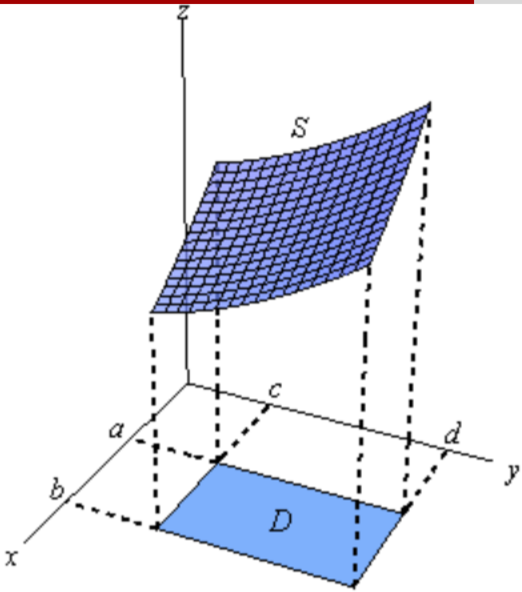
Chapter 5: Surface integral

Lecturer: Assoc. Prof. Nguyễn Duy Tân
email: tan.nguyenduy@hust.edu.vn

Faculty of Mathematics and Informatics, HUST

February, 2025

Contents


z
S
c
a
d
b
y
D
x

4.1.1. Definition

- Let $f(x, y, z)$ be a function defined on a surface S.
- Split S into n smaller pieces. Let $\Delta S_1, \dots, \Delta S_n$ be the areas of these pieces. Let d_i be the diameter of ΔS_i.
- In each ΔS_i, take $M_i(x_i^*, y_i^*, z_i^*)$ and define the Riemann sum

$$\sum_{i=1}^{n} f(M_i)\Delta S_i.$$

- If $\max d_i \to 0$ and the sum $\sum_{i=1}^{n} f(M_i)\Delta S_i$ approaches to a finite limit, not depending on S_i and M_i, the the limit is called the integral of $f(x, y, z)$ over S, and is denoted by

$$\iint_S f(x, y, z)dS.$$

- If S is smooth and $f(x, y, z)$ is continuous on S then the integral exists.
- The area of S is equal to $\iint\limits_S dS$.
- Surface integrals of type I have similar properties to line integrals of type I: Linearity, additivity, monotonicity.

4.1.2. Calculation

- Let S be a surface defined by $z = z(x, y)$, where (x, y) is in a closed and bounded region D.
- Assume that $z(x, y)$ has continuous partial derivatives on D.
- If $f(x, y, z)$ is a continuous function on S.
- Then

$$\iint\limits_S f(x, y, z)dS = \iint\limits_D f(x, y, z(x, y))\sqrt{1 + (z'_x)^2 + (z'_y)^2}dxdy.$$

- Similar formulae hold the case $x = x(y, z)$ or $y = y(x, z)$.

Example (Final 20182)

Evaluate the integral $\iint\limits_S \sqrt{1+x^2+y^2}dS$, where S is the surface $2z = x^2+y^2$, $0 \le x, y \le 1$.

- $z = (x^2+y^2)/2$, $z'_x = x$, $z'_y = y$.
- $I = \iint\limits_D \sqrt{1+x^2+y^2}\sqrt{1+x^2+y^2}dxdy$, $D\colon 0 \le x, y \le 1$.
- $I = \int\limits_0^1 dx \int\limits_0^1 (1+x^2+y^2)dy = \int\limits_0^1 (\frac{4}{3}+x^2)dx = \frac{5}{3}$.

