Introduction to Communications Engineering

IT4593E

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ONE LOVE. ONE FUTURE.

Thông tin chung

- Tên học phần: **Nhập môn kỹ thuật truyền thông**
- Mã học phần: **IT4593E**
- Khối lượng: **2 TC (2-1-0-4)**
- Lý thuyết và bài tập: **10 buổi lý thuyết, 5 buổi bài tập**
- Đánh giá học phần:

  30% QT (kiểm tra + bài tập/project + chuyên cần-quiz)

  70% CK (trắc nghiệm + tự luận)
- Tài liệu tham khảo:
  - Lecture slides
  - Lecture notes
  - Textbooks, ví dụ ***Communication Systems Engineering***, 2nd Edition, by John G. Proakis Masoud Salehi
  - Internet

Lec 08: Nyquist Criterion for NO ISI

Nyquist Criterion

Given the function $x(t) = p(t) * q(t)$

NO ISI condition:

$$x\left(t_0 + iT\right) = 1 \quad \text{if} \quad i = 0$$
$$x\left(t_0 + iT\right) = 0 \quad \text{if} \quad i \neq 0$$

For simplicity, assume $t_0=0$ (discuss later).

The NO ISI condition becomes

$$\boxed{\begin{array}{l} x(iT) = 1 \quad \text{if} \quad i = 0 \\ x(iT) = 0 \quad \text{if} \quad i \neq 0 \end{array}}$$

We call this the Nyquist Criterion in the time domain.

Nyquist Criterion

Nyquist's Second Theorem

If a function $x(t)$ satisfies the Nyquist Criterion in the time domain:

$$x(iT) = 1 \quad \text{if } \ i = 0 \qquad\qquad x(iT) = 0 \quad \text{if } \ i \neq 0$$

We can represent: $$x(t)\sum_i \delta(t - iT) = \delta(t)$$

Consequently (taking Fourier transform): $$X(f) * \frac{1}{T}\left[\sum_n \delta\left(f - \frac{n}{T}\right)\right] = 1$$

This is the Nyquist Criterion in the frequency domain:

$$\boxed{\sum_n X\left(f - \frac{n}{T}\right) = T}$$

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For a function x(t), to check the Nyquist Criterion in the frequency domain, we:

- consider all versions of $X(f)$ centered around frequencies that are multiples of $1/T$
- add these versions

The result must be a constant across the frequency axis:

$$\boxed{\sum_{n} X\left(f - \frac{n}{T}\right) = T}$$

Nyquist Criterion

$$\boxed{\begin{array}{ll} x(iT) = 1 & \text{if } \; i = 0 \\ x(iT) = 0 & \text{if } \; i \neq 0 \end{array}}$$

$$\boxed{\sum_{n} X\left(f - \frac{n}{T}\right) = T}$$

Which functions $x(t)$ satisfy this criterion?

Consider:

- Functions $x(t)$ characterized by a spectrum $X(f)$ (result of Fourier transform) with an infinite frequency domain.
- Functions $x(t)$ characterized by a spectrum $X(f)$ (result of Fourier transform) with a finite frequency domain.

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Functions $x(t)$ characterized by a spectrum $X(f)$ (result of Fourier transform) with an infinite frequency domain.

Many such functions x(t) can be found!!!

Among them, we already know functions of the form $x(t)=p(t)*q(t)$ with:

- $p(t)$ = orthonormal vector with time domain $[0,T]$
- $q(t) = p(T-t)$

definitely satisfy the Nyquist Criterion.

