Linear Algebra

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Chapter 2: Matrices - Determinants - System of Linear Equations

Matrices

Definition

A matrix is a rectangular array of numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{n2} & \cdots & a_{mn} \end{pmatrix}$$

Most of this lecture focuses on real and complex matrices, that is, matrices whose elements are real numbers or complex numbers, respectively.

i) The size of a matrix is defined by the number of its rows and columns. A matrix with m rows and n columns is called an $m \times n$ matrix.
ii) Matrices with a single row are called row vectors, and those with a single column are called column vectors.
iii) $m = n \Rightarrow$ square matrix.

Operations on matrices

1) Matrix Adddition

$$(a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}.$$

2) Scalar Multiplication

$$k(a_{ij})_{m\times n} = (ka_{ij})_{m\times n}.$$

3) Transposition

$$(a_{ij})^T_{m\times n} = (a_{ji})_{n\times m}.$$

4) Matrix Multiplication: Let $A \in M_{m,n}, B \in M_{n,k}$. The matrix C defined by

$$c_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

is called the product of A and B, denoted $C = AB$.

Properties

1. $$\begin{cases} A+B=B+A \\ A+0=0+A=A \\ A+(-A)=(-A)+A=0 \\ (A+B)+C=A+(B+C) \end{cases}$$

2. $$\begin{cases} k(A+B)=kA+kB \\ (k+h)A=kA+hA=A \\ k(hA)=(kh)A \\ 1.A=A \\ 0.A=0 \end{cases}$$

3. $$\begin{cases} (AB)C=A(BC) \\ AI=IA=A \\ \text{where } I \text{ is the identity matrix} \\ \text{Note that } AB \neq BA \end{cases}$$

4. $$\begin{cases} A(B+C)=AB+AC) \\ (A+B)C=AC+BC \\ (AB)C=A(BC) \\ k(BC)=(kB)C=B(kC) \end{cases}$$

5. $(AB)^T = B^T A^T$

Examples

Example

Find the matrix X such that:

a) $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + 2X = \begin{bmatrix} 1 & -2 \\ 5 & 7 \end{bmatrix}$

b) $\frac{1}{2}X - \begin{bmatrix} 1 & -3 & 2 \\ 3 & -4 & 1 \\ 2 & -5 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 6 \\ 1 & 2 & 5 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 6 \\ -2 & 9 & 2 \\ -4 & -8 & 6 \end{bmatrix}$

Example

Compute A^n, where

a) $A = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}$

b) $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$

Invertible Matrix

Definition

An n-by-n square matrix A is called invertible (also nonsingular or nondegenerate) if there exists an n-by-n square matrix B such that

$$AB = BA = I$$

If this is the case, then the matrix B is uniquely determined by A and is called the inverse of A, denoted by A^{-1}.

Example

Find the inverse matrix of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

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Determinant

Determinant

The determinant of a matrix A is denoted $\det A$, or $|A|$.

1) 1-square matrix $\det(a) = a$,
2) 2-square matrix $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$,
3) 3-square matrix

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}.$$

4) n-square matrix, $n \geq 3$:

$$\det(a_{ij})_{n \times n} = a_{11} \det M_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det M_{1n},$$

where M_{ij} are minors.

Determinant

Properties

1) $\det I = 1$.
2) $\det A = \det A^T$,i.e., if a property is true for columns, so is it for rows.
3) The Laplace formula:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det M_{ij},$$

   where M_{ij} are the minors.
4) $\det(AB) = \det A \det B \Rightarrow \det(A^{-1}) = \frac{1}{\det A}$.
5) If A is a triangular matrix, then its determinant equals the product of the diagonal entries: $\det A = a_{11} a_{22} \cdots a_{nn}$.
6) If in a matrix, any row or column has all elements equal to zero, then the determinant of that matrix is 0.

Determinant

Properties

7) Adding a scalar multiple of one column to another column does not change the value of the determinant.
8) If two columns or rows of a matrix are identical, then its determinant is 0.
9) Interchanging any pair of columns or rows of a matrix multiplies its determinant by -1.
10) The determinant is an n-linear function

$$\begin{vmatrix} \lambda\alpha_1 + \mu\beta_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ \lambda\alpha_n + \mu\beta_n & a_{n2} & \dots & a_{nn} \end{vmatrix} =$$

$$\lambda \begin{vmatrix} \alpha_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_n & a_{n2} & \dots & a_{nn} \end{vmatrix} + \mu \begin{vmatrix} \beta_1 & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots \\ \beta_n & a_{n2} & \dots & a_{nn} \end{vmatrix} \Rightarrow \det(cA) = c^n \det A$$

Determinant

The elementary row operation

Row-reduce the matrix to "upper triangular" form:

i) Multiplying a row of matrix by a number c multiplies its determinant by the same number,

ii) Adding a multiple of one row of a matrix to another row does not change the determinant,

iii) Interchanging two rows of a matrix changes the sign of the determinant.

