Multiple Integrals

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Bibliography

- James Stewart, Calculus Early Transcendentals, Brooks Cole Cengage Learning, 2012.
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  2. Integrals depending on a parameter:
  3. Line Integrals: Chapter 16,
  4. Surface Integrals: Chapter 16,
  5. Vector Calculus: Chapter 16,
  6. Series: Chapter 11.
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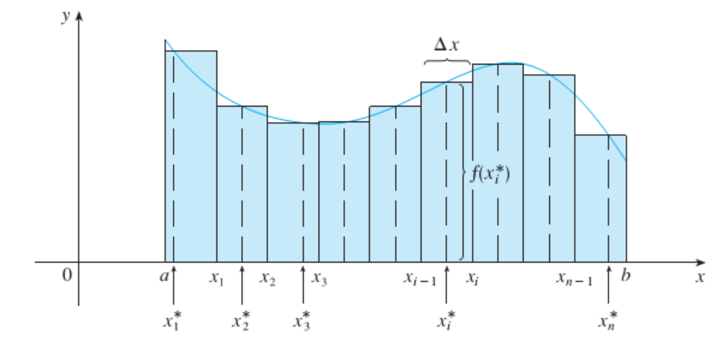
Multiple Integrals

Double Integrals

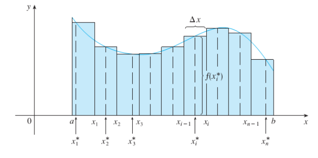
Review of the Definite Integral



$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Double Integrals

Review of the Definite Integral



1. divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$
2. choose sample points x_i^* in these subintervals,
3. form the Riemann sum $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$
4. take the limit $V = \int\limits_a^b f(x)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

Volumes and Double Integrals

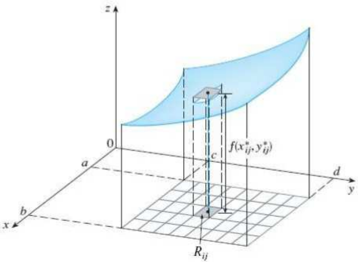


FIGURE 4

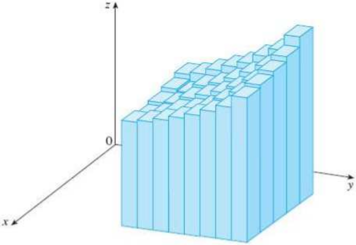
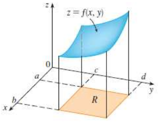
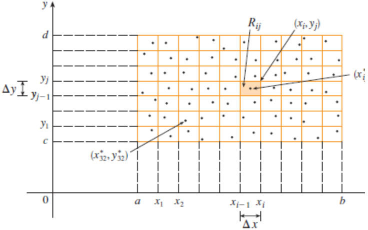


FIGURE 5

Goal: find the volume of $S := \{(x, y, z) \in \mathbb{R}^3 | 0 \le z \le f(x, y), (x, y) \in \mathbb{R}.\}$

Volumes and Double Integrals





1. divide $[a, b]$ into m subintervals and $[c, d]$ into n subintervals, each of equal width.
2. choose sample points $(x_{ij}^*, y_{ij}^*) \in R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$,
$$V(R_{ij}) \approx f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y.$$
3. Riemann Sum $V = \sum_{i=1}^{m}\sum_{j=1}^{n} R_{ij} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y = S_{m,n}$.
4. take the limit $\lim\limits_{m,n\to\infty} S_{mn}$.

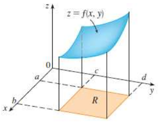
Double Integrals

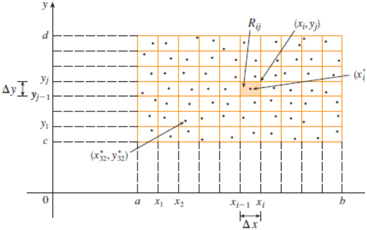
Definition

The double integral of f over the rectangle R is

$$\iint_R f(x,y)\,dxdy = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

if this limit exists.



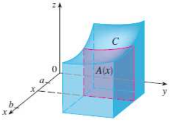


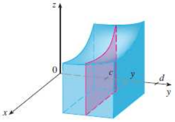
Fubini's Theorem

Theorem (Fubini's Theorem)

If f is continuous on the rectangle $R = \{(x, y) | a \leq x \leq b, c \leq x \leq d\}$, then

$$\iint_R f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$



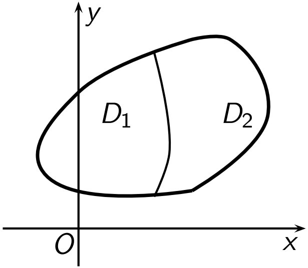


- $\iint\limits_R f(x,y)dxdy = V = \int\limits_a^b A(x)dx$, where
- $A(x) = \int\limits_c^d f(x,y)dy.$

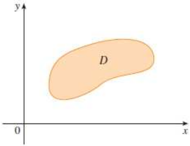
Triple Integrals

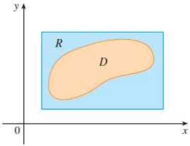
Properties

1. $\iint\limits_R [f(x,y) + g(x,y)]\,dxdy = \iint\limits_R f(x,y)\,dxdy + \iint\limits_R g(x,y)\,dxdy.$
2. $\iint\limits_R \alpha f(x,y)\,dxdy = \alpha \iint\limits_R f(x,y)dxdy.$
3. $\iint\limits_D f(x,y)\,dxdy = \iint\limits_{D_1} f(x,y)\,dxdy + \iint\limits_{D_2} f(x,y)\,dxdy.$



Double Integrals over general regions





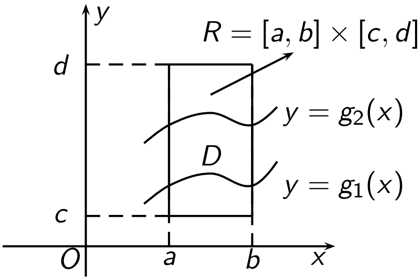
$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \in R \setminus D. \end{cases}$$

and

$$\iint_D f(x,y)dxdy = \iint_R F(x,y)dxdy.$$

Double Integrals over plane regions of type I

$$D = \{(x, y) | a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}.$$



By Fubini's Theorem,

$$\iint_D f(x, y) dxdy = \iint_R F(x, y) dxdy$$

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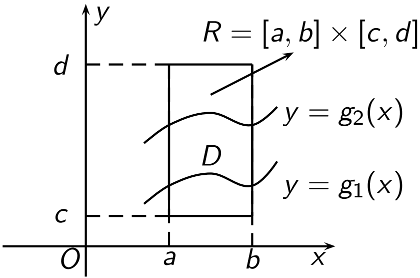

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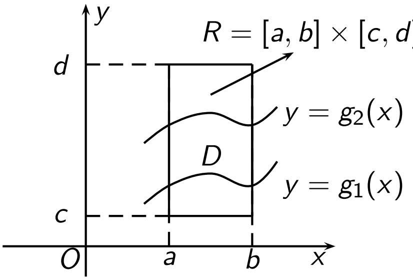


By Fubini's Theorem,

$$\iint_D f(x,y)dxdy = \iint_R F(x,y)dxdy$$
$$= \int_a^b dx \int_c^d F(x,y)dy = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy.$$

Double Integrals over general regions

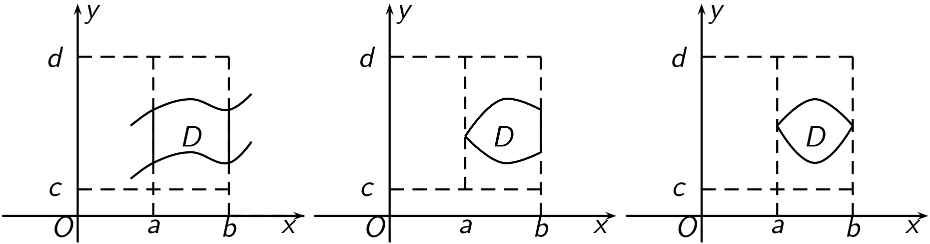
Double Integrals over plane regions of type I

If f is continuous on a type I region D such that

$$D = \{(x,y) | a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x,y)dxdy = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy.$$



Double Integrals over general regions

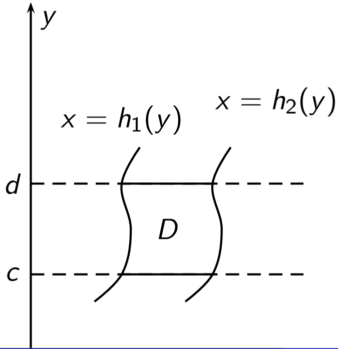
Double Integrals over plane regions of type II

If f is continuous on a type II region D such that

$$D = \{(x, y) | c \le y \le d,\ \ h_1(y) \le x \le h_2(y)\}$$

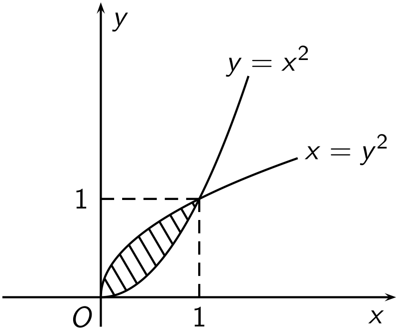
then

$$\iint_D f(x, y)dxdy = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x, y)dx.$$



Example

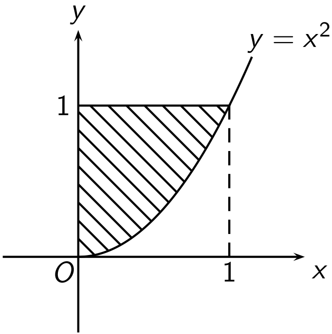
Evaluate $\iint\limits_D x^2 (y - x)\, dxdy$ where D is the region bounded by $y = x^2$ and $x = y^2$.



