Introduction to Communications Engineering

IT4593E

Đỗ Công Thuần, Ph.D.

Dept. of CE, SoICT, HUST

Email: thuandc@soict.hust.edu.vn

ONE LOVE. ONE FUTURE.

Thông tin chung

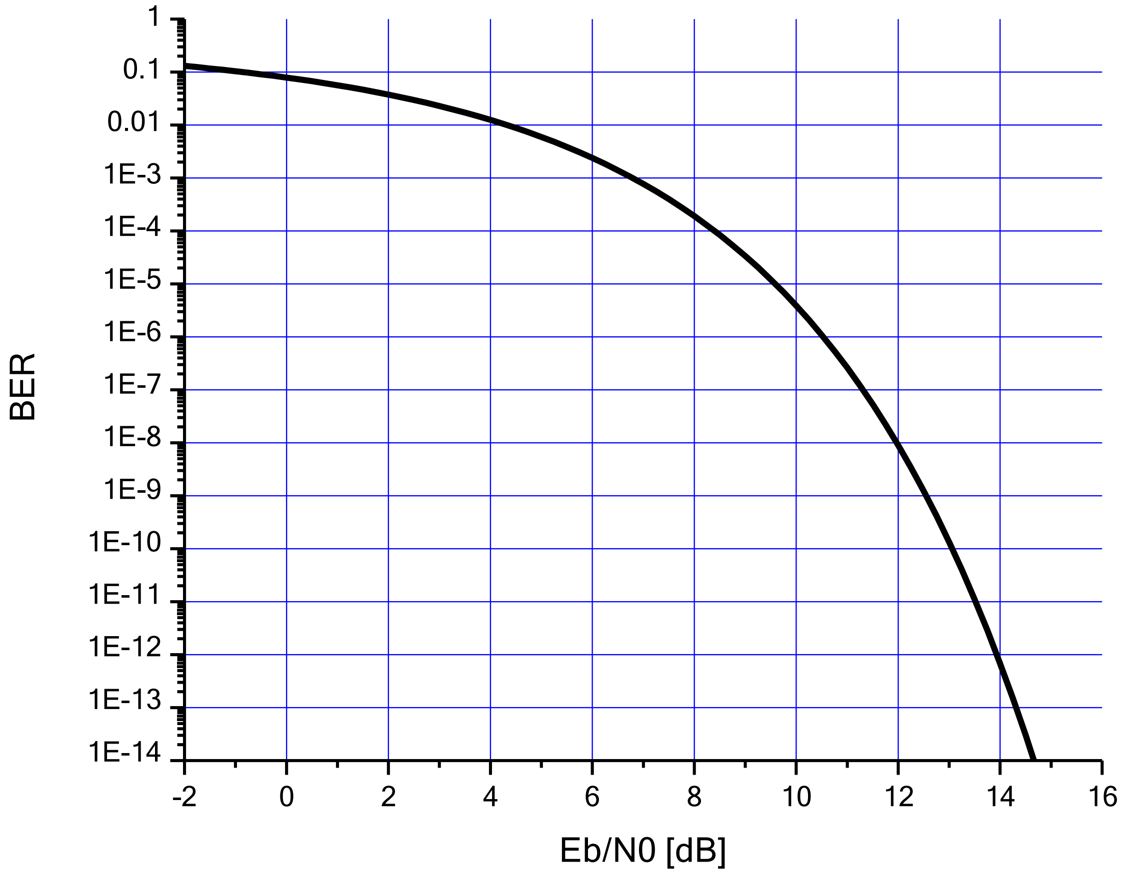
- Tên học phần: **Nhập môn kỹ thuật truyền thông**
- Mã học phần: **IT4593E**
- Khối lượng: **2 TC (2-1-0-4)**
- Lý thuyết và bài tập: **10 buổi lý thuyết, 5 buổi bài tập**
- Đánh giá học phần:

  30% QT (kiểm tra + bài tập/project + chuyên cần-quiz)

  70% CK (trắc nghiệm + tự luận)
- Tài liệu tham khảo:
  - Lecture slides
  - Lecture notes
  - Textbooks, ví dụ ***Communication Systems Engineering***, 2nd Edition, by John G. Proakis Masoud Salehi
  - Internet

Lec 06: Calculating and Evaluating Signal Spectra

Calculating SER/BER for Binary Antipodal Signals



$$P_b(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Different signal spaces (different transmitted waveforms) but with the same vector space have the same BER!

As in the previous example, BER does not depend on the orthonormal signal, such as the two types of orthonormal signals below.

$$b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$b_1(t) = \sqrt{\frac{2}{T}} P_T(t) \cos(2\pi f_0 t)$$

BER Comparison: Example

Comparison between antipodal signal space and orthogonal signal space:

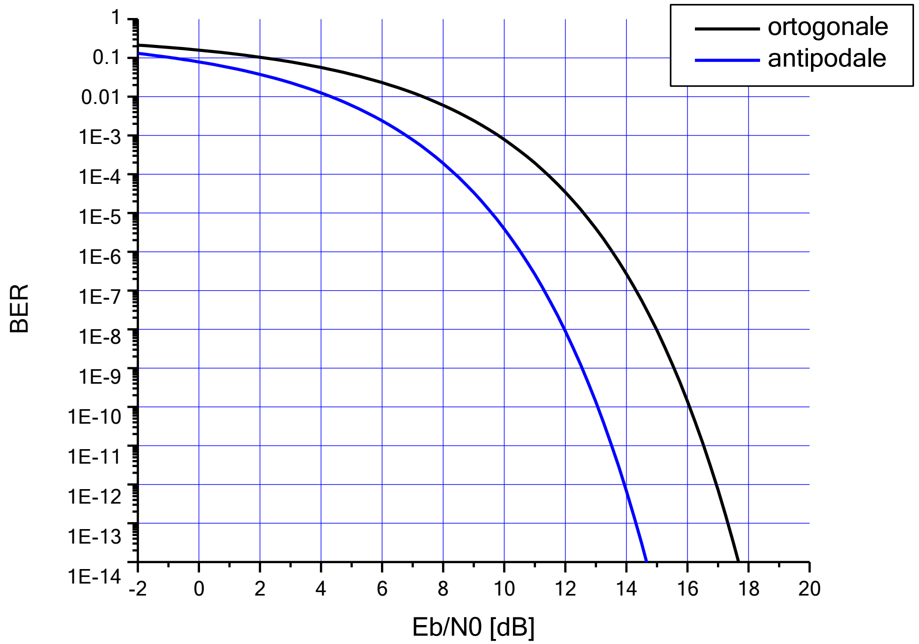
$$P_b(e)|_{antipodal} = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right) \qquad P_b(e)|_{orthogonal} = \frac{1}{2} erfc\left(\sqrt{\frac{1}{2}\frac{E_b}{N_0}}\right)$$

Antipodal space has better performance

- For a fixed BER, a system with antipodal signal space requires lower E_b/N_0
- For a fixed E_b/N_0, the system will have a lower BER

