UNIT 7

Z-TRANSPFORM

PhD. Nguyễn Hồng Quang
Assoc Prof. Trịnh Văn Loan
PhD. Đoàn Phong Tùng
Department of Computer Engineering

❑ Contents

❑ Learning Objectives

After completing this lesson, you will have a grasp of the following concepts:

- The definition and significance of the Z-transform in signal processing.
- Methods for determining the region of convergence for the Z-transform of discrete-time signals.

1. Definition of Z transform

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

$$X(z) \equiv Z\{x(n)\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \cdots + x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

- The Z-transform exists only for values of Z that make the series converge.
- The region of convergence (ROC) of $X(z)$ is the set of values of Z for which X(z) has finite values.

Example

- Compute the Z-transform and the region of convergence for the following signal:

$$x_1(n) = \{\mathbf{1}, 2, 5, 7, 0, 1\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

ROC is the Z plane except $z = 0$

Example

- Compute the Z-transform and the region of convergence for the following signal:

$$x_2(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$X_2(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n$$

$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A}, |A| < 1$$

$$\boxed{X(z) = \frac{1}{1-\frac{1}{2z}} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{Z}{Z-\frac{1}{2}} \qquad \text{ROC: } |z| > \frac{1}{2}}$$

Observation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

- The Z-transform is simply an alternative representation of a signal..
- The coefficient of z^{-n} in the transform is the value of the signal at time n.
- he exponent of Z contains information about the time needed to determine the samples of the signal.
- With the Z-transform, we can represent signals with infinite duration in the time domain by a finite form in the Z domain.

2. Region of convergence of the Z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

- Region of Convergence – ROC is the range of values of for which the geometric series in the definition of the Z-transform converges.
- The Cauchy criterion is commonly used to determine the ROC. Specifically, a series of the form:

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

converges if the condition $\lim_{n\to\infty} |u_n|^{1/n} < 1$ is satisfied.

Determine ROC

$$X(z) = X_1(z) + X_2(z) = \sum_{n=-\infty}^{-1} x(n)z^{-n} + \sum_{n=0}^{\infty} x(n)z^{-n}$$

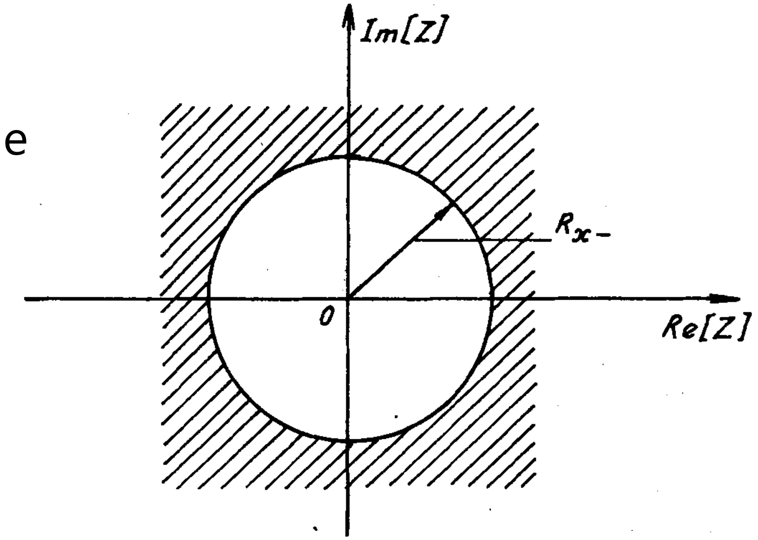
- Applying Caushy criterion we obtain

$$\lim_{n\to\infty} |x(n)z^{-n}|^{1/n} < 1 \implies \lim_{n\to\infty} |x(n)|^{1/n}|z^{-1}| < 1$$