Change The Order of Integration

Example

Evaluate $I = \iint\limits_D xe^{y^2} dx$, where $D = \{(x, y) : 0 \leq x \leq 1, x^2 \leq y \leq 1\}$.



Change The Order of Integration

Exercise

Change the order of integration $I = \int\limits_a^b dx \int\limits_{f_1(x)}^{f_2(x)} f(x,y)dy$.

1. From the iterated integral, sketch the region of integration,
2. Divide it into regions of type II, for instance,
$$D_i = \{(x,y) | c_i \le y \le d_i,\ g_i(y) \le x \le h_i(x)\},$$
3. $$\int\limits_a^b dx \int\limits_{f_1(x)}^{f_2(x)} f(x,y)dy = \sum_i \int\limits_{c_i}^{d_i} dy \int\limits_{g_i(y)}^{h_i(y)} f(x,y)dx.$$

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Exercise

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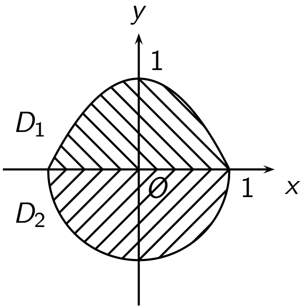
3. $$\int\limits_a^b dx \int\limits_{f_1(x)}^{f_2(x)} f(x,y)dy = \sum_i \int\limits_{c_i}^{d_i} dy \int\limits_{g_i(y)}^{h_i(y)} f(x,y)dx.$$

Similar for $\int\limits_c^d dy \int\limits_{f_1(y)}^{f_2(y)} f(x,y)dx.$

Change The Order of Integration

Example

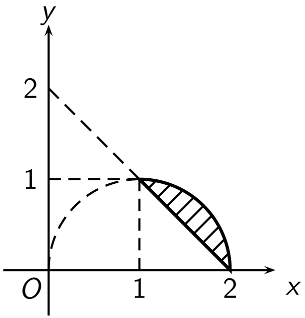
Change the order of integration $\int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{1-x^2} f(x, y)\, dy$.



Change The Order of Integration

Example

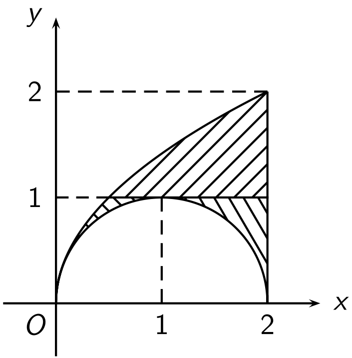
Change the order of integration $\int\limits_0^1 dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx$.



Change The Order of Integration

Example

Change the order of integration $\int\limits_0^2 dx \int\limits_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)\,dy$.

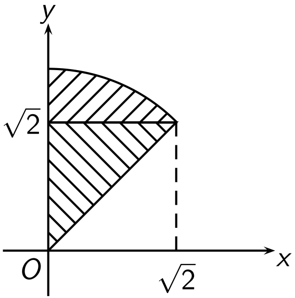


Change The Order of Integration

Example

Change the order of integration

$$\int_0^{\sqrt{2}} dy \int_0^{y} f(x,y)\,dx + \int_{\sqrt{2}}^{2} dy \int_0^{\sqrt{4-y^2}} f(x,y)\,dx.$$



Double Integrals involving Absolute Value Functions

Evaluate $\iint\limits_{D} |f(x,y)|\, dxdy$. The curve $f(x,y) = 0$ divides D into two parts,

$$D^{+} = D \cap \{f(x,y) \geq 0\}, D^{-} = D \cap \{f(x,y) \leq 0\}.$$

$$\boxed{\iint\limits_{D} |f(x,y)|\, dxdy = \iint\limits_{D^{+}} f(x,y)\, dxdy - \iint\limits_{D^{-}} f(x,y)\, dxdy} \tag{1}$$

Double Integrals involving Absolute Value Functions

Evaluate $\iint\limits_D |f(x,y)|\,dxdy$. The curve $f(x,y)=0$ divides D into two parts,

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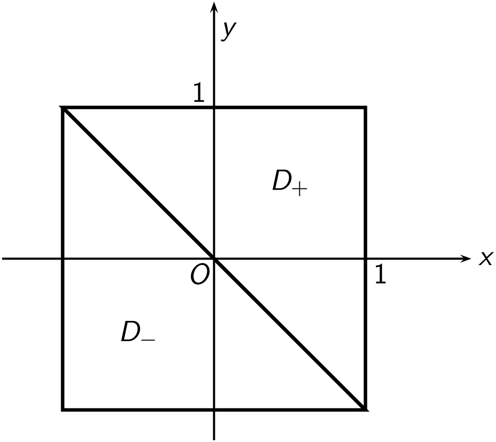
Algorithm

1. Sketch the curve $f(x,y)=0$ to find D^+, D^-.
2. Apply formula (1).

Double Integrals involving Absolute Value Functions

Example

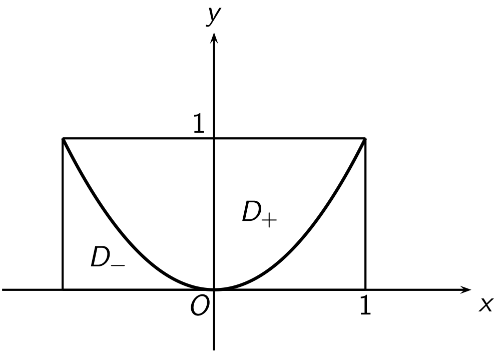
Evaluate $\iint\limits_D |x+y|dxdy, D : \left\{(x,y) \in \mathbb{R}^2 \, ||x \leq 1| \, , |y| \leq 1\right\}.$



Double Integrals involving Absolute Value Functions

Example

Evaluate $\iint\limits_D \sqrt{|y-x^2|}dxdy, D: \{(x,y) \in \mathbb{R}^2 \,||x| \leq 1, 0 \leq y \leq 1\}$.



Solving Double Integrals Using Symmetry

Theorem

If

1. *$f(x, y)$ is an odd function with respect to y and,*
2. *D is symmetric with respect to $x-$ axis*

then

$$\iint\limits_D f(x, y)\, dxdy = 0.$$

Solving Double Integrals Using Symmetry

Theorem

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then

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Theorem

If

1. $f(x, y)$ *is an even function with respect to* y *and,*
2. D *is symmetric with respect to* $x-$ *axis*

$$\iint\limits_{D} f(x, y)\, dxdy = 2 \iint\limits_{D^+} f(x, y)\, dxdy.$$

Solving Double Integrals Using Symmetry

Theorem

If

1. $f(-x,-y) = -f(x,y)$,
2. *D is symmetric with respect to the origin,*

then $\iint\limits_D f(x,y)\,dxdy = 0$.

Solving Double Integrals Using Symmetry

Theorem

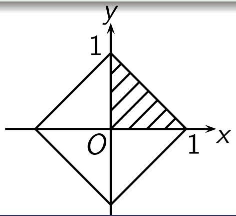
If

1. $f(-x,-y) = -f(x,y)$,
2. *D is symmetric with respect to the origin,*

then $\iint\limits_D f(x,y)\,dxdy = 0$.

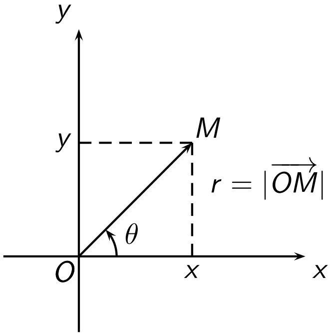
Example

Evaluate $\iint\limits_{|x|+|y|\leq 1} |x|+|y| dxdy$.