BER Comparison

$$P_b(e)|_{antipodal} = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right) \qquad P_b(e)|_{orthogonal} = \frac{1}{2} erfc\left(\sqrt{\frac{1}{2}\frac{E_b}{N_0}}\right)$$



BER Comparison

Fixed E_b/N_0 =12 dB:

- Antipodal space achieves performance $P_b(e)$ =1e-8
- While orthogonal space achieves performance $P_b(e)$ =5e-5 (higher value → worse performance)

To achieve $P_b(e)=$1e-6:

- Antipodal space requires: E_b/N_0 = 10.6 dB;
- While orthogonal space requires: E_b/N_0 =13.6 dB

(Antipodal space gains 3 dB, note that: this ratio corresponds to the received signal power)

BER Comparison

Example: Line-of-sight (LoS) transmission has received signal power as follows:

$$P_R = P_T \frac{G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2}$$ (Friis transmission equation)

Antipodal space has $P_b(e)=$1e-6 with received signal power needing to be only half that of orthogonal space.

With the same transmit power, the transmission distance for antipodal space will be greater than that for orthogonal space by $\sqrt{2}$

Or, for the same transmission distance, we can reduce the transmit power by half if using antipodal space (or benefit from using smaller transmit/receive antennas).

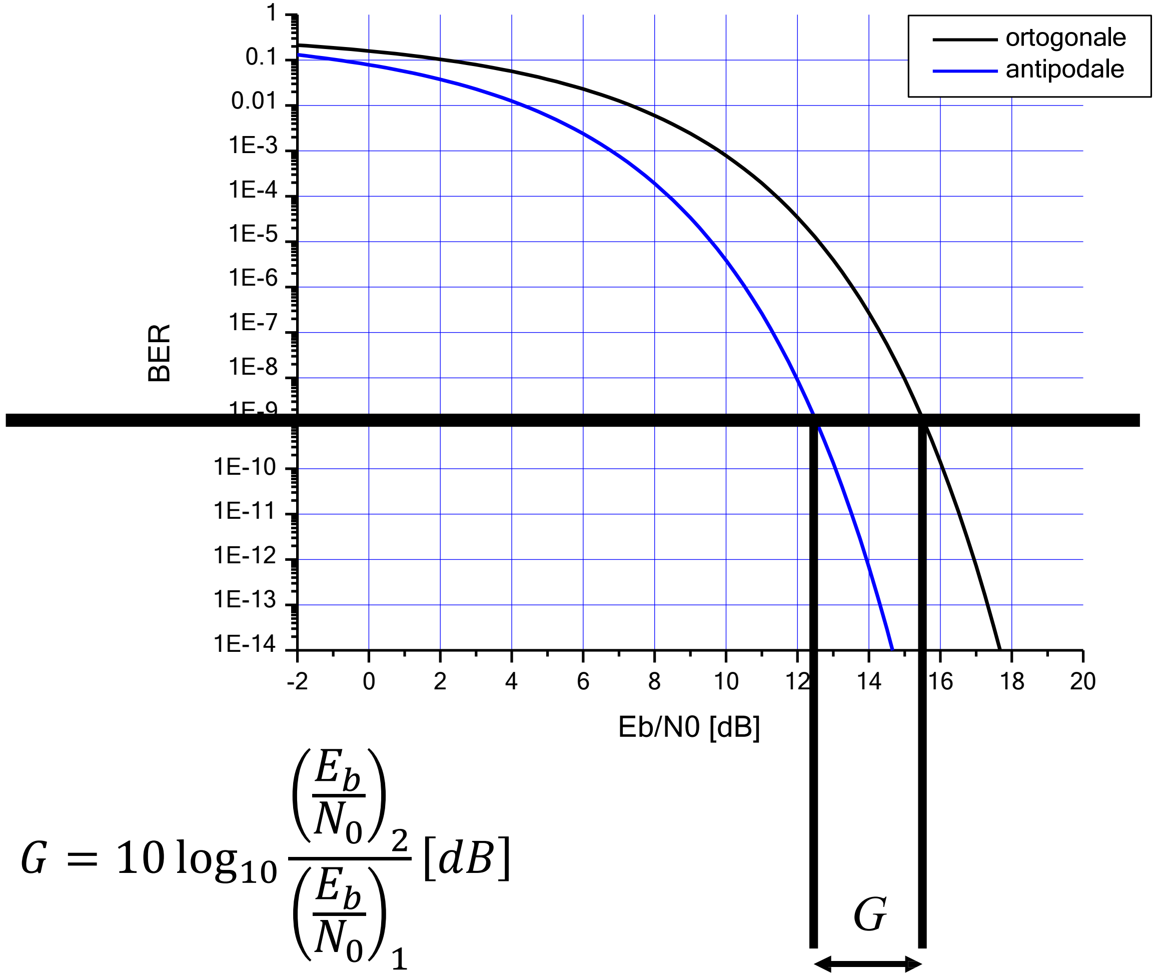
BER Comparison

Generalization for two spaces M_1 and M_2 with performance:

$$P_b(e)|_1 \approx \ erfc\left(\sqrt{y_1 \frac{E_b}{N_0}}\right) \qquad P_b(e)|_2 \approx \ erfc\left(\sqrt{y_2 \frac{E_b}{N_0}}\right)$$

If $y_1 > y_2$ then space M_1 has better performance (lower BER)

BER Comparison



$$G = 10\log_{10}\frac{\left(\frac{E_b}{N_0}\right)_2}{\left(\frac{E_b}{N_0}\right)_1}[dB]$$

Asymptotic Performance

Asymptotic assumption ($E_b/N_0 \rightarrow \infty$) with very small noise variance.

When an error occurs, it almost always occurs in the adjacent Voronoi region (of the signals with the smallest distance).

It can be shown that the asymptotic symbol error probability is approximately

$$\boxed{P_S(e) \approx \frac{1}{2} A_{\min} \, erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}} \right)}$$

$$\boxed{d_{\min} = \min_{\underline{s}_1\, \underline{s}_j \in M} d_E(\underline{s}_1, \underline{s}_j)}$$

$$\boxed{A_{\min} = \text{multiplicity} = \text{number of signals } \underline{s}_j \text{ with } d_{\mathrm{E}}(\underline{s}_j\ \underline{s}_1) = d_{\min}}$$

Similarly for BER:

$$\boxed{P_b(e) \approx \frac{1}{2}\frac{w_{\min}}{k}\, erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right)}$$