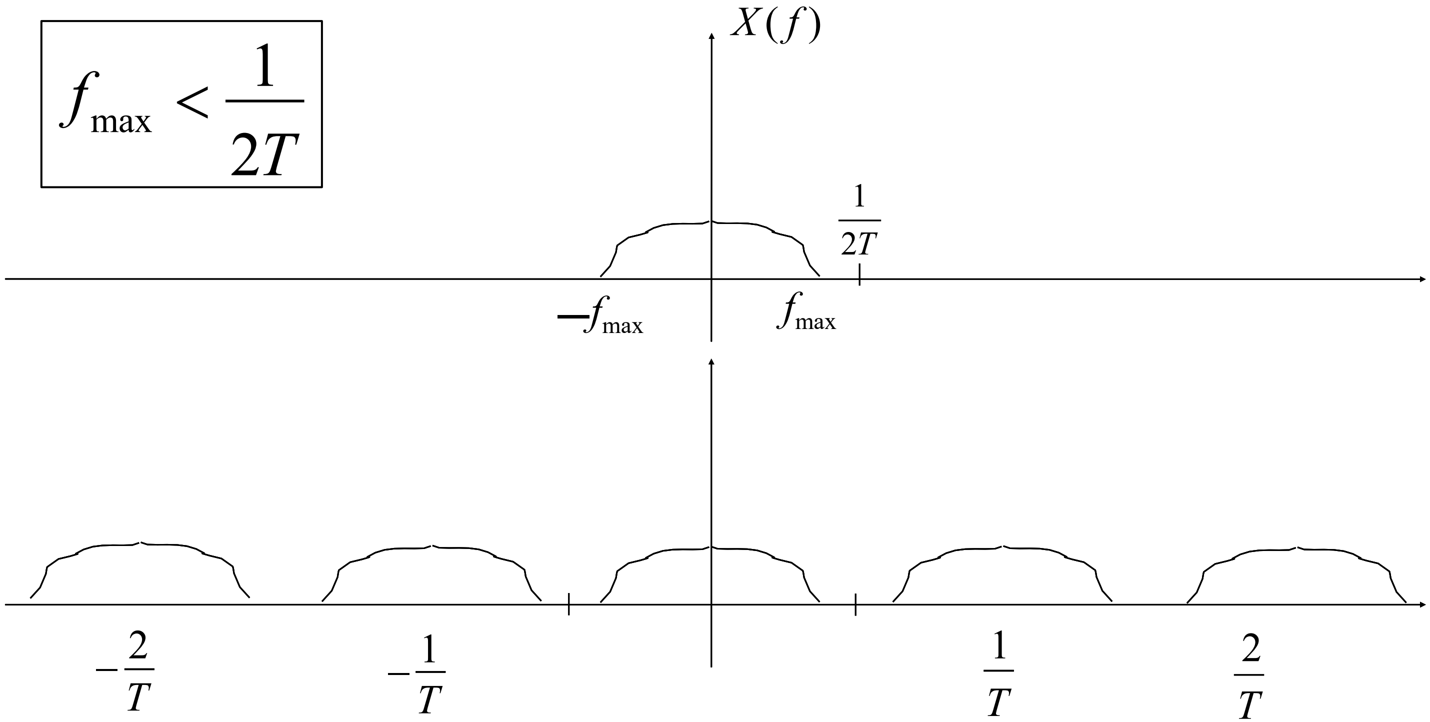
Nyquist Criterion

Functions $x(t)$ characterized by a spectrum $X(f)$ (result of Fourier transform) with a finite frequency domain $[-f_{max}, f_{max}]$

Does such a function x(t) exist?

Nyquist Criterion, Case 1

Case 1: $\boxed{f_{\max} < \dfrac{1}{2T}}$



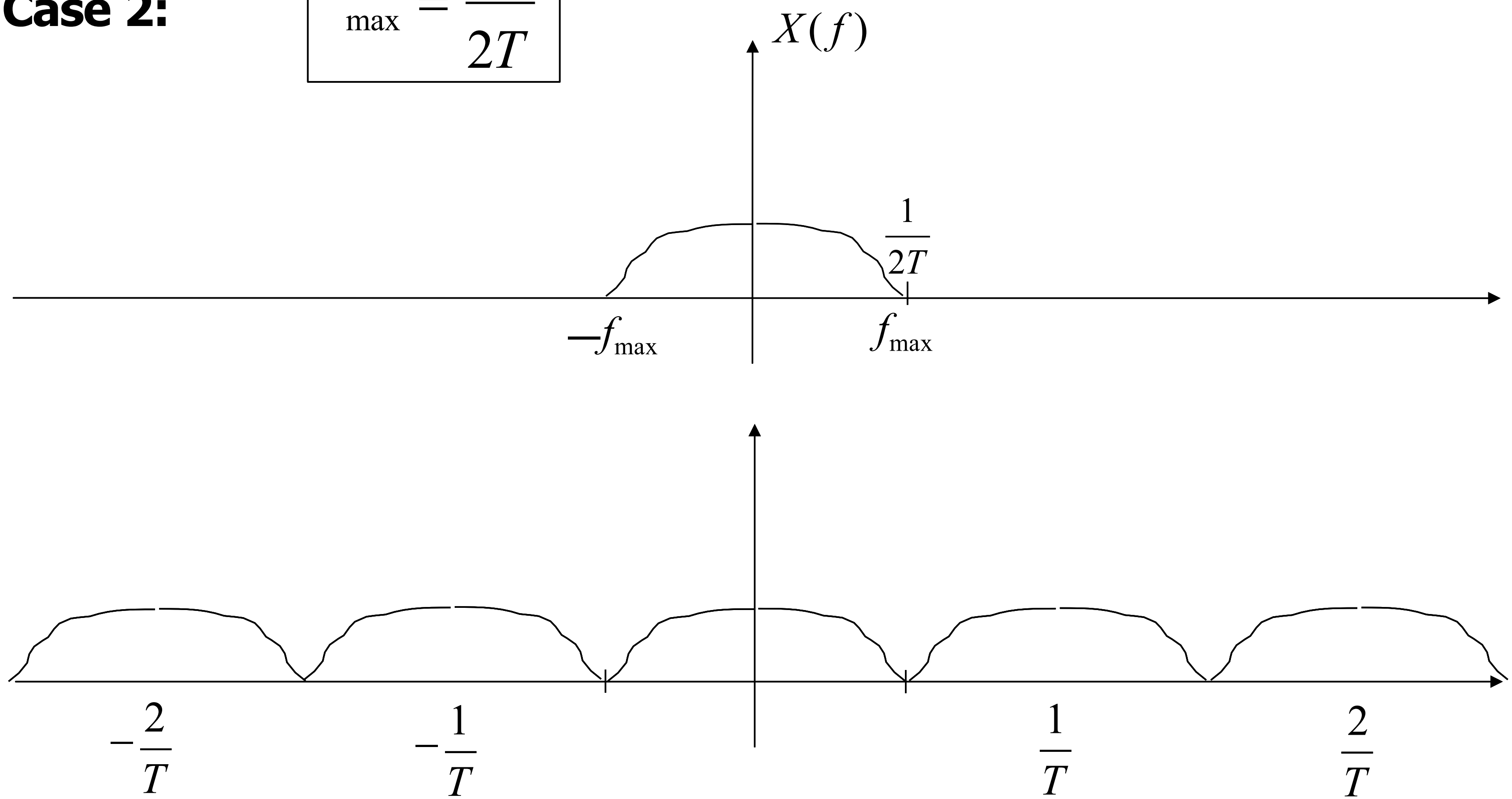
In this case, it is impossible to find a function x(t) satisfying the Nyquist Criterion in the frequency domain, because there are gaps (holes) at frequencies that are multiples of $n/2T$) (When summing shifted versions, the gaps cannot be filled to yield a constant T)

