Hanoi University of Science and Technology
School of Applied Mathematics and Informatics

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EXERCISES ON LINEAR ALGEBRA

CHAPTER I

Sets – Maps – Complex numbers

Exercise 1. Let $f(x), g(x)$ be two functions defined on $\mathbb{R}$. We denote by $A = \{x \in \mathbb{R} | f(x) = 0\}$, $B = \{x \in \mathbb{R} | g(x) = 0\}$. Show the solutions the following equations through A, B

a) $f(x)g(x) = 0$

b) $[f(x)]^2 + [g(x)]^2 = 0$

Exercise 2. Let A, B be two sets such that $A = [3; 6), B = (1; 5), C = [2; 4]$. Determine the following set $(A \cap B) \backslash C$.

Exercise 3. Let A, B, C, D be four sets. Prove that

a) $A \cap (B \backslash C) = (A \cap B) \backslash (A \cap C)$.

b) $A \cup (B \backslash A) = A \cup B$.

c) $(A \backslash B) \cap (C \backslash D) = (A \cap C) \backslash (B \cup D)$

Exercise 4. Let f, g be two maps such that

$$f: \quad \mathbb{R}\backslash\{0\} \to \mathbb{R}, \quad x \mapsto \frac{1}{x}$$

$$g: \quad \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{2x}{1+x^2}$$

a) Which of the maps are injective, surjective? Determine $g(\mathbb{R})$.

b) Determine the following map $h = g \circ f$.

Exercise 5. Let $f: X \to Y$ be a map. Prove that

a) $f(A \cup B) = f(A) \cup f(B);\ A, B \subset X$.

a) $f(A \cap B) \subset f(A) \cap f(B);\ A, B \subset X$. Give the examples that prove the opposite is false?

b) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B);\ A, B \subset Y$

c) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B);\ A, B \subset Y$

d) $f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B);\ A, B \subset Y$

e) Prove that f is injective if and only if $f(A \cap B) = f(A) \cap f(B);\ \forall A, B \subset X$

Exercise 6. Let $f: \mathbb{R} \to \mathbb{R}$ be a map defined by $f(x) = x^2 + 4x - 5, \quad \forall x \in \mathbb{R}$, and the set $A = \{x \in \mathbb{R}\ | -3 \leq x \leq 3\}$. Determine the following sets $f(A), f^{-1}(A)$.

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Exercise 7. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined by $f(x, y) = (x + y, x - y)$ and the set $A = \{(x, y) \in \mathrm{R}^2 | x^2 + y^2 = 9\}$. Determine the following sets $f(A)$ and $f^{-1}(A)$.

Exercise 8. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined by $f(x; y) = (x^2 - y;\ x + y)$. Determine whether the map f is injective or surjective? Why?

Exercise 9. Show the canonical form of the following complex numbers

a) $(1 + i\sqrt{3})^9$ b) $\frac{(1+i)^{21}}{(1-i)^{13}}$ c) $(2 + i\sqrt{12})^5(\sqrt{3} - i)^{11}$

Exercise 10. Find complex solutions of the following equations

a) $z^2 + z + 1 = 0$ b) $z^2 + 2iz - 5 = 0$ c) $z^4 - 3iz^2 + 4 = 0$

d) $z^6 - 7z^3 - 8 = 0$ e) $\overline{z^7} = \frac{1024}{z^3}$ f) $z^8(\sqrt{3} + i) = 1 - i$ g) $\boldsymbol{iz^2 - (1 + 8i)z + 7 + 17i = 0}$

Exercise 11. Let $\epsilon_1, \ldots, \epsilon_{2014}$ be the different 2014-roots of the complex number 1. Compute $A = \sum_{i=1}^{2014} \epsilon_i^2$

Exercise 12. Given the equation $\frac{(x+1)^9 - 1}{x} = 0$

a) Solve the above equation.
b) Find the modulus of the solutions.
c) Compute the product of its solutions and $\prod_{k=1}^{8} sin\frac{k\pi}{9}$.