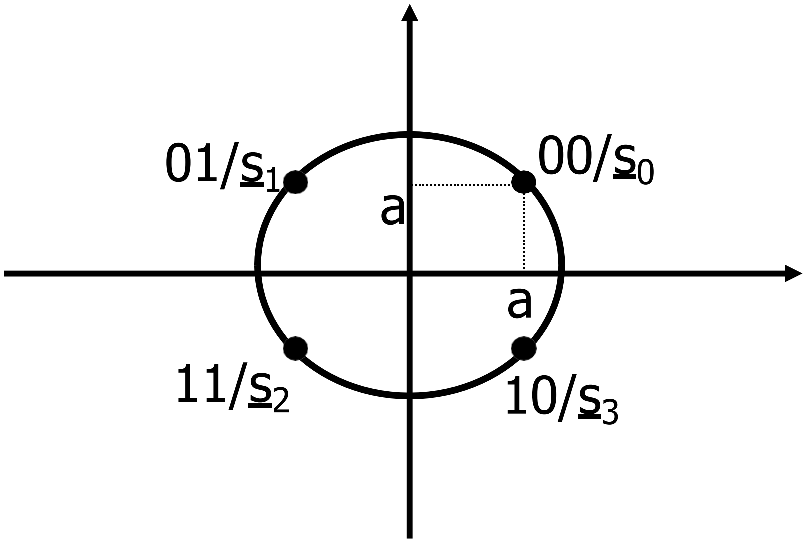
Where:

$$\boxed{w_{\min} = \text{input multiplicity} = \sum_{\underline{s}_j :\, d_E(\underline{s}_1, \underline{s}_j) = d_{\min}} d_H(\underline{v}_1, \underline{v}_j)}$$

These formulas are not just upper bounds, but can be considered approximations of the actual probabilities in the high SNR case.

Example

4-PSK constellation



$$d_{\min} = 2a \qquad A_{\min} = 2 \qquad w_{\min} = 2$$

$$P_S(e) \approx erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right) = erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_b(e) \approx \frac{1}{2} erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Gray Labelling

Consider asymptotic BER approximation: $$P_b(e) \approx \frac{1}{2}\frac{w_{\min}}{k}\, erfc\left(\sqrt{\frac{d^2_{\min}}{4N_0}}\right)$$

We have $$A_{\min} \le w_{\min}$$

$$\boxed{A_{\min} = \text{multiplicity} = \text{number of signals } \underline{s}_j \text{ with } d_{\mathrm{E}}(\underline{s}_j\ \underline{s}_1) = d_{\min}}$$

$$\boxed{w_{\min} = \text{input multiplicity} = \sum_{\underline{s}_j : d_E(\underline{s}_1, \underline{s}_j) = d_{\min}} d_H(\underline{v}_1, \underline{v}_j)}$$

In the optimal case: $$A_{\min} = w_{\min}$$

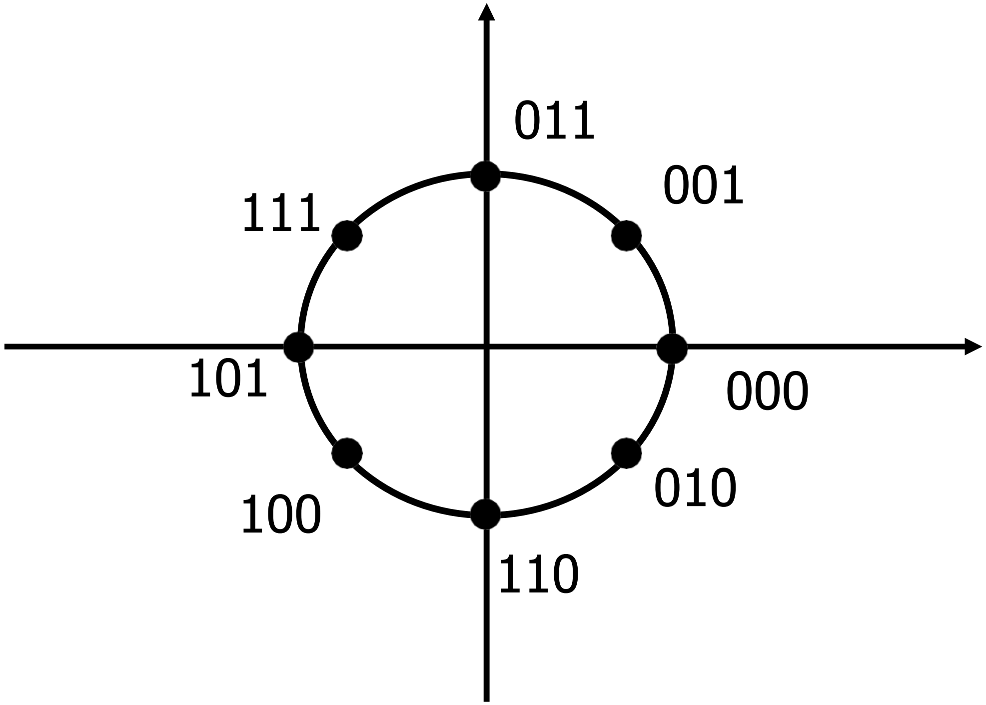
SOICT

For signal $\underline{s}_i$ associated with vector via mapping $\underline{v}_i = e^{-1}(\underline{s}_i)$.

All "**adjacent**" signals $\underline{s}_i$ (with minimum d_{min} to $\underline{s}_i$) are associated with binary vectors that have a Hamming distance of 1 from $\underline{v}_i$.

In this way, asymptotic BER is minimized.

Example:



Calculating, Estimating Signal Spectrum

PSD = Power Spectral Density

Spectral Properties

We want to study the frequency characteristics of the transmitted signal $s(t)$ by calculating its power spectral density (PSD) $G_s(f)$ and defining the appropriate bandwidth of the transmitted signal: the frequency region containing (a large part of the value of) $G_s(f)$.

1-D Bipolar Signal Space

Consider the following signal space:

$$M = \{s_1(t) = +AP_T(t)\,,\ s_2(t) = -AP_T(t)\,\}$$

This is a one-dimensional signal space with (d=1), with orthonormal basis:

$$B = \left\{ \mathrm{b}_1(\mathrm{t}) = \frac{1}{\sqrt{\mathrm{T}}} \mathrm{P}_\mathrm{T}(\mathrm{t}) \right\}$$

Therefore, the signals in M have vector representations as follows:

$$M = \{\underline{s_1} = (+\alpha)\,,\ \underline{s_2} = (-\alpha)\} \qquad \alpha = A\sqrt{T}$$

SOICT

Consider the transmitted signal:

In the first cycle *[0,T]* we transmit

$$s_1(t) = +\alpha b_1(t) \quad \text{or} \quad s_2(t) = -\alpha b_1(t)$$

In any cycle *[nT,(n+1)T]* we transmit

$$s_1(t-nT) = +\alpha b_1(t-nT) \quad \text{or} \quad s_2(t-nT) = -\alpha b_1(t-nT)$$

