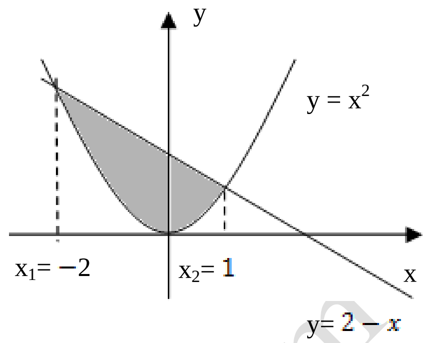
BÀI TẬP TÍCH PHÂN KÉP

(*Lưu ý*: *Tài liệu đã cho chưa được thẩm định nên có phần chưa chính xác hoàn toàn*)



Tính các tích phân kép:

a) $I = \iint\limits_D (x^2 + xy)\,dxdy;$ với $D: \begin{cases} y = x^2 \\ y = 2 - x \end{cases}$

Cho $x^2 = 2 - x$ để xác định giao điểm của hai đường:

$y = x^2$ và $y = 2 - x$

$x^2 + x = 0$ có 2 nghiệm: $x_1 = 1$ và $x_2 = -2$

Vậy: $-2 \leq x \leq 1$

Vì đường y = 2 – x nằm trên đường $y = x^2$ $\Rightarrow$ $x^2 \leq y^2 \leq -2$

$$\Rightarrow I = \int_{-2}^{1} dx \int_{x^2}^{2-x} (x^2 + xy)dy$$

Xét: $\int_{x^2}^{2-x} (x^2 + xy)dy$

Tính tích phân theo y, coi x như hằng số:

$$I_1 = \int_{x^2}^{2-x} (x^2 + xy)dy = x^2 + y + x.\frac{y^2}{2}\Big|_{x^2}^{2-x}$$

$$\left(x^2(2-x) + x\frac{(2-x)^2}{2}\right) - \left(x^2x^2 + x\frac{(x^2)^2}{2}\right)$$

$$= 2x^2 - x^3 + \frac{x}{2}(4 - 4x^2 + x^4) - x^4 - x.\frac{x^4}{2}$$

$$= 2x^2 - x^3 + 2x - 2x^3 + x^5 - x^4 - \frac{x^5}{2}$$

$$= \frac{x^5}{2} - x^4 - 3x^3 + 2x^2 + 2x$$

$$\Rightarrow \quad I = \int_{-2}^{1} \left(\frac{x^5}{2} - x^4 - 3x^3 + 2x^2 + 2x\right)dx$$

$$= \frac{1}{2}.\frac{x^6}{6} - \frac{x^5}{5} - 3.\frac{x^4}{4} + 2.\frac{x^3}{3} + 2\frac{x^2}{2}\Big|_{-2}^{1}$$

$$= \left(\frac{1}{12} - \frac{1}{5} - \frac{3}{4} + \frac{2}{3} + 1\right) - \left(\frac{64}{12} - \frac{32}{5} - \frac{3.16}{4} + \frac{2.8}{3} + 4\right)$$

$$= \frac{5}{8} + \frac{8}{5} = \frac{12}{5}$$

b) $I = \iint\limits_D xy\,dxdy;$ với $D: \begin{cases} x = \sqrt{y};\ x = 2\sqrt{y} \\ y = 1 \end{cases}$

Để cho "dễ nhìn, quen mặt", ta đặt x = y

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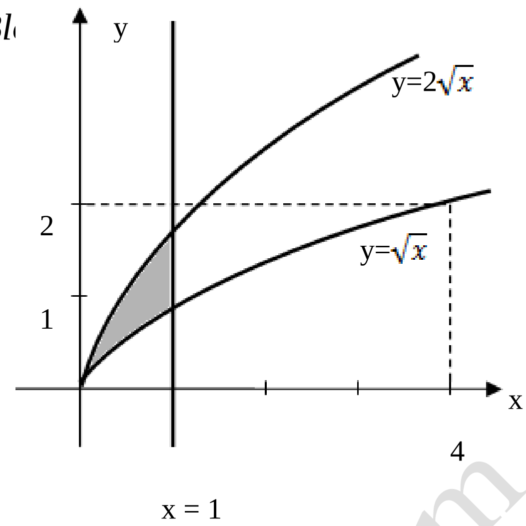


$\Rightarrow$ dx = dy.

