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Artificial Intelligence

Lecturer 10 – First Order Logic

School of Information and Communication Technology - HUST

First Order Logic

- Syntax
- Semantic
- Inference
  - Resolution

First Order Logic (FOL)

- First Order Logic is about
  - Objects
  - Relations
  - Facts
- The world is made of objects
  - *Objects* are things with individual identities and properties to distinguish them
  - Various *relations* hold among objects. Some of these relations are functional
  - Every fact involving objects and their relations are either *true* or *false*

FOL

- Syntax
- Semantic
- Inference
    - Resolution

FOL Syntax

- Symbols
  - Variables: x, y, z,…
  - Constants: a, b, c, …
  - Function symbols (with arities): f, g, h, …
  - Relation symbols (with arities): p, r, r
  - Logical connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
  - Quantifiers: $\exists, \forall$

FOL Syntax

- Variables, constants and function symbols are used to build terms
  - X, Bill, FatherOf(X), …
- Relations and terms are used to build predicates
  - Tall(FatherOf(Bill)), Odd(X), Married(Tom,Marry), Loves(Y,MotherOf(Y)), …
- Predicates and logical connective are used to build sentences
  - Even(4), $\forall$X. Even(X) $\Rightarrow$ Odd(X+1), $\exists$X. X > 0

FOL Syntax

- Terms
  - Variables are terms
  - Constants are terms
  - If $t_1,\ldots, t_n$ are terms and f is a function symbol with arity n then $f(t_1,\ldots, t_n)$ is a term

FOL Syntax

- Predicates
  - If $t_1,\ldots, t_n$ are terms and p is a relation symbol with arity n then $p(t_1,\ldots, t_n)$ is a predicate

FOL Syntax

- Sentences
  - True, False are sentences
  - Predicates are sentences
  - If α, β are sentences then the followings are sentences

$$\exists x.\alpha, \forall x.\alpha, (\alpha), \neg\alpha, \alpha \wedge \beta, \alpha \vee \beta, \alpha \Rightarrow \beta, \alpha \Leftrightarrow \beta$$

FOL Formal grammar

Sentence	::=	AtomicS | ComplexS
AtomicS	::=	**True** | **False** | RelationSymb(Term, . . .) | Term = Term
ComplexS	::=	(Sentence) | Sentence Connective Sentence | ¬Sentence
	|	Quantifier Sentence
Term	::=	FunctionSymb(Term, . . .) | ConstantSymb | Variable
Connective	::=	$\wedge$ | $\vee$ | $\rightarrow$ | $\leftrightarrow$
Quantifier	::=	$\forall$ Variable | $\exists$ Variable
Variable	::=	$a \mid b \mid \cdots \mid x \mid y \mid \cdots$
ConstantSymb	::=	$A \mid B \mid \cdots \mid John \mid 0 \mid 1 \mid \cdots \mid \pi \mid \ldots$
FunctionSymb	::=	$F \mid G \mid \cdots \mid Cosine \mid Height \mid FatherOf \mid + \mid \ldots$
RelationSymb	::=	$P \mid Q \mid \cdots \mid Red \mid Brother \mid Apple \mid > \mid \cdots$

FOL

- Syntax
- Semantic
- Inference
  - Resolution

FOL Semantic

- Variables
  - Objects
- Constants
  - Entities
- Function symbol
  - Function from objects to objects
- Relation symbol
  - Relation between objects
- Quantifiers
  - $\exists x.P$ true if P is true under some value of x
  - $\forall x.P$ true if P is true under every value of x
- Logical connectives
  - Similar to Propositional Logic

FOL Semantic

- Interpretation (D, σ)
  - D is a set of objects, called *domain* or *universe*
  - σ is a mapping from variables to D
  - C^D is a member of D for each constant C
  - F^D is a mapping form D^n to D for each function symbol F with arity n
  - R^D is a relation over D^n for each relation symbol R with arity n