- Assuming $\lim_{n\to\infty} |x(n)|^{1/n} = R_{x-}$
- $X_2(z)$ converge if z satisfies

$$|z| > R_{x-}$$

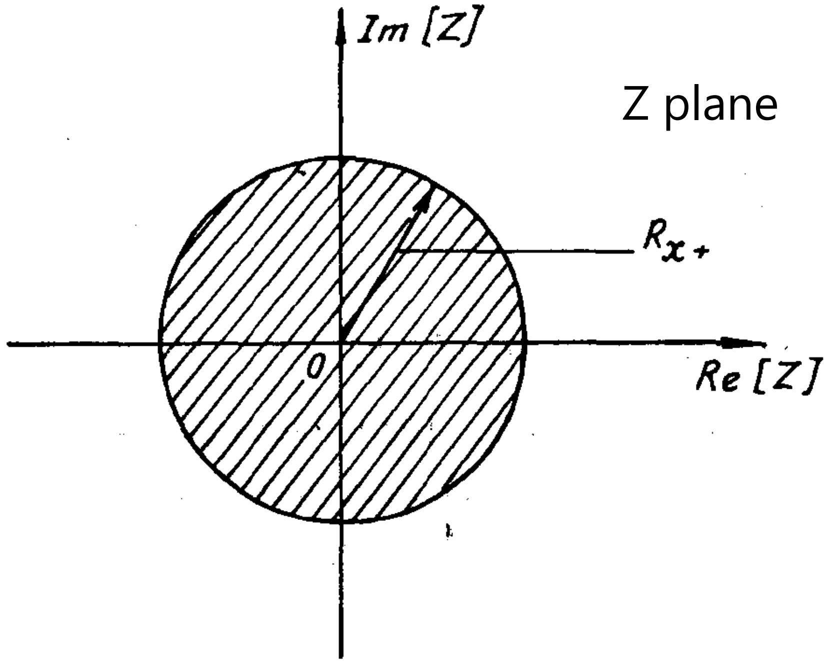
Z plane



ROC of $X_1(Z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$

- Non-causal signal
- $X_1(z)$ converge if z satisfies $|z| < R_{x+}$, where

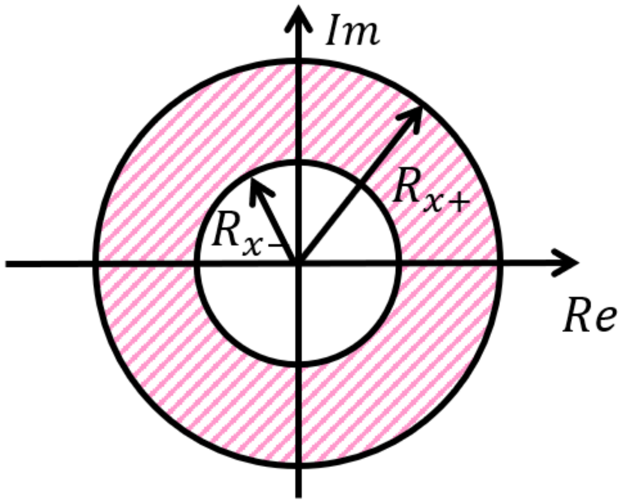
$$R_{x+} = \frac{1}{\lim_{n\to\infty} |x(-n)|^{1/n}}$$



ROC of Z transform

- In the general case, ROC of Z transform is

$$0 \le R_{x-} < |z| < R_{x+} \le \infty$$



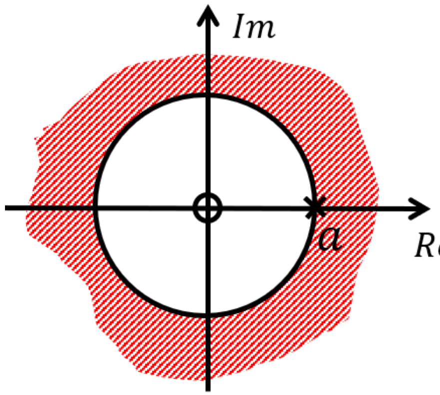
- Example: Given the signal $x(n) = a^n u(n)$, determine the Z transform and its ROC.

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

  - ROC: $|z| > |a|, R_{x-} = |a|, R_{x+} = \infty$
  - Zero point: $z = 0$
  - Polar point: $z = a$
  - ROC excludes pole

