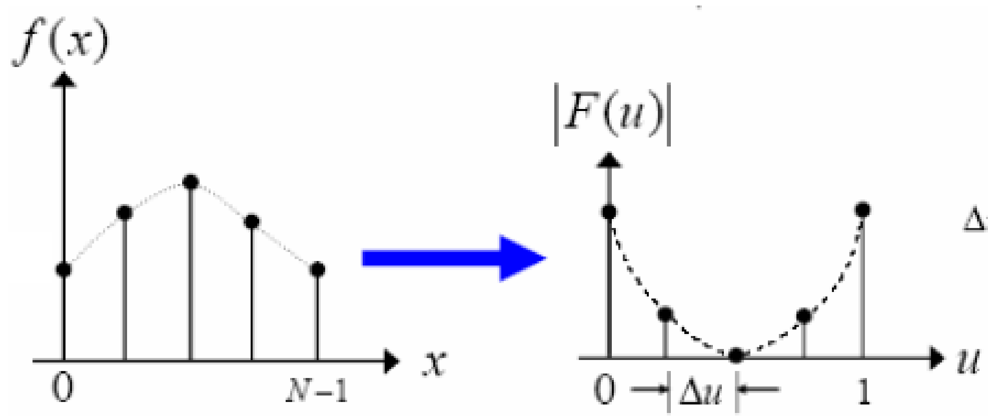
- One dimensional Discrete Fourier transform (DFT)

$$W_N = \exp(-\frac{j2\pi}{N}) = e^{-\frac{j2\pi}{N}}$$

$$\boxed{W_N = \exp\left(-j\frac{2\pi}{N}\right)}$$



W_N是复平面单位园上的点也叫旋转因子（Twiddle factor）



$$\Delta u = \frac{1}{N}$$

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- One dimensional Discrete Fourier transform (DFT)

Using the Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, the DFT can be rewritten as

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left[\cos\frac{2\pi ux}{N} + j\sin\frac{2\pi ux}{N}\right]$$

where $u = 0,1,\cdots,M-1$

- Fourier spectrum $|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$
- Phase angle $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
- Power spectrum $P(u) = |F(u)|^2 = R^2(u) + I^2(u)$

一些基本概念的定义

DFT:

$$F(u)=\frac{1}{M}\sum_{x=0}^{M-1}f(x)e^{-j2\pi ux/M} \qquad u=0,1,\cdots,M-1$$

Inverse-DFT:

$$f(x)=\sum_{x=0}^{M-1}F(u)e^{j2\pi ux/M}$$

$$=F(0)+\sum_{x=1}^{M-1}F(u)e^{j2\pi ux/M}, \qquad x=0,1,\cdots,M-1$$

$$F(0)=\frac{1}{M}\sum_{x=0}^{M-1}f(x)$$

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用矩阵表示Fourier 变换和反变换

$$\begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(M-2) \\ F(M-1) \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^1 & W_M^2 & \cdots & W_M^{M-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_M^{M-2} & W_M^{2(M-2)} & \cdots & W_M^{(M-1)(M-2)} \\ 1 & W_M^{M-1} & W_M^{2(M-1)} & \cdots & W_M^{(M-1)(M-1)} \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-2) \\ f(M-1) \end{bmatrix}$$

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-2) \\ f(M-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{-1} & W_M^{-2} & \cdots & W_M^{-(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_M^{-(M-2)} & W_M^{-2(M-2)} & \cdots & W_M^{-(M-1)(M-2)} \\ 1 & W_M^{-(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)(M-1)} \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(M-2) \\ F(M-1) \end{bmatrix}$$

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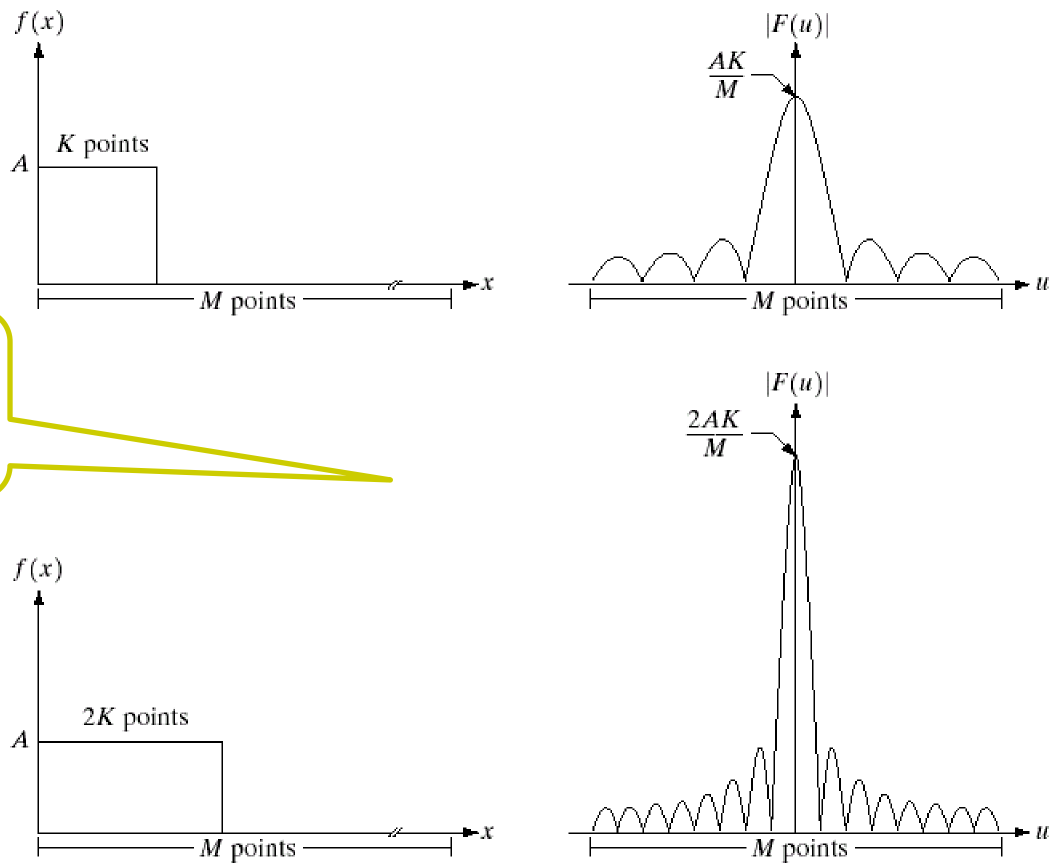
Example. Fourier spectrum of two simple functions



a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

自己做实验验证

特征:

(1) 当曲线下的面积在*x*域加倍时（整体能量增大）, 频率谱的“高度”也加倍;

(2) 当函数的长度加倍时, 相同长度区域内的零点数量也加倍. 极限情况?

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解释: 图a函数的**离散傅里叶变换**为:

$$F(u)=\frac{1}{M}\sum_{x=0}^{K-1}Ae^{-j2\pi ux/M}=\frac{A}{M}\sum_{x=0}^{K-1}(r^u)^x \qquad (r=e^{-j2\pi/M})$$

易见, 当$u=0$时, $r^u=1$, 故而 $F(0)=\dfrac{A}{M}\cdot\sum_{x=0}^{K-1}1=\dfrac{AK}{M}$

显然非零点增多时振幅也升高。若$u\neq 0$, 则$r^u\neq 1$, 对$u=1, 2, \ldots, M-1$,

$$F(u)=\frac{A}{M}\cdot\sum_{x=0}^{K-1}(r^u)^x=\frac{A}{M}\cdot\frac{1-r^{uK}}{1-r^u}$$

利用欧拉公式可知: $r=\cos(2\pi/M)-j\sin(2\pi/M)$.

所以, 当uK是M的倍数时, 就有$r^{uk}=1$(当然这时也有$r^{u2K}=1$), 从而 $F(u)=0$. 如果图a中函数$f(x)$非零点的个数是K时, $F(u)=0$的点数是n个, 那么, 当$f(x)$的非零点数是$2K$时, $F(u)=0$的点数应该是$2n$个.

频率域及相关的概念

利用欧拉公式: $e^{j\theta} = \cos\theta + j\sin\theta$, 有

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)[\cos(j2\pi ux/N) - j\sin(2\pi ux/N)]$$

其中$u = 0, 1, 2, \ldots, N-1$.

变量u (频率）确定了变换的频率成分, u的取值范围称为频率域(给定一个u 上述公式可以计算出离散信号中包含了“多少”这个频率的谐波). 对每一个u, F(u)称为变换的频率分量(也叫振幅)。

注意: 一个恰当的比喻是将傅里叶变换比作一个玻璃棱镜。棱镜是可以将光分成不同颜色成分的物理仪器, 每个成分的颜色由波长(或频率)决定。傅里叶变换可看作是“数学的棱镜”, 将函数基于频率分成不同的成分。 当我们考虑光时, 主要讨论它的光谱和频率谱线。同样, 傅里叶变换使我们能够通过频率成分来分析一个函数。扩展了分析的手段。

上面是非常重要的概念。

谱的概念

注意到傅里叶变换后的函数是在复数域内, 也可以表示为

$$F(u) = R(u) + iI(u)$$

或极坐标的形式: $F(u) = |F(u)|e^{j\phi(u)}$.

我们把量$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$称为傅里叶变换的幅度(Magnitude)或者谱(Spectrum). 这是在图像处理（特别是增强）中要经常用到的量. 谱可以表示原函数(或图像)对某一频谱分量的贡献.

$$\phi(u) = \arctan\left[\frac{I(u)}{R(u)}\right]$$

称为变换的相角（phase angle) 或者相位谱（Phase spectrum）, 用来表示原函数中某一频谱分量的起始位置*.

另外, 一个重要的量是功率谱（Power spectrum）(有时也叫能量谱、谱密度（Spectral density))

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

关于变量的说明:

记号 $f(x)$ (x = 0, 1, …, M-1) 表示从连续函数中取M个样点, 这些点不一定选取为区间[0, M-1]中的整数点。通常用x_0 (任意位置的)表示第一个取样点，Δx是取样间隔。所以，f(x) 理解为

$$f(x) \triangleq f(x_0 + x\Delta x)$$

其中: x = 0, 1, …, M-1.

同理, 变量u有相似的解释, 但序列通常总是从0频率开始. 因此, u的取值序列为$u = 0, \Delta u, 2\Delta u, \ldots, (M-1)\Delta u$. $F(u)$理解为:

$$F(u) \triangleq F(u\Delta u)$$

其中: u = 0, 1, …, M-1.

值得注意的是, 当M固定时, Δx和Δu之间有如下的反比关系:

$$\Delta u = \frac{1}{M\Delta x}$$

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小结:

- 连续函数的两种Fourier级数展开是等价的
- 离散信号的傅立叶变换实际上就是计算这个信号（向量）在一组特殊基底下线性组合的系数
- 一些概念：谱(spectrum)、相角(phase angle)、能量谱(Power spectrum)

DFT:

$$F(u)=\frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad u=0,1,\cdots,M-1$$

Inverse-DFT:

$$f(x)=\sum_{x=0}^{M-1} F(u)e^{j2\pi ux/M}$$

$$=F(0)+\sum_{x=1}^{M-1} F(u)e^{j2\pi ux/M}, \qquad x=0,1,\cdots,M-1$$

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用矩阵表示一个变元n维向量的Fourier 变换和反变换

$$\begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(M-2) \\ F(M-1) \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^1 & W_M^2 & \cdots & W_M^{M-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_M^{M-2} & W_M^{2(M-2)} & \cdots & W_M^{(M-1)(M-2)} \\ 1 & W_M^{M-1} & W_M^{2(M-1)} & \cdots & W_M^{(M-1)(M-1)} \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-2) \\ f(M-1) \end{bmatrix}$$

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-2) \\ f(M-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{-1} & W_M^{-2} & \cdots & W_M^{-(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_M^{-(M-2)} & W_M^{-2(M-2)} & \cdots & W_M^{-(M-1)(M-2)} \\ 1 & W_M^{-(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)(M-1)} \end{bmatrix} \begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(M-2) \\ F(M-1) \end{bmatrix}$$

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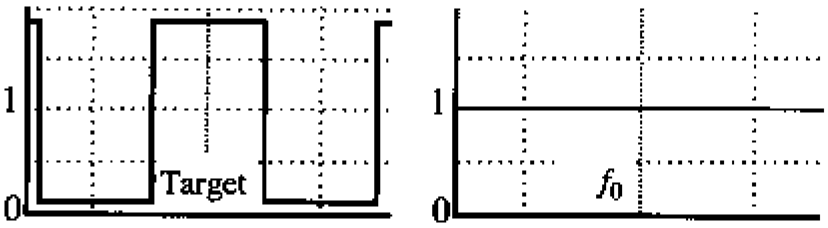
用基向量线性组合的方式表示Fourier反变换

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(M-2) \\ f(M-1) \end{bmatrix} = F(0)\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + F(1)\begin{bmatrix} 1 \\ W_M^{-1} \\ \vdots \\ W_M^{-(M-2)} \\ W_M^{-(M-1)} \end{bmatrix} + F(2)\begin{bmatrix} 1 \\ W_M^{-2} \\ \vdots \\ W_M^{-2(M-2)} \\ W_M^{-2(M-1)} \end{bmatrix} + \cdots + F(M-1)\begin{bmatrix} 1 \\ W_M^{-(M-1)} \\ \vdots \\ W_M^{-(M-1)(M-2)} \\ W_M^{-(M-1)(M-1)} \end{bmatrix}$$

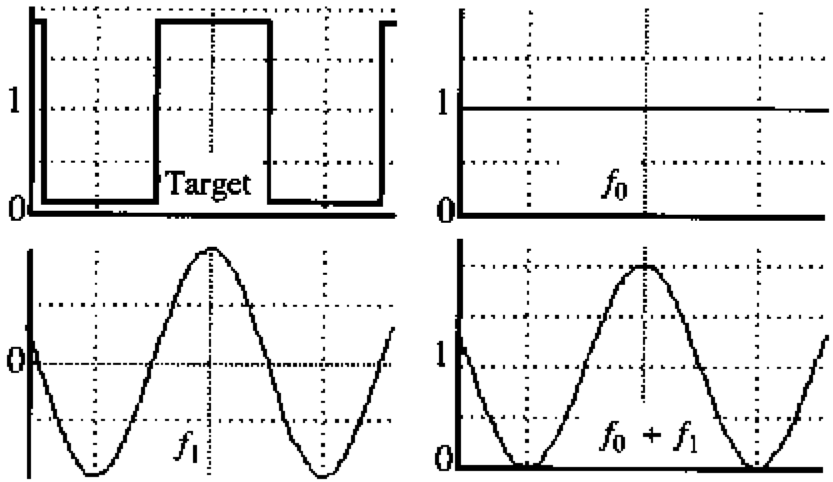
$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} = \sum_{x=0}^{M-1} f(x)W_M^{xu} \quad u = 0,1,\cdots,M-1$$

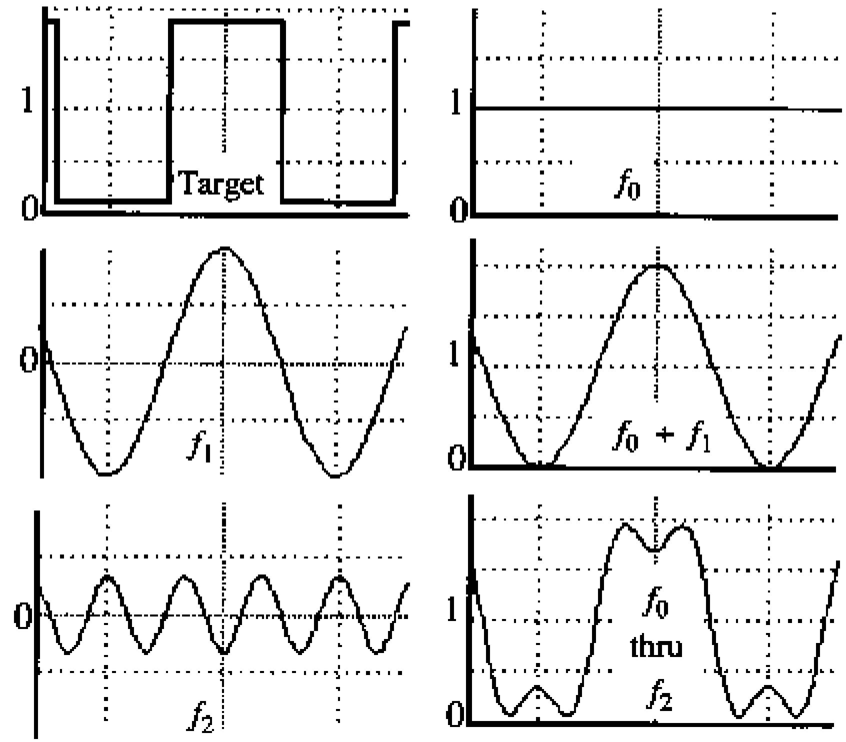
$$W_M = \exp(-j2\pi/M) = e^{-j2\pi/M}$$

A Sum of Sine Waves

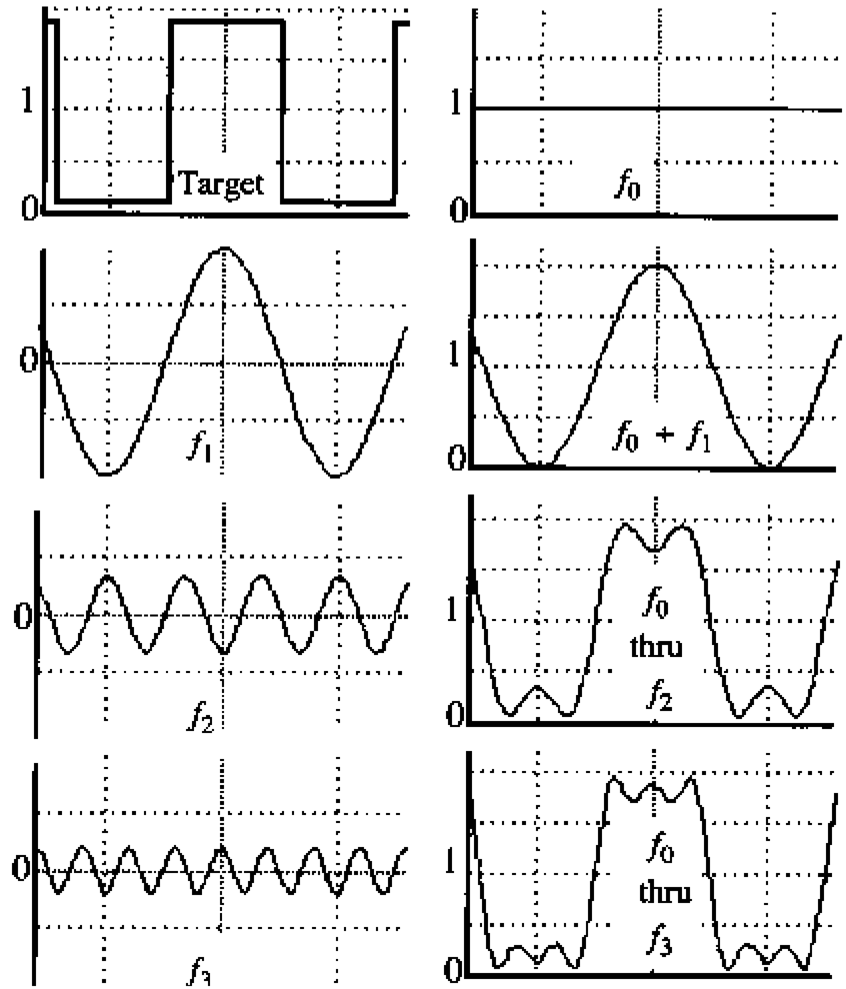


A Sum of Sine Waves

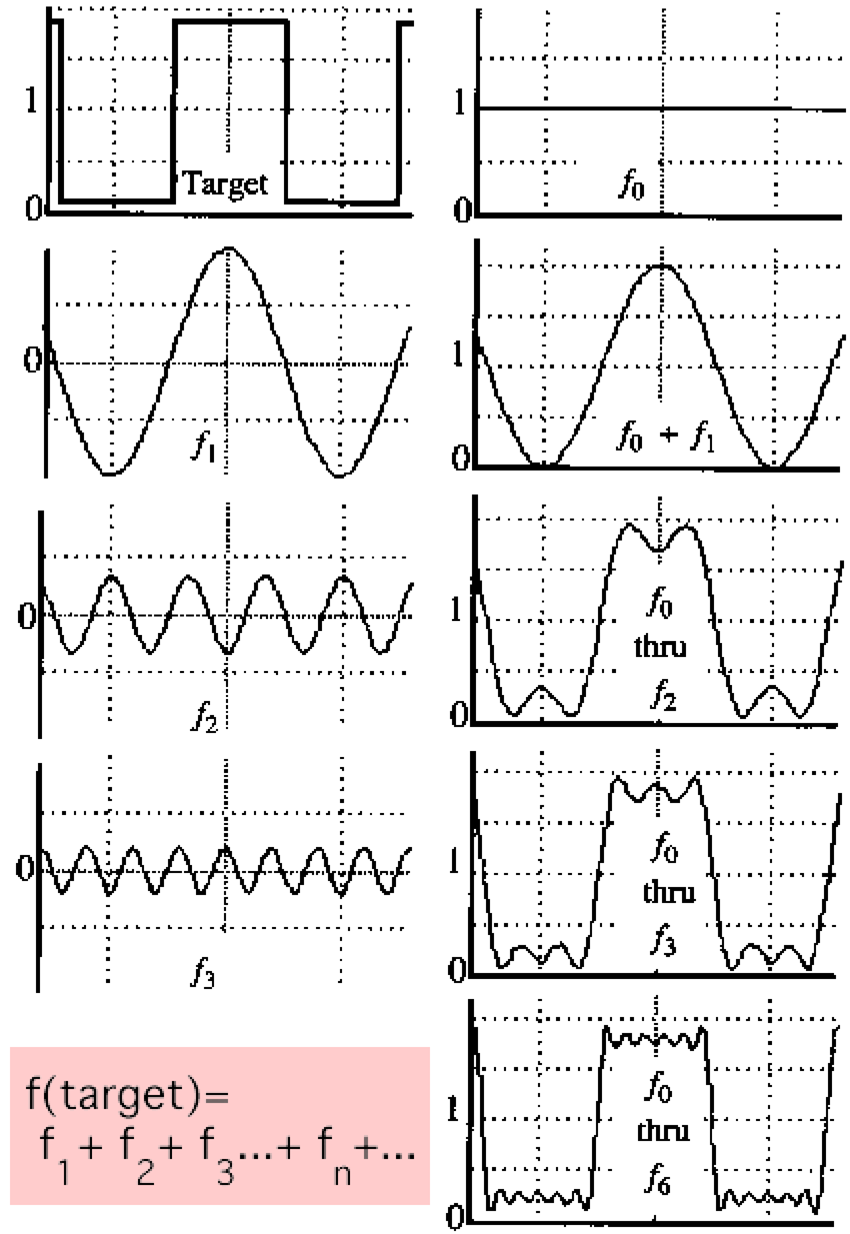




1
0
Target
1
0
f_0
0
f_1
1
0
$f_0 + f_1$
0
f_2
1
0
f_0
thru
f_2


Target
f_0
f_1
$f_0 + f_1$
f_2
f_0 thru f_2
f_3
f_0 thru f_3

A Sum of Sine Waves



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

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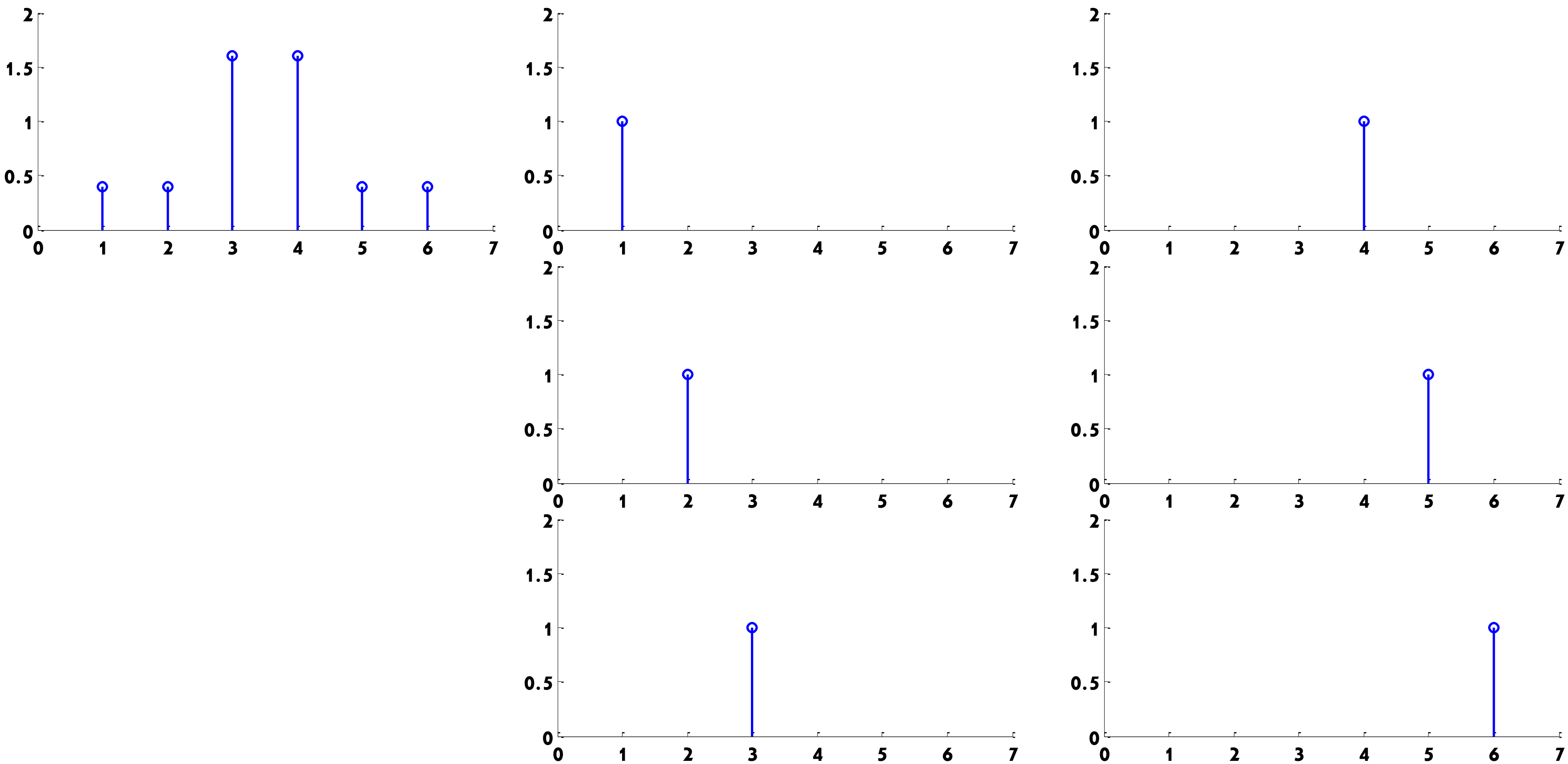

At least, you have to remember this:

DFT:

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad u = 0,1,\cdots, M-1$$

Inverse-DFT:

$$f(x) = \sum_{x=0}^{M-1} F(u)e^{j2\pi ux/M}$$

$$= F(0) + \sum_{x=1}^{M-1} F(u)e^{j2\pi ux/M}, \qquad x = 0,1,\cdots, M-1$$

$$F(0) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)$$

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- **Two dimensional Discrete Fourier transform (DFT)**

二维（变元）傅里叶变换本质上是一个变元情形向两个方向的简单扩展。

DFT:

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)W_{N}^{ux}W_{M}^{vy},$$

$$u=0,1,\cdots,M-1;\ v=0,1,\cdots,N\text{-}1.$$

Inverse DFT:

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)W_{N}^{-ux}W_{M}^{-vy},$$

$$x=0,1,\cdots,M-1;\ y=0,1,\cdots,N\text{-}1.$$

Here,

$$W_{N}^{ux}=\exp\left(\frac{-j2\pi ux}{M}\right)=e^{-j2\pi ux/M}$$

$$W_{M}^{vy}=\exp\left(\frac{-j2\pi vy}{n}\right)=e^{-j2\pi vy/N}$$

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- **Two dimensional Discrete Fourier transform (DFT)**

Since F(u, v) is complex, we have

$$F(u,v) = R(u,v) + jI(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

Here,

- **Fourier spectrum** $$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$
- **Phase angle** $$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$
- **Power spectrum** $$P(u,v) = R^2(u,v) + I^2(u,v)$$

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- **Two dimensional Discrete Fourier transform (DFT)**

二维（变元）傅里叶变换本质上是一维（变元）情形向两个方向的简单扩展。

DFT: $$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

Inverse DFT: $$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$F(0,0)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)$$

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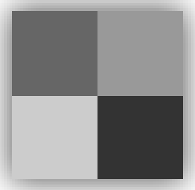
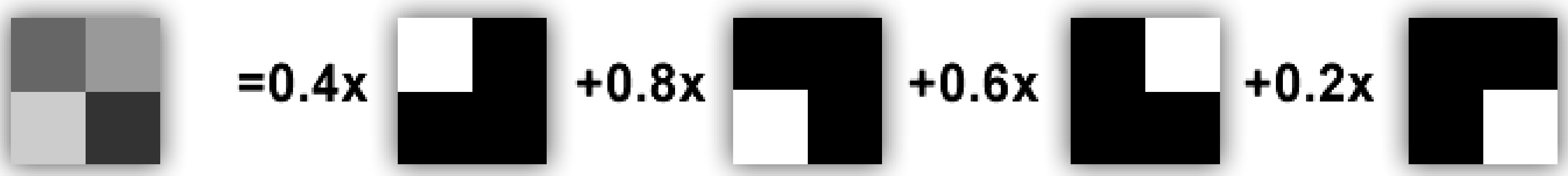
A simple example

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}$$



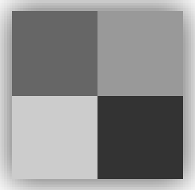
A simple example

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix} = 0.4\times\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.8\times\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0.6\times\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0.2\times\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



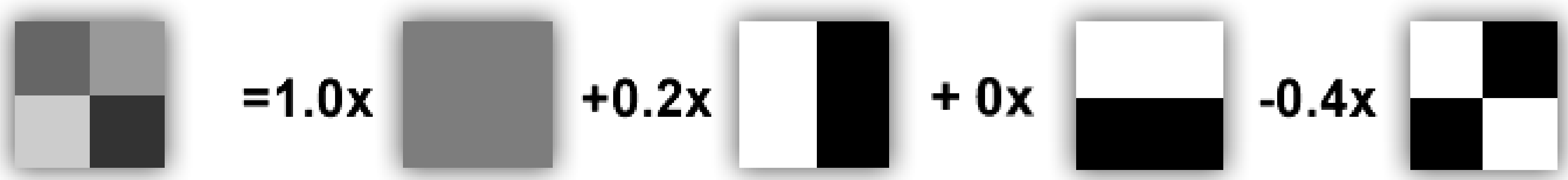
A simple example

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix}$$



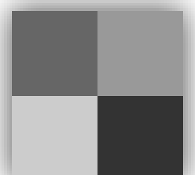
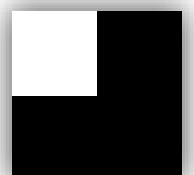
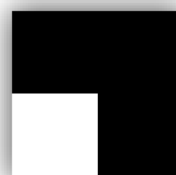
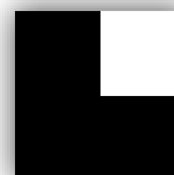
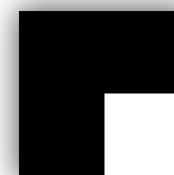
A simple example

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix} = 1.0 \times \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} + 0.2 \times \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix} + 0 \times \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix} - 0.4 \times \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

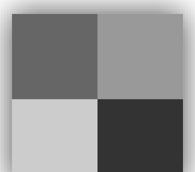
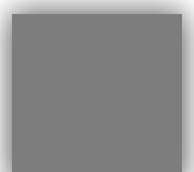
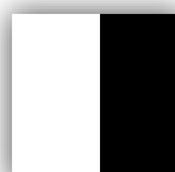
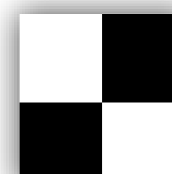


A simple example

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix} = 0.4\times\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0.8\times\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0.6\times\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0.2\times\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

 =0.4x  +0.8x  +0.6x  +0.2x 

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix} = 1.0\times\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} + 0.2\times\begin{bmatrix} 0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix} + 0\times\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix} - 0.4\times\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

 =1.0x  +0.2x  + 0x  -0.4x 

DCT Basis —— Fourier基底的一个变种

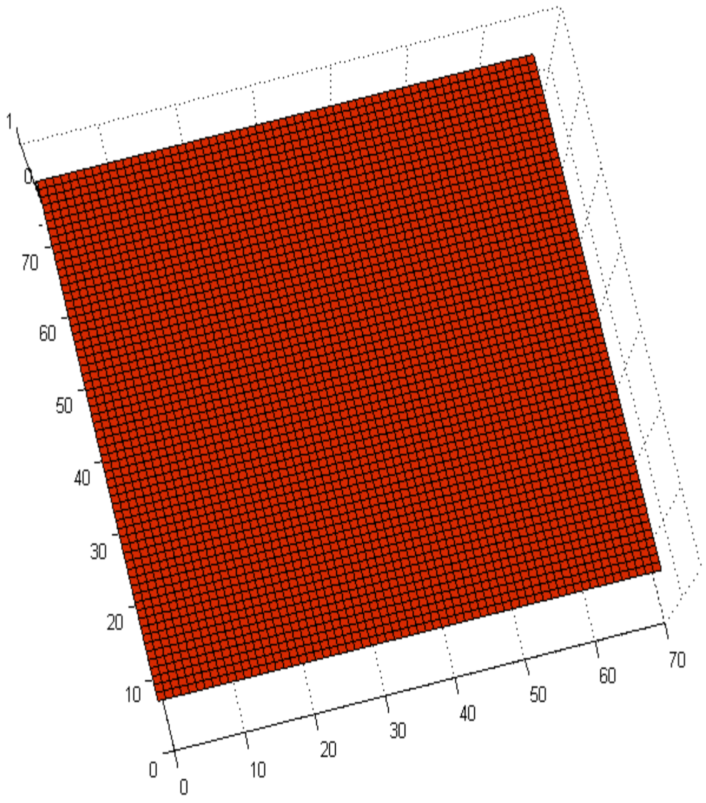
- $$DCT\_Basis_{u,v}(x,y)$$

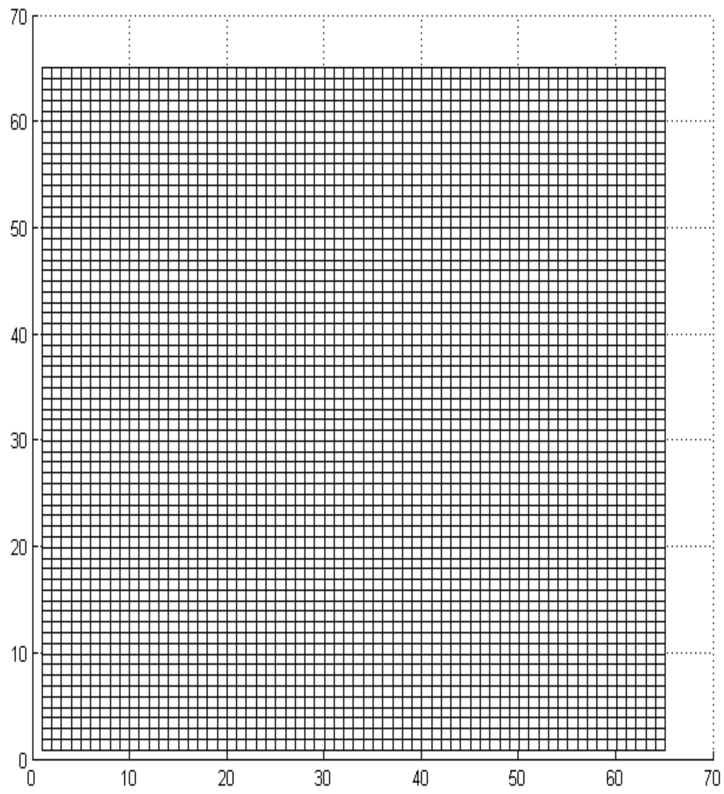
$$= \cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2M}\right]$$