Exercise 13. Let $f: \mathbb{C} \to \mathbb{C}$ be a map defined by $f(z) = iz^2 + (4 - i)z - 9i$, where i is the imaginary unit. Determine $f^{-1}(\{7\})$.

Exercise 14. Let z_1, z_2 be two complex solutions of the equation $z^2 - z + ai = 0$, where a is a real and i is the imaginary unit. Find a such that $|z_1^2 - z_2^2| = 1$.

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CHAPTER II

Matrix – Determinant – System of linear equations

Exercise 1. Given the following matrices $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -1 \\ 0 & 3 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$,

$C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.

Find : $A + BC, A^T B - C, A(BC), (A + 3B)(B - C)$.

Exercise 2. Let $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ be two matrices and I be the identity matrix of size 2.

a) Compute $F = A^2 - 3A$.

b) Find matrix X that satisfies $(A^2 + 5E)X = B^T(3A - A^2)$.

Exercise 3. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 1 \\ 3 & -5 & 3 \end{bmatrix}$ be a matrix and $f(x) = 3x^2 - 2x + 5$ be a function. Compute $f(A)$.

Exercise 4. Compute A^n, where

a) $A = \begin{bmatrix} \text{cosa} & \text{-sina} \\ \text{sina} & \text{cosa} \end{bmatrix}$.

b) $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$.

Exercise 5. Find all square matrices of size 2 that satisfy

a) $X^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

b) $X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 6.

a) Prove that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that satisfies the following equation

$$x^2 - (a + d)x + ad - bc = 0$$

b) Prove that if A is a square matrix of size 2 then $A^k = 0, (k > 2) \Leftrightarrow A^2 = 0$.

Exercise 7. Prove the following equalities by using the properties of determinant

a) $\begin{vmatrix} a_1 + b_1x & a_1 - b_1x & c_1 \\ a_2 + b_2x & a_2 - b_2x & c_2 \\ a_3 + b_3x & a_3 - b_3x & c_3 \end{vmatrix} = -2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

b) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

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Exercise 8. Compute the following determinants

a) $A = \begin{vmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{vmatrix}$ b) $B = \begin{vmatrix} a+b & ab & a^2+b^2 \\ b+c & bc & b^2+c^2 \\ c+a & ca & a^2+c^2 \end{vmatrix}$

c) $D = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}$

Exercise 9.

a) Let A be an antisymmetric matrix of an odd size. Prove that $\det(A) = 0$

b) Let A be a square matrix of size 2021. Prove that $\det(A - A^T) = 0$

Exercise 10. Determine the rank of the following matrices

a) $A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{bmatrix}$ b) $B = \begin{bmatrix} 4 & 3 & -5 & 2 & 3 \\ 8 & 6 & -7 & 4 & 2 \\ 4 & 3 & -8 & 2 & 7 \\ 4 & 3 & 1 & 2 & -5 \\ 8 & 6 & -1 & 4 & -6 \end{bmatrix}$

Exercise 11. Find m such that the rank of matrix $A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 2 & 2 & 1 \\ 1 & 0 & 4 & m \end{bmatrix}$ is 2

Exercise 12. Find the inverse of the following matrices

a) $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ b) $B = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & 1 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Exercise 13. Find a such that matrix $A = \begin{bmatrix} a+1 & -1 & a \\ 3 & a+1 & 3 \\ a-1 & 0 & a-1 \end{bmatrix}$ is invertible.

Exercise 14. Let A be a square matrix of size n. Prove that if A satisfies $a_k A^k + a_{k-1}A^{k-1} + \cdots + a_1 A + a_0 E = 0$, where $a_i \in \mathbb{R}$, $a_0 \neq 0$, then A is invertible.