$$R_{x-} = \lim_{n\to\infty} |x(n)|^{1/n}$$

$$R_{x+} = \frac{1}{\lim_{n\to\infty} |x(-n)|^{1/n}}$$



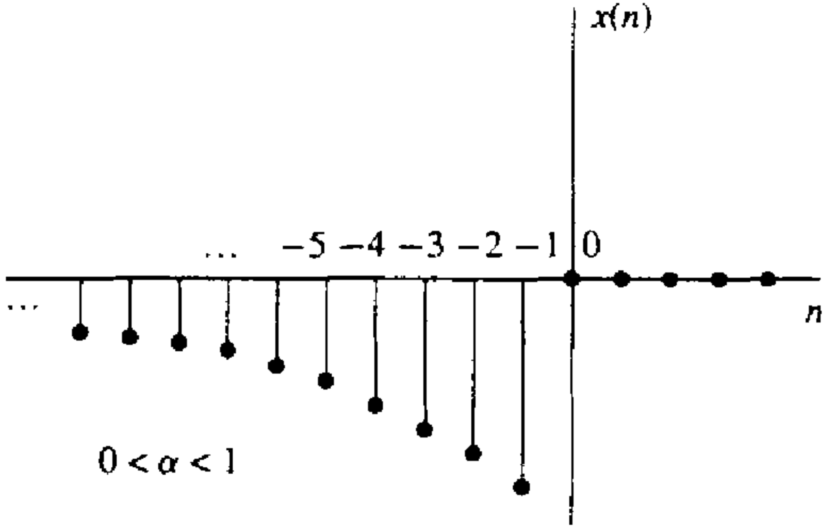
Example

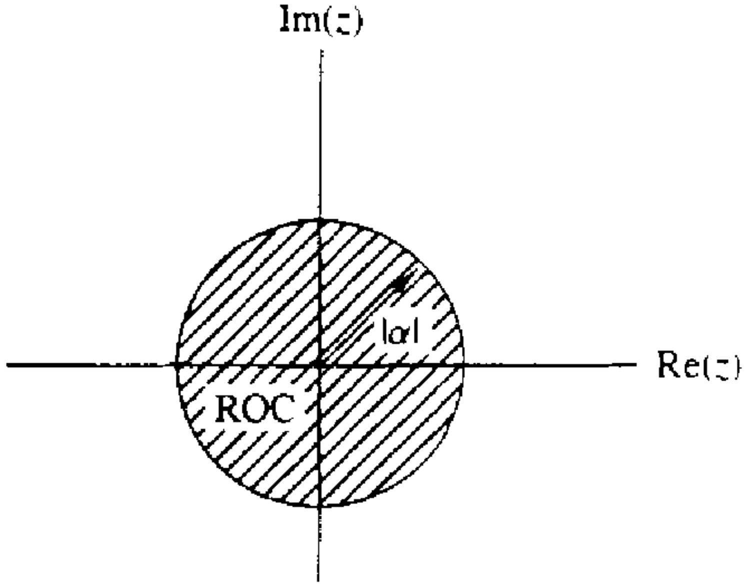
- Determine Z-transform of the following signal

$$x(n) = -\alpha^{n}u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -\alpha^{n} & n \leq -1 \end{cases}$$

ROC: $|z| < |\alpha|$

$$x(n) = -\alpha^{n}u(-n-1) \xleftrightarrow{z} X(z) = \frac{1}{1-\alpha z^{-1}} = \frac{Z}{Z-a}$$