$\cos\alpha\cos\beta = \frac{1}{2}(\cos\alpha + \cos\beta)$

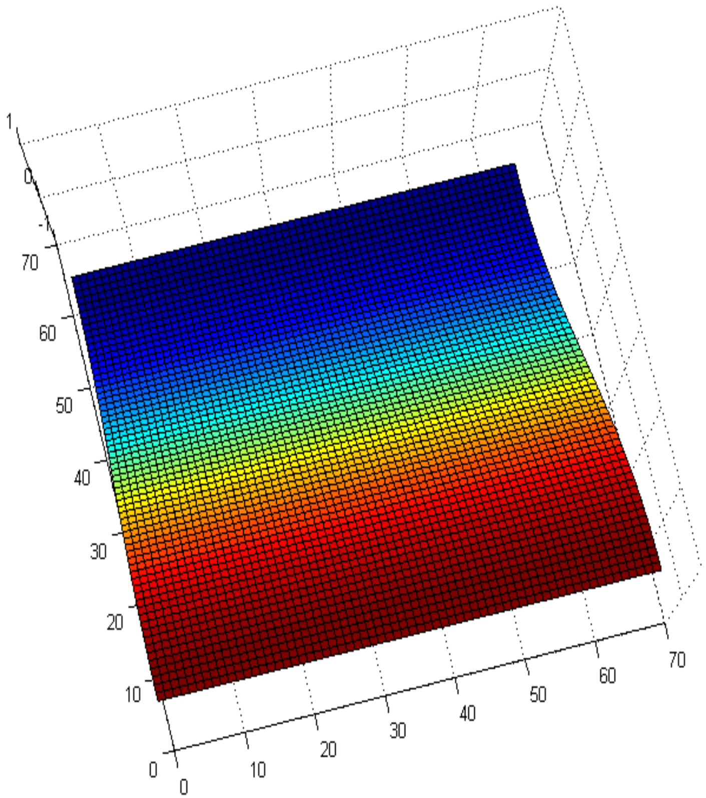
$$= \frac{1}{2}\cos\left[\frac{\pi(2x+1)Mu + \pi(2y+1)Nv}{2MN}\right] + \frac{1}{2}\cos\left[\frac{\pi(2x+1)Mu - \pi(2y+1)Nv}{2MN}\right]$$

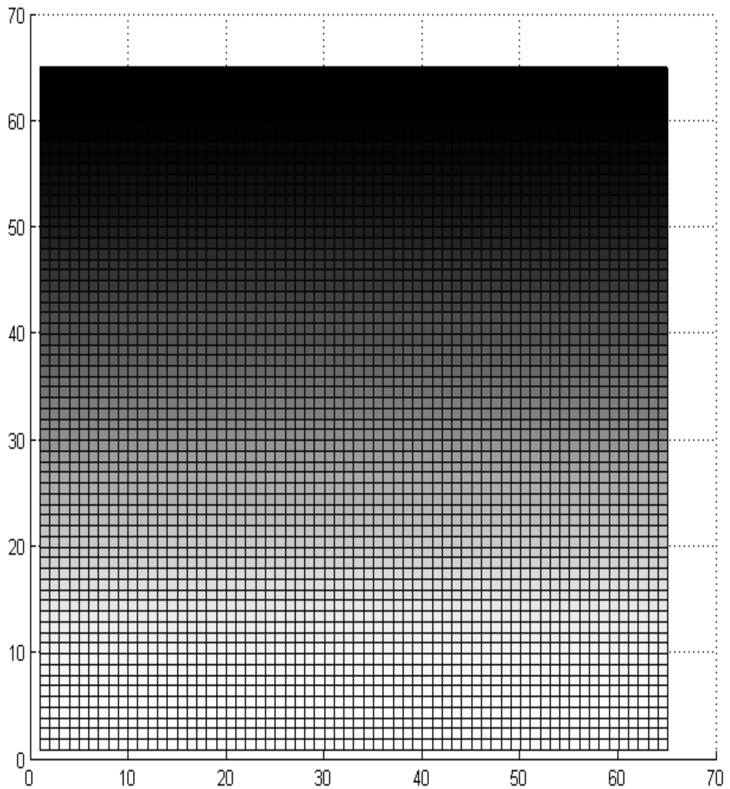
DCT Basis (u=0, v=0)



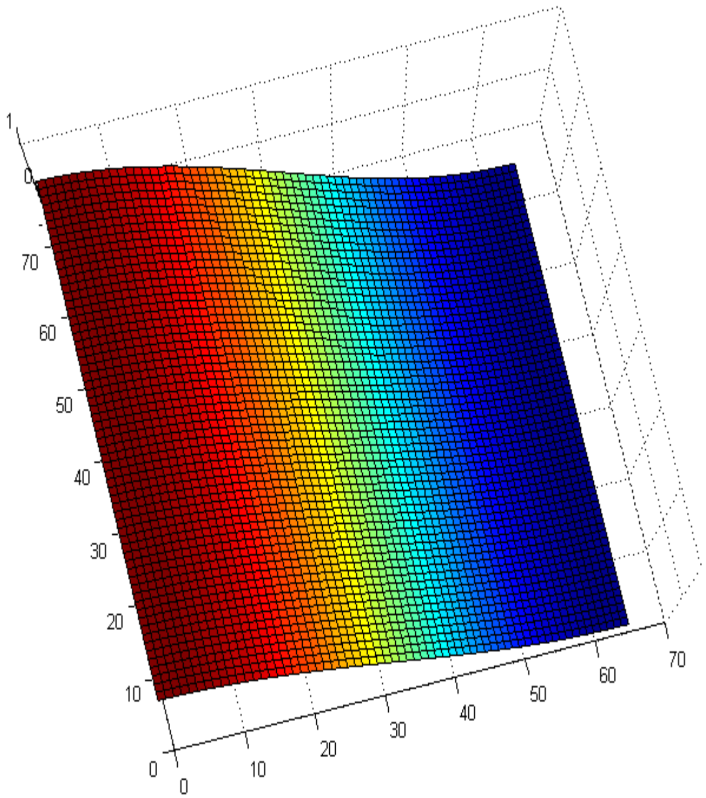


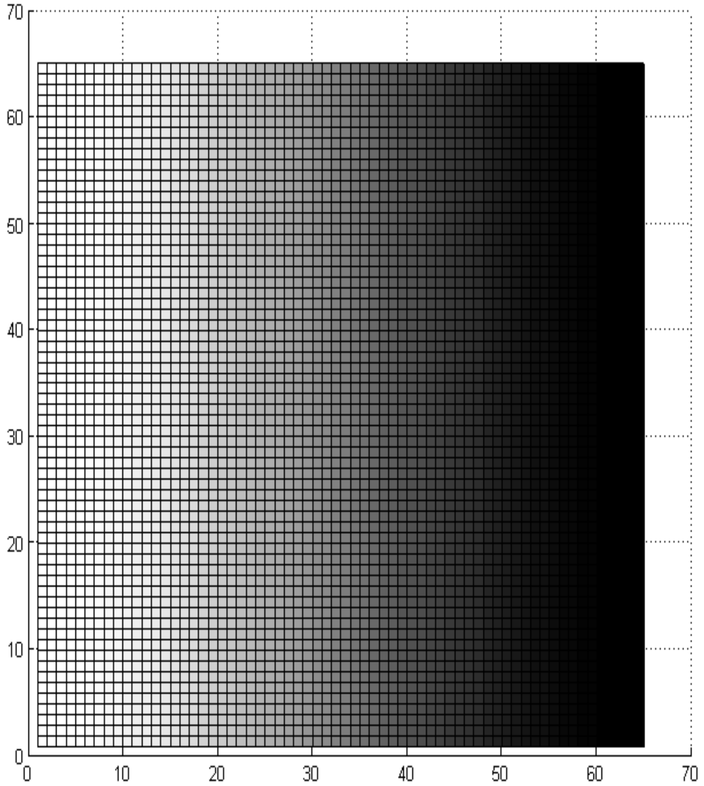
DCT Basis (u=1, v=0)



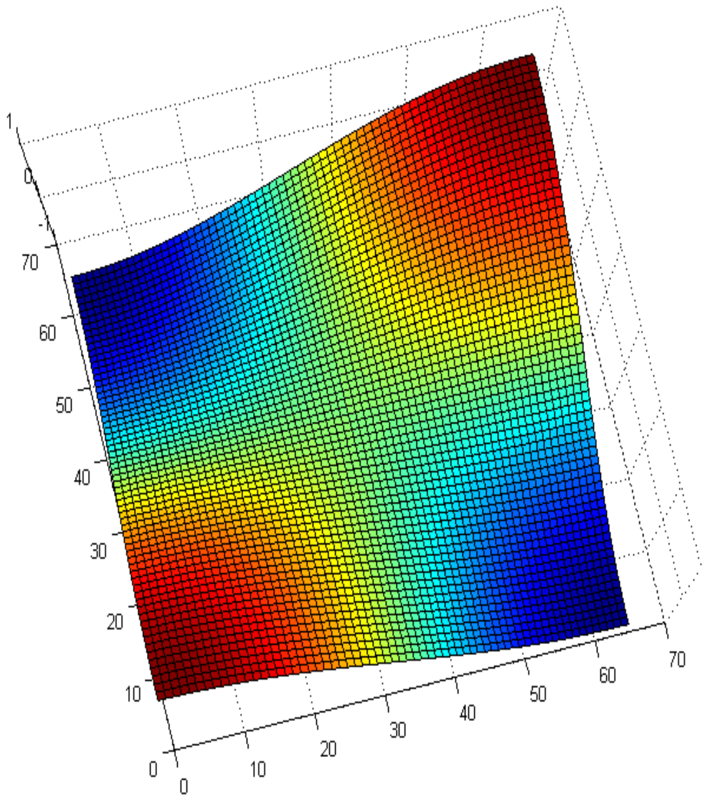


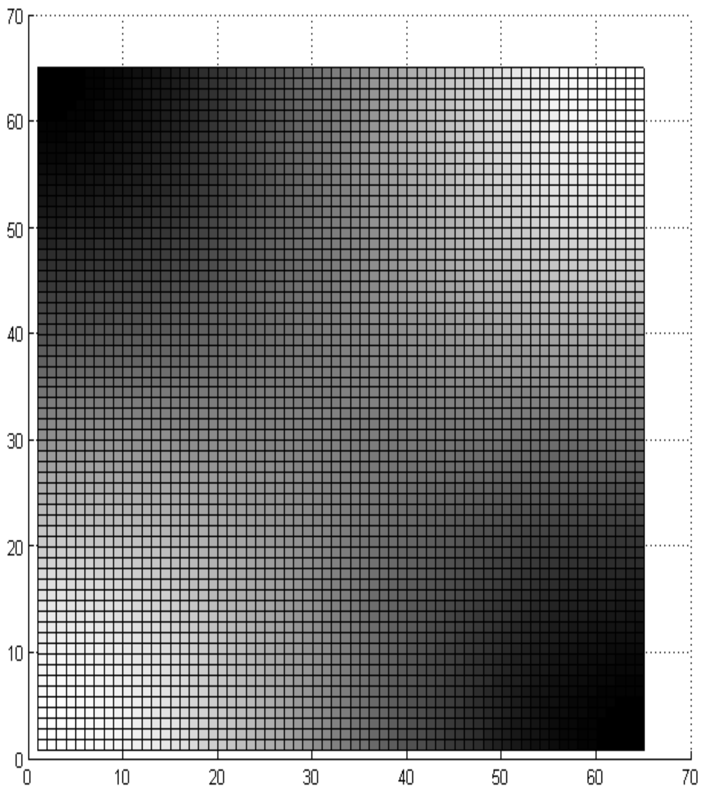
DCT Basis (u=0, v=1)



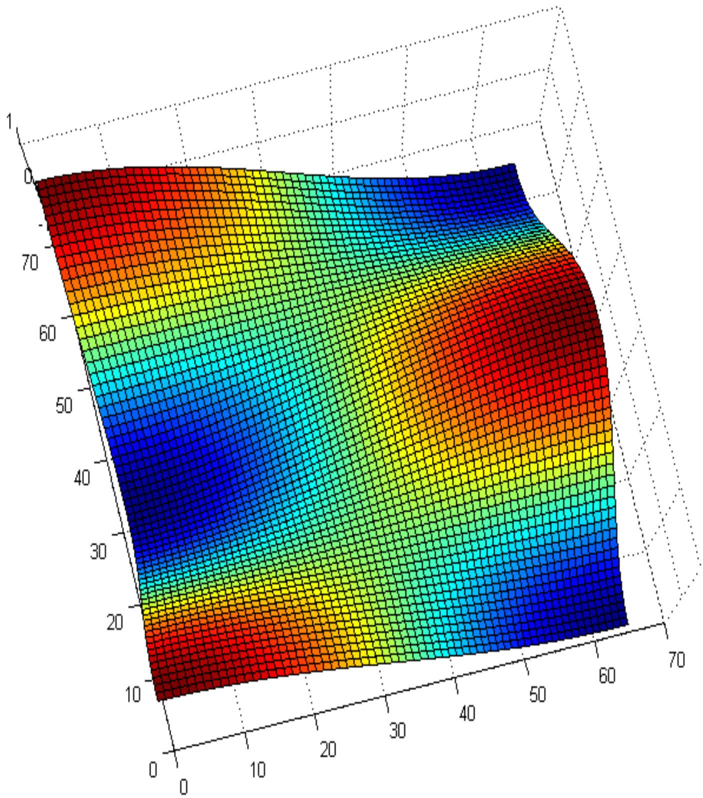


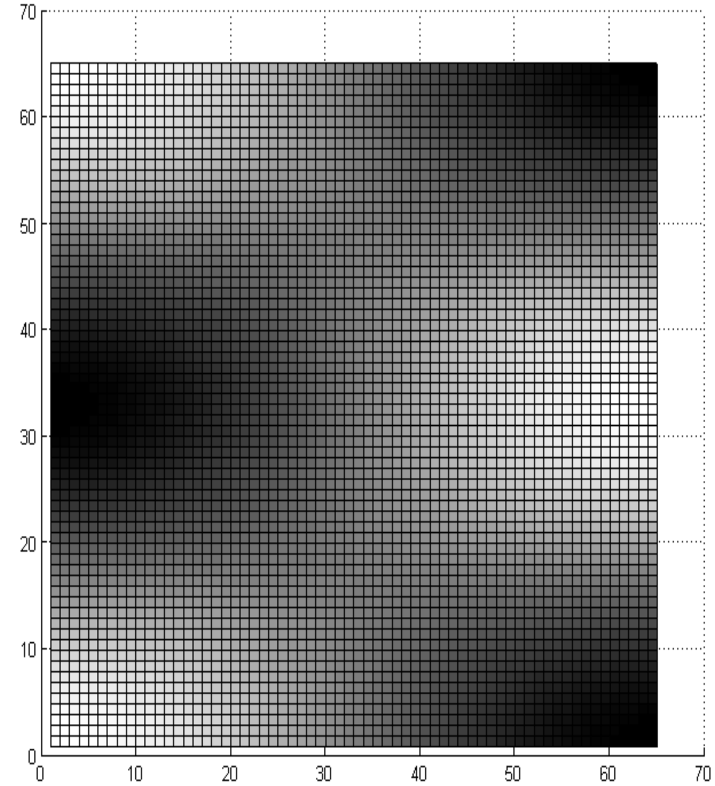
DCT Basis (u=1, v=1)



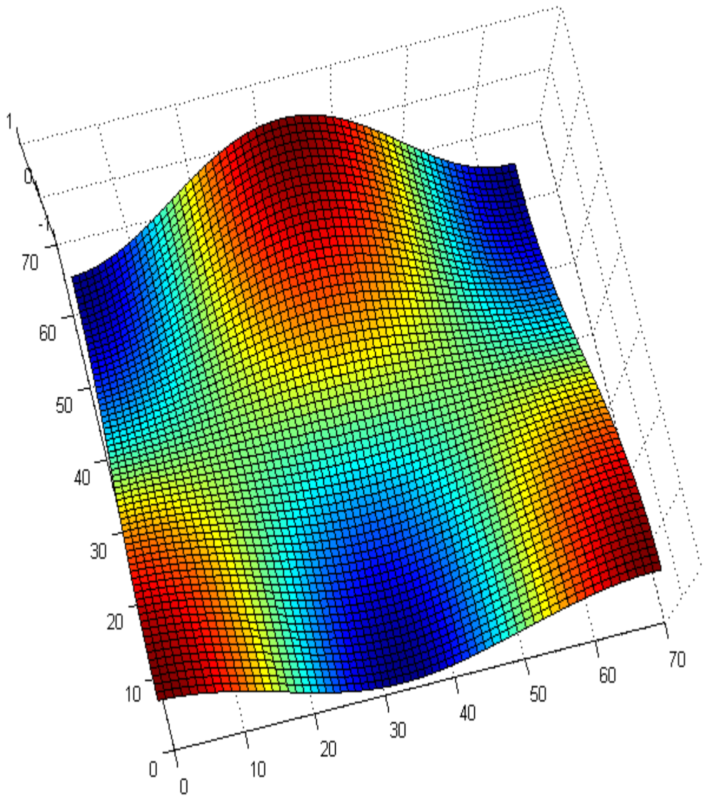


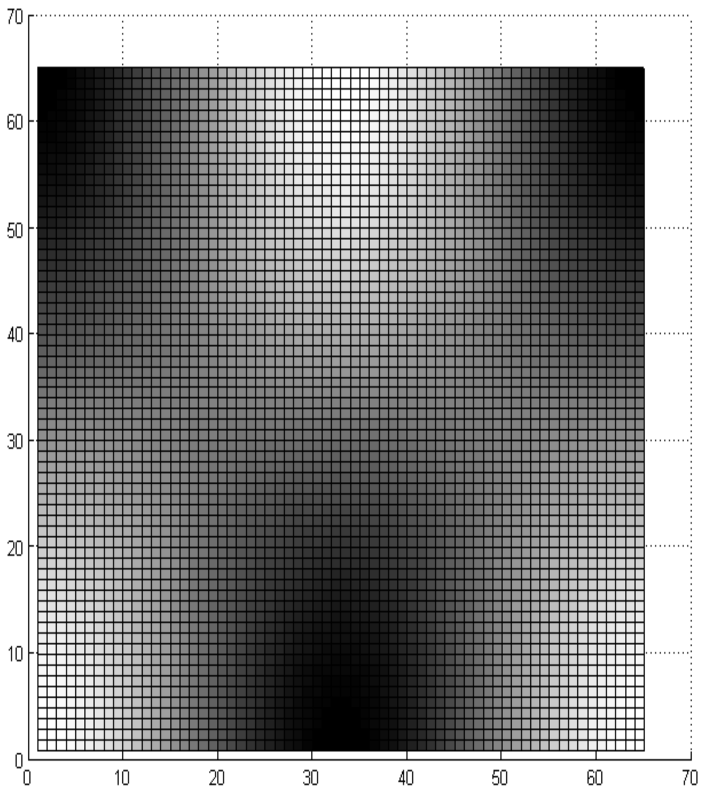
DCT Basis (u=2, v=1)



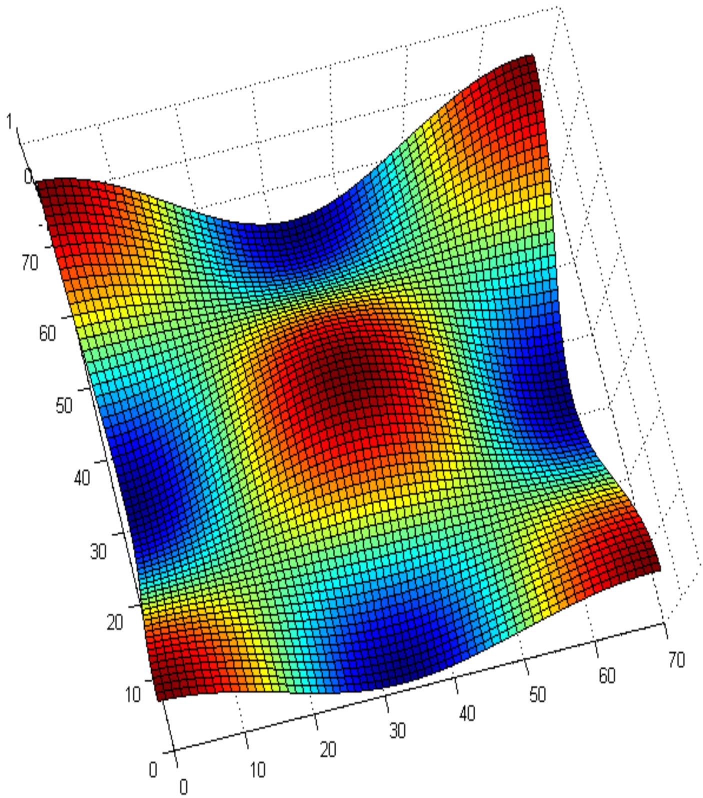


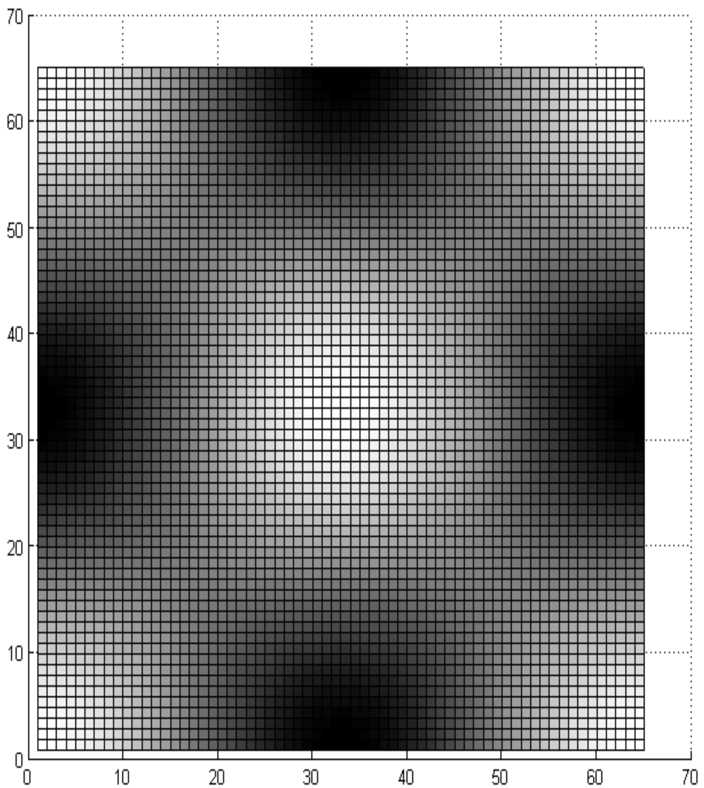
DCT Basis (u=1, v=2)



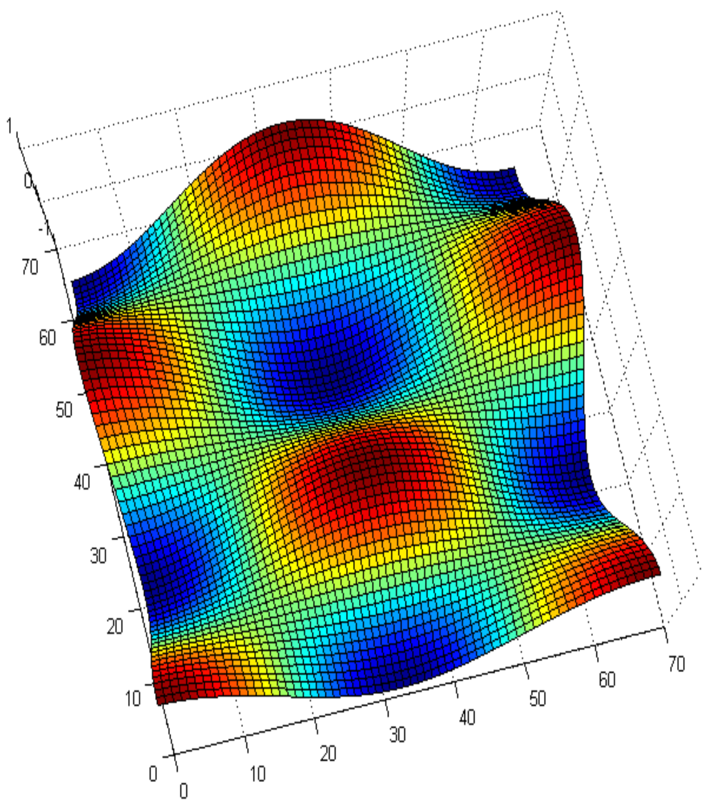


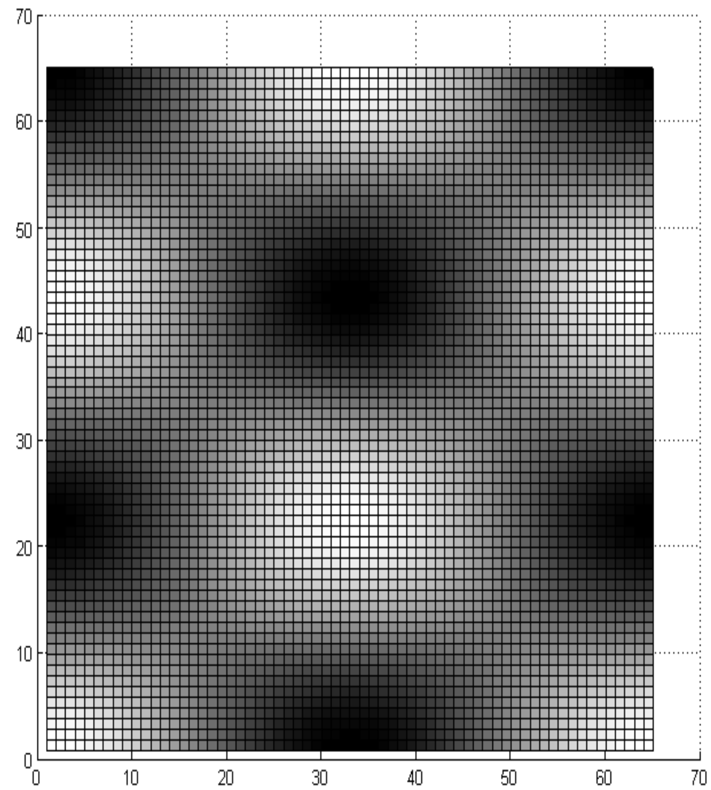
DCT Basis (u=2, v=2)





DCT Basis (u=3, v=2)





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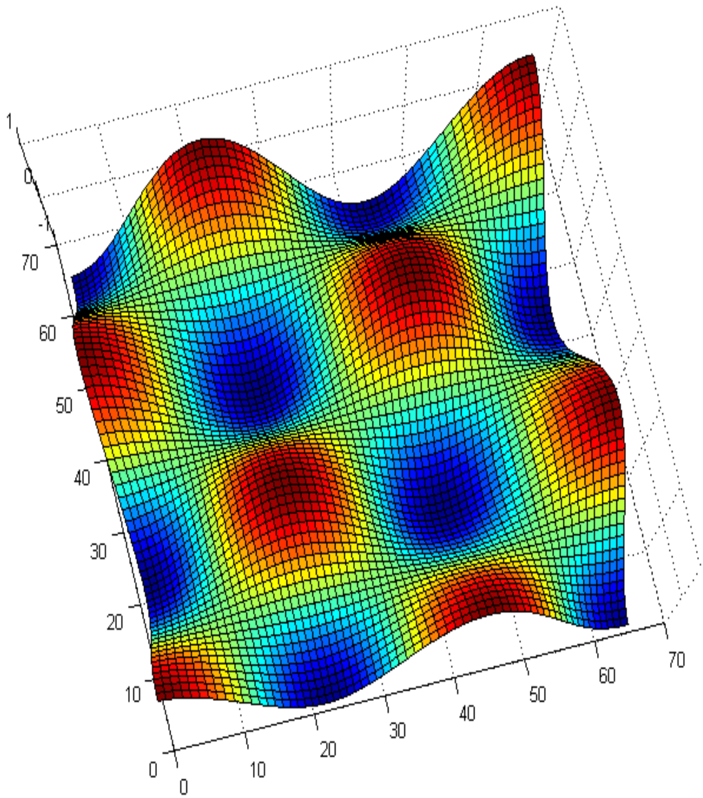
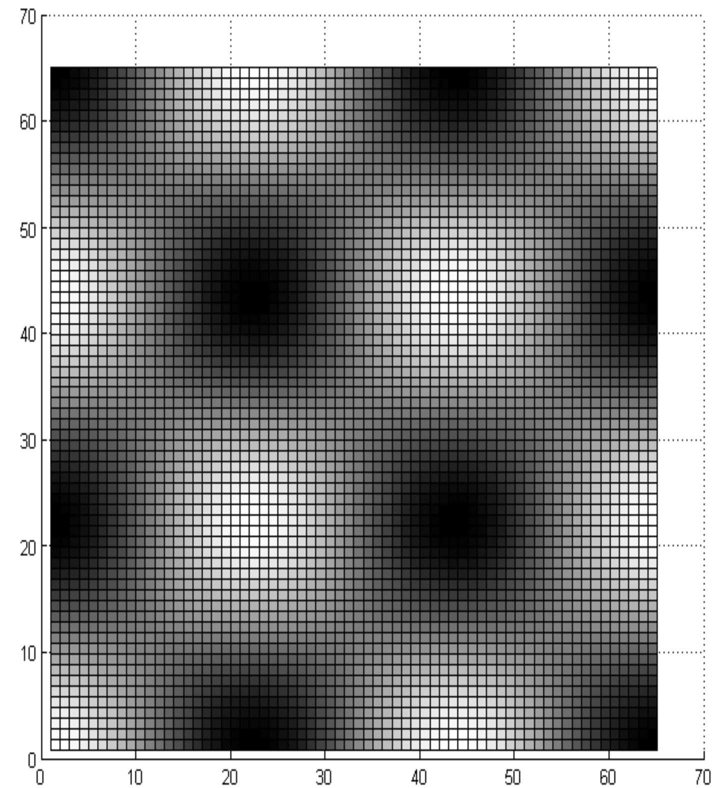
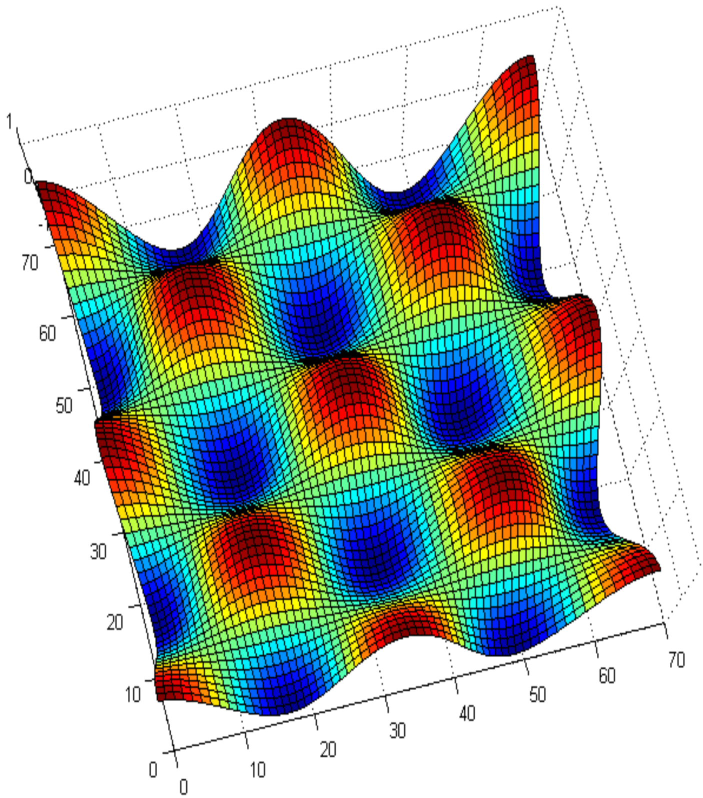
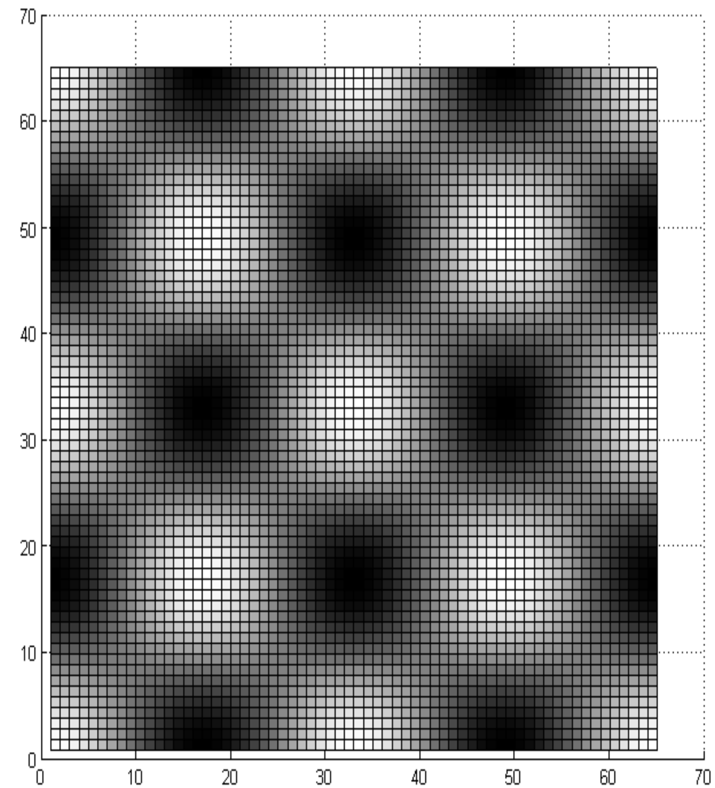
DCT Basis (u=3, v=3)



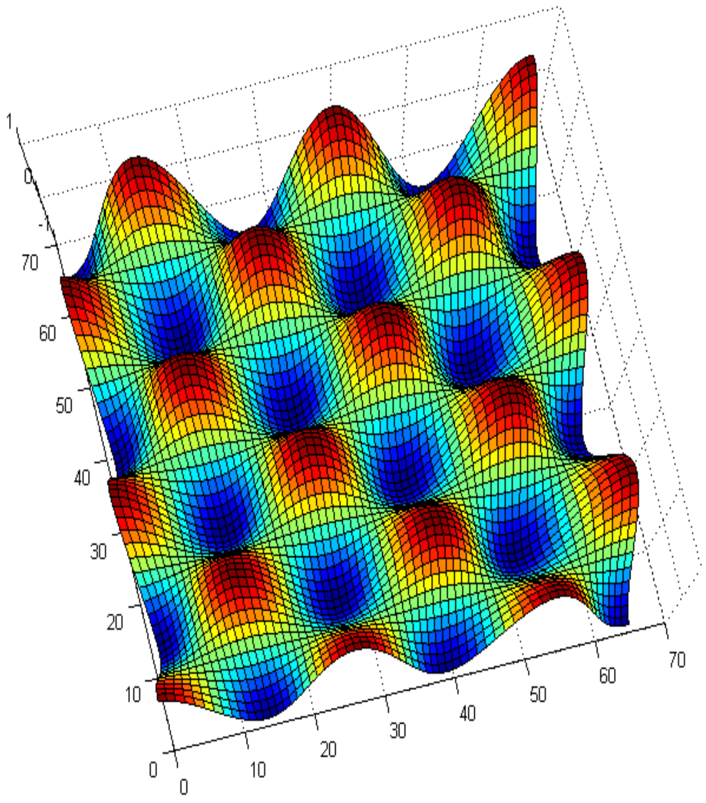


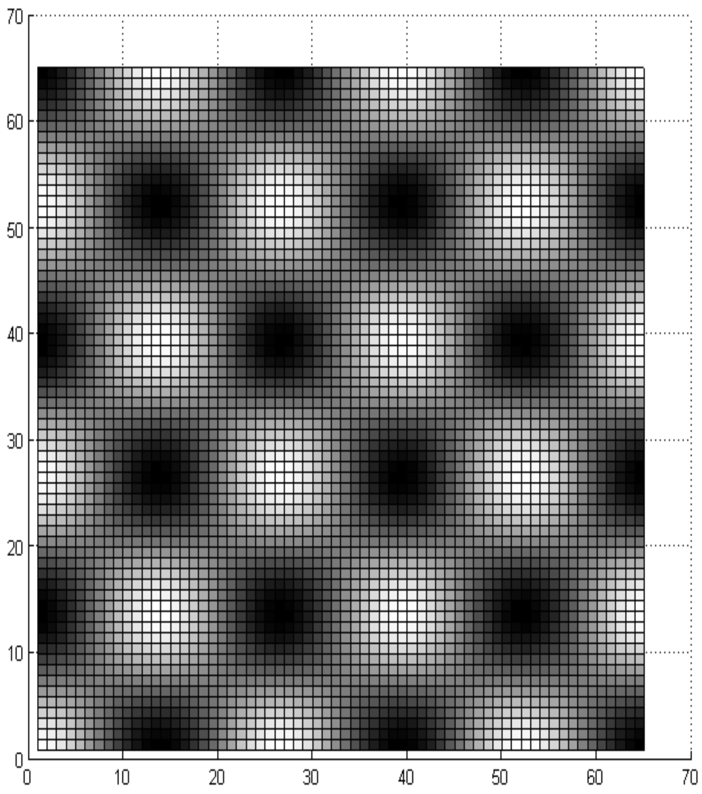
DCT Basis (u=4, v=4)



