

Line Integrals

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Line Integrals

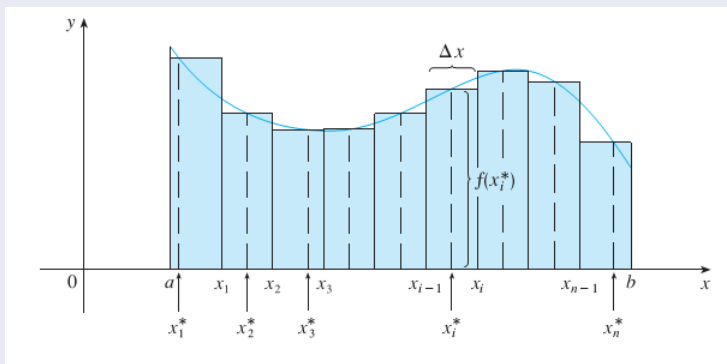
- 1 Line Integrals of scalar Fields
- 2 Line Integrals of vector Fields
 - Green's Theorem
 - Applications of Line Integrals
 - Independence of Path

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Line Integrals

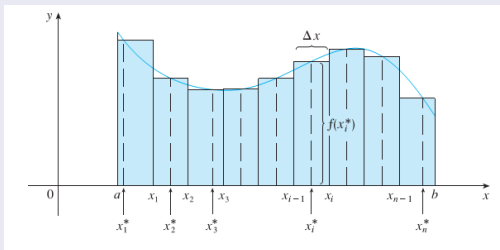
Review of the Definite Integral



$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

Line Integrals of scalar Fields

Review of the Definite Integral



- ① divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$
- ② choose sample points x_i^* in these subintervals,
- ③ form the Riemann sum $\sum_{i=1}^n f(x_i^*)\Delta x$
- ④ take the limit $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$

Line Integrals of scalar Fields

- Line integrals are integrated over a curve C instead of over an interval $[a, b]$.
- "curve integrals" would be better terminology.

Line Integrals of scalar Fields

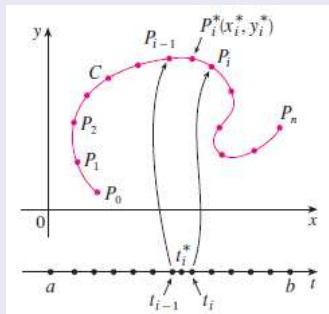
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Definition

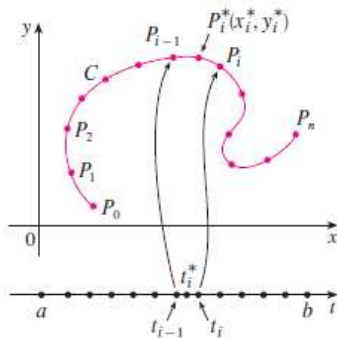
Let C be a curve given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \leq t \leq b$.

- 1 divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$
- 2 choose sample points $P_i^*(x_i^*, y_i^*) \in P_{i-1}P_i$,
- 3 form the Riemann sum $\sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$
- 4 take the limit

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$



Line Integrals of scalar Fields



$$\begin{aligned}
 \int_C f(x, y) ds &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i \\
 &= \int_a^b f(x(t), y(t)) \sqrt{(x'_t)^2 + (y'_t)^2} dt = \int_a^b f(r(t)) |r'(t)| dt.
 \end{aligned}$$

Formulations

① If C is given by $x = x(t), y = y(t), a \leq t \leq b$, then

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{x'^2(t) + y'^2(t)} dt.$$

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- ② If C is given by $y = y(x), a \leq x \leq b$ then

$$\int_C f(x, y) ds = \int_a^b f(x, y(x)) \sqrt{1 + y'^2(x)} dx.$$

- ③ If C is given by $x = x(y), c \leq y \leq d$ then

$$\int_C f(x, y) ds = \int_c^d f(x(y), y) \sqrt{1 + x'^2(y)} dy.$$

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Example

Evaluate $\int_C (x - y) ds$, where C is the circle $x^2 + y^2 = 2x$.