Observation

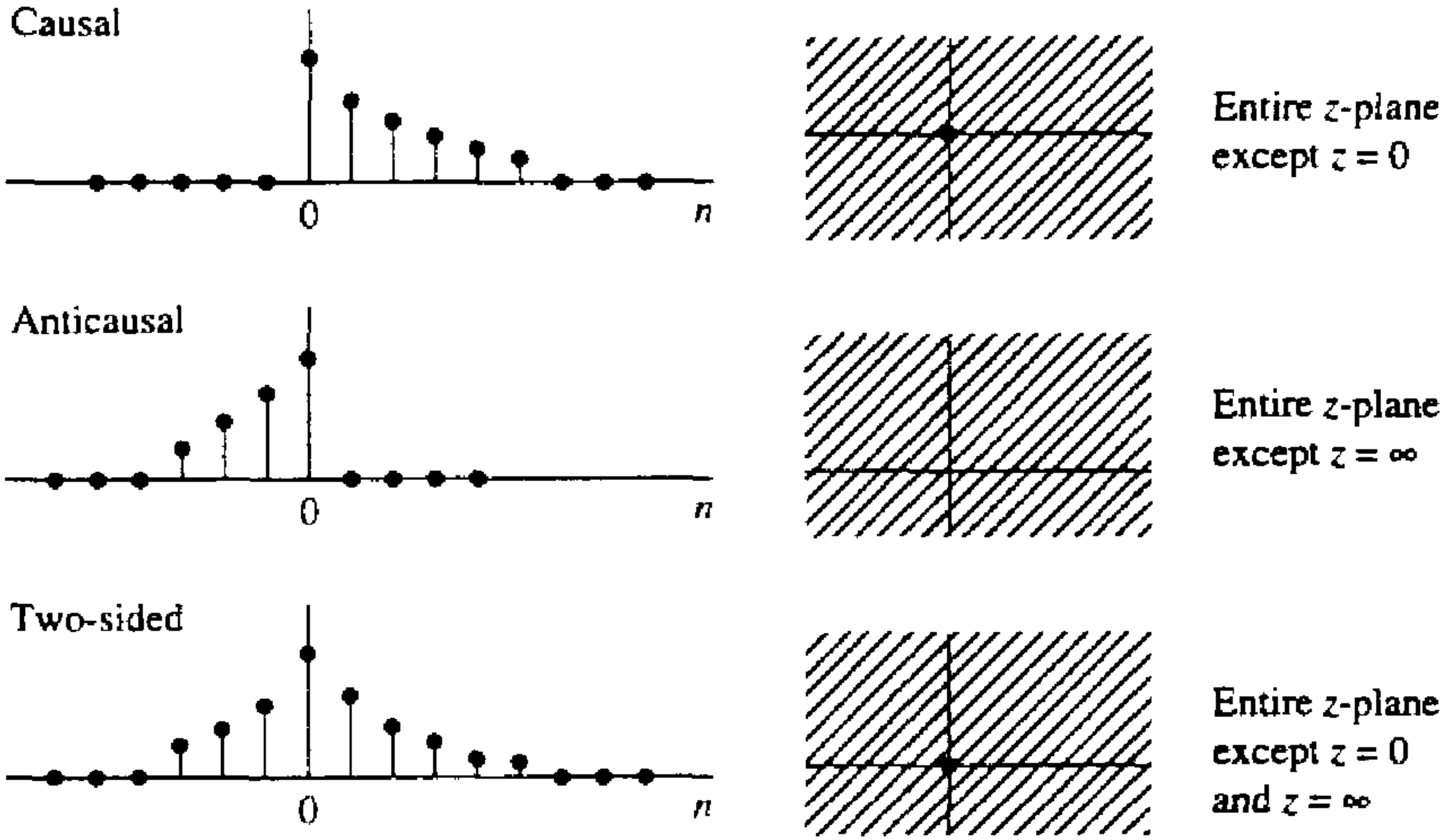
- The causal signal $\alpha^{n}u(n)$ and the anti-causal signal $-\alpha^{n}u(-n-1)$ have the same Z-transform expression.
- Therefore, the Z-transform expression alone is not sufficient to determine the signals in the time domain.