Example

Find the determinant of the matrix $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{pmatrix}$.

Determinants of some matrices of special form

Vandermonde Matrix

The Vandermonde matrix of order n:

$$V_n(a_1, a_2, ..., a_n) = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_1^2 & a_2^2 & \dots & a_{n-1}^2 & a_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_{n-1}^{n-1} & a_n^{n-1} \end{bmatrix}$$

Lemma

$$\det V_n(a_1, a_2, ..., a_n) = \prod_{1 \leqslant i < j \leqslant n} (a_j - a_i).$$

Determinants of some matrices of special form

The Cauchy matrix

The Cauchy matrix of order n, $A = (a_{ij})$, where $a_{ij} = \frac{1}{x_i + y_j}$.

Lemma

$$\det A = \frac{\prod\limits_{i>j}(x_i - x_j)(y_i - y_j)}{\prod\limits_{i,j}(x_i + x_j)}$$

Determinants of some matrices of special form

The Frobenius matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{pmatrix}$$

or the friend matrix of the polynomial

$$p(\lambda) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \dots - a_0.$$

Lemma

$$\det(\lambda I - A) = p(\lambda)$$

Determinants of some matrices of special form

The tridiagonal matrix

$$\begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 \dots & 0 & 0 & 0 & \\ 0 & c_2 & a_3 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-2} & b_{n-2} & 0 \\ 0 & 0 & 0 & \dots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & c_{n-1} & a_n \end{pmatrix}$$

Lemma

$$\Delta_k = a_k\Delta_{k-1} - b_{k-1}c_k\Delta_{k-2},\ k \geq 2,\ \textit{where } \Delta_k = \det(a_{ij})_{i,j=1}^k.$$

The tridiagonal matrix

Denote by

$$(a_1 \ldots a_n) = \begin{vmatrix} a_1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & a_2 & 1 \ldots & 0 & 0 & 0 & \\ 0 & -1 & a_3 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-2} & 1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & a_{n-1} & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & a_n \end{vmatrix}$$

then

$$\frac{(a_1 a_2 \ldots a_n)}{(a_2 a_3 \ldots a_n)} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

Determinants of some matrices of special form

Block matrix

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are m-square and n-square matrices, respectively.

Theorem

$$\begin{vmatrix} DA_{11} & DA_{12} \\ A_{21} & A_{22} \end{vmatrix} = |D|.|A| \text{ and } \begin{vmatrix} A_{11} & A_{12} \\ A_{21} + BA_{11} & A_{22} + BA_{12} \end{vmatrix} = |A|.$$

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Rank of a matrix

Definition

Let A be an $m \times n$ matrix and p an integer with $0 < p \leq \min\{m, n\}$. A $p \times p$ minor of A is the determinant of a $p \times p$ matrix obtained from A by deleting $m - p$ rows and $n - p$ columns.

$$A \begin{pmatrix} i_1 & \dots & i_p \\ k_1 & \dots & k_p \end{pmatrix} = \begin{vmatrix} a_{i_1 k_1} & a_{i_1 k_2} & \dots & a_{i_1 k_p} \\ \vdots & \vdots & \dots & \vdots \\ a_{i_p k_1} & a_{i_p k_2} & \dots & a_{i_p k_p} \end{vmatrix}$$

Definition

The rank of a matrix is the order of the largest non-zero minor, denoted by $r(A)$ or $\rho(A)$.

Rank of a matrix

Row echelon form

A matrix is in row echelon form if

1) all nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes,
2) the leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Theorem

Rank of a row echelon form is the number of nonzero rows.

Rank of a matrix

Transformation to row echelon form

By means of a finite sequence of elementary row operations, also called Gaussian elimination, any matrix can be transformed to row echelon form.

1) Change the positions of two rows.
2) Multiply a row by a nonzero scalar.
3) Add to one row a scalar multiple of another.

Example

$$A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{bmatrix}$$

Rank of a matrix

Transformation to row echelon form

By means of a finite sequence of elementary row operations, also called Gaussian elimination, any matrix can be transformed to row echelon form.