Optional (Stewart)

- Assume S is given (parametrized) by $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$, với $(s,t) \in D$.
- Let $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$, and

$$\vec{r}_s' = (x_s', y_s', z_s') \text{ and } \vec{r}_t' = (x_t', y_t', z_t').$$

- Then

$$\iint\limits_S f(x,y,z)dS = \iint\limits_D f(x(s,t), y(s,t), z(s,t))|\vec{r}_s' \times \vec{r}_t'|dsdt.$$

Example (Optional)

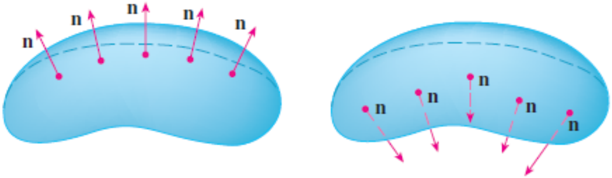
Evaluate $\iint\limits_S zdS$, where S is the surface defined by $x^2+y^2=1$ and $0 \le z \le 1+x$.

- Parametric equations for S:
$$x=\cos\theta, \quad y=\sin\theta, z=z,$$
where $(\theta, z) \in D$: $0 \le \theta \le 2\pi$, $0 \le z \le 1+\cos\theta$.
- Hence
$$\vec{r}(\theta,\phi)=\cos\theta\vec{i}+\sin\theta\vec{j}+z\vec{k}$$
and
$$\vec{r}_\theta=-\sin\theta\vec{i}+\cos\theta\vec{j}, \quad \vec{r}_z=\vec{k}$$
- One can compute that
$$|\vec{r}_\theta\times\vec{r}_z|=1.$$
- $\iint\limits_S zdS=\iint\limits_D z|\vec{r}_\theta\times\vec{r}_z|d\theta dz=\int_0^{2\pi}d\theta\int_0^{1+\cos\theta}zdz=\dfrac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2d\theta=\dfrac{3\pi}{2}.$

Some past exam questions

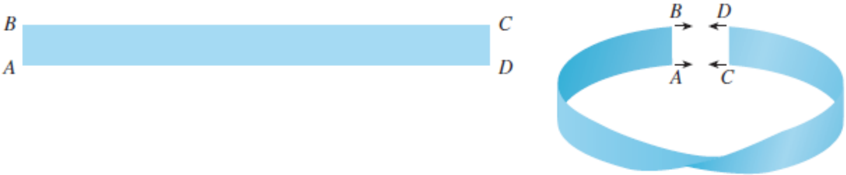
- (Final 20192) Evaluate $\iint\limits_S dS$, where S is the surface $z = \dfrac{2}{3}(x^{3/2} + y^{3/2})$ with $0 \leq x \leq 2, 0 \leq y \leq 1$.
- (Final 20192) Evaluate $\iint\limits_S y^2 z dS$, where S is the surface bounded by $z = \sqrt{x^2 + y^2}$, $z = 1$, and $z = 2$.
- (Final 20193) Evaluate $\iint\limits_S z\sqrt{x^2 + y^2} dS$, where S is defined by $z = \sqrt{x^2 + y^2}$ with $1 \leq z \leq 2$.

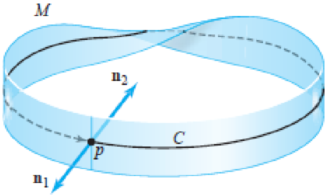
4.2.1. Definition



- We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point). There are two unit normal vectors $\vec{n}$ and $-\vec{n}$ at (x, y, z).
- If it is possible to choose a unit normal vector $\vec{n}$ at every such point (x, y, z) so that $\vec{n}$ varies continuously over S, then S is called an oriented surface and the given choice of $\vec{n}$ provides S with an orientation. For any orientable surface, there are two possible orientations.
- For a closed surface, that is, a surface that is the boundary of a solid region E, the convention is that the positive orientation is the one for which the normal vectors point outward from E, and inward-pointing normals give the negative orientation.

