Linear Algebra

Dr. Bui Xuan Dieu

School of Applied Mathematics and Informatics,
Hanoi University of Science and Technology

Chapter 5: Quadratic Form- Euclidean Space

1. Bilinear form - Quadratic form
2. Reduction of Binary Quadratic Forms
   - Lagrange reduction
3. Euclidean space
   - The inner product (dot product)
   - Gram-Schmidt process
   - Projection of a vector onto subspace
4. Orthogonal Diagonalization
   - Orthogonal Diagonalization of Symmetric Matrices
   - Orthogonal diagonalization of Quadratic Forms
   - Quadratic curve in plane
   - Quadratic surface classification
   - Orthogonal transform to constrained extrema problems

Bilinear form - Quadratic form

Definition

A bilinear form on a vector space V is a bilinear map $\varphi : V \times V \to \mathbb{R}$. In other words, it is linear in each argument separately:

$$\begin{cases} \varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \text{ and } \varphi(kx, y) = k\varphi(x, y) \\ \varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \text{ and } \varphi(x, ky) = k\varphi(x, y) \end{cases}$$

The bilinear form φ is called symmetric if

$$\varphi(x, y) = \varphi(y, x) \text{ for all } x, y \in V.$$

Definition

Let φ be a symmetric bilinear form on V. The mao $H : V \to \mathbb{R}$ defined by $H(x) = \varphi(x, x)$ is called a quadratic form corresponding to φ.

Coordinate representation

Let $\varphi : V \times V \to \mathbb{R}$ be a bilinear form, where V is an n-dimensional vector space with basis $S = \{e_1, e_2, \ldots, e_n\}$. Then,

$$\begin{aligned} \varphi(x, y) = \varphi\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j,\right) &= \sum_{i,j=1}^{n} \varphi(e_i, e_j) x_i y_j \\ &= \sum_{i,j=1}^{n} a_{ij} x_i y_j = [x]^T A [y]. \end{aligned} \tag{1}$$

Remark: φ is symmetric $\Leftrightarrow A = [a_{ij}] = [\varphi(e_i, e_j)]$ is a symmetric matrix.

Definition

The matrix $A = [a_{ij}] = [\varphi(e_i, e_j)]$ is called the matrix of the bilinear form φ (or the quadratic form H) w.r.t the basis S.

1) $\varphi(x, y) = [x]^T A [y]$: the coordinate representation of φ.

2) $H(x, x) = [x]^T A [x]$: the coordinate representation of H.

Coordinate representation

Example

Assume that

i) f is a bilinear form on a 3-dimensional vector space V and the matrix of f w.r.t a basis $\mathcal{B}$ is $A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & -2 \\ 3 & 4 & 5 \end{bmatrix}$

ii) $h : V \to V$ is a linear transformation and the matrix of h w.r.t. the basis $\mathcal{B}$ is $B = \begin{bmatrix} -1 & 1 & 1 \\ -3 & -4 & 2 \\ 1 & -2 & -3 \end{bmatrix}$

Prove that the map $g(x, y) = f(x, h(y))$ is a bilinear form on V. Find its matrix w.r.t the basis $\mathcal{B}$.

Classification of Quadratic Forms

A quadratic form $\varphi(x,x)$ is said to be

positive definite if	$\varphi(x,x) > 0$	for all $x \in V, x \neq 0$
positive semidefinite if	$\varphi(x,x) \geq 0$	for all $x \in V, x \neq 0$
negative definite if	$\varphi(x,x) < 0$	for all $x \in V, x \neq 0$
negative semidefinite if	$\varphi(x,x) \leq 0$	for all $x \in V, x \neq 0$
indefinite if	$\varphi(x,x) < 0, \varphi(y,y) > 0$	for some $x, y \in V$

Eigenvalues and definiteness

Let A be the matrix of the quadratic form $\varphi(x,x) : \mathbb{R}^n \to \mathbb{R}$ w.r.t. the some basis of $\mathbb{R}^n$. Then φ is

1) positive definite if and only if all eigenvalues of A are strictly positive.
2) negative definite if and only if all eigenvalues of A are strictly negative.
3) positive semidefinite if and only if all eigenvalues of A are nonnegative.
4) negative semidefinite if and only if all eigenvalues of A are nonpositive.