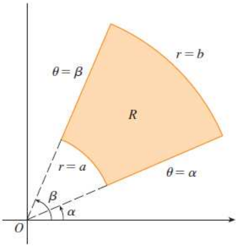
Double Integrals in Polar Coordinates

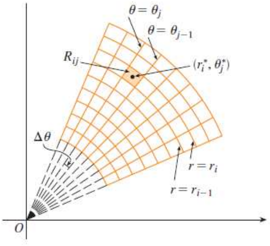
The polar coordinate of a point M is a pair (r, θ), where $\begin{cases} r = \left|\overrightarrow{OM}\right| \\ \theta = \widehat{\overrightarrow{OM}, Ox}. \end{cases}$



Polar coordinates vs rectangular coordinates: $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$

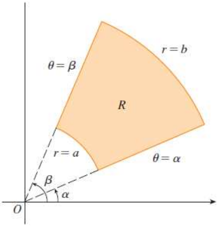
Double Integrals in Polar Coordinates

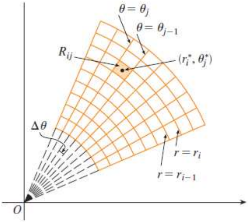




$$\begin{aligned}\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta.\end{aligned}$$

Double Integrals in Polar Coordinates





$$\iint_R f(x,y)\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^* \Delta r \Delta\theta$$

$$= \iint_R f(r\cos\theta, r\sin\theta)\boxed{r}\,drd\theta.$$

Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates

If f is continuous on a polar region of the form $\begin{cases} \theta_1 \leq \theta \leq \theta_2 \\ r_1(\theta) \leq r \leq r_2(\theta) \end{cases}$,

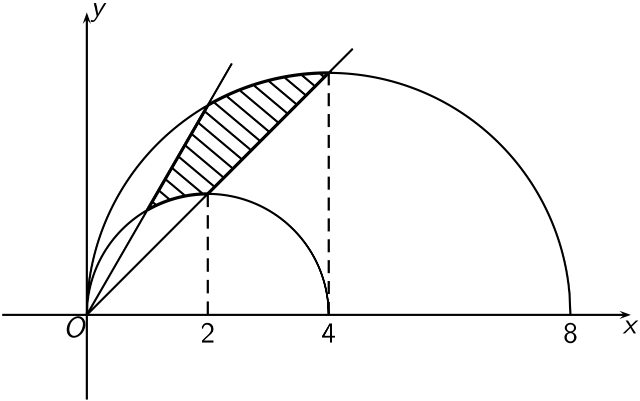
then

$$I = \int_{\theta_1}^{\theta_2} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) \boxed{r} dr$$

Double Integrals in Polar Coordinates

Example

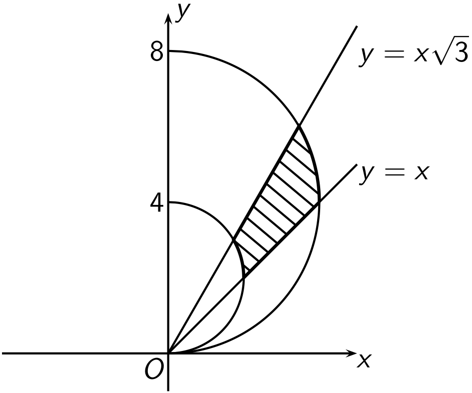
Evaluate $I = \iint\limits_D dxdy$, where $D : \begin{cases} 4x \leq x^2 + y^2 \leq 8x, \\ x \leq y \leq \sqrt{3}x. \end{cases}$



Double Integrals in Polar Coordinates

Example

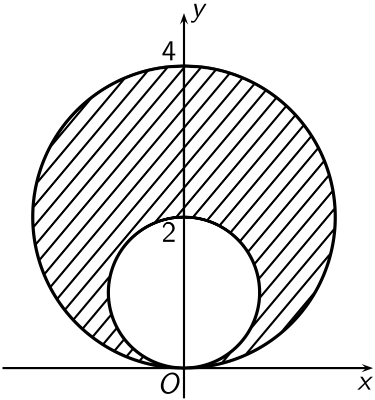
Evaluate $\iint\limits_D \frac{dxdy}{(x^2+y^2)^2}$, where $D : \begin{cases} 4y \le x^2 + y^2 \le 8y \\ x \le y \le x\sqrt{3}. \end{cases}$



Double Integrals in Polar Coordinates

Example

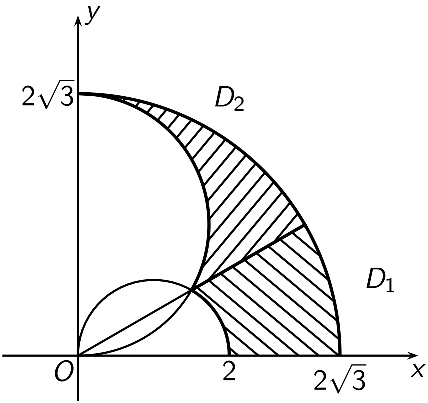
Evaluate $\iint\limits_D xy^2dxdy$ where D is bounded by $\begin{cases} x^2+(y-1)^2=1 \\ x^2+y^2-4y=0. \end{cases}$



Double Integrals in Polar Coordinates

Example

Evaluate $\iint\limits_D \frac{xy}{x^2+y^2}dxdy$, where $D : \begin{cases} x^2+y^2 \le 12, x^2+y^2 \ge 2x \\ x^2+y^2 \ge 2\sqrt{3}y, x \ge 0, y \ge 0. \end{cases}$



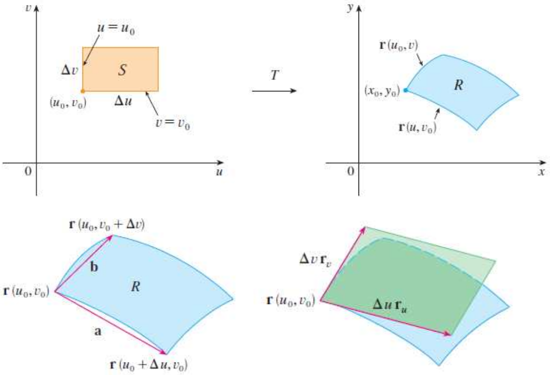
Change of variables in double integrals

- Back in Calculus I, $\int\limits_a^b f(x)dx = \int\limits_c^d f(x(t))\boxed{x'(t)}dt$, where $x = x(t)$.
- Similarly, $\iint\limits_R f(x,y)dxdy = \iint\limits_S f(x(u,v),y(u,v))\boxed{factor}dudv$

Example

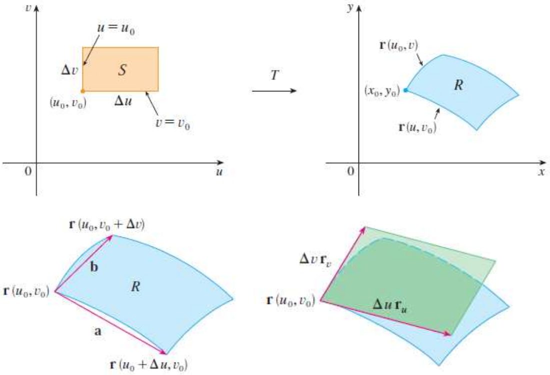
Consider the transformation $T: \begin{cases} x = x(u,v) = u^2 - v^2, \\ y = y(u,v) = 2uv. \end{cases}$
Find the image of the square $S = \{(u,v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$.

Change of variables in double integrals



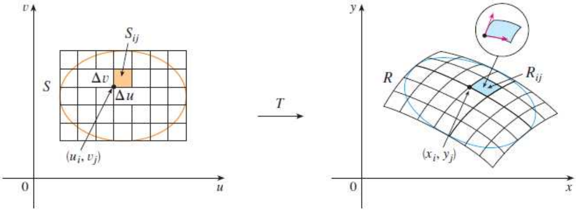
$$\Delta A \approx |r_u \times r_v| \Delta u \Delta v = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} \Delta u \Delta v,$$

Change of variables in double integrals



$$\Delta A \approx |r_u \times r_v| \Delta u \Delta v = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} \Delta u \Delta v, \quad J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}.$$

Change of variables in double integrals



$$\iint\limits_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta A$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x(u_i,v_j),y(u_i,v_j)) \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} \Delta u \Delta v$$

$$= \iint\limits_S f(x(u,v),y(u,v))|J|dudv.$$

Change of variables in double integrals

Change of variables

Let $T:\begin{cases} x = x(u,v), \\ y = y(u,v) \end{cases}, S \to R,$

- $x(u,v), y(u,v)$ has continuous partial derivatives on S,
- the transformation is a $1-1$ map.
- the Jacobian $J = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} \neq 0$ on S.

