Surface Integrals

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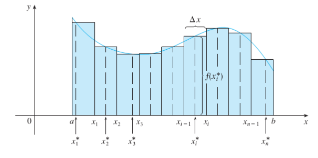
Surface Integrals

Surface Integrals of scalar Fields

Review of the Definite Integral



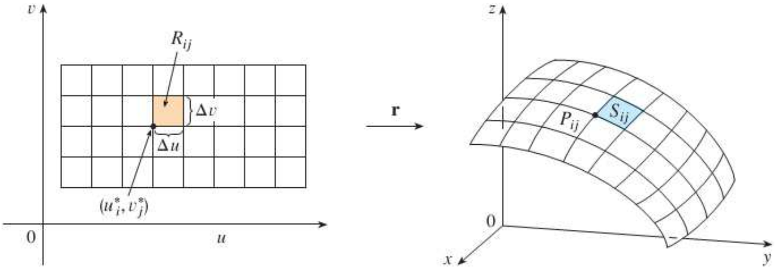
i) divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$

ii) choose sample points x_i^* in these subintervals,

iii) form the Riemann sum $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

iv) take the limit $\int\limits_a^b f(x)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

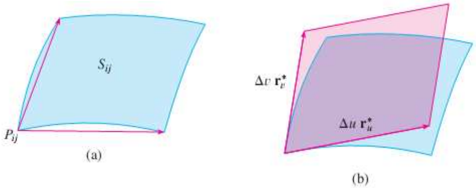
Surface Area



Consider the surface $r(u,v) = x(u,v).\vec{i} + y(u,v).\vec{j} + z(u,v).\vec{k}, (u,v) \in D$.

- Divide D into subrectangles R_{ij} and let (u_i^*, v_j^*) be the lower left corner R_{ij}.
- The part S_{ij} of the surface that corresponds to R_{ij} is called a patch. Let $r_u^* = r_u(P_{ij}^*), r_v^* = r_v(P_{ij}^*)$.

Surface Area



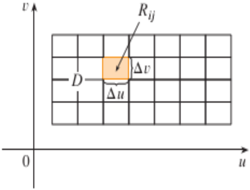
Each patch S_{ij} can be approximated by the parallelogram determined by the vectors $\Delta u r_u^*$ and $\Delta v r_v^*$ with area $|(\Delta u r_u^*) \times (\Delta v r_v^*)| = |r_u^* \times r_v^*|\Delta u \Delta v$ so an approximation to the area of is

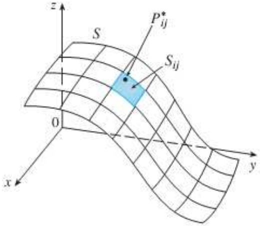
$$\sum_{i=1}^{m} \sum_{j=1}^{n} |r_u^* \times r_v^*|\Delta u \Delta v.$$

Definition

$$A(S) = \iint_D |r_u \times r_v| dudv.$$

Surface Integrals of scalar Fields





Let S be $r(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}, (u,v) \in [a,b] \times [c,d]$.

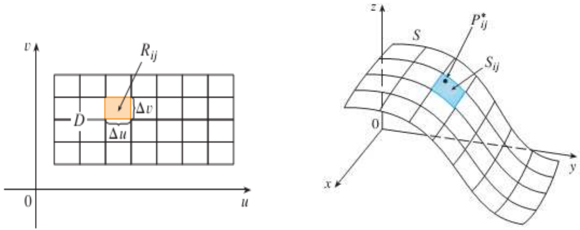
i) divide S in to patches by dividing $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ and dividing $[c,d]$ into n subintervals, each of equal length.

ii) choose sample points P^*_{ij} in each patch,

iii) form the Riemann sum $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(P^*_{ij})\Delta S_{ij}$

iv) take the limit $\iint\limits_S f(x,y,z)dS = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta S_{ij}$.

Formulations



In our discussion of surface area, we made the approximation

$$\Delta S_{ij} \approx |r_u^* \times r_v^*| \Delta u \Delta v.$$

Therefore

$$\iint_S f(x, y, z) dS = \iint_D f(x(u, v), y(u, v), z(u, v)) |r_u \times r_v| dudv.$$

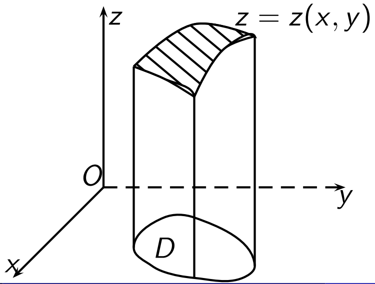
Formulations

Any surface with equation $z = z(x, y)$ can be parameterized $x = x, y = y, z = z(x, y)$ and $r_x \times r_y = \sqrt{1 + (z'_x)^2 + \left(z'_y\right)^2}$. Therefore, if