We can represent the mathematical form of the transmitted signal as follows:

$$\boxed{s(t) = \sum_{n=0}^{+\infty} a[n]p(t - nT)}$$

$$\boxed{a[n] \in \{+\alpha, -\alpha\}}$$

$$\boxed{p(t) = b_1(t)}$$

For any 1-D signal space

$$M = \{\underline{s_1} = (\alpha_1), \underline{s_2} = (\alpha_2), \ldots, \underline{s_m} = (\alpha_m)\} \subseteq R$$

with orthonormal signal: $b_1(t)$

The transmitted signal will have the form:

$$\boxed{s(t) = \sum_{n=0}^{+\infty} a[n] p(t - nT)}$$

$$\boxed{a[n] \in \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_m\}}$$

$$\boxed{p(t) = b_1(t)}$$

The sequence $a[n]$ is a stationary sequence of random variables, consisting of statistically independent symbols with uniform probability distribution:

$$P\left(a[n]=\alpha_i\right)=\frac{1}{m}$$

Mean: $$\mu_a = \frac{1}{m}\sum_{i=1}^{m}\alpha_i$$

Variance: $$\sigma_a^2 = \frac{1}{m}\sum_{i=1}^{m}\left(\alpha_i - \mu_a\right)^2$$

Transmitted Waveform

In the next part, we will calculate the power spectral density of the transmitted signal generated by a 1-D vector space:

$$s(t) = \sum_{n=-\infty}^{+\infty} a[n] p(t - nT)$$

$s(t)$ is a random process, for which we want to calculate its power spectral density (PSD):

$$G_s(f) = \int_{-\infty}^{+\infty} R_s(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

This provides information about the distribution of the signal's power in the frequency domain:

$$P(s) = R_s(0) = \int_{-\infty}^{+\infty} G_s(f)\, df$$

Theory of Power Spectral Density

Given a random process: $$s(t) = \sum_{n=-\infty}^{+\infty} a[n]p(t-nT)$$

With

- $a[n]$ is a stationary sequence of random variables, with

$$M_a(i)=E[a[n]a[n+i]]$$

- $p(t)$ is a real signal with Fourier transform $P(f)$

The power spectral density is calculated: $$\boxed{G_s(f) = S_a(f)\frac{|P(f)|^2}{T}}$$

where $S_a(f) = \sum_i M_a(i)e^{-j2\pi fiT}$

Proof (self-reference)

$$G_s(f) = \int_{-\infty}^{+\infty} R_s(\tau) e^{-j2\pi f\tau} d\tau$$

The random process is a cyclostationary process. We calculate:

$$R_s(\tau) = \frac{1}{T}\int_0^T M_{SS}(t+\tau, t)\, dt$$

where: $M_{SS}(t+\tau, t) = E[S(t+\tau)S(t)]$
(autocorrelation)

$$M_{SS}(t+\tau,t) = E[S(t+\tau)S(t)] = E\left[\sum_{m=-\infty}^{+\infty} a[m]p(t+\tau-mT)\sum_{n=-\infty}^{+\infty} a[n]p(t-nT)\right] =$$

$$= \sum_{m=-\infty}^{+\infty}\ \sum_{n=-\infty}^{+\infty} E\big[a[m]a[n]\big]\,p(t+\tau-mT)\ p(t-nT) =$$

$$= \sum_{m=-\infty}^{+\infty}\ \sum_{n=-\infty}^{+\infty} M_a(m-n)\,p(t+\tau-mT)\,p(t-nT) =$$

where $i=(m-n)$ $[\ m = n+i\ ]$

$$\boxed{M_{SS}(t+\tau,t) = \sum_{i=-\infty}^{+\infty}\ \sum_{n=-\infty}^{+\infty} M_a(i)\,p(t+\tau-nT-iT)\,p(t-nT)}$$

$$M_{SS}(t+\tau,t)=\sum_{i=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}M_a(i)p(t+\tau-nT-iT)p(t-nT)$$

$$R_s(\tau)=\frac{1}{T}\int_0^T M_{SS}(t+\tau,t)\,dt=\frac{1}{T}\int_0^T\left[\sum_{i=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}M_a(i)p(t+\tau-nT-iT)p(t-nT)\right]dt=$$

$$=\frac{1}{T}\sum_{i=-\infty}^{+\infty}M_a(i)\sum_{n=-\infty}^{+\infty}\int_0^T\left[p(t+\tau-nT-iT)p(t-nT)\right]dt=$$

where $t'=(t-nT)$

$$=\frac{1}{T}\sum_{i=-\infty}^{+\infty}M_a(i)\sum_{n=-\infty}^{+\infty}\int_{-nT}^{-(n-1)T}\left[p(t'+\tau-iT)p(t')\right]dt'$$

$$\boxed{R_s(\tau)=\frac{1}{T}\sum_{i=-\infty}^{+\infty}M_a(i)\int_{-\infty}^{+\infty}\left[p(t'+\tau-iT)p(t')\right]dt'}$$

$$R_s(\tau) = \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty} \left[p(t' + \tau - iT)\, p(t') \right] dt'$$

$$G_s(f) = \int_{-\infty}^{+\infty} R_s(\tau) e^{-j2\pi f\tau} d\tau = \int_{-\infty}^{+\infty} \left[\frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty} \left[p(t' + \tau - iT)\, p(t') \right] dt' \right] e^{-j2\pi f\tau} d\tau =$$

$$= \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[p(t' + \tau - iT)\, p(t') e^{-j2\pi f\tau} \right] dt'\, d\tau =$$

$$t'' = t' + \tau - iT \qquad\qquad \tau = t'' - t' + iT$$

$$e^{-j2\pi f\tau} = e^{-j2\pi ft''}\, e^{j2\pi ft'}\, e^{-j2\pi fiT}$$

$$G_s(f) = \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[p(t'')\, p(t') e^{-j2\pi ft''}\, e^{j2\pi ft'}\, e^{-j2\pi fiT} \right] dt'\, dt'' =$$

$$G_s(f) = \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[p(t'')p(t')e^{-j2\pi ft''}\ e^{j2\pi ft'}\ e^{-j2\pi fiT} \right] dt'\,dt'' =$$

$$= \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) e^{-j2\pi fiT} \int_{-\infty}^{+\infty} p(t'')\, e^{-j2\pi ft''} dt'' \int_{-\infty}^{+\infty} p(t')\ \ e^{j2\pi ft'}\ dt' =$$

$$= \frac{1}{T} S_a(f) P(f) P^*(f)$$

$$\boxed{G_v(f) = S_a(f)\,\frac{\left|P(f)\right|^2}{T}}$$

Case 1: Statistically independent symbols with zero mean

<u>Typical case:</u>

Assume the sequence $a[n]$ has the characteristics:

- Statistically independent
- Zero mean: $\mu_a = 0$

Corresponds to a 1-D antipodal signal space centered at the origin

We have:

$$\text{for } i \neq 0 \quad M_a(i) = E(a[n+i]a[n]) = E(a[n+i])E(a[n]) = 0$$

$$\text{for } i = 0 \quad M_a(i) = E\left(a[n]^2\right) = \sigma_a^2$$

With:

$$S_a(f) = \sum_i M_a(i) e^{-j2\pi fiT} = \sigma_a^2$$

Power spectral density (PSD):

$$G_s(f) = S_a(f) \frac{|P(f)|^2}{T}$$

Simplifies to:

$$\boxed{G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}}$$

For a 1-D antipodal signal space centered at the origin, the power spectral density (PSD) of the transmitted signal is proportional to $|P(f)|^2$