$\Rightarrow y = \sqrt{x}; \quad y = 2\sqrt{x} \quad ; \quad x = 1$

Tích phân I không đổi.

Vẽ $y = \sqrt{x}$:

x = 0, y = 0;

x = 1, y = 1;

x = 4, y = 2.

Vẽ $y = \sqrt{2}x$

Vậy: D: $\begin{cases} 0 \le x \le 1 \\ \sqrt{x} \le y \le 2\sqrt{x} \end{cases}$

nên:

$$I = \int_0^1 dx \int_{\sqrt{x}}^{2\sqrt{x}} (xy)dy \qquad \text{xét } I_1 = \int_{\sqrt{x}}^{2\sqrt{x}} (xy)dy$$

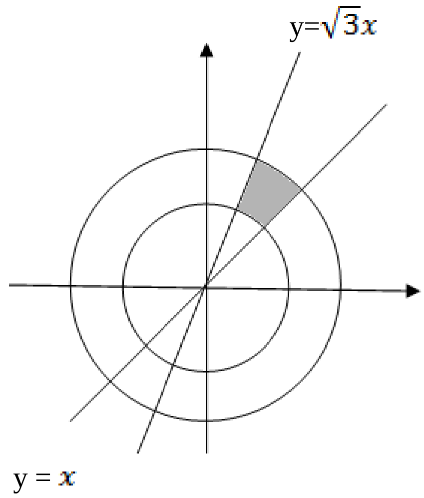
$$\Leftrightarrow I_1 = x.\frac{y^2}{2}\bigg|_{\sqrt{x}}^{2\sqrt{x}}$$

$$= x.\frac{(2\sqrt{x})^2}{2} - x.\frac{(\sqrt{x})^2}{2}$$

$$= \frac{x}{2}.4.x - \frac{x}{2}.x$$

$$= 2x^2 - \frac{x^2}{2} = \frac{3x^2}{2}$$

$$I = \int_0^1 \frac{3x^2}{2} dx = \frac{3}{2}.\frac{x^2}{3}\bigg|_0^1 = \frac{1}{2}$$



c) $\boldsymbol{I = \iint_D (x+y)\,dxdy;}$ ***với D:*** $\begin{cases} \boldsymbol{1 \le x^2 + y^2 \le 4} \\ \boldsymbol{x \le y \le \sqrt{3}.x} \end{cases}$

đổi trục tọa độ: x = rcosφ

y = rsinφ

$\Rightarrow 1 \le r^2\cos^2\varphi + r^2\sin^2\varphi \le 4$

$\Leftrightarrow 1 \le r^2(\cos^2\varphi + \sin^2\varphi) \le 4$

$\Leftrightarrow 1 \le r^2 \le 4$ $\quad ((\cos^2\varphi + \sin^2\varphi) = 1)$

$\Leftrightarrow 1^2 \le r^2 \le 2^2$ $\quad \Rightarrow 1 \le r \le 2$

$x \le y \le \sqrt{3}x$ *lấy dấu = để tính 2 con φ*

$x = y \quad ; \qquad tg\varphi = \frac{sin\varphi}{cos\varphi} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r}.\frac{r}{x} = \frac{y}{x} = 1$ *(vì x = y)*

$\Rightarrow \varphi = \frac{\pi}{4} \qquad$ *bấm máy* $shiftan^{-1}(1) = 45^0 = \frac{45\pi}{180} = \frac{\pi}{4}$

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$y = \sqrt{3}x$; $tg\varphi = \frac{y}{x} = \frac{\sqrt{3}x}{x} = \sqrt{3}$