- the surface S is given by $z = z(x, y)$,
- the projection of S onto Oxy is D,

then

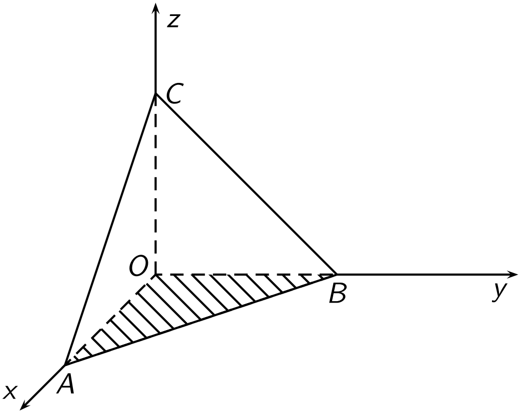
$$\iint\limits_S f(x, y, z)dS = \iint\limits_D f(x, y, z(x, y))\sqrt{1 + (z'_x)^2 + \left(z'_y\right)^2}dxdy.$$



Surface Integrals of scalar Fields

Example

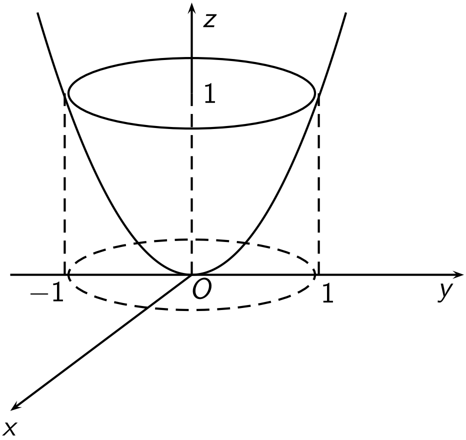
Evaluate $\iint\limits_S \left(z + 2x + \frac{4y}{3}\right) dS$, where
$S = \{(x,y,z) \,|\, \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, x, y, z \geqslant 0\}$



Surface Integrals of scalar Fields

Example

Evaluate $\iint\limits_S (x^2 + y^2)\, dS$, where $S = \{(x, y, z)\,|z = x^2 + y^2, 0 \leqslant z \leqslant 1\}$.

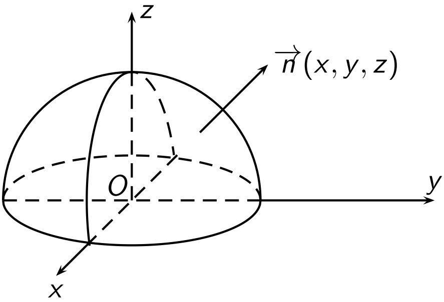


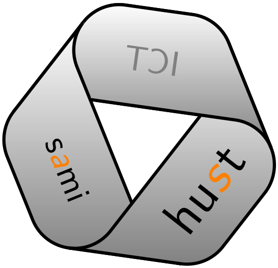
Surface Integrals

Oriented Surfaces (Two-sided Surfaces)

Let S be a surface. At every point, there are two unit normal vectors $\overrightarrow{n}$ and $-\overrightarrow{n}$.

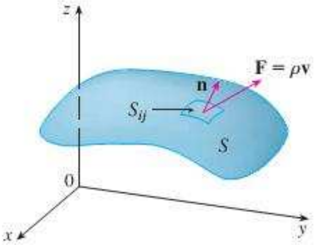
- If it is possible to choose a unit normal vector at every such point so that it varies continuously over S, then S is called an oriented surface. There are two possible orientations for any orientable surface.
- Conversely, S is called a nonorientable surface.





Surface Integrals of vector Fields

- S is an oriented surface with unit normal vector $\vec{n}$,
- a fluid with density $\rho(x, y, z)$ and velocity field $\vec{v}(x, y, z)$ flowing through S.
- **Problem:** Evaluate the mass of fluid per unit time crossing S.