Exercise 15. Given $A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 3 & 4 \\ 3 & 1 & -1 \end{bmatrix}$; $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ 0 & 3 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 12 & 10 \\ 6 & 16 & 7 \end{bmatrix}$. Find matrix X that satisfies $AX + B = C^T$

Exercise 16. Solve the following systems of linear equations

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a) $\begin{cases} 3x_1 - 5x_2 + 2x_3 + 4x_4 = 2 \\ 7x_1 - 4x_2 + x_3 + 3x_4 = 5 \\ 5x_1 + 7x_2 - 4x_3 - 6x_4 = 3 \end{cases}$ b) $\begin{cases} 3x_1 - x_2 + 3x_3 = 1 \\ -4x_1 + 2x_2 + x_3 = 3 \\ -2x_1 + x_2 + 4x_3 = 4 \\ 10x_1 - 5x_2 - 6x_3 = -10 \end{cases}$

c) $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 1 \\ 3x_1 - x_2 + x_3 = 2 \\ 5x_1 + 2x_2 + 5x_3 = 3 \\ x_1 - 4x_2 - 3x_3 = 1 \end{cases}$

Exercise 17. Solve the following systems of linear equations by using Gauss' method

a) $\begin{cases} x + 2y - z + 3t = 12 \\ 2x + 5y - z + 11t = 49 \\ 3x + 6y - 4z + 13t = 49 \\ x + 2y - 2z + 9t = 33 \end{cases}$ b) $\begin{cases} x+2y+3z+4t=-4 \\ 3x+7y+10z+11t=-11 \\ x+2y+4z+2t=-3 \\ x+2y+2z+7=-6 \end{cases}$

Exercise 18. Find a such that the system of linear equations $\begin{cases} (a+5)x + 3y + (2a+1)z = 0 \\ ax + (a-1)y + 4z = 0 \\ (a+5)x + (a+2)y + 5z = 0 \end{cases}$ has nontrivial solutions.

Exercise 19. Find m such that the system of linear equations $\begin{cases} mx_1 + 2x_2 - x_3 = 3 \\ x_1 + mx_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 + x_3 = -m \end{cases}$ has a unique solution

Exercise 20. Given the system of linear equations $\begin{cases} x_1 + 2x_2 - x_3 + mx_4 = 4 \\ -x_1 - x_2 + 3x_3 + 2x_4 = k \\ 2x_1 - x_2 - 3x_3 + (m-1)x_4 = 3 \\ x_1 + x_2 + x_3 + 2mx_4 = 5 \end{cases}$

a) Solve the system of linear equations when $m = 2, k = 5$

b) Find m, k such that the system has a unique solution

c) Find m, k such that the system has infinitely many solutions

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CHAPTER III

Vector space

Exercise 1. Let V be a set with the following operations. Determine whether V is a vector space?

a) $V = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$$

$k(x, y, z) = (|k|x, |k|y, |k|z)$, where $k \in \mathbb{R}$

b) $V = \{x = (x_1, x_2) \mid x_1 > 0, x_2 > 0\} \subset \mathbb{R}^2$

$$(x_1, x_2) + (y_1, y_2) = (x_1y_1, x_2y_2)$$

$k(x_1, x_2) = ({x_1}^k, {x_2}^k)$, where $k \in \mathbb{R}$

Exercise 2. Prove that the following subset of each vector space is a vector subspace

a) Given set $E = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | 2x_1 - 5x_2 + 3x_3 = 0\}$

b) The set of symmetric matrices of the square matrices on size n

Exercise 3. Let V_1, V_2 be two vector subspaces of vector space V.

a) Prove that $V_1 \cap V_2$ is a vector subspace of V

b) Let $V_1 + V_2 = \{u_1 + u_2 | u_1 \in V_1, u_2 \in V_2\}$. Prove that $V_1 + V_2$ is a vector subspace of V

Exercise 4. Given a vector space V. Let vector set $\{u_1, u_2, \cdots, u_n, u_{n+1}\}$ be a linearly dependent and $\{u_1, u_2, \cdots, u_n\}$ be linearly independent. Prove that u_{n+1} is a linear combination of vectors $u_1, u_2, \dots, u_n$

Exercise 5. Determine whether the following vector sets are linearly independent in R^3?

a) $v_1 = (4; -2; 6), v_2 = (-6; 3; -9)$.
b) $v_1 = (2; 3; -1), v_2 = (3; -1; 5), v_3 = (-1; 3; -4)$.
c) $v_1 = (1; 2; 3), v_2 = (3; 6; 7), v_3 = (-3; 1; 3), v_4 = (0; 4; 2)$.

