	Hanoi University of Science and Technology **Faculty of Mathematics and Informatics**	
	Course: Calculus 2	**Course ID:** MI1124E
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1.1 Week 1

Exercise 1. Find an equation of the tangent line and normal line to the curves

(a) $y = e^{1-x^2}$ at the intersection of this curve and the line $y = 1$

(b) $\begin{cases} x = 2t - \cos(\pi t) \\ y = 2t + \sin(\pi t) \end{cases}$ at the point A corresponding to $t = 1/2$

(c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$ at the point $M(8; 1)$

Exercise 2. Evaluate the curvature at the arbitrary point of

(a) $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ $(a > 0)$.

(b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ $(a > 0)$

(c) $r = ae^{b\varphi}, (a, b > 0)$

Exercise 3. Evaluate the curvature of the curve $y = \ln x$ at the point with positive abscissa $x > 0$. At what point does the curve have maximum curvature? If $x \to \infty$, how is the curvature?

Exercise 4. Find the envelope of the family of the following curves:

(a) $y = \frac{x}{c} + c^2$

(b) $cx^2 - 3y - c^3 + 2 = 0$

(c) $y = c^2(x - c)^2$

(d) $4x \sin c + y \cos c = 1$

Exercise 5. Suppose that $\vec{p}(t), \vec{q}(t), \alpha(t)$ are differentiable functions. Prove that

(a) $\frac{d}{dt}(\vec{p}(t) + \vec{q}(t)) = \frac{d\vec{p}(t)}{dt} + \frac{d\vec{q}(t)}{dt}$

(b) $\frac{d}{dt}(\alpha(t)\vec{p}(t)) = \alpha(t)\frac{d\vec{p}(t)}{dt} + \alpha'(t)\vec{p}(t)$

(c) $\frac{d}{dt}(\vec{p}(t)\vec{q}(t)) = \vec{p}(t)\frac{d\vec{q}(t)}{dt} + \frac{d\vec{p}(t)}{dt}\vec{q}(t)$

(d) $\frac{d}{dt}(\vec{p}(t) \times \vec{q}(t)) = \vec{p}(t) \times \frac{d\vec{q}(t)}{dt} + \frac{d\vec{p}(t)}{dt} \times \vec{q}(t)$

Exercise 6. The curve C is given by $\vec{r}(t)$. Suppose that $\vec{r}(t)$ is a differentiable function and $\vec{r}'(t)$ is always perpendicular to $\vec{r}(t)$. Prove that C lies on the sphere with center the origin.

1.2 Week 2

Exercise 7. Find an equation of the tangent line and normal plane of the curve

(a) $\begin{cases} x = a\sin^2 t \\ y = b\sin t\cos t \\ z = c\cos^2 t \end{cases}$ at the point corresponding to $t = \frac{\pi}{4}$, $(a, b, c > 0)$

(b) $\begin{cases} x = 4\sin^2 t \\ y = 4\cos t \\ z = 2\sin t + 1 \end{cases}$ at $M(1; -2\sqrt{3}; 2)$

Exercise 8. Evaluate the curvature of the curve

(a) $\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$ at the point corresponding to $t = \frac{\pi}{2}$

(b) $\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \\ z = t \end{cases}$ at the point corresponding to $t = \pi$

(c) Evaluate the curvature at the point $M(1; 0; -1)$ of the intersection curve of the cylinder $4x^2 + y^2 = 4$ and the plane $x - 3z = 4$

Exercise 9. Find an equation of the tangent plane and normal line of the surface

(a) $x^2 - 4y^2 + 2z^2 = 6$ at the point $(2; 2; 3)$

(b) $z = 2x^2 + 4y^2$ at $(2; 1; 12)$

(c) $\ln(2x + y^2) + 3z^3 = 3$ at the point $(0; -1; 1)$

(d) $x^2 + 2y^3 - yz = 0$ at $(1; 1; 3)$

Exercise 10. Find an equation of the tangent line and normal plane of the curve

(a) $\begin{cases} x^2 + y^2 = 10 \\ y^2 + z^2 = 25 \end{cases}$ at $A(1; 3; 4)$

(b) $\begin{cases} 2x^2 + 3y^2 + z^2 = 47 \\ x^2 + 2y^2 = z \end{cases}$ at $B(-2; 1; 6)$