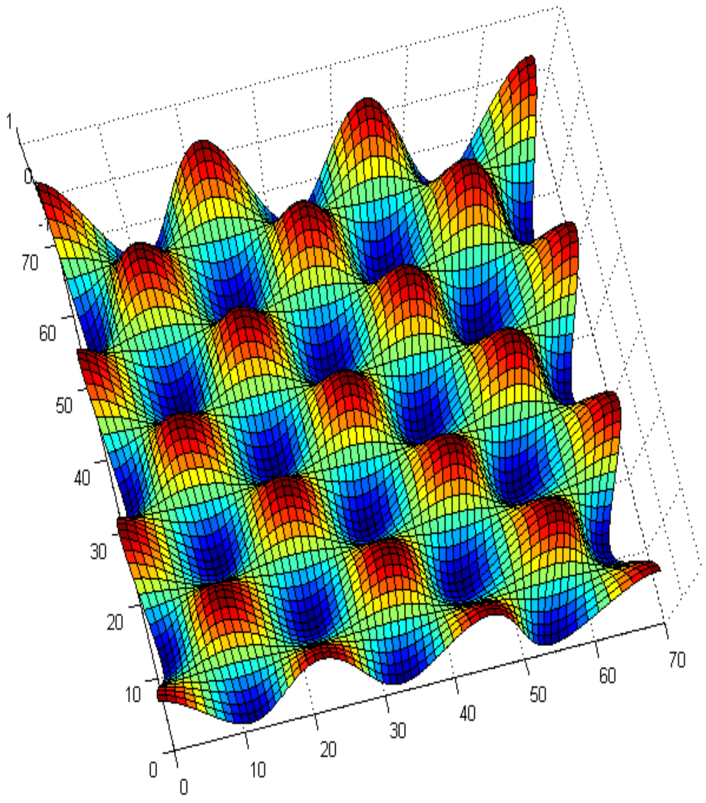


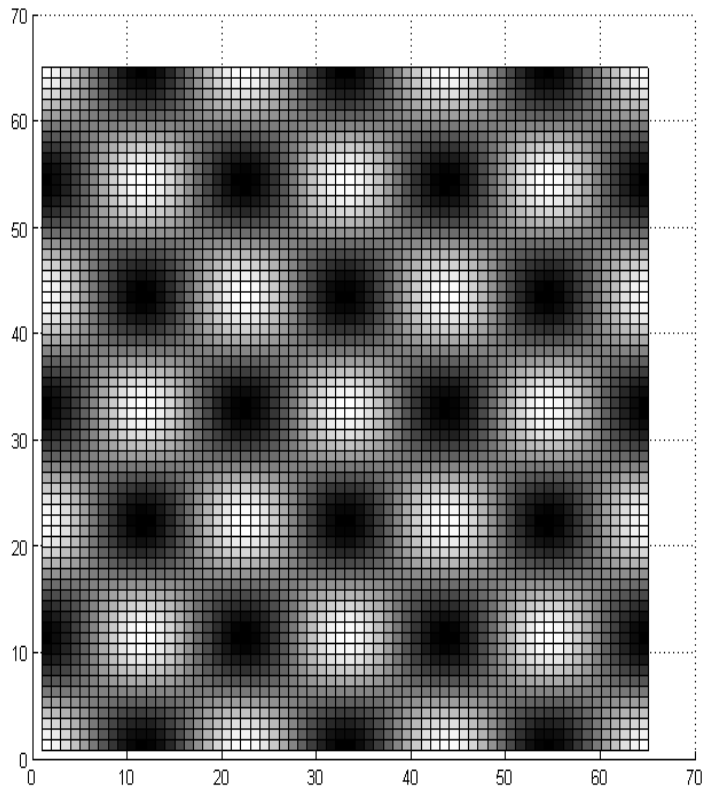
DCT Basis (u=5, v=5)



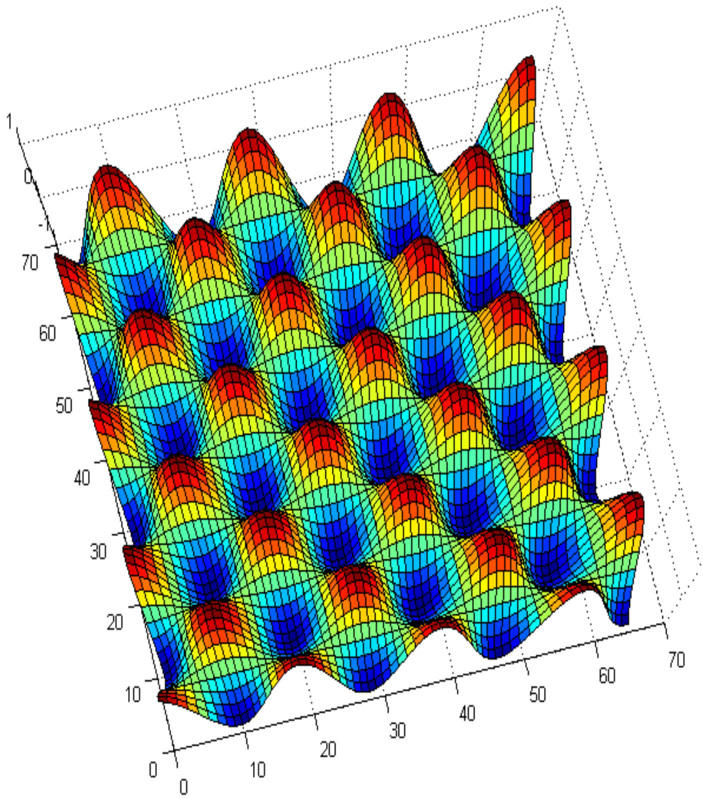


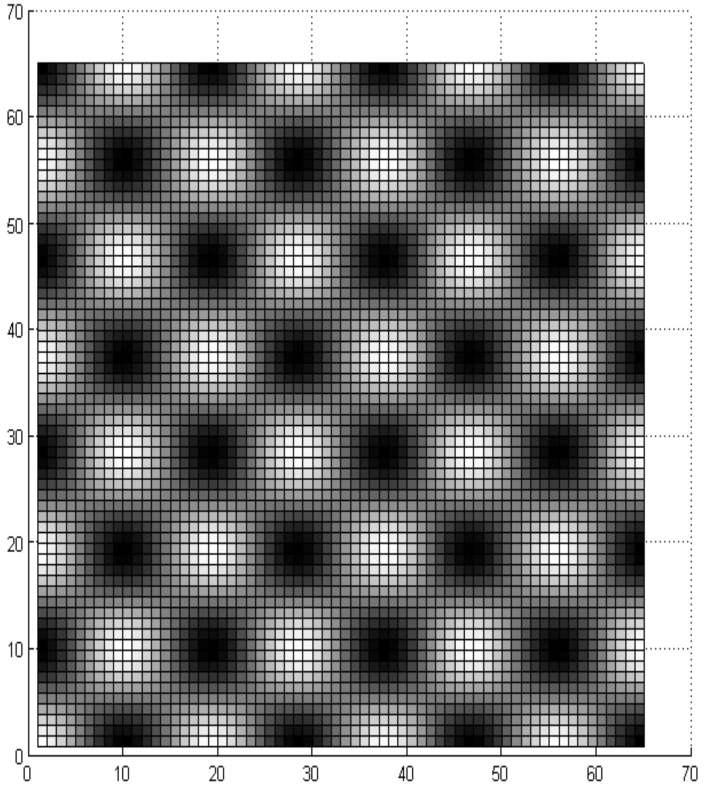
DCT Basis (u=6, v=6)



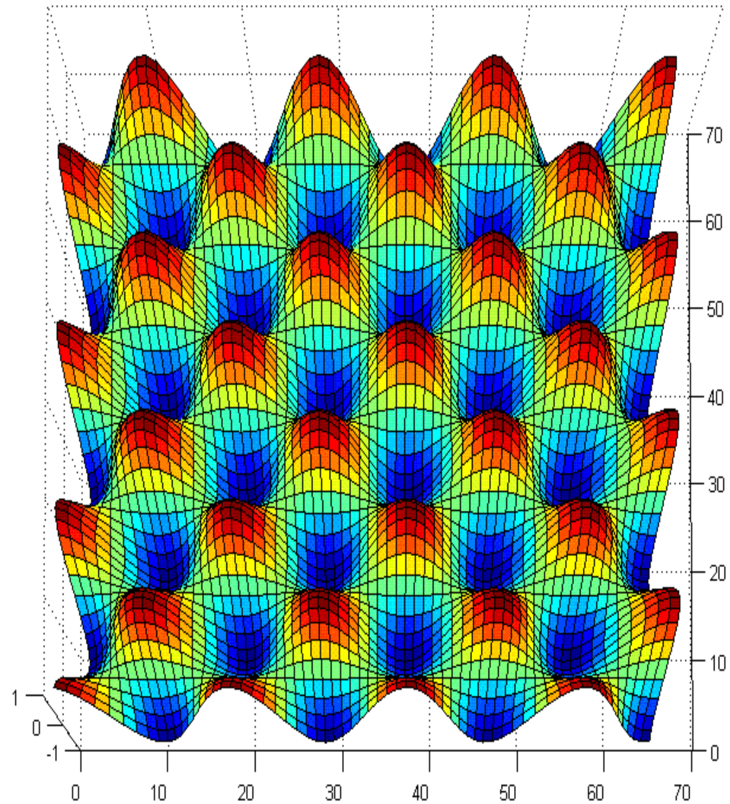


DCT Basis (u=7, v=7)



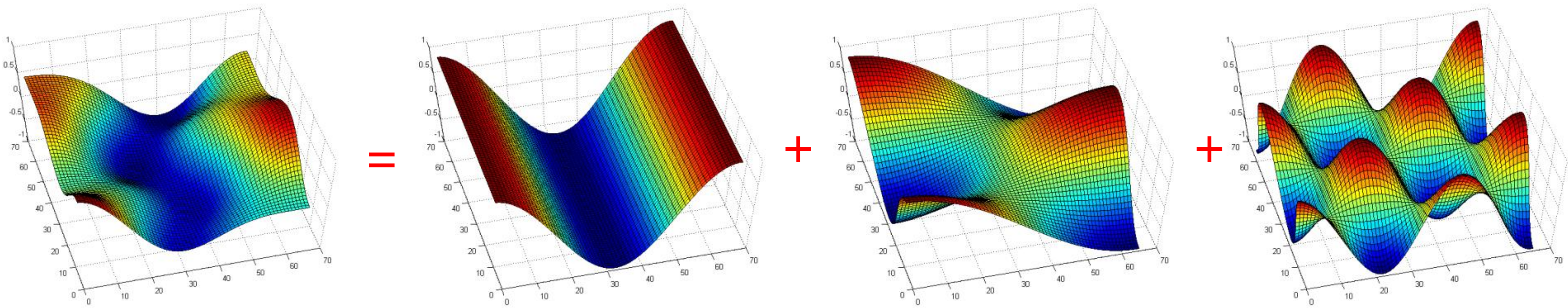


DCT Basis (u=7, v=7) Animation

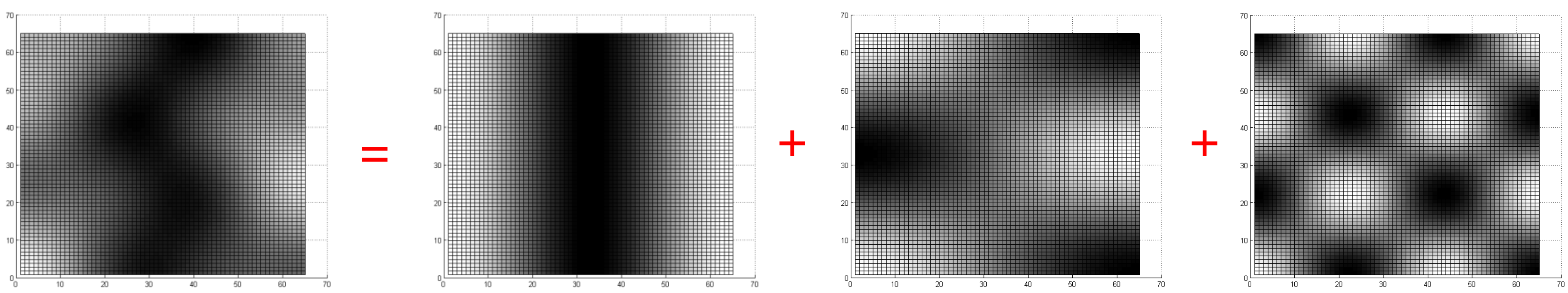


Example

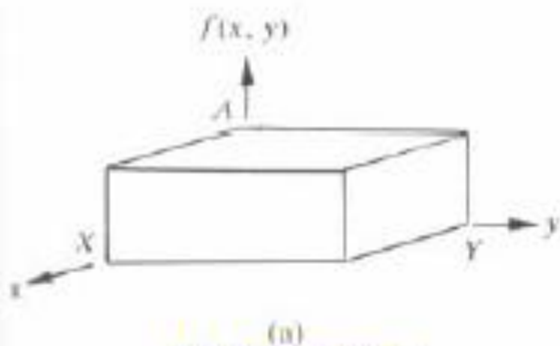
u=0, v=2 **u=2, v=1** **u=3, v=3**



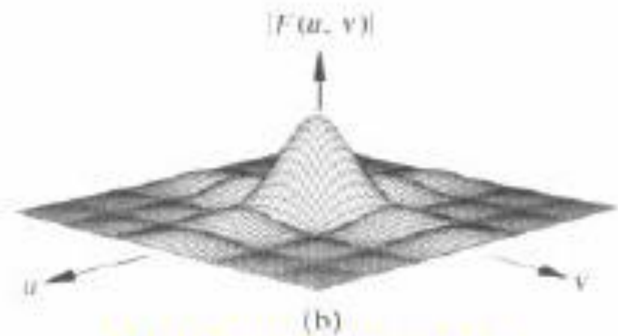
0.5 **0.3** **0.2**



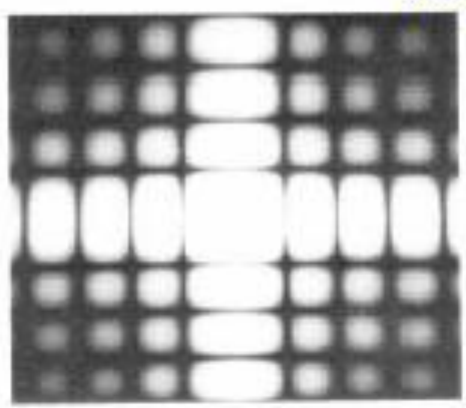
$$F(0,0)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)$$



A 2-D function



The Fourier spectrum



The spectrum displayed as an intensity function

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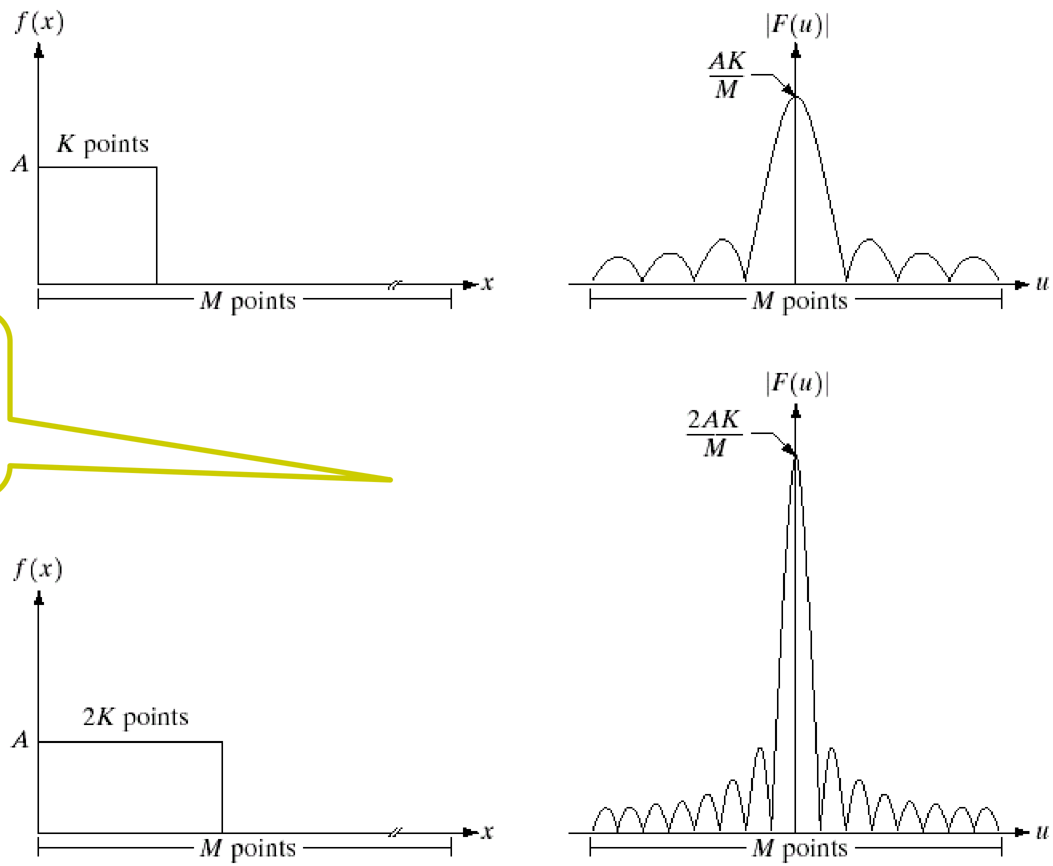
Example. Fourier spectrum of two simple functions



a b
c d

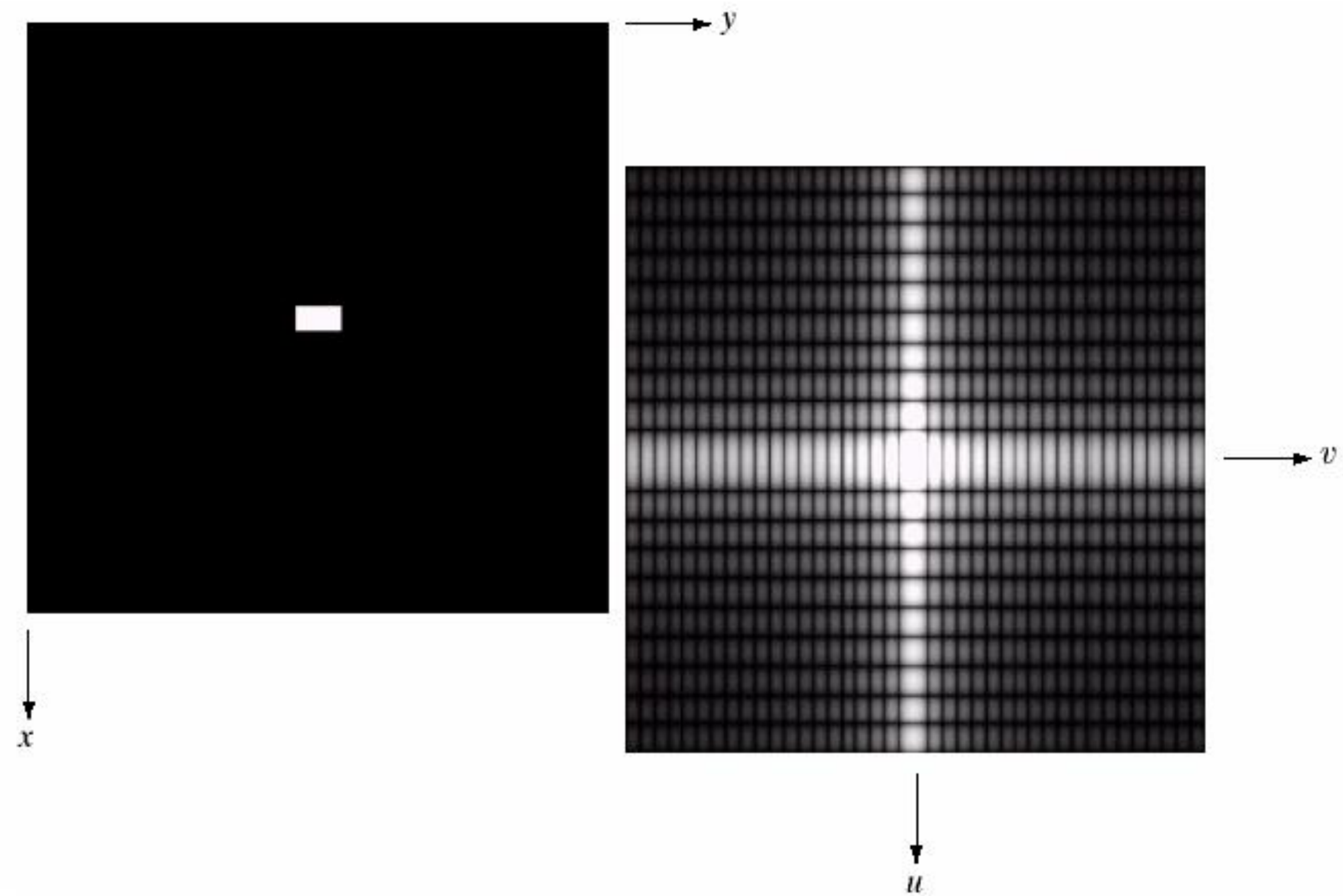
FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

自己做实验验证

特征:

(1) 当曲线下的面积在*x*域加倍时（整体能量增大），频率谱的“高度”也加倍;

(2) 当函数的长度加倍时, 相同长度区域内的零点数量也加倍. 极限情况?



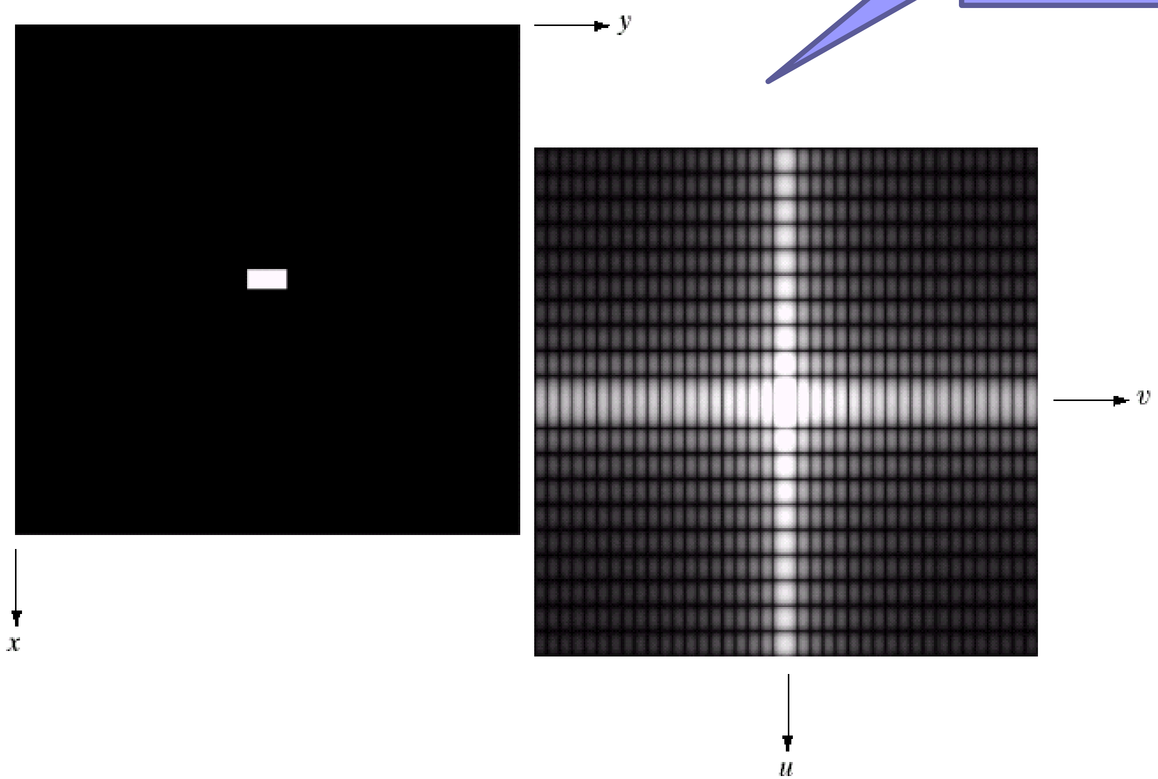
The signal is sharper in X direction, hence the spectrum is wider in u-direction

- **Display Fourier transform image**
  - Calculate Fourier transform and its spectrum of an image
  - Calculate *log transformation* of the spectrum
  - Scaling

谱图象的绘制还需注意另外的问题

a b

FIGURE 4.3
(a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels.
(b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



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- **Properties of DFT**
  - **Periodicity and conjugate symmetry**

Periodicity

$$F(u, v) = F(u+M, v) = F(u, v+N) = F(u+M, v+N)$$

Because of $e^{-j2\pi x} = 1$ and,

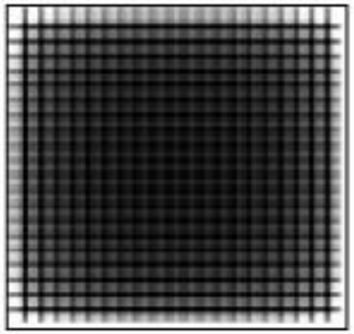
$$e^{-j2\pi((u+M)x/M+vy/N)} = e^{-j2\pi(ux/M+vy/N)}$$

Inverse DFT has the same property

$$f(x, y) = f(x+M, y) = f(x, y+N) = f(x+M, y+N)$$

$\triangle u=2\pi/N$, so the period = $N\triangle u=2\pi$. Generally, $F(u, v)$ is shown only for the range $[0, 2\pi]$

$f(x,y)$



$F(u,v)$

Properties of DFT

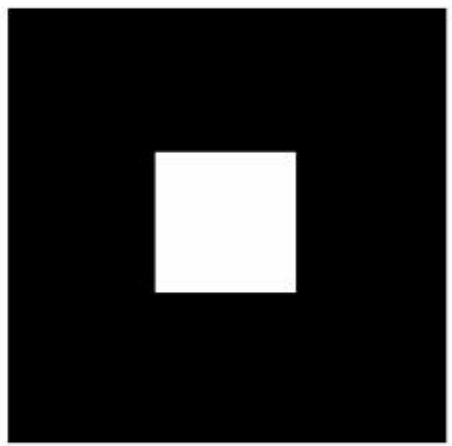
- **Conjugate symmetry（共轭对称）： if $f(x)$ is real**

$$F(u, v) = F^*(-u, -v)$$

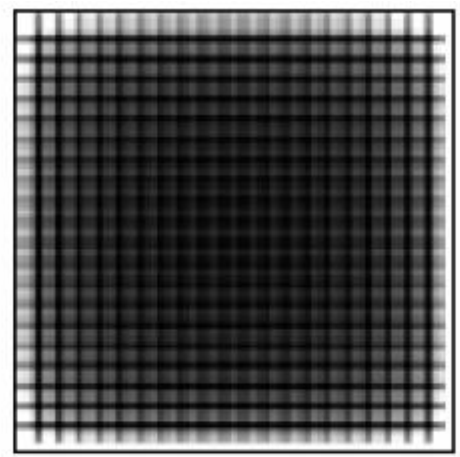
$$|F(u, v)| = |F(-u, -v)|$$

共轭的定义为: 若一个复数$Z = x + j\,y$, 则其共轭为$Z^* = x - jy$, 且 $|Z| = |Z^*|$.

white square



$f(x, y)$



$F(u, v)$

Properties of DFT

Translation

Translation in the image plane

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\, e^{-j2\pi(u_0x/M+v_0y/N)}$$

影响 phase
不影响 gain plot

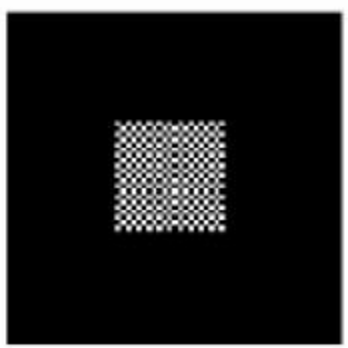
Translation in the frequency plane

$$f(x,y)\, e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

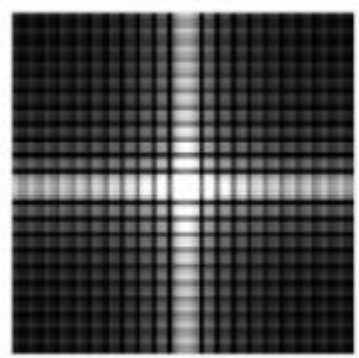
$$\mathcal{F}\left[f(x,y)(-1)^{x+y}\right] = F\left(u-M/2,\ v-N/2\right)$$

Due to periodicity, conjugate symmetry, usually image $f(x, y)$ is multiplied by $(-1)^{x+y}$ before DFT to shift the coordinate origin to $(u-M/2, v-N/2)$.

f(x, y) is a white square, so $f(x, y)(-1)^{x+y}$ = Chess board



$f(x,y)(-1)^{x+y}$



$F(u-N/2, v-N/2)$

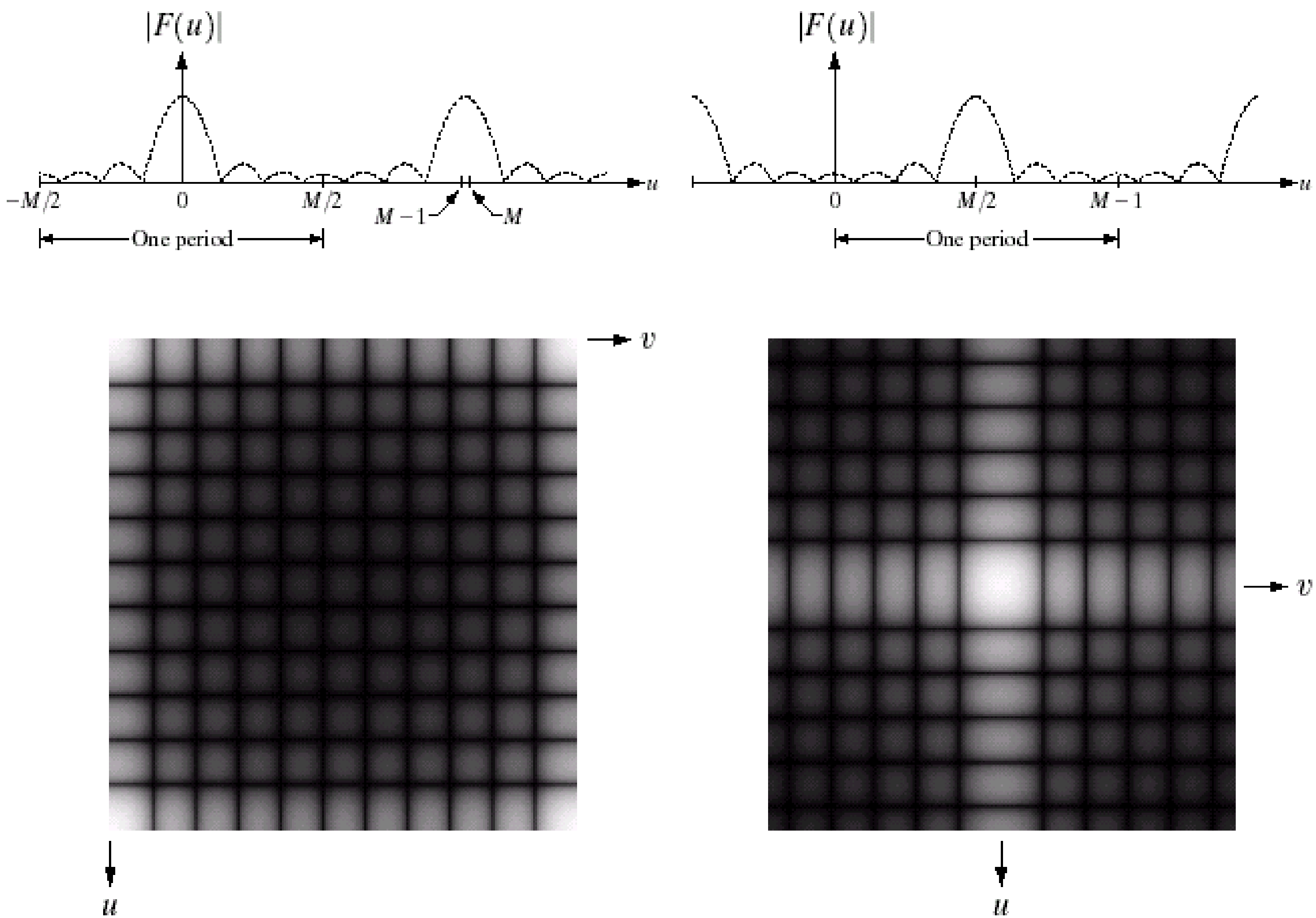
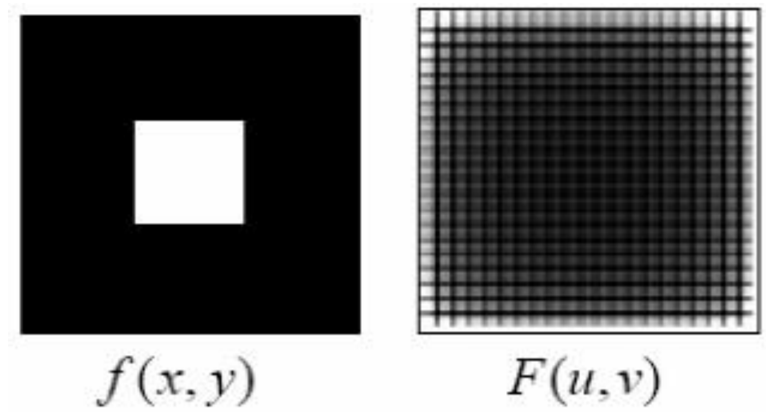
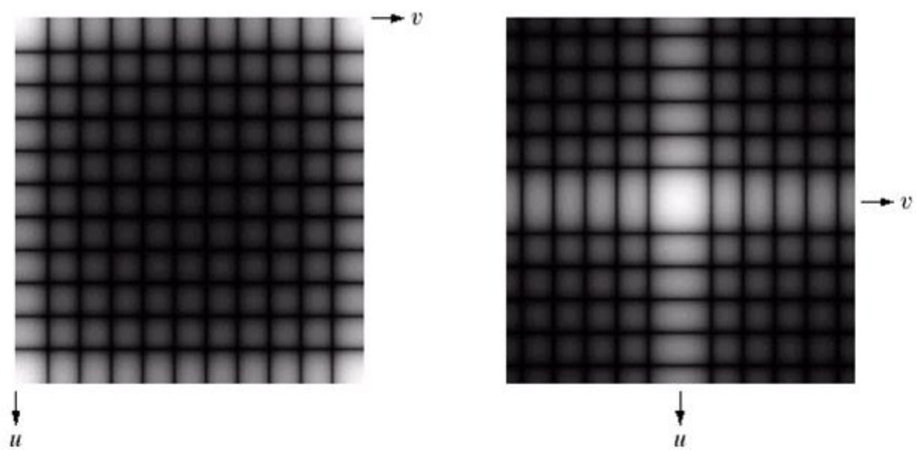


FIGURE 4.34 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Shifted spectrum showing a full period in the same interval. (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions. (d) Centered Fourier spectrum.