Exercise 6. Determine whether the vector set $B = \{u_1 = 1 + 2x, u_2 = 3x - x^2, u_3 = 2 - x + x^2\}$ is linearly independent in the vector space $P_2[x]$?

Exercise 7. Consider $\mathbb{R}^3$,prove that $B = \{v_1 = (1; 1; 1), v_2 = (1; 1; 2), v_3 = (1; 2; 3)\}$ is a basis. Determine the transformation matrix from the standard basis of R^3 to this basis. Find the coordinate vector of $x = (6; 9; 14)$ with respect to this basis by two ways.

Exercise 8. Prove that $B = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$and find $[v]_B$

a) $v_1 = (2; 1; 1), v_2 = (6; 2; 0), v_3 = (7; 0; 7), v = (15; 3; 1)$.

b) $v_1 = (0; 1; 1), v_2 = (2; 3; 0), v_3 = (1; 0; 1), v = (2; 3; 0)$.

Exercise 9. Find a basis and the dimension of the following vector space which is generated by the following vector set

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a) $v_1 = (2; 1; 3; 4), v_2 = (1; 2; 0; 1), v_3 = (-1; 1; -3; 0)$ in $\mathbb{R}^4$.

b) $v_1 = (2; 0; 1; 3; -1), v_2 = (1; 1; 0; -1; 1), v_3 = (0; -2; 1; 5; -3), v_4 = (1; -3; 2; 9; -5)$ in $\mathbb{R}^5$.

Exercise 10. Consider $\mathbb{R}^4$, given vectors $v_1 = (1; 0; 1; 0), v_2 = (0; 1; -1; 1), v_3 = (1; 1; 1; 2), v_4 = (0; 0; 1; 1)$. Let $V_1 = span\{v_1, v_2\}, V_2 = span\{v_3, v_4\}$. Find a basis and the dimension of vector spaces $V_1 + V_2, V_1 \cap V_2$

Exercise 11. Consider $\mathbb{R}^4$, given vectors $u_1 = (1; 3; -2; 1), u_2 = (-2; 3; 1; 1), u_3 = (2; 1; 0; 1), u = (1; -1; -3; m)$. Find m such that $u \in Span\{u_1, u_2, u_3\}$

Exercise 12. Consider $P_3[x]$, given vectors $v_1 = 1, v_2 = 1 + x, v_3 = x + x^2, v_4 = x^2 + x^3$

a) Prove that $B = \{v_1, v_2, v_3\}$ is the basis of $P_3[x]$

b) Find the coordinate of vector $v = 2 + 3x - x^2$ with respect to this basis

c) find the coordinate of vector $v = a_0 + a_1x + a_2x^2$ with respect to this basis

Exercise 13. Consider $P_3[x]$, given a vector set containing $v_1 = 1 + x^2 + x^3$, $v_2 = x - x^2 + 2x^3$, $v_3 = 2 + x + 3x^3$, $v_4 = -1 + x - x^2 + 2x^3$.

a) Find the rank of this vector set

b) Find a basis of space $Span\{v_1, v_2, v_3, v_4\}$

Exercise 14. Find a basis and the dimension of solutions space of the following system of linear equations

a) $\begin{cases} x_1 - x_2 + 2x_3 + 2x_4 - x_5 = 0 \\ x_1 - 2x_2 + 3x_3 - x_4 + 5x_5 = 0 \\ 2x_1 + x_2 + x_3 + x_4 + 3x_5 = 0 \\ 3x_1 - x_2 - 2x_3 - x_4 + x_5 = 0 \end{cases}$

b) $\begin{cases} 2x_1 - x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \\ 4x_1 - 2x_2 + 5x_3 + x_4 + 7x_5 = 0 \\ 2x_1 - x_2 + x_3 + 8x_4 + 2x_5 = 0 \end{cases}$

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CHAPTER IV.