1.3 Week 3

Exercise 11. Calculate the iterated integral by first reversing the order of integration

(a) $\int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{1-x^2} f(x, y)dy$

(b) $\int\limits_{0}^{1} dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x, y)dx$

(c) $\int\limits_0^2 dx \int\limits_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)dy$

(d) $\int\limits_0^{\frac{\pi}{2}} dy \int\limits_{\sin y}^{1+y^2} f(x,y)dx$

(e) $\int\limits_0^{\sqrt{2}} dy \int\limits_0^{y} f(x,y)dx + \int\limits_{\sqrt{2}}^{2} dy \int\limits_0^{\sqrt{4-y^2}} f(x,y)dx$

Exercise 12. Calculate the value of the multiple integral

(a) $\iint\limits_{\mathcal{D}} \frac{y}{1+xy} dxdy, \mathcal{D} = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1; 0 \le y \le 2\}$

(b) $\iint\limits_{\mathcal{D}} x^2(y-x)dxdy$, where $\mathcal{D}$ is a region bounded by two curves $y = x^2$ and $x = y^2$

(c) $\iint\limits_{\mathcal{D}} 2xydxdy$, where $\mathcal{D}$ is bounded by the curves $x = y^2, x = -1, y = 0$ and $y = 1$

(d) $\iint\limits_{\mathcal{D}} (x+y)dxdy$, where $\mathcal{D}$ is bounded by $x^2 + y^2 \le 1, \sqrt{x} + \sqrt{y} \ge 1$

(e) $\iint\limits_{\mathcal{D}} |x+y|dxdy$, where $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : |x| \le 1; |y| \le 1\}$

(f) $\iint\limits_{|x|+|y|\le 1} (|x|+|y|)dxdy$

(g) $\int\limits_0^1 dx \int\limits_0^{1-x^2} \dfrac{xe^{3y}}{1-y} dy$

1.4 Week 4

Exercise 13. Find the limits of integration in the polar coordinates of $\iint\limits_{\mathcal{D}} f(x,y)dxdy$, where $\mathcal{D}$ is a domain

(a) $a^2 \le x^2 + y^2 \le b^2$

(b) $x^2 + y^2 \ge 4x, x^2 + y^2 \le 8x, y \ge x, y \le \sqrt{3}x$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1,\ y \ge 0, (a, b > 0)$

(d) $x^2 + y^2 \le 2x, x^2 + y^2 \le 2y$

Exercise 14. Evaluate the given integral by changing to polar coordinates

(a) $\int\limits_0^R dx \int\limits_0^{\sqrt{R^2-x^2}} \ln(1 + x^2 + y^2)dy, \quad (R > 0)$

(b) $\iint\limits_{\mathcal{D}} xydxdy$, where $\mathcal{D}$ is a half of surface: $(x-2)^2 + y^2 \le 1, y \ge 0$

(c) $\iint\limits_{\mathcal{D}} (\sin y + 3x)dxdy$, where $\mathcal{D}$ is a surface: $(x-2)^2 + y^2 \le 1$

(d) $\iint\limits_{\mathcal{D}} |x+y|dxdy$, where $\mathcal{D}$ is surface: $x^2 + y^2 \le 1$

(e) $\iint\limits_{\mathcal{D}} xy^2dxdy$, where D is a region bounded by the circles $x^2 + (y-1)^2 = 1$ and $x^2 + y^2 - 4y = 0$.

Exercise 15. Rewrite the following integral in terms of two variables u and v: $\int\limits_0^1 dx \int\limits_{-x}^{x} f(x,y)dy$, if we let $u = x + y$, and $v = x - y$.