Bấm máy $\Rightarrow$ $\varphi = \frac{\pi}{3} = 60^0$ $\Rightarrow$ $\frac{\pi}{4} \le \varphi \le \frac{\pi}{3}$

$$I = \iint_D (rcos\varphi + rsin\varphi)rdrd\varphi$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_1^2 (rcos\varphi + rsin\varphi)rdr$$

$$\text{xét } I_1 = \int_1^2 (cos\varphi + sin\varphi)r^2\,dr$$

$$= (cos\varphi + sin\varphi).\frac{r^3}{3}\Big|_1^2 = (cos\varphi + sin\varphi).\left[\frac{8}{3} - \frac{1}{3}\right]$$

$$= \frac{7}{3}(cos\varphi + sin\varphi)$$

$$I = \frac{7}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (cos\varphi + sin\varphi)d\varphi = \frac{7}{3}(sin\varphi - cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{7}{3}\left[\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] = \frac{7}{3}\left(\frac{\sqrt{3}-1}{2}\right) = \frac{7(\sqrt{3}-1)}{6}$$

d) $I = \iint_D \sqrt{4 - x^2 - y^2}\,dxdy;$ ***với D:*** $\begin{cases} x^2 + y^2 = 2x \\ y \le 0 \end{cases}$

$x^2 + y^2 - 2x = 0$

$\Leftrightarrow x^2 - 2x + 1 - 1 + y^2 = 0$

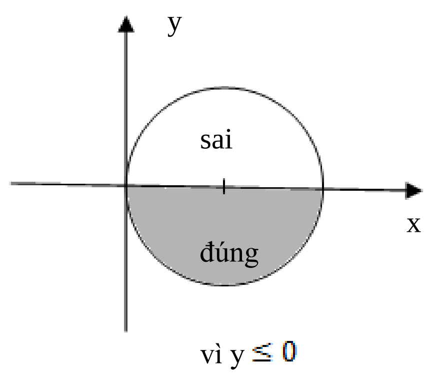
$\Leftrightarrow (x-1)^2 - y^2 = 1^2$

đặt: $x - 1 = rcos\varphi$

$\Rightarrow r^2cos^2\varphi + r^2sin^2\varphi = 1^2$

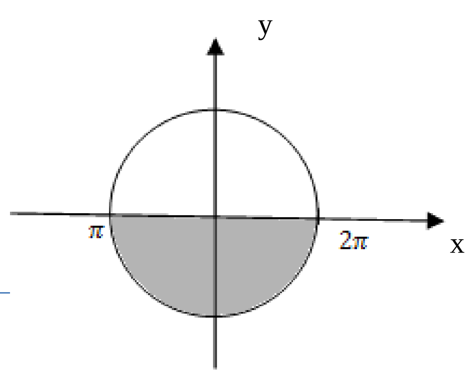
$y = rsin\varphi \Rightarrow r^2 = 1^2$

vậy: $0 \le r \le 1$



$y \le 0 \ \Rightarrow rsin\varphi \le 0 \Leftrightarrow sin\varphi \le 0$ (vì *r* không âm)

$\Rightarrow \pi \le \varphi \le 2\pi$ (thì $\sin\varphi \le 0$)



$$\Rightarrow \quad I = \int_{\pi}^{2\pi} d\varphi \int_0^1 \left(\sqrt{4 - x^2 - y^2}\right) r dr$$

Xét:

$$xét\ I_1 = \int_0^1 \left(\sqrt{4 - (rcos\varphi + 1)^2 - (rsin\varphi)^2}\right) rdr$$

$$= \int_0^1 \sqrt{4 - (r^2cos^2\varphi + rcos\varphi + 1) - r^2sin^2\varphi}\ rdr$$

$$= \int_0^1 \sqrt{4 - r^2cos^2\varphi - rcos\varphi - 1 - r^2sin^2\varphi}\ rdr$$

$$= \int_0^1 \sqrt{3 - 2rcos\varphi - r^2}\ rdr$$

Đặt lại: $x = rcos\varphi$

$y = rsin\varphi \Rightarrow r^2 = 2rcos\varphi \Leftrightarrow r = 2cos\varphi$

$\Rightarrow \quad 0 \le r \le 2cos\varphi$

$\pi \le \varphi \le 2\pi$

$$\Rightarrow \quad I = \int_{\pi}^{2\pi} d\varphi \int_0^{2cos\varphi} \left(\sqrt{4 - r^2cos^2\varphi - r^2sin^2\varphi}\right) rdr$$

$$I = \int_{\pi}^{2\pi} d\varphi \int_0^{2cos\varphi} \sqrt{4 - r^2}\ rdr$$

$$xét\ I_2 = \int_0^{2cos\varphi} \sqrt{4 - r^2}\ rdr$$

$$= \frac{1}{2}\int_0^{2cos\varphi} \sqrt{4 - r^2}\ d(r)^2 \qquad = \frac{1}{2}\int_0^{2cos\varphi} \sqrt{4 - r^2}\ d(4 - r^2)$$