$$\sum_n X\left(f - \frac{n}{T}\right) = T$$

Nyquist Criterion, Case 2

Case 2:

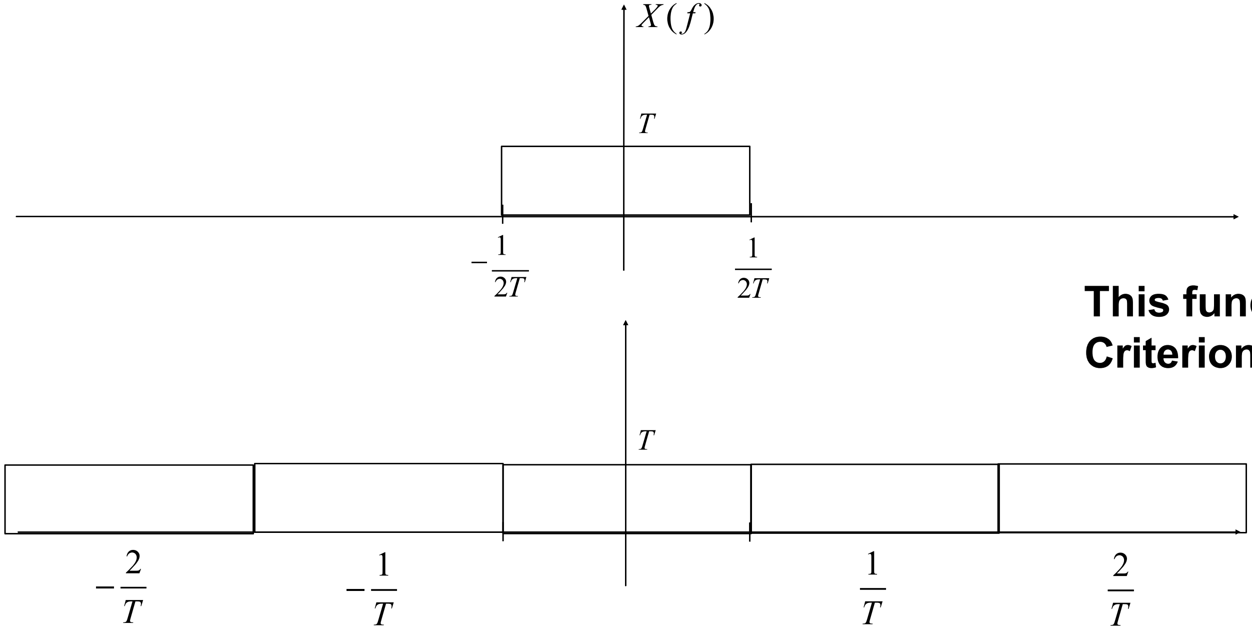
$$\boxed{\text{max} = \frac{1}{2T}}$$



Nyquist Criterion, Case 2

Case 2: $$\boxed{f_{\max} = \frac{1}{2T}}$$

One solution: the **ideal low pass filter**

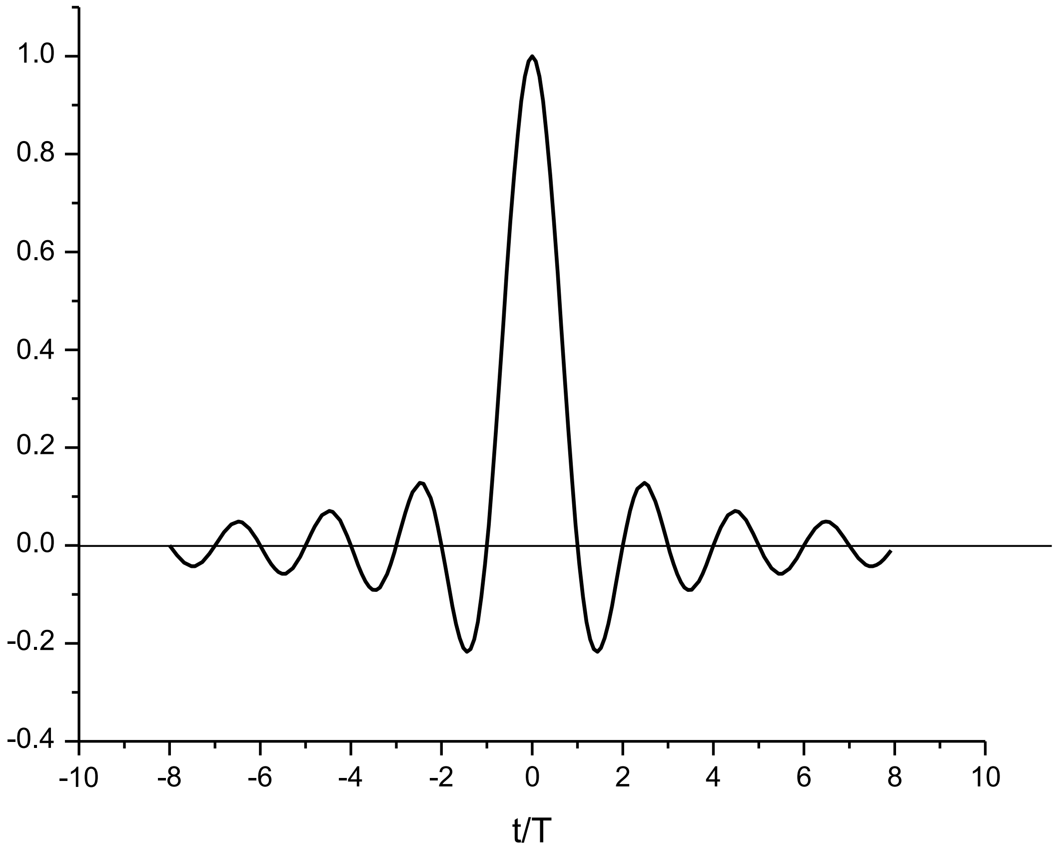


