Ordinary differential equations

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May 15, 2023

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Motivation

Mathematical models of many phenomena in physics, biology, economy,. . . result in ordinary differential equations.

Models of population growth

Reasonable assumption: the population grows at a rate proportional to the size of the population, under ideal conditions: unlimited environment, adequate nutrition, absence of predators, immunity from disease.

$$\frac{dP}{dt} = kP,$$

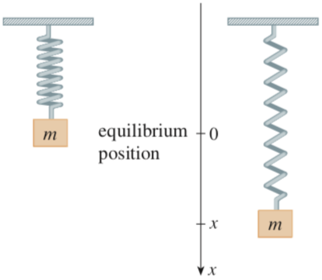
P: the number of individuals, t: time, k: proportionality constant.
$P(t) > 0$ for all t.
More realistic model: a given environment has limited resource

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

M: carrying capacity.

Spring – mass system

Assumption: no external resisting forces (air resistance or friction)



Newton's second law: force equals mass times acceleration

$$m\frac{d^2x}{dt^2} = -kx$$

x: the displacement from the equilibrium, k: stiffness (spring constant).

Psychologists interested in learning theory study **learning curves**. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative dP/dt represents the rate at which performance improves.

(a) When do you think P increases most rapidly? What happens to dP/dt as t increases? Explain.

(b) If M is the maximum level of performance of which the learner is capable, explain why the differential equation

$$\frac{dP}{dt} = k(M - P) \qquad k \text{ a positive constant}$$

is a reasonable model for learning.

(c) Make a rough sketch of a possible solution of this differential equation.

Further examples

1. Falling objects: $v' + \dfrac{k}{m}v = g$, g: graviational constant, kv: air resistance force, v: velocity.
2. Electrical circuits: $\dfrac{dI}{dt} + \dfrac{R}{L}I = \dfrac{E}{L}$, I: current, R: resistance, L: inductor, E: electromotive force.
3. Oscillation equation of a pendulum: $x''(t) + \dfrac{g}{l}\sin x = 0$, l: length of a string.
4. Continuously compounded interest.

Basic concepts

Definition

An ordinary differential equation is an equation involving an unknown function (of one variable) and its derivatives.

$$F(x, y, y', y'', \dots, y^{(n)}) = 0,$$

where x is a variable, $y = y(x)$ is the function in search, and $y', y'', \dots, y^{(n)}$ are the derivatives of y.

Definition

The order of an ODE is the order of the highest derivative appearing in the equation.

Example

1. $y''' - 3xy' + y^2 = 0.$
2. $y'y'' - y^3 \cos x + xy = 0.$
3. $xy'' - (1 + x^2)y' + 5y = \tan x.$
4. $\sin y \dfrac{dy}{dx} - 2x^3 y + x^4 = 0.$
5. $e^x \dfrac{d^3 u}{dx^3} + 2\left(\dfrac{du}{dx}\right)^2 = x^3.$

Definition

A linear ODE is an ODE where F is linear with respect to $y, y', y'', \ldots, y^{(n)}$.
General form

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x),$$

where $a_1(x), \ldots, a_{n-1}(x), a_n(x), f(x)$ are given functions.

Definition

A solution of an ODE is a function $y = y(x)$, $x \in I$, which satisfies the equation identically for all $x \in I$.

General solution of an ODE is the set of all solutions depending on parameters, which can be found once additional conditions are given.

A particular solution of an ODE is any solution obtained from the general solution by specifying values of the parameters.

A singular solution of an ODE is a solution that cannot be obtained from the general solution.

Example

1. $y' = f(x)$, the general solution is $y = \int f(x)dx + C$.
2. Oscillation equation of a spring $mx'' + kx = 0$.
   - General solution $x(t) = C_1 \cos \omega t + C_2 \sin \sin \omega t$.
   - Observe at the time of release $t = 0$: e.g. $x(0) = A_0$, $x'(0) = 0$.
     We obtain a particular solution $x(t) = A_0 \cos(\omega t)$, $C_1 = A_0, C_2 = 0$.

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General form: $f(x, y, y') = 0$ or $y' = f(x, y)$.
The solutions can be given in explicit / implicit forms, or by parametrization

- Explicitly: general solution $y = \varphi(x, C)$; particular solution $y = \varphi(x, C_0)$.
- Implicitly: general integral $\Phi(x, y, C) = 0$; particular integral $\Phi(x, y, C_0) = 0$.
- Parametrization: $\begin{cases} x = x(t, C) \\ y = y(t, C). \end{cases}$

Existence and Uniqueness theorem

Initial value problem (Cauchy problem)

$$\begin{cases} y' = f(x, y), \quad x \in U_\varepsilon(x_0), \\ y(x_0) = y_0. \end{cases} \tag{1}$$

Theorem

Assume that $f(x, y)\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ *is* **continuous** *on* D, *and* $(x_0, y_0) \in D$. *Then, in some interval* $(x_0 - h, x_0 + h)$, *there exists one solution* $y = y(x)$ *of the IVP* (1).