Linear map

Exercise 1. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x_1, x_2, x_3) = (3x_1 + x_2 - x_3, 2x_1 + x_3)$

a) Prove that f is a linear map.

b) Find the matrix of f with respect to two standard bases.

c) Find a basis of $Kerf$.

Exercise 2. Consider the map $f: P_2[x] \to P_4[x]$ defined by $f(p) = p + x^2 p, \forall p \in P_{2.}[x]$

a) Prove that f is a linear map

b) Find the matrix of f with respect to the standard bases $E_1 = \{1, x, x^2\}$ in $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ in $P_4[x]$

c) Find the matrix of f with respect to the bases $E_1{}' = \{1 + x, 2x, 1 + x^2\}$ in $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ in $P_4[x]$

Exercise 3. Consider the map $f: P_2[x] \to P_2[x]$ that satisfies $f(1 - x^2) = -3 + 3x - 6x^2, f(3x + 2x^2) = 17 + x + 16x^2,\ f(2 + 6x + 3x^2) = 32 + 7x + 25x^2$.

a) Find the matrix of f with respect to standard basis in $P_2[x]$. Compute $f(1 + x^2)$

b) Determine m that vector $v = 1 + x + mx^2$ in Imf

Exercise 4. Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ be a matrix of linear map $f: P_2[x] \to P_2[x]$ with respect to basis $B = \{v_1, v_2, v_3\}$, where $v_1 = 3x + 3x^2, v_2 = -1 + 3x + 2x^2, v_3 = 3 + 7x + 2x^2$

a) Find $f(v_1), f(v_2), f(v_3)$ b) Find $f(1 + x^2)$

Exercise 5. Consider $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 - x_2 + x_3, -x_1 + x_2 + x_3)$. Find the matrix of f with respect to the basis $B = \{v_1 = (1; 0; 0), v_2 = (1; 1; 0), v_3 = (1; 1; 1)\}$

Exercise 6. Consider a linear map $f: P_2[x] \to P_2[x]$ that satisfies
$f(1 - x^2) = -3 + 3x - 6x^2, f(3x + 2x^2) = 17 + x + 16x^2,\ f(2 + 6x + 3x^2) = 32 + 7x + 25x^2$.

a) Find the matrix of f with respect to the standard basis in $P_2[x]$. Compute $f(1 + x^2)$.

b) Determine m that vector $v = 1 + x + mx^2$ in Imf

Exercise 7. Let $A = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 1 & 6 & 2 & 1 \\ -3 & 0 & 7 & 1 \end{bmatrix}$ be a matrix of a linear map $\mathbb{R}^4 \to \mathbb{R}^3$ with respect to the two standard bases $B = \{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^4$ and $B' = \{u_1, u_2, u_3\}$ in $\mathbb{R}^3$ where $v_1 =$

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$(0;1;1;1), v_2 = (2;1;-1;-1), v_3 = (1;4;-1;2), v_4 = (6;9;4;2)$ và $u_1 = (0;8;8), u_2 = (-7;8;1), u_3 = (-6;9;1)$.

a) Find $[f(v_1)]_{B''}, [f(v_2)]_{B''}, [f(v_3)]_{B''}, [f(v_4)]_{B'}$.

b) Find $f(v_1), f(v_2), f(v_3), f(v_4)$

c) Find $f(2;2;0;0)$.

Exercise 8. Consider a linear operator in $P_2[x]$ defined by $f(1+2x) = -19 + 12x + 2x^2; f(2+x) = -14 + 9x + x^2; f(x^2) = 4 - 2x - 2x^2$

Find the matrix of f with respect to the basis in $P_2[x]$ and find $rank(f)$

Exercise 9. Consider a linear operator in $\mathbb{R}^3$ defined by $f(x_1;x_2;x_3) = (x_1 - 2x_2 + x_3; x_1 + x_2 - x_3; mx_1 - x_2 + x_3)$, where m is a paramater. Determine the matrix of f with respect to the standard basis of f and find m that f is surjective

Exercise 10. Find eigenvalues and a basis of eigenvector spaces of the following matrices

a) $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ b) $B = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ c) $C = \begin{bmatrix} 2 & -1 & 0 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

d) $D = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix}$ e) $E = \begin{bmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{bmatrix}$

Exercise 11. Consider a linear operator $f: P_2[x] \to P_2[x]$ defined by $f(a_0 + a_1x + a_2x^2) = (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2$.

a) Find eigenvalues of f

b) Find eigenvectors with respect to the above eigenvalues.