Classification of Quadratic Forms

Exercise

Determine the definiteness of the following quadratic form on $\mathbb{R}^3$.

i) $\omega_1(x_1, x_2, x_3) = x_1^2 + 5x_2^2 - 4x_3^2 + 2x_1x_2 - 4x_1x_3$,

ii) $\omega_2(x_1, x_2, x_3) = x_1x_2 + 4x_1x_3 + x_2x_3$,

iii) $\omega_3 = 5x^2 + 2y^2 + z^2 - 6xy + 2xz - 2yz$.

Exercise

Find a such that the following quadratic forms are positive definite:

a) $5x_1^2 + x_2^2 + ax_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$.

b) $2x_1^2 + x_2^2 + 3x_3^2 + 2ax_1x_2 + 2x_1x_3$.

c) $x_1^2 + x_2^2 + 5x_3^2 + 2ax_1x_2 - 2x_1x_3 + 4x_2x_3$.

Chapter 5: Quadratic Form- Euclidean Space

1 Bilinear form - Quadratic form

2 Reduction of Binary Quadratic Forms
- Lagrange reduction

3 Euclidean space
- The inner product (dot product)
- Gram-Schmidt process
- Projection of a vector onto subspace

4 Orthogonal Diagonalization
- Orthogonal Diagonalization of Symmetric Matrices
- Orthogonal diagonalization of Quadratic Forms
- Quadratic curve in plane
- Quadratic surface classification
- Orthogonal transform to constrained extrema problems

Reduction of Binary Quadratic Forms

Definition (diagonal form)

The following form

$$\varphi(x, x) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \ldots + \alpha_n x_n^2$$

is called the diagonal form of the quadratic form w.r.t. some basis S of V. The corresponding symmetric matrix is diagonal $A = \mathrm{diag}[\alpha_1, \ldots, \alpha_n]$.

Lagrange reduction

Let $Q(x,x) = \sum\limits_{i,j=1}^{n} a_{ij}x_i x_j$, where $a_{ij} = a_{ji}$.

1) If there is some $a_{ii} \neq 0$, says $a_{11} \neq 0$:

$$\begin{aligned} Q &= \left(a_{11}x_1^2 + 2a_{12}x_1x_2 + \ldots + 2a_{1n}x_1x_n\right) + \ldots + a_{nn}x_n^2 \\ &= \frac{1}{a_{11}}\left(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n\right)^2 + Q_1, \end{aligned}$$

   where Q_1 does not contain x_1.

2) Let $y_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, y_2 = x_2, \ldots, y_n = x_n$, then $Q = \frac{1}{a_{11}}y_1^2 + Q_1$, where Q_1 does not contain y_1.
3) Continue the procedure with Q_1.
4) If $a_{ii} = 0 \forall k \Rightarrow \exists a_{ij} \neq 0$. Let

$$x_i = y_i + y_j, x_j = y_i - y_j, x_k = y_k, k \neq i,j$$

   then $2a_{ij}x_ix_j = 2a_{ij}\left(y_i^2 - y_j^2\right) \Rightarrow$ Continue the procedure.

Lagrange reduction

Exercise

Lagrange reduction of quadratic forms to canonical (diagonal) form

a) $\omega_1(x_1, x_2, x_3) = x_1^2 + 5x_2^2 - 4x_3^2 + 2x_1x_2 - 4x_1x_3,$

b) $\omega_2(x_1, x_2, x_3) = x_1x_2 + 4x_1x_3 + x_2x_3,$

c) $\omega_3 = 5x^2 + 2y^2 + z^2 - 6xy + 2xz - 2yz.$

d) $5x_1^2 + x_2^2 + ax_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3.$

e) $2x_1^2 + x_2^2 + 3x_3^2 + 2ax_1x_2 + 2x_1x_3.$

f) $x_1^2 + x_2^2 + 5x_3^2 + 2ax_1x_2 - 2x_1x_3 + 4x_2x_3.$

Classification of Quadratic Forms

Example

Find a such that $\omega = 5x_1^2 + x_2^2 + ax_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$ is positive definite.