SOICT

Example: Signal Space 1

Assume:

$$M = \{s_1(t) = +AP_T(t)\,,\, s_2(t) = -AP_T(t)\}$$

This is a 1-D space (d=1), with orthonormal signal

$$B = \left\{ \mathrm{b}_1(\mathrm{t}) = \frac{1}{\sqrt{\mathrm{T}}}\mathrm{P}_{\mathrm{T}}(\mathrm{t}) \right\}$$

The signal space is represented in vector form as:

$$M = \{\underline{s_1} = (+\alpha)\,,\, \underline{s_2} = (-\alpha)\} \qquad \alpha = A\sqrt{T}$$

Transmitted signal form: $$s(t) = \sum_{n=0}^{+\infty} a[n]p(t - nT)$$

where $a[n] \in \{+\alpha, -\alpha\}$

$$p(t) = b_1(t)$$

With the sequence *a[n]* having:

Mean: $\mu_a = 0.5\,(-\alpha + \alpha) = 0$

Variance: $\sigma_a^2 = 0.5\,(\alpha^2 + \alpha^2) = \alpha^2 = A^2T$

The sequence $a[n]$ consists of statistically independent symbols with zero mean

Power spectral density (PSD) is:

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$$

SOICT

We have: $p(t) = b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$

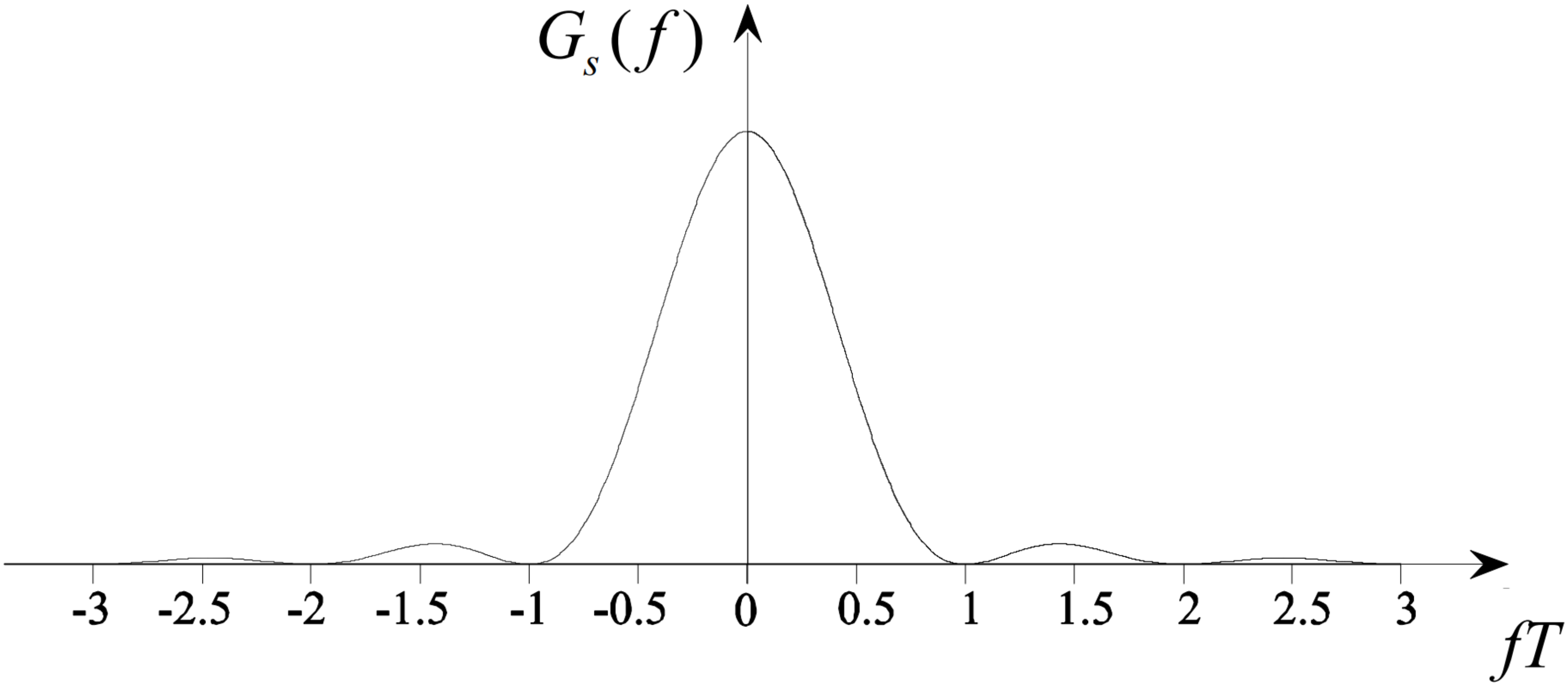
We define the *sinc(x)* function: $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)}$

Fourier transform of *p(t)* is:

$$P(f) = \sqrt{T}\mathrm{sinc}(fT)e^{-j\pi f \mathrm{T}} = \sqrt{T}\,\frac{\sin(\pi fT)}{(\pi fT)}\,e^{-j\pi f \mathrm{T}}$$

Therefore the PSD of this signal space is:

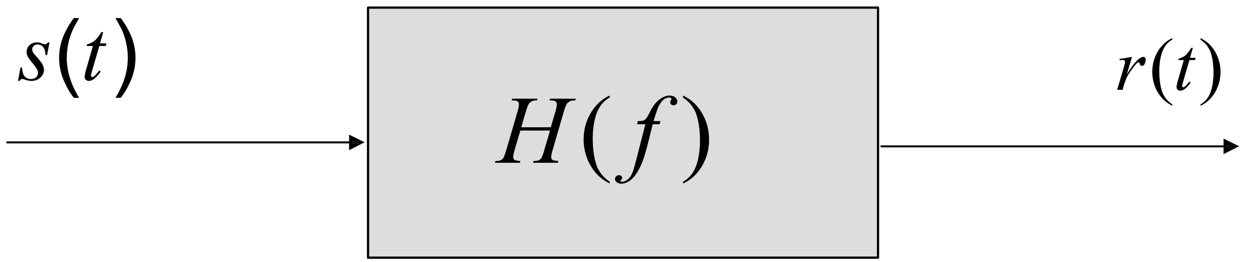
$$\boxed{G_s(f) = A^2T\left(\frac{\sin(\pi fT)}{(\pi fT)}\right)^2}$$