Then $\iint\limits_D f(x,y)dxdy = \iint\limits_S f(x(u,v),y(u,v))|J|dudv.$

Change of variables in double integrals

Change of variables

Let $T : \begin{cases} x = x(u,v), \\ y = y(u,v) \end{cases}, S \to R,$

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Then $\iint\limits_D f(x,y)dxdy = \iint\limits_S f(x(u,v), y(u,v))|J|dudv.$

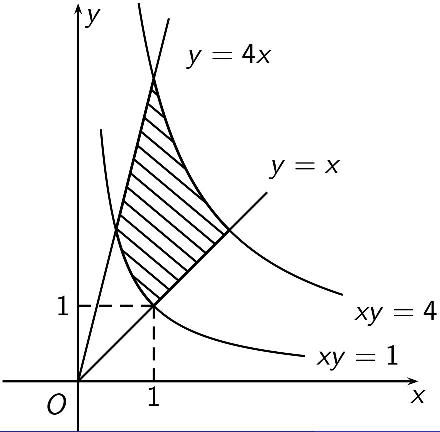
Example

Evaluate $I = \iint\limits_D (4x^2 - 2y^2)\, dxdy$, where $D : \begin{cases} 1 \le xy \le 4 \\ x \le y \le 4x. \end{cases}$

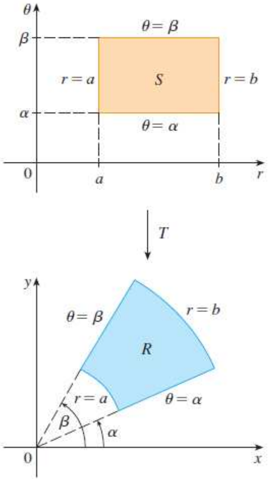
Change of variables in double integrals

Example

Evaluate $I = \iint\limits_D (4x^2 - 2y^2)\, dxdy$, where $D : \begin{cases} 1 \le xy \le 4 \\ x \le y \le 4x. \end{cases}$



Polar coordinate transformation



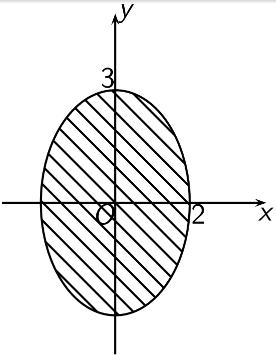
1. $\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases}$
2. $J = \frac{D(x,y)}{D(r,\theta)} = \begin{vmatrix} x'_r & x'_\theta, \\ y'_r & y'_\theta \end{vmatrix} = r$
3. $\iint\limits_R f(x,y)dxdy = \iint\limits_S f(r\cos\theta, r\sin\theta)\boxed{r}drd\theta.$

Polar coordinates

If $D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, then $\begin{cases} x = ar\cos\varphi \\ y = br\sin\varphi \end{cases}$, $\boxed{J = abr}$

Example

Evaluate $\iint\limits_D |9x^2 - 4y^2|\, dxdy$, where $D: \frac{x^2}{4} + \frac{y^2}{9} \leq 1$.

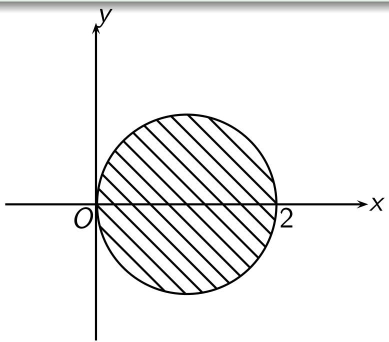


Polar Coordinates

If $D : (x-a)^2 + (y-b)^2 \leq R^2$, then $\begin{cases} x = a + r\cos\varphi \\ y = b + r\sin\varphi \end{cases}$, $\boxed{J = r}$

Example

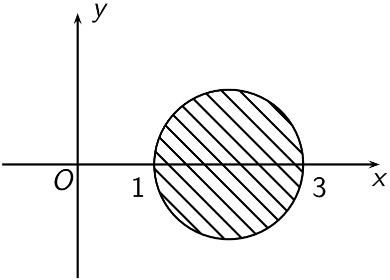
Evaluate $\int\limits_0^2 dx \int\limits_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \sqrt{2x - x^2 - y^2}\,dy$.



Polar Coordinates

Example

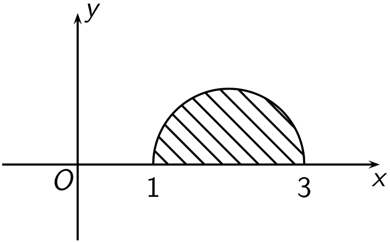
Evaluate $\iint\limits_{D} xy\mathrm{d}x\mathrm{d}y$, where $D : (x-2)^2 + y^2 \leq 1$.



Polar Coordinates

Example

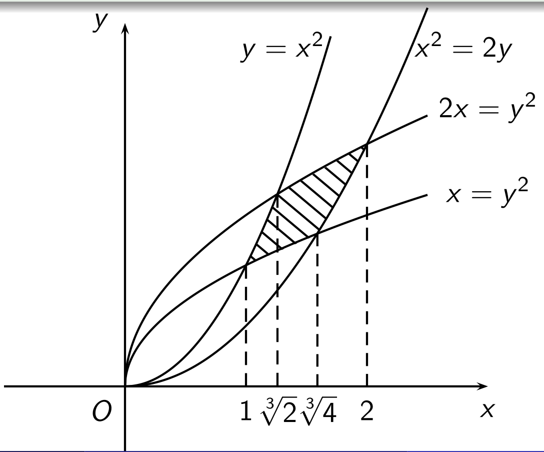
Evaluate $\iint\limits_D xy\mathrm{d}x\mathrm{d}y$, where $D : (x-2)^2 + y^2 \le 1, y \ge 0$



Applications of Double Integrals

Example

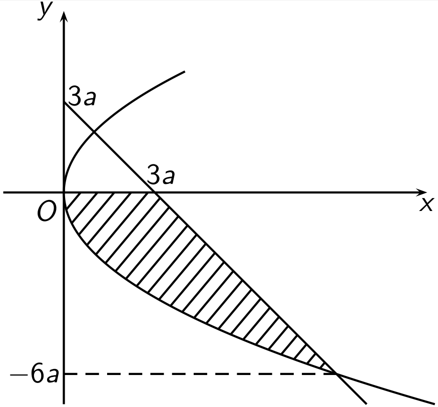
Compute the area of the domain bounded by $\begin{cases} y^2 = x, y^2 = 2x \\ x^2 = y, x^2 = 2y. \end{cases}$



Applications of Double Integrals

Example

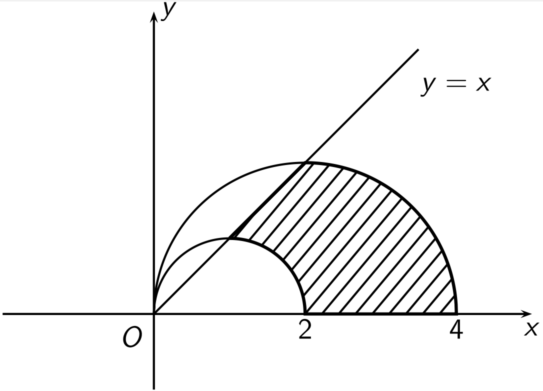
Compute the area of the domain D bounded by $\begin{cases} y = 0, y^2 = 4ax \\ x + y = 3a, \ \ (a > 0) \end{cases}.$



Applications of Double Integrals

Example

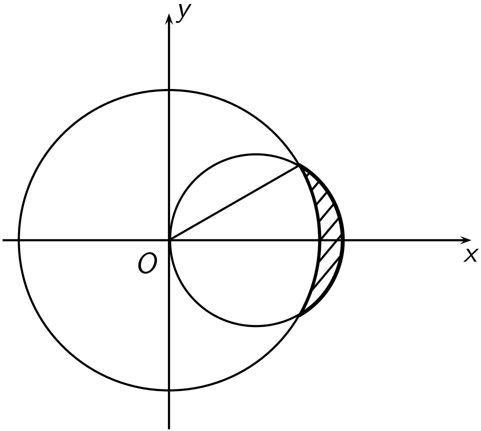
Compute the area of the domain D bounded by
$$\begin{cases} x^2+y^2=2x, x^2+y^2=4x \\ x=y, y=0. \end{cases}$$



Applications of Double Integrals

Example

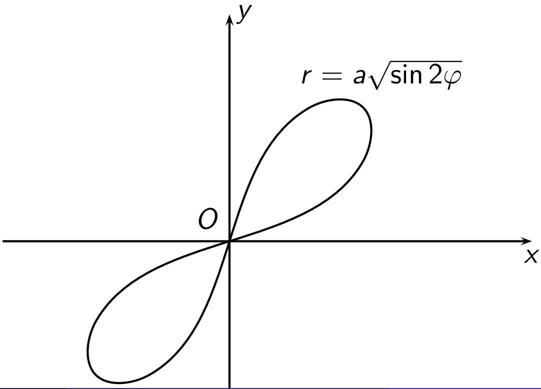
Compute the area of the domain D bounded by $r = 1, r = \frac{2}{\sqrt{3}}\cos\varphi$.



