Linear Algebra

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Chapter 4: Linear Mapping

1. Linear Mapping
   - Definition, Examples
   - Image, Kernel, Injective, Surjective, Bijective
   - Matrix of linear map
   - Eigenvalues, Eigenvectors

Linear Mapping

Definition

Let V and W be vector spaces. A function $T : V \to W$ is is said to be a linear map if

i) $T(u+v) = T(u) + T(v), \forall u, v \in V$

ii) $T(ku) = kT(u), \forall k \in \mathbb{R}, u \in V$

First consequences

a) $T(0) = 0$.

b) $T(-v) = -T(v), \forall v \in V$.

c) $T(u-v) = T(u) - T(v), \forall u, v \in V$.

Kernel, image and the rankâĂŞnullity theorem

Definition

Let $T : V \to W$ be a linear map. We define the kernel and the image or range of T by

$$\mathrm{Ker}(T) := \{x | x \in V, T(x) = 0\}$$

and

$$\mathrm{Im}(T) := \{y | y \in W, \exists x \in V, T(x) = y\} = \{T(x) | x \in V\}.$$

Properties

i) $\mathrm{Ker}(T)$ is a subspace of V.

ii) $\mathrm{Im}(T)$ is a subspace of W.

iii) $\dim \mathrm{Ker}(T) + \dim \mathrm{Im}(T) = \dim V$ (the rank-nullity theorem).

Kernel, image and the rankâĂŞnullity theorem

Lemma

Let $T : V \to W$ *be a linear map and* $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ *is a basis of* V*. Then* $\operatorname{Im}(T) = \operatorname{span}\{f(e_1), f(e_2), \dots, f(e_n)\}$.

Lemma

Let $f : V \to W$ *be a linear map. Then*

a) f *is injective if and only if* $\operatorname{Ker} f = \{0\}$.

b) f *is surjective if and only if* $\operatorname{Im} f = W$.

Corollary

Let V, V' *be n-dimensional spaces and* $f : V \to V'$ *be a linear map. The following are equivalent*

a) f *is injective.* b) f *is surjective.* c) f *is bijective.*

Quotient space

Let W be a subspace of V. We define $M_v = \{x \in V | x - v \in W\}$ and

$$V/W = \{M_v | v \in V\}.$$

The operations $\begin{cases} \lambda M_v = M_{\lambda v}, \\ M_v + M_{v'} = M_{v+v'}. \end{cases}$

Definition

The space V/W is called the quotient space of V modulo W.

The map

$$p : V \to V/W, p(v) = M_v,$$

is called the projection. Obviously, $\operatorname{Ker} p = W$ and $\operatorname{Im} p = V/W$.

Lemma

$\dim(V/W) = \dim V - \dim W$

Matrix of linear map

Problem

Let $T : V \to W$ be a linear map from n-dimensional space V to m-dimensional space W. Suppose that $\mathcal{B}$ is a basis of V and $\mathcal{B}'$ is a basis of W, where

$$\mathcal{B} = \{u_1, u_2, \ldots, u_n\}, \mathcal{B}' = \{v_1, v_2, \ldots, v_m\}$$

Find the relation between $[T(x)]_{\mathcal{B}'}$ and $[x]_{\mathcal{B}}$.

Definition

A $m \times n$ matrix A satisfies

$$[T(x)]_{\mathcal{B}'} = A.[x]_{\mathcal{B}}, \forall x \in V,$$

if exists, is called the matrix of the linear map $T : V \to W$ with respect to the bases $\mathcal{B}$ of V and $\mathcal{B}'$ of W.

Matrix of linear map

Theorem

The matrix of the linear map $T : V \to W$ with respect to the bases $\mathcal{B}$ of V and $\mathcal{B}'$ of W is uniquely determined by

$$A = \left[[T(u_1)]_{\mathcal{B}'}, [T(u_2)]_{\mathcal{B}'}, \ldots, [T(u_n)]_{\mathcal{B}'}, \right].$$

Example

Let the function $f : P_2[x] \to P_4[x]$ defined as: $f(p) = p + x^2 p, \forall p \in P_2.$

a) Prove that f is a linear map.

b) Find the matrix of f with respect to the bases $E_1 = \{1, x, x^2\}$ of $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ of $P_4[x]$.

c) Find the matrix of f with respect to the bases $E_1' = \{1 + x, 2x, 1 + x^2\}$ of $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ of $P_4[x]$.