Conclusion

- This is a baseband spectrum (concentrated around the origin = DC)
- Spectrum is zero at frequencies that are multiples of $1/T$
- Main lobe has width $2/T$, from $-1/T$ to $+1/T$
- All side lobes have width $1/T$ with decreasing intensity

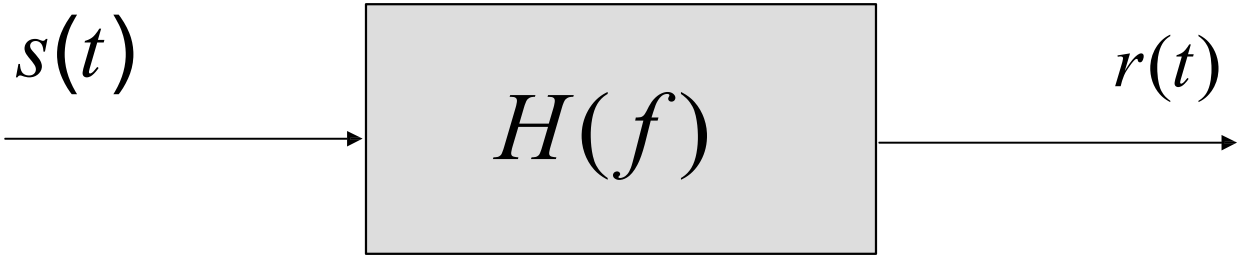
This signal space is suitable for channels with “low-pass frequency response”



The received signal spectrum (excluding noise) is:

$$G_r(f) = |H(f)|^2 \, G_s(f)$$

Since $G_s(f)$ is unlimited in the frequency axis, only an ideal channel with frequency response $H(f)=1$ (for all f) will not cause signal distortion.



$$G_r(f) = |H(f)|^2 \, G_s(f)$$

For a channel with frequency response $H(f)$

We must design a transmitted signal such that its spectrum $G_s(f)$ is concentrated around the frequency region where $H(f)$ is "good".

In this way, the received signal will approximate the transmitted signal.

Signal Space 2

The transmitted signal has the form: $s(t) = \sum_{n=0}^{+\infty} a[n] p(t - nT)$

where $a[n] \in \{+\alpha, -\alpha\}$

$p(t) = b_1(t)$

With the sequence $a[n]$ having:

Mean: $\mu_a = 0.5\,(-\alpha + \alpha) = 0$

Variance: $\sigma_a^2 = 0.5\,(\alpha^2 + \alpha^2) = \alpha^2 = A^2 \frac{T}{2}$

Signal Space 2

Power spectral density (PSD) has the form:

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$$

where $p(t) = b_1(t) = \sqrt{\frac{2}{T}} P_T(t) \cos\left(2\pi f_0 t\right)$

Its Fourier transform is:

$$P(f) = \left[\sqrt{2T} \frac{\sin(\pi f T)}{(\pi f T)} e^{-j\pi f \mathrm{T}} \right] * \left[\frac{1}{2} \left(\delta(f - f_0) + \delta(f + f_0) \right) \right] =$$

$$= \sqrt{\frac{T}{2}} \left[\left(\frac{\sin(\pi(f - f_0)T)}{(\pi(f - f_0)T)} \right) + \left(\frac{\sin(\pi(f + f_0)T)}{(\pi(f + f_0)T)} \right) \right] e^{-j\pi f \mathrm{T}}$$

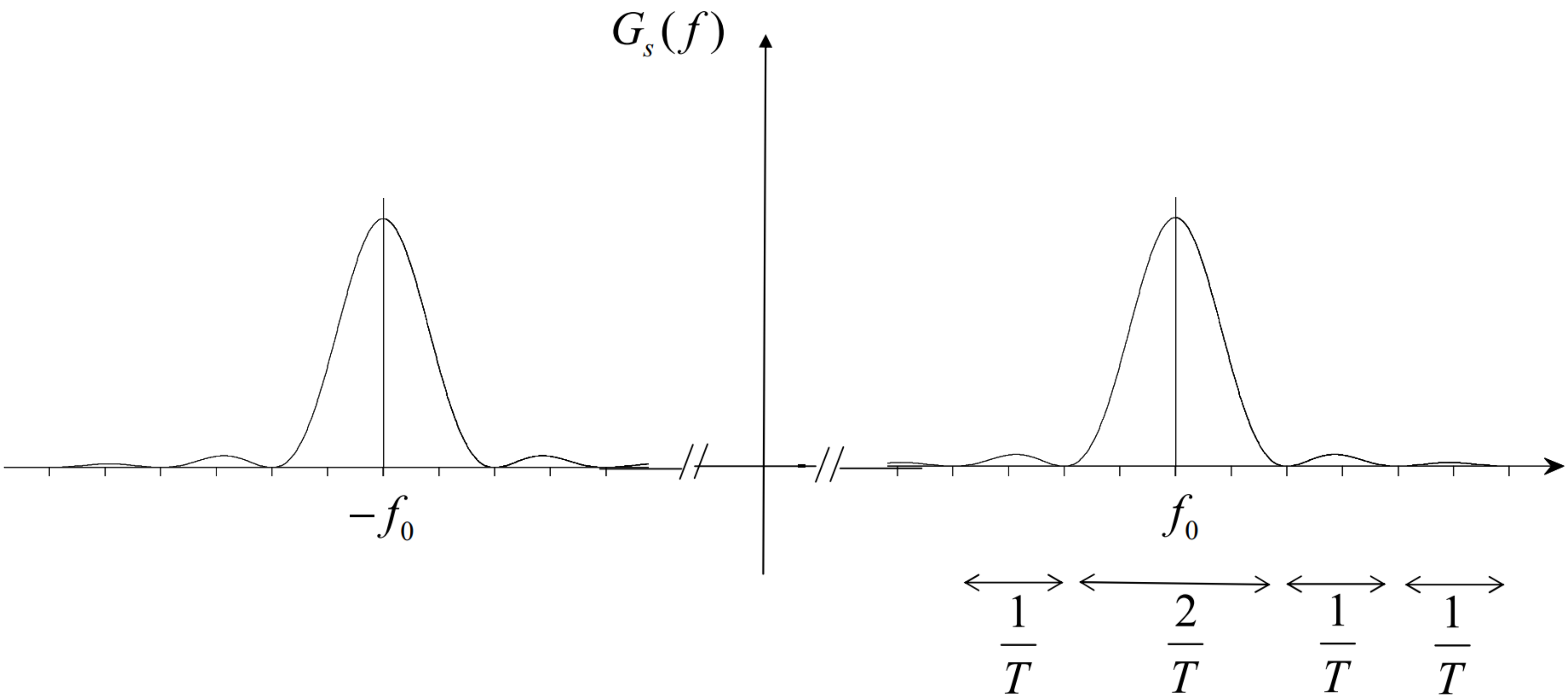
We have

$$|P(f)|^2 = \frac{T}{2}\left[\left(\frac{\sin(\pi(f-f_0)T)}{(\pi(f-f_0)T)}\right)^2 + \left(\frac{\sin(\pi(f+f_0)T)}{(\pi(f+f_0)T)}\right)^2\right]$$

