Calculus 2 Midterm mock exam Solution

Q1. (1pt)

- **(0.5pt)** Prove that there is an intersection between two lines:

  Parametric form of d_1: $\begin{cases} x = -2-2t \\ y = 4+2t \\ z = 2+t \end{cases}$

  Consider the set of equations: $\begin{cases} -2-2t = 1+3t' \\ 4+2t = 2-4t' \\ 2+t = -1 \end{cases}$

  This set of equations has a solution $\begin{cases} t = -3 \\ t' = 1 \end{cases}$

  so d_1 and d_2 intersect at point $A(4,-2,-1)$

- **(0.5pt)** Find the intersection angle

  Direction vector of d_1 and d_2 are $\vec{u}_{d_1}(-2,2,1)$ and $\vec{u}_{d_2}(3,-4,0)$, respectively.

  The angle between d_1 and d_2: $\cos\alpha = \dfrac{|\vec{u}_{d_1}.\vec{u}_{d_2}|}{||\vec{u}_{d_1}||.||\vec{u}_{d_2}||} = \dfrac{14}{15}$

  $\Rightarrow \alpha = \arccos\dfrac{14}{15}$

Q2. (1pt)

- **(0.25pt)** Find t

  We have: $\vec{r}(t) = \left(e^{-1-t}\cos t, e^t\sin t, e^{t+1}\right)$

Point $\left(\frac{1}{e}, 0, e\right)$ belongs to curve $\vec{r}(t)$: $\begin{cases} e^{-1-t}\cos t = \frac{1}{e} \\ e^{t}\sin t = 0 \\ e^{t+1} = e \end{cases} \Rightarrow \quad t = 0$

- **(0.75pt)** Find the formula of the tangent equation

  Since $\vec{r}(t) = \left(e^{-1-t}\cos t, e^{t}\sin t, e^{t+1}\right)$

  $\Rightarrow \vec{r}'(t) = \left(-e^{-1-t}(\cos t + \sin t), e^{t}(\sin t + \cos t), e^{t+1}\right)$

  $\Rightarrow \vec{r}'(0) = \left(-\frac{1}{e}, 1, e\right)$

  So the tangent line of the curve $\vec{r}(t)$ at the point $\left(\frac{1}{e}, 0, e\right)$ is:

$$\frac{x - \frac{1}{e}}{-\frac{1}{e}} = \frac{y}{1} = \frac{z - e}{e}$$

Q3. (1pt)

- $\begin{cases} x(t) = \frac{1}{3}t^3 \\ y(t) = 2t \\ z(t) = t^2 \end{cases} \Leftrightarrow \begin{cases} x'(t) = t^2 \\ y'(t) = 2 \\ z'(t) = 2t \end{cases}$

- Length of the curve is:

  $L = \int_0^2 \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}dt = \int_0^2 \sqrt{t^4 + 4t^2 + 4}dt$

  $= \int_0^2 (t^2 + 2)dt = \frac{20}{3}$

Q4. (1pt)

- **(0.25pt)** Find t

We have $\vec{r}(t) = \big(3\sin t, 4t, 3\cos t\big)$

Point $\big(0, 4\pi, -3\big)$ belongs to $\vec{r}(t)$: $\begin{cases} 3\sin t = 0 \\ 4t = 4\pi \\ 3\cos t = -3 \end{cases} \Rightarrow t = \pi$

- **(0.75pt)** Curvature at point $\big(0, 4\pi, -3\big)$

  Since $\vec{r}(t) = \big(3\sin t, 4t, 3\cos t\big)$

  $\Rightarrow \begin{cases} \vec{r}'(t) = \big(3\cos t, 4, -3\sin t\big) \\ \vec{r}''(t) = \big(-3\sin t, 0, -3\cos t\big) \end{cases} \Leftrightarrow \begin{cases} \vec{r}'(\pi) = \big(-3, 4, 0\big) \\ \vec{r}''(\pi) = \big(0, 0, 3\big) \end{cases}$

  $\Rightarrow \vec{r}'(t) \times \vec{r}''(t) = (12, 9, 0)$

  $\Rightarrow$ The curvature of the curve $\vec{r}(t)$ at point $\big(0, 4\pi, -3\big)$ is:

$$\mathscr{K} = \frac{\big|\vec{r}'(\pi) \times \vec{r}''(\pi)\big|}{\big|\vec{r}'(\pi)\big|^3} = \frac{\sqrt{(12)^2 + (9)^2}}{\sqrt{(-3)^2 + 4^2}^3} = \frac{3}{25}$$