Möbius strip





Vector fields

- A vector field in $\mathbb{R}^2$ is a map $\vec{F}$ assigning each point $M(x,y)$ in $D \subset \mathbb{R}^2$ a vector $\vec{F}(M) \in \mathbb{R}^2$:

$$\vec{F}(x,y) = (P(x,y), Q(x,y)) = P(x,y)\vec{i} + Q(x,y)\vec{j},$$

  or $\vec{F} = P\vec{i} + Q\vec{j}$.
- A vector field in $\mathbb{R}^3$ is a map $\vec{F}$ assigning each point $M(x,y,z)$ in $E \subset \mathbb{R}^3$ a vector $\vec{F}(M) \in \mathbb{R}^3$:

$$\begin{aligned}\vec{F}(x,y,z) &= (P(x,y,z), Q(x,y,z), R(x,y,z)) \\ &= P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k},\end{aligned}$$

  or $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$.

Surface integral of vector fields

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the surface integral of F over S is

$$\iint\limits_S \vec{F} \cdot dS = \iint\limits_S \vec{F} \cdot \vec{n}\, dS.$$

This integral is also called the flux of $\vec{F}$ of across S.
It is also denoted by

$$\iint\limits_S P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy.$$

Remarks

- Surface integral of vector fields depends on the orientation of S.
- Surface integral of vector fields have the following properties: linearity, additivity.
- Suppose that S is an oriented surface with unit normal vector $\vec{n}$, and suppose a fluid with density $\rho(x, y, z)$ and velocity field $\vec{v}(x, y, z)$ flowing through S. Then the rate of the flow (mass per unit time) is equal to

$$\iint\limits_S \rho\vec{v} \cdot \vec{n}\, dS = \iint\limits_S \rho\vec{v} \cdot d\vec{S}.$$

Optional (Stewart)

- Assume that S has parametric equations $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$, với $(s,t) \in D$.
- Let $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$, and

$$\vec{r}'_s = (x'_s, y'_s, z'_s) \text{ và } \vec{r}'_t = (x'_t, y'_t, z'_t).$$

- Assume that S is oriented by the unit normal vector

$$\vec{n} = \frac{\vec{r}'_s \times \vec{r}'_t}{|\vec{r}'_s \times \vec{r}'_t|}.$$

- Then

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_S \vec{F} \cdot \frac{\vec{r}'_s \times \vec{r}'_t}{|\vec{r}'_s \times \vec{r}'_t|} |\vec{r}'_s \times \vec{r}'_t| dsdt = \iint_D \vec{F}(x(s,t), y(s,t), z(s,t)) \cdot (\vec{r}'_s \times \vec{r}'_t) dsdt.$$

- Assume that S has the equation $z = z(x, y)$, where (x, y) is in a closed bounded region D. Then $x = x$, $y = y$, $z = z(x, y)$ and $\vec{r}(x, y) = (x, y, z(x, y))$.
- We have $\vec{r}'_x = (1, 0, z'_x)$, $\vec{r}'_y = (0, 1, z'_y)$ and

$$\vec{r}'_x \times \vec{r}'_y = (-z'_x, -z'_y, 1) = -z'_x\vec{i} - z'_y\vec{j} + \vec{k}.$$

- So

$$\vec{F} \cdot (\vec{r'}_x \times \vec{r'}_y) = (P, Q, R) \cdot (-z'_x, -z'_y, 1) = -Pz'_x - Qz'_y + R.$$

- Assume that the orientation on S is given by the unit normal vector which has the same direction as $\vec{r}'_x \times \vec{r}'_y$ (S is with upward orientation). Then

$$\boxed{\iint\limits_S \vec{F} \cdot \vec{n}\, dS = \iint\limits_S Pdydz + Qdzdx + Rdxdy = \iint\limits_D (-Pz'_x - Qz'_y + R)dxdy.}$$

- Suppose further that $P = Q = 0$, then $\vec{F} = R\vec{k}$ (S is still with upward orientation) and we have

$$\iint\limits_S \vec{F} \cdot \vec{n}\, dS = \iint\limits_S R(x, y, z)dxdy = \iint\limits_D R(x, y, z(x, y))dxdy.$$