Line Integrals

Properties

- Line Integrals of scalar Fields do not depend on the direction of C .

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- Physical interpretation: The mass of C is $\int_C \rho(x, y) ds$, where $\rho(x, y)$ is the density function.
- The length of C is $l = \int_C ds$.
- Linearity and additivity.

Line Integrals

line integrals of f along C with respect to x and y

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$$\int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i$$

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3

$$\int_C f(x, y) dx = - \int_C f(x, y) dx, \quad \int_C f(x, y) dy = - \int_C f(x, y) dy.$$

Line Integrals

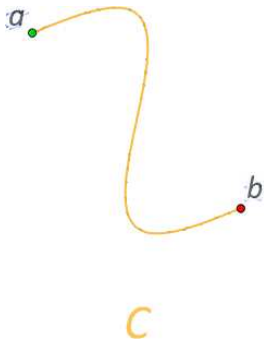
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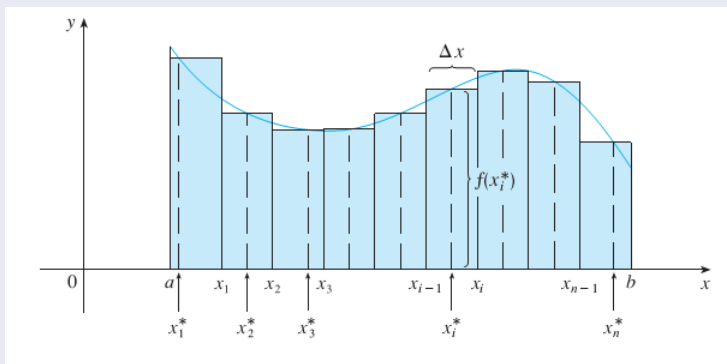
Line Integrals of vector Fields

Suppose that $F = P\vec{i} + Q\vec{j}$ is a continuous force field. Compute the work done by this force in moving a particle along a smooth curve C .



Line Integrals

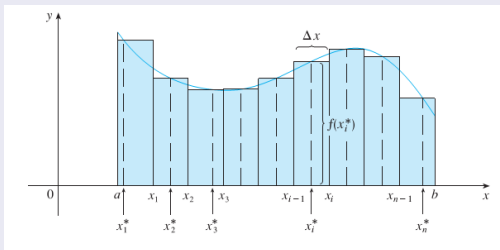
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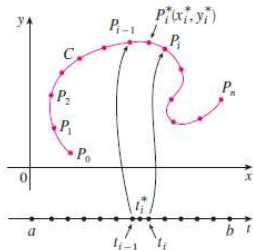
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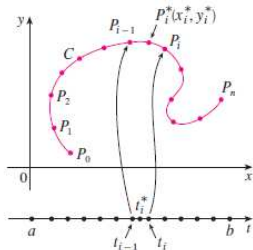
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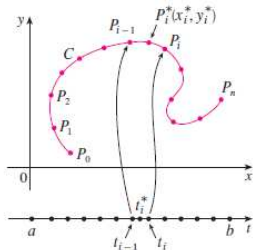


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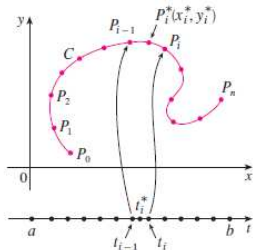


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The work done by the force F in moving the particle from P_{i-1} to P_i is approximately

$$F(x_i^*, y_i^*) \cdot [\Delta s_i T(t_i^*)] = [F(x_i^*, y_i^*) \cdot T(t_i^*)] \Delta s_i.$$

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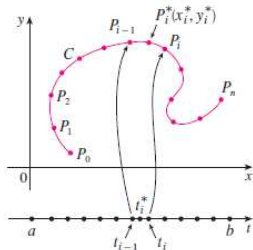
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The total work is approximately

$$\textcircled{3} \quad S_n = \sum_{i=1}^n [F(x_i^*, y_i^*) \cdot T(t_i^*)] \Delta s_i.$$