对频谱图像的认识?



应用FFT的性质



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	Spatial Domain[†]		Frequency Domain[†]
1)	$f(x, y)$ real	$\Leftrightarrow$	$F^*(u, v) = F(-u, -v)$
2)	$f(x, y)$ imaginary	$\Leftrightarrow$	$F^*(-u, -v) = -F(u, v)$
3)	$f(x, y)$ real	$\Leftrightarrow$	$R(u, v)$ even; $I(u, v)$ odd
4)	$f(x, y)$ imaginary	$\Leftrightarrow$	$R(u, v)$ odd; $I(u, v)$ even
5)	$f(-x, -y)$ real	$\Leftrightarrow$	$F^*(u, v)$ complex
6)	$f(-x, -y)$ complex	$\Leftrightarrow$	$F(-u, -v)$ complex
7)	$f^*(x, y)$ complex	$\Leftrightarrow$	$F^*(-u - v)$ complex
8)	$f(x, y)$ real and even	$\Leftrightarrow$	$F(u, v)$ real and even
9)	$f(x, y)$ real and odd	$\Leftrightarrow$	$F(u, v)$ imaginary and odd
10)	$f(x, y)$ imaginary and even	$\Leftrightarrow$	$F(u, v)$ imaginary and even
11)	$f(x, y)$ imaginary and odd	$\Leftrightarrow$	$F(u, v)$ real and odd
12)	$f(x, y)$ complex and even	$\Leftrightarrow$	$F(u, v)$ complex and even
13)	$f(x, y)$ complex and odd	$\Leftrightarrow$	$F(u, v)$ complex and odd

[†]Recall that x, y, u, and v are *discrete* (integer) variables, with x and u in the range $[0, M - 1]$, and y, and v in the range $[0, N - 1]$. To say that a complex function is *even* means that its real *and* imaginary parts are even, and similarly for an odd complex function.

TABLE 4.1 Some symmetry properties of the 2-D DFT and its inverse. $R(u, v)$ and $I(u, v)$ are the real and imaginary parts of $F(u, v)$, respectively. The term *complex* indicates that a function has nonzero real and imaginary parts.

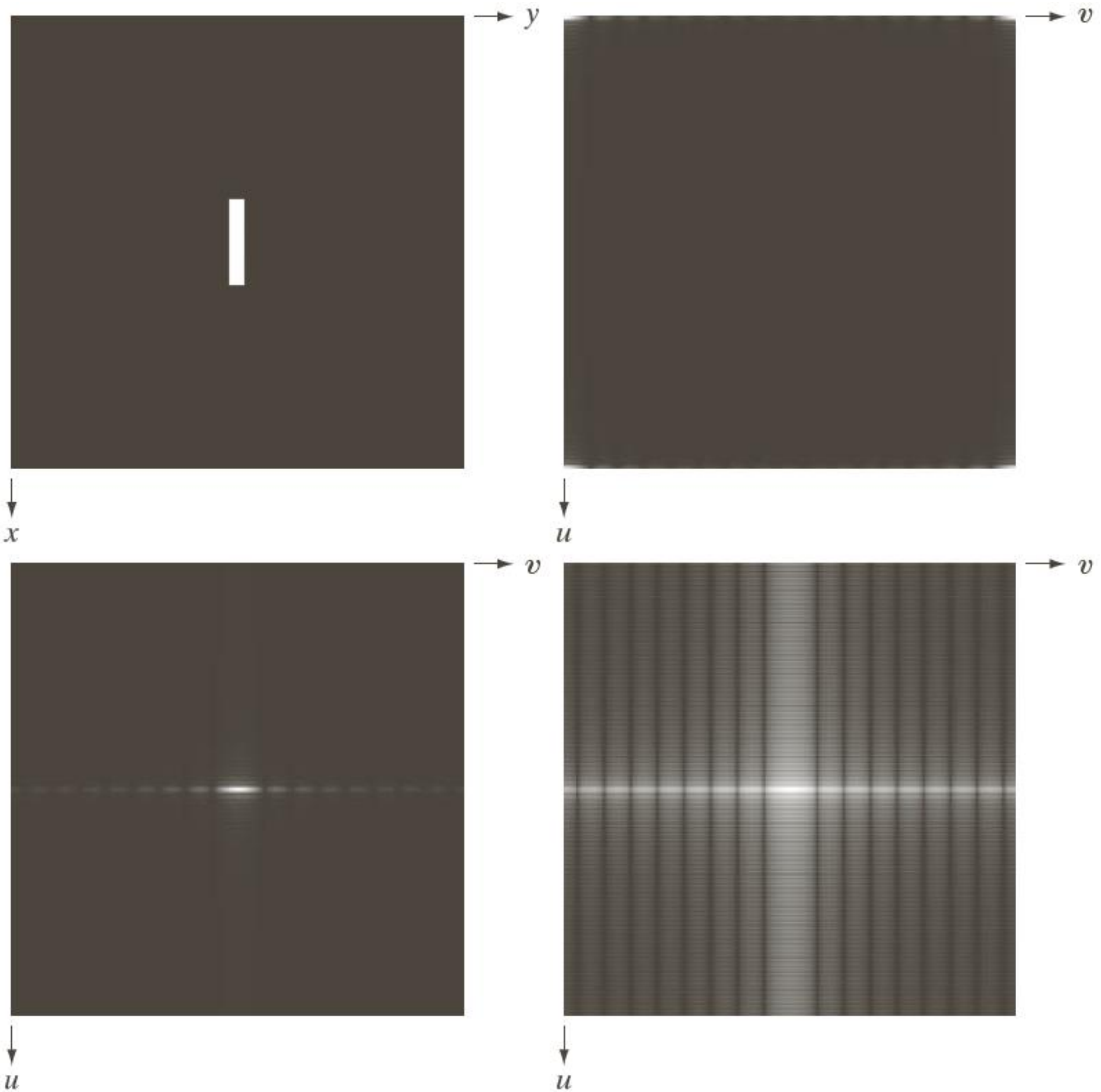
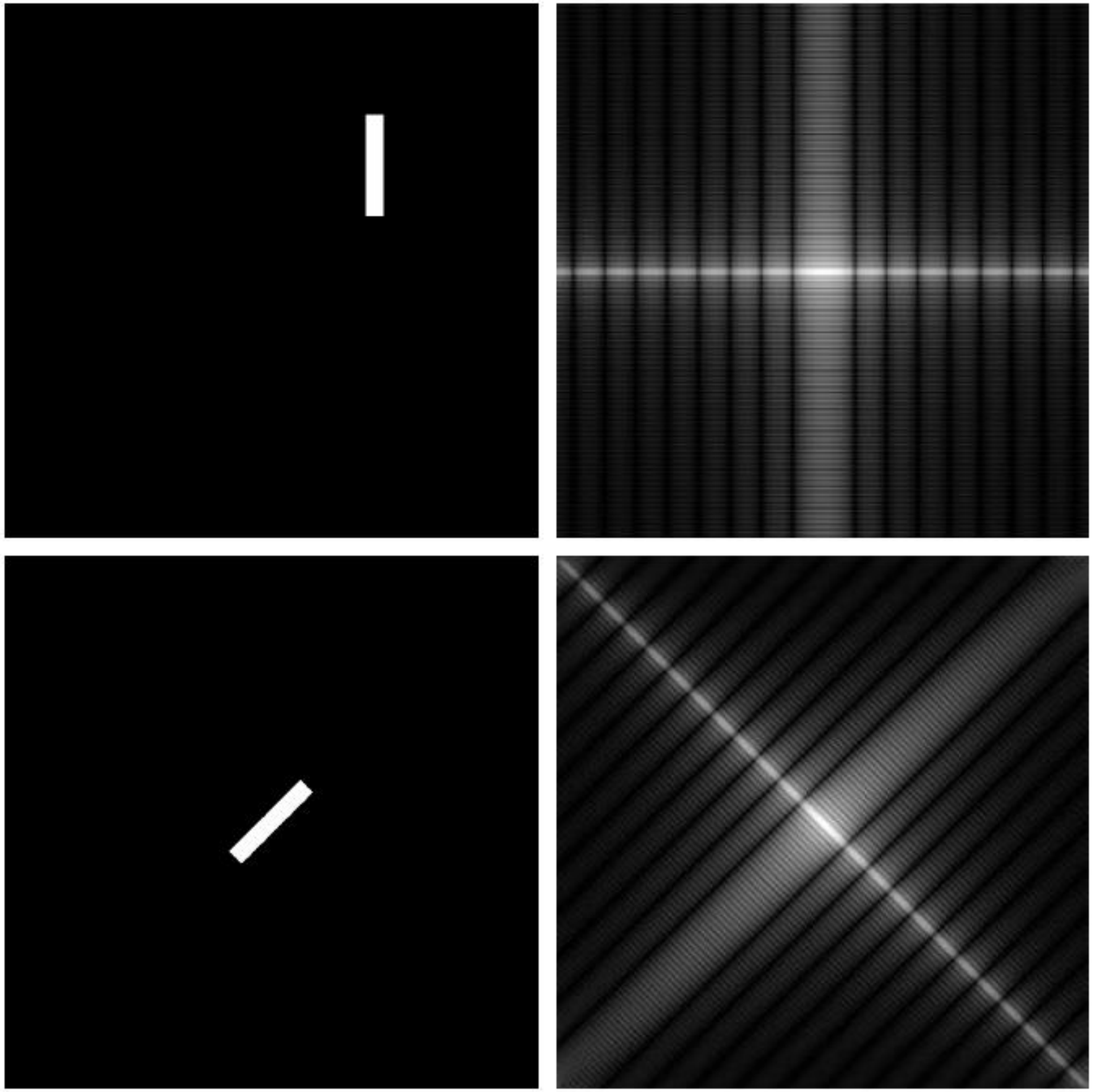
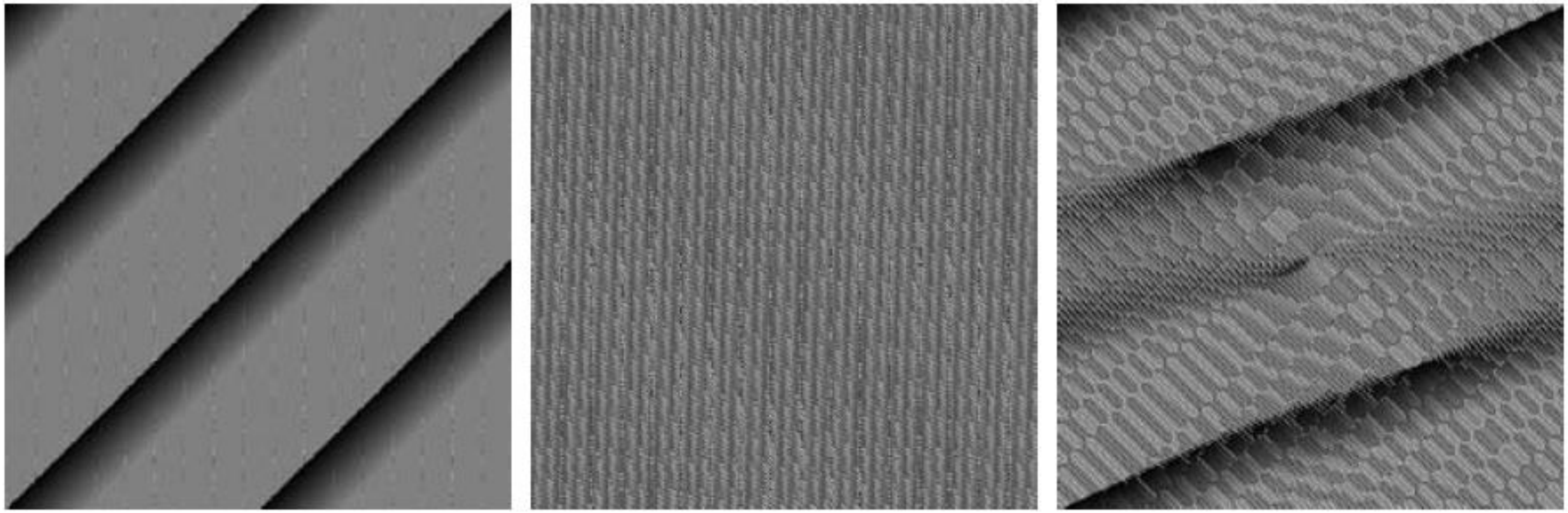


FIGURE 4.24 (a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.



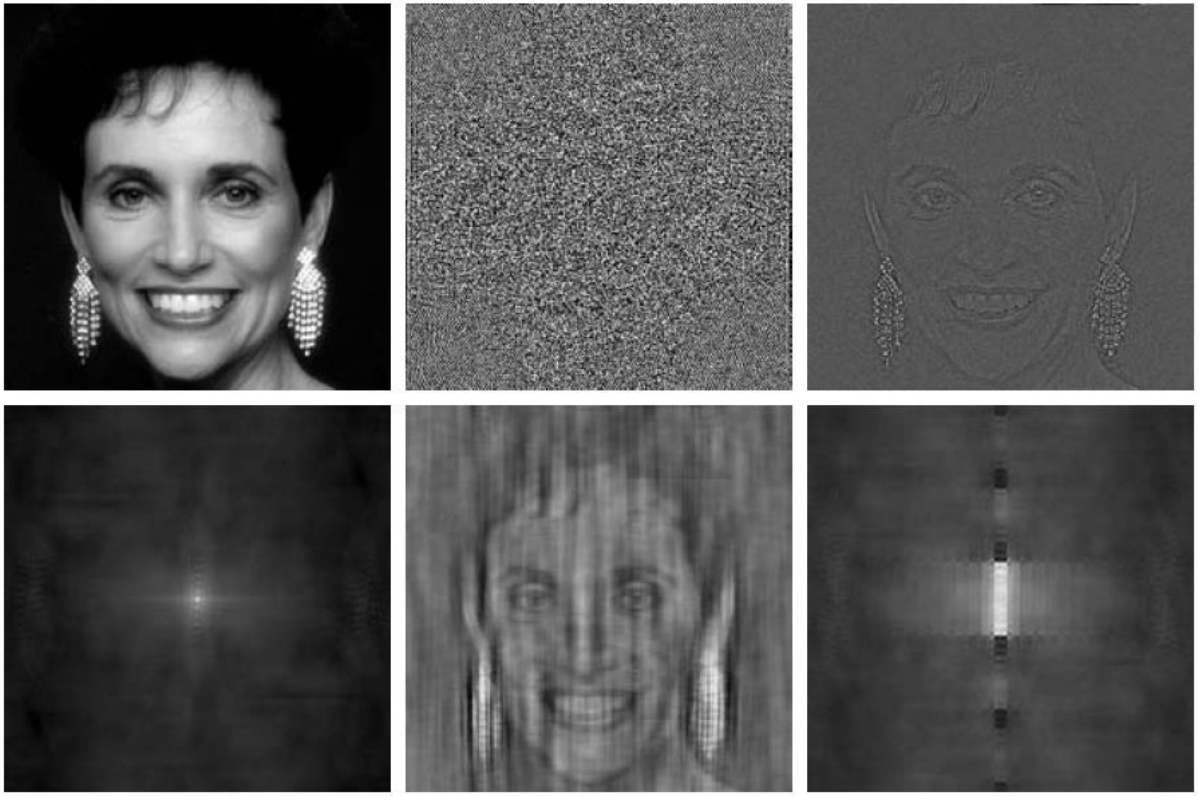
a	b
c	d

FIGURE 4.25
(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).



a b c

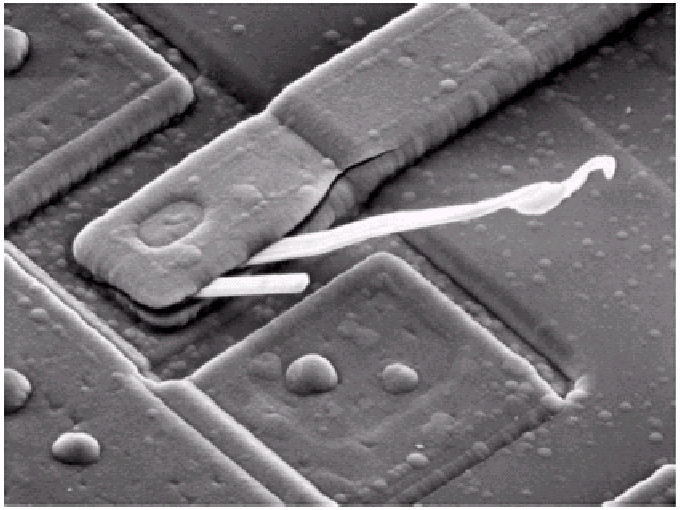
FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

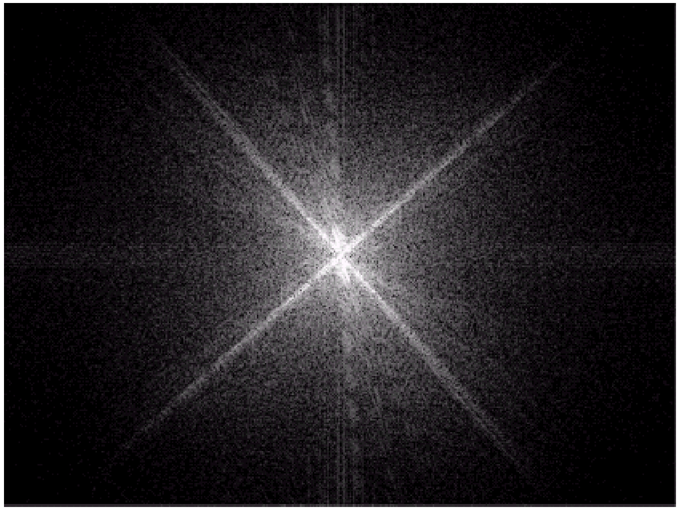


a	b	c
d	e	f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

例4.3 集成电路的扫描电子显微镜图像, 放大2500倍.





注意: ±45°角的两个强边缘和热感应不足引起的两个白色氧化突起.

高频和低频部分, 能量分布的一般情况.

频率域的基本性质

傅里叶变换每个频谱分量都要涉及到图像空间中所有的象素, 所以一般来说, 频谱信息中很难看出空间的信息. 但由于频率反映的是空间强度的变化率, 如低频对应着图像变化慢的部分, 高频对应着图像变化快的部分. 所以, 在某种意义上两者之间仍然有不可分割的联系, 尽管这些联系是"总体"的.

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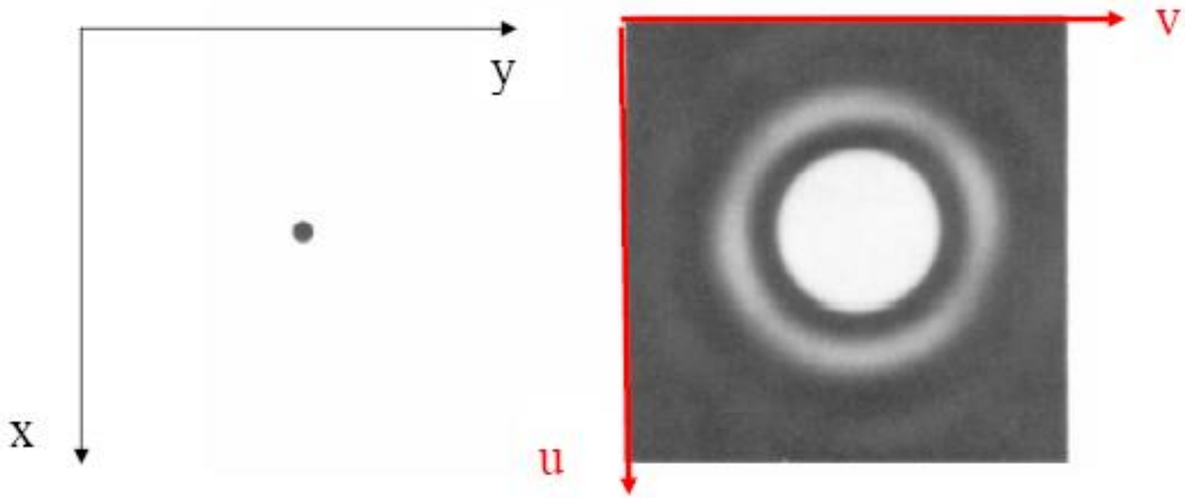
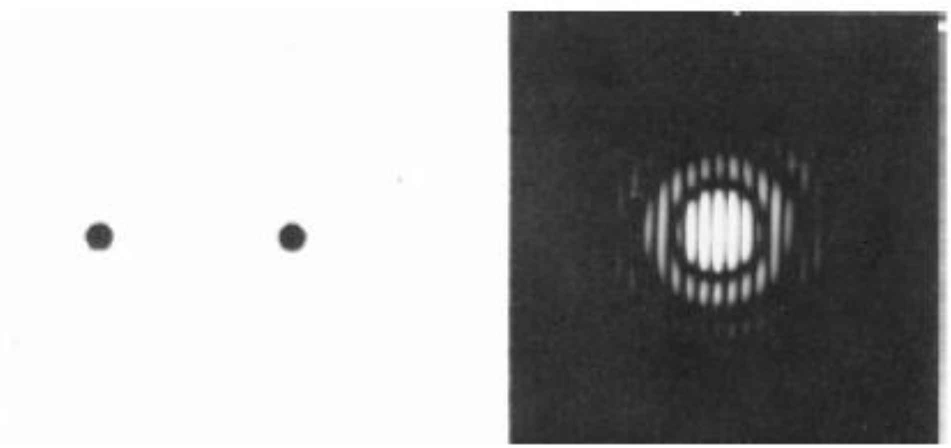
Examples of DFT





Identical gain plot with phase difference to result in ‘interference pattern’ similar to wave interference

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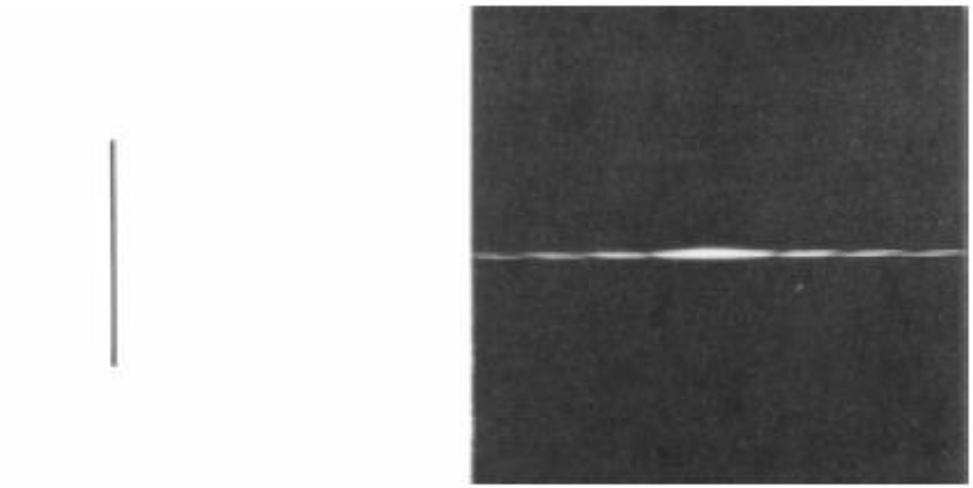
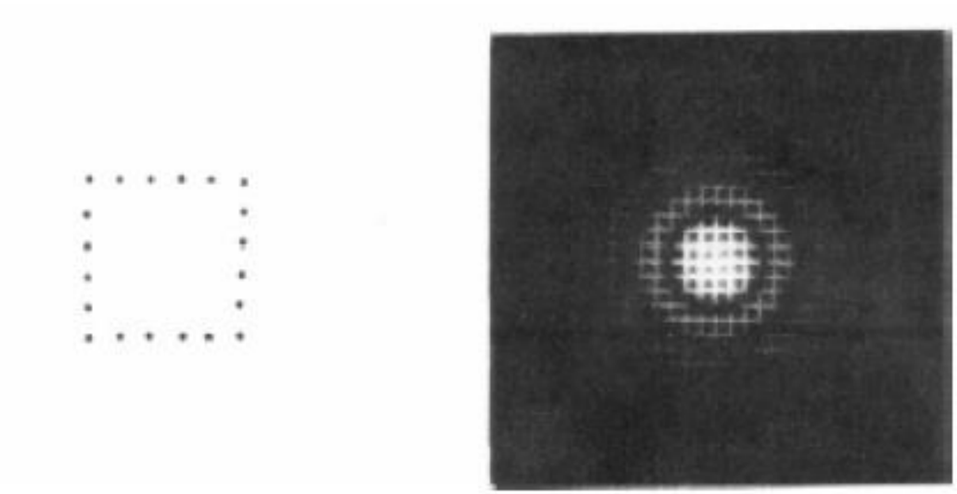
- **Examples of DFT**



The 2-D signal varies radically along y-axis, hence has an infinite bandwidth (spectrum) along the v-direction.



Identical gain plot with phase difference to result in 'interference pattern' similar to wave interference

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- **Filtering in the frequency domain**

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

If let

$$F(u,v)\Rightarrow\tilde{F}(u,v),$$

we will be filtered image of $f(x, y)$

$$\tilde{f}(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{F}(u,v)e^{j2\pi(ux/M+vy/N)}$$

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Procedure of Filtering in the frequency domain

f(x, y) —— given image; F(u,v) ——Fourier transform of f(x, y)

H(u,v) —— the frequency transfer function of the filter

g(x, y) ——the output image. It can easily be shown that

G(u,v)=H(u,v) F(u,v)

To perform this task and obtain the output image g(x, y) , one can follow the steps given below

(1) Multiply f(x, y) by $(-1)^{x+y}$ to obtain $f(x, y)(-1)^{x+y}$

(2) Find $\mathcal{F}\left[f(x,y)(-1)^{x+y}\right] = F\left(u - M/2,\ v - N/2\right)$

(3) Obtain $G\left(u - M/2,\ v - N/2\right) = H\left(u - M/2,\ v - N/2\right) F\left(u - M/2,\ v - N/2\right)$

(4) Computer inverse transform of $G\left(u - M/2,\ v - N/2\right)$ to obtain g(x, y) $(-1)^{x+y}$

(5) Obtain g(x, y) by multiply the result of (4) with $(-1)^{x+y}$.

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- **Notch filter using DFT （频率域的陷波滤波器）**

A notch filter has a frequency transfer function of the following form

$$H(u,v)=\begin{cases}0, & \text{if } (u,v)=(\bar{u},\bar{v}) \\ 1, & \text{otherwise}\end{cases}$$

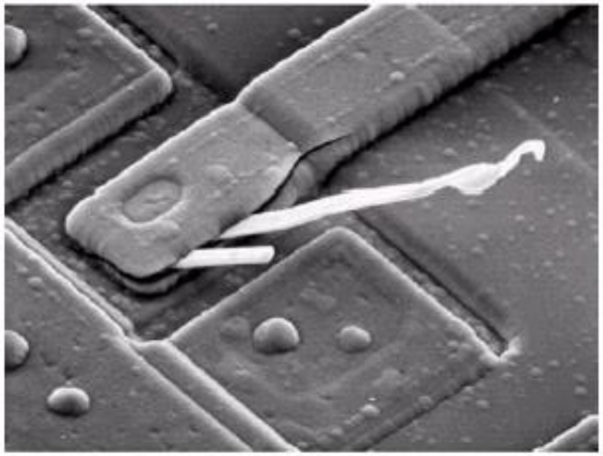
Such a filter filters out the frequency component at a specific frequency (u, v) and leave other frequency components unchanged.

For instance, if we want to remove dc bias of an image such that its average intensity be zero, we can apply the following notch filter

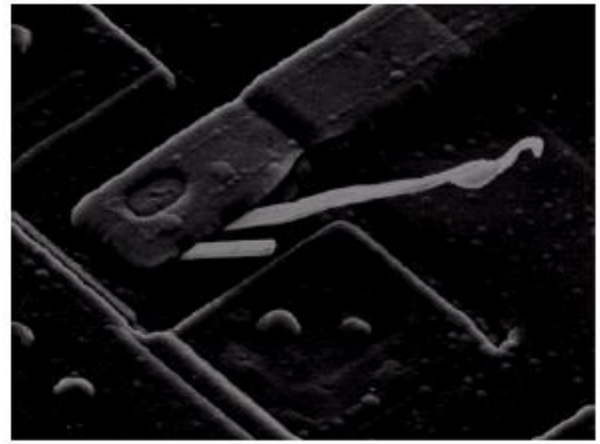
$$H(u,v)=\begin{cases}0, & \text{if } (u,v)=(0,0) \\ 1, & \text{otherwise}\end{cases}$$

If, however, the original image has been shifted by M/2 and N/2, then the following should be used instead

$$H(u,v)=\begin{cases}0, & \text{if } (u,v)=\left(\frac{M}{2},\frac{N}{2}\right) \\ 1, & \text{otherwise}\end{cases}$$



Original image



dc-bias = 0

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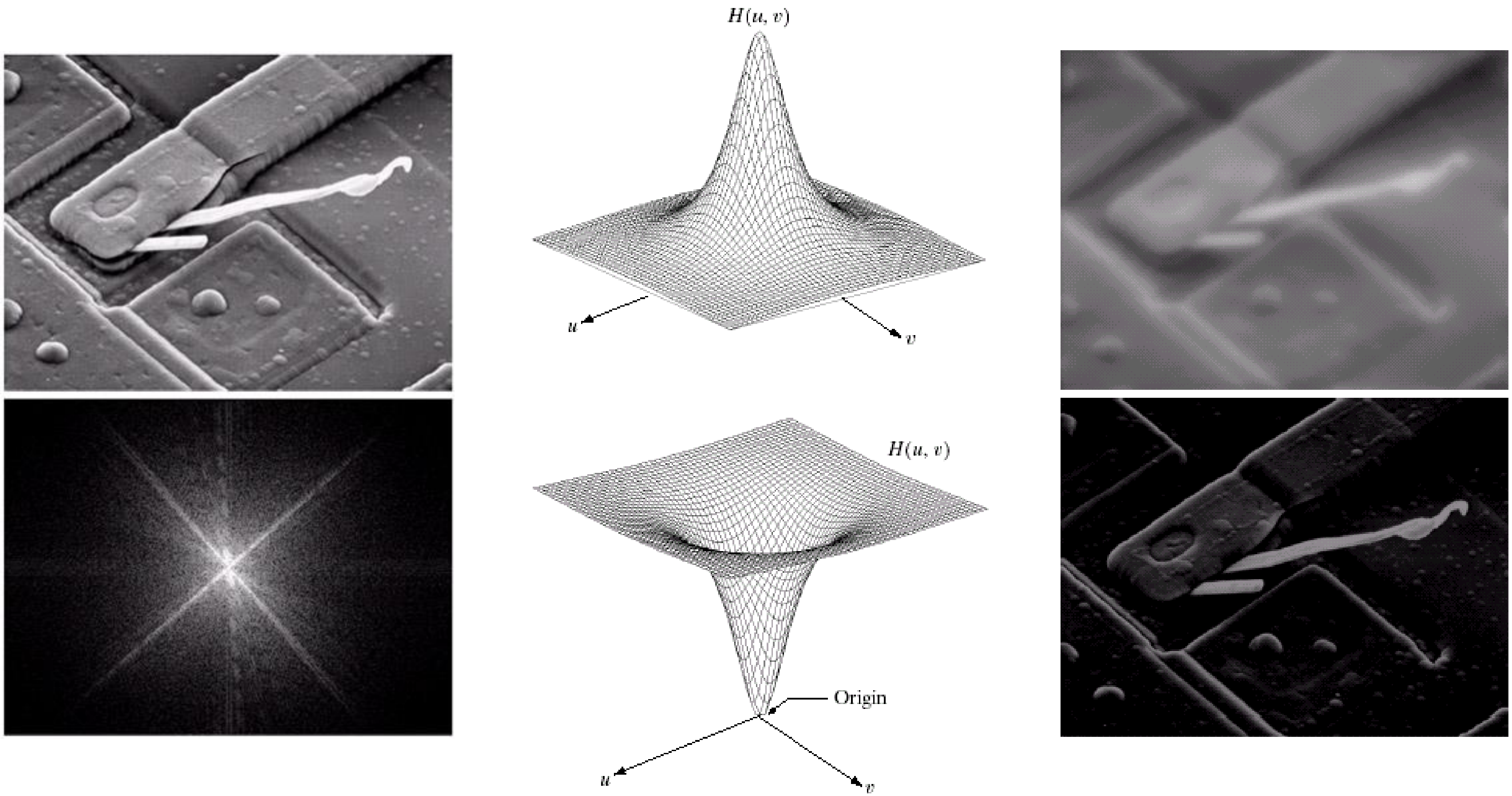
Low-pass and high pass filters



a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

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- **Filtering in the spatial domain or frequency domain?**
  - **Several concepts**

    Impulse (脉冲): 相当于脉冲矩阵，即在坐标 (x, y) 处为1，其他位置为0的矩阵或图像信号

    impulse response of a filter (滤波器的脉冲响应): 滤波器作用于脉冲信号后的结果（或者输出）。此量给出了滤波器的重要性质，在不会混淆的情况下，滤波器的脉冲响应可以等同于滤波器本身。

  - **Relationship between filtering in spatial domain and frequency domain**

    结论: 空间域的滤波器, 和频率域的滤波器组成了一组傅里叶变换对.也就是说, 给出在频率域的滤波器（低通）, 通过傅里叶反变换就可以得到在空间域相应的滤波器（平滑）, 反之亦然.

    这个最基本的联系是由著名的卷积定理建立起来的.

Spatial Correlation and Convolution (2D)

Origin $f(x, y)$

0	0	0	0	0
0	0	0	0	0
0	0	1	0	0
0	0	0	0	0
0	0	0	0	0

$w(x, y)$

1	2	3
4	5	6
7	8	9

(a)

Padded f

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(b)

Initial position for w

1	2	3	0	0	0	0	0	0
4	5	6	0	0	0	0	0	0
7	8	9	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(c)

Full correlation result

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	9	8	7	0	0	0
0	0	0	6	5	4	0	0	0
0	0	0	3	2	1	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(d)

Cropped correlation result

0	0	0	0	0
0	9	8	7	0
0	6	5	4	0
0	3	2	1	0
0	0	0	0	0

(e)

Rotated w

9	8	7	0	0	0	0	0	0
6	5	4	0	0	0	0	0	0
3	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(f)

Full convolution result

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	2	3	0	0	0
0	0	0	4	5	6	0	0	0
0	0	0	7	8	9	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(g)

Cropped convolution result

0	0	0	0	0
0	1	2	3	0
0	4	5	6	0
0	7	8	9	0
0	0	0	0	0

(h)

FIGURE 3.30 Correlation (middle row) and convolution (last row) of a 2-D filter with a 2-D discrete, unit impulse. The 0s are shown in gray to simplify visual analysis.

两个$M \times N$的离散函数（矩阵）$f(x, y)$和$h(x, y)$的卷积定义为

$$f(x,y)*h(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$$

$f(x, y)$ — 要处理的图像， $h(x, y)$ — *filter* （**mask**）

空间域滤波的本质上就是用选好的掩模$h(x, y)$ (**mask**), 经过一定的处理, 与给定的图像$f(x, y)$作卷积.

例如: 平滑滤波的**mask**

$$\frac{1}{9}\times\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array} \qquad \frac{1}{16}\times\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline\end{array}$$

卷积定理

$F(u,v)$—$f(x,y)$的傅里叶变换

$H(u,v)$—$h(x,y)$的傅里叶变换

则有

$$f(x,y)*h(x,y) \Leftrightarrow F(u,v)\cdot H(u,v) \qquad (4.2.31)$$

$$f(x,y)h(x,y) \Leftrightarrow F(u,v)*H(u,v) \qquad (4.2.32)$$

其实从上述公式就已经可以得出我们这一小节的结果论

定义在坐标(x_0, y_0)（x_0和y_0均为正整数）处强度为A的冲激(脉冲)函数（矩阵）

$A\delta(x-x_0, y-y_0)$——在(x_0, y_0)处为A，其他处为零。

满足

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)A\delta(x-x_0, y-y_0) = As(x_0, y_0) \qquad (4.2.37)$$

用这样的冲激函数和滤波器作卷积就得到前面的结论.