Exercise 12. Find P such that P diagonalizes A and determine $P^{-1}AP$

a) $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$ b) $B = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ d) $D = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

Find A^n

Exercise 13. Is matrix A diagonal? If yes, find the diagonal matrix

a) $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ b) $B = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$ c) $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

Exercise 14. Find a basis of $\mathbb{R}^3$ that the matrix of $f: \mathbb{R}^3 \to \mathbb{R}^3$ with respect to this basis is diagonal

a) $f(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + 2x_2 + x_3, x_1 + x_2 + 2x_3)$.

b) $f(x_1, x_2, x_3) = (2x_1 - x_2 - x_3, x_1 - x_2, -x_1 + x_2 + 2x_3)$

Exercise 15. Consider a linear operator in $\mathbb{R}^3$ defined by $f(1;2;-1) = (4;-2;-6), f(1;1;2) = (5;5;0), f(1;0;0) = (1;2;1)$

a) Find m that $u = (6;-3;m) \in Im(f)$

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b) Find eigenvalues and eigenvectors of f

Exercise 16. Consider a linear map $f: P_2[x] \to P_2[x]$ with matrix $A = \begin{bmatrix} -3 & 1 & 2 \\ 6 & 0 & -3 \\ -10 & 2 & 6 \end{bmatrix}$ with respect to standard basis $\{1, x, x^2\}$ of $P_2[x]$

a) Find $f(1 + x + x^2)$. Find m that $v = 1 - x + mx^2$ in $Kerf$

b) Find a basis of $P_2[x]$ that the matrix of f with respect to this basis is diagonal.

CHAPTER V.

Quadratic form, Euclide space

Exercise 1. Let ω_i be a quadratic form in $\mathbb{R}^3$

$\omega_1(x_1, x_2, x_3) = {x_1}^2 + 5{x_2}^2 - 4{x_3}^2 + 2x_1x_2 - 4x_1x_3$. $\omega_2(x_1, x_2, x_3) = x_1x_2 + 4x_1x_3 + x_2x_3$.

a) Convert quadratic form to canonical form by using Lagrange reduction

b) Is quadratic form positive definite or negative definite?

Exercise 2. Determine a such that the following quadratic forms are definite?

a) $5{x_1}^2 + {x_2}^2 + a{x_3}^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$

b) $2{x_1}^2 + {x_2}^2 + 3{x_3}^2 + 2ax_1x_2 + 2x_1x_3$

c) ${x_1}^2 + {x_2}^2 + 5{x_3}^2 + 2ax_1x_2 - 2x_1x_3 + 4x_2x_3$.

Exercise 3. Given a bilinear form in $\mathbb{R}^3$ defined by $< (x_1, x_2, x_3), (y_1, y_2, y_3) > = 2x_1y_1 + x_1y_2 + x_2y_1 + ax_2y_2 - 2x_2y_3 - 2x_3y_2 + 3x_3y_3$,where a is a parameter. Find the matrix of this bilinear form with respect to standard basis of $\mathbb{R}^3$ and determine a such that the bilinear form is an inner product in $\mathbb{R}^3$

Exercise 4. Given a bilinear from in $\mathbb{R}^3$ defined as $f(x, y) = (x_1, x_2, x_3)A(y_1, y_2, y_3)^t$, where $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & 4 \\ -1 & a^2 & 2a \end{bmatrix}$ and $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$. Determine a such that $f(x, y)$ is an inner product in $\mathbb{R}^3$.