This function sastifies the Nyquist Criterion in the frequency domain:

$$\sum_n X\left(f - \frac{n}{T}\right) = T$$

Ideal Low Pass Filter

Ideal low pass filter: $$\boxed{x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)}}$$

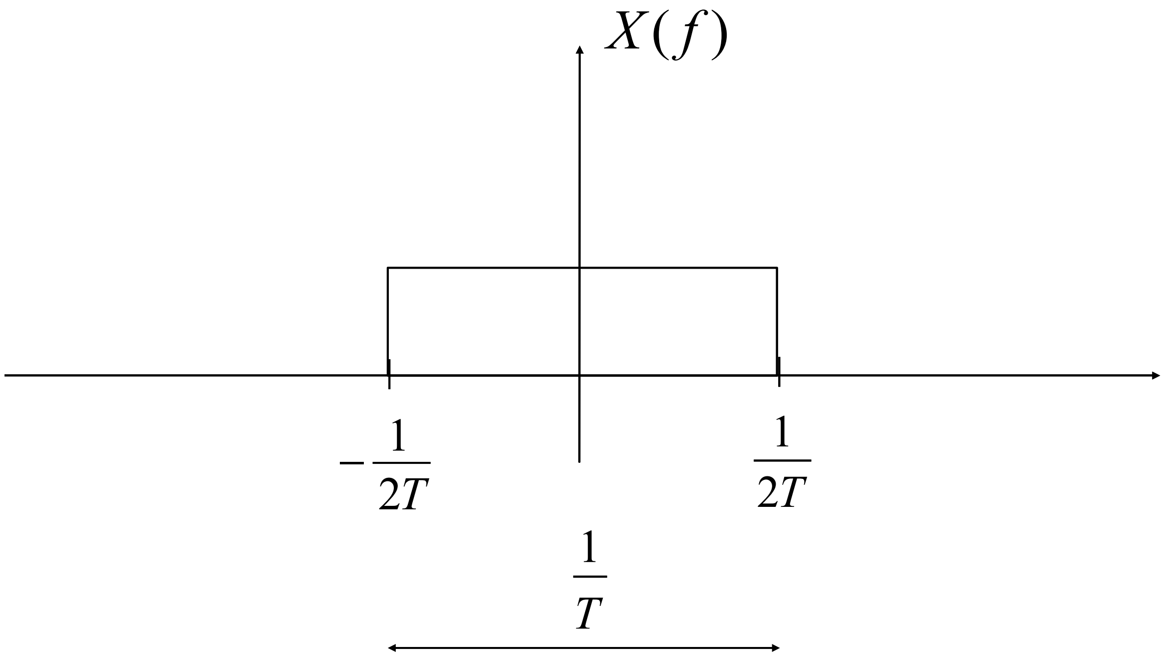


Sastifies the Nyquist Criterion in the time domain:

$$x(iT) = 1 \quad \text{if} \quad i = 0$$

$$x(iT) = 0 \quad \text{if} \quad i \neq 0$$

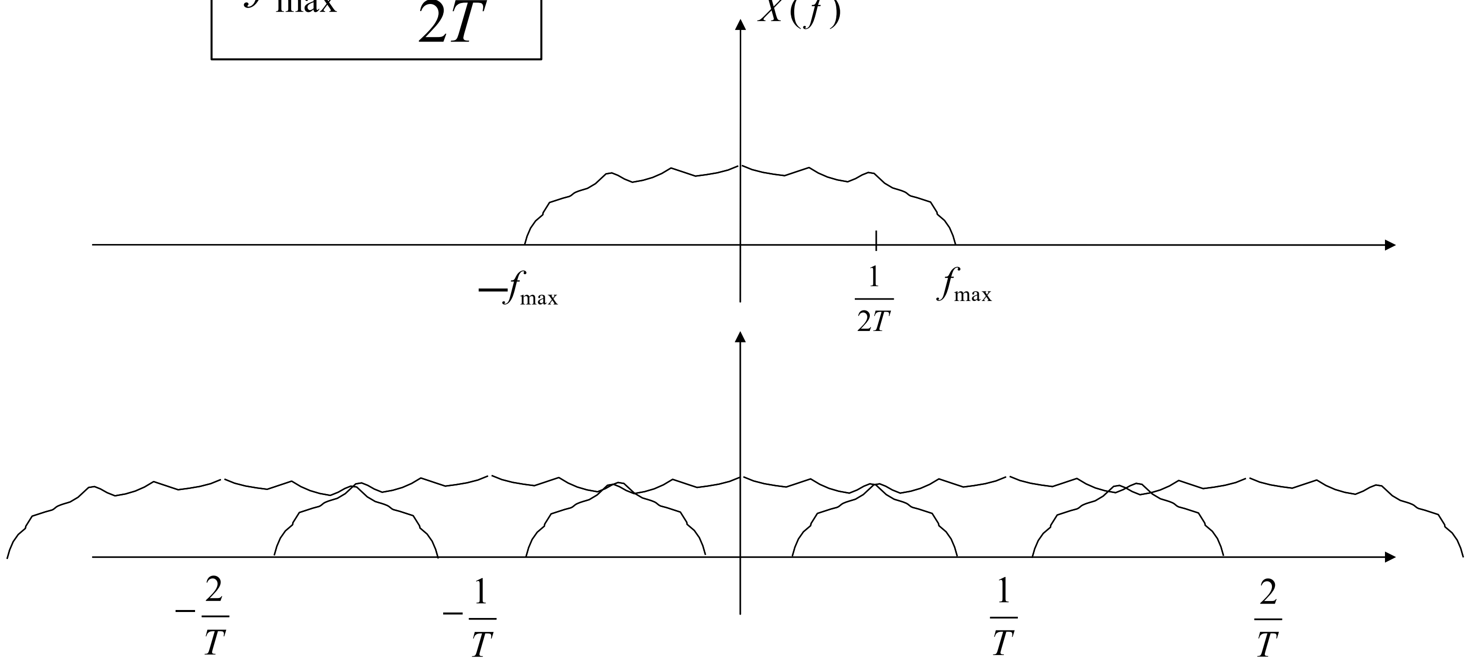
Frequency domain:



This is a waveform that satisfies the criterion with optimal occupied bandwidth.

Nyquist Criterion, Case 3

Case 3: $\boxed{f_{\max} > \dfrac{1}{2T}}$



There exist many functions $x(t)$ satisfying the Nyquist Criterion condition in this case. (The excess bandwidth allows shaping $X(f)$ so that the sum of shifted versions is constant).

$$\sum_{n} X\left(f - \frac{n}{T}\right) = T$$

Raised Cosine Filters

Example (very important in applications)

Raised Cosine Filters

$$\boxed{x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)} \frac{\cos(\alpha \pi t / T)}{1 - (2\alpha t / T)^2}}$$

"Roll-off" factor:

$$0 \le \alpha \le 1$$

Note:

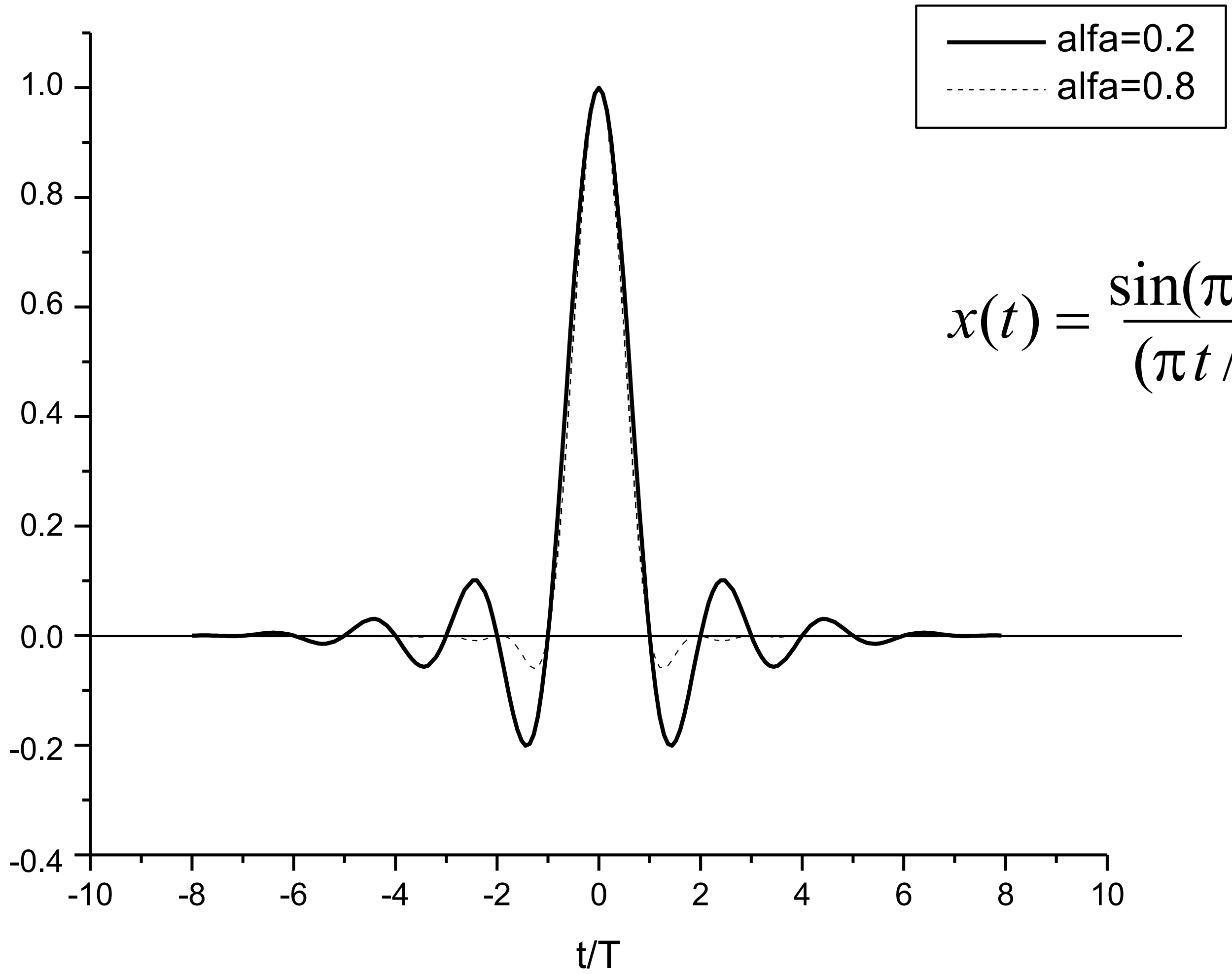
1. Definitely satisfies the Nyquist Criterion in the time domain:

$$x(iT) = 1 \quad \text{if} \quad i = 0$$

$$x(iT) = 0 \quad \text{if} \quad i \neq 0$$

2. With α=0 we have the ideal low pass filter

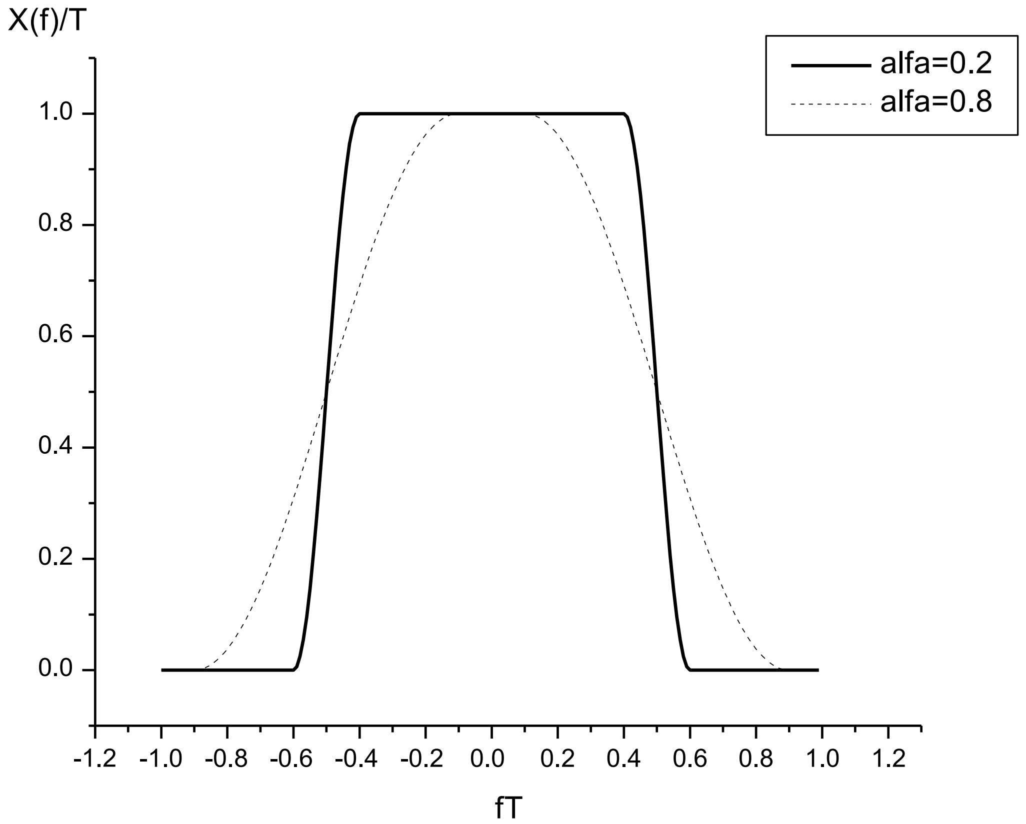
Raised Cosine Filters



$$x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)} \frac{\cos(\alpha \pi t / T)}{1 - (2\alpha t / T)^2}$$