Exercise 16. Evaluate the given integrals

(a) $\iint\limits_{\mathcal{D}} \dfrac{2xy+1}{\sqrt{1+x^2+y^2}}dxdy$, where $\mathcal{D} : x^2 + y^2 \leq 1$

(b) $\iint\limits_{\mathcal{D}} \dfrac{dxdy}{(x^2+y^2)^2}$, where $\mathcal{D} : \begin{cases} y \leq x^2 + y^2 \leq 2y \\ x \leq y \leq \sqrt{3}x \end{cases}$

(c) $\iint\limits_{\mathcal{D}} \dfrac{xy}{x^2+y^2}dxdy$, where $\mathcal{D} : \begin{cases} 2x \leq x^2 + y^2 \leq 12 \\ x^2 + y^2 \geq 2\sqrt{3}y \\ x \geq 0, y \geq 0 \end{cases}$

(d) $\iint\limits_{\mathcal{D}} |9x^2 - 4y^2|dxdy$, where $\mathcal{D} : \dfrac{x^2}{4} + \dfrac{y^2}{9} \leq 1$

(e) $\iint\limits_{\mathcal{D}} (3x + 2xy)dxdy$, where $\mathcal{D} : \begin{cases} 1 \leq xy \leq 9 \\ y \leq x \leq 4y \end{cases}$

1.5 Week 5

Exercise 17. Evaluate the triple integral

(a) $\iiint\limits_V zdxdydz$, where the region V is bounded by $\begin{cases} 0 \leq x \leq 1 \\ x \leq y \leq 2x \\ 0 \leq z \leq \sqrt{5 - x^2 - y^2} \end{cases}$

(b) $\iiint\limits_V (3xy^2 - 4xyz)dxdydz$, where the region V is bounded by $\begin{cases} 1 \leq y \leq 2 \\ 0 \leq xy \leq 2 \\ 0 \leq z \leq 2 \end{cases}$

(c) $\iiint\limits_V xye^{yz^2}dxdydz$, where the region V is bounded by $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \\ x^2 \leq z \leq 1 \end{cases}$

(d) $\iiint\limits_V (x^2 + y^2)dxdydz$, where the region V is bounded by $\begin{cases} x^2 + y^2 + z^2 \leq 1 \\ x^2 + y^2 - z^2 \leq 0 \end{cases}$

Exercise 18. $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, where

(a) V is bounded by the cylinder $x^2 + y^2 = 2x$ and planes: $y = 0, z = 0, z = a$, $(y \geq 0, a > 0)$

(b) V is a half of the sphere $x^2 + y^2 + z^2 \leq a^2, z \geq 0, (a > 0)$

(c) V l is a half of the elipsoid $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leq 1, z \geq 0, (a, b > 0)$

Exercise 19. $\iiint\limits_V ydxdydz$, where V is a region bounded by the cone: $y = \sqrt{x^2+z^2}$ and the plane $y = h, (h > 0)$

Exercise 20. Evaluate the triple integral

(a) $\iiint\limits_V \dfrac{x^2}{a^2} dxdydz$, where $V : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \ (a, b, c > 0)$

(b) $\iiint\limits_V (x^2 + y^2 + z^2) dxdydz$, where $V : \begin{cases} 1 \leq x^2 + y^2 + z^2 \leq 4 \\ x^2 + y^2 \leq z^2 \end{cases}$

(c) $\iiint\limits_V \sqrt{x^2 + y^2} dxdydz$, where V is a region bounded by $x^2 + y^2 = z^2, z = -1$

(d) $\iiint\limits_V \dfrac{dxdydz}{[x^2 + y^2 + (z-2)^2]^2}$, where $V : \begin{cases} x^2 + y^2 \leq 1 \\ |z| \leq 1 \end{cases}$

(e) $\iiint\limits_V \sqrt{x^2 + y^2 + z^2} dxdydz$, where V is the region bounded by $x^2 + y^2 + z^2 \leq z$

1.6 Week 6

Exercise 21. Evaluate the area of the region $\mathcal{D}$ bounded by the curves

(a) $\begin{cases} y^2 = x, y^2 = 2x \\ x^2 = y, x^2 = 2y \end{cases}$

(b) $\begin{cases} y = 0, y^2 = 4ax \\ x + y = 3a, y \leq 0, (a > 0). \end{cases}$

(c) $\begin{cases} 2x \leq x^2 + y^2 \leq 4x \\ 0 \leq y \leq x \end{cases}$

(d) $r \geq 1, r \leq \frac{2}{\sqrt{3}} \cos \varphi$

(e) $x^2 + (\alpha x - y)^2 \leq 4$ is a constant $\forall \alpha \in \mathbb{R}$