If additionally $\dfrac{\partial f}{\partial y}(x, y)$ **is continuous** *in* D *then the solution is* **unique**.

General form of first order ODEs:

$$F(x, y, y') = 0.$$

1. Equations without y: $F(x, y') = 0$.
2. Equations without x: $F(y, y') = 0$.

Equations without y: $F(x, y') = 0$

Assume that we can transform the given equation to one of the following forms:

- $y' = f(x)$. It implies $y = \int f(x)dx$.
- $x = f(y')$. We look for solutions given by parametrization

$$\begin{cases} y' = t \\ x = f(t) \end{cases} \Rightarrow \begin{cases} x = f(t) \\ dy = y'(x)x'(t)dt = tf'(t)dt \end{cases} \Rightarrow \begin{cases} x = f(t) \\ y = \int tf'(t) \end{cases}$$

Example

Solve the equation $x = (y')^2 + 4y' - 3$.

Set $y'(x) = t \Rightarrow \begin{cases} x = t^2 + 4t - 3 \\ dy = y'(x)dx = t(2t+4)dt = (2t^2 + 4t)dt \end{cases}$

$$\Rightarrow \begin{cases} x = t^2 + 4t - 3 \\ y = \dfrac{2t^3}{3} + 2t^2 + C. \end{cases}$$

Equations without x: $F(y, y') = 0$

Assume that we can transform the given equation to one of the following forms:

- $y' = f(y) \Rightarrow dx = \dfrac{dy}{f(y)} \Rightarrow x = \displaystyle\int \frac{dy}{f(y)}$.
  Besides that, constant solutions $y = C$ which satisfy $f(C) = 0$.
- $y = f(y')$. We look for solutions given by parametrization

$$\begin{cases} y' = t \\ y = f(t) \end{cases} \Rightarrow \begin{cases} y = f(t) \\ dx = \dfrac{dy}{y'(x)} = \dfrac{f'(t)}{t} dt \end{cases} \Rightarrow \begin{cases} y = f(t) \\ x = \displaystyle\int \frac{f'(t)}{t} dt. \end{cases}$$

Separable equations

General form: $\boxed{f(x)dx = g(y)dy}$

Integrating both sides of the equation:

$$\int f(x)dx = \int g(y)dy \Rightarrow F(x) = G(y) + C,$$

where $F(x)$, $G(y)$ are antiderivatives of $f(x)$, $g(y)$ respectively.

Example (20182)

Solve the ODE $y' = 2xy^2$.

- $y = 0$ is a solution of the equation.
- $y \neq 0$, the equation becomes $\dfrac{dy}{y^2} = 2xdx$. Integrating both sides, we get

$$\int \frac{dy}{y^2} = \int 2xdx \Rightarrow -\frac{1}{y} = x^2 + C.$$

Hence, the solutions are $y = -\dfrac{1}{x^2 + C}$ and $y = 0$.

Example (20181)

Solve the following problem $y' = 3 + xy + x + 3y, y(0) = 1$.

The equation is equivalent to $\dfrac{dy}{dx} = (x+3)(y+1)$.

- $y + 1 = 0 \Rightarrow y = -1$ does not satisfy the condition $y(0) = 1$, hence it is not a solution.
- $y + 1 \neq 0$, ta có

$$\begin{aligned}\frac{dy}{y+1} = (x+3)dx \Rightarrow \int \frac{dy}{y+1} &= \int (x+3)dx \\ &\Rightarrow \ln|y+1| = \frac{x^2}{2} + 3x + \ln|C|, C \neq 0 \\ &\Rightarrow y + 1 = Ce^{\frac{x^2}{2}+3x}.\end{aligned}$$

Plugging the condition in the solution, we obtain $C = 2$.

Hence, the solution is $y + 1 = 2e^{\frac{x^2}{2}+3x}$.

Homogeneous equations

General form: $\boxed{\dfrac{dy}{dx} = f(x, y)}$ where $f(tx, ty) = f(x, y)$.

Or $y' = g\left(\dfrac{y}{x}\right)$

We transform it into a separable equation as follows:

- Making a substitution $y = ux$.
- The resulting equation is $x\dfrac{du}{dx} = g(u) - u \Rightarrow u(x)$.
- Substituting back we get $y(x)$.

Example (20181)

Solve the following problem $y' = \dfrac{-x+2y}{x}$, $y(1) = 2$.