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$$\textcircled{4} \quad W = \lim_{n \rightarrow \infty} S_n = \int_C F(x, y) \cdot T(x, y) ds.$$

Line Integrals of vector Fields

Definition

Let F be a continuous vector field defined on a smooth curve C given by a vector function $r(t)$, $a \leq t \leq b$. Then the line integral of F along C is

$$\int_C F \cdot T ds$$

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It is sometimes denoted by

$$\int_C F \cdot T ds = \int_C P(x, y) dx + Q(x, y) dy.$$

Line Integrals of vector Fields

Properties

- Line integrals of vector fields depend on the direction of the curve, i.e., $\int_{-C} P(x, y) dx + Q(x, y) dy = - \int_C P(x, y) dx + Q(x, y) dy$.
- Linearity and additivity.

Formulations

- If $C = \widetilde{AB}$ is given by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, A, B correspond to $t = a, t = b$, then

$$\int_C P dx + Q dy = \int_a^b [P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t)] dt.$$

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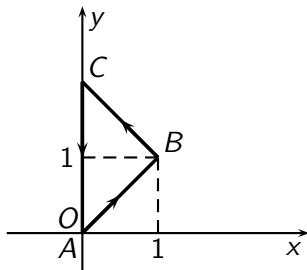
3. If $C = \widetilde{AB}$ is given by $x = x(y)$, A, B correspond to $y = c, y = d$, then

$$\int_C Pdx + Qdy = \int_c^d [P(x(y), y)x'(y) + Q(x(y), y)] dy.$$

Line Integrals of vector Fields

Example

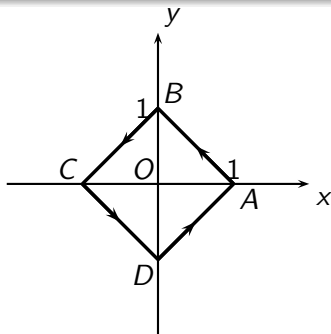
Evaluate $\int_{ABCA} 2(x^2 + y^2) dx + x(4y + 3) dy$, where $ABCA$ is the quadrangular curve, $A(0, 0)$, $B(1, 1)$, $C(0, 2)$.



Line Integrals of vector Fields

Example

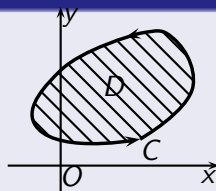
Evaluate $\int_{ABCD} \frac{dx+dy}{|x|+|y|}$, where $ABCD$ is the triangular curve, $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$.



Green's Theorem

Closed Curve Orientation

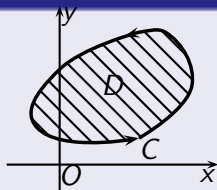
We use the convention that the positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C .



Green's Theorem

Closed Curve Orientation

We use the convention that the positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C .



Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

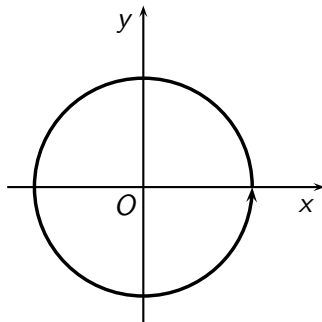
$$\int_C P(x, y) dx + Q(x, y) dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Green's Theorem

Example

Evaluate the integral $\int_C (xy + x + y) dx + (xy + x - y) dy$, where C is the positively oriented circle $x^2 + y^2 = R^2$ by

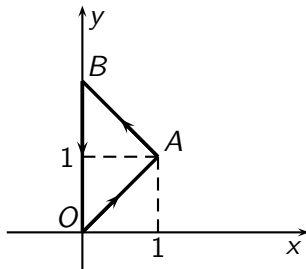
- computing it directly and
- Green's Theorem then compare the results,



Green's Theorem

Example

Evaluate $\oint_{OABO} e^x [(1 - \cos y) dx - (y - \sin y) dy]$, where $OABO$ is the triangle, $O(0, 0)$, $A(1, 1)$, $B(0, 2)$.