(f) $\begin{cases} x + y \geq 1 \\ x + 2y \leq 2 \\ y \geq 0, 0 \leq z \leq 2 - x - y \end{cases}$

Exercise 22. Evaluate the area of the region $\mathcal{D}$ is bounded by the curves $(a > 0)$

(a) $(x^2 + y^2)^2 = 2a^2 xy$

(b) $r = a(1 + \cos \varphi)$

Exercise 23. Evaluate the volume of the region is bounded by the surfaces

(a) $\begin{cases} z = 4 - x^2 - y^2 \\ 2z = 2 + x^2 + y^2 \end{cases}$

(b) $|x - y| + |x + 3y| + |x + y + z| \leq 1.$

(c) $z = 1 + x^2 + y^2$, surface $x^2 + 4y^2 = 4$ and the plane Oxy.

(d) $az = x^2 + y^2, z = \sqrt{x^2 + y^2}, (a > 0).$

Exercise 24. Evaluate the area of a part of the sphere $x^2 + y^2 + z^2 = 4a^2$ lying inside the surface $x^2 + y^2 - 2ay = 0$, $(a > 0)$.

1.7 Week 7

Exercise 25. Evaluate

(a) $\lim\limits_{y\to 0}\int\limits_{y}^{1+y}\frac{dx}{1+x^2+y^2}$

(b) $\lim\limits_{y\to 0}\int\limits_{0}^{2}x^2\cos xy dx$

Exercise 26. Evaluate

(a) $I(y)=\int\limits_0^1 \arctan\frac{x}{y}dx$

(b) $J(y)=\int\limits_0^1 \ln(x^2+y^2)dx$

(c) $K=\int\limits_0^1\frac{x^b-x^a}{\ln x}dx$, $0<a<b$.

Exercise 27. Show that the integral

(a) $I(y)=\int\limits_1^\infty \sin(yx)dx$ is convergent if $y=0$ and is divergent if $y\neq 0$.

(b) $I(y)=\int\limits_0^\infty \frac{\cos\alpha x}{x^2+1}$ is uniformly convergent on $\mathbb{R}$.

(c) $I(y)=\int\limits_0^1 x^{-y}dx=\int\limits_1^\infty t^{y-2}dt$ is convergent if $y<1$ and divergent if $y\geq 1$.

(d) $I(y)=\int\limits_0^{+\infty} e^{-yx}\frac{\sin x}{x}$ is uniformly convergent on $[0,+\infty)$.

Exercise 28. (a) Evaluate $I(y)=\int\limits_0^{+\infty} ye^{-yx}dx(y>0)$

(b) Prove that $I(y)$ converges to 1 uniformly on $[y_0,+\infty)$ for all $y_0>0$.

(c) Explain why $I(y)$ is not uniformly convergent on $(0;+\infty)$.

Exercise 29. Prove that

(a) $\int\limits_0^\infty e^{-x^2}=\frac{\sqrt{\pi}}{2}$

(b) $\int\limits_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$

(c) $\int\limits_0^\infty \sin(x^2)dx=\int\limits_0^\infty\cos(x^2)dx=\frac{\sqrt{2\pi}}{4}$

(d) $\int\limits_0^{+\infty}\frac{\sin x}{x}e^{-yx}dx=\frac{\pi}{2}-\arctan y$

(e) $\int\limits_0^\infty \frac{\sin yx}{x(1+x^2)}dx=\frac{\pi}{2}(1-e^{-y}),\quad y\geq 0$

(f) $\int\limits_0^\infty \frac{1-\cos yx}{x^2}=\frac{|y|\pi}{2}$

(g) $\int\limits_0^\infty \frac{x\sin yx}{a^2+x^2}dx=\frac{\pi}{2}e^{-ay},\quad a,y\geq 0.$

(h) $\int\limits_0^\infty e^{-yx^2}dx=\frac{\sqrt{\pi}}{2\sqrt{y}},\quad y>0$

(i) $\int\limits_0^{+\infty}\left(e^{-\frac{a}{x^2}}-e^{-\frac{b}{x^2}}\right)dx=\sqrt{\pi b}-\sqrt{\pi a}$, $(a,b>0)$.