Q5. (1pt)

- **(0.5pt)** Change order of variables

$$D: \begin{cases} 0 \le y \le 3 \\ \sqrt{\dfrac{y}{3}} \le x \le 1 \end{cases} \Leftrightarrow \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 3x^2 \end{cases}$$

- **(0.5pt)** Calculate the integrals

$$I = \int_0^3 dy \int_{\sqrt{\frac{y}{3}}}^1 \sqrt{x^3+1}\, dx = \int_0^1 dx \int_0^{3x^2} \sqrt{x^3+1}\, dy$$

$$= \int_0^1 \left(y\sqrt{x^3+1}\right)\Bigg|_{y=0}^{y=3x^2} dx = \int_0^1 3x^2\sqrt{x^3+1}\, dx$$

$$= \int_0^1 \sqrt{x^3+1}\, d(x^3+1) = \frac{2}{3}(x^3+1)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{2}{3}(2\sqrt{2}-1)$$

Q6. (1pt)

- **(0.5pt)** Determine the region

  We divide D into 2 regions:

  $$\begin{cases} D_1 : -1 \le x \le 1, 0 \le y \le x^2 : y - x^2 \le 0 \\ D_2 : -1 \le x \le 1, x^2 \le y \le 2 : y - x^2 \le 0 \end{cases}$$

- **(0.5pt)** Calculate the integrals

  $$I = \iint\limits_{D_1} \sqrt{x^2-y} dxdy + \iint\limits_{D_2} \sqrt{x^2-y} dxdy$$
  $$= \int_{-1}^1 dx \int_0^{x^2} \sqrt{x^2-y} dy + \int_{-1}^1 dx \int_{x^2}^2 \sqrt{y-x^2} dy$$
  $$= \frac{2}{3}\int_{-1}^1 (|x|^3 + (2-x^2)^{\frac{3}{2}}) dx$$
  $$= \frac{1}{3} + \frac{2}{3}\int_{-1}^1 (2-x^2)^{\frac{3}{2}} dx$$

  Let $x = \sqrt{2}\sin t \Rightarrow dx = \sqrt{2}\cos t dt$

  Hence,

  $$I = \frac{1}{3} + \frac{4}{3}\int_0^{\frac{\pi}{4}} 4\cos^4 t\, dt = \frac{1}{3} + \frac{4}{3}\int_0^{\frac{\pi}{4}} 4.\frac{1+2\cos 2t}{4} dt = \frac{\pi}{2} + \frac{5}{3}$$

Q7. (1pt)

- **(0.5pt)** Change of variables Let $\begin{cases} u = x + y \\ y = y \end{cases} \Rightarrow \begin{cases} x = u - y \\ y = y \end{cases}$

  We change the variables from (x,y) to (u,y) so $J = \begin{vmatrix} x'_u & x'_y \\ y'_u & y'_y \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

  and the new region $D : \{0 \le x \le 1, 0 \le y \le 1 - x\} = \{0 \le y \le 1, 0 \le x \le 1 - y\}$

  $\Rightarrow D' : \{0 \le y \le 1, 0 \le u - y \le 1 - y\} = \{0 \le y \le 1, y \le u \le 1\}$

  $= \{0 \le u \le 1, 0 \le y \le u\}$

- **(0.5pt)** Calculate the left-hand side integrals

  Hence,

$$\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}} dy = \int_0^1 du \int_0^u e^{\frac{y}{u}} dy$$
$$= \int_0^1 u.e^{\frac{y}{u}}\Big|_0^u du$$
$$= \int_0^1 u.(e-1) du = \frac{e-1}{2} \quad \text{(q.e.d)}$$

Q8. (1pt)

- **(0.5pt)** Change of variables to polar coordinates

  Consider $\begin{cases} x = rcos\phi \\ y = rsin\phi \end{cases} \Rightarrow D : \begin{cases} r^2 \le 4rsin\phi \\ r^2 \le 4rcos\phi \end{cases} \Leftrightarrow D : \begin{cases} r \le 4sin\phi \\ r \le 4cos\phi \end{cases}$