Applications of Double Integrals

Example

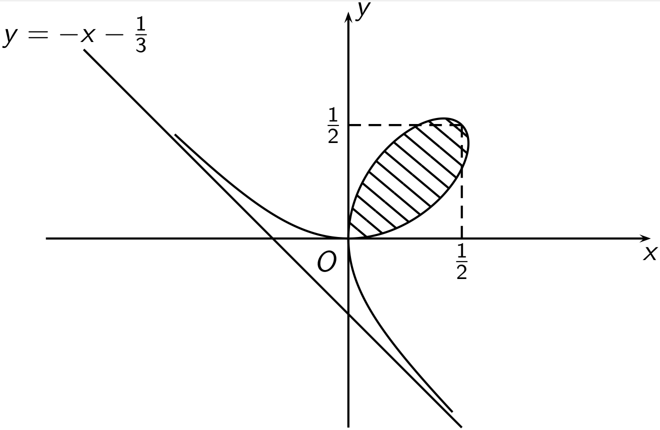
Compute the area of the domain D bounded by
$(x^2+y^2)^2 = 2a^2xy \ \ (a>0)$.



Applications of Double Integrals

Example

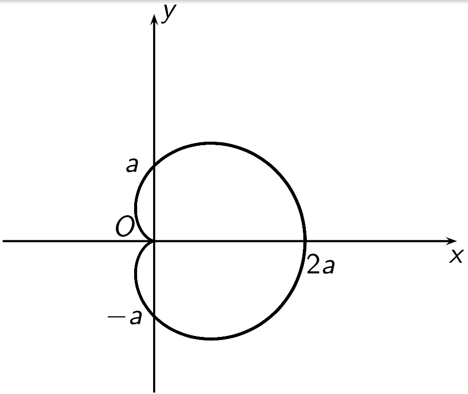
Compute the area of the domain D bounded by $x^3 + y^3 = axy \ \ (a > 0)$ (Descartes leaf)



Applications of Double Integrals

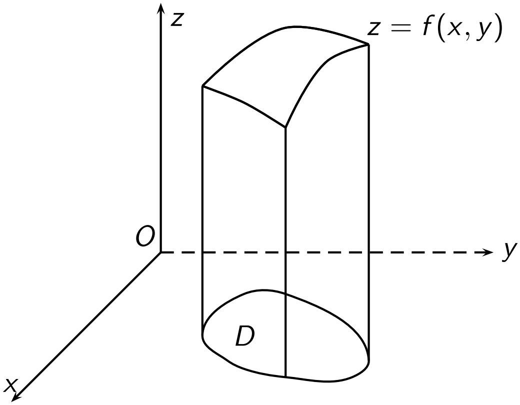
Example

Compute the area of the domain D bounded by $r = a\,(1 + \cos\varphi) \quad (a > 0)$ (Cardioids)



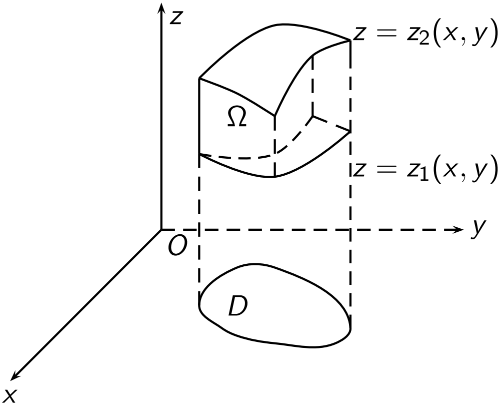
Applications of Double Integrals

$$\Omega : \begin{cases} 0 \leq z \leq f(x,y), \\ (x,y) \in D \end{cases} \Rightarrow V(\Omega) = \iint\limits_D f(x,y)\, dxdy.$$



Volume of cylindrical objects

$$V : \begin{cases} z_1(x,y) \le z \le z_2(x,y), \\ (x,y) \in D \end{cases} \quad \Rightarrow V(\Omega) = \iint\limits_D (z_2(x,y) - z_1(x,y))dxdy.$$

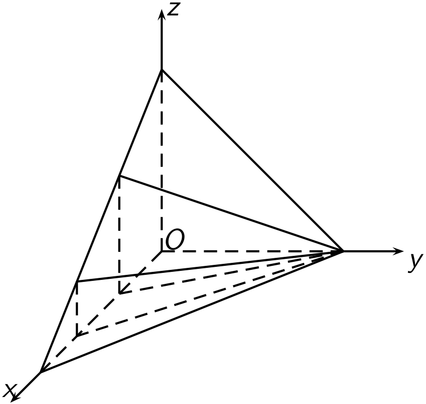


Applications of Double Integrals

Example

Compute the volume of the object given by

$$\begin{cases} 3x + y \geq 1, y \geq 0 \\ 3x + 2y \leq 2, 0 \leq z \leq 1 - x - y. \end{cases}$$

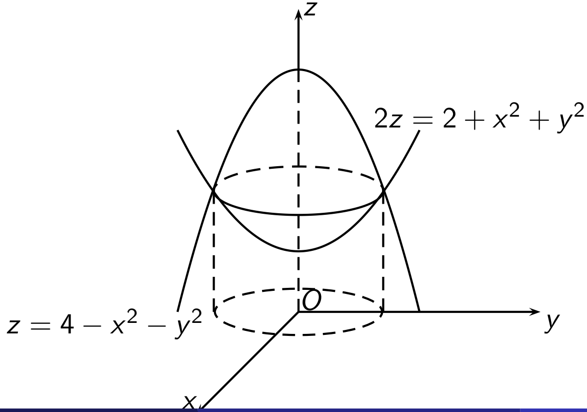


Applications of Double Integrals

Example

Compute the volume of the object bounded by the surfaces

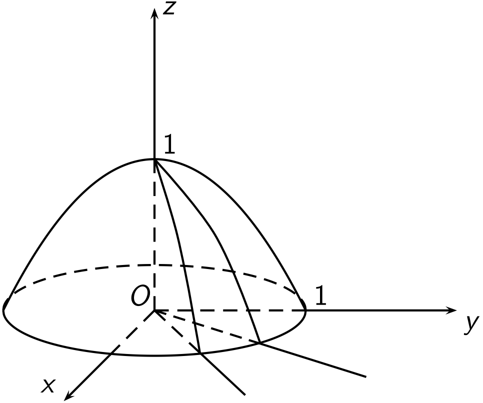
$$\begin{cases} z = 4 - x^2 - y^2 \\ 2z = 2 + x^2 + y^2. \end{cases}$$



Applications of Double Integrals

Example

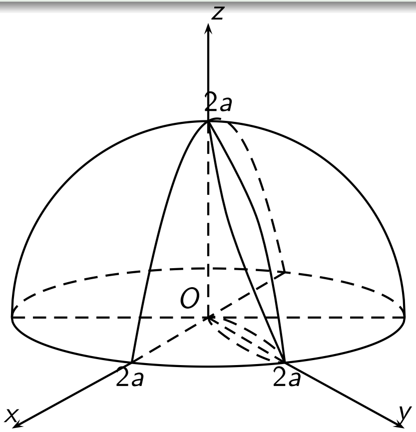
Compute the volume of the object given by $V : \begin{cases} 0 \leq z \leq 1 - x^2 - y^2 \\ y \geq x, y \leq \sqrt{3}x \end{cases}$



Applications of Double Integrals

Example

Compute the volume of the object given by $V : \begin{cases} x^2 + y^2 + z^2 \leq 4a^2 \\ x^2 + y^2 - 2ay \leq 0 \end{cases}$

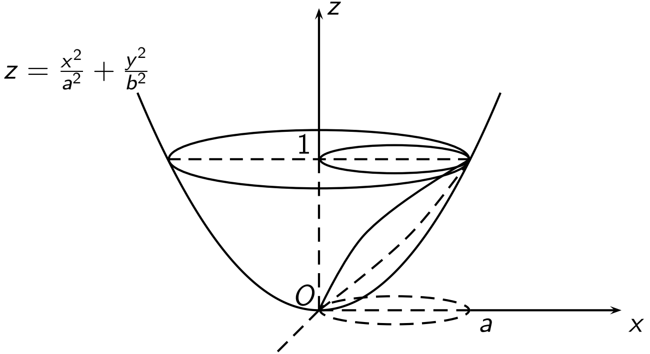


Applications of Double Integrals

Example

Compute the volume of the object bounded by the surfaces

$$\begin{cases} z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}, z = 0 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2x}{a} \end{cases}$$