(j) $\int\limits_0^{+\infty}\frac{\arctan\frac{x}{a}-\arctan\frac{x}{b}}{x}dx=\frac{\pi}{2}\ln\frac{b}{a}$, $(a,b>0)$.

Exercise 30. Explain why

$$\lim_{y\to 0^+}\left(\int_0^{+\infty} ye^{-yx}dx\right) \neq \int_0^{+\infty}\left(\lim_{y\to 0^+} ye^{-yx}\right)dx.$$

Exercise 31. Evaluate $(a, b, \alpha, \beta > 0)$:

(a) $\int_0^{+\infty} \frac{e^{-\alpha x}-e^{-\beta x}}{x}dx$

(b) $\int_0^{+\infty} \frac{e^{-\alpha x^2}-e^{-\beta x^2}}{x^2}dx$

(c) $\int_0^{+\infty} \frac{dx}{(x^2+y)^{n+1}}$

(d) $\int_0^{+\infty} \frac{\sin bx-\sin cx}{x}e^{-ax}dx$

(e) $\int_0^{+\infty} \frac{\cos bx-\cos cx}{x}e^{-ax}dx$

(f) $\int_0^{+\infty} e^{-ax}\cos(yx)dx$

(g) $\int_0^{+\infty} e^{-x^2}\cos(yx)dx$

(h) $\int_{-\infty}^{+\infty} \frac{\arctan(x+y)}{1+x^2}dx$

(i) $\int_0^{+\infty} \frac{e^{-ax^2}-e^{-bx^2}}{x}dx$

(j) $\int_0^{+\infty} \frac{e^{-ax^3}-e^{-bx^3}}{x}dx$

(k) $\int_0^{+\infty} \frac{e^{-ax^2}-\cos bx}{x^2}dx$

(l) $\int_0^{\pi} \ln(1+y\cos x)dx$

(m) $\int_0^{+\infty} e^{-x^2}\sin ax dx$

(n) $\int_0^{+\infty} \frac{\sin xy}{x}dx, \quad y \geq 0$

(o) $\int_0^{+\infty} \int_0^{+\infty} e^{-ax^2}\cos bx dx$

(p) $\int_0^{+\infty} x^{2n}e^{-x^2}\cos bx dx, \quad n\in\mathbb{N}$

(q) $\int_0^{+\infty} \frac{\sin ax\cos bx}{x}dx$

(r) $\int_0^{+\infty} \frac{\sin ax\sin bx}{x}dx$

3.3. Euler Integrals

Exercise 32. Evaluate

(a) $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^4 x dx$

(b) $\int_0^{a} x^{2n}\sqrt{a^2-x^2}dx, (a>0)$

(c) $\int_0^{+\infty} x^{10}e^{-x^2}dx$

(d) $\int_0^{+\infty} \frac{\sqrt{x}}{(1+x^2)^2}dx$

(e) $\int_0^{+\infty} \frac{1}{1+x^3}dx$

(f) $\int_0^{+\infty} \frac{x^{n+1}}{(1+x^n)}dx, (2<n\in\mathbb{N})$

(g) $\int_0^{1} \frac{1}{\sqrt[n]{1-x^n}}dx, n\in\mathbb{N}^*$

(h) $\int_0^{+\infty} \frac{x^4}{(1+x^3)^2}dx$

Exercise 33. On which intervals the following function

$$I(y) = \int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2} dx$$

is continuous.

Exercise 34. Find $\lim\limits_{y\to 1} \int\limits_0^y \frac{\arctan x}{x^2 + y^2} dx.$

Exercise 35. The integral $I(y) = \int\limits_0^1 \frac{yf(x)}{x^2 + y^2} dx$ where $f(x)$ is a possitive function, and continuous on $[0, 1]$.

Exercise 36. Given a function $f(y) = \int\limits_0^{\frac{\pi}{2}} \ln(\sin^2 x + y^2 \cos^2 x) dx$. Evaluate $f'(1)$.