Divide D into 2 regions

$D_1 : \begin{cases} 0 \leq \phi \leq \dfrac{\pi}{4} \\ 0 \leq r \leq 4sin\phi \end{cases}$ and $D_2 : \begin{cases} \dfrac{\pi}{4} \leq \phi \leq \dfrac{\pi}{2} \\ 0 \leq r \leq 4cos\phi \end{cases}$

Jacobian Determinant: $|J| = r$

- **(0.5pt)** Calculate the mass

$$M = \iint\limits_D \sqrt{x^2+y^2}dxdy = \int_0^{\frac{\pi}{4}} d\phi \int_0^{4sin\phi} r^2dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\phi \int_0^{4cos\phi} r^2dr$$

$$= \int_0^{\frac{\pi}{4}} \frac{64\sin^3\phi}{3}d\phi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64\cos^3\phi}{3}d\phi$$

$$= \frac{64}{3}.\left(\int_0^{\frac{\pi}{4}} (1-\cos^2\phi).\sin\phi d\phi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\sin^2\phi).\cos\phi d\phi\right)$$

$$= \frac{64}{3}.\left[\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Bigg|_0^{\frac{\pi}{4}} + \left(\sin\phi - \frac{\sin^3\phi}{3}\right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$$

$$= \frac{32(8-5\sqrt{2})}{9}$$

Q9. (1pt)

- **(0.5pt)** Change of variables

$$I = \iiint\limits_V (x^2+y^2)dxdydz = \iint\limits_D (x^2+y^2).(2-x^2-y^2-\sqrt{x^2+y^2})dxdy$$

where D is the region bounded by $\sqrt{x^2+y^2} \leq 2-x^2-y^2$

Change the variables to cylindrical/polar coordinates:

$$\begin{cases} x = r.\cos\varphi \\ y = r.\sin\varphi \end{cases} \Rightarrow |J| = r$$

and $D : \{\sqrt{r^2} \leq 2 - x^2 - y^2\} \Rightarrow D : \{r \leq 1, 0 \leq \phi \leq 2\pi\}$

- **(0.5pt)** Calculate the integrals

  Hence,

$$\begin{aligned} I &= \iint\limits_D (x^2+y^2).(2-x^2-y^2-\sqrt{x^2+y^2})dxdy \\ &= \iint\limits_D r^2.(2-r^2-r).r\,drd\varphi \\ &= \int_0^{2\pi} d\varphi \int_0^1 r^2.(2-r^2-r).r\,dr \\ &= \varphi\Big|_0^{2\pi}.\left(\frac{r^4}{2}-\frac{r^6}{6}-\frac{r^5}{5}\right)\Big|_0^1 \\ &= 2\pi.\frac{2}{15} = \frac{4\pi}{15} \end{aligned}$$

Q10. (1pt)

- **(0.5pt)** Change of variables to cylindrical coordinates

  The volume of the solid bounded by $x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{1}{2}} = 4$ and the coordinate planes is:

$$V = \iint\limits_D (4 - x^{\frac{3}{2}} - y^{\frac{3}{2}})^2$$

  where D is a region bounded by $x^{\frac{4}{3}} + y^{\frac{4}{3}} = 4$ and lie in the first quadrant.

Change the variables: $\begin{cases} x = r\cos^{\frac{4}{3}}\varphi \\ y = r\sin^{\frac{4}{3}}\varphi \end{cases}$ where $D = \{0 \le r \le 2, 0 \le \varphi \le \pi/2\}$

$\Rightarrow |J| = \dfrac{4}{3} r \sin^{\frac{1}{3}}\varphi \cos^{\frac{1}{3}}\varphi$

- **(0.5pt)** Calculate the integrals

  We have:

$$\begin{aligned} V &= \iint\limits_D (4 - x^{\frac{3}{2}} - y^{\frac{3}{2}})^2 = \iint\limits_D (4-r^2)^2.|J|drd\varphi \\ &= \iint\limits_D (4-r^2)^2.\frac{4}{3} r\sin^{\frac{1}{3}}\varphi\cos^{\frac{1}{3}}\varphi drd\varphi \\ &= \int\limits_0^{\pi/2} \left(\frac{\sin 2\varphi}{2}\right)^{\frac{1}{3}} d\varphi \int\limits_0^2 (4 - r^{\frac{3}{2}})rdr \\ &= \frac{\sqrt{3}\pi\Gamma(\frac{5}{6})}{\sqrt[3]{2}\Gamma(\frac{1}{3})}.(8 - \frac{16\sqrt{2}}{7}) \end{aligned}$$