→ **A discrete-time signal $x(n)$ is uniquely defined by two components:**

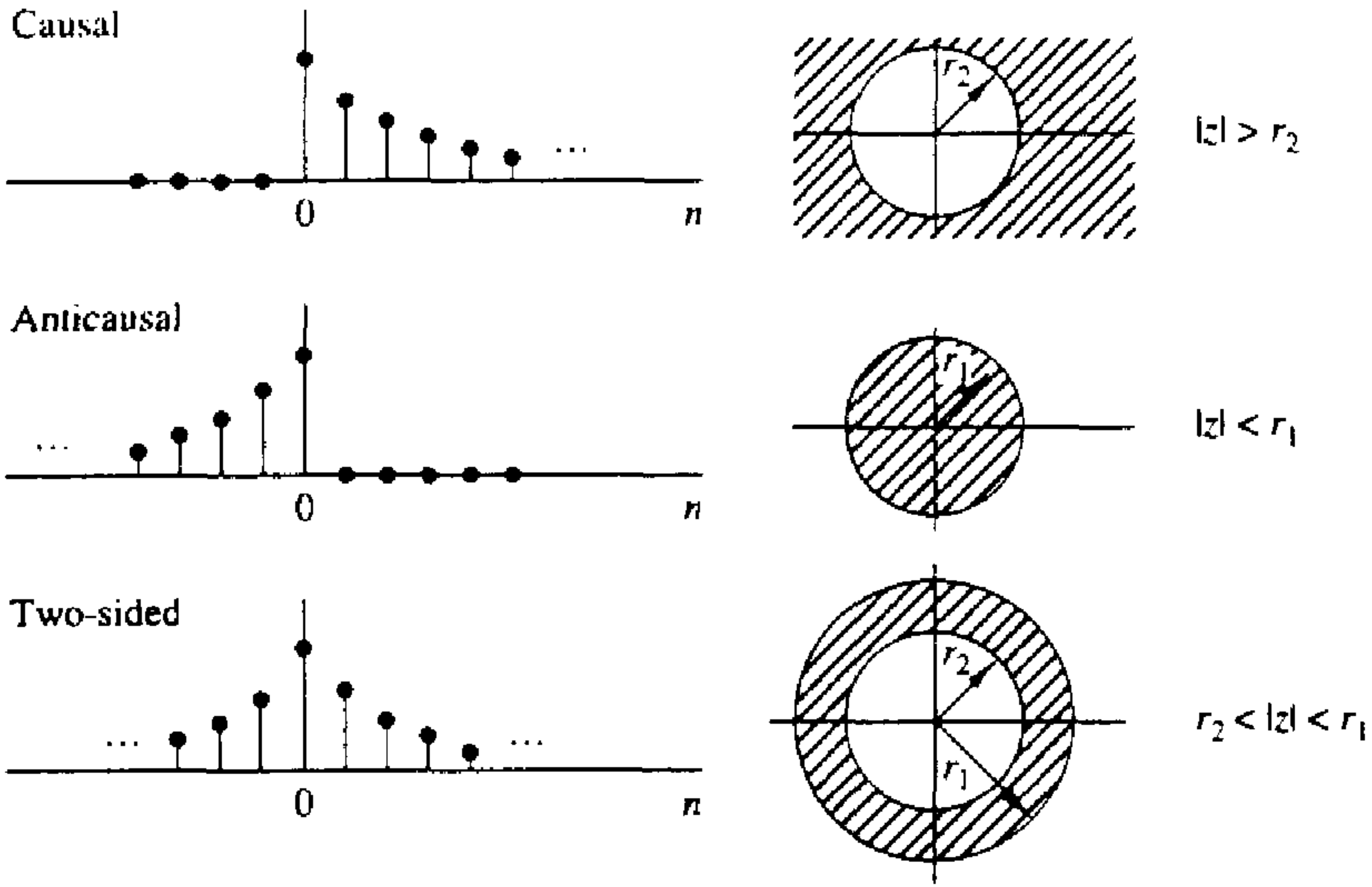
❖ **$X(z)$**

❖ **its region of convergence (ROC)**

Summary: Finite-length signal



Summary: Infinite-length signal



4. Summary

- The Z-transform allows for the representation of a signal in the complex domain. An important advantage of the Z-transform is that it allows for the representation of infinite-length signals as finite-length forms.
- The region of convergence (ROC) of the Z-transform is the range of values of Z for which the infinite power series in the Z-transform definition converges. Together with the Z-transform expression, the ROC uniquely determines the corresponding discrete-time signal $x(n)$ in the time domain.

5. Assignment

- Assignment 1

  - Determine Z-transform and its ROC of the following signals:

$$a.\ x_1(n) = \{1, \underset{\uparrow}{2}, 5, 7, 0, 1\}$$

$$b.\ x_2(n) = \delta(n)$$

$$c.\ x_3(n) = \delta(n-k),\ k > 0$$

$$d.\ x_4(n) = \delta(n+k),\ k > 0$$

Homework

- Assignment 2
  - Determine Z-transform and its ROC of the following signals:

    a. $x(n) = (\cos \omega_0 n)u(n)$

    b. $x(n) = (\sin \omega_0 n)u(n)$

    c. $x(n) = (3^{n+1} - 1)u(n)$

    d. $x(n) = 2^{-n}u(n) + 3^{n+1}u(n)$

Homework

- Assignment 3
  - Calculate the Z-transform and ROC of the following signals. Then comment on the changes in the ROC:

    a. $x(n) = 2^n u(n)$

    b. $y_1(n) = 3^n x(n)$

    c. $y_2(n) = \left(\frac{1}{3}\right)^n x(n)$

    d. $y_3(n) = e^{j\pi n/2} x(n)$

Homework

- Assignment 4
  - Calculate the Z-transform and ROC of the following signals:

    a. $x(n) = a^n(\cos \omega_0 n)u(n)$

    b. $x(n) = a^n(\sin \omega_0 n)u(n)$

    c. Ramp signal $u_r(n)$

The next unit 8

PROPERTIES OF THE Z-TRANSFORM

References:

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wishing you all the best in your studies!