Method 1: Lagrange reduction:

$$\omega = 5\left(x_1 + \frac{1}{5}x_2 - \frac{2}{5}x_3\right)^2 + \frac{1}{5}(x_2 - 3x_3)^2 + (a-2)\,x_3^2$$

ω is positive definite if and only if $a > 2$.

Method 2: ω is positive definite $\Leftrightarrow$ all the eigenvalues of A are strictly positive.

Sylvester's law of inertia

Sylvester's law of inertia

A real quadratic form Q in n variables can by a suitable change of basis be brought to the diagonal form

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_{ii}x_i^2$$

with each $a_{ii} \in \{0, 1, -1\}$. The number of coefficients of a given sign is an invariant of Q, i.e., does not depend on a particular choice of diagonalizing basis.

1) The number of $+1s$, denoted n_+, is called the positive index of inertia of A,
2) The number of $-1s$, denoted n_-, is called the negative index of inertia of A.

Chapter 5: Quadratic Form- Euclidean Space

1. Bilinear form - Quadratic form
2. Reduction of Binary Quadratic Forms
   - Lagrange reduction
3. Euclidean space
   - The inner product (dot product)
   - Gram-Schmidt process
   - Projection of a vector onto subspace
4. Orthogonal Diagonalization
   - Orthogonal Diagonalization of Symmetric Matrices
   - Orthogonal diagonalization of Quadratic Forms
   - Quadratic curve in plane
   - Quadratic surface classification
   - Orthogonal transform to constrained extrema problems

The inner product (dot product)

Definition

Let V be a linear space. An inner product (dot product) on V is a map $V \times V \to R$ that satisfies the following axioms:

1) $u \cdot v = v \cdot u,$
2) $(u + v) \cdot w = u \cdot w + v \cdot w,$
3) $(ku) \cdot v = k(u \cdot v),$
4) $u \cdot u > \geq 0, u \cdot u = 0$ *if and only if* $u = 0.$

Remark: Inner product is a symmetric bilinear form and the corresponding quadratic form is positive definite.

Length - Distance - Orthogonality

Length of a vector

The *length* (or *norm*) of a vector $v \in V$ is defined by $\|v\| = \sqrt{v \cdot v}$.

Distance

The distance between two vectors u and v is defined by $d(u, v) = \|u - v\|$.

Orthogonality

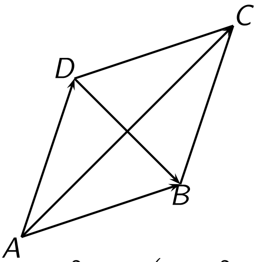
Two vectors u and v are called *orthgonal*, denoted by $u \perp v$, if

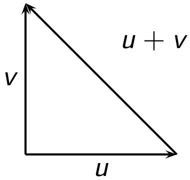
$$u \cdot v = 0.$$

Parallelogram formulas - Pythagorean theorem

Let V be an Euclidean space.

$$\begin{cases} \|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right), \\ u \perp v \Leftrightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2, \forall u, v \in V. \end{cases}$$





$$\left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{DB}\right|^2 = 2\left(\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AD}\right|^2\right)$$

The inner product (dot product)

Exercise

Determine if the following are inner products on $P_3[x]$?

a) $p \cdot q = p(0)q(0) + p(1)q(1) + p(2)q(2)$

b) $p \cdot q = p(0)q(0) + p(1)q(1) + p(2)q(2) + p(3)q(3)$

c) $p \cdot q = \int\limits_{-1}^{1} p(x)q(x)dx.$

In case it is, compute $p \cdot q$, where
$p = 2 - 3x + 5x^2 - x^3, q = 4 + x - 3x^2 + 2x^3.$

The inner product (dot product)

Exercise

Let $\mathcal{B} = \{e_1, e_2, ..., e_n\}$ be a basis of an n-dimensional vector space V. If $u, v \in V$, we have $\begin{cases} u = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n, \\ v = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n \end{cases}$

$$\Rightarrow u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

a) Prove that this is an inner product on V.

b) Apply for $V = \mathbb{R}^3$, where $e_1 = (1, 0, 1)\,, e_2 = (1, 1, -1)\,, e_3 = (0, 1, 1)\,, u = (2, -1, -2)\,, v = (2, 0, 5)$ and compute $u \cdot v$.

c) Apply for $V = P_2[x]$, where $\mathcal{B} = \{1, x, x^2\}\,, u = 2 + 3x^2, v = 6 - 3x - 3x^2$ and compute $u \cdot v$.

d) Apply for $V = P_2[x]$, where $\mathcal{B} = \{1 + x, 2x, x - x^2\}\,, u = 2 + 3x^2, v = 6 - 3x - 3x^2$ and compute $u \cdot v$.