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} w_1 & w_2 & w_3\\ w_4 & w_5 & w_6\\ w_7 & w_8 & w_9 \end{bmatrix}$$

$$\begin{bmatrix} w_9 & w_8 & w_7\\ w_6 & w_5 & w_4\\ w_3 & w_2 & w_1 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&w_1&w_2&w_3&0&0&0\\ 0&0&0&w_4&w_5&w_6&0&0&0\\ 0&0&0&w_7&w_8&w_9&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

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附页：卷积定理证明概要

两个M × N的离散函数$f(x, y)$和$h(x, y)$的卷积（convolution）定义为

$$f(x,y)*h(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$

两边同时做FT

$$\mathrm{F}\left[f(x,y)*h(x,y)\right]=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)\mathrm{F}\left[h(x-m,y-n)\right]$$

利用FT的平移性质 (4.6-4) ——very important skill

$$\mathrm{F}\left[f(x,y)*h(x,y)\right]=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)H(u,v)\mathrm{e}^{-j2\pi(\frac{um}{M}+\frac{vn}{N})}$$

$$=F(u,v)H(u,v)$$

$$f(x, y) * h(x, y) \Leftrightarrow F(u, v) \cdot H(u, v) \qquad (4.2.31)$$

$$f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v) \qquad (4.2.32)$$

Comparison of Eqs. (4.2.31) and (4.2.32) indicates that, in theory, the computational efficiency is the same no matter the filtering operation is performed in the frequency domain or spatial domain. However, it is more intuitive to design a filter in the frequency domain, while the spatial domain of a filter generally uses a mask of a smaller size. Therefore, it is a common practice that a filter is designed in the frequency domain and implemented in the spatial domain (as a convolution mask).

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Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, e^{-j2\pi(ux/M + vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v)\, e^{j2\pi(ux/M + vy/N)}$
3) Polar representation	$F(u, v) = \lvert F(u, v)\rvert e^{j\phi(u,v)}$
4) Spectrum	$\lvert F(u, v)\rvert = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$ $R = \text{Real}(F); \quad I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
6) Power spectrum	$P(u, v) = \lvert F(u, v)\rvert^2$
7) Average value	$\bar{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

(*Continued*)

TABLE 4.2
Summary of DFT definitions and corresponding expressions.

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N)$ $= F(u + k_1M, v + k_2N)$ $f(x, y) = f(x + k_1M, y) = f(x, y + k_2N)$ $= f(x + k_1M, y + k_2N)$
9) Convolution	$f(x, y) \bigstar h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$
10) Correlation	$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v)e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

TABLE 4.2
(*Continued*)

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Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

(*Continued*)

TABLE 4.3
Summary of DFT pairs. The closed-form expressions in 12 and 13 are valid only for continuous variables. They can be used with discrete variables by sampling the closed-form, continuous expressions.

TABLE 4.3
(Continued)

Name	DFT Pairs
7) Correlation theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F^*(u, v)\, H(u, v)$ $f^*(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$
8) Discrete unit impulse	$\delta(x, y) \Leftrightarrow 1$
9) Rectangle	$\text{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $j\dfrac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0)\right]$
11) Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $\dfrac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0)\right]$
The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.	
12) *Differentiation* (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)	$\left(\dfrac{\partial}{\partial t}\right)^m\left(\dfrac{\partial}{\partial z}\right)^n f(t, z) \Leftrightarrow (j2\pi\mu)^m(j2\pi\nu)^n F(\mu, \nu)$ $\dfrac{\partial^m f(t, z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu, \nu);\ \dfrac{\partial^n f(t, z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu, \nu)$
13) *Gaussian*	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

†Assumes that the functions have been extended by zero padding. Convolution and correlation are associative, commutative, and distributive.

- **Gaussian Low-pass Filter (高斯低通濾波器)**

◆ A Gaussian Filter is unique in that it has the same function form both in the spatial domain and frequency domain. The impulse response of a 1-D Gaussian filter is

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2} \qquad (4.2.39)$$

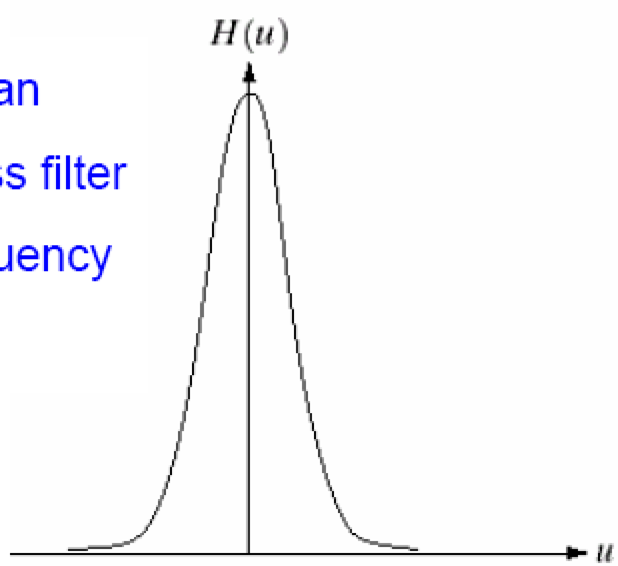
and its frequency transfer function is given by

$$H(u) = Ae^{-u^2/2\sigma^2} \qquad (4.2.38)$$

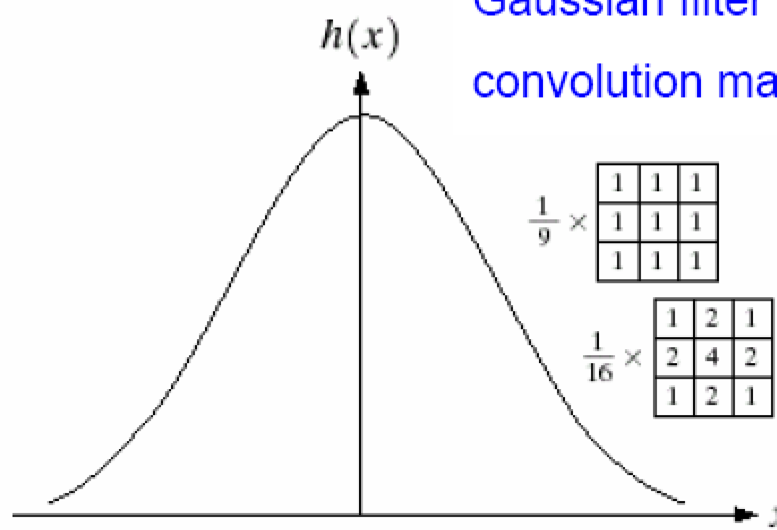
where σ is the standard deviation, and the cut-off frequency is 1/σ.

◆ **The bandwidth of *H(u)* increases with 1/σ, and the effect of noise filtering decreases.**

Gaussian low-pass filter in Frequency domain



Gaussian filter as a convolution mask



- 2-D Gaussian Low-pass Filter

$$H(u,v) = Ae^{\frac{-(u^2+v^2)}{2\sigma^2}}$$

and its impulse response be

$$h(x,y) = \sqrt{2\pi}\sigma Ae^{-2\pi^2\sigma^2(x^2+y^2)}$$

Gaussian Function has two important features which make it very useful in image processing:

① **Both Gaussian function itself and its Fourier Transform are real Gaussian functions;**

② 这一对傅利叶变换相互间有某种反比特性, 即当***H*(*u*)**有较宽的轮廓(大的σ值)时, ***h*(*x*)**有较窄的轮廓, 反之亦然.

当接近无限时, ***H*(*u*)**趋于常数量, ***h*(*x*)**则趋于冲激函数.

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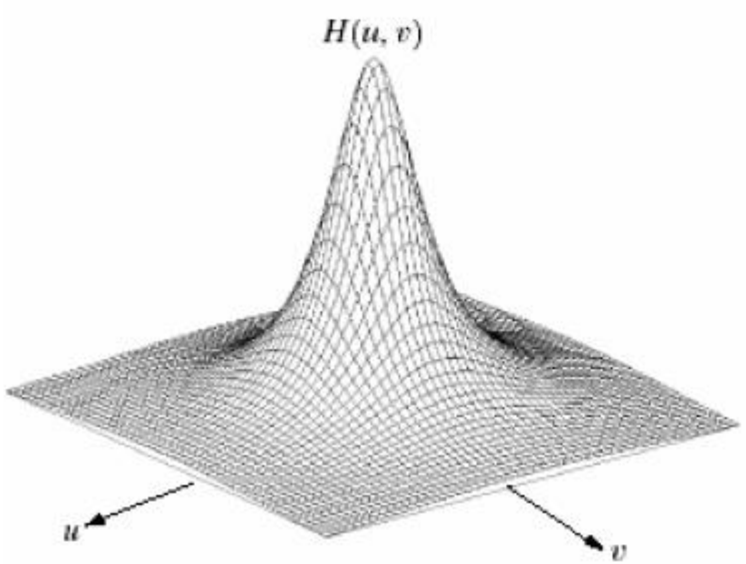
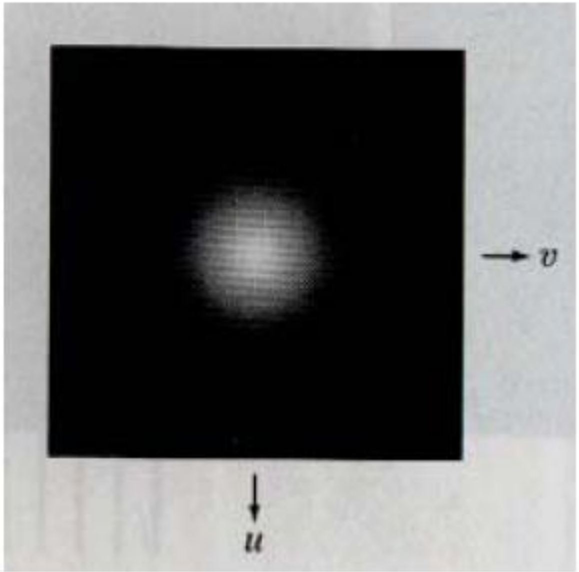
- Gaussian filter is a rotationally symmetric, single lobe function.

Meaning

(1) it has no directionality(各向同性), and (2) values of elements in its spatial convolution mask decreases with the distance from its center, a nice local property（良好的局部性质） for a convolution mask.



Gaussian is single lobe



gain plot indicated by intensity

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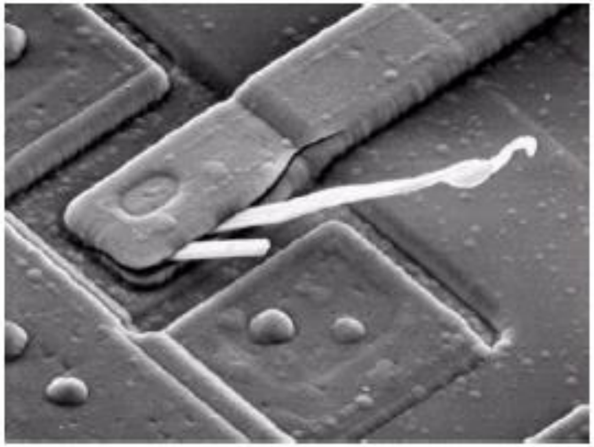
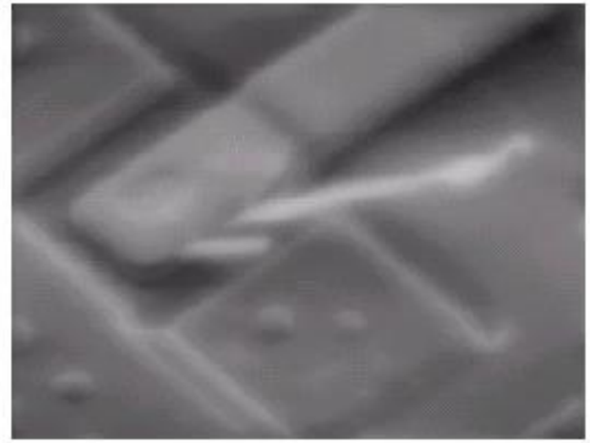


Original image



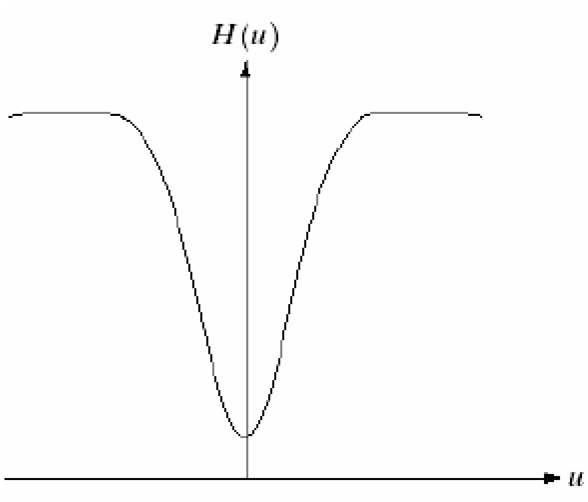
after Gaussian filtering

- Gaussian high-pass Filter
  - Gaussian high-pass filter can be created as the difference of two Gaussian low-pass filter, such as

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2} \tag{4.2.40}$$

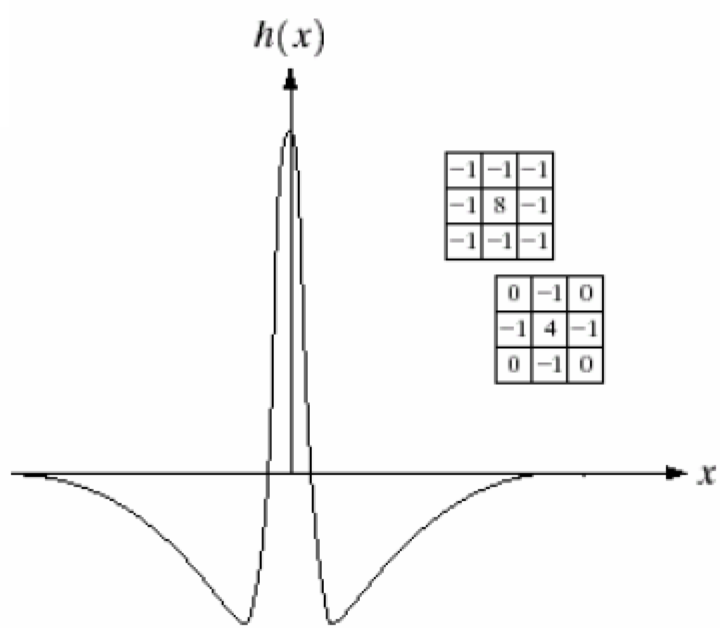
whose impulse transform is

$$h(x)=\sqrt{2\pi}\left(\sigma_1 A e^{-2\pi^2\sigma_1^2x^2}-\sigma_2 B e^{-2\pi^2\sigma_2^2x^2}\right)$$



Gaussian high-pass filter

In the frequency domain



Gaussian high-pass filter

as a convolution mask

Example 4.15: obtaining a frequency domain filter from a small special mask

a b

FIGURE 4.38 (a) Image of a building, and (b) its spectrum.

(b) = spectrum image of (a)

Example 4.15: obtaining a frequency domain filter from a small special mask

Considering simple Sobel vertical edge detector

-1	-2	-1
0	0	0
1	2	1

-1	0	1
-2	0	2
-1	0	1

Sobel horizontal (L) and vertical (R) edge detector

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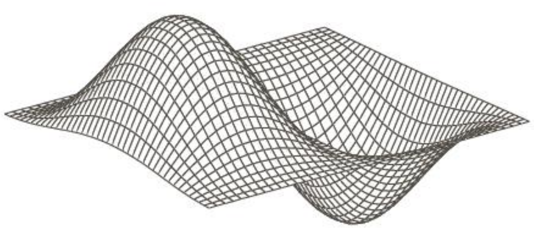
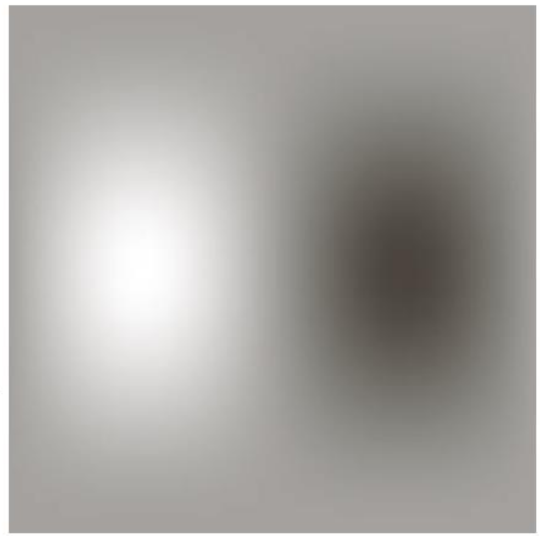
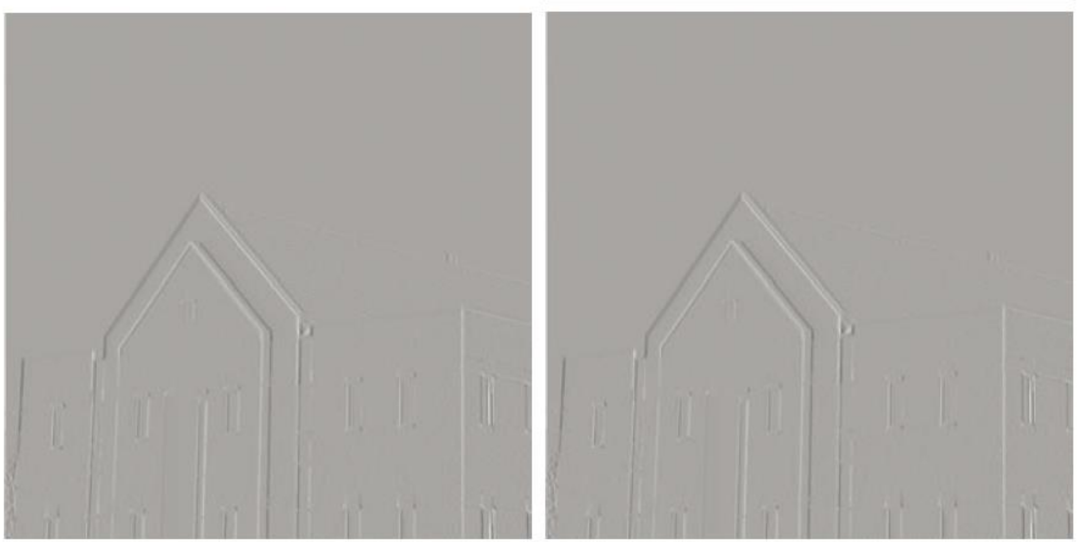
Example 4.15:

−1	0	1
−2	0	2
−1	0	1







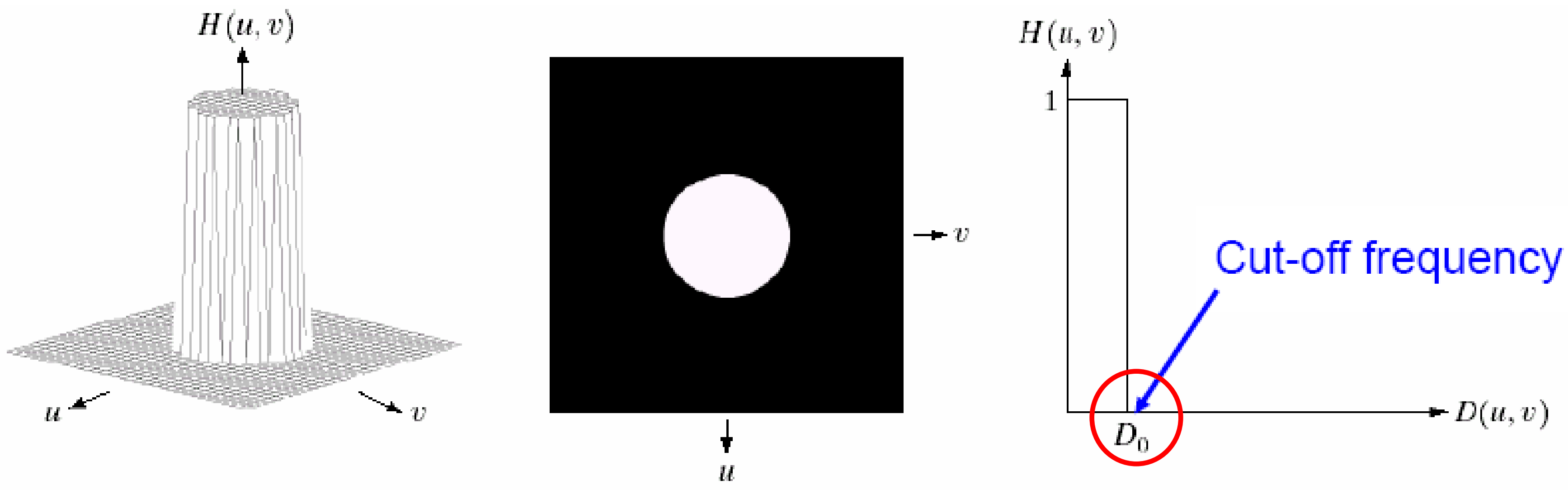
a b
c d

FIGURE 4.39 (a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.

4.8 Smoothing (low-pass) Filter in the Frequency Domain

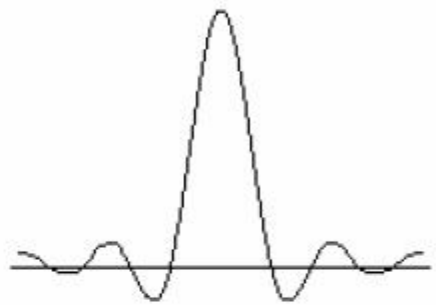
- **Ideal Low-pass Filter**

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)> D_0\end{cases}\quad \text{where } D(u,v)\triangleq\sqrt{\left(u-\frac{M}{2}\right)^2+\left(v-\frac{N}{2}\right)^2}$$

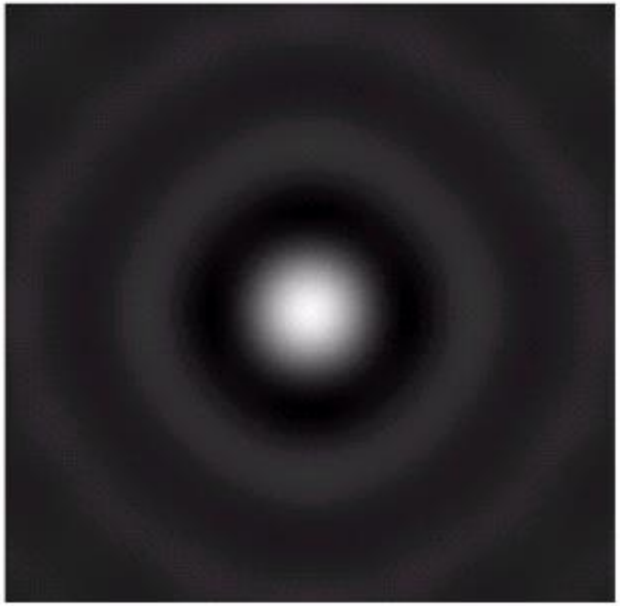


Gain plot of an ideal low-pass filter

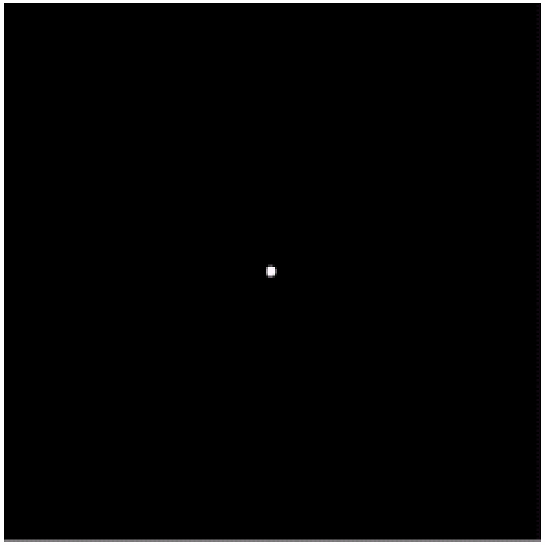
➢ **An ideal low-pass filter is non-causal in the space domain, it cannot be implemented as a convolution mask, but it can be carried out by DFT and inverse DFT.**

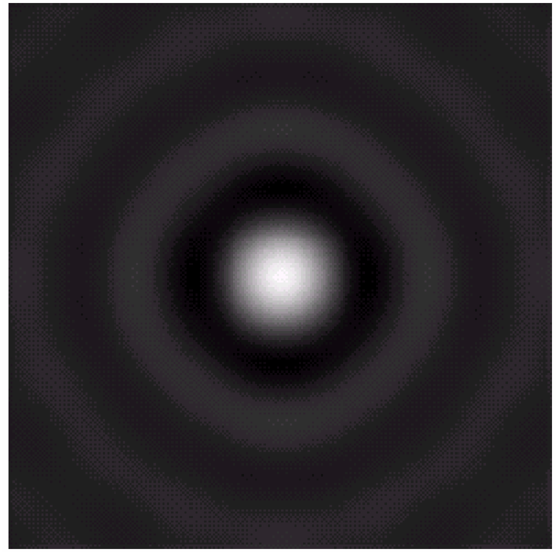


Impulse response of an 1-D ideal low-pass filter

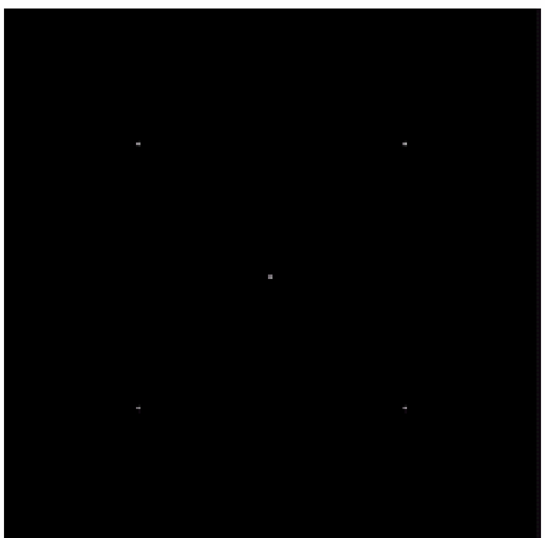


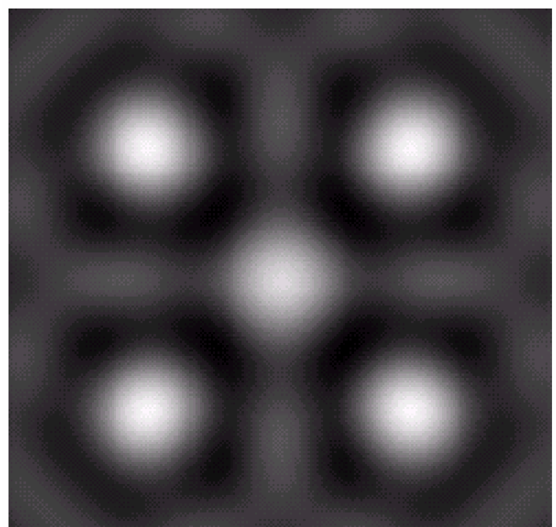
Gain plot of 2-D ideal low-pass filter shows ringing effect





ILPF with radius 5 in the frequency domain shows ringing effect





ILPF with FIVE circles of radius 5 in the frequency domain shows superimposed ringing effect

- The cutoff frequency is sometimes determined by the percentage power to be removed.
- If (100-α)% of the total power is to be taken away from an image by a low-pass filter, where the total power P_T is defined as

$$P_T = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|H(u,v)|^2 \qquad (4.3.4)$$

then the remaining power in the image meets

$$\alpha = \frac{100}{P_T}\sum_{u=0}\sum_{v=0}|H(u,v)|^2 \qquad (4.3.5)$$

总和取处于圆内或边界上所有的(*u*, *v*)值.

值得注意:

① 理想的低通滤波器在空间域无法硬件实现;

② 滤波模糊和振铃现象.

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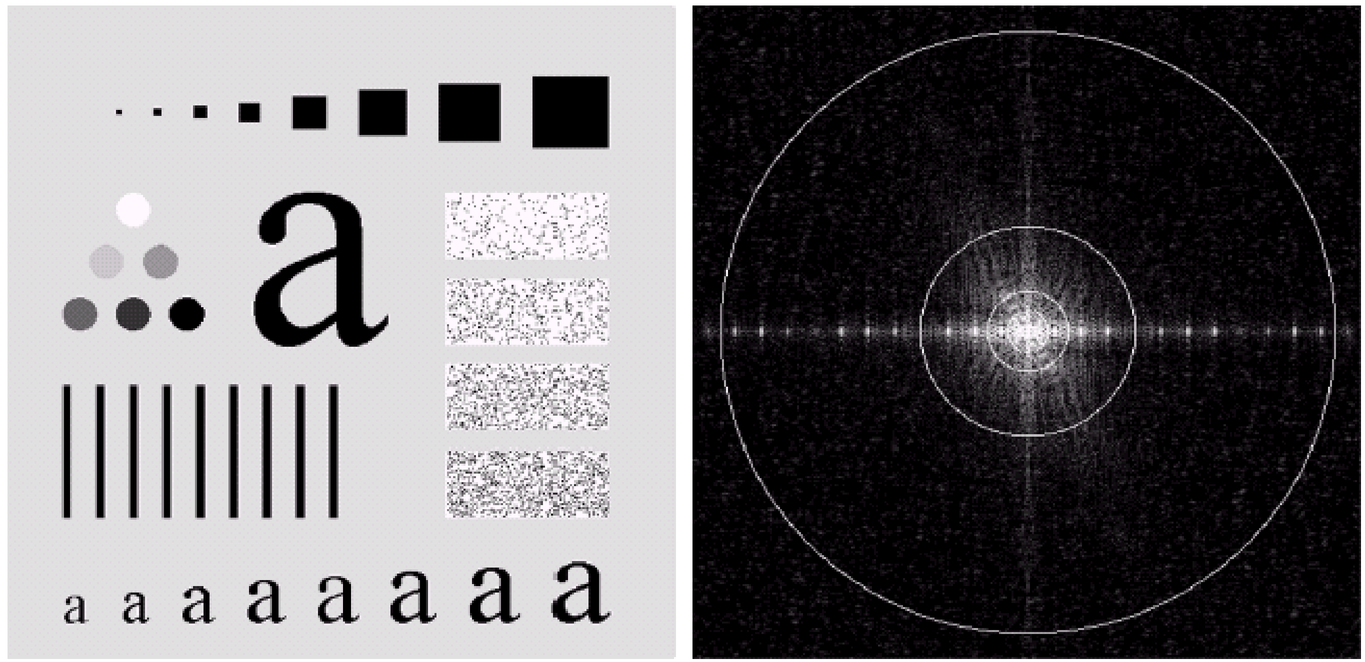
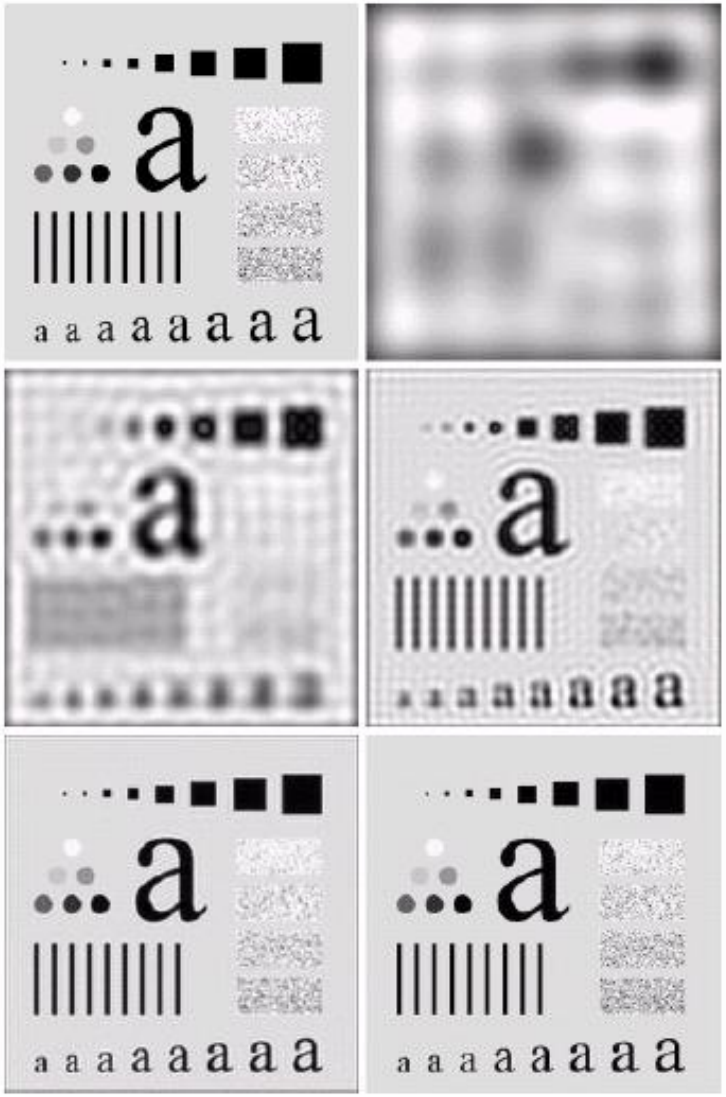
例4.4 图像功率作为距频谱矩形中心距离的函数



a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

这些圆周包围的图像功率的百分比分别是92.0%、94.6%、96.4%、98%和99.5%.



a	b
c	d
e	f

(a) Original image

(b) 8% power removed

(c) 5.4% power removed

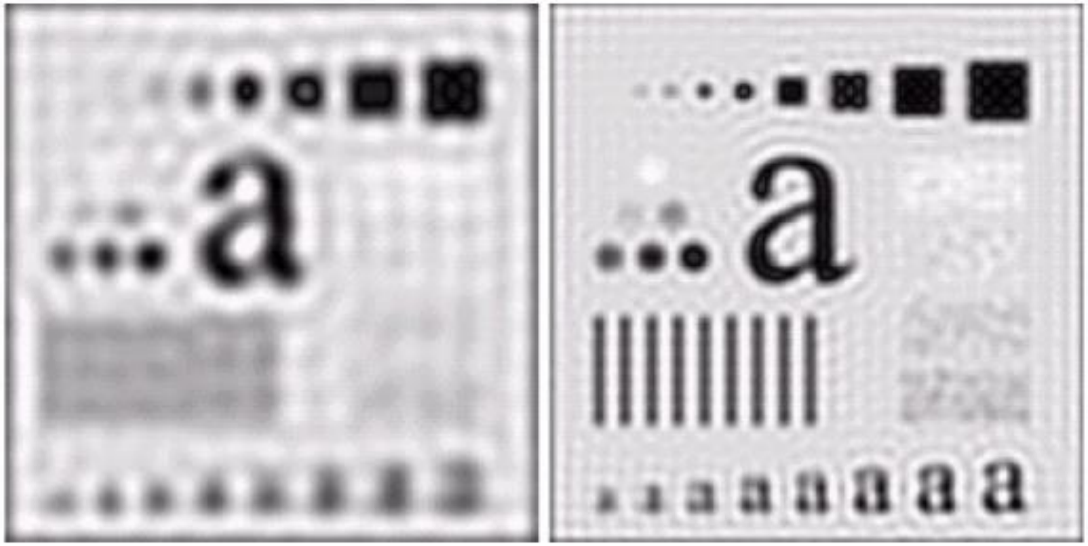
(d) 3.6% power removed

(e) 2% power removed

(f) 0.5% power removed

当截止频率半径较小时，有明显的模糊和振铃效应，为什么？这和对应的空间滤波器的形状有关，特别是出现大量的负值.

Ringing effect



The blurring effect in (c) and (d) is no doubt due to ideal low-pass filter. The ringing effect is also a result of ideal low-pass filtering.

Let h(x, y) be the 2-D impulse response of an ideal low-pass filter, f(x, y) be the input image, and g(x, y) be the output image, then the output image

$$g(x,y) = \frac{1}{MN} \sum_{u=-M/2}^{M/2} \sum_{v=-N/2}^{N/2} f(u,v)h(x-u,y-v)$$

is a super-position of the impulse responses of the ideal low-pass filter which has ringing effect, hence the result is a super-position of ringing patterns.

