Lời giải bài tập tuần 4

1 Bài tập phép tịnh tiến và phân thức đơn giản

1.

a. $\mathcal{L}^{-1}\left\{\dfrac{3s}{s^3-1}\right\}(t)$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}(t) - \mathcal{L}^{-1}\{\dfrac{s-1}{s^2+s+1}\right\}(t)$

$= e^t - \mathcal{L}^{-1}\left\{\dfrac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}} - \dfrac{\frac{3}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right\}(t)$

$= e^t - e^{-\frac{1}{2}t}.cos(\dfrac{\sqrt{3}}{2e}t) + \sqrt{3}.e^{\frac{-1}{2}t}.sin(\dfrac{\sqrt{3}}{2}t)$

b. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2-4s+5)^2}\right\}(t)$

$= \mathcal{L}^{-1}\left\{\dfrac{1}{((s-2)^2+1)^2}\right\}(t)$

$=^{2t} .\dfrac{1}{2}.(sintt - tcost)$

c. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s-2)^2+9}\right\}(t)$

$= \mathcal{L}^{-1}\left\{\dfrac{s-2}{(s-2)^2+3^2} - \dfrac{2}{(s+2)^2+3^2}\right\}(t)$

$= e^{2t}.cos3t - \dfrac{2}{3}.e^{2t}.sin3t$

2.

a. $(t) = e^{-2t}.sin(3\pi t)$

$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\left\{e^{-2t}.sin(3\pi t)\right\}(s)$

$= \dfrac{3\pi}{(s+2)^2+9\pi^2}$ với $s > -2$

b. $f(t) = e^{\frac{-t}{2}}.cos2\left(t - \dfrac{\pi}{8}\right)$

$$= \frac{1}{\sqrt{2}}.e^{\frac{-t}{2}}.(cos2t + sin2t)$$

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\left\{\frac{1}{\sqrt{2}}.e^{\frac{-t}{2}}.(cos2t + sin2t)\right\}(s)$$

$$= \frac{1}{\sqrt{2}}.\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+4} + \frac{1}{\sqrt{2}}.\frac{2}{(s+\frac{1}{2})^2+4} \quad (s > -\frac{1}{2})$$

c. $f(t) = e^t.cos^2t$

$$= \frac{1}{2}e^t + \frac{1}{2}.e^t.cos2t$$

$$\Rightarrow \mathcal{L}\{f(t)\}(s) = \mathcal{L}\left\{\frac{1}{2}.e^t + \frac{1}{2}.e^t.cos2t\right\}(s)$$

$$= \mathcal{L}\{f(t)\}(s) = \mathcal{L}\left\{\frac{1}{2}e^t + \frac{1}{2}e^t.cos2t\right\}(s)$$

$$= \mathcal{L}\left\{\frac{1}{2}e^t\right\}(s) + \mathcal{L}\left\{\frac{1}{2}.e^t.cos2t\right\}(s)$$

$$\frac{1}{2}.\frac{1}{s-1} + \frac{1}{2}.\frac{s-1}{(s-1)^2+4} \quad (s > 1)$$

d. $f(t) = (t) = e^t.(sin^4t + cos^4t)$

$$\Rightarrow f(t) = e^t.\left(\frac{1}{4}.cos4t + \frac{3}{4}\right)$$

$$\Rightarrow f(t) = \frac{1}{4}e^t.cos4t + \frac{3}{4}.e^t$$

$$\Rightarrow \mathcal{L}\{f(t)\}(s) = \mathcal{L}\left\{\frac{1}{4}e^t.cos4t + \frac{3}{4}.e^t\right\}(s)$$

$$\Rightarrow \mathcal{L}\{f(t)\}(s) = \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}.\frac{s-1}{(s-1)^2+4} + \frac{3}{4}.\frac{1}{s-1} \quad s > 1$$

3.

a. $x^3 - 2x'' + 16x = 0$ với $x(0) = x'(0) = 0; x''(0) = 20$

Tác động phép biến đổi Laplace vào hai vế của phương trình đại số đã cho để được phương trình đại số:

$$[s^3X(s) - s^2x(0) - sx'(0) - x''(0)] - 2[s^2X(s) - s.x(0) - x'(0)] + 16X(s) = 0$$

$$\Rightarrow X(s).(s^3 - 2s^2 + 16) = 20$$

$$\Rightarrow X(s) = \frac{20}{s^3 - 2s^2 + 16}$$

$$\Rightarrow X(s) = \frac{1}{s+2} - \frac{s-6}{s^2 - 4s + 8}$$

$$\Rightarrow X(s) = \frac{1}{s+2} - \frac{s-2}{(s-s)^2 + 4} - s.\frac{2}{(s-2)^2 + 4}$$