Orthogonal and orthonormal set

Orthogonal and orthonormal set

a) A set of vectors $(e_1, e_2, \ldots, e_k)$ in an inner product space is called pairwise orthogonal if each pairing of them is orthogonal, i.e.,

$$< e_i, e_j >= 0 \text{ náž£u } i \neq j.$$

Such a set is called an orthogonal set.

b) A set of vectors $(e_1, e_2, \ldots, e_k)$ form an orthonormal set if all vectors in the set are mutually orthogonal and all of unit length,i.e.,

$$< e_i, e_j >= \delta_{ij} = \begin{cases} 0 \text{ náž£u } i \neq j \\ 1 \text{ náž£u } i = j \end{cases}$$

Orthogonal and orthonormal set

Proposition

i) An orthogonal set of non-zero vectors is linearly independent.

ii) If $(e_1, e_2, \ldots, e_k)$ *are an orthogonal set of non-zero vectors, then* $\left(\frac{e_1}{\|e_1\|}, \frac{e_2}{\|e_2\|}, \ldots, \frac{e_k}{\|e_k\|}\right)$ *is an orthonormal set.*

Gram-Schmidt process

Theorem

Let V be an inner product space and $S = \{u_1, u_2, ..., u_n\}$ be linearly vectors in V. There is an orthonormal set $S' = \{v_1, v_2, ..., v_n\}$, such that $\text{span}\,\{u_1, u_2, ..., u_k\} = \text{span}\,\{v_1, v_2, ..., v_k\}$ *for all $k = 1, 2, ..., n$.*

Gram-Schmidt process

1) Let $v_1 = u_1$.
2) Let $v_2 = u_2 + tv_1$ such that $v_2 \cdot v_1 = 0 \Rightarrow t = -u_2 \cdot v_1$.
3) Continue this procedure.
4) Let $v_k = u_k + t_1 v_1 + ... + t_{k-1} v_{k-1}$ such that $v_k \cdot v_j = 0, \forall j = \overline{1, k-1} \Rightarrow t_j = -u_k \cdot v_j$.
5) Continue this procedure until $k = n$ we construct orthogonal set $S' = \{v_1, v_2, ..., v_n\}$.
6) Orthonormalization the set $S' = \{v_1, v_2, ..., v_n\}$ to get the result.

GramâĂŞSchmidt process

Example

Apply the Gram-Schmidt process to the vectors $\{u_1, u_2, u_3, u_4\}$, where

$$u_1 = (1,1,1,1)\,, u_2 = (0,1,1,1)\,, u_3 = (0,0,1,1)\,, u_4 = (0,0,0,1)\,.$$

Example

Let the inner product on $P_2[x]$ be defined as $p \cdot q = \int\limits_{-1}^{1} p(x)q(x)dx$, where $p, q \in P_2[x]$.

a) Apply the GramâĂŞSchmidt process to the basis $\mathcal{B} = \{1, x, x^2\}$ to get an orthonormal basis $\mathcal{A}$.

b) Find the change of basis matrix for converting the basis $\mathcal{B}$ to the basis $\mathcal{A}$

c) Find the coordinate vector $[r]_{\mathcal{A}}$ if $r = 2 - 3x + 3x^2$

Orthogonal complement

Definition

Let U, V be subspaces of an Euclide space E.

a) *We say that a vector $v \in E$* orthogonal *(or* perpendicular*) with U and write $v \perp U$, if $v \perp u$ for all $u \in U$.*

b) *We say that U* orthogonal *(or* perpendicular*) with V and write $U \perp V$, if $u \perp v$ for all $u \in U, v \in V$.*

Definition

Let U be a subspace of an Euclidean space E. We define the orthogonal complement $W^{\perp}$ to be

$$U^{\perp} = \{v \in E | v \perp U\}.$$

i) $U^{\perp}$ is a subspace of E.

ii) If E is a finite dimensional space, then $\dim E = \dim U + \dim U^{\perp}$.

