ĐÁP ÁN BÀI TẬP TUẦN 3

Bài tập 1. Tính các biến đổi Laplace ngược sau.

1. $\mathfrak{L}^{-1}\left\{\frac{s}{(s-2)^2(s-1)}\right\}$

   - Ta có
   $\frac{s}{(s-2)^2(s-1} = \frac{2(s-1)-(s-2)}{(s-2)^2(s-1)} = \frac{s}{(s-1)^2} - \frac{1}{(s-2)(s-1)} = \frac{2}{(s-2)^2} + \frac{1}{s-1} - \frac{1}{s-2}$
   - Áp dụng tính chất $\mathfrak{L}^{-1}\left\{e^{-at}f(t)\right\} = F(s+a)\ (a>0)$
   $\mathfrak{L}^{-1}\left\{\frac{2}{(s-1)^2}\right\} = 2e^{2t}t$
   $\mathfrak{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$
   $\mathfrak{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$
   $\Longrightarrow \mathfrak{L}^{-1}\left\{\frac{s}{(s-2)^2(s-1)}\right\}(t) = 2e^{2t} + e^{t} - e^{2t}$

2. $\mathfrak{L}^{-1}\left\{\frac{3s}{s^3-1}\right\}$

   - $$\begin{aligned}&\mathfrak{L}^{-1}\left\{\frac{3s}{s^3-1}\right\}\\ =&\mathfrak{L}^{-1}\left\{\frac{s^2+s+1-(s-1)^2}{(s-1)(s^2+s+1)}\right\}\\ =&\mathfrak{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathfrak{L}^{-1}\left\{\frac{s-1}{s^2+s+1}\right\}\end{aligned}$$
   - Ta có $\mathfrak{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$
   - Xét $\mathfrak{L}^{-1}\left\{\frac{s-1}{s^2+s+1}\right\} = \mathfrak{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{s})^2+\frac{3}{4}}\right\} - \mathfrak{L}^{-1}\left\{\frac{\frac{3}{2}}{(s+\frac{1}{s})^2+\frac{3}{4}}\right\}$
   - $\mathfrak{L}^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{s})^2+\frac{3}{4}}\right\} = e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$
   - $\mathfrak{L}^{-1}\left\{\frac{\frac{3}{2}}{(s+\frac{1}{s})^2+\frac{3}{4}}\right\} = e^{\frac{1}{2}t}\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)$
   - Vậy $\mathfrak{L}^{-1}\left\{\frac{3s}{s^2+1}\right\}(t) = e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - e^{\frac{1}{2}t}\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)$

3. $\mathfrak{L}\left\{t\cos^3 t\right\}(s)$

   - Áp dụng tính chất $\mathfrak{L}^{-1}\left\{tf(t)\right\}(s) = -F'(s)$

- Ta có

$$\begin{aligned}
&\mathfrak{L}\left\{t\cos^3 t\right\}(s)\\
=&\mathfrak{L}\left\{\frac{3\cos t+\cos 3t}{4}\right\}(s)\\
=&\frac{3s}{4(s^2+1)}+\frac{s}{4(s^2+9)}
\end{aligned}$$

- Suy ra $\mathfrak{L}^{-1}\left\{tf(t)\right\}(s)=-\left(\frac{3(1-s^2)}{4(s^2+1)^2}+\frac{9-s^2}{4(s^2+9)}\right)$
- $u(t-a)=\left[\begin{array}{lll} t & , & t\geq a\\ 0 & , & t<a\end{array}\right.$
- $\mathfrak{L}\{u(s-a)\}(s)=\int\limits_0^\infty u(t-a)e^{-st}\mathrm{d}t=\int\limits_0^\infty e^{-st}\mathrm{d}t=-\frac{e^{-st}}{s}\Big|_a^\infty=\frac{e^{-as}}{s}$

Bài tập 2. 1. $x^(3)-2x''-16x=0$

- Ta có

$$\begin{aligned}
&\mathfrak{L}\left\{x\right\}(s)=X(s)\\
&\mathfrak{L}\left\{x''\right\}(s)=s^2X(s)-sx(0)-x'(0)=s^2X(s)\\
&\mathfrak{L}\left\{x^(3)\right\}(s)=s^3X(s)-s^2x(0)-sx'(0)-x''(0)=s^3X(0)-20
\end{aligned}$$