Exercise 37. Prove that the integral depending on the parameter

$$I(y) = \int_{-\infty}^{+\infty} \frac{\arctan(x + y)}{1 + x^2} dx$$

is a continuous and is differentiable function with respect to y. Evaluate $I'(y)$ and then find the formula for $I(y)$.

Exercise 38. Evaluate the following integrals (with $a, b, \alpha, \beta > 0$ and $n \in \mathbb{Z}^+$):

(a) $\int\limits_0^1 \frac{x^b - x^a}{\ln x} dx$

(b) $\int\limits_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx$

(c) $\int\limits_0^{+\infty} e^{-ax} \frac{\sin(bx) - \sin(cx)}{x} dx$

(d) $\int\limits_0^1 x^\alpha (\ln x)^n dx$

(e) $\int\limits_0^{+\infty} \frac{dx}{(x^2 + y)^{n+1}}$

(f) $\int\limits_0^{\frac{\pi}{2}} \ln(1 + y \sin^2 x) dx$, with $y > -1$

Exercise 39. Evaluate the following integrals

(a) $\int\limits_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x dx$

(b) $\int\limits_1^{+\infty} \frac{(\ln x)^4}{x^2} dx$

(c) $\int\limits_0^{+\infty} x^{10} e^{-x^2} dx$

(d) $\int\limits_0^{+\infty} \frac{\sqrt{x}}{(1 + x^2)^2} dx$

(e) $\int\limits_0^{+\infty} \frac{1}{1 + x^3} dx$

(f) $\int\limits_0^{+\infty} \frac{x^{n+1}}{(1 + x^n)^2} dx, (2 < n \in \mathbb{N})$

(g) $\int\limits_{-\infty}^0 e^{2x} \sqrt[3]{1 - e^{3x}} dx$

(h) $\int\limits_0^a x^{2n} \sqrt{a^2 - x^2} dx, \ (a > 0, n \in \mathbb{N})$

(i) $\int\limits_0^1 \frac{1}{\sqrt[n]{1 - x^n}} dx, (2 \leq n \in \mathbb{N})$

Exercise 40. Evaluate the following line integrals

(a) $\int\limits_C (3x - y)ds$, where C is a half of the circle $y = \sqrt{9 - x^2}$

(b) $\int\limits_C (x - y)ds$, where C is a circle $x^2 + y^2 = 2x$

(c) $\int\limits_C y^2 ds$, where C is a curve $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ $(0 \le t \le 2\pi, a > 0)$

(d) $\int\limits_C \sqrt{x^2 + y^2} ds$, where C is a curve $\begin{cases} x = a(\cos t + t\sin t) \\ y = a(\sin t - t\cos t) \end{cases}$ $(0 \le t \le 2\pi, a > 0)$

Exercise 41. Evaluate the following line integrals

(a) $\int\limits_{AB} (x^2 - 2xy)dx + (2xy - y^2)dy$, where AB is a part of parabol $y = x^2$ from $A(1;1)$ to $B(2;4)$.

(b) $\int\limits_C (2x - y)dx + xdy$, where C is a curve $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ whose direction is increasing direction of the parameter t, $(0 \le t \le 2\pi, a > 0)$.

(c) $\int\limits_{ABCA} 2(x^2 + y^2)dx + x(4y + 3)dy$, where $ABCA$ is a broken line through the points $A(0;0)$, $B(1;1)$, $C(0;2)$

(d) $\int\limits_{ABCDA} \frac{dx+dy}{|x|+|y|}$, where $ABCDA$ is a broken line through the points $A(1;0)$, $B(0;1)$, $C(-1;0)$, $D(0;-1)$

(e) $\int\limits_C \frac{\sqrt[4]{x^2+y^2}dx}{2} + dy$, where C is curve $\begin{cases} x = t\sin\sqrt{t} \\ y = t\cos\sqrt{t}, (0 \le t \le \frac{\pi^2}{4}). \end{cases}$

Exercise 42. Evaluate the following line integral

$$\int\limits_C (xy + x + y)dx + (xy + x - y)dy$$

in two ways: by computing it directly, and by Green's formula, then compare the results, where C is a curve:

(a) $x^2 + y^2 = R^2$ (b) $x^2 + y^2 = 2x$ (c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a, b > 0)$