$$= -\frac{1}{2}.\frac{(4 - r^2)^{\frac{1}{2}+1}}{\frac{1}{2} + 1} = -\frac{1}{2}.\frac{2}{3}(4 - r^2)^{\frac{3}{2}} \quad = -\frac{1}{3}(4 - r^2)^{\frac{3}{2}}\Big|_0^{2cos\varphi}$$

$$= -\frac{1}{3}.\left[(4 - 4cos^2\varphi)^{\frac{3}{2}} - (4 - 0)^{\frac{3}{2}}\right] \quad = -\frac{1}{3}\left[(4 - 4cos^2\varphi)^{\frac{3}{2}} - 8\right]$$

$$= -\frac{1}{3}.(4 - 4cos^2\varphi)\sqrt{(4 - 4cos^2\varphi)} + \frac{3}{8}$$

$$= -\frac{1}{3}.4(1 - cos^2\varphi).2.(1 - cos^2\varphi) + \frac{3}{8}$$

$$= -\frac{42}{3}sin^2\varphi|sin\varphi| + \frac{3}{8}$$

$$\Rightarrow \quad I = \int_{\pi}^{2\pi} \left(\frac{8}{3}sin^3\varphi + \frac{3}{8}\right) d\varphi \qquad vì\ |sin\varphi| = -sin\varphi\ (trong\ \pi \to 2\pi)$$

$$= \frac{8}{3}\int_{\pi}^{2\pi}(sin^2\varphi sin\varphi + \ 1)d\varphi$$

$$= -\frac{8}{3}\int_{\pi}^{2\pi}[1 - cos^2\varphi]dcos\varphi + \frac{8}{3}\varphi\Big|_{\pi}^{2\pi}$$

$$= -\frac{8}{3}\left(cos\varphi - \frac{cos^3\varphi}{3}\right)\Big|_{\pi}^{2\pi} + \frac{8}{3}\varphi\Big|_{\pi}^{2\pi} = \frac{24}{9}(\pi - 1)$$

e) $I = \iint_D (x + 1)\,dxdy;$ ***với D:*** $\begin{cases} x^2 + y^2 \le 4 \\ y \ge -x\,;\, y \le 0 \end{cases}$

Đổi biến: D: $\begin{cases} x = rcos\varphi \\ y = rsin\varphi \end{cases}$

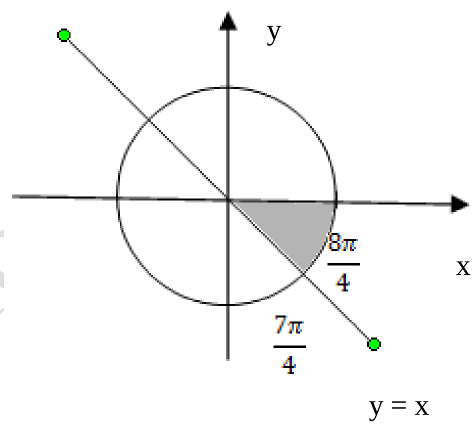
$\Rightarrow x^2 + y^2 \le 4$

$\Rightarrow r^2 \le 4 \quad \Rightarrow r^2 \le 2^2$

$\Leftrightarrow 0 \le r \le 2$

Nhìn vào hình:

$\frac{7\pi}{4} \le \varphi \le 2\pi$ hay $-\frac{\pi}{4} \le \varphi \le 0$



$I = \iint_D (rcos\varphi \ + 1)\,d\varphi rdr$

$$I = \int_{-\frac{\pi}{4}}^{0} d\varphi \int_0^2 (rcos\varphi + 1)rdr\,;$$

$$Xét\ I_1 = \int_0^2 (r.rcos\varphi + 1r)dr$$

$$= \frac{r^3}{3}cos\varphi + \frac{r^2}{2}\Big|_0^2 = \frac{8}{3}cos\varphi + 2$$

$$\Rightarrow \quad I = \int_{-\frac{\pi}{4}}^{0}\left(\frac{8}{3}cos\varphi + 2\right)d\varphi = \int_{-\frac{\pi}{4}}^{0}\frac{8}{3}cos\varphi d\varphi + \int_{-\frac{\pi}{4}}^{0} 2d\varphi$$