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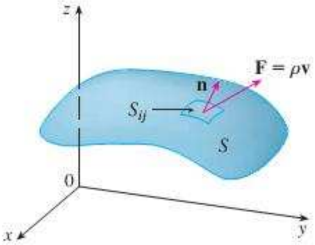

The mass of fluid per unit time crossing S_{ij} is approximated by

$$(\rho\vec{v} \cdot \vec{n})\Delta(S_{ij}) = (\vec{F} \cdot \vec{n})\Delta(S_{ij}).$$

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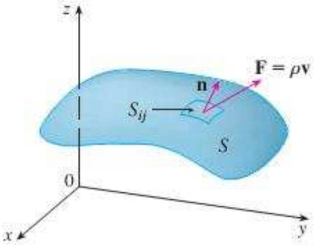

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$$(\rho\vec{v} \cdot \vec{n})\Delta(S_{ij}) = (\vec{F} \cdot \vec{n})\Delta(S_{ij}).$$

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_S \vec{F}(x, y, z) \cdot \vec{n}(x, y, z) dS.$$

Surface Integrals of vector Fields

Definition

If $\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint\limits_S Pdydz + Qdzdx + Rdxdy := \iint\limits_S \vec{F} \cdot \vec{n} dS.$$

This integral is also called the flux of $\vec{F}$ across S.

Surface Integrals of vector Fields

Surface Integrals of scalar Fields

$$\iint_S f(x,y,z)dS = \iint_D f(x(u,v),y(u,v),z(u,v))|r_u \times r_v|dudv.$$

Surface Integrals of vector Fields

Surface Integrals of scalar Fields

$$\iint_S f(x,y,z)dS = \iint_D f(x(u,v),y(u,v),z(u,v))|r_u \times r_v|dudv.$$

Let S be given by $r(u,v)$, then a normal vector is $\vec{N} = r_u \times r_v = (A,B,C)$.

Surface Integrals of vector Fields

Surface Integrals of scalar Fields

$$\iint_S f(x,y,z)dS = \iint_D f(x(u,v),y(u,v),z(u,v))|r_u \times r_v|dudv.$$

Let S be given by $r(u,v)$, then a normal vector is $\vec{N} = r_u \times r_v = (A,B,C)$.

- If $N \upuparrows n$, then $n = \left(\frac{A}{|r_u \times r_v|}, \frac{B}{|r_u \times r_v|}, \frac{C}{|r_u \times r_v|}\right)$ Therefore,

$$\iint_S Pdydz + Qdzdx + Rdxdy = \iint_D (AP + BQ + CR)dudv.$$

Surface Integrals of vector Fields

Surface Integrals of scalar Fields

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x(u,v),y(u,v),z(u,v))|r_u \times r_v|dudv.$$

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- If $N \uparrow\downarrow n$, then

$$\iint\limits_S Pdydz + Qdzdx + Rdxdy = -\iint\limits_D (AP+BQ+CR)dudv.$$

Surface Integrals of vector Fields

If $P = Q = 0$ and $\begin{cases} S : z = z(x, y), \\ (x, y) \in D \end{cases}$

Surface Integrals of vector Fields

If $P = Q = 0$ and $\begin{cases} S : z = z(x, y), \\ (x, y) \in D \end{cases} \Rightarrow \vec{N} = (-z'_x, -z'_y, 1).$

Formulations

- If $\widehat{(\vec{n}, Oz)} < \frac{\pi}{2}$, then $\iint\limits_S Rdxdy = \iint\limits_D R\left(x, y, z\left(x, y\right)\right) dxdy.$
- If $\widehat{(\vec{n}, Oz)} > \frac{\pi}{2}$, then $\iint\limits_S Rdxdy = -\iint\limits_D R\left(x, y, z\left(x, y\right)\right) dxdy.$

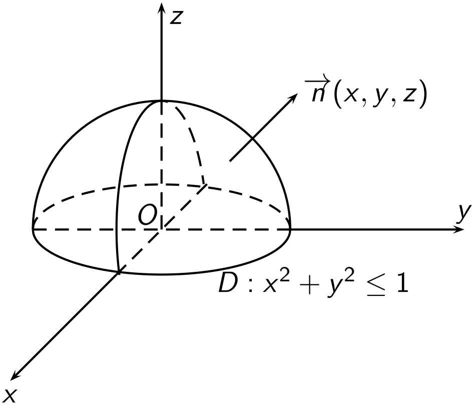
Remark:

$$I = \underbrace{\iint\limits_S Pdydz}_{I_1} + \underbrace{\iint\limits_S Qdzdx}_{I_2} + \underbrace{\iint\limits_S Rdxdy}_{I_3}.$$

Surface Integrals of vector Fields

Example

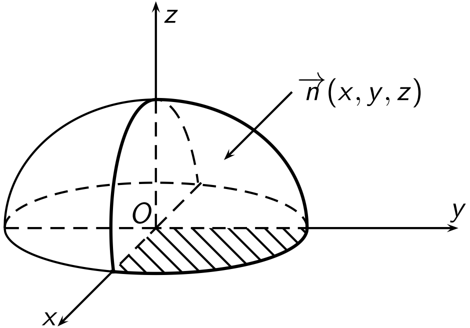
Evaluate $\iint\limits_S z\left(x^2+y^2\right)dxdy$, where S is a half of the sphere $x^2+y^2+z^2=1, z \geqslant 0$, with the outer-pointing normal vector.