Again

滤波函数 $h(x, y)$ 和其傅里叶变换$H(u, v)$ 之间的倒数关系(宽窄互置)以及卷积性质可以从数学上说明理想滤波器的模糊和振铃现象.

中心成分的大小和截止频率成反比, 决定了模糊的程度.距原点中心每单位距离的同心圆的多少和截至频率亦成反比, 决定了振铃的程度.

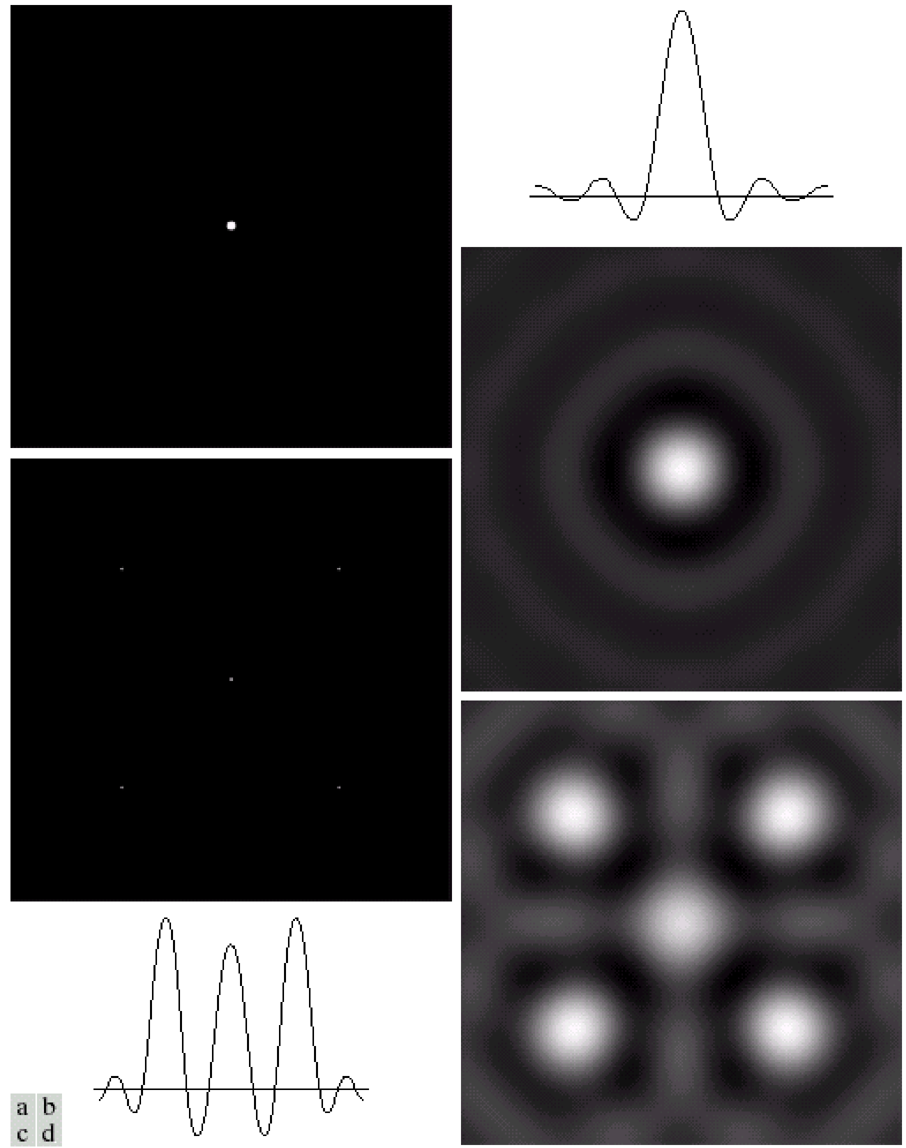


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

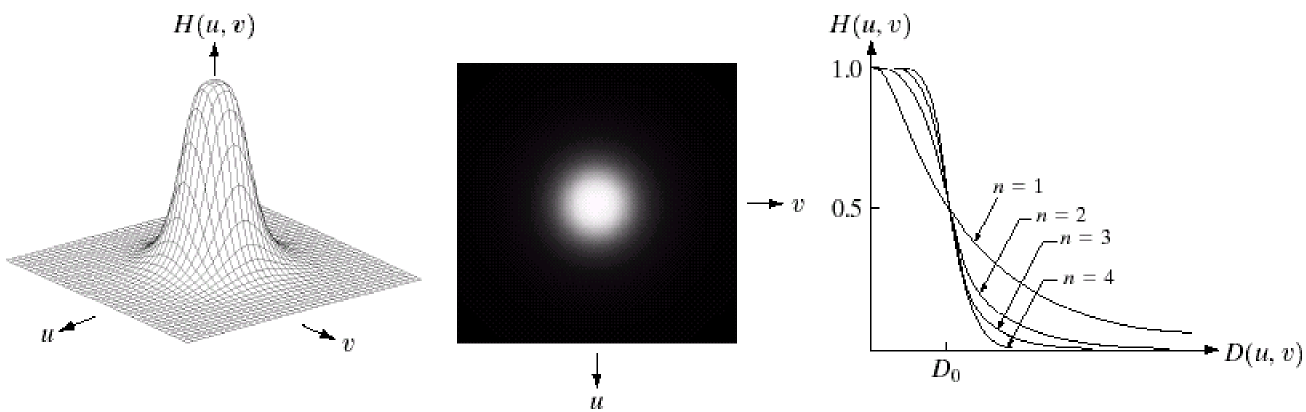
● Butterworth (巴特沃斯) Low-pass filters

The transfer function of a Butterworth low-pass filter (BLPF) of order *n*, and with the cutoff frequency at a distance D_0 from the origin, is defined as

$$H(u,v) = \frac{1}{1 + \left[D(u,v)/D_0\right]^{2n}}$$

Here: $D(u, v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$

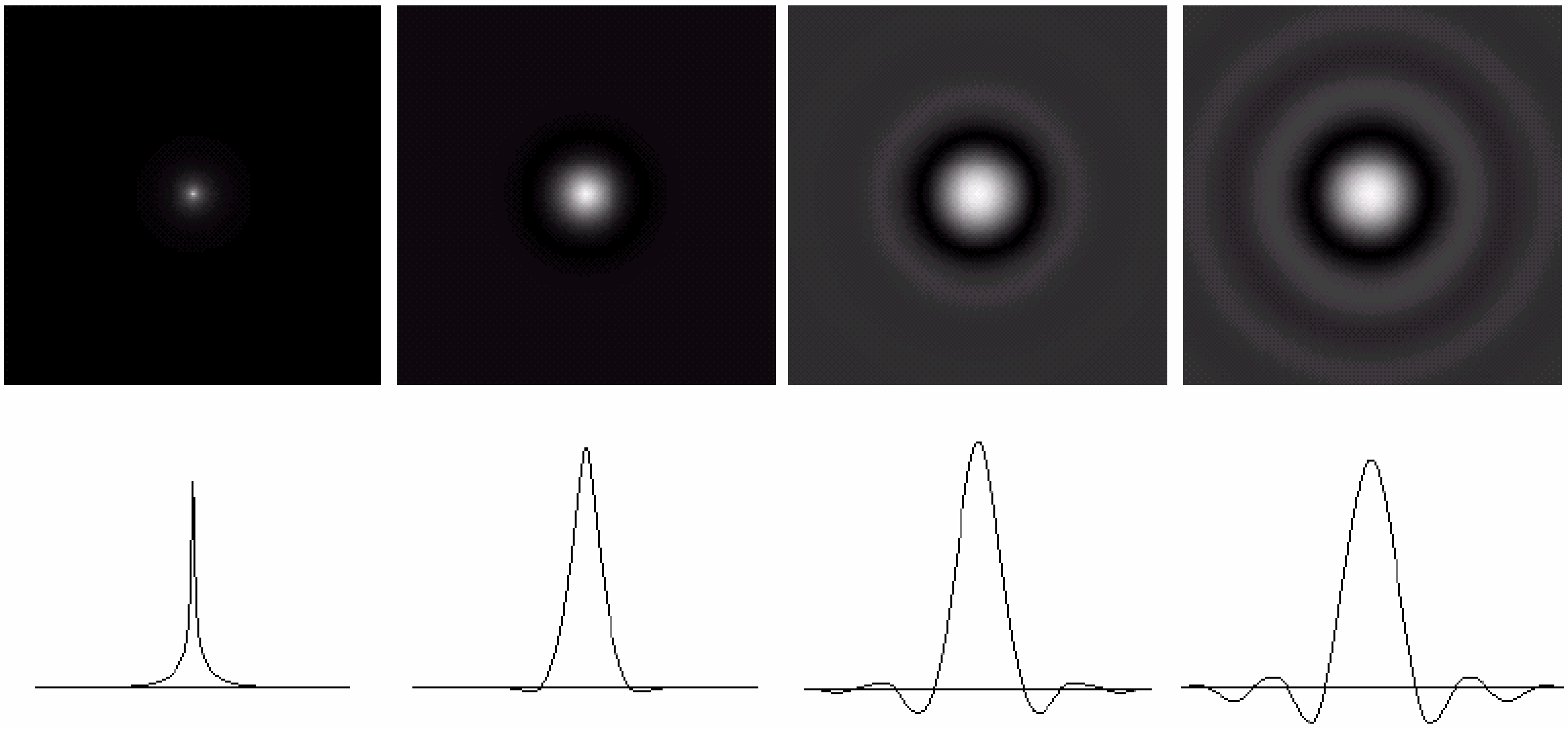
在通带和被滤除的频率之间没有明显的截断（Cutoff）。截止频率在这里是定义一个位置, 使得频率$H(u, v)$的幅度降到最大值的一部分, 例如这里的50%.



a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

◆ When n is large, the Butterworth filter approximates an ideal low-pass filter, and the ringing effect occurs in accompany with blurring effect, as is shown below.



Impulse responses of Butterworth filters with a cut-off frequency = 5

(a) n=1 (b) n=2 (c) n=5 (d) n=20

Figure 4.15

2^{nd} order (n=2)

Butterworth filtering

with a cutoff frequency

at

(a)15,

(b)30,

(c) 80 and

(d) 230

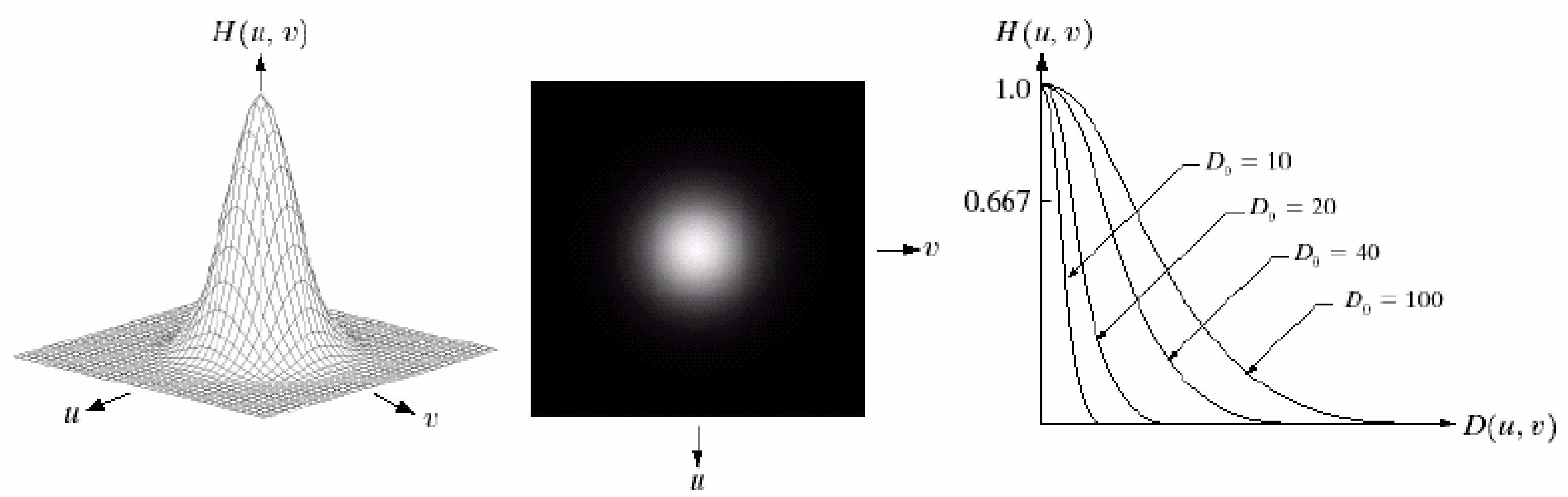
Ring effect is not

noticeable.

- Gaussian low-pass filtering

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

where D_0 is the cutoff frequency



Gain plot of a Gaussian low-pass filter

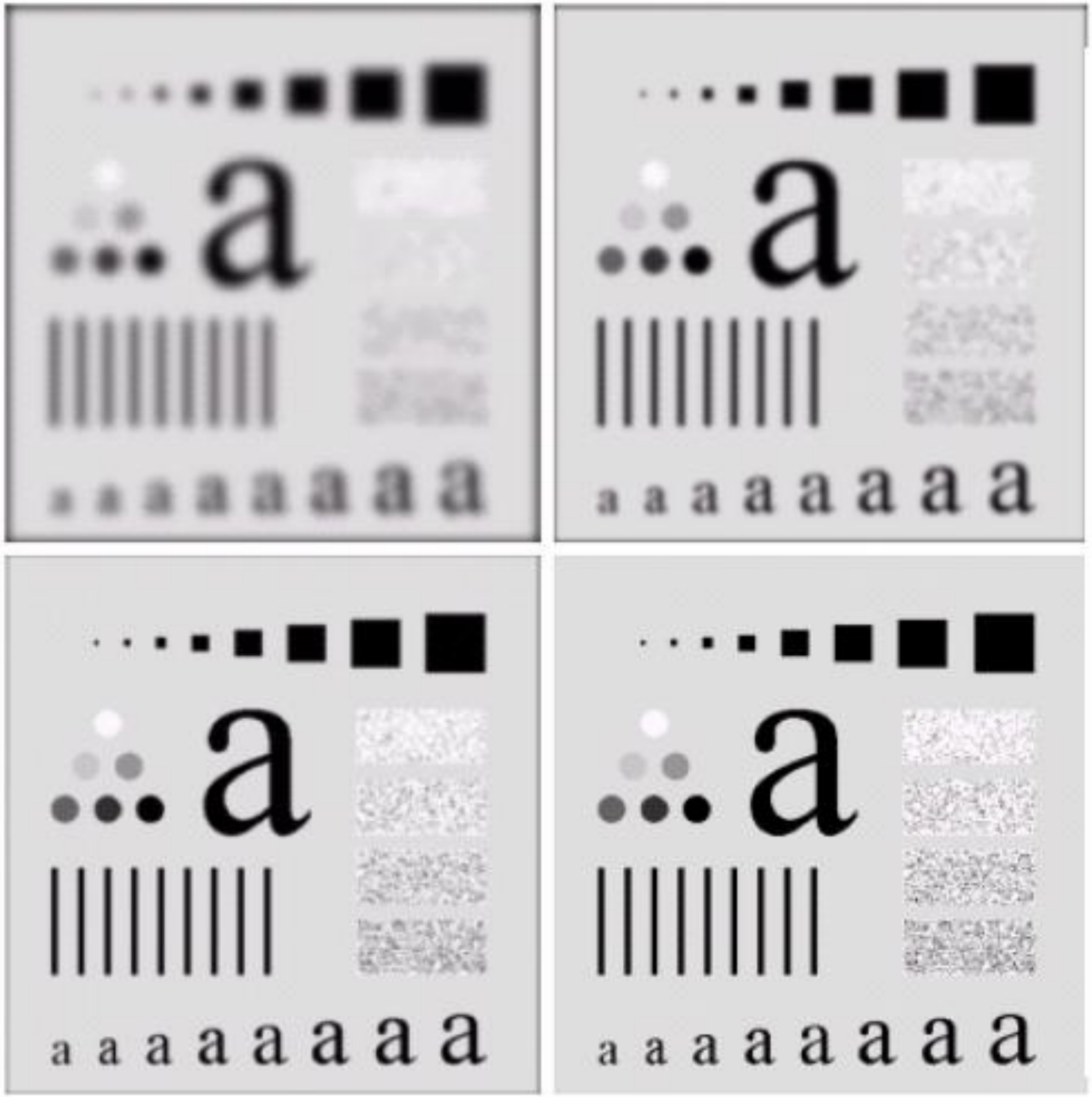


Figure 4.18

Gaussain filtering with a cutoff frequency at (a)15, (b)30, (c) 80 and (d) 230

Results resemble Butterworth filtering

在同样的截止频率条件下, GLPF不如2 阶BLPF的模糊效果好.

从截面图也可以看出来, 前者不如后者的紧凑. 但高斯滤波器完全保证没有振铃, 这在很多人为误差无法接受的应用中(如医学图像)是非常重要的. 而在需要严格控制高低频之间截止频率过渡的情况下, BLPF 是合适的选择.

当然代价是会出现振铃.

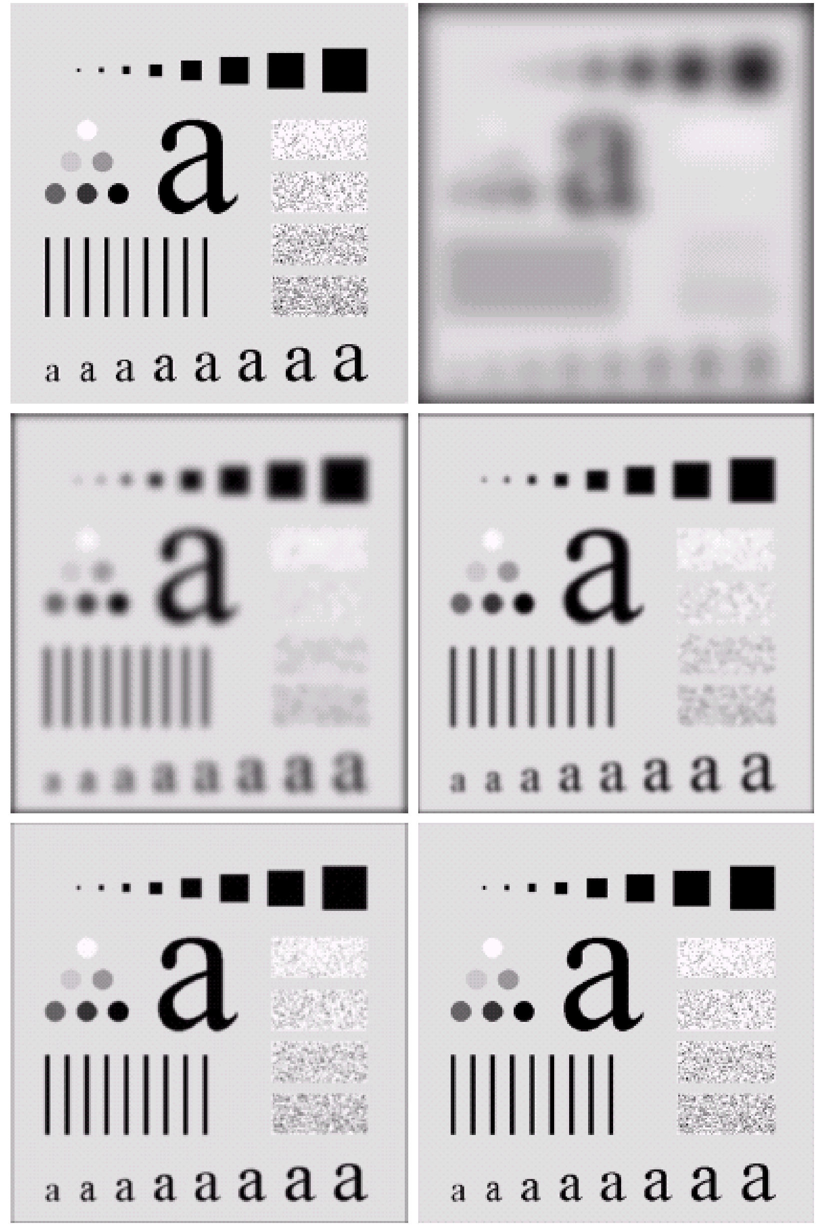
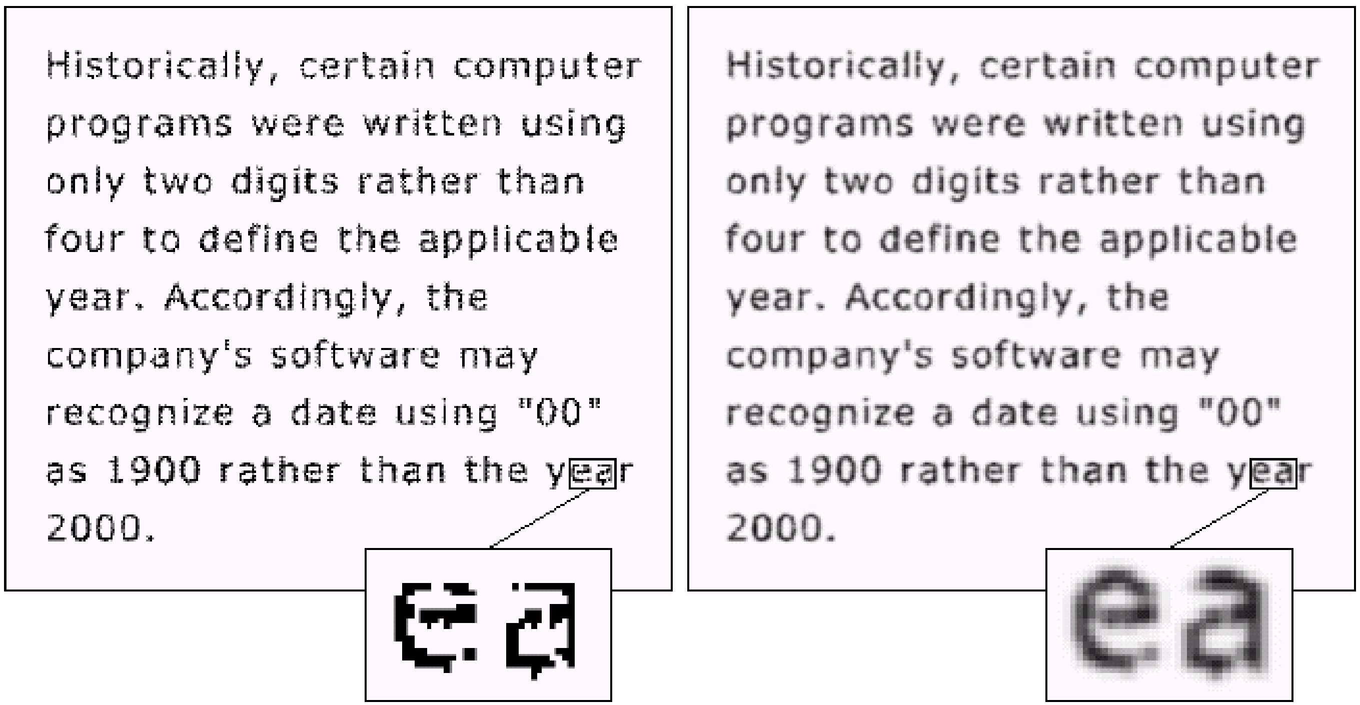


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

a b
c d
e f

低通滤波的其它例子

低分辨率文本字符修复(Gaussian Low-pass filter)



a b

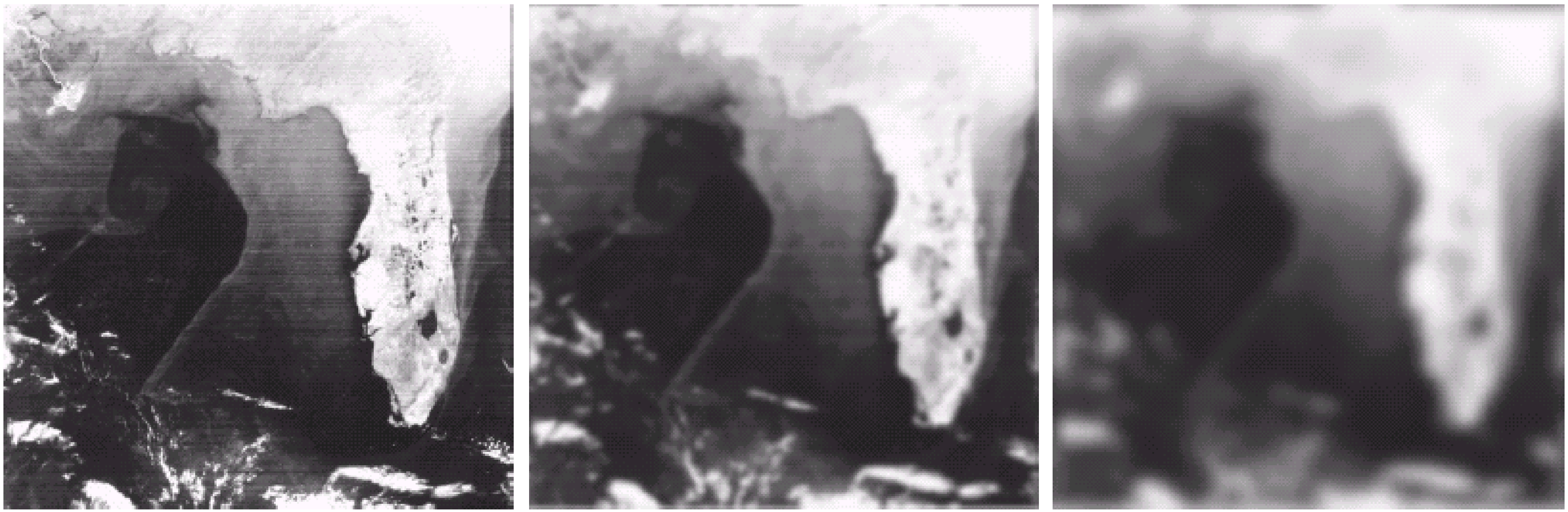
FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

面部柔和（鱼尾纹没有了，但图像也相对模糊了）

a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

去除背景噪声



a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

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TABLE 4.4
Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

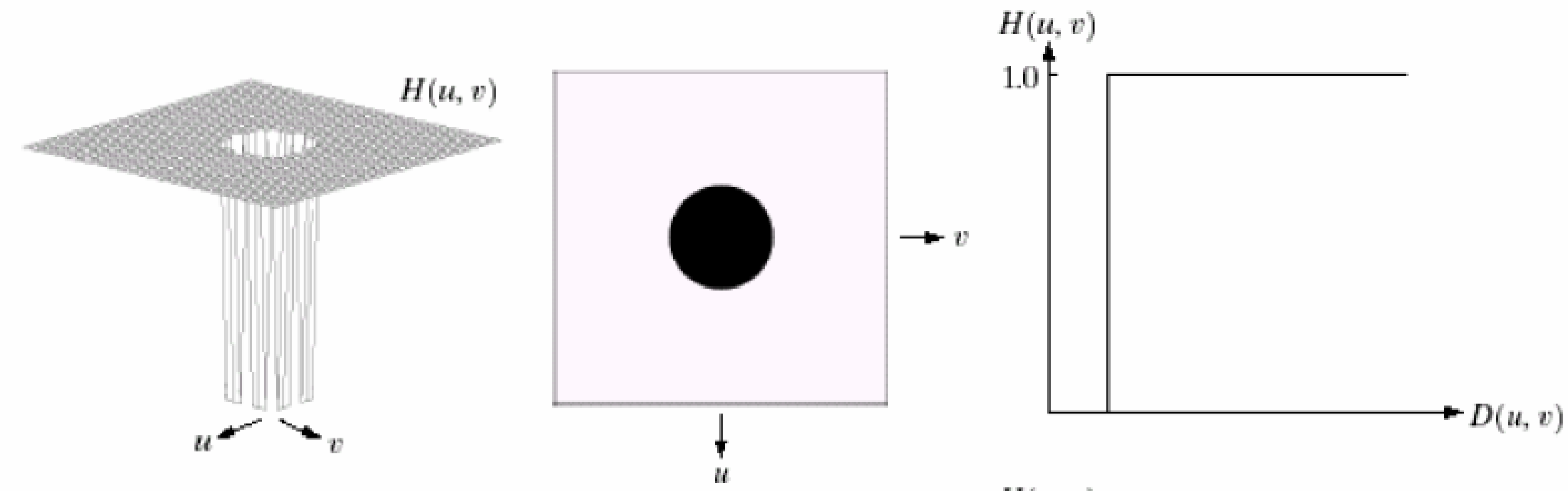
Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

4.9 Frequency domain sharpening filters

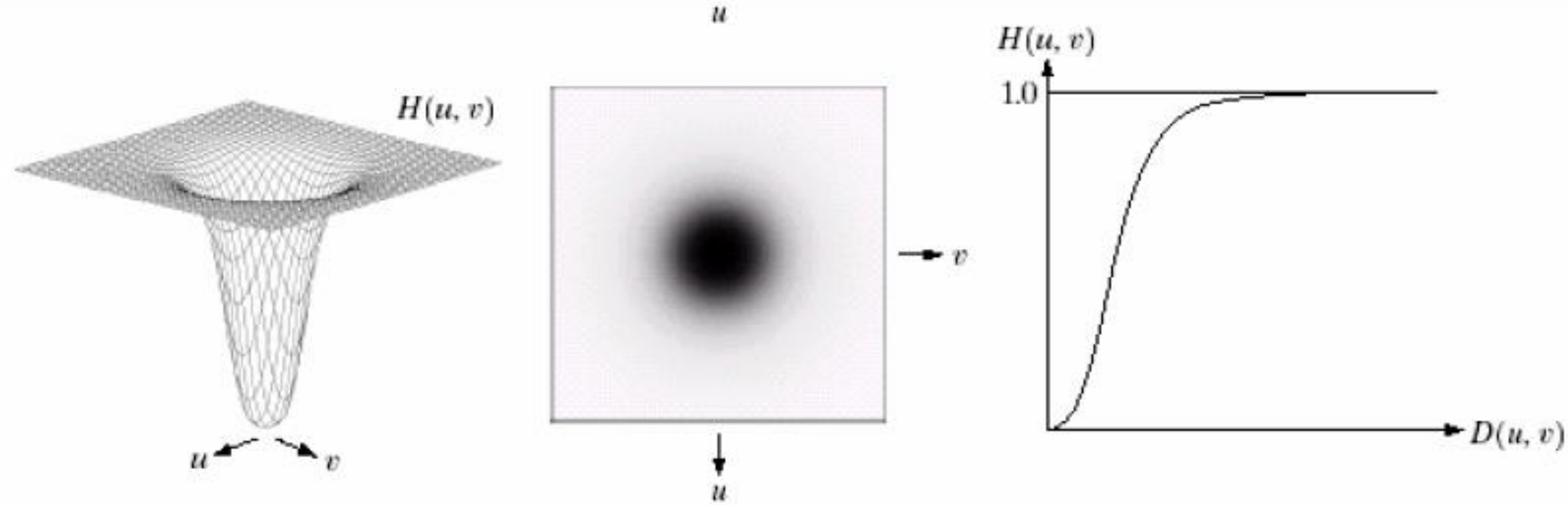
- Again, sharpening is achieved by using a high-pass filter (edge detector)

$$H_{Hp}(u,v) = 1 - H_{Lp}(u,v)$$

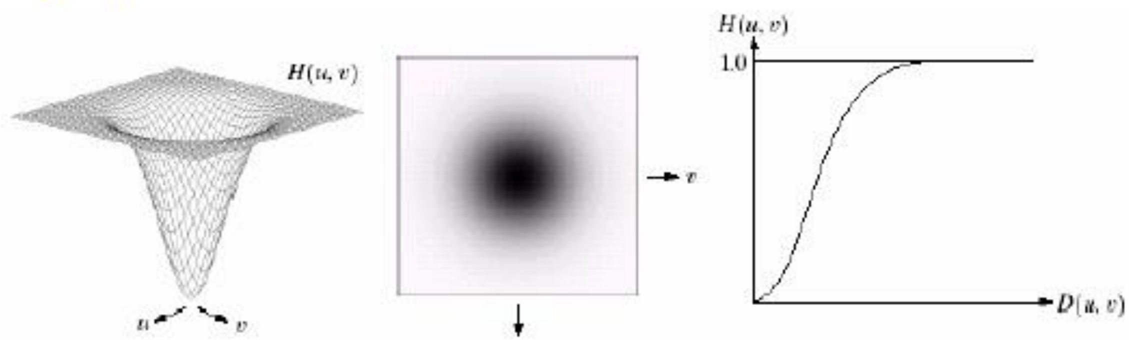
- Ideal high-pass filter

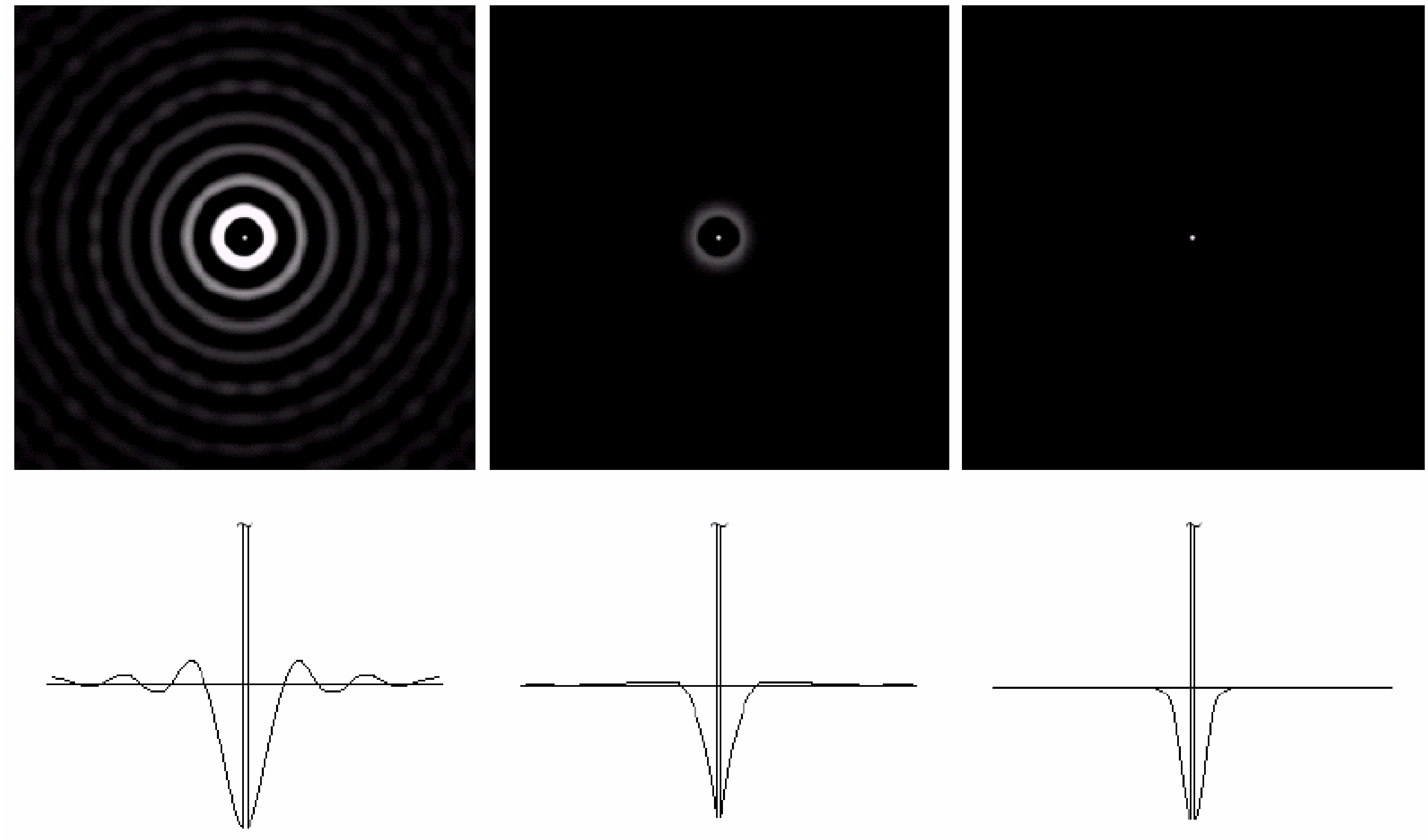


- Butterworth high-pass filter



- Gaussian high-pass filter

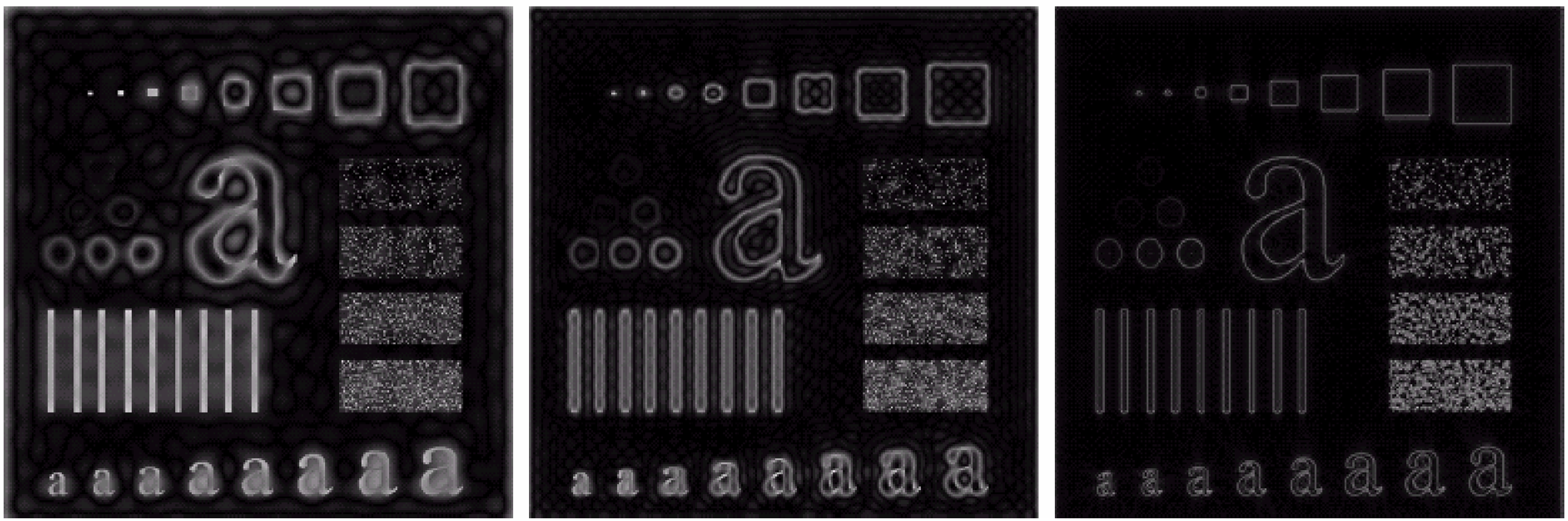




Impulse response of high-pass filter, from left: ILPF, BLP, GLP

Please notice that ringing effect occurs in the ideal high-pass filter.