Orthogonal complement

Example

Let V be an n-dimensional Euclidean space and V_1 be an m-dimensional subspace of V. Let $V_2 = \{u \in V | \, u \perp v, \forall v \in V_1\}$.

a) Prove that V_2 is a subspace of V.

b) Prove that V_1 and V_2 be orthogonal complement.

c) Find $\dim V_2$

Example

Let

$$v_1 = (1, 1, 0, 0, 0)\,, v_2 = (0, 1, -1, 2, 1)\,, v_3 = (2, 3, -1, 2, 1)$$

and $V = \{x \in \mathbb{R}^5 | \, x \perp v_i, i = 1, 2, 3\}$

a) Prove that V is a subspace of $\mathbb{R}^5$.

b) Find $\dim V$.

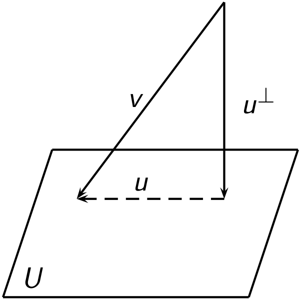
Projection of a vector onto a subspace

Theorem

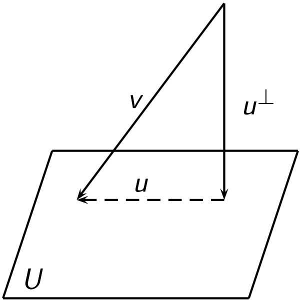
Let U be a subspace of an Euclidean space E. Each vector $v \in E$ has a unique representation

$$v = u + u^{\perp}, \text{ where } u \in U, u^{\perp} \in U^{\perp}.$$

u is called the projection of v onto subspace U.



Projection of a vector onto a subspace



Theorem

Let U be a subspace of an Euclidean space E and $S = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of U. The projection of v onto U is

$$u = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \cdots + (v \cdot u_n)u_n.$$

Projection of a vector onto a subspace

Example

Let $v_1 = (6, 3, -3, 6)$, $v_2 = (5, 1, -3, 1)$. Find the projection of $v = (1, 2, 3, 4)$ onto $U = \mathrm{span}(v_1, v_2)$.

Projection of a vector onto a subspace

Example

Let $v_1 = (6, 3, -3, 6)\,, v_2 = (5, 1, -3, 1)$. Find the projection of $v = (1, 2, 3, 4)$ onto $U = \text{span}(v_1, v_2)$.

Method 1: Apply the Gram-Schmidt process vectors $\{v_1, v_2\}$ to get an orthonormal basis of W:

$$u_1 = \frac{1}{3\sqrt{10}}(6, 3, -3, 6)\,, \quad u_2 = \frac{1}{\sqrt{260}}(9, -3, -7, -11)\,.$$

Apply the formula

$$u = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 = \left(\frac{-9}{26}, \frac{21}{13}, \frac{10}{13}, \frac{115}{26}\right).$$

Method 2: Decompose $v = u + u^{\perp}$, where $u \in U, u^{\perp} \perp W$. It leads to a system of two linear equations.

Projection of a vector onto a subspace

Vector projection

The vector projection of a vector v onto a nonzero vector $u \Leftrightarrow$ the projection of v onto $U = \text{span}\,(u)$

Method 1: U has an orthonormal basis $S = \left\{u_1 = \frac{u}{\|u\|}\right\}$. Therefore,

$$w_1 = (v \cdot u_1)u_1 = \frac{v \cdot u}{\|u\|^2}u$$

is the projection of v onto u.

Method 2: Decompose $v = w_1 + w_2$, where $w_1 \in U, w_2 \perp U$. Base on conditions $w_1 \in W, w_2 \perp W$ to find w_1, w_2.

Example

Find the projection of $u = (1, 3, -2, 4)$ onto $v = (2, -2, 4, 5)$

Projection of a vector onto a subspace

Let W be a subspace of a finite Euclidean space V and $v \in V$. Prove that

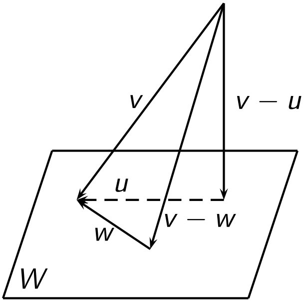
a) There exists $u \in W$ such that $(v-u) \perp W$.

b) Then $\|v-u\| \leq \|v-w\|, \forall w \in W$.