$$= \frac{8}{3}sin\varphi + 2\varphi\Big|_{-\frac{\pi}{4}}^{0} = \frac{8}{3}.0 + 2.0 - \left[\frac{8}{3}\sin\left(-\frac{\pi}{4}\right) + 2\left(-\frac{\pi}{4}\right)\right]$$

$$= 0 - \left(-\frac{8}{3}.\frac{\sqrt{2}}{2} - \frac{\pi}{2}\right) = \frac{4\sqrt{2}}{3} + \frac{\pi}{2}$$

f) $I = \iint_D x\,dxdy;$ ***với D:*** $\begin{cases} 2y \le x^2 + y^2 \le 4y \\ y \ge x \\ x \ge 0 \end{cases}$

$2y = x^2 + y^2 \Leftrightarrow x^2 + y^2 - 2y = 0$

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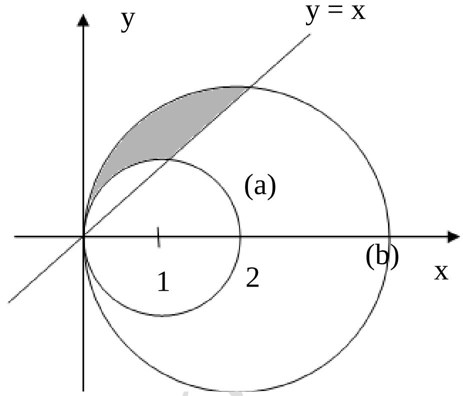
$$\Leftrightarrow x^2 + y^2 - 2y + 1 - 1 = 0$$

$\Leftrightarrow x^2 + (y-1)^2 = 1^2$ (a) Đường tròn tâm I(0, 1) bán kính r = 1.

$$x^2 + y^2 = 4y$$

$$\Leftrightarrow x^2 + y^2 - 4y + 2^2 - 2^2 = 0$$

$\Leftrightarrow x^2 + (y-2)^2 = 2^2$ (b) Đường tròn tâm $I_1(0, 2)$ bán kính $r_1 = 2$.



$$\begin{cases} x = r\cos\varphi \\ y = r\cos\varphi \end{cases}$$

$$2y \le x^2 + y^2 \Leftrightarrow 2r\sin\varphi \le r^2$$

$$\Leftrightarrow \quad r^2 \ge 2r\sin\varphi \Leftrightarrow r \ge 2\sin\varphi$$

$$x^2 + y^2 \le 4y \quad \Leftrightarrow \quad r^2 \le 4\sin\varphi$$

Kết hợp: $2\sin\varphi \le r \le 4\sin\varphi$

y ≥ x ⇔ cận dưới $\varphi_1 = \frac{\pi}{4}$

x ≥ 0 ⇔ cận trên $\varphi = \frac{\pi}{2}$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\varphi \int_{2\sin\varphi}^{4\sin\varphi} (r\cos\varphi).r dr$$

Xét

$$I_1 = \int_{2\sin\varphi}^{4\sin\varphi} \cos\varphi.r^2 dr = \cos\varphi.\frac{r^3}{3}\Big|_{2\sin\varphi}^{4\sin\varphi}$$

$$= \cos\varphi.\frac{64\sin^3\varphi}{3} - \cos\varphi.\frac{8\sin^3\varphi}{3}$$

$$= \cos\varphi.\sin^3\varphi\left(\frac{64}{3} - \frac{8}{3}\right) = \frac{56}{3}\cos\varphi.\sin^3\varphi$$

$$\Rightarrow \quad I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{56}{3}\cos\varphi.\sin^3\varphi = \frac{56}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3\varphi d(\sin\varphi)$$

$$= \frac{56}{3}.\frac{\sin^4\varphi}{4}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{56}{3.4}.\sin^4\frac{\pi}{2} - \frac{56}{3.4}\sin^4\frac{\pi}{4}$$

$$= \frac{56}{12} - \frac{56}{3.4.4} = \frac{56.4}{3.4.4} - \frac{56}{3.4.4} = \frac{168}{3.4.4} = \frac{21}{3.2} = \frac{7}{2}$$

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