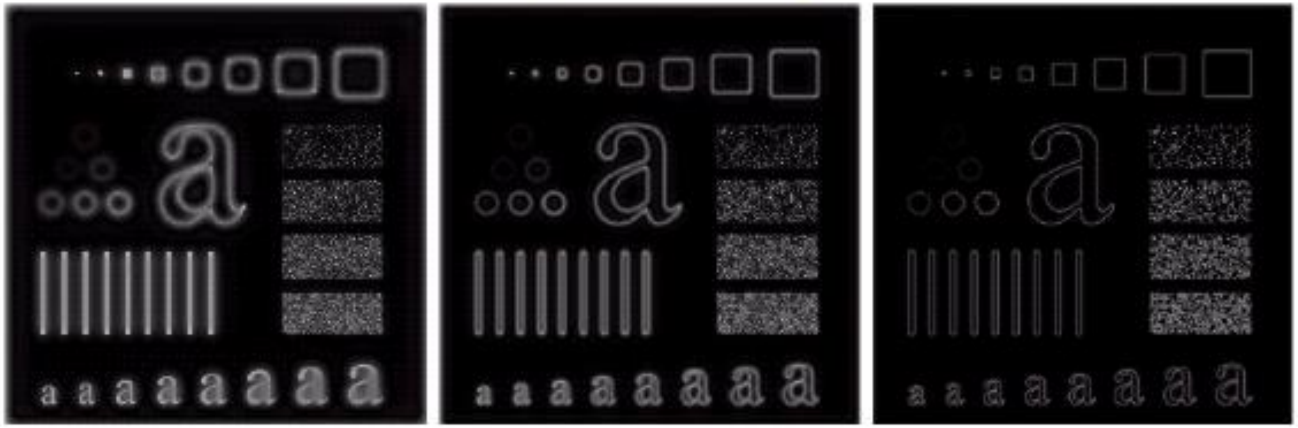
Figure 4.24 Results of ideal high-pass filtering



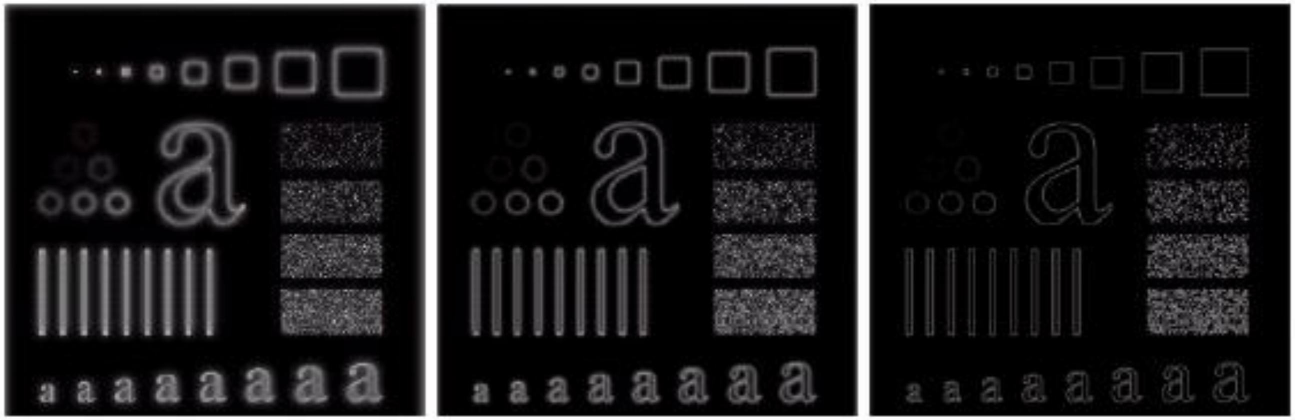
(a) $D_0 = 15$ (a) $D_0 = 30$ (a) $D_0 = 80$

- Ringing effect occurs again in the high-pass filtering

- Result of Butterworth high-pass filtering



- Result of Gaussian high-pass filtering



a b c

FIGURE 4.57 (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b). (Original image courtesy of the U.S. National Institute of Standards and Technology.)

Butterworth high-pass filter of 4 orders

TABLE 4.5

Highpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$	$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$

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- **Laplacian in the frequency domain**

It can be shown that **(不知道的请下课后自己补习)**

$$\frac{d^n f(x)}{dx^n} \Leftrightarrow (ju)^n F(u)$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \Leftrightarrow -C(u^2 + v^2)F(u,v)$$

That is, in the frequency the Laplacian operator acts like a filter

$$\boldsymbol{H(u, v) = -C(u^2 + v^2)}$$

When shifting the center of the image to the spatial coordinate (-M/2, –N/2), the Laplacian filter becomes

$$\boldsymbol{H(u, v) = -C\,[(u - M/2)^2 + (v - N/2)^2]}$$

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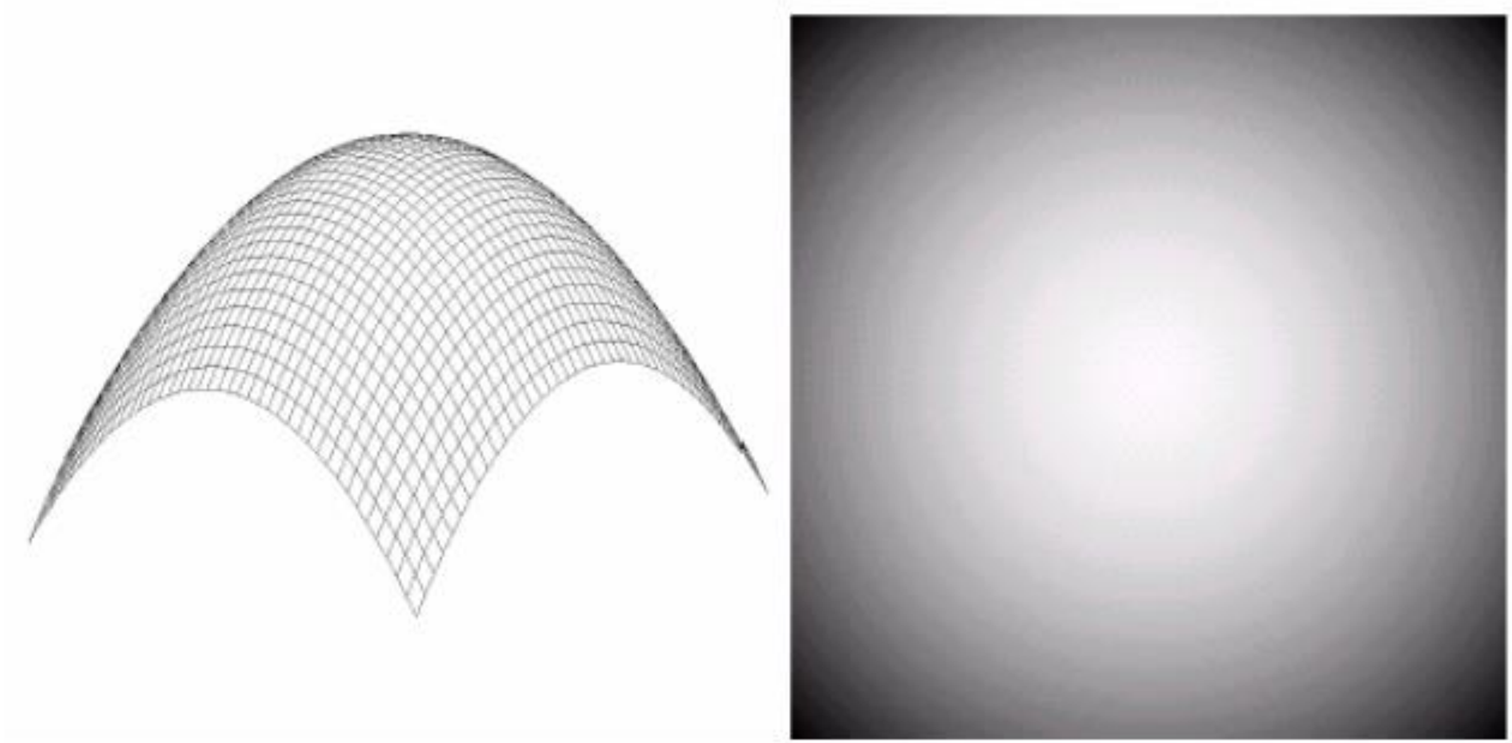
$$\text{Hence, } \nabla^2 f(x,y) \Leftrightarrow -\left[\left(u-\frac{M}{2}\right)^2+\left(v-\frac{N}{2}\right)^2\right]F(u,v)$$



Gain plot of the Laplcian filter

- Image enhancement by Laplacian

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

In the frequency domain, it becomes

$$g(x,y) = \mathcal{F}^{-1}\left\{\left[1 - \left\{(u - M/2)^2 + (v - N/2)^2\right\}\right] F(u,v)\right\}$$

original

sharpened

- High boost filtering in the frequency domain

High boost filter = A • Original image – low-pass filtered image

= (A-1) • Original image –high-pass filtered image

$$=(A-1)f(x,y)+f_{Hp}(x,y)$$

In the frequency domain, the above equation becomes

$$H_{Hb}(u,v)=(A-1)+H_{Hp}(u,v)$$

where $H_{Hp}(u,v)$ can be any of the aforementioned high-pass filter.

Further, we can have even more general representation

$$H_{Hb}(u,v)=k1+k2\star H_{Hp}(u,v)$$

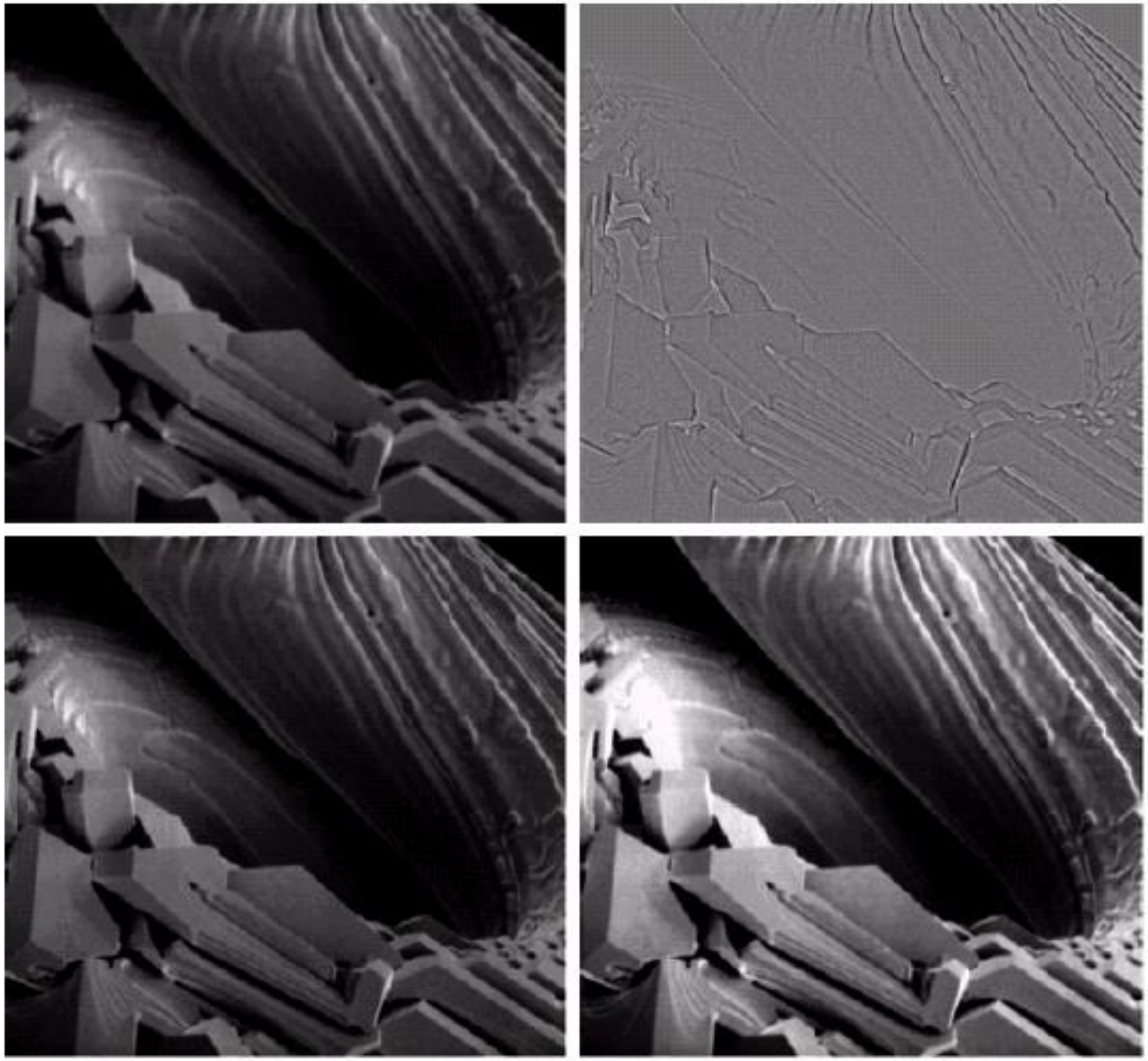


Figure 4.29

(a) original image

(b) edges detected by high-pass filter

(c) high-boot filtered with A = 2.0

(d) high-boot filtered with A = 2.7

4.10 同态滤波器－Homomorphic Filter (skip)

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在2.3.4节介绍过图像的“照射—反射”模型, 一幅图像$f(x, y)$其形式可以被表示成照射和反射两部分的乘积:

$$f(x, y) = i(x, y)\, r(x, y)$$

这个结果也可以用来研发一种频率域的图像处理方法, 同时达到灰度压缩和对比度增强的效果。但上式中的乘积傅里叶变换无法将之分开, 故需要作一些特殊的处理。整个处理过程如下:

定义: $Z(x, y) = \ln f(x, y) = \ln i(x, y) + \ln r(x, y)$

则有 $\Im\{ z(x, y) \} = \Im\{ \ln f(x, y) \}$

$= \Im\{ \ln i(x, y) \} + \Im\{ \ln r(x, y) \}$

或者 $Z(u, v) = F_i(u, v) + F_r(u, v)$

4.10 Homomorphic Filtering

这时, 可利用频谱域的滤波器对结果进行滤波. 用滤波器$H(u, v)$处理$Z(u, v)$.

$$S(u, v) = H(u, v)Z(u, v)$$

$$= H(u, v) F_i(u, v) + H(u, v)F_r(u, v)$$

处理后的图像可以再通过傅里叶反变换以及指数变换得到:

$$S(u, v) = \mathfrak{I}^{-1}\{ H(u, v)Z(u, v) \}$$

$$= \mathfrak{I}^{-1}\{ H(u, v) F_i(u, v) \} + \mathfrak{I}^{-1}\{ H(u, v)F_r(u, v) \}$$

令 $i'(u, v) = \mathfrak{I}^{-1}\{ H(u, v) F_i(u, v) \}$,

$$r'(u, v) = \mathfrak{I}^{-1}\{ H(u, v)F_r(u, v) \}$$

则, $S(x, y) = i'(x, y) + r'(x, y)$.

在进行对数反变换(指数变换), 就得到了增强后的图像

$$g(x, y) = e^{S(x, y)} = e^{i'(x, y)} \cdot e^{r'(x, y)} = i_0(x, y)\ r_0(x, y)$$

其中: $i_0(x, y)$和$r_0(x, y)$分别为输出图像的照射分量和反射分量.

4.10 同态滤波器

同态滤波流程图(关键是把照明和反射分量分开)

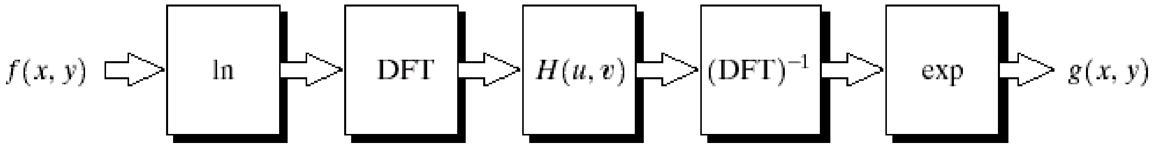

FIGURE 4.31 Homomorphic filtering approach for image enhancement.

图像照明分量通常以空间域的慢变化为特征, 而反射分量往往造成突变, 特别在不同物体的连接部分.这些特征导致图像对数的傅里叶变换的低频成分与照明分量相对应, 而高频成分与反射分量联系在一起.尽管这些联系只是大体上的近似, 但对某类图像增强有指导性的意义.

许多控制可以通过用同态滤波器对照射分量和反射分量滤波来实现.此时选择的滤波器函数$H(u, v)$能以不同的方式影响傅式变换的高频和低频分量.

此处选定的滤波函数同时有减少低频(照明)的贡献和增加高频(反射)的贡献

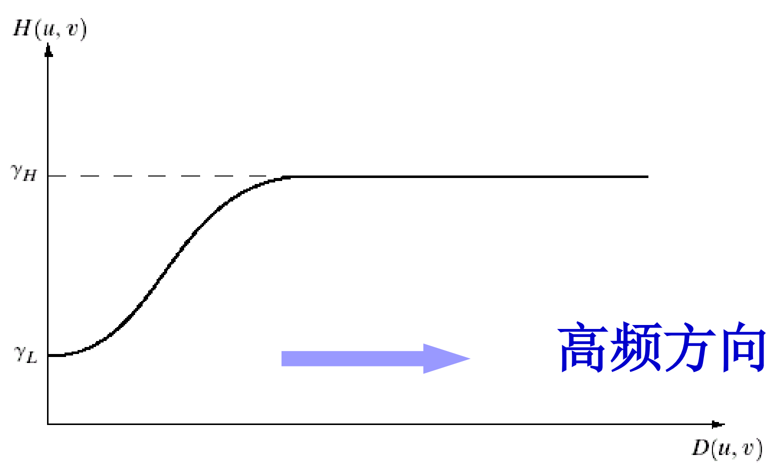

FIGURE 4.32 Cross section of a circularly symmetric filter function. $D(u, v)$ is the distance from the origin of the centered transform.

4.10 同态滤波器

例4.10 同态滤波增强

a b

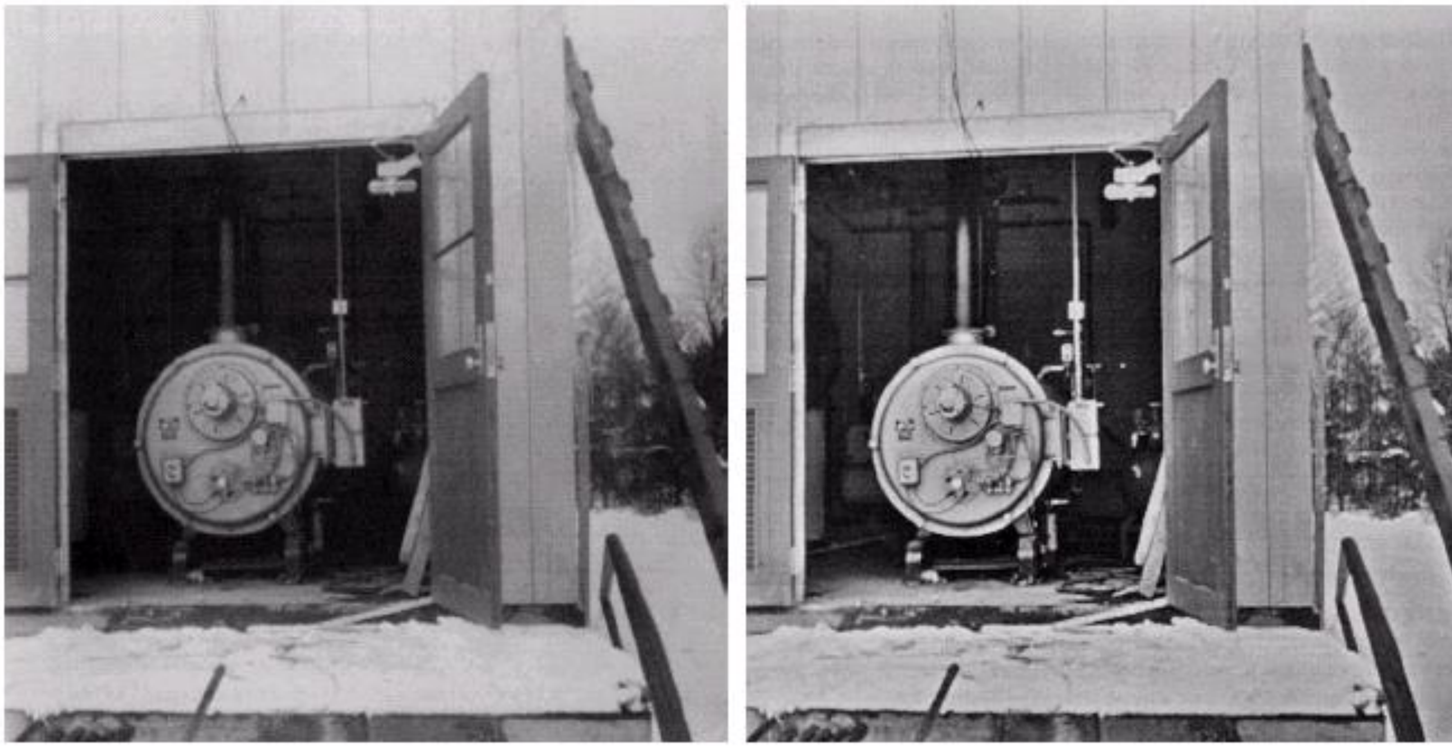

FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)

PS的结果

我们同态滤波结果

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- **Summary of Filtering in the frequency domain**

f(x,y) —original image

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{j2\pi(ux/M+vy/N)}$$

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

Filtering: finding filter H(u,v)

G(u,v)=H(u,v) F(u,v)

Inverse DFT: G(u,v) $\Longrightarrow$ g(x,y) — filtered image of *f*(*x*,*y*)

4.11 Implementation of DFT and inverse DFT

There are some tricks/algorithms to speed up the implementation of DFT and inverse DFT.

- Separability

The 2-D DFT can be implemented as TWO 1-D DFT as follows

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j\left(\frac{2\pi ux}{M}+\frac{2\pi vy}{N}\right)}$$

$$=\frac{1}{M}\sum_{x=0}^{M-1}f(x,y)e^{-j\left(\frac{2\pi ux}{M}\right)}\bullet\sum_{y=0}^{N-1}f(x,y)e^{-j\left(\frac{2\pi vy}{N}\right)}$$

$$=\frac{1}{M}\sum_{x=0}^{M-1}F(x,v)e^{-j\left(\frac{2\pi ux}{M}\right)}$$

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Therefore the 2-D DFT can be separated into two 1-D DFT as follows

(1) Obtain $F(x,v)=\sum_{y=0}^{N-1} f(x,y)e^{-j\left(\frac{2\pi vy}{N}\right)}$ using a 1-D DFT along y-axis

(i.e., row operation)

(2) Obtain $F(u,v)=\frac{1}{M}\sum_{x=0}^{M-1} F(x,v)e^{-j\left(\frac{2\pi ux}{M}\right)}$ using a 1-D DFT along x-axis

(i.e., column operation)

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Therefore the 2-D DFT can be separated into two 1-D DFT as follows

(1) Obtain $F(x,v)=\sum_{y=0}^{N-1} f(x,y)e^{-j\left(\frac{2\pi vy}{N}\right)}$ using a 1-D DFT along y-axis (i.e., row operation)

(2) Obtain $F(u,v)=\frac{1}{M}\sum_{x=0}^{M-1} F(x,v)e^{-j\left(\frac{2\pi ux}{M}\right)}$ using a 1-D DFT along x-axis (i.e., column operation)

■ Computing inverse DFT using DFT

主要思路是基于傅里叶变换的共轭性质, 利用正变换公式计算反变换. 傅里叶变换和反变换

$$F(u)=\frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}$$

$$f(x)=\sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M}$$

上式两边取复共轭并除以M, 有

$$\frac{1}{M}f^*(x)=\frac{1}{M}\sum_{u=0}^{M-1} F^*(u)e^{-j2\pi ux/M}$$

上式的形式和前向变换一样, 左边再取复共轭并乘以M就是对***F*(*u*)**反变换的函数.

Question : why do this?

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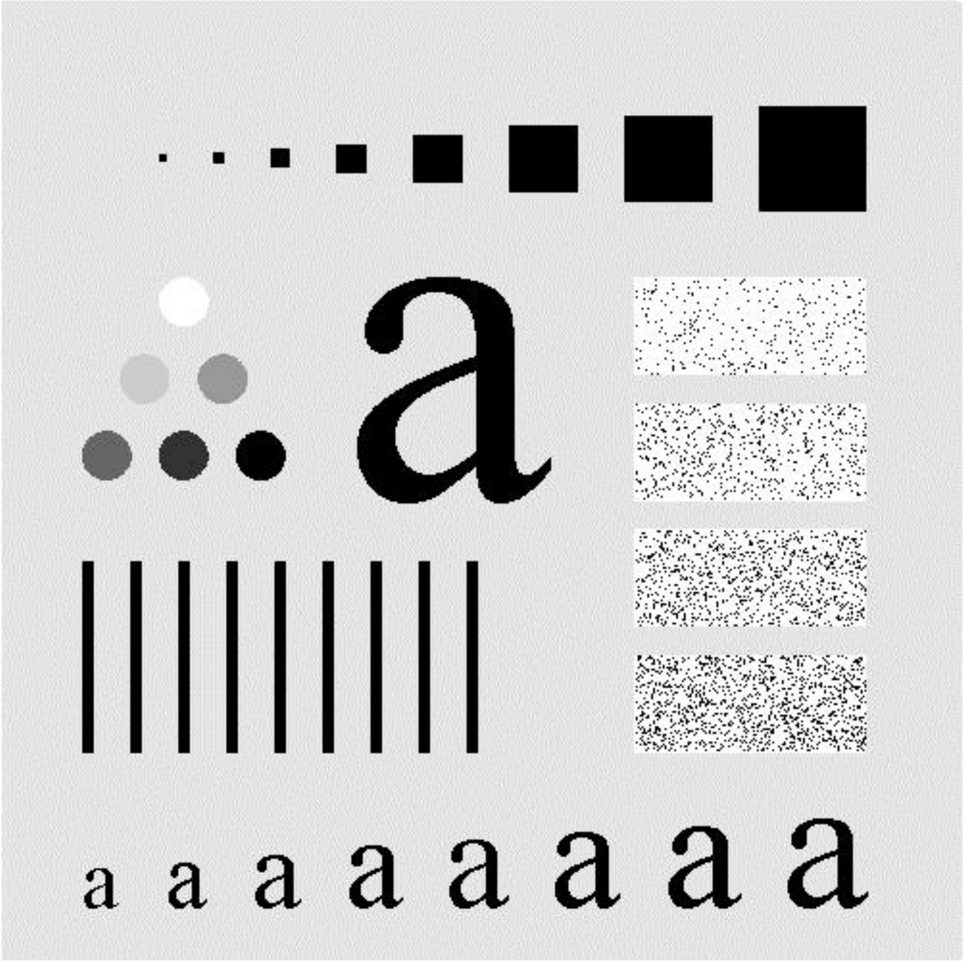
■ Boundary problems

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 0 & 5 & 7 & 0 \\ 2 & 2 & 5 & 2 \end{bmatrix}$$

$$\frac{1}{9} \times \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \qquad \frac{1}{16} \times \begin{array}{|c|c|c|} \hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array}$$

3X3 box filter — weighted averaging filter

- **Boundary problems**



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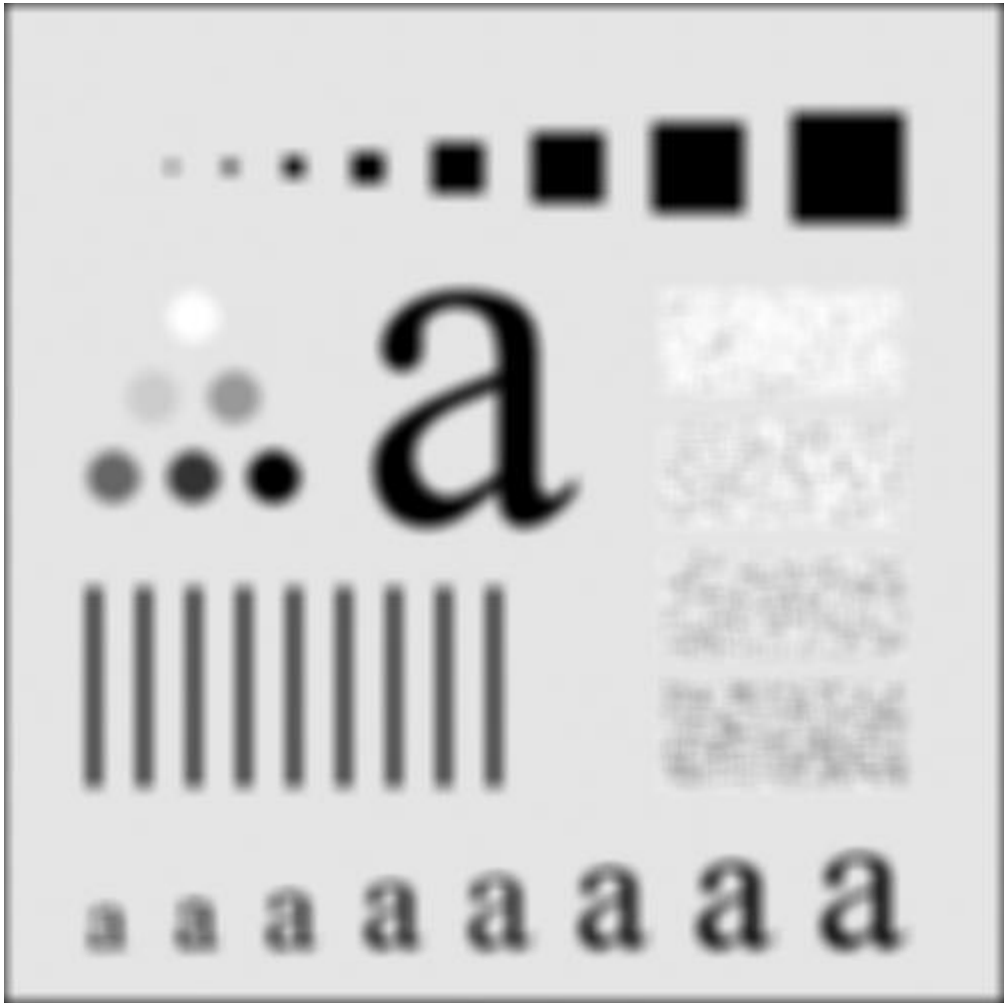
Boundary problems

Zero padding



- **Boundary problems**

periodic padding

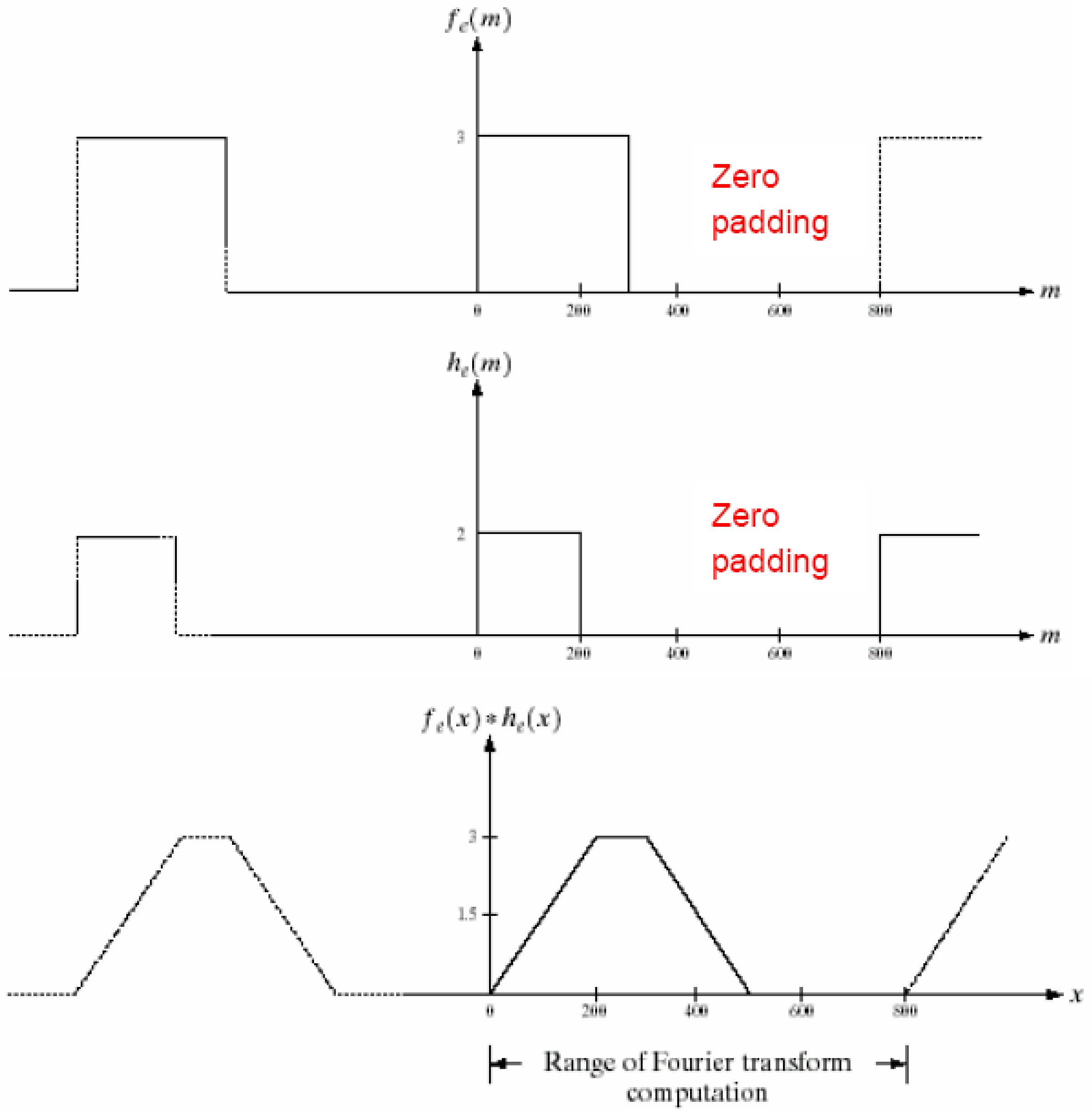
◆ zero padding to get rid of wraparound error

Let the number of sampling data in f(x) and h(x) be A and B, respectively, to avoid wraparound error, we need to add at least B-1 and A-1 zeros to the sampling data f(x) and h(x), respectively, so that

$$f_{ZP}(x)=\begin{cases} f(x) & 0\le x\le A-1 \\ 0 & A\le x\le P,\ P\ge A+B-1 \end{cases}$$ Zero padding

and

$$h_{ZP}(x)=\begin{cases} h(x) & 0\le x\le B-1 \\ 0 & B\le x\le P,\ P\ge A+B-1 \end{cases}$$ Zero padding



The resultant resolution in DFT doubles without wraparound error

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- Increasing resolution by zero padding in 2-D image

Let the size of two images before convolution be AXB and CXD, to avoid wraparound error, these two images should be zero-padded to the size of PXQ, where $P \geq A+C-1$ and $Q \geq B+D-1$, and

$$f_{ZP}(x,y)=\begin{cases} f(x,y) & 0 \leq x \leq A-1 \text{ and } 0 \leq y \leq B-1 \\ 0 & A \leq x \leq P,\ B \leq y \leq Q \end{cases} \tag{4.6.25}$$

and

$$h_{ZP}(x,y)=\begin{cases} h(x,y) & 0 \leq x \leq C-1 \text{ and } 0 \leq y \leq D-1 \\ 0 & C \leq x \leq P,\ D \leq y \leq Q \end{cases} \tag{4.6.26}$$

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利用傅立叶变换在频率域计算卷积（滤波）的正确步骤

(1) 对两个函数作零延拓至适当的长度;

(2) 做两个序列的傅里叶变换（长度和延拓后的一致）;

(3) 将两个序列的傅里叶变换相乘;

(4) 计算乘积的傅里叶反变换.