1) Change the positions of two rows.
2) Multiply a row by a nonzero scalar.
3) Add to one row a scalar multiple of another.

Example

$$A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{bmatrix} \to \ldots \to \begin{bmatrix} 1 & 3 & 5 & -1 \\ 0 & -7 & -11 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \Rightarrow r(A) = 4.$$

Inverse of a matrix

Definition

Let A be an n-by-n square matrix. If there exists an n-by-n square matrix B such that

$$AB = BA = I_n,$$

then the matrix B is called the inverse of A, denoted by A^{-1}.

The uniqueness of A^{-1}

The inverse of a matrix A, if exists, is unique.

The existence of A^{-1}

If $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} . C^T,$$

where $C = [c_{ij}]_{n \times n}$ and $c_{ij} = (-1)^{i+j} \det M_{ij}$.

Inverse of a matrix

Properties

1) $(A^{-1})^{-1} = A$.
2) $(kA)^{-1} = \frac{1}{k}A^{-1}$.
3) $(AB)^{-1} = B^{-1}A^{-1}$.
4) $(A^T)^{-1} = (A^{-1})^T$.

Gauss-Jordan elimination

1) Adjoin the identity matrix to the right side of A,
2) Apply row operations to this matrix until the left side is reduced to I.
3) When A is reduced to I, then I becomes A^{-1}.

Examples

Example

Find the inverse of the matrices

a) $B = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & 1 \end{bmatrix}$ b) $C = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Example

Prove that if A is a skew-symmetric (or antisymmetric or antimetric) matrix of order n, where n is odd, then $\det(A) = 0$.

Example

Let $A = [a_{ij}]_{n\times n}$ be a complex matrix such that $a_{ij} = -\overline{a_{ji}}$. Prove that $\det(A)$ is a real number.

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System of linear equations

Definition

A general system of m linear equations with n unknowns can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1}$$

where $x_1, x_2, \ldots, x_n$ *are the unknowns,* $a_{11}, a_{12}, \ldots, a_{mn}$ *are the coefficients of the system, and* $b_1, b_2, \ldots, b_m$ *are the constant terms.*

The matrix equation

$$Ax = b,$$

where $A = [a_{ij}]_{m\times n}, x = [x_1, x_2, \ldots, x_n]^T$ and $b = [b_1, b_2, \ldots, b_m]^T$.

Cramer's rule

Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows:

$$Ax = b, \tag{2}$$

where the $n \times n$ matrix A has a nonzero determinant.

Theorem (Cramer'rule)

The system (2) has a unique solution, whose individual values for the unknowns are given by:

$$x_j = \frac{\det A_j}{\det A},$$

where A_j is the matrix formed by replacing the j-th column of A by the column vector b.

Kronecker-Capelli Theorem

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (3)$$

Kronecker-Capelli Theorem

The system (3) has solution if and only if $r\left(\overline{A}\right) = r(A)$, where $\overline{A} = [A|b]$.

Corollary

1) If $r\left(\overline{A}\right) \neq r(A)$, then the system (3) has no solution.

2) If $r\left(\overline{A}\right) = r(A) = n$, then the system (3) has a unique solution.

3) If $r\left(\overline{A}\right) = r(A) < n$, then the system (3) has infinitely many solutions.

Gaussian elimination

Consider the augmented matrix $\overline{A}$. This matrix is then modified using elementary row operations until it reaches reduced row echelon form.

S1. Swap the positions of two rows.

S2. Multiply a row by a nonzero scalar.

S3. Add to one row a scalar multiple of another.

Example

a) $\begin{cases} x_1 - 2x_2 + x_3 & = 4 \\ 2x_1 + x_2 - x_3 & = 0 \\ -x_1 + x_2 + x_3 & = -1 \end{cases}$

b) $\begin{cases} 3x_1 - 5x_2 - 7x_3 & = 1 \\ x_1 + 2x_2 + 3x_3 & = 2 \\ -2x_1 + x_2 + 5x_3 & = 2. \end{cases}$

c) $\begin{cases} ax_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + ax_2 + x_3 + x_4 = a \\ x_1 + x_2 + ax_3 + x_4 = a^2 \end{cases}$

d) $\begin{cases} (2-a)x_1 + x_2 + x_3 = 0 \\ x_1 + (2-a)x_2 + x_3 = 0 \\ x_1 + x_2 + (2-a)x_3 = 0 \end{cases}$