Geometric meaning

i) u is the projection of v onto W.

ii) The the perpendicular distance is the shortest one.



Chapter 5: Quadratic Form- Euclidean Space

1. Bilinear form - Quadratic form
2. Reduction of Binary Quadratic Forms
   - Lagrange reduction
3. Euclidean space
   - The inner product (dot product)
   - Gram-Schmidt process
   - Projection of a vector onto subspace
4. Orthogonal Diagonalization
   - Orthogonal Diagonalization of Symmetric Matrices
   - Orthogonal diagonalization of Quadratic Forms
   - Quadratic curve in plane
   - Quadratic surface classification
   - Orthogonal transform to constrained extrema problems

Orthogonal Diagonalization of Symmetric Matrices

Definition

An orthogonal matrix is a square matrix whose columns and rows are orthonormal vectors,i.e., $PP^T = P^TP = I$.

Definition

A square matrix A is orthogonally diagonalizable if there is an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix.

Necessary and sufficient conditions

A matrix is orthogonally diagonalizable if and only if it is symmetric.

Theorem

i) *All eigenvalues of a symmetric matrix are real.*

ii) *Eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal.*

Orthogonal Diagonalization of Symmetric Matrices

Method for orthogonal diagonalization of a symmetric matrix

1) Find eigenvalues of A.
2) Find the eigenspace for each eigenvalue and its basis.
3) Apply Gram-Schmidt process to find an orthogonal basis for each eigenspace.
4) Together, these orthogonal bases of eigenspaces form an orthogonal basis $\{f_1, f_2, ..., f_n\}$ of $\mathbb{R}^n$ corresponding to eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.
5) Let $P = [[f_1], [f_2], \ldots, [f_n]]$ as columns, then

$$A' = P^{-1}AP = P^tAP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Orthogonal Diagonalization of Symmetric Matrices

Exercise

Orthogonal Diagonalization of the following Symmetric Matrices

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

b) $B = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

Orthogonal diagonalization of Quadratic Forms

Let Q be a quadratic form.

1) Find the symmetric matrix A which represents Q.
2) Let P be the matrix that orthogonally diagonalizes A.
3) Then

$$\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = P \begin{bmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{bmatrix}$$

is the required orthogonal change of coordinates and

$$Q = \lambda_1 \xi_1^2 + \lambda_2 \xi_2^2 + \dots + \lambda_n \xi_n^2.$$

Example

Orthogonal diagonalization of $Q = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$.

Orthogonal diagonalization of Quadratic Forms

Exercise

Orthogonal diagonalization of the following quadratic forms

a) $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$
b) $7x_1^2 - 7x_2^2 + 48x_1x_2$
c) $2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_2x_3$
d) $5x_1^2 + x_2^2 + x_3^2 - 6x_1x_2 + 2x_1x_3 - 2x_2x_3$.

Quadratic curve classification

The general bivariate quadratic curve can be written

$$ax^2 + 2bxy + cy^2 + 2dx + 2fy + g = 0,$$

where the left hand-side is a sum of function P and a quadratic form Q:

$$\begin{cases} Q = ax^2 + 2bxy + cy^2, \\ P = 2dx + 2fy + g. \end{cases}$$

1) Orthogonal diagonalization of $Q \Rightarrow \lambda_1\xi_1^2 + \lambda_2\xi_2^2 + r\lambda_1 + s\lambda_2 + t = 0$.
2) The quadratic curve then can be classified as: parabola, hyperbola, ellipse or circle depending on the context.

Example

Quadratic curve classification $x^2 + 2xy + y^2 + 8x + y = 0$.

Remark

Only Orthogonal diagonalization can be used to classify quadratic curves, since it preserves the length.