Surface Integrals of vector Fields

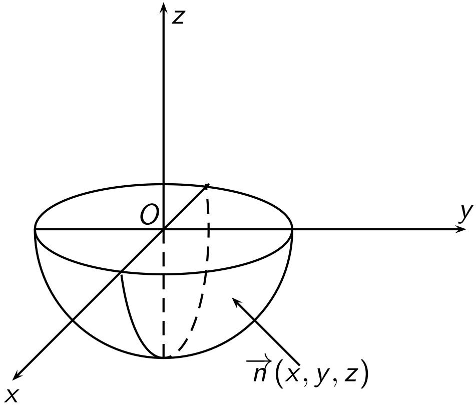
Example

Evaluate $\iint\limits_S ydxdz + z^2dxdy$, where S is the surface $x^2 + \frac{y^2}{4} + z^2 = 1, x \geqslant 0, y \geqslant 0, z \geqslant 0$ is oriented downward.



Example

Evaluate $\iint\limits_S x^2y^2zdxdy$, where S is the surface $x^2+y^2+z^2=R^2, z \leq 0$ and is oriented upward.



Surface Integrals of vector Fields

Ostrogradsky-Gauss Theorem (The Divergence Theorem)

$$\iint_S P dydz + Q dzdx + R dxdy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz.$$

Surface Integrals of vector Fields

Ostrogradsky-Gauss Theorem (The Divergence Theorem)

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Example

Evaluate the following integrals, where S is the surface $x^2 + y^2 + z^2 = a^2$ with outward orientation.

a. $\iint\limits_S xdydz + ydzdx + zdxdy$

b. $\iint\limits_S x^3dydz + y^3dzdx + z^3dxdy$.

The Divergence Theorem

Example

Evaluate the following integrals, where S is the surface $x^2 + y^2 + z^2 = a^2$ with outward orientation.

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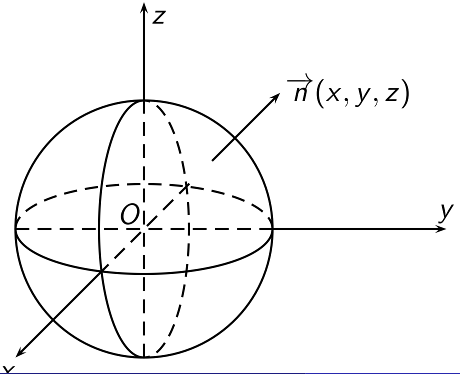
b. $\iint\limits_S x^3dydz + y^3dzdx + z^3dxdy$.



The Divergence Theorem

- If S is inward oriented, then
$$\iint\limits_{S} Pdydz + Qdzdx + Rdxdy = -\iiint\limits_{V} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$
- If S is not closed, then we use the "close off" technique.

The Divergence Theorem

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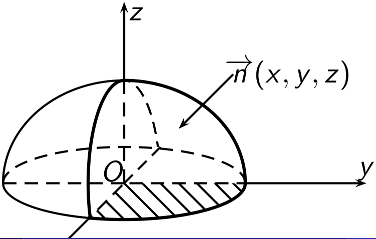
- If S is not closed, then we use the "close off" technique.

Example

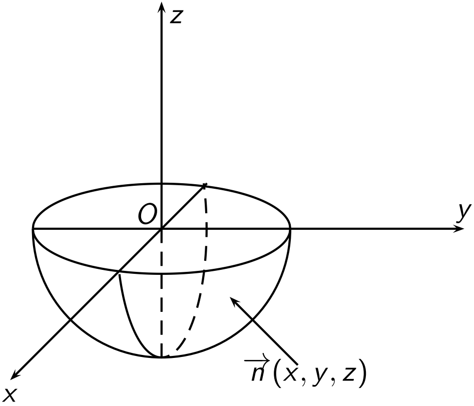
Evaluate $\iint\limits_S ydxdz + z^2dxdy$, where $S : x^2 + \frac{y^2}{4} + z^2 = 1, x, y, z \geq 0$, is oriented downward.