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4.11.3 快速傅里叶变换

已经有成熟的结果，但一直是重要的研究课题.

先看看直接利用定义公式计算傅里叶变换的代价. 在一维情形, **M**个点的傅里叶变换(**DFT**)需要M^2次乘法运算. 这个计算量在很多实际问题中是无法承受的.

快速傅里叶变换（**FFT**）的出现后, 把这个运算降低为$M\log_2 M$. 正是由于**FFT**的发展, 才最终使得傅里叶变换成为信号处理的一种最基础的工具。例如, 当**M=1024**时, 未经优化的方法需要10^6次操作, 而快速算法只需要10^4, **100:1**的优势. 当点数和维数增大时, 优势更加明显。

参考09级数媒班学生的报告

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4.11.3 快速傅里叶变换

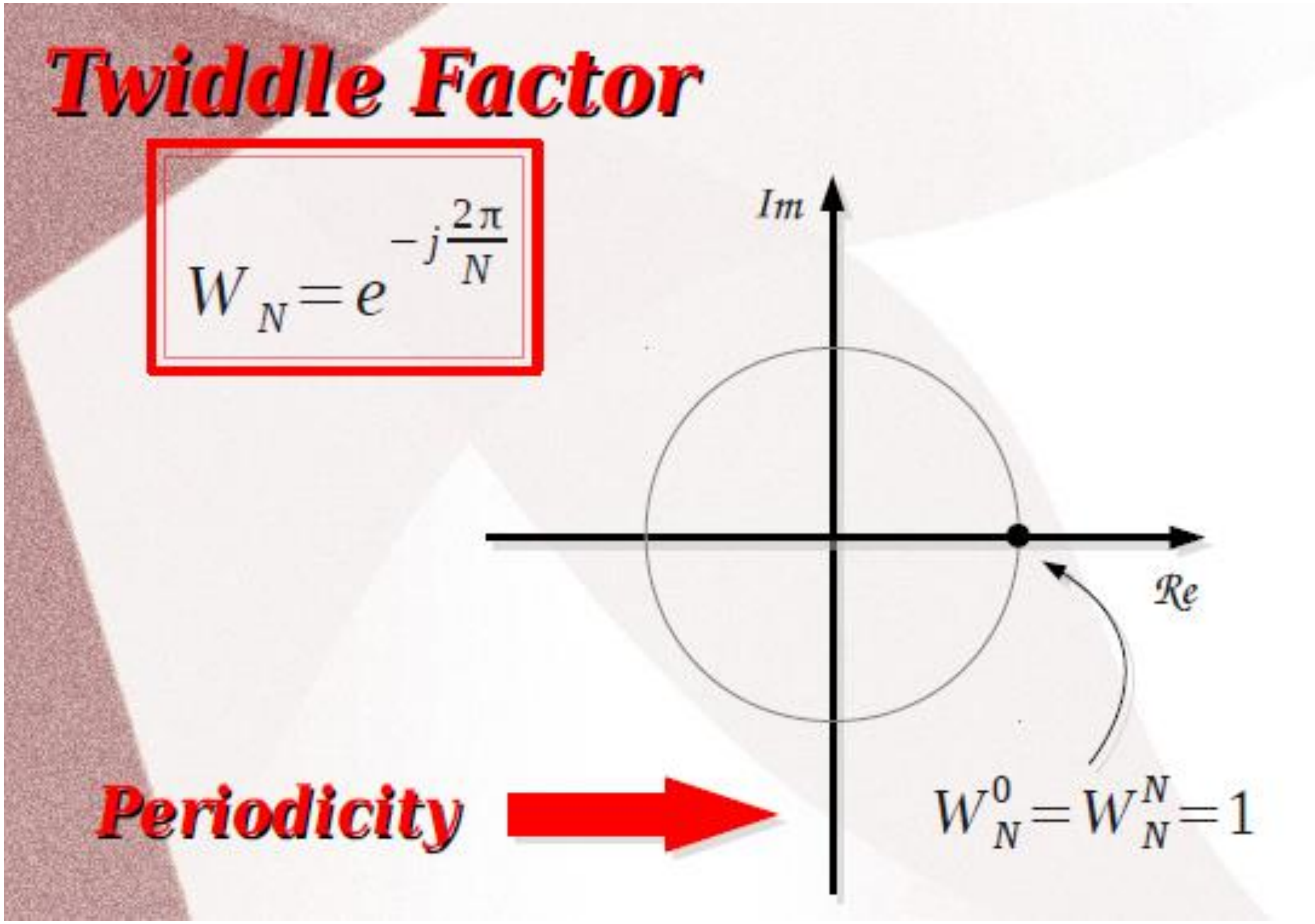
快速傅里叶变换的基本思想: **FFT**基于逐次倍乘法(**Successive doubling method**). 这个方法的主要思想是利用傅里叶变换(基底)的性质:

$$W_M = e^{j2\pi/M} = e^{j2\pi/(2M/2)}$$

将2M个数据的傅里叶变换转化为2组M个数据的傅里叶变换. 这样, 原来$4M^2$的运算量就降低为$2M^2$的运算量了. 依次类推, 便可得到快速算法。

在很多场合W_M被称为**Twiddle Factor**（旋转因子）。

4.11.3 快速傅里叶变换



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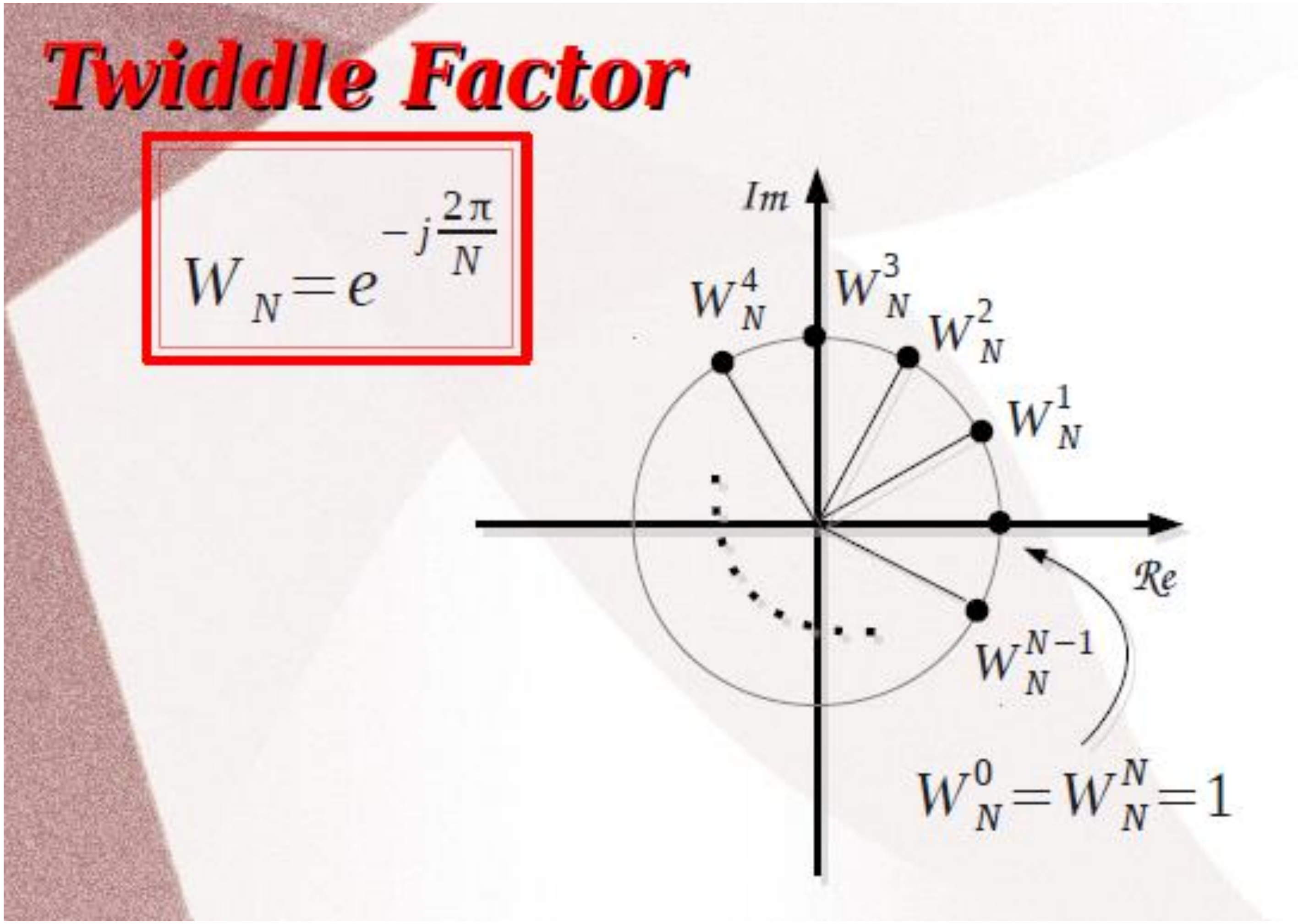
4.11.3 快速傅里叶变换



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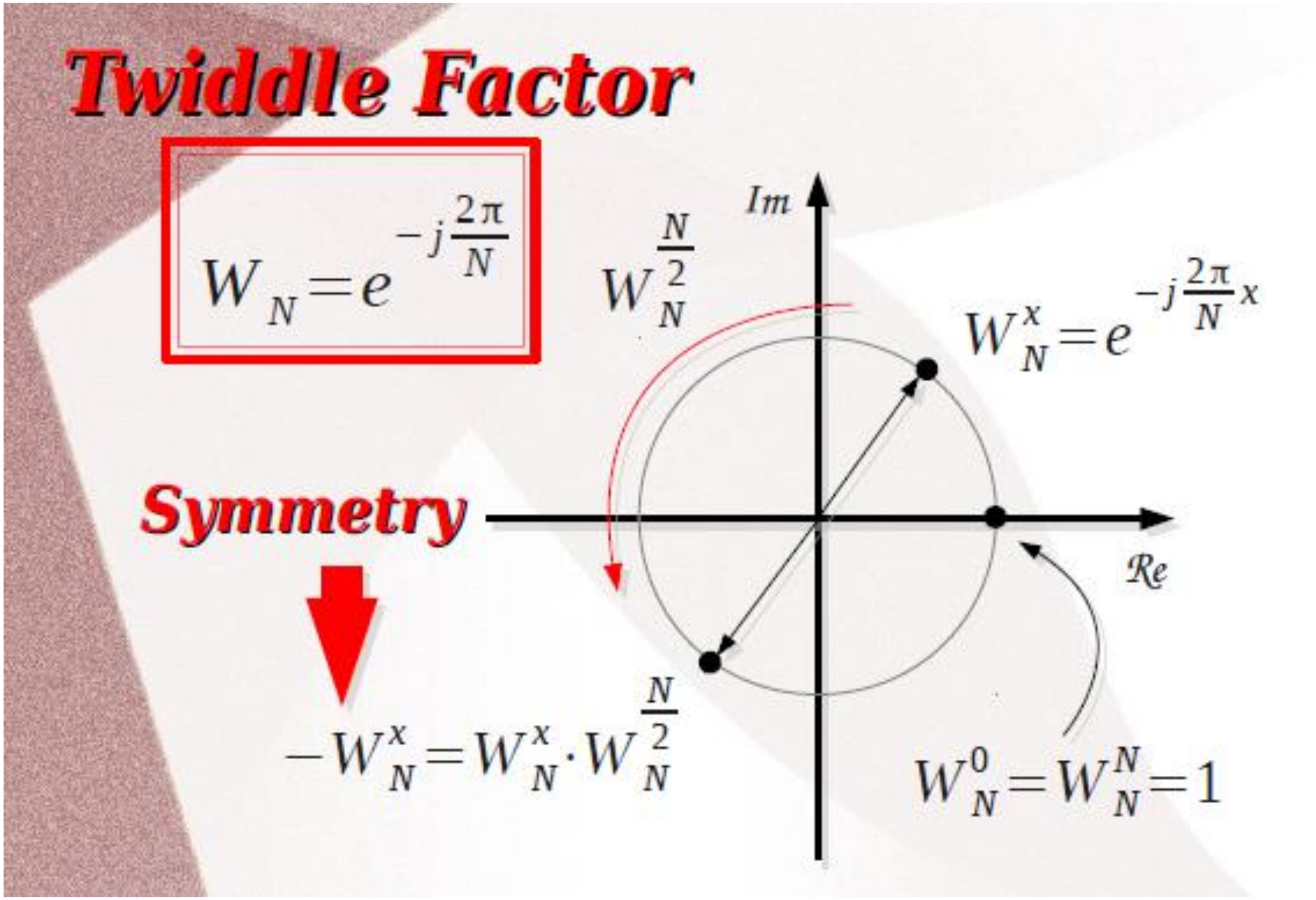
4.11.3 快速傅里叶变换



4.11.3 快速傅里叶变换

Twiddle Factor partial conclusion

$$W_N = e^{-j\frac{2\pi}{N}}$$

***Periodicity :** $W_N^0 = W_N^N = 1$

***Symmetry :** $-W_N^x = W_N^{x+\frac{N}{2}}$

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4.11.3 快速傅里叶变换

以一维为例作简单介绍, 二维情形可以通过两次一维计算实现. 将傅里叶变换

$$F(u)=\frac{1}{M}\sum_{x=0}^{M-1}f(x)e^{-j2\pi ux/M}$$

$$\rightarrow \quad F(u)=\frac{1}{M}\sum_{x=0}^{M-1}f(x)W_M^{ux} \qquad (W_M=e^{-j2\pi/M})$$

仅考虑M具有2的幂次方**M=2ⁿ**的形式, **n**为正整数. 故**M**还可以表示为**2K, K = M/2**也是正整数. 从而

$$F(u)=\frac{1}{2K}\sum_{x=0}^{2K-1}f(x)W_{2K}^{ux}$$

$$=\frac{1}{2}\left[\frac{1}{K}\sum_{x=0}^{K-1}f(2x)W_{2K}^{u(2x)}+\frac{1}{K}\sum_{x=0}^{K-1}f(2x+1)W_{2K}^{u(2x+1)}\right]$$

4.11.3 快速傅里叶变换

注意: $W_K = e^{-j2\pi/K}$, K = M/2, 且 $W_{2K}^{2ux} = (e^{-j2\,2\pi/2K})^{ux} = W_K^{ux}$, 故有

$$F(u)=\frac{1}{2}\left[\frac{1}{K}\sum_{x=0}^{K-1}f(2x)W_K^{ux}+\frac{1}{K}\sum_{x=0}^{K-1}f(2x+1)W_K^{ux}W_{2K}^{u}\right]$$

对 u = 0, 1, …, K-1, 定义

$$F_{even}(u)=\frac{1}{K}\sum_{x=0}^{K-1}f(2x)W_K^{ux} \qquad \textbf{(4.6.41)}$$

$$F_{odd}(u)=\frac{1}{K}\sum_{x=0}^{K-1}f(2x+1)W_K^{ux} \qquad \textbf{(4.6.42)}$$

$$F(u)=\frac{1}{2}\left[F_{even}(u)+F_{odd}(u)W_{2K}^{u}\right] \qquad \textbf{(4.6.43)}$$

又因为对任意的正整数K, 有 $W_K^{u+K} = W_K^u$ 和 $W_{2K}^{u+K} = -W_{2K}^u$.
从上面几个公式可以得到

$$F(u+K)=\frac{1}{2}\left[F_{even}(u)-F_{odd}(u)W_{2K}^{u}\right] \qquad \textbf{(4.6.44)}$$

4.11.3 快速傅里叶变换

分析:

- 观察(4.6.41)一(4.6.44)式, 易见一个M点的傅里叶变换可以把原始表达式分成“前”/“后”两部分来计算.
- “前”一半M/2个点上的变换值, 可以通过计算(4.6.41)、(4.6.42)然后带入(4.6.43)得到; “后”一半可以直接从(4.6.41)得到, 不需要另外的变换运算.
- 注意(4.6.41)和(4.6.42). 它们都可以看成是$K = M/2$个点的傅里叶变换,, $K = 2^{n-1}$, 它们当然可以作类似的处理, 得到另外4个需要计算的$K/2$个点的傅里叶变换.
- 如此循环推导下去, 直到最后剩下若干组两个点对. 用归纳法可以证明, FFT的计算复杂性是$M\log_2 M$.
- 参考09级数媒班学生的报告

◆ The FFT algorithm

Assume that there are M sampling data, K=M/2, so we need to find M values of F(u), u = 0, 1, 2, …, M-1.

STEP 1: Solve K terms of $F_{odd}(u)$ using (4.6.42)

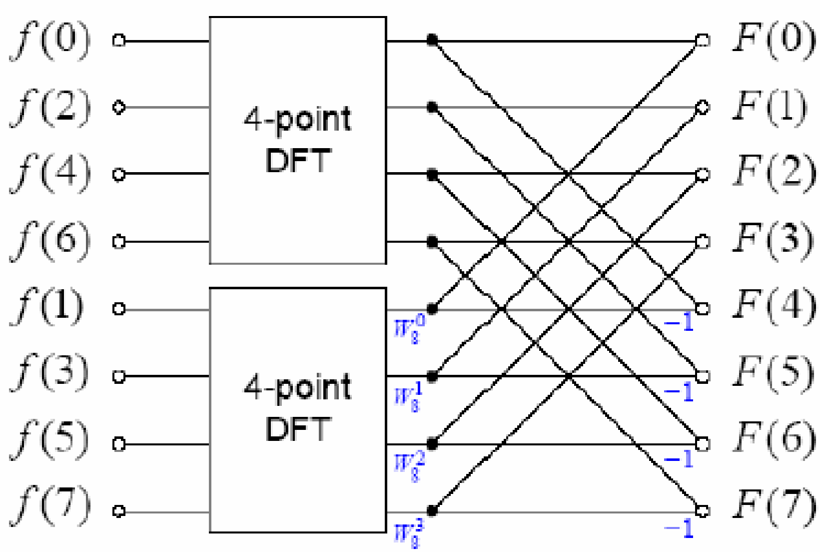
STEP 2: Solve K terms of $F_{even}(u)$ using (4.6.41)

STEP 3: Solve K terms of F(u) for u = 0, 1, 2, …, K-1 using (4.6.43).

STEP 4: Solve K terms of F(u) for u = K, K+1, …, M-1 using (4.6.44)

Example

Let M=8, K=4, the FFT is solved in the manner depicted on the right



4.11 傅里叶变换的实现

曲线表示了FFT相对于直接实现DFT的计算优势, 该优势随着n的增长而快速增长.

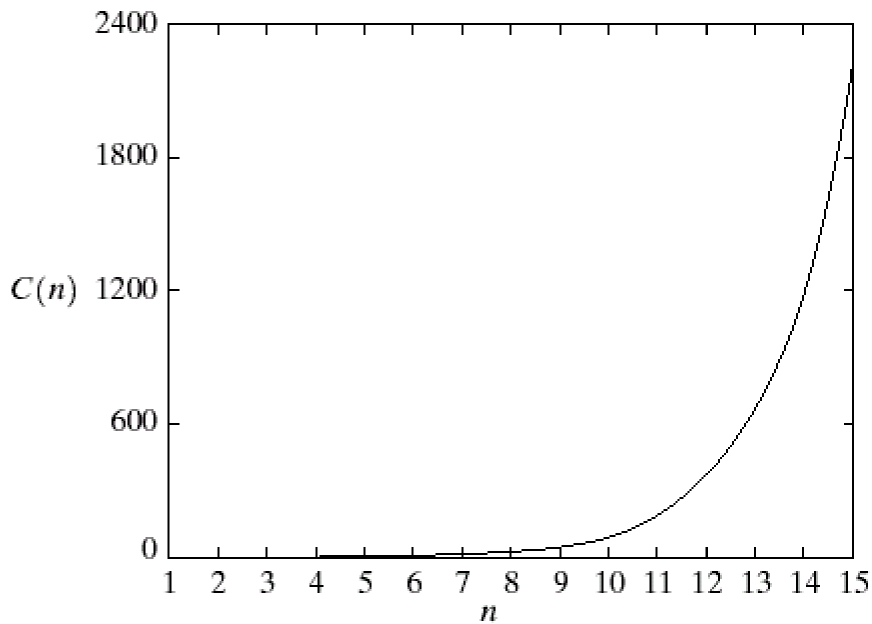


FIGURE 4.42 Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of *n*.

$$C(M)=\frac{M^2}{M\log_2 M}$$

表示直接用公式计算FT的计算量和FFT计算量的比。

因为$M=2^n$，故上图中的$C(n)=2^n/n$。

第四章要点

中山大學

◆富里叶变换的基本概念和公式

◆卷积定理→建立了空间域滤波器和频谱域滤波其之间的关系；

◆几个简单的频谱域滤波器：理想、高斯、巴特沃思等

◆注意边界的处理（**padding**）

◆快速傅立叶变换（**FFT**）

09数媒同学关于FFT的 presentation

http://gitl.sysu.edu.cn/dip/presentation/dmp4fft.pdf

第四章附录一

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Correlation Operation（相关运算）和图像配准（registration）、物体识别

- Convolution and correlation theorem
  - The cross correlation of two functions f(x, y) and h(x, y) is defined as

$$f(x,y)\circ h(x,y)\triangleq\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{m=0}^{N-1}f^*(m,n)h(x+m,y+n)\qquad(4.6.30)$$

In dealing with images, all f(x, y) are real, hence f*(x, y) = f(x, y), thus

$$\begin{aligned}f(x,y)\circ h(x,y)&\triangleq\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^*(m,n)h(x+m,y+n)\\&=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x+m,y+n)\qquad(4.6.31)\end{aligned}$$

which is very similar to the convolution operation

$$f(x,y)*h(x,y)\triangleq\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)\qquad(4.6.27)$$

except the difference in the argument of h(s, t).

Spatial Correlation and Convolution (2D)

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Origin $f(x, y)$

0	0	0	0	0
0	0	0	0	0
0	0	1	0	0
0	0	0	0	0
0	0	0	0	0

$w(x, y)$

1	2	3
4	5	6
7	8	9

(a)

Padded f

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(b)

Initial position for w

1	2	3	0	0	0	0	0	0
4	5	6	0	0	0	0	0	0
7	8	9	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(c)

Full correlation result

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	9	8	7	0	0	0
0	0	0	6	5	4	0	0	0
0	0	0	3	2	1	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(d)

Cropped correlation result

0	0	0	0	0
0	9	8	7	0
0	6	5	4	0
0	3	2	1	0
0	0	0	0	0

(e)

Rotated w

9	8	7	0	0	0	0	0	0
6	5	4	0	0	0	0	0	0
3	2	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(f)

Full convolution result

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	1	2	3	0	0	0
0	0	0	4	5	6	0	0	0
0	0	0	7	8	9	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

(g)

Cropped convolution result

0	0	0	0	0
0	1	2	3	0
0	4	5	6	0
0	7	8	9	0
0	0	0	0	0

(h)

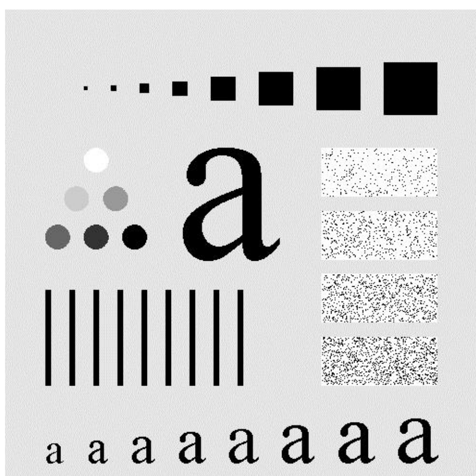
FIGURE 3.30 Correlation (middle row) and convolution (last row) of a 2-D filter with a 2-D discrete, unit impulse. The 0s are shown in gray to simplify visual analysis.

Due to this similarity, it can easily be shown that

$$\mathscr{F}[f(x,y)\circ h(x,y)] = F^*(u,v)H(u,v) \tag{4.6.31}$$

and

$$\mathscr{F}[f^*(x,y)h(x,y)] = F(u,v)\circ H(u,v) \tag{4.6.32}$$



$$\begin{bmatrix} w_1 & w_2 & w_3 \\ w_4 & w_5 & w_6 \\ w_7 & w_8 & w_9 \end{bmatrix}$$

$$\begin{bmatrix} w_9 & w_8 & w_7 \\ w_6 & w_5 & w_4 \\ w_3 & w_2 & w_1 \end{bmatrix}$$

◆ Use of cross correlation in pattern matching **(registration)**

$h(x, y)$ — template of a smaller size;

$f(x, y)$ — test image of a larger size which contains a portion of image that copies $h(x, y)$ at the same size and orientation;

Problem — to find the required shifting distance (x_0, y_0) of $h(x, y)$ so that it completely overlaps that portion of $f(x, y)$.

This problem can be solve using the cross correlation function:

◆ Use of cross correlation in pattern matching (registration)

This problem can be solve using the cross correlation function:

$$\operatorname{Arg}\ \max f(x,y)\circ h(x,y)$$

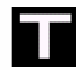
$$= \operatorname{Arg}\ \max \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x+m,y+n)$$

That is, to find the position $(x, y) = (x_0, y_0)$ at which following function $g(x,y)$ has maximum cross correlation.

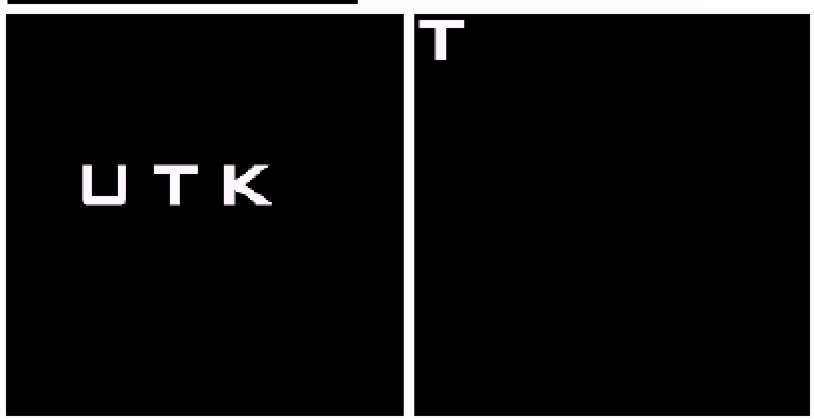
$$g(x,y) \triangleq f(x,y)\circ h(x,y)$$

For example, we are given a test image containing three letters ‘U’, ‘T’, ‘K’ (on the left) and a template containing a letter ‘T’ (on the right).

Input image 256X256



Template 38X42

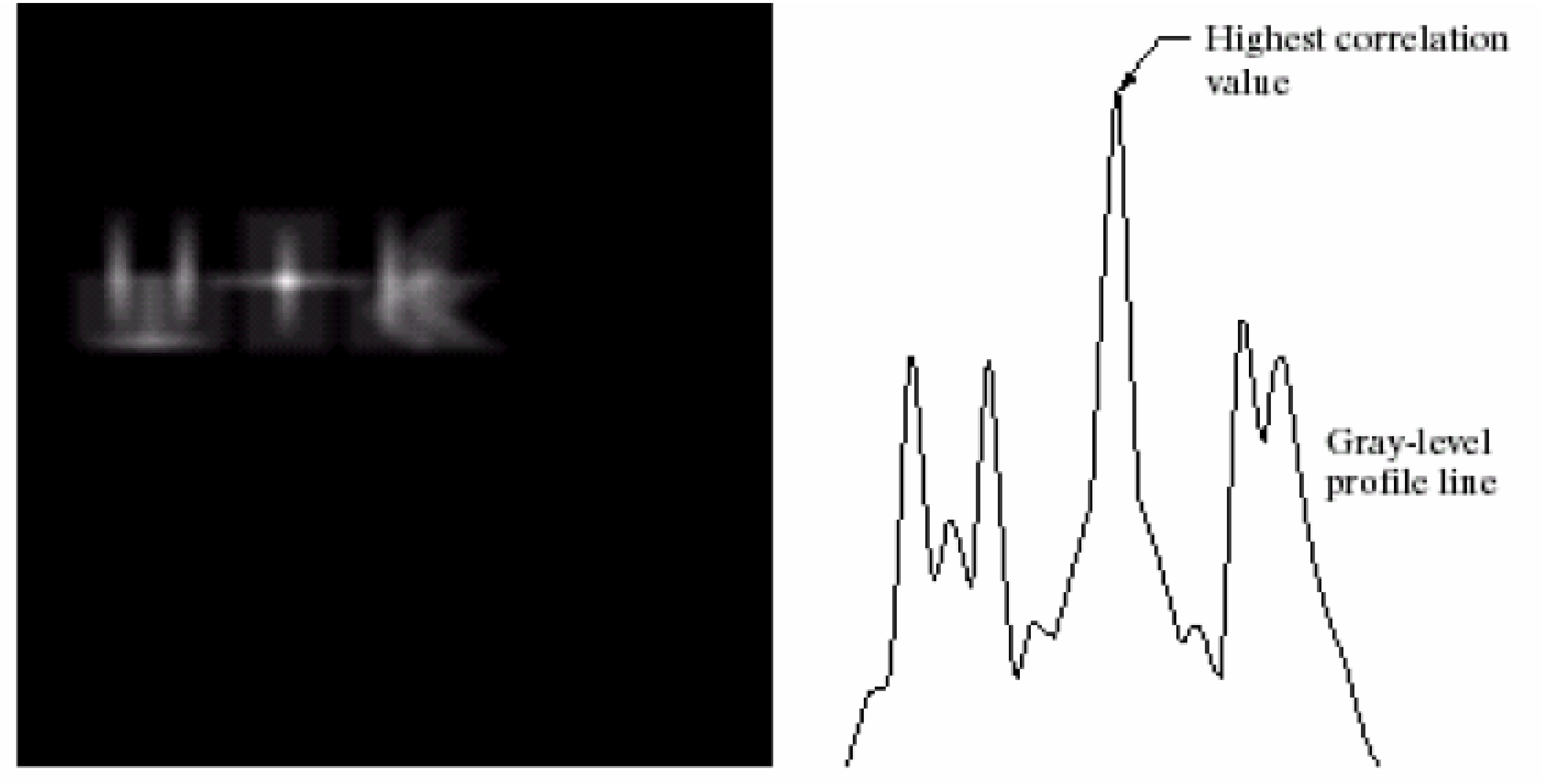


The need and rule of zero padding is the same as convolution.

293X297

256+38-1 256+42-1

zero padding input image and template



Result of cross correlation

◆ Autocorrelation function （自相关函数）

When h(x, y) and f(x, y) are identical, the cross correlation function becomes the auto-correlation function

$$f(x,y)\circ f(x,y) \triangleq \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n) f(x+m, y+n)$$

Using Eq. (4.6.31) and (4.6.32), it can easily be shown that

$$\mathscr{F}\left[f(x,y)\circ f(x,y)\right] = F^*(u,v)F(u,v) = \left|F(u,v)^2\right| \qquad (4.6.33)$$

$$\mathscr{F}\left[\left|f(x,y)\right|^2\right] = \mathscr{F}\left[f^*(x,y)f(x,y)\right] = F(u,v)\circ F(u,v) \qquad (4.6.34)$$

Eq. (4.6.34) shows Auto-correlation of F(u,v) equals the power spectrum of f(t).

Notice

- 补充题：信息隐藏
- 习题: 4.1、4.8、4.9、4.15、4.22
- 选择题: 4.16、4.27
- 选作: 验证傅里叶变换和反变换的周期效应以及对卷积(滤波)结果的影响.
- Project: 04-01 ~ 04-05
- 纠错: p131

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第四章预备知识小测验：

1、给出在[0, T]区间上定义的函数*f(x)*傅立叶级数收敛的条件。（给出结论即可，不需要证明）

2、给定下列函数 *f(x)*

$$f(x)=\begin{cases}1, & x\in[0,\frac{1}{2}]\\ -1, & x\in(\frac{1}{2},1]\end{cases}$$

① 将函数*f(x)*展开成傅立叶级数，计算各项相关的傅立叶系数

② 指出出下标增大时傅立叶系数的变化趋势；

$$f(x)=\sum_{n=-\infty}^{+\infty}c_n e^{j\frac{2\pi}{T}nx} \qquad \mathrm{T}=1$$

一些概念：

1、定义在有限区间[0，T]上的函数$f(x)$

2、奇延拓——关于原点对称、偶延拓——关于纵坐标对称、

3、周期延拓——

4、周期函数的积分区间问题

5、函数$f(x)$的三角级数展开 （和教材上的相同，只是定义区间有所不同）

$$f(x)=a_0+\sum_{n=1}^{+\infty}\{a_n\cos(\tfrac{2\pi}{T}nx)+b_n\sin(\tfrac{2\pi}{T}nx)\}$$

$$a_0=\frac{1}{T}\int_0^T f(x)dx$$

$$a_n=\frac{2}{T}\int_0^T f(x)\cos\left(\tfrac{2\pi}{T}nx\right)dx$$

$$b_n=\frac{2}{T}\int_0^T f(x)\sin\left(\tfrac{2\pi}{T}nx\right)dx$$

6、复数形式的傅立叶级数

$$f(x)=\sum_{n=-\infty}^{+\infty}c_n e^{j\frac{2\pi}{T}nx}$$

$$c_n=\frac{1}{T}\int_0^T f(x)e^{-j2\pi nx}dx$$

复数形式和实数形式等价（高等数学），系数可以相互转换：

$$c_0=a_0,\ c_n=\frac{a_n-jb_n}{2},\ c_{-n}=\frac{a_n+jb_n}{2},\ n=1,2,3,\cdots\cdots$$

测验题：高等数学下册P243有类似的结果