4.1.2. Evaluation of surface integrals of vector fields

- Assume that S has the equation $z = z(x, y)$, where $(x, y) \in D$. Assume that $z(x, y)$ has the continuous first partial derivatives in D.
- Let $R(x, y, z)$ be a continuous function in S.
- If the angle between $\vec{n}$ with Oz is less than or equal to 90° then

$$\iint\limits_S R(x, y, z)dxdy = \iint\limits_D R(x, y, z(x, y))dxdy.$$

- If the angle between $\vec{n}$ with Oz is more than 90° then

$$\iint\limits_S R(x, y, z)dxdy = -\iint\limits_D R(x, y, z(x, y))dxdy.$$

- Similar formulae for $\iint\limits_S Pdydz$, $\iint\limits_S Qdxdz$.

Example (Final 20192)

Evaluate $\iint\limits_S y^2 z dxdy$, where S is part of the surface $z^2 = x^2 + y^2$ bounded by $z = 1$ and $z = 2$ with upward direction.

- Equation of S: $z = \sqrt{x^2 + y^2}$, $(x, y) \in D$, where D: $1 \le x^2 + y^2 \le 4$.
- The angle between the unit normal vector and Oz is less than 90. So

$$I = \iint\limits_D y^2 \sqrt{x^2 + y^2} dxdy.$$

- Let $x = r\cos\varphi$, $y = r\sin\varphi$, D': $1 \le r \le 2$, $0 \le \varphi \le 2\pi$. Jacobi $J = r$.
- $I = \int\limits_0^{2\pi} d\varphi \int\limits_1^2 r^2 \sin\varphi^2 r \cdot r dr = \int\limits_0^{2\pi} \frac{1 - \cos 2\varphi}{2} d\varphi \int\limits_1^2 r^4 dr = \pi \cdot \frac{31}{5} = \frac{31\pi}{5}$.

Example

Evaluate $I = \iint\limits_S xdydz + ydzdx + zdxdy$, where S is the sphere $x^2 + y^2 + z^2 = R^2$ with outward direction.

- The unit normal vector at $M(x, y, z)$ with the outward direction is $\vec{n} = (\frac{x}{R}, \frac{y}{R}, \frac{z}{R})$.
- $\vec{F} = (x, y, z)$ and $\vec{F} \cdot \vec{n} = (x^2 + y^2 + z^2)/R$.
- $I = \iint\limits_S \vec{F} \cdot \vec{n} dS = \iint\limits_S \frac{x^2 + y^2 + z^2}{R} dS = R \iint\limits_S dS = R \cdot Area(S) = 4\pi R^3$.

Example

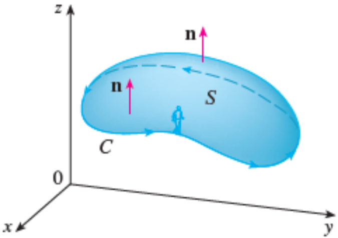
Evaluate $I = \iint\limits_S (y-z)dydz + (z-x)dzdx + (x-y)dxdy$, where S is the surface $x^2+y^2=z^2$ bounded by $0 \le z \le h$ with the outward direction.

- The unit normal vector at $M(x,y,z)$ in S with outward direction is $\vec{n} = (x,y,-z)/\sqrt{x^2+y^2+z^2}$.
- $\vec{F} = (y-z)\vec{i} + (z-x)\vec{j} + (z-x)\vec{k}$ và $\vec{F}\cdot\vec{n} = (2yz-2xz)/\sqrt{x^2+y^2+z^2}$.
- $I = \int\limits_S \vec{F}\cdot\vec{n}dS = \int\limits_S \dfrac{2yz-2xy}{\sqrt{x^2+y^2+z^2}}dS$.
- The equation of S is $z=\sqrt{x^2+y^2}$ where $(x,y)\in D$ and $D\colon x^2+y^2\le h^2$.
- $I = \iint\limits_D 2\dfrac{(y-x)\sqrt{x^2+y^2}}{\sqrt{2(x^2+y^2)}}\sqrt{1+\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2+y^2}}dxdy = 0$.