The Divergence Theorem

Example

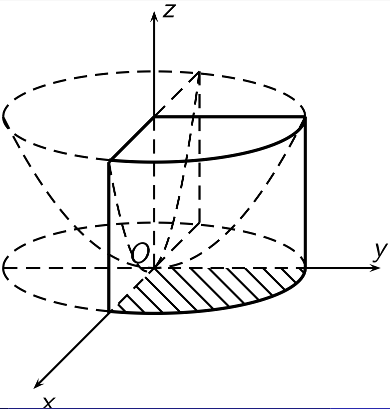
Evaluate $\iint\limits_S x^2y^2zdxdy$, where S is the surface $x^2 + y^2 + z^2 = R^2, z \leq 0$ and is oriented upward.



The Divergence Theorem

Example

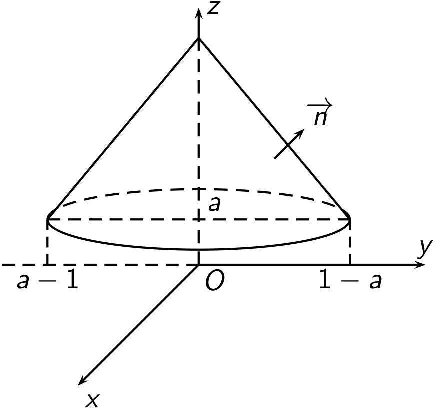
Evaluate $\iint\limits_S y^2 z dxdy + xzdydz + x^2 ydxdz$, where S is the boundary of the domain $x \geq 0, y \geq 0, x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2$ which is outward oriented.



The Divergence Theorem

Example

Evaluate $\iint\limits_S xdydz + ydzdx + zdxdy$, where S the boundary of the domain $(z-1)^2 \leqslant x^2 + y^2, a \leqslant z \leqslant 1, a > 0$ which is outward oriented.

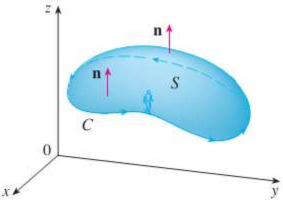


Surface Integrals of vector Fields

Stokes' Formula

$$\int_C Pdx + Qdy + Rdz$$
$$= \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy,$$

where the orientation of S induces the positive orientation of the boundary curve C shown in the figure.



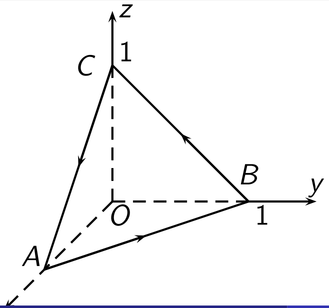
Stokes' Formula

Example

Evaluate $\int\limits_L \vec{F} \cdot d\vec{r} = \int\limits_L Pdx + Qdy + Rdz$, where

$$\vec{F}(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k},$$

L is the triangle ABC, $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ oriented counterclockwise as viewed from above.



Stokes' Theorem

Example

Use Stokes' Theorem to evaluate $\int\limits_C F\cdot dr = \int\limits_C Pdx + Qdy + Rdz$. In each case C is oriented counterclockwise as viewed from above.

1. $F(x,y,z) = (x+y^2)\mathbf{i} + (y+z^2)\mathbf{j} + (z+x^2)\mathbf{k}$, C is the triangle with vertices $(1,0,0),(0,1,0)$ and $(0,0,1)$.
2. $F(x,y,z) = \mathbf{i} + (x+yz)\mathbf{j} + (xy-\sqrt{z})\mathbf{k}$, C is the boundary of the part of the plane $3x+2y+z=1$ in the first octant.
3. $F(x,y,z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$, C is the circle $x^2+y^2=16, z=5$.
4. $F(x,y,z) = xy\mathbf{i} + 2z\mathbf{j} + 3y\mathbf{k}$, C is the curve of intersection of the plane $x+z=5$ and the cylinder $x^2+y^2=9$.

Surface Integrals of vector Fields

Relation between Surface Integrals of scalar and vector Fields

$$\iint\limits_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]\, dS$$
$$= \iint\limits_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy,$$

where $n = (\cos\alpha, \cos\beta, \cos\gamma)$ is the unit normal vector of S.

Relation between two kind of Surface Integrals

Example

Let S be the part of the sphere $x^2 + y^2 + z^2 = 1$ that contained in the cylinder $x^2 + x + z^2 = 0, y \geq 0$. S is outward oriented. Prove that

$$\iint\limits_S (x-y)dxdy + (y-z)dydz + (z-x)dxdz = 0.$$