Exercise 43. Evaluate the following line integrals

(a) $$\oint\limits_{x^2+y^2=2x} x^2(y + \frac{x}{4})dy - y^2(x + \frac{y}{4})dx$$

(b) $\oint\limits_{OABO} e^x[(1 - \cos y)dx - (y - \sin y)dy]$, where $OABO$ is a broken line through the points $O(0;0)$, $A(1;1)$, $B(0;2)$

(c) $\oint\limits_{x^2+y^2=2x} (xy + e^x \sin x + x + y)dx - (xy - e^{-y} + x - \sin y)dy$

(d) $\oint\limits_C (xy^4 + x^2 + y\cos(xy))dx + (\frac{x^3}{3} + xy^2 - x + x\cos(xy))dy$, where C is a circle $\begin{cases} x = a\cos t \\ y = a\sin t, \end{cases}$ $(a > 0)$

Exercise 44. Using the line integral of the second kind in order to compute the area of the region bounded by an arch of the cycloid: $x = a(t - \sin t), y = a(1 - \cos t)$ and x-axis, $(a > 0)$.

Exercise 45. Evaluate the following line integrals

(a) $\int\limits_{(-2;-1)}^{(3;0)} (x^4 + 4xy^3)dx + (6x^2y^2 - 5y^4)dy$ (b) $\int\limits_{(1;\pi)}^{(2;2\pi)} (1 - \frac{y^2}{x^2}\cos\frac{y}{x})dx + (\sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x})dy$

Exercise 46. Evaluate the line integral

$$I = \int\limits_L (3x^2y^2 + \frac{2}{4x^2+1})dx + (3x^3y + \frac{2}{y^3+4})dy$$

where L is a curve $y = \sqrt{1 - x^4}$ from $A(1;0)$ to $B(-1;0)$.

Exercise 47. Find the constant α such that the following integral is an independent of path in the domain

$$\int\limits_{AB} \frac{(1-y^2)dx + (1-x^2)dy}{(1+xy)^\alpha}.$$

Exercise 48. Find the constants a, b such that $(y^2 + axy + y\sin(xy))dx + (x^2 + bxy + x\sin(xy))dy$ is total differetial of the function $u(x,y)$. Find the function $u(x,y)$.

Exercise 49. Find the function $h(x)$ such that the integral

$$\int\limits_{AB} h(x)[(1+xy)dx + (xy + x^2)dy]$$

is independent of the path in the domain. With the function $h(x)$ found, evaluate the integral above from $A(2;0)$ to $B(1;2)$.

Exercise 50. Find the function $h(xy)$ in order to the integral

$$\int\limits_{AB} h(xy)[(y + x^3y^2)dx + (x + x^2y^3)dy]$$

is not dependent on the path in the domain. With the function $h(xy)$ just found, find the above integral from $A(1;1)$ to $B(2;3)$.

Exercise 51. Evaluate the following surface integrals

(a) $\iint\limits_S (z + 2x + \frac{4y}{3})dS$, where $S = \{(x,y,z) : \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, x \geq 0, y \geq 0, z \geq 0\}$

(b) $\iint\limits_S (x^2+y^2)dS$, where $S = \{(x,y,z) : z = x^2+y^2; 0 \le z \le 1\}$

(c) $\iint\limits_S zdS$, where $S = \{(x,y,z) : y = x+z^2, 0 \le x \le 1, 0 \le z \le 1\}$

(d) $\iint\limits_S \dfrac{dS}{(1+x+y+z)^2}$, where S is a boundary of the tetrahedron $x+y+z \le 2, x \ge 0, y \ge 0, z \ge 0$

Exercise 52. Evaluate the following surface integrals

(a) $\iint\limits_S z(x^2+y^2)dxdy$, where S is a half of a sphere: $x^2+y^2+z^2 = 1,\ z \ge 0$, with the outward normal vector.

(b) $\iint\limits_S ydzdx + z^2dxdy$, where S is the sphere $x^2 + \dfrac{y^2}{4} + z^2 = 1, x \ge 0, y \ge 0, z \ge 0$, with downward orientation.