Some practice problems

- (Final 2081) Evaluate $\iint\limits_S z(x^2+y^2)dxdy$, where S is the surface $z^2 = x^2+y^2$, bounded by $0 \le z \le 1$, with the outward direction.
- (Final 20171) Evaluate $\iint\limits_S z^2\sqrt{2x-x^2-y^2}dxdy$, where S is the surface $z = \sqrt{2x-x^2-y^2}$, with the upward direction.

4.2.3. Stokes' theorem



Let S be a smooth surface with a smooth bounding curve C. Then for any continuously differentiable vector function $F(x, y, z) = (P, Q, R)$

$$\oint\limits_C Pdx + Qdy + Rdz =$$

$$= \iint\limits_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

Example (Final 2018)

Evaluate the line integral $\oint\limits_C (y^2+z^2)dx + (z^2+x^2)dy + (x^2+y^2)dz$, where C is the intersection of the sphere $x^2+y^2+z^2=4$ and the cone $z=\sqrt{x^2+(y-1)^2}$, with the clockwise direction looking from O.

- $P = y^2+z^2$, $Q = z^2+x^2$, $R = x^2+y^2$.
- $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2+z^2 & z^2+x^2 & x^2+y^2 \end{vmatrix} = \vec{i}(2y-2z)+\vec{j}(2z-2x)+\vec{k}(2x-2y) = \vec{F}.$
- Let S be part of the sphere $x^2+y^2+z^2=4$ lying inside the cone with the outward direction.
- The unit normal vector of S at $M(x,y,z)$ is $\vec{n} = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2})$.

- By Stokes' theorem

$$
\begin{aligned}
I &= \iint_S (2y - 2z)dydz + (2z - 2x)dzdx + (2x - 2y)dxdy \\
&= \iint_S \vec{F} \cdot \vec{n}\, dS \\
&= \iint_S [\frac{x}{2}(2y - 2z) + \frac{y}{2}(2z - 2x) + \frac{z}{2}(2x - 2y)]dS = 0.
\end{aligned}
$$

Example

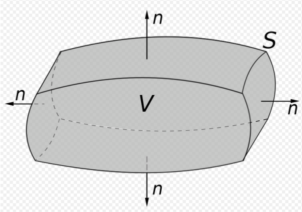
Use Stokes' Theorem to evaluate the line integral $\oint\limits_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$, where C is the curve formed by intersection of the sphere $x^2+y^2+z^2=1$ with the plane $x+2y+2z=0$, and C is oriented counterclockwise as viewed from above.

- Let S be the part of the plane $x+2y+2z=0$ that lies inside the sphere $x^2+y^2+z^2\le 1$, i.e. $S: x+2y+2z=0,\ x^2+y^2+z^2\le 1$, with upward orientation. Then S is a disk whose boundary is C and the orientation on S is compatible with the orientation on C.
- The unit upward normal vector to the surface S is $\vec{n}$ is $\mathrm{n} = \frac{1\cdot\mathrm{i}+2\cdot\mathrm{j}+2\cdot\mathrm{k}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\mathrm{i}+\frac{2}{3}\mathrm{j}+\frac{2}{3}\mathrm{k}$.
- $P=y+2z,\quad Q=x+2z,\quad R=x+2y.$
- So $\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{k}=\mathrm{j}.$
- Using Stokes' Theorem, we have

$$\oint\limits_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz = \iint\limits_S \mathrm{j}\cdot\vec{n} = \frac{2}{3}\iint\limits_S dS.$$

- The integral is $I=\frac{2}{3}\iint\limits_S dS = \frac{2}{3}\cdot\pi\cdot 1^2 = \frac{2\pi}{3}$.