Raised cosine filters

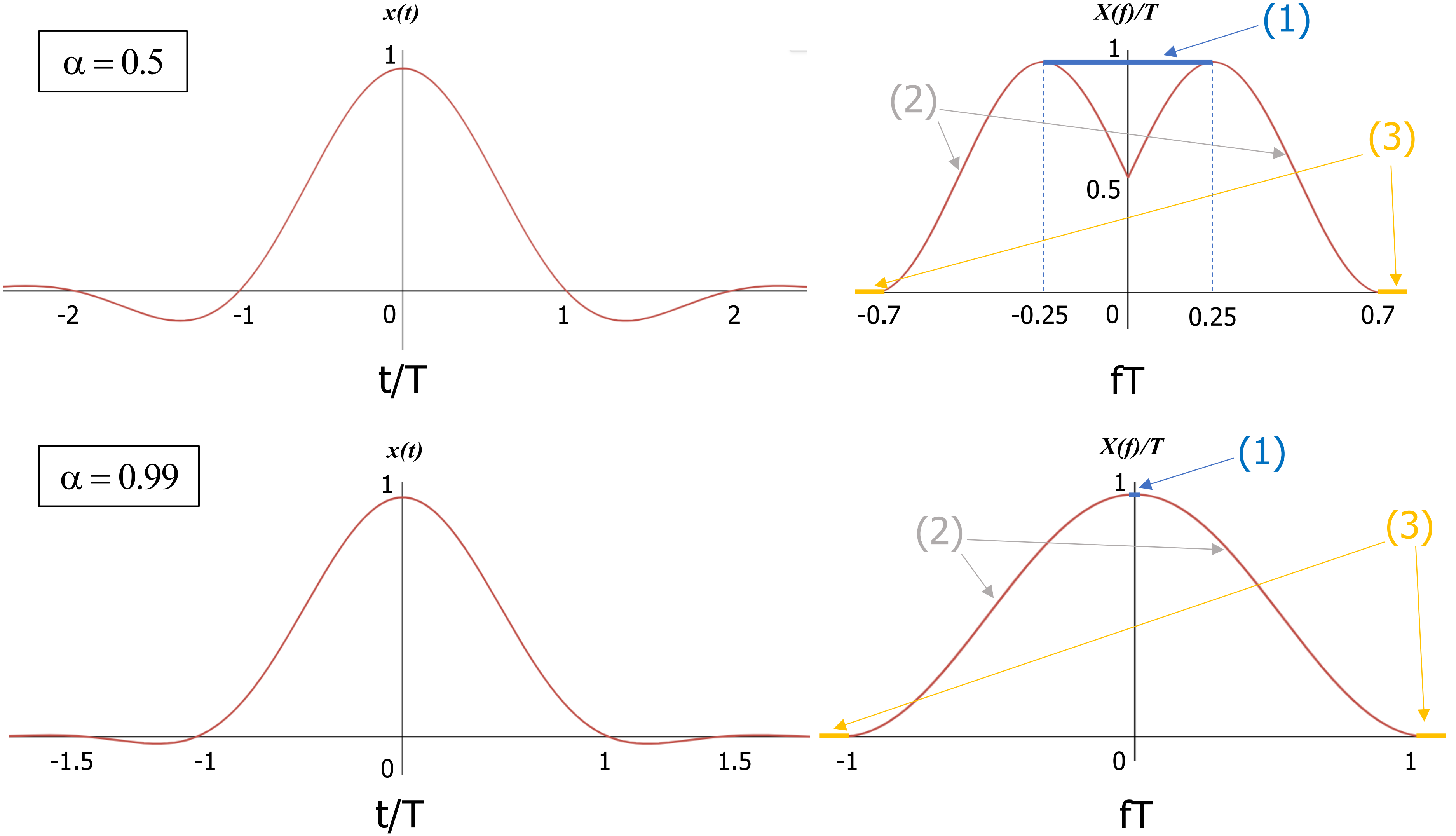
Frequency response:



$$X(f) = T \quad \text{for} \quad |f| \le \frac{(1-\alpha)}{2T}$$

$$X(f) = \frac{T}{2}\left[1 - \sin\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1}{2T}\right)\right)\right] \quad \text{for} \quad \frac{(1-\alpha)}{2T} \le |f| \le \frac{(1+\alpha)}{2T}$$

$$X(f) = 0 \quad \text{for} \quad |f| \le \frac{(1+\alpha)}{2T}$$



$$x(t) = \frac{\sin(\pi t/T)}{(\pi t/T)} \frac{\cos(\alpha\pi t/T)}{1-(2\alpha t/T)^2}$$

$X(f) = T$	for	$\lvert f \rvert \le \frac{(1-\alpha)}{2T}$	(1)
$X(f) = \frac{T}{2}\left[1 - \sin\left(\frac{\pi T}{\alpha}\left(\lvert f \rvert - \frac{1}{2T}\right)\right)\right]$	for	$\frac{(1-\alpha)}{2T} \le \lvert f \rvert \le \frac{(1+\alpha)}{2T}$	(2)
$X(f) = 0$	for	$\lvert f \rvert \le \frac{(1+\alpha)}{2T}$	(3)

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Raised Cosine Filters

Raised cosine filters have the time domain representation

$$x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)} \frac{\cos(\alpha \pi t / T)}{1 - (2\alpha t / T)^2}$$