Set $y = x.u$, the equation becomes
$xu' + u = -1 + 2u \Leftrightarrow x\dfrac{du}{dx} = u - 1$.
$y(1) = 2$ so $u(1) = \dfrac{y(1)}{1} = 2$. $u = 1$ does not satisfy the condition, hence it is not a solution of the problem.
$u \neq 1$, the equation can be rewritten as

$$\frac{du}{u-1} = \frac{dx}{x} \Rightarrow \int \frac{du}{u-1} = \int \frac{dx}{x}$$
$$\Rightarrow \ln|u-1| = \ln|x| + \ln|C|, (C \neq 0) \Rightarrow \frac{y}{x} - 1 = Cx.$$

Using $y(1) = 2$, we get $C = 1$.The solution is $y = x(x+1)$.

Exact differential equations

General form

$$\boxed{P(x,y)dx + Q(x,y)dy = 0,}$$

where $P(x,y), Q(x,y)$ are continuous functions and have continuous first partial derivatives on some rectangle D of the plane and $\frac{\partial \mathbf{P}}{\partial \mathbf{y}} = \frac{\partial \mathbf{Q}}{\partial \mathbf{x}}$.

Under these conditions, we can find a function $u(x,y)$ such that

$$P = \frac{\partial u}{\partial x}, Q = \frac{\partial u}{\partial y}.$$

The equation reads $du = 0$, hence the general solution is given implicitly by:

$$u(x,y) = C.$$

$u(x,y)$ is given by:

$$u(x,y) = \int_{x_0}^{x} P(x,y_0)dx + \int_{y_0}^{y} Q(x,y)dy$$
$$= \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy.$$

where $(x_0, y_0) \in D$.

Example (CK20181)

Solve the ODE $(3x^2 - 6xy)dx - (3x^2 + 4y^3)dy = 0$.

- $P(x, y) = 3x^2 - 6xy$, $Q = -3x^2 - 4y^3$. $P'_y = Q'_x = -6x \Rightarrow$ exact differential equation.
- The general integral is given by:

$$
\begin{aligned}
u(x, y) &= \int_0^x P(x, 0)dx + \int_0^y Q(x, y)dy = C \\
&\Leftrightarrow \int_0^x 3x^2 dx + \int_0^y (-3x^2 - 4y^3)dy = C \\
&\Leftrightarrow x^3 - (3x^2 y + y^4) \Big|_0^y = C \\
&\Leftrightarrow x^3 - 3x^2 y - y^4 = C.
\end{aligned}
$$

We can also find $u(x, y)$ by solving the system
$$\begin{cases} u'_x = 3x^2 - 6xy \\ u'_y = -3x^2 - 4y^3. \end{cases}$$
The first equation yields that

$$u = \int (3x^2 - 6xy)dx = x^3 - 3x^2y + C(y).$$

Plugging into the second equation, we get

$$u'_y = -3x^2 + C'(y) = -3x^2 - 4y^3,$$

we obtain $C'(y) = -4y^3 \Rightarrow C(y) = -y^4$.
Hence, $u = x^3 - 3x^2y - y^4$, the general integral is
$x^3 - 3x^2y - y^4 = C$.

Integrating factor

In general, the equation $P(x, y)dx + Q(x, y)dy = 0$ is not an exact DE.
A function $\alpha(x, y)$ is called integrating factor if

$$\alpha(x, y)[P(x, y)dx + Q(x, y)dy] = 0$$

is an exact DE, which means $\dfrac{\partial(\alpha P)}{\partial y} = \dfrac{\partial(\alpha Q)}{\partial x}$.

Particular cases of the integrating factor

- If $\dfrac{Q'_x - P'_y}{Q} = \varphi(x) \Rightarrow \alpha(x, y) = \alpha(x) = e^{-\int \varphi(x)dx}$
- If $\dfrac{Q'_x - P'_y}{P} = \psi(y) \Rightarrow \alpha(x, y) = \alpha(y) = e^{\int \psi(y)dy}$

Example (20182)

Solve the problem $e^y dx + (9y + 4xe^y)dy = 0, y(1) = 0.$

$P = e^y, Q = 9y + 4xe^y \Rightarrow \dfrac{Q'_x - P'_y}{P} = \dfrac{4e^y - e^y}{e^y} = 3$ hence, an integrating factor is $\alpha(y) = e^{3y}$.