The power spectral density (PSD) is:

$$G_s(f) = \frac{1}{4}A^2T\left[\left(\frac{\sin(\pi(f-f_0)T)}{(\pi(f-f_0)T)}\right)^2 + \left(\frac{\sin(\pi(f+f_0)T)}{(\pi(f+f_0)T)}\right)^2\right]$$

$$G_s(f) = \frac{1}{4}A^2T\left[\left(\frac{\sin(\pi(f-f_0)T)}{\left(\pi(f-f_0)T\right)}\right)^2 + \left(\frac{\sin(\pi(f+f_0)T)}{\left(\pi(f+f_0)T\right)}\right)^2\right]$$



Conclusion

- This is a bandpass spectrum (with center frequency $f_0 \neq 0$)
- Spectral zeros at frequencies offset from f_0 by multiples of $1/T$
- Main lobe has width $2/T$ centered at f_0
- Other lobes have width $1/T$ and decreasing intensity

Linear Modulation

Generally, for $s(t) = \sum_n a[n] p(t - nT)$

With spectrum $G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$

If we consider $s'(t) = \sum_n a_n p'(t - nT)$

where $p'(t) = p(t)\cos(2\pi f_0 t)$

Then the signal spectrum: $G_{s'}(f) = \frac{1}{4}[G_s(f - f_0) + G_s(f + f_0)]$

The frequency spectrum is shifted around f_0

Signal Space 3

Consider $M = \left\{ s_1(t) = +A\frac{\sin(\pi t/T)}{(\pi t/T)}, s_2(t) = -A\frac{\sin(\pi t/T)}{(\pi t/T)} \right\}$

This is a 1-D signal space with orthonormal basis

$$B = \left\{ b_1(t) = \frac{1}{\sqrt{T}}\frac{\sin(\pi t/T)}{(\pi t/T)} \right\}$$

The signal space is equivalent to the vector set:

$$M = \{\underline{s_1} = (+\alpha), \underline{s_2} = (-\alpha)\} \qquad \alpha = A\sqrt{T}$$

Transmitted signal $$s(t) = \sum_{n=0}^{+\infty} a[n]p(t - nT)$$

where $$a[n] \in \{+\alpha, -\alpha\}$$

$$p(t) = b_1(t)$$

The sequence *a[n]* has:

Mean: $$\mu_a = 0.5\,(-\alpha + \alpha) = 0$$

Variance: $$\sigma_a^2 = 0.5\,(\alpha^2 + \alpha^2) = \alpha^2 = A^2T$$

Power spectral density (PSD) has the form

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$$

where $$p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t / T)}{(\pi t / T)}$$

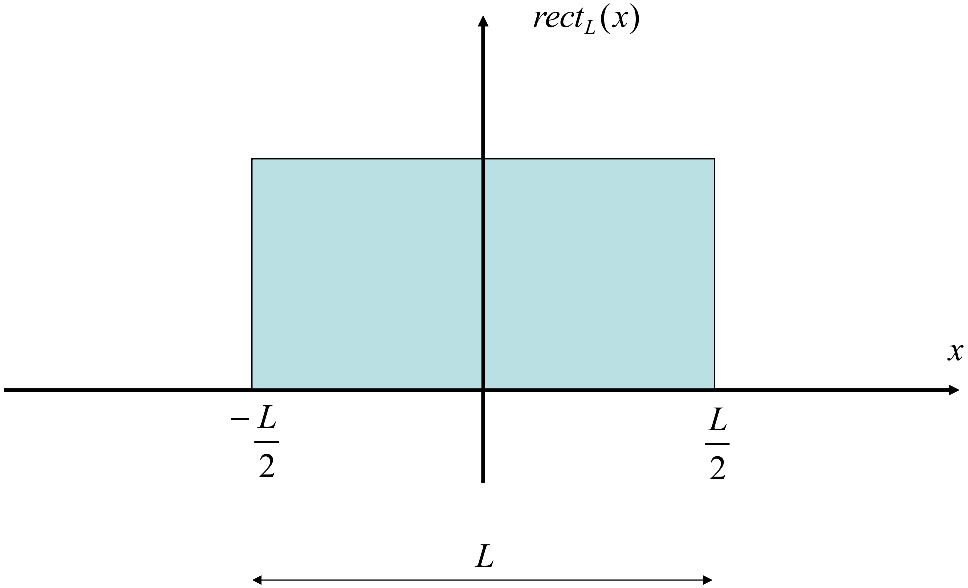
Its Fourier transform is $$P(f) = \sqrt{T} rect_{\frac{1}{T}}(f)$$

p(t): is a low-pass filter with constant Fourier transform between *-1/(2T)* and *+1/(2T)*.

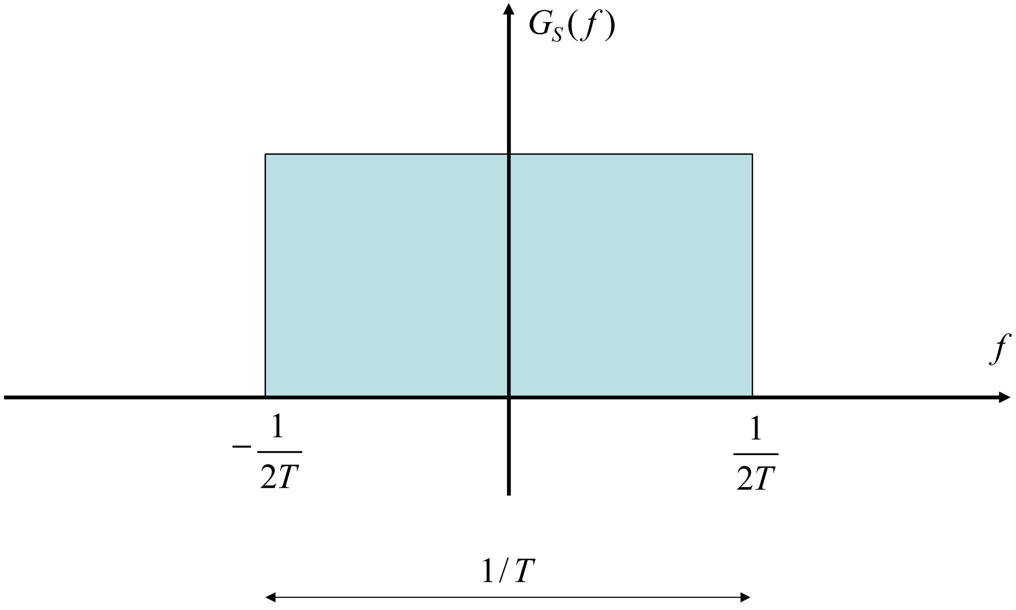
$$p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t / T)}{(\pi t / T)}$$

$$P(f) = \sqrt{T} rect_{\frac{1}{T}}(f)$$

$$|P(f)|^2 = T\, rect_{\frac{1}{T}}(f)$$



Power spectral density (PSD) is: $G_s(f) = A^2 T \, rect_{\frac{1}{T}}(f)$



This is a baseband spectrum

Case 2: Signal Space with Non-Zero Mean

$$\boxed{G_s(f) = S_a(f)\,\frac{|P(f)|^2}{T}}$$

$$S_a(f) = \sum_i M_a(i) e^{-j2\pi f i T}$$

Consider a 1-D signal space not centered at the origin (non-zero mean):