- Phương trình trở thành

$$\begin{aligned}
&s^3X(s)-20+2s^2X(s)-16X(s)=0\\
\Longleftrightarrow&(s^3+2s^2-16)X(s)=20\\
\Longleftrightarrow&X(s)=\frac{20}{s^3+2s^2-16}\\
\Longleftrightarrow&X(s)=\frac{20}{(s-2)(s^2+4s+8)}\\
\Longleftrightarrow&X(s)=\frac{1}{s-2}-\frac{s+2}{(s+2)^2+2^2}-\frac{4}{(s+2)^2+2^2}
\end{aligned}$$

- $\mathfrak{L}^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$
- $\mathfrak{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2^2}\right\}=e^{-2t}\cos 2t$
- $\mathfrak{L}^{-1}\left\{\frac{4}{(s+2)^2+2^2}\right\}=2e^{-2t}\sin 2t$

- Vậy $x = e^{2t} - e^{-2t}\cos 2t - 2e^{-2t}\sin 2t$

2. $x^{(4)} - x = 0$

- $\mathfrak{L}\{x\}(s) = X(s)$
- $\mathfrak{L}\{x^{(4)}\}(s) = s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x^{(3)}(0) = s^4X(s) - s^3$
- Phương trình trở thành

$$\begin{aligned} &s^4X(s) - s^3 - X(s) = 0 \\ \Longrightarrow &X(s) = \frac{s^3}{s^4-1} = \frac{1}{2}\frac{s(s^2+1+s^2-1}{s^4-1} = \frac{1}{s}\left(\frac{s}{s^2+1} + \frac{s}{s^2-1}\right) \end{aligned}$$

- $\mathfrak{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\} = \cos t$
- $\mathfrak{L}^{-1}\left\{\dfrac{s}{s^2-1}\right\} = \cosh t$
- Vậy $x = \dfrac{1}{2}(\cos t + \cosh t)$

3. • $\begin{cases} \mathfrak{L}\{x\}(s) = X(s) \\ \mathfrak{L}\{y\}(s) = Y(s) \\ \mathfrak{L}\{x'\}(s) = sX(s) - x(0) = sX(s) \\ \mathfrak{L}\{y'\}(s) = sY(s) - y(0) = sY(s) \end{cases}$

- Hệ phương trình đảo cho phép biến đổi Laplace là
$\begin{cases} sX(s) - 2Y(s) = \dfrac{1}{s} \\ sY(s) + 2X(s) = \dfrac{1}{s^2} \end{cases}$

$$\begin{aligned} \Longrightarrow &s^2X(s) + 4X(s) = 1 + \frac{2}{s^2} \\ \Longleftrightarrow &X(s) = \frac{s^2+2}{s^2(s^2+4)} \\ \longrightarrow &Y(s) = \frac{-1}{s(s^2+4)} \end{aligned}$$

- Ta có

$$X(s) = \frac{s^2+2}{s^2(s^2+4)} = \frac{1}{2}\frac{s^2+4+s^2}{s^2(s^2+4)} = \frac{1}{2}\left(\frac{1}{s^2} + \frac{1}{s^2+4}\right)$$

$$\longrightarrow x(t) = \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)$$
$$Y(s) = \frac{-1}{s(s^2+4)} = \frac{1}{4}\left(\frac{s}{s^2+4} - \frac{1}{s}\right)$$
$$\Longrightarrow y(t) = \frac{1}{4}(\cos 2t - 1)$$

- Vậy $\begin{cases} x(t) = \dfrac{1}{2}\left(t + \dfrac{\sin 2t}{2}\right) \\ y(t) = \dfrac{1}{4}(\cos 2t - 1). \end{cases}$

4. $\begin{cases} \mathfrak{L}\{z\}(s) = Z(s) \\ \mathfrak{L}\{y\}(s) = Y(s) \\ \mathfrak{L}\{z'\}(s) = sZ(s) - 1 \\ \mathfrak{L}\{y'\}(s) = sY(s) - 1 \end{cases}$

- Ảnh của hệ pt qua phép biến đổi Laplace là
$$\begin{cases} sY(s) - 1 + y(s) + z(s) = \dfrac{1}{s-1} \\ sZ(s) - 1 + sY(s) - 2Z(s) = \dfrac{2}{s-1} \end{cases} \Longleftrightarrow \begin{cases} (s+1)Y(s) - Z(s) = \dfrac{s}{s-1} \\ 3Y(s) + (s-2)Z(s) = \dfrac{s+1}{s-1} \end{cases}$$
$$\Longrightarrow Y(s) = \frac{1}{s-1}, Z(s) = \frac{1}{s-1}$$

- Do đó $\begin{cases} y(t) = e^t \\ z(t) = e^t \end{cases}$