Multiplying through by e^{3y}, we obtain
$e^{4y}dx + (9ye^{3y} + 4xe^{4y})dy = 0$ (exact equation).

$$u(x,y) = \int_{x_0}^{x} P(x,y_0)dx + \int_{y_0}^{y} Q(x,y)dy = C$$

In particular $y(x_0) = y_0$, we get $C = 0$.
The integral of the problem is

$$\int_{1}^{x} dx + \int_{0}^{y} (9ye^{3y} + 4xe^{4y})dy = 0.$$

Linear equations

General form

$$\boxed{y' + p(x)y = q(x),}$$

where $p(x), q(x)$ are continuous function on $I \subset \mathbb{R}$.

$$y' + p(x)y = q(x) \Rightarrow \big(p(x)y - q(x)\big)dx + dy = 0.$$

- $P = p(x)y - q(x), Q = 1, \dfrac{Q'_x - P'_y}{Q} = -p(x)$, an integrating factor is $\alpha(x) = e^{\int p(x)dx}$.
- Multiplying both sides by $\alpha(x)$, we get

$$\begin{aligned}(y' + p(x)y)e^{\int p(x)dx} &= q(x)e^{\int p(x)dx} \\ \Leftrightarrow \left(ye^{\int p(x)dx}\right)' &= q(x)e^{\int p(x)dx} \\ &\Rightarrow y = e^{-\int p(x)dx}\Big(\int q(x)e^{\int p(x)dx}dx + C\Big).\end{aligned}$$

The general solution is given by

$$\boxed{y = \left(\int q(x) e^{\int p(x)dx} dx + C \right) e^{-\int p(x)dx}}.$$

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Example (20182)

Solve the ODE $y' - 2y \tan x = 2 \sin 2x$.

$$y = e^{\int 2\tan x dx} \left(\int 2 \sin 2x e^{-\int 2\tan x dx} dx + C \right)$$

$$\Rightarrow y = \frac{1}{\cos^2 x} \left(\int 2 \sin 2x \cos^3 x dx + C \right)$$

$$\Rightarrow y = \frac{C - \cos^4 x}{\cos^2 x}.$$

Structure of the general solutions of linear equations

The equation $y' + p(x)y = q(x)$ has the general solution

$$y = \left(\int q(x)e^{\int p(x)dx} dx + C \right) e^{-\int p(x)dx} = y^* + \bar{y},$$

where

- $\bar{y} = Ce^{-\int p(x)dx}$ is the general solution of the corresponding homogeneous equation $y' + p(x)y = 0$.
- y^* is a particular solution of the given inhomogeneous equation.

  Variation of constants:
  We look for y^* of the form $y^* = C(x)e^{-\int p(x)dx}$ and substitute in the equation to find $C(x)$.

Bernoulli equations

General form:

$$\boxed{y' + p(x)y = q(x)y^{\alpha},\ \alpha \neq 0, 1.}$$

1. Verify whether $y = 0$ is a solution.
2. $y \neq 0$, set $v = y^{1-\alpha}$, the equation becomes

$$v' + (1-\alpha)p(x)v = (1-\alpha)q(x),$$

which is a linear equation.

The resulting equation has the general solution given by

$$v = \bar{v} + v^* = \Big(\int (1-\alpha)q(x)e^{\int (1-\alpha)p(x)dx}dx + C\Big)e^{-\int (1-\alpha)p(x)dx}.$$

Substitute back, we have $y = v^{\frac{1}{1-\alpha}}$.

Example

Solve the ODE $y' + xy = x^3y^3$.

Bernoulli equation, $\alpha = 3$.

- $y = 0$ is a solution.
- $y \neq 0$. Dividing both sides by y^3, we obtain $\dfrac{y'}{y^3} + x\dfrac{1}{y^2} = x^3$.

Set $z = \dfrac{1}{y^2} \Rightarrow z' = -\dfrac{2y'}{y^3}$, the equation becomes

$$-\frac{z'}{2} + xz = x^3 \Leftrightarrow z' - 2xz = -2x^3.$$

(linear equation in z).
The general solution is

$$\begin{aligned}\frac{1}{y^2} &= e^{\int 2xdx}\left(C - 2\int x^3 e^{-\int 2xdx} dx\right)\\ &= e^{x^2}\left(C - \int x^2 e^{-x^2} d(x^2)\right)\\ &= e^{x^2}\left(C + (x^2+1)e^{-x^2}\right).\end{aligned}$$

Solve the following ODEs

1. $xy' = y \ln \dfrac{y}{x}$.
2. $y' - 2y \tan x + y^2 \sin^2 x = 0$.
3. $2y'\sqrt{x} = \sqrt{1 - y^2}$.
4. $(e^x + y + \sin y)dx + (e^y + x + x \cos y)dy = 0$.
5. $y' = \sin(y - x - 1)$.
6. $(x^2 + y)dx + (x - 2y)dy = 0$.
7. $y' + y \cos x = \sin x \cos x$.
8. $y' + \dfrac{y}{x} = x^2 y^4$.
9. $\tan y dx - x \ln x dy = 0$.