Quadratic curve classification

Exercise

Classify the following quadratic curves

a) $2x^2 - 4xy - y^2 + 8 = 0$

b) $x^2 + 2xy + y^2 + 8x + y = 0$

c) $11x^2 + 24xy + 4y^2 - 15 = 0$

d) $2x^2 + 4xy + 5y^2 = 24$

e) $x^2 + xy - y^2 = 18$

f) $x^2 - 8xy + 10y^2 = 10.$

Quadratic surface classification

The general bivariate quadratic surface can be written

$$ax^2 + by^2 + cz^2 + 2rxy + 2sxz + 2tyz + 2ex + 2gy + 2hz + d = 0,$$

where the left hand-side is a sum of function P and a quadratic form Q:

$$\begin{cases} Q = ax^2 + by^2 + cz^2 + 2rxy + 2sxz + 2tyz, \\ P = 2ex + 2gy + 2hz + d. \end{cases}$$

1) Orthogonal diagonalization of Q
$$\Rightarrow \lambda_1\xi_1^2 + \lambda_2\xi_2^2 + \lambda_3\xi_3^2 + m\xi_1 + n\xi_2 + p\xi_3 + q = 0.$$
2) The quadratic surface then can be classified as: paraboloid, hyperboloid, cylinder, ellipsoid or sphere depending on the context.

Example

Classify the quadratic surface $x^2 + y^2 + z^2 + 2xy = 4.$

Quadratic curve classification

Exercise

Classify the following quadratic surfaces

a) $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 = 4$

b) $5x^2 + 2y^2 + z^2 - 6xy + 2xz - 2yz = 1$

c) $2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 = 16$

d) $7x^2 - 7y^2 + 24xy + 50x - 100y - 175 = 0$

e) $7x^2 + 7y^2 + 10z^2 - 2xy - 4xz + 4yz - 12x + 12y + 60z = 24$

f) $2xy + 2yz + 2xz - 6x - 6y - 4z = 0$

Apply the orthogonal transform to constrained extrema problems

Problem

Let $Q = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ be a quadratic form. Find extrema of Q subject to the constraint $x^T x = x_1^2 + x_2^2 + ... + x_n^2 = 1$.

By orthogonal transform $x = P\xi$, Q becomes

$$Q = \lambda_1 \xi_1^2 + \lambda_2 \xi_2^2 + \cdots + \lambda_n \xi_n^2.$$

Suppose $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$, then

$$\lambda_1 \sum_{i=1}^{n} \xi_i^2 \leq Q \leq \lambda_n \sum_{i=1}^{n} \xi_i^2$$

$x = P\xi \Rightarrow x^T x = (P\xi)^T (P\xi) = \xi^T P^t P\xi = \xi^T \xi = 1 \Rightarrow \lambda_1 \leq Q \leq \lambda_n$

i) Q attains the maximum value λ_n at $\xi^M = (0, 0, ..., 1) \Leftrightarrow x = P\xi^M$.

ii) Q attains the minimum value λ_1 at $\xi^m = (1, 0, ..., 0) \Leftrightarrow x = P\xi^m$.

Orthogonal transform to constrained extrema problems

Example

Let $Q(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$.

a) Find $\max\limits_{x_1^2+x_2^2+x_3^2=1} Q(x_1, x_2, x_3)$, $\min\limits_{x_1^2+x_2^2+x_3^2=1} Q(x_1, x_2, x_3)$.

b) Find $\max\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3)$, $\min\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3)$.

Orthogonal transform to constrained extrema problems

Example

Let $Q(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$.

a) Find $\max\limits_{x_1^2+x_2^2+x_3^2=1} Q(x_1, x_2, x_3)$, $\min\limits_{x_1^2+x_2^2+x_3^2=1} Q(x_1, x_2, x_3)$.

b) Find $\max\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3)$, $\min\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3)$.

Remark

Problem: *Let* $Q = \sum\limits_{i,j=1}^{n} a_{ij}x_ix_j$ *be a quadratic form. Find extrema of* Q *subject to the constraint* $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \ldots + \frac{x_n^2}{a_n^2} = 1$.

Solution: *Let* $y_i = \frac{x_i}{a_i}$, *then* $Q = \sum\limits_{i,j=1}^{n} b_{ij}y_iy_j$, *and the constraint becomes* $y_1^2 + y_2^2 + \ldots + y_n^2 = 1 \Rightarrow$ *back to the above Problem.*