Meaning of the matrix of linear maps

$$\begin{tikzcd}
x \arrow[r, "\text{Directly}"] \arrow[d, "(1)"'] & T(x) \\
{[x]_{\mathcal{B}}} \arrow[r, "\text{Multiply } A[x]_{\mathcal{B}}", "\text{Indirectly (2)}"'] & {[T(x)]_{\mathcal{B}'}} \arrow[u, "(3)"']
\end{tikzcd}$$

Compute $T(x)$ indirectly through 3 steps

1) Find the coordinate vector $[x]_{\mathcal{B}}$.
2) Compute $[T(x)]_{\mathcal{B}'} = [T(x)]_{\mathcal{B}'}$.
3) From $[T(x)]_{\mathcal{B}'}$, it follows $T(x)$.

Meaning of the matrix of linear maps

i) It provides a method to compute $T(x)$ via computers.

ii) We can choose $\mathcal{B}$ vÃă $\mathcal{B}'$ such that the matrix A is as simple as possible. Then it can provided some important information about the linear mapp.

Transformation matrix

Matrix similarity

Two n-by-n matrices A and B are called similar if $B = P^{-1}AP$ for some invertible n-by-n matrix P.

Theorem

Let $T : V \to V$ be a linear map over finite dimensional space V. If

i) A is the transformation matrix of T w.r.t the basis $\mathcal{B}$ and

ii) A' is the transformation matrix of T w.r.t the basis $\mathcal{B}'$

then

$$A' = P^{-1}AP,$$

where P is the matrix that change the basis from $\mathcal{B}$ to $\mathcal{B}'$.

Rank of linear transformation

Definition

The rank of a linear transformation $T : V \to W$ *is the dimension of its image, written rank* $\operatorname{rank}(T)$*:*

$$\operatorname{rank}(T) = \dim \operatorname{Im}(T)$$

Rank-Nullity Theorem

$$n = \dim V = \dim \operatorname{Im}(T) + \dim \operatorname{Ker}(T) = \dim \operatorname{rank}(T) + \dim \operatorname{Ker}(T).$$

Example

Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Prove that $\operatorname{rank}(AB) \leq \min \{\operatorname{rank} A, \operatorname{rank} B\}$.

Rank of linear transformation vs Rank of matrix

An $n \times m$ matrix A can be used to define a linear transformation $L_A : \mathbb{R}^m \to \mathbb{R}^n$ given by $L_A(v) = Av$. If we do this,

$$\text{rank}(A) = \text{rank}(L_A), \quad nullity(A) = \text{Ker}(L_A).$$

COnversely,

Theorem

The rank of a linear transformation $T : V \to W$ equals to the rank of its matrix w.r.t any bases $\mathcal{B}$ of V and $\mathcal{B}'$ of W.

Corollary

$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$

Eigenvalues and Eigenvectors of a matrix

Definition

Let A be an n-square matrix. A real number λ is called an eigenvalue of A if the equation

$$Ax = \lambda x, x \in \mathbb{R}^n$$

has at least a non-trivial solution $x = (x_1, x_2, \ldots, x_n) \neq (0, 0, \ldots, 0)$.

The equation $Ax = \lambda x$ can be written in the following form

$$(A - \lambda I)x = 0 \tag{1}$$

λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0 \tag{2}$$

Definition

The equation (2) is called the characteristic equation of A, and the polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of A.

Eigenvalues and eigenvectors of linear transformation

Definition

Let V be a linear space. A real number λ is called an eigenvalue of the linear transformation $T: V \to V$ if the equation $T(x) = \lambda x$ has at least a non-trivial solution $x \neq 0$.

Eigenspace

i) If λ is an eigenvalue of the matrix A, then

$$E = \{v \in \mathbb{R}^n : (A - \lambda I)v = 0\} = nullity(A - \lambda I)$$

is called the eigenspace of A.

ii) Similarly, if λ is an eigenvalue of the linear transformation $f : V \to V$, then

$$E = \{v \in V : (f - \lambda \,\mathrm{Id}_V)v = 0\} = \mathrm{Ker}(A - \lambda I)$$

is called the eigenspace of f.

Eigenvalues and eigenvectors

Matrix vs Linear transformation

Let A be the matrix of T w.r.t. a basis $\mathcal{B}$ of V. Then,

i) λ is an eigenvalue of $T \Leftrightarrow \lambda$ is an eigenvalue of A.

ii) v is an eigenvector of T w.r.t the eigenvalue λ if and only if $[v]_{\mathcal{B}}$ is an eigenvector of A w.r.t. λ.