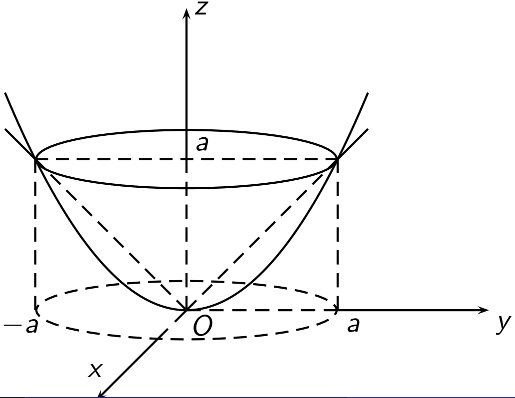


Applications of Double Integrals

Example

Compute the volume of the object bounded by the surfaces

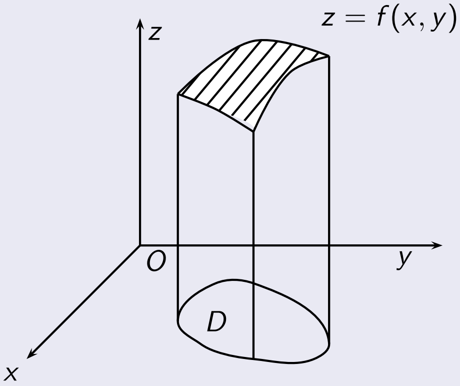
$$V : \begin{cases} az = x^2 + y^2 \\ z = \sqrt{x^2 + y^2} \end{cases}$$



Applications of Double Integrals

Area of a curved surface

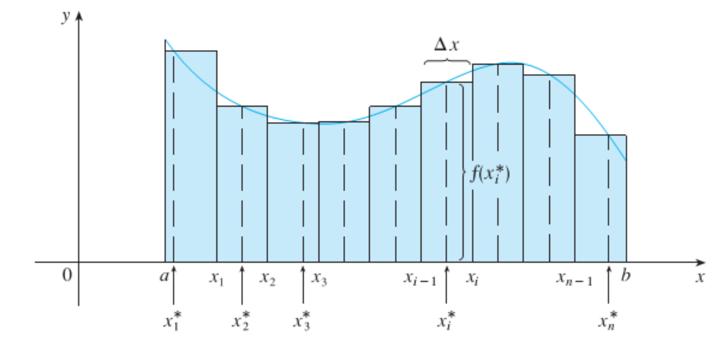
$$S = \iint\limits_D \sqrt{1 + z_x'^2 + z_y'^2}\,dxdy.$$



Multiple Integrals

Triple Integrals

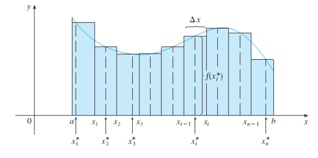
Review of the Definite Integral



$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Triple Integrals

Review of the Definite Integral



1. divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$
2. choose sample points x_i^* in these subintervals,
3. form the Riemann sum $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$
4. take the limit $V = \int\limits_a^b f(x)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

Volumes and Double Integrals

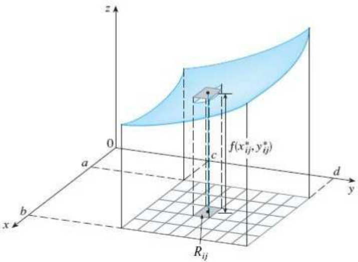


FIGURE 4

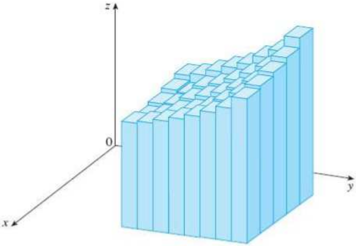
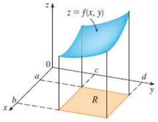
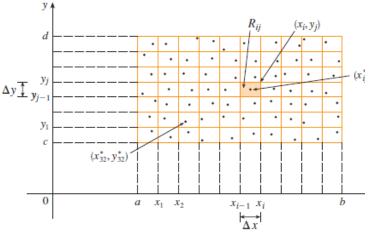


FIGURE 5

Goal: find the volume of $S := \{(x, y, z) \in \mathbb{R}^3 | 0 \le z \le f(x, y), (x, y) \in R.\}$

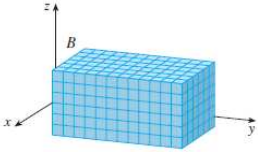
Volumes and Double Integrals

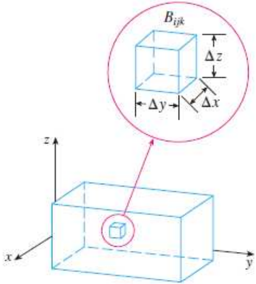




1. divide $[a, b]$ into m subintervals and $[c, d]$ into n subintervals, each of equal width.
2. choose sample points $(x_{ij}^*, y_{ij}^*) \in R_{ij} = [x_{i-1}, x_i] \times [y_{i-1}, y_i]$,
$$V(R_{ij}) \approx f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y.$$
3. Riemann Sum $V = \sum_{i=1}^{m}\sum_{j=1}^{n} R_{ij} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x\Delta y = S_{m,n}$.
4. take the limit $\lim\limits_{m,n\to\infty} S_{mn}$.

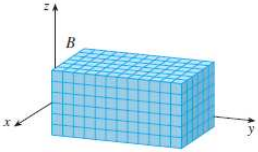
Triple Integrals over Rectangular Box

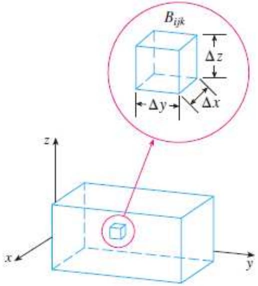




1. divide B into sub-boxes by
   - dividing $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx,
   - dividing $[c, d]$ into m subintervals $[y_{j-1}, y_j]$ of equal width Δy,
   - dividing $[r, s]$ into n subintervals $[z_{k-1}, z_k]$ of equal width Δz.

Triple Integrals over Rectangular Box





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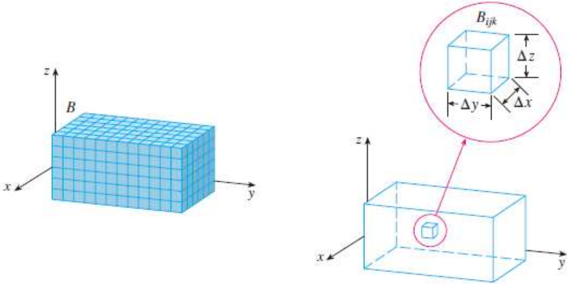
   - dividing $[r, s]$ into n subintervals $[z_{k-1}, z_k]$ of equal width Δz.
2. Choose sample points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each box B_{ijk}. Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.

Triple Integrals over Rectangular Box



1. divide B into sub-boxes by
   - dividing $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx,
   - dividing $[c, d]$ into m subintervals $[y_{j-1}, y_j]$ of equal width Δy,
   - dividing $[r, s]$ into n subintervals $[z_{k-1}, z_k]$ of equal width Δz.
2. Choose sample points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each box B_{ijk}. Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.
3. form the triple Riemann sum $\sum\limits_{i=1}^{l} \sum\limits_{j=1}^{m} \sum\limits_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$.

Triple Integrals over Rectangular Box

Definition

$$\iiint\limits_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V.$$

Again, the triple integral always exists if is continuous.

Triple Integrals over Rectangular Box

Definition

$$\iiint\limits_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V.$$

Again, the triple integral always exists if is continuous.

Theorem (Fubini's Theorem)

If f is continuous on the rectangular box $B = [a,b]\times[c,d]\times[r,s]$ then

$$\iiint\limits_B f(x,y,z)dxdydz = \int_a^b dx \int_c^d dy \int_r^s f(x,y,z)dz.$$

Triple Integrals over General Regions

If E is a general bounded solid region, choose

$$B = [a, b] \times [c, d] \times [r, s] \supset V$$

and define

$$F(x, y, z) = \begin{cases} f(x, y, z), & \text{if } (x, y, z) \in E, \\ 0, & \text{if } (x, y, z) \in B \setminus E. \end{cases}$$

Triple Integrals over General Regions

If E is a general bounded solid region, choose

$$B = [a, b] \times [c, d] \times [r, s] \supset V$$

and define

$$F(x, y, z) = \begin{cases} f(x, y, z), & \text{if } (x, y, z) \in E, \\ 0, & \text{if } (x, y, z) \in B \setminus E. \end{cases}$$

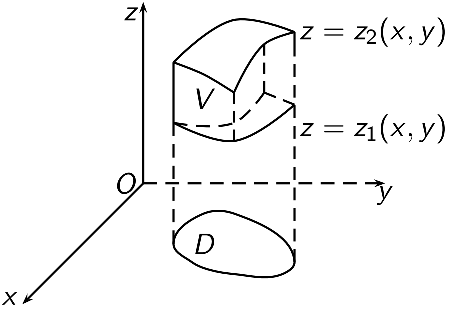
Definition

$$\iiint\limits_E f(x, y, z) dV = \iiint_B F(x, y, z) dV.$$

Triple Integrals over Regions of type I

If $V : \begin{cases} z_1(x,y) \le z \le z_2(x,y), \\ (x,y) \in D \end{cases}$ then

$$I = \iiint\limits_V f(x,y,z)\,dxdydz = \iint\limits_D dxdy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz$$



Triple Integrals over Regions of type I

Reduce the triple integral into a double integral

1. Find the projection of V onto Oxy.
2. Find $\begin{cases} \text{the lower boundary } z = z_1(x, y), \\ \text{the upper boundary } z = z_2(x, y) \end{cases}$ of V.
3. Apply the formula

$$I = \iiint\limits_V f(x, y, z)\, dxdydz = \iint\limits_D dxdy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz$$

.