FOL Semantic

- Given an interpretation (D, σ), semantic of a term/sentence α is denoted

$$[\alpha]_a^D$$

- Interpretation of terms

$$
\begin{aligned}
[\![x]\!]_\sigma^{\mathcal{D}} &:= \sigma(x) \\
[\![C]\!]_\sigma^{\mathcal{D}} &:= C^{\mathcal{D}} \\
[\![F(t_1,\ldots,t_n)]\!]_\sigma^{\mathcal{D}} &:= F^{\mathcal{D}}([\![t_1]\!]_\sigma^{\mathcal{D}},\ldots,[\![t_n]\!]_\sigma^{\mathcal{D}})
\end{aligned}
$$

FOL Semantic

- Interpretation of sentence

$$
\begin{array}{llll}
[\![R(t_1,\ldots,t_n)]\!]_\sigma^\mathcal{D} & := & \mathit{True} & \text{iff} \quad \langle [\![t_1]\!]_\sigma^\mathcal{D},\ldots,[\![t_n]\!]_\sigma^\mathcal{D}\rangle \in R^\mathcal{D} \\
[\![\neg\varphi]\!]_\sigma^\mathcal{D} & := & \mathit{True}/\mathit{False} & \text{iff} \quad [\![\varphi]\!]_\sigma^\mathcal{D} = \mathit{False}/\mathit{True} \\
[\![\varphi_1 \vee \varphi_2]\!]_\sigma^\mathcal{D} & := & \mathit{True} & \text{iff} \quad [\![\varphi_1]\!]_\sigma^\mathcal{D} = \mathit{True} \text{ or } [\![\varphi_2]\!]_\sigma^\mathcal{D} = \mathit{True} \\
[\![\exists x\, \varphi]\!]_\sigma^\mathcal{D} & := & \mathit{True} & \text{iff} \quad [\![\varphi]\!]_{\sigma'}^\mathcal{D} = \mathit{True} \text{ for some } \sigma' \text{ the}
\end{array}
$$

$$
\begin{array}{lll}
[\![\varphi_1 \wedge \varphi_2]\!]_\sigma^\mathcal{D} & := & [\![\neg(\neg\varphi_1 \vee \neg\varphi_2)]\!]_\sigma^\mathcal{D} \\
[\![\varphi_1 \rightarrow \varphi_2]\!]_\sigma^\mathcal{D} & := & [\![\neg\varphi_1 \vee \varphi_2]\!]_\sigma^\mathcal{D} \\
[\![\varphi_1 \leftrightarrow \varphi_2]\!]_\sigma^\mathcal{D} & := & [\![(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)]\!]_\sigma^\mathcal{D} \\
[\![\forall x \varphi]\!]_\sigma^\mathcal{D} & := & [\![\neg\exists x\, \neg\varphi]\!]_\sigma^\mathcal{D}
\end{array}
$$

Example

- Symbols
  - Variables: x,y,z, …
  - Constants: 0,1,2, …
  - Function symbols: +,*
  - Relation symbols: >, =
- Semantic
  - Universe: N (natural numbers)
  - The meaning of symbols
    - Constants: the meaning of 0 is *the number zero, …*
    - Function symbols: the meaning of + is *the natural number addition*, …
    - Relation symbols: the meaning of > is *the relation greater than*, …

FOL Semantic

- Satisfiability
  - A sentence α is satisfiable if it is true under some interpretation (D, σ)
- Model
  - An interpretation (D, σ) is a model of a sentence α if α is true under (D, σ)
  - Then we write (D, σ) |= α
- A sentence is valid if every interpretation is its mode
- A sentence α is valid in D if (D, σ) |= α for all σ
- A sentence is unsatisfiable if it has no model

Example

- Consider the universe N of natural numbers
  - $\exists x. x + 1 > 5$ is satisfiable
  - $\forall x. x + 1 > 0$ is valid is N
  - $\exists x. 2x + 1 = 6$ is unsatisfiable