(c) $\iint\limits_S x^2y^2zdxdy$, where S is the surface $x^2+y^2+z^2 = R^2, z \le 0$ and has upward orientation.

(d) $\iint\limits_S (y+z)dxdy$, where S is the surface $z = 4-4x^2-y^2$ and $z \ge 0$.

(e) $\iint\limits_S x^3dydz + y^3dzdx + z^3dxdy$, where S is the sphere $x^2+y^2+z^2 = R^2$ and is oriented downward.

(f) $\iint\limits_S y^2zdxdy + xzdydz + x^2ydzdx$, where S is outside of the domain

$$\begin{cases} x^2+y^2 \le 1, 0 \le z \le x^2+y^2 \\ x \ge 0, y \ge 0 \end{cases}$$

(g) $\iint\limits_S xdydz + ydzdx + zdxdy$, where S is outside of the solid

$$\begin{cases} (z-1)^2 \ge x^2+y^2 \\ a \le z \le 1 \end{cases}$$

Exercise 53. Use Stokes Theorem to evaluate the line integral

$$\int\limits_C (x+y^2)dx + (y+z^2)dy + (z+x^2)dz,$$

where C is a boundary of the triangle $(1;0;0), (0;1;0), (0;0;1)$, oriented counterclockwise when viewed from above.

Exercise 54. Given the sphere S: $x^2+y^2+z^2 = 1$ lies inside the cylinder

$$\begin{cases} x^2+x+z^2 = 0 \\ y \ge 0, \end{cases}$$

which is oriented outward. Prove that: $\iint\limits_S (x-y)dxdy + (y-z)dydz + (z-x)dzdx = 0.$

Exercise 55. Find the directional derivative of the function $u = x^3 + 2y^3 + 3z^2 + 2xyz$ at $A(2;1;1)$ in the direction of $\vec{AB}$, where $B(3;2;3)$.

Exercise 56. Compute the length of vector $\overrightarrow{\text{grad}} u$, with $u = x^3 + y^3 + z^3 - 3xyz$ at $A(2;1;1)$. When is $\overrightarrow{\text{grad}} u$ perpendicular to Oz, and when is $\overrightarrow{\text{grad}} u = 0$?

Exercise 57. Evaulate $\overrightarrow{\text{grad}} u$, with

$$u = r^2 + \frac{1}{r} + \ln r \text{ where } r = \sqrt{x^2 + y^2 + z^2}$$

Exercise 58. Find the directions in which the rate of change of the function

$$u = x \sin z - y \cos z$$

at the origin $O(0,0,0)$ is maximum?

Exercise 59. Evaluate the angle of two vectors $\overrightarrow{\text{grad}} z$ at $(3;4)$ of the fucntions:

$$z = \sqrt{x^2 + y^2}$$

$$z = x - 3y + \sqrt{3xy}$$

Exercise 60. Which of the following fields are conservative and find their potential functions.

(a) $\vec{F} = 5(x^2 - 4xy)\vec{i} + (3x^2 - 2y)\vec{j} + \vec{k}$

(b) $\vec{F} = (yz - 3x^2)\vec{i} + xz\vec{j} + (xy + 2)\vec{k}$

(c) $\vec{F} = (x + y)\vec{i} + (x + z)\vec{j} + (z + y)\vec{k}$

(d) $\vec{F} = C\dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{(x^2 + y^2 + z^2)^3}}, C \neq 0$ is a constant

(e) $\vec{F} = (\arctan z + 4xyz)\vec{i} + (2x^2z - 3y^2)\vec{j} + (\frac{x}{1+z^2} + 2x^2y)\vec{k}$

Exercise 61. Let $\vec{F} = xz^2\vec{i} + yx^2\vec{j} + zy^2\vec{k}$. Find the flux of F across the surface $S : x^2 + y^2 + z^2 = 1$, with the outward direction.

Exercise 62. Let $\vec{F} = x(y+z)\vec{i} + y(z+x)\vec{j} + z(x+y)\vec{k}$, L be an intersection of the cylinder $x^2+y^2+y = 0$ and the half of sphere $x^2 + y^2 + z^2 = 2, z \geq 0$. Prove that the circulation of $\vec{F}$ around L equals to 0.