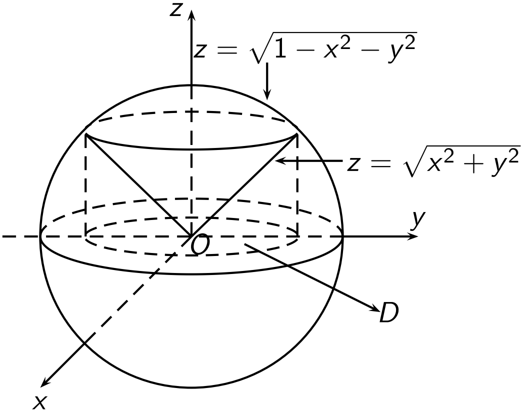
The idea:

$$\boxed{\text{Triple Integrals} \Rightarrow \text{Double Integrals} \Rightarrow \text{Iterated integrals}}$$

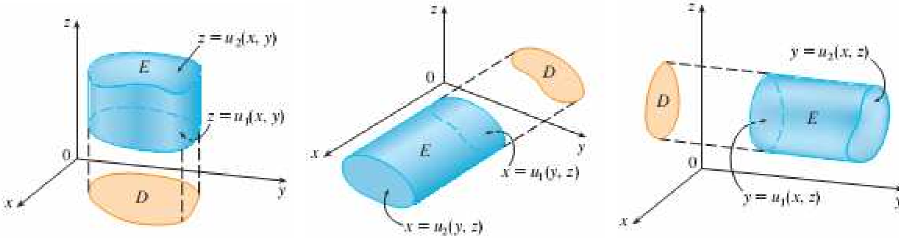
Triple Integrals over General Regions

Example

Evaluate $\iiint\limits_V (x^2+y^2)\, dxdydz$, where $V : \sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2}$.



Triple Integrals over General Regions



By the same sort of argument,

1. Type II: $\iiint\limits_E f(x,y,z)dxdydz = \iint\limits_D \left[\int\limits_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dx\right] dydz.$

2. Type III: $\iiint\limits_E f(x,y,z)dxdydz = \iint\limits_D \left[\int\limits_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dy\right] dxdz.$

Solving triple integrals using symmetry

Theorem

If

1. V *is symmetric with respect to* $z = 0$,
2. $f(x, y, z)$ *is an odd function with respect to* z

then $\iiint\limits_V f(x, y, z)\, dxdydz = 0.$

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Theorem

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then $\iiint\limits_V f(x, y, z)\, dxdydz = 2\iiint\limits_{V^+} f(x, y, z)\, dxdydz$.

Note: The role of x, y, z can be interchangeable.

Change of Variables in Triple Integrals

1. Calculus 1, $\int_a^b f(x)dx = \int_c^d f(x(u))\boxed{x'(u)}du$, where $x = x(u)$.

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3. $$\iiint\limits_B f(x,y,z)dxdydz =$$
$$\iint\limits_S f(x(u,v,w),y(u,v,w),z(u,v,w))\boxed{\text{factor}}dudvdw,$$

where $\begin{cases} x = x(u,v,w), \\ y = y(u,v,w), \\ z = z(u,v,w). \end{cases}$

Change of Variables in Triple Integrals

Consider the transformation: $T : \begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w). \end{cases}$, $V' \to V$ satisfies

1. T is a $1-1$ map.
2. $x(u, v, w), y(u, v, w), z(u, v, w)$ are continuous and have continuous partial derivatives on V'.
3. The Jacobian determinant

$$J = \frac{D(x, y, z)}{D(u, v, w)} = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix} \neq 0 \text{ in } V'.$$

Then

$$\iiint\limits_V f(x, y, z)dxdydz = \iiint\limits_{V'} f\left(x(u, v, w), y(u, v, w), z(u, v, w)\right)|J|dudvdw. \tag{1}$$

Change of Variables in Triple Integrals

Example

Evaluate $\iiint\limits_V (x+y+z)dxdydz$, where V is bounded by

$$\begin{cases} x+y+z=\pm 3 \\ x+2y-z=\pm 1. \\ x+4y+z=\pm 2 \end{cases}$$

Change of Variables in Triple Integrals

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Evaluate $\iiint\limits_V (x+y+z)dxdydz$, where V is bounded by

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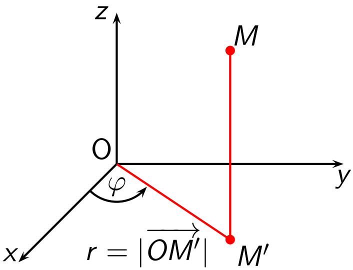
Consider the transformation $\begin{cases} u=x+y+z \\ v=x+2y-z \\ \mathrm{w}=x+4y+z \end{cases}$

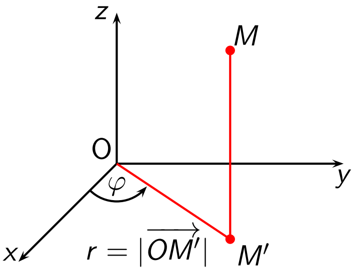
$$J^{-1}=\frac{D(u,v,\mathrm{w})}{D(x,y,z)}=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{vmatrix}=6 \Rightarrow J=\frac{1}{6}.$$

Triple Integrals in Cylindrical Coordinates



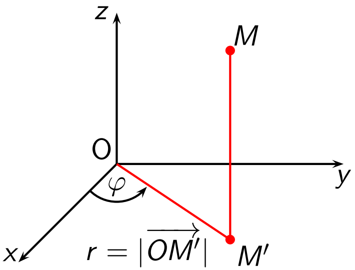
- Cylindrical Coordinate $M(r, \varphi, z)$, where (r, φ) is the polar coordinate of M'.

Triple Integrals in Cylindrical Coordinates



- Cylindrical Coordinate $M(r, \varphi, z)$, where (r, φ) is the polar coordinate of M'.
- Cylindrical vs rectangular coor. $x = r\cos\varphi, y = r\sin\varphi, z = z$.

Triple Integrals in Cylindrical Coordinates



- Cylindrical Coordinate $M(r, \varphi, z)$, where (r, φ) is the polar coordinate of M'.
- Cylindrical vs rectangular coor. $x = r\cos\varphi, y = r\sin\varphi, z = z$.

The transformation

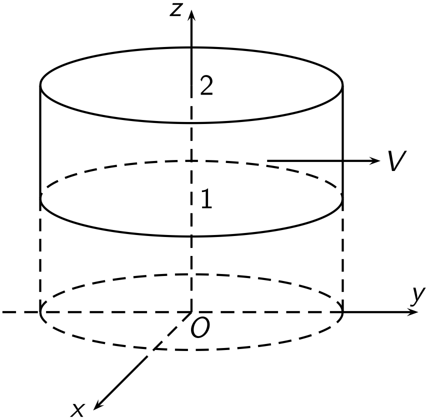
Taking the transformation $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z. \end{cases}$ then $J = \frac{D(x,y,z)}{D(r,\varphi,z)} = r$ and

$$I = \iiint\limits_V f(x, y, z)dxdydz = \iiint\limits_{V_{r\varphi z}} f(r\cos\varphi, r\sin\varphi, z)rdrd\varphi dz.$$

Triple Integrals in Cylindrical Coordinates

Example

Evaluate $\iiint\limits_V (x^2+y^2)\, dxdydz$, where $V : \begin{cases} x^2+y^2 \leq 1 \\ 1 \leq z \leq 2 \end{cases}$



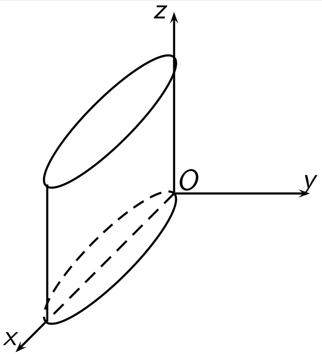
Triple Integrals in Cylindrical Coordinates