$$\Rightarrow x(t) = e^{-2t} - e^{2t}cos2t - 2.e^{2t}.sin2t$$

b. $x^{(4)} - x = 0 vix(0), x'(0) = 0 = x''(0) = x^{(3)}$

Tác động phép biến đổi Laplace vào hai vế của phương trình đã cho ta được

$$s^4.X(s) - s^3.x(0) - s.x'(0) - s.x''(0) - x^{(0)} - X(s) = 0$$

$$\Rightarrow X(s).(s^4 - 1) - s^3 = 0$$

$$\Rightarrow X(s) = \frac{s^3}{s^4 - 1}$$

$$\Rightarrow X(s) = \frac{1}{4.(s-1)} + \frac{1}{4(s+1)} + \frac{s}{2(s^2+1)}$$

$$\Rightarrow x(t) = \frac{1}{4}.e^t + \frac{1}{4}.e^{-t} + \frac{1}{2}.cost$$

c. $y^{(3)} - 2y'' + y' = 4$ với $y(0) = 1; y'(0) = 1; y'(0) = 2; y''(0) = -2$

Tác động phép biến đổi Laplace vào hai vế của phương trình đã cho ta được:

$$[s^3.Y(s) - s^2.y(0) - s.y'(0) - y''(0)] - 2.[s^2.Y(s) - s.y(0) - y'(0)] + s.Y(s) - y(0) = \frac{4}{s}$$

$$\Rightarrow Y(s).(s^3 - 2s^2 + s) - (s^2 + 5) = \frac{4}{s}$$

$$\Rightarrow Y(s) = \frac{4 + s^3 - 5s}{s^3 - 2s^2 + s}$$

$$\Rightarrow Y(s) = \frac{s^2 + s - 4}{s^2.(s-1)}$$

$$\Rightarrow Y(s) = \frac{-2}{s-1} + \frac{3}{s} + \frac{4}{s^2}$$

$$\Rightarrow y(t) = -2.e^t + 3 + 4t$$

d. $x^{(3)} + x'' - 6x' = 0$ với $x(0) = 1; x'(0) = 2; x''(0) = 3$

Tác động phép biến đổi Laplace bậc hai vế của phương trình đã cho ta được:

$$[s^3.X(s) - s^2.x(0) - s.x'(0) - x''(0)] + [s^2.X(s) - s.x(s) - x'(0)] - 6.[s.X(s) - x(0)] = 0$$

$$\Rightarrow X(s).(s^3 + s^2 - 6s) - s^2 - 2s - 3 - s - 2 + 6 = 0$$

$$\Rightarrow X(s) = \frac{s^2 + 3s - 1}{s^3 + s^2 - 6s}$$

$$\Rightarrow X(s) = \frac{1}{6s} + \frac{9}{10(s-2)} - \frac{1}{15.(s+3)}$$

$$\Rightarrow x(t) = \frac{1}{6} + \frac{9}{10}.e^{2t} - \frac{1}{15}.e^{-3t}$$

2 Bài tập đạo hàm, tích phân, tích các phép biến đổi

1.

a. $\mathcal{L}\left\{\frac{1}{(s^2+b^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\}$

$= \frac{1}{b^2}.sinbt * sinbt$

$= \frac{1}{b^2}.\int_0^t cos(b(2u-t)) - cos(bt)\,\text{du}$

$= \frac{1}{2b^3}.(sin(bt) - bt.cos(bt))$

b. $\mathcal{L}\left\{t.e^{-t}sin^2t\right\} = -\frac{d}{ds}\left(\mathcal{L}\{e^{-t}.sin^2t\}\right)$

$= \frac{-d}{ds}.\left(\mathcal{L}\{sin^2t\}(s+1)\right)$

$= -\frac{d}{ds}\left(\mathcal{L}\{\frac{1}{2} - \frac{1}{2}.cos2t\}(s-1)\right) = \frac{-d}{ds}\left(\frac{1}{2s+2} - \frac{1}{2}.\frac{s+1}{(s+1)^2+4}\right)$

$= \frac{1}{2(s+1)^2} - \frac{s^2+2s+3}{2(s^2+2s+5)^2}$

c. $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^3}\right\} = t.\mathcal{L}^{-1}\left\{\int_s^{+\infty}\frac{\sigma}{(\sigma^2+1)^3}d\sigma\right\}$

$= \frac{t}{4}.\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

$$= \frac{t}{4}.\frac{1}{2.1^3}.(sint(1.t) - 1.t.cos(1.t)$$

$$= \frac{t}{8}.(sint - tcost)$$

2. $t.x'' + (t-1).x' + x = 0; \quad x(0) = 0$

$$\mathcal{L}\{tx''\} = -\frac{d}{ds}(s^2.X(s) - s.x(0) - x'(0)) = -(2s.x(s) + s^2.x'(s))$$

$$\Rightarrow \mathcal{L}\{(t-1)x'\} = -\frac{d}{ds}(\mathcal{L}\{s'\}) - \mathcal{L}\{x'\}$$

$$= -\frac{d}{ds}(s.x(s) - x(0)) - (s.x(s) - x(0)))$$

$$= -(sx'(s) + (s+1).x(s))$$

Phương trình trở thành:

$$-(2s.X(s) + s^2.X'(s)) - (sX' + (s+1).X(s)) + X(s) = 0$$

$$\Leftrightarrow (s^2 + s)X'(s) + 3s.X(s) = 0$$

$$\Leftrightarrow \frac{X'(s)}{X(s)} = -\frac{3}{s+1}$$

$$\Leftrightarrow ln|X(s)| = ln|C.(s+1)^{-3}|, \quad C \neq 0$$

$$\Leftrightarrow X(s) = \frac{C}{(s+1)^3}$$

$$\Rightarrow x(t) = C.e^{-t}.\frac{t^2}{2}$$

b. $\quad tx'' + (t-3)x' + 2x = 0, \quad x(0) = 0$

Ta có: $\mathcal{L}\{tx''\} = \frac{d}{ds}(\mathcal{L}\{x''\}) = -\frac{d}{ds}\left[s^2.X(s) - s.x(0) - x'(0)\right]$

$$\Leftrightarrow \mathcal{L}\{(t-3)x'\} = -\tfrac{d}{ds}(\mathcal{L}\{x'\}) - 3\mathcal{L}\{x'\}$$

$$= -\tfrac{d}{ds}(s.X(s) - x(0)) - 3(s.X(s) - x(0))$$

$$= -\left[8X'(s) + (3s+1).X(s)\right]$$

Phương trình:

$$-[2sX(s)+s^2.X'(s)]-[sX'(s)+(3s+1)X(s)]+2X(s)=0$$

$$\Leftrightarrow (s^2+s).X'(s)+(5s-1).X(s)=0$$

$$\Leftrightarrow \frac{X'(s)}{X(s)}=\frac{1-5s}{s(s+1)}=\frac{1}{s}-\frac{6}{s+1}$$

$$\Leftrightarrow ln|X(s)|=ln|C.s(s+1)^{-6}| \quad (x\neq 0)$$

$$\Leftrightarrow X(s)=\frac{C}{(s+1)^5}-\frac{C}{(s+1)^6}$$

$$\Rightarrow x(t)=C.e^{-t}.\left[\frac{t^4}{4!}-\frac{t^5}{5!}\right] \quad (c\neq 0)$$

3.

a. $f(t)=[u(t)-u(t-\frac{\pi}{2})].t+\frac{\pi}{2}.u(t-\frac{\pi}{2})$

$$\Rightarrow \mathcal{L}\{f(t)\}=\frac{1}{s^2}-e^{\frac{\pi}{2}S}.\left(\frac{1}{s^2}+\frac{\pi}{2s}\right)+\frac{\pi}{2}.\frac{e^{\frac{-\pi}{2}.s}}{s}$$

$$=\left(1-e^{-\frac{\pi}{2}.s}\right).\frac{1}{s^2}$$

Và: $s^2.Y(s)+4.Y(s)=(1-e^{-\frac{\pi}{2}.s}).\frac{1}{s^2}$

$$\Leftrightarrow Y(s)=(1-e^{-\frac{\pi}{2}.s}).\frac{1}{s^2.(s^2+4)}=\frac{1-e^{-\frac{\pi}{2}.t}}{4}.(\frac{1}{s^2}-\frac{1}{s^2+4})$$

$$\Rightarrow y(t)=\frac{1}{4}.\mathcal{L}^{-1}\{\frac{1}{s^2}-\frac{1}{s^2+4}\}(t)-u(t-\frac{\pi}{2}).\mathcal{L}^{-1}\{\frac{1}{s^2}-\frac{1}{s^2+4}\}(t-\frac{\pi}{2})$$

$$=\frac{1}{4}.(t-\frac{1}{2}.sin2t)-\frac{1}{4}.u(t-\frac{\pi}{2}).[t-\frac{\pi}{2}-\frac{1}{2}.sin(2t-\pi)]$$

$$=\begin{cases}\frac{1}{4}t-\frac{1}{8}sin2t, & t<\frac{\pi}{2}\\ \frac{\pi}{8}-\frac{1}{2}sin2t, & t\geq\frac{\pi}{2}\end{cases}$$

b. $f(t)=u(t)-u(t-\pi)$

$$\Rightarrow \mathcal{L}\{f(t)\}=\frac{e^{-s}-e^{-\pi s}}{s}$$

$$\Rightarrow (s^2+16).X(s)=\frac{e^{-s}-e^{-\pi s}}{s}$$

$$\Leftrightarrow X(s) = \frac{e^{-s} - e^{-\pi s}}{s.(s^2+16)}$$

$$\Rightarrow x(t) = \frac{1}{16}.u(t).\mathcal{L}\{\frac{1}{s} - \frac{1}{s^2+16}\}(t) - \frac{1}{16}.u(t-\pi).\mathcal{L}\{\frac{1}{s} - \frac{1}{s^2+16}\}(t-\pi)$$

$$= \frac{1}{16}.u(t).(1 - \frac{1}{4}sin4t) - \frac{1}{16}.u(t-\pi).(1 - \frac{1}{4}.sin(4(t-\pi))$$

$$= \begin{cases} \frac{1}{16} - \frac{1}{64}.sin4t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$$