The sequence *a[n]* is stationary with symbols:

- Statistically independent
- Non-zero mean $\mu_a \neq 0$

We have: $M_a(i) = E(a[n+i]a[n]) = \mu_a^2 + \delta_i \sigma_a^2$

$$\delta_i = 1 \text{ for } i{=}0 \qquad \delta_i = 0 \text{ for } i \neq 0$$

SOICT

Therefore: $S_a(f) = \sigma_a^2 + \mu_{a\,i}^2 \sum_{n=-\infty}^{+\infty} e^{-j2\pi fiT} = \sigma_a^2 + \frac{\mu_a^2}{T} \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T}\right)$

Hint: Fourier series

Therefore, the PSD is:

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} + \frac{\mu_a^2}{T^2} \sum_{n=-\infty}^{+\infty} \left| P\left(\frac{n}{T}\right) \right|^2 \delta\left(f - \frac{n}{T}\right)$$

We can have Dirac impulses at frequencies that are multiples of 1/T

Signal Space 4

Consider the signal space:

$$M = \{s_1(t) = 0 \, , \, s_2(t) = AP_T(t)\}$$

This is a 1-D signal space (d=1), with orthonormal basis:

$$B = \left\{ b_1(t) = \frac{1}{\sqrt{T}} P_T(t) \right\}$$

The signal space is transformed into the vector space:

$$M = \{\underline{s_1} = (0) \, , \, \underline{s_2} = (+\alpha)\} \qquad \alpha = A\sqrt{T}$$

Transmitted signal has the form: $s(t) = \sum_{n=0}^{+\infty} a[n] p(t - nT)$

where $a[n] \in \{0, +\alpha\}$

$$p(t) = b_1(t)$$

The sequence $a[n]$ has:

Mean: $\mu_a = \frac{\alpha}{2}$

Variance: $\sigma_a^2 = \frac{\alpha^2}{4}$

We have

$$\left|P(f)\right|^2 = T\left(\frac{\sin(\pi f T)}{(\pi f T)}\right)^2$$

The power spectral density (PSD) is:

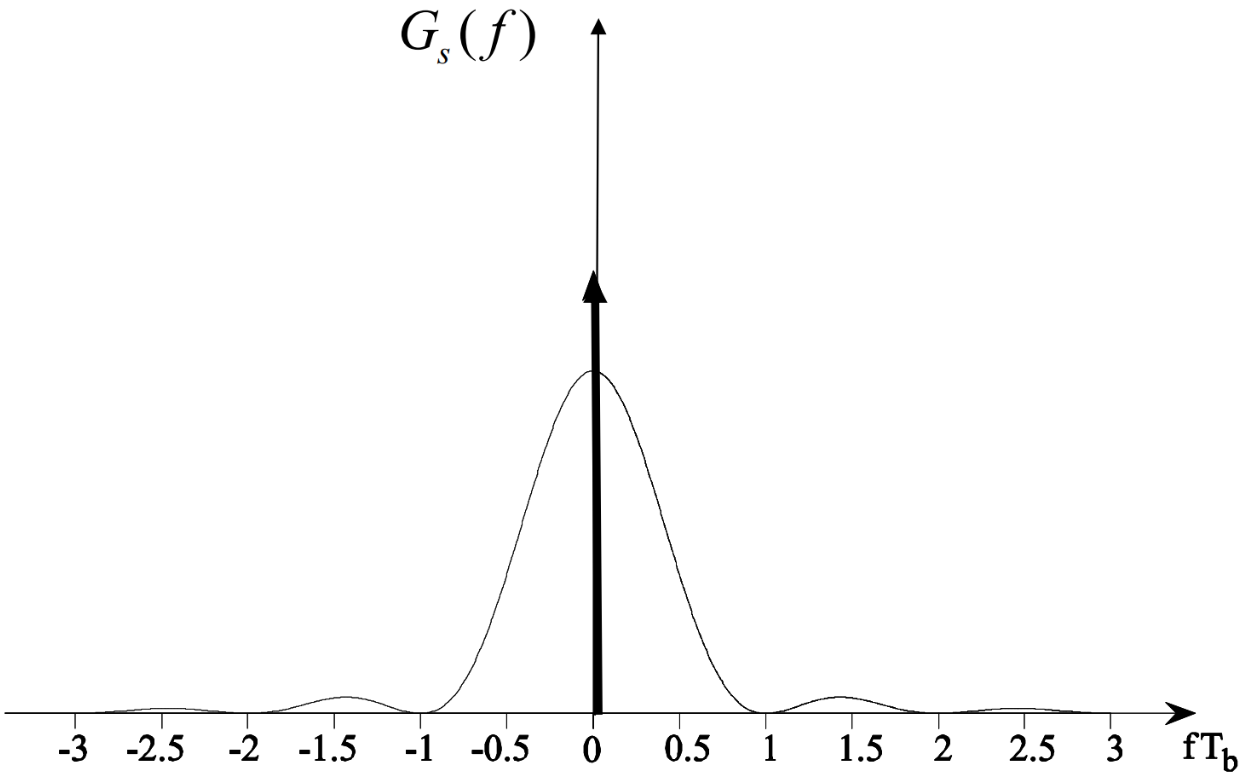
$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} + \frac{\mu_a^2}{T^2} \sum_{n=-\infty}^{+\infty} \left|P\left(\frac{n}{T}\right)\right|^2 \delta\left(f - \frac{n}{T}\right)$$

$$\left|P\left(\frac{n}{T}\right)\right|^2 = T \quad \text{if} \quad i = 0$$

$$\left|P\left(\frac{n}{T}\right)\right|^2 = 0 \quad \text{if} \quad i \neq 0$$

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} + \frac{\mu_a^2}{T^2} T\delta(f)$$

$$G_s(f) = \frac{A^2}{4} T \left(\frac{\sin(\pi f \mathrm{T})}{(\pi f \mathrm{T})} \right)^2 + \frac{A^2}{4} \delta(f)$$



Conclusion

- This is a **baseband spectrum** (energy concentrated around the origin = DC)
- Spectral zeros at frequenciest hat are multiples of $1/T$
- Main lobe has width $2/T$ between $-1/T$ and + $1/T$
- All side lobes have a width of $1/T$ and decreasing intensity
- There is a Dirac impulse at the origin