Diagonalizable matrix

Diagonalizable matrix

A square matrix A is called diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Diagonalizable linear map

A linear map f: $V \to V$ is called diagonalizable if there exists an ordered basis of V with respect to which f is represented by a diagonal matrix.

i) Diagonalization is the process of finding a corresponding diagonal matrix for a diagonalizable matrix or linear map.

ii) A square matrix that is not diagonalizable is called defective.

Diagonalizable matrix

Necessary and sufficient conditions

A matrix A is diagonalizable if and only if

(i)

$$P_A(X) = (-1)^n (X-\lambda_1)^{s_1}(X-\lambda_2)^{s_2}\dots(X-\lambda_m)^{s_m},$$

where $s_1 + s_2 + \cdots + s_m = n$,

(ii) $\text{rank}(A - \lambda_i I) = n - s_i \ (i = 1, 2, \dots, m)$.

Corollary

An $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Corollary

An $n \times n$ matrix A is diagonalizable if it has n distinct eigenvalues.

Diagonalization

The algorithm

1) Find n linearly independent eigenvectors of the matrix A:

$$p_1, p_2, \ldots, p_n$$

2) Writing P as a block matrix of its column vectors $p_1, p_2, \ldots, p_n$.
3) Then

$$P^{-1}AP = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n],$$

where $\lambda_i (i = 1, 2, \ldots, n)$ are eigenvalues corresponding to eigenvectors p_i.

Example

Diagonalization the matrix $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

Trace of a matrix

Definition

The trace of an n-by-n square matrix A is defined to be the sum of the elements on the main diagonal,i.e., $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$.

Properties

1) The trace is a linear mapping. That is,
   i) $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$,
   ii) $\operatorname{tr}(cA) = c\operatorname{tr}(A)$.
2) $\operatorname{tr}(A) = \operatorname{tr}(A^T)$,
3) $\operatorname{tr}(AB) = \sum_{i,j} a_{ij}b_{ji} = \operatorname{tr} BA \Rightarrow \operatorname{tr}(PAP^{-1}) = \operatorname{tr}(P^{-1}AP) = \operatorname{tr} A$,
4) The trace of a matrix is the sum of the (complex) eigenvalues, and it is invariant with respect to a change of basis.

Trace of a matrix

Properties

5) The trace is invariant under cyclic permutations, i.e., $\operatorname{tr}(ABCD) = \operatorname{tr}(BCDA) = \operatorname{tr}(CDAB) = \operatorname{tr}(DABC)$, however, $\operatorname{tr} ABC \neq \operatorname{tr} ACB$.
6) If A is symmetric and B is antisymmetric, then $\operatorname{tr} AB = 0$.
7) If A is an n-by-n matrix with real or complex entries and if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A (with multiplicities), then
   i) $\operatorname{tr} A = \sum\limits_{i=1}^{n} \lambda_i$,
   ii) In contrast, $\det A = \prod\limits_{i=1}^{n} \lambda_i$.

   More generally, $\operatorname{tr} A^k = \sum\limits_{i=1}^{n} \lambda_i^k$.
8) If A is an idempotent matrix ($A^2 = A$), then $\operatorname{tr}(A) = \operatorname{rank}(A)$.
9) The trace of a nilpotent matrix ($A^k = 0$ for some $k > 0$) is zero.

Minimal Polynomial

Definition

The minimal polynomial m_A of an $n \times n$ matrix A is the monic polynomial P of least degree such that $P(A) = 0$.

Theorem

Any other polynomial Q with $Q(A) = 0$ is a (polynomial) multiple of m_A.

Theorem (Cayley - Hamilton)

Each matrix is a root of its characteristic polynomial.

Eigenvalues, eigenvectors, characteristic polynomials

Lemma

If λ is a root of multiplicity p of the characteristic polynomial of the matrix A, then

a. $\dim \operatorname{Ker}(A-\lambda I) \leq p$.

b. $1 \leq n-\operatorname{rank}(A-\lambda I) \leq p$.