Ostrogradsky' theorem (Divergence theorem)



Let V be a closed bounded solid in $\mathbb{R}^3$, with the boundary S with the outward direction. Let P, Q, R be functions in V with continuous partial derivatives. Then

$$\iint\limits_S Pdydz + Qdzdx + Rdxdy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz.$$

Note

The volume V is equal to

$$V = \frac{1}{3}\iint\limits_{S} x dydz + y dzdx + z dxdy.$$

Example

Evaluate the surface integral $I = \iint\limits_S xdydz + ydzdx + zdxdy$, where S is the sphere $x^2 + y^2 + z^2 = R^2$ with the outward direction.

- $P = x$, $Q = y$, $R = z$.
- By the divergence theorem $I = \iiint\limits_V (1 + 1 + 1)dxdydz$, where V is the solid sphere $x^2 + y^2 + z^2 \leq R^2$.
- $I = 3 \iiint\limits_V dxdydz = 3\dfrac{4}{3}\pi R^3 = 4\pi R^3$.

Example (Final 20173)

Evaluate $\iint\limits_S (3x+2y+z)^3(dydz+dzdx+dxdy)$, where S is the surface $9x^2+4y^2+z^2=1$ with the outward direction.

- By the divergence theorem $I=\iiint\limits_V 3(3+2+1)(3x+2y+z)^2dxdydz=$ $18\iiint\limits_V (3x+2y+z)^2dxdydz = 18\iiint (9x^2+4y^2+z^2)dxdydz$, where $V: 9x^2+4y^2+z^2\le 1$.
- Let $x=\frac{1}{3}r\cos\varphi\sin\theta,\ y=\frac{1}{2}r\sin\varphi\sin\theta,\ z=r\cos\theta,\ |J|=\frac{1}{6}r^2\sin\theta,\ 0\le r\le 1,\ 0\le\varphi\le 2\pi,$ $0\le\theta\le\pi$.
- $I=10\int\limits_0^1 dr\int\limits_0^{2\pi}d\varphi\int\limits_0^{\pi}r^2\frac{1}{6}r^2\sin\theta d\theta = 3\cdot 2\pi\int\limits_0^{\pi}\sin\theta d\theta\int\limits_0^1 r^4dr = 6\pi\cdot 2\cdot\frac{1}{5}=\frac{12\pi}{5}$.

Example (Final 20152)

Evaluate the integral $\iint\limits_S (x^3+y)dydz + (y^3+2z)dzdx + zdxdy$, where S is the sphere $x^2+y^2+z^2=1$, $z \geq 0$ with the outward direction.

- Add the surface S': $z=0$ $(x^2+y^2 \leq 1)$, with the downward direction
- By the divergence theorem

$$\iint\limits_{S\cup S'} = \iiint\limits_V (3x^2+3y^2+1)dxdydz = 3\iiint\limits_V (x^2+y^2)dxdydz + \frac{2}{3}\pi,$$

  where $V: x^2+y^2+z^2 \leq 1$, $z \geq 0$.
- Spherical coordinates: $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$, $|J| = r^2\sin\theta$, $0 \leq r \leq 1$, $0 \leq \varphi \leq 2\pi$, $0 \leq \theta \leq \pi/2$.

- $\iiint\limits_V (x^2+y^2)dxdydz = \int\limits_0^1 dr \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi/2} (r^2\sin^2\theta) r^2 \sin\theta d\theta = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 r^4 dr \int\limits_0^{\pi/2} \sin^3\theta d\theta = \frac{1}{5}\cdot 2\pi \cdot \frac{2}{3} = \frac{4\pi}{15}$.
- $\iint\limits_{S\cup S'} = 3\cdot\frac{4\pi}{15} + \frac{2\pi}{3} = \frac{22\pi}{15}$.
- Evaluate $\iint\limits_{S'}$. The unit normal vector of S' is $\vec{n} = (0,0,-1)$.
- $\vec{F}\bullet\vec{n} = -z$.
- $\iint\limits_{S'} = \iint\limits_{S'} -zdS = 0$.
- So $\iint\limits_{S} = \frac{22\pi}{15}$.

Example (Final 20152)

Evaluate the integral $\iint\limits_S (x^3+y)dydz + (y^3+2z)dzdx + zdxdy$, where S is the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$ with the outward orientation.