Green's Theorem

- If ∂D is negatively oriented, then

$$\int_C Pdx + Qdy = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

- If C is not closed, then we "closed off" the curve, applying Green's Theorem, and then subtract the integral over the piece with glued on.

Green's Theorem

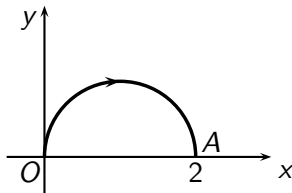
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Example

Evaluate $\int_C (xy + x + y) dx + (xy + x - y) dy$, where C is a half of the circle $x^2 + y^2 = 2x, y \geq 0$, traced from $O(0,0)$ to $A(2,0)$.



Applications of Line Integrals

Area of a Domain

If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then $A(D) = \iint_D 1 dx dy = \int_{\partial D} P dx + Q dy$.

Applications of Line Integrals

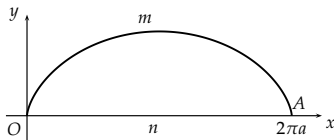
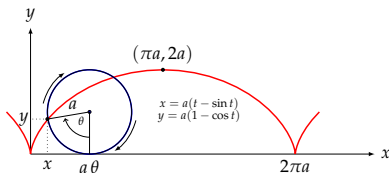
Area of a Domain

If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then $A(D) = \iint_D 1 dx dy = \int_{\partial D} P dx + Q dy$.

Example

Find the area of the domain bounded by an arch of the cycloid

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases} \quad \text{and } Ox \quad (a > 0).$$



Independence of Path

Assume that D is a simple domain, P, Q and their partial derivatives are continuous on \overline{D} . Then the following assertions are equivalent:

1. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for all $(x, y) \in D$.
2. $\int_L Pdx + Qdy = 0$ for all closed curve L contained in D .
3. $\int_C Pdx + Qdy$ is independent of path.
4. $F = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is conservative, i.e.,

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4. $F = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is conservative, i.e., $\exists u(x, y)$ s.t. $\nabla u = F$.
The function u is computed by:

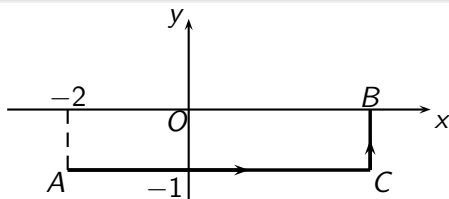
$$\begin{aligned} u(x, y) &= \int_{x_0}^x P(x, y_0)dx + \int_{y_0}^y Q(x, y)dy \\ &= \int_{x_0}^x P(x, y)dx + \int_{y_0}^y Q(x_0, y)dy. \end{aligned}$$

Independence of Path

- 1 Check the condition $P'_y = Q'_x$.
- 2 Choose the path such that the integration is simplest. It might be the line connects A and B , or zigzags whose line segments parallel to axis.
- 3 If we can find u such that $du = Pdx + Qdy$, then $I = u(B) - u(A)$.

Example

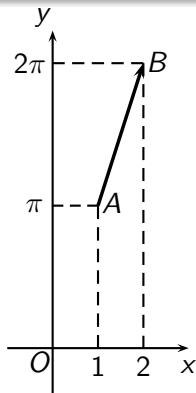
Evaluate $\int_{(-2,-1)}^{(3,0)} (x^4 + 4xy^3) dx + (6x^2y^2 - 5y^4) dy$.



Independence of Path

Example

Evaluate $\int_{(1,\pi)}^{(2,2\pi)} \left(1 - \frac{y^2}{x^2} \cos \frac{y}{x}\right) dx + \left(\sin \frac{y}{x} + \frac{y}{x} \cos \frac{y}{x}\right) dy$.



Independence of path

Equivalence

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4. \vec{F} is conservative, i.e. $\exists u(x, y, z)$ st. $\nabla u = \vec{F}$.

$$u(x, y) = \int_{x_0}^x P(x, y_0, z_0)dx + \int_{y_0}^y Q(x, y, z_0)dy + \int_{z_0}^z R(x, y, z)dz + C.$$