In the Case of Time Delay

Until now, we have considered the case with $t_0=0$

$$\rho[n] = y(t_0 + nT) \quad \text{with} \quad t_0 = 0 \quad \Rightarrow \quad \begin{array}{ll} x(iT) = 1 & \text{if} \quad i = 0 \\ x(iT) = 0 & \text{if} \quad i \neq 0 \end{array}$$

For a function $x(t)$ satisfying the criterion with $t_0=0$, the function $x'(t) = x(t-t_0)$ satisfies the condition for any t_0

(Note that, at the receiver, the symbol synchronization circuit can always determine t_0 accurately)

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Transmit (TX) and Receive (RX) Filters

We examine the properties of the function $x(t)$, where

$$x(t)=p(t)*q(t)$$

The matched filter characterized by $q(t)$ is defined as:

$$q(t)=p(T-t)$$

$$Q(f) = P(f)^{*}\, e^{-j2\pi fT}$$

Transmit (TX) and Receive (RX) Filters

If $x(t)$ is an ideal low-pass filter, what are the representations of the filters $p(t)$ and $q(t)$?

If $p(t)$ is an even function, $p(t)=p(-t)$

We have $$q(t) = p(T-t)=p(t-T)$$

We can $$q(t) = p(t)$$

The delay T is determined by the synchronization circuits.

We have $$X(f) = P(f)\ Q(f)$$

If $q(t)=p(t)$ hence $Q(f)=P(f)$, and

$$X(f) = P(f)^2 \quad \rightarrow \quad P(f) = Q(f) = \sqrt{X(f)}$$

We decompose the function $x(t)$ into two similar functions, referred to as the transmit filter $p(t)$ and the receive filter $q(t)$

Ideal Low-Pass TX Filter (time domain)

With the ideal low-pass filter: $$x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)}$$

We have: $$\boxed{p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t / T)}{\left(\pi t / T\right)}}$$

Root Raised Cosine (RRC) TX Filter (time domain)

Raised cosine filter: $$x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)} \frac{\cos(\alpha \pi t / T)}{1-(2\alpha t / T)^2}$$

We have:

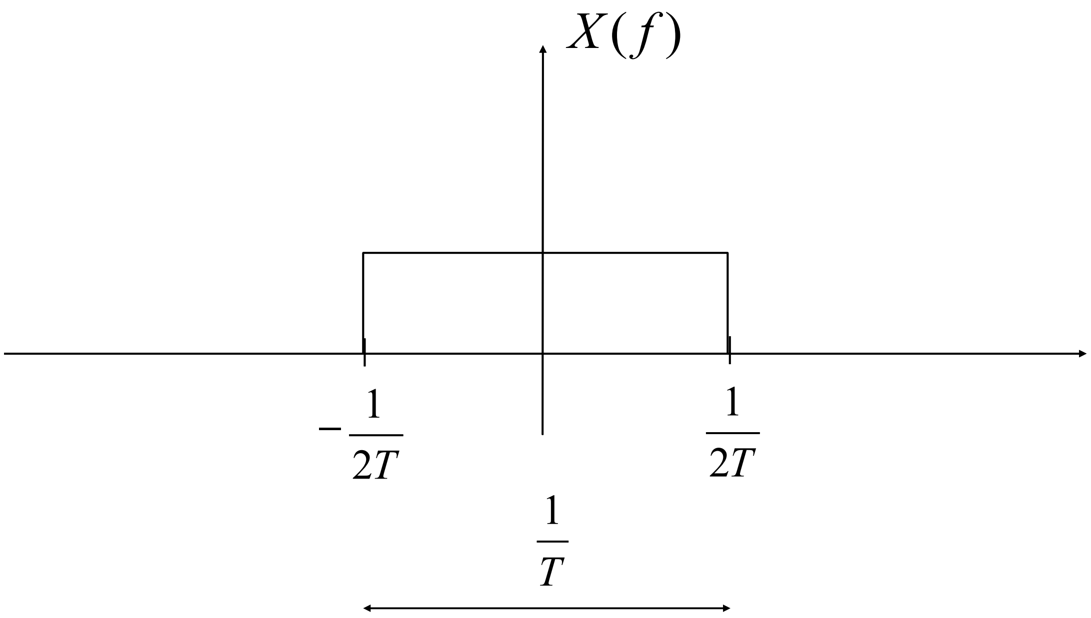
Root Raised Cosine (RRC) filter

$$\boxed{p(t) = \frac{1}{\sqrt{T}} \frac{\sin\left(\pi \frac{t}{T}(1-\alpha)\right) + 4\alpha \frac{t}{T} \cos\left(\pi \frac{t}{T}(1+\alpha)\right)}{\pi \frac{t}{T}\left(1-\left(4\alpha \frac{t}{T}\right)^2\right)}}$$

Ideal Low-Pass TX Filter (freq domain)

TX filter $p(t)$: ideal low-pass filter

Minimum occupied bandwidth: $\boxed{\frac{1}{2T}}$



Root Raised Cosine (RRC) TX Filter (freq domain)

TX filter $p(t)$: root raised cosine filter

Occupied bandwidth: $\boxed{\frac{1}{2T}(1+\alpha)}$