Exercise 5. Consider that V is n-dimensional vector space with a basis $B = \{e_1, e_2, \dots, e_n\}$. Given vectors u, v of V, where $u = a_1e_1 + a_2e_2 + \cdots + a_ne_n; v = b_1e_1 + b_2e_2 + \cdots + b_ne_n$. Let $< u, v > = a_1b_1 + \cdots + a_nb_n$

a) Prove that $< u, v >$ is an inner product.

b) When $V = \mathbb{R}^3$ with $e_1 = (1; 0; 1), e_2 = (1; 1; -1), e_3 = (0; 1; 1), u = (2; -1; -2), v = (2; 0; 5)$. Compute $< u, v >$.

c) When $V = P_2[x]$ with $B = \{1; x; x^2\}, u = 2 + 3x^2, v = 6 - 3x - 3x^2$. Compute $< u, v >$.

d) When $V = P_2[x]$ with $B = \{1 + x; 2x; x - x^2\}, u = 2 + 3x^2, v = 6 - 3x - 3x^2$. Compute $< u, v >$.

Exercise 6. Determine where $< p, q >$ is an inner product in the vector space $P_3[x]$

a) $< p, q > = p(0)q(0) + p(1)q(1) + p(2)q(2)$

b) $< p, q > = p(0)q(0) + p(1)q(1) + p(2)q(2) + p(3)q(3)$

c) $< p, q > = \int_{-1}^{1} p(x)q(x)dx$

Compute $< p, q >$ when it is an inner product with $p = 2 - 3x + 5x^2 - x^3. q = 4 + x - 3x^2 + 2x^3$

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Exercise 7. Given Euclide space V. Prove that

a) $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$.

b) $u \perp v \Leftrightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2, \forall u, v \in V$.

Exercise 8. Let $B = \{(1;1;-2),(2;0;1),(1;2;3)\}$ be a basis of space $\mathbb{R}^3$with the conventional inner product. Apply Gram-Schmidt process on B to obtain the orthonormal basis B'. Find coordinate of vector $u - (5;8;6)$ with respect to B'.

Exercise 9. Find the orthogonal projection of vector u onto $Span\{v\}$

a) $u = (1;3;-2;4), v = (2;-2;4;5)$

b) $u = (4;1;2;3;-3), v = (-1;-2;5;1;4)$

Exercise 10. Given space $\mathbb{R}^3$ with the conventional inner product and vectors $u = (3;-2;1)$, $v_1(2;2;1)$, $v_2 = (2;5;4)$. Let $W = span\{v_1, v_2\}$. Find the orthogonal projection of vector u onto W.

Exercise 11. Given space $\mathbb{R}^3$with the conventional inner product and vectors $u = (1;2;-1), v = (3;6;3)$. Let $H = \{w \in \mathbb{R}^3 | w \perp u\}$

a) Find an orthonormal basis of space H

b) Find the orthogonal projection of vector v onto H

Exercise 12. Consider space $\mathbb{R}^4$ with the conventional inner product. Given $u_1 = (6;3;-3;6), u_2 = (5;1;-3;1)$. Find the orthonormal basis of $Span\{u_1, u_2\}$

Exercise 13. Let $< p, q >= \int_{-1}^{1} p(x)q(x)dx$ be an inner product in $P_2[x]$, where $p, q \in P_2[x]$

a) Apply Gram-Schmidt process on $B = \{1; x; x^2\}$ to obtain the orthonormal basis A

b) Determine the transformation matrix from B to A

c) Find $[r]_A$where $r = 2 - 3x + 3x^2$

Exercise 14. Orthogonally diagonalize the following matrices

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ b) $B = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d) $D = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

Exercise 15. Convert the following quadratic forms to canonical forms by orthogonal diagonalization

a) ${x_1}^2 + {x_2}^2 + {x_3}^2 + 2x_1x_2$

b) $7{x_1}^2 - 7{x_2}^2 + 48x_1x_2$

c) $7{x_1}^2 + 6{x_2}^2 + 5{x_3}^2 - 4x_1x_2 + 4x_2x_3$