Lemma

If A is an n-by-n matrix with real or complex entries and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (with multiplicities), then

i) *The eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}$ (with multiplicities),*

ii) *The eigenvalues of A^2 are $\lambda_1^2, \cdots, \lambda_n^2$ (with multiplicities),*

iii) *The eigenvalues of A^p are $\lambda_1^p, \cdots, \lambda_n^p$ (with multiplicities).*

Eigenvalues, eigenvectors, characteristic polynomials

Lemma

If A is an n-by-n matrix with complex entries and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (with multiplicities), then

$$\det f(A) = f(\lambda_1) f(\lambda_2) \ldots f(\lambda_n),$$

where $f(X)$ is any polynomial with complex coefficients.

Lemma

If $P_A(\lambda) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{s_i}$ is the characteristic polynomial of A, then

$$P_{f(A)}(\lambda) = \prod_{i=1}^{k} (f(\lambda_i - \lambda))^{s_i},$$

where $f(A)$ is any polynomial.

Some special matrices

Symmetric and antisymmetric matrices

i) A matrix A is called symmetric if $A^T = A$.

ii) A matrix A is called antisymmetric if $A^T = -A$.

Properties

i) If A is an antisymmetric matrix, then A^2 is a symmetric matrix.

ii) All nonzero eigenvalues of an antisymmetric matrix are pure imaginary.

iii) The rank of an antisymmetric matrix is an even number.

iv) If A is an antisymmetric matrix, then $I + A$ is an invertible matrix.

v) If A is an antisymmetric, invertible, then A^{-1} is also an antisymmetric matrix.

Some special matrices

Nilpotent matrix

A nilpotent matrix is a square matrix$A$ such that $A^k = 0$ for some positive integer k. The smallest such k is sometimes called the index of A.

Properties

1) A is nilpotent if and only if $P_A(\lambda) = \lambda^n$.
2) A is nilpotent if and only if $\mathrm{tr}(A^p) = 0$ for all $p = 1, 2, \ldots, n$.
3) A is nilpotent if and only if the only eigenvalue for A is 0.
4) If A is nilpotent, then $I - A$ is invertible.

Some special matrices

Idempotent matrix - Projection

i) An idempotent matrix is a matrix which, when multiplied by itself, yields itself, i.e., $A^2 = A$.

ii) A linear map $P : V \to V$ is a projection if $P^2 = P$.

Properties

1) There is a basis of V s.t the matrix of P is $\operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$.
2) If P is a projection, then $\operatorname{rank} P = \operatorname{tr} P$.
3) If P is a projection, then $I - P$ is also a projection. Moreover, $\operatorname{Ker}(I - P) = \operatorname{Im} P$ and $\operatorname{Im}(I - P) = \operatorname{Ker} P$.
4) The following are equivalent:

   a. A idempotent.

   b. $\mathbb{C}^n = \operatorname{Im} A + \operatorname{Ker} A$.

   c. $\operatorname{Ker} A = \operatorname{Im}(I - A)$

   d. $\operatorname{rank}(A) + \operatorname{rank}(I - A) = n$

   e. $\operatorname{Im}(A) \cap \operatorname{Im}(I - A) = \{0\}$

Idempotent matrix - Projection

Properties

5) A is idempotent $\Leftrightarrow \operatorname{rank}(A) = \operatorname{tr}(A)$ and $\operatorname{rank}(I - A) = \operatorname{tr}(I - A)$.
6) If $AB = A$ and $BA = B$, then A, B are idempotent.
7) If A is idempotent, then $(A + I)^k = I + (2^k - 1)A \forall k \in \mathbb{N}$.
8) Let A, B be idempotent and $I - (A + B)$ invertible, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.
9) Let P_1 and P_2 be projection. Then,
    a) $P_1 + P_2$ is a projection if and only if $P_1 P_2 = P_2 P_1 = 0$.
    b) $P_1 - P_2$ is a projection if and only if $P_1 P_2 = P_2 P_1 = P_2$.

Some special matrices

Involutory matrix

An involutory matrix is a matrix that is its own inverse, i.e., $A^2 = I$.

Properties

1) P is an idempotent matrix if and only if $2P - I$ is an involutory matrix.
2) If A isan involutory matrix, then $A \sim \text{diag}(\pm 1, \ldots, \pm 1)$.
3) If A isan involutory matrix, then $\mathbb{R}^n = \text{Ker}(A + I) \oplus \text{Ker}(A - I)$.
4) A matrix A can be represented as a product of two involutory matrices if and only if $A \sim A^{-1}$.
5) A is an involutory matrix if and only if $\frac{1}{2}(I + A)$ is an idempotent matrix.