- $\vec{F} = \langle x^3+y, y^3+2z, z\rangle$. The outward unit normal vector of S is $\vec{n} = \langle x, y, z\rangle$.
- Hence $\vec{F}\cdot\vec{n} = x^4+xy+y^4+2yz+z^2$.
- The integral is $I = \iint\limits_S \vec{F}\cdot\vec{n}\,dS = \iint\limits_S (x^4+xy+y^4+2yz+z^2)dS$.
- $S\colon z = \sqrt{1-x^2-y^2}$, where $(x,y)\in D\colon x^2+y^2\leq 1$.
- $dS = \sqrt{1+(z_x)^2+(z_y)^2}dxdy = (1/\sqrt{1-x^2-y^2})dxdy$.

- Hence

$$I = \iint\limits_{D} (x^4 + xy + y^4 + 2y\sqrt{1 - x^2 - y^2} + 1 - x^2 - y^2)\frac{1}{\sqrt{1 - x^2 - y^2}}dxdy$$

$$= \iint\limits_{D} (x^4 + y^4 + 1 - x^2 - y^2)\frac{1}{\sqrt{1 - x^2 - y^2}}dxdy$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_0^1 (r^4\cos^4\varphi + r^4\sin^4\varphi + 1 - r^2)\frac{1}{\sqrt{1 - r^2}}dr$$

$$= \int\limits_0^{2\pi} (\cos^4\varphi + \sin^4\varphi)d\varphi \int\limits_0^1 \frac{r^5}{\sqrt{1 - r^2}}rdr + \int\limits_0^{2\pi} d\varphi \int\limits_0^1 \sqrt{1 - r^2}rdr = \frac{3\pi}{2}\frac{8}{15} + 2\pi\frac{1}{3} = \frac{22\pi}{15}.$$

Optional: Example (Final 20152) revisted

Evaluate the integral $\iint\limits_S (x^3+y)dydz + (y^3+2z)dzdx + zdxdy$, where S is the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$ with the outward orientation.

- $\vec{F} = \langle x^3+y, y^3+2z, z\rangle$. The outward unit normal vector of S is $\vec{n} = \langle x, y, z\rangle$.
- Hence $\vec{F}\cdot\vec{n} = x^4+xy+y^4+2yz+z^2$.
- The integral is $I = \iint\limits_S \vec{F}\cdot\vec{n}\,dS = \iint\limits_S (x^4+xy+y^4+2yz+z^2)dS$.
- Parametrize S: $x=\cos\varphi\sin\theta$, $y=\sin\varphi\sin\theta$, $z=\cos\theta$, where $D: 0\leq\varphi\leq 2\pi$, $0\leq\theta\leq\pi/2$. (Then $|\vec{r}'_\varphi \times \vec{r}'_\theta| = \sin\theta$.)
- Hence

$$I = \iint\limits_D (\cos^4\varphi\sin^4\theta + \cos\varphi\sin\varphi\sin^2\theta + \sin^4\varphi\sin^4\theta + 2\sin\varphi\sin\theta\cos\theta + \cos^2\theta)\sin\theta d\varphi d\theta$$

$$= \int_0^{2\pi}(\cos^4\varphi+\sin^4\varphi)d\varphi\int_0^{\pi/2}\sin^5\theta d\theta + \int_0^{2\pi}d\varphi\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta = \frac{3\pi}{2}\frac{8}{15} + 2\pi\frac{1}{3} = \frac{22\pi}{15}.$$

Some past exam problems

- (Final 20192) Evaluate $\iint\limits_S xy^3 dydz + (x^2 + z^2)dxdy$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, with the outward direction
- (Final 20162) Evaluate $\iint\limits_S (3xy^2 + x)dydz + (y^3 + 2xz)dzdx + (6x^2z + xy)dxdy$, where S is the paraboloid $z = x^2 + y^2$ bounded by $z \leq 4$, with the downward direction.