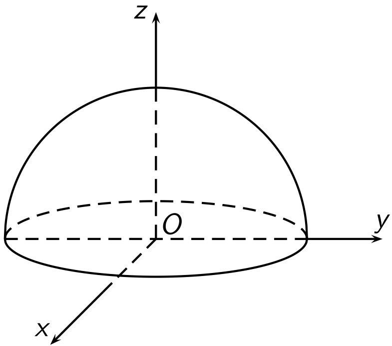
Example

Evaluate $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, where:

a) V is bounded by: $x^2+y^2=2x$ and $z=0, z=a \ (a>0)$.

b) V is a half of the sphere $x^2+y^2+z^2 \le a^2, z \ge 0 \ (a>0)$

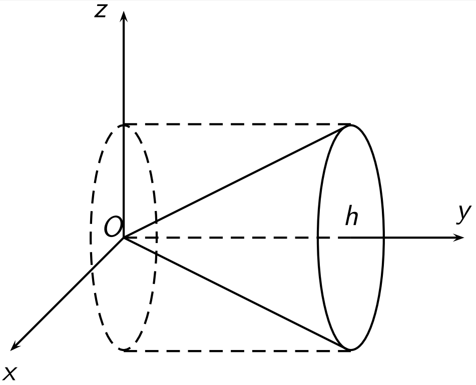




Triple Integrals in Cylindrical Coordinates

Example

Evaluate $I = \iiint\limits_V y dx dy dz$, where V is bounded by: $\begin{cases} y = \sqrt{z^2 + x^2} \\ y = h. \end{cases}$

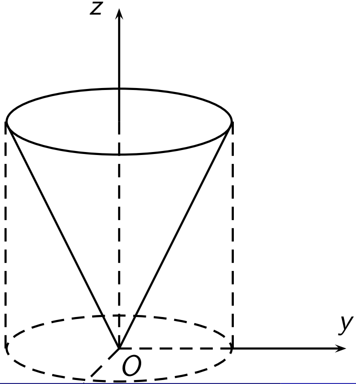


Triple Integrals in Cylindrical Coordinates

Example

Evaluate $I = \iiint\limits_V \sqrt{x^2+y^2}\,dxdydz$ where V is bounded by:

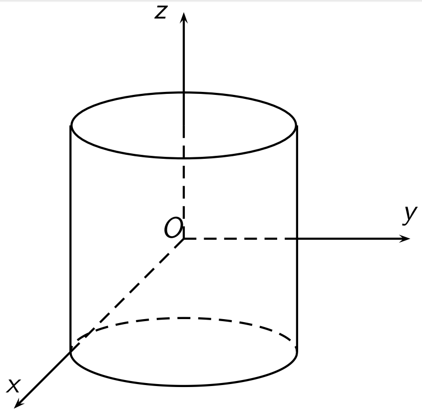
$$\begin{cases} x^2+y^2=z^2 \\ z=1. \end{cases}$$



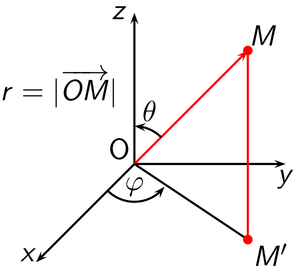
Triple Integrals in Cylindrical Coordinates

Example

Evaluate $\iiint\limits_V \frac{dxdydz}{\sqrt{x^2+y^2+(z-2)^2}}$, where $V : \begin{cases} x^2+y^2 \le 1 \\ |z| \le 1. \end{cases}$



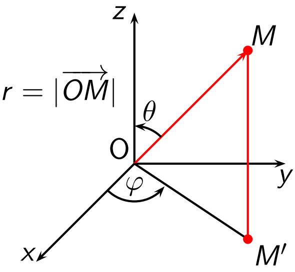
Triple Integrals in Spherical Coordinates



- The spherical coordinate of a point M is an ordered triple (r, θ, φ), where

$$r = \left|\overrightarrow{OM}\right|, \theta = \left(\widehat{Oz, \overrightarrow{OM}}\right), \varphi = \left(\widehat{Ox, \overrightarrow{OM'}}\right)$$

Triple Integrals in Spherical Coordinates



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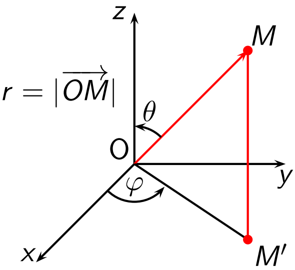
Triple Integrals in Spherical Coordinates



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$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta.$

The transformation

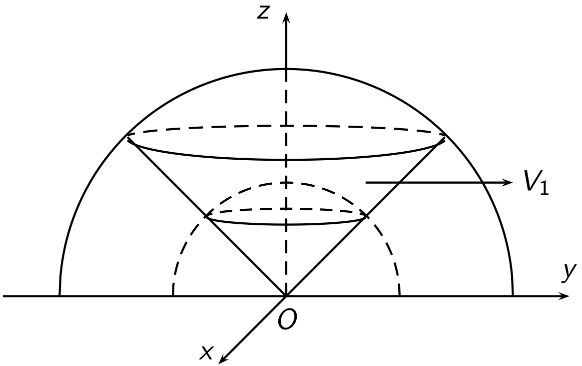
Let $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta, \end{cases}$ then $J = \frac{D(x,y,z)}{D(r,\theta,\varphi)} = -r^2\sin\theta$ and

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V_{r\theta\varphi}} f(\cdots, \cdots, \cdots)r^2\sin\theta drd\theta d\varphi.$$

Triple Integrals in Spherical Coordinates

Example

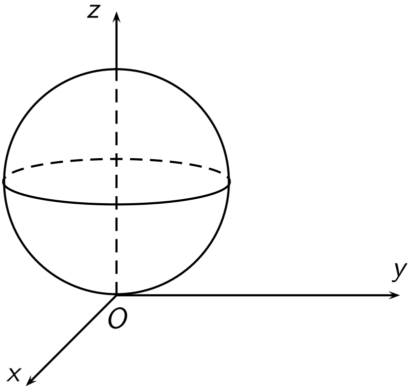
Evaluate $\iiint\limits_V (x^2+y^2+z^2)\, dxdydz$, where $V : \begin{cases} 1 \leq x^2+y^2+z^2 \leq 4 \\ x^2+y^2 \leq z^2. \end{cases}$



Triple Integrals in Spherical Coordinates

Example

Evaluate $\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dxdydz$, where $V:\ x^2+y^2+z^2 \le z$.



Spherical and Cylindrical Coordinates - Special Cases

1. $V : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \Rightarrow \begin{cases} x = ar\sin\theta\cos\varphi \\ y = br\sin\theta\sin\varphi \\ z = cr\cos\theta \end{cases}, J = -abcr^2\sin\theta$

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2. $V: (x-a)^2+(y-b)^2+(z-c)^2 \le R^2$

$$\Rightarrow \begin{cases} x = a + r\sin\theta\cos\varphi \\ y = b + r\sin\theta\sin\varphi \\ z = c + r\cos\theta \end{cases}, J = -r^2\sin\theta$$

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3. $V: \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1 \Rightarrow \begin{cases} z = bz' \\ x = ar\cos\varphi \\ y = ar\sin\varphi \end{cases}$

Spherical and Cylindrical Coordinates - Special Cases

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Evaluate $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, where $V : \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1, z \geq 0.$

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Cylindrical Coordinate.

Let $\begin{cases} z = bz' \\ x = ar\cos\varphi \\ y = ar\sin\varphi \end{cases}$, then

Spherical Coordinate.

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Let $\begin{cases} z = bz' \\ x = ar\cos\varphi \\ y = ar\sin\varphi \end{cases}$, then

$J = a^2br$ and $\begin{cases} 0 \leq \varphi \leq 2\pi, \\ 0 \leq r \leq 1, \\ 0 \leq z' \leq \sqrt{1-r^2}. \end{cases}$

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Spherical and Cylindrical Coordinates - Special Cases

Example

Evaluate $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, where $V : \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1, z \geq 0.$

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$$J = a^2 br^2 \sin\theta \text{ and } \begin{cases} 0 \leq \varphi \leq 2\pi, \\ 0 \leq \theta \leq \frac{\pi}{2}, \\ 0 \leq r \leq 1. \end{cases}$$

$$I = \frac{2\pi a^3 b^2}{15}.$$

Volume of a solid Object

$$V = \iiint\limits_V dxdydz.$$

Example

Compute the volume of the domain V bounded by $\begin{cases} x+y+z=\pm 3 \\ x+2y-z=\pm 1 \\ x+4y+z=\pm 2. \end{cases}$

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Change of variables $\begin{cases} u = x+y+z \\ v = x+2y-z \\ \mathrm{w} = x+4y+z \end{cases}$

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$$J^{-1} = \frac{D(u,v,\mathrm{w})}{D(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 6$$

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$$J^{-1} = \frac{D(u,v,\mathrm{w})}{D(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 6 \Rightarrow J = \frac{1}{6},$$

Volume of a solid Object

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