PRACTICE TEST FINAL EXAMINATION FOR CALCULUS 3

Semester: 2022.2

Elitech programs Total times: 90 minutes

Attention: Neither material nor document is allowed

Prob 1. (2 points) Examine for convergence or divergence

a) $\sum_{n=0}^{\infty}\left(\dfrac{2n^4+10}{3n^4+2}\right)^n$. b) $\sum_{n=2}^{\infty}\dfrac{\cos(n\pi)}{n+3}$.

Prob 2. (1 point) Find the domain of convergence of the following series of function

$$\sum_{n=1}^{\infty}\frac{n}{n^2-2}\left(\frac{2x-1}{x}\right)^{2n}$$

Prob 3. (1 point) Given $f(x)=x, \forall x\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$, is a periodic function with period $T=\pi$. Determine the Fourier expansion of $f(x)$.

Prob 4. (2 points) The Predator-Prey model (or Lotka-Volterra model) is a system of first-order differential equations used to describe the dynamics of biological systems, in which two species, one as predator and another as prey (for example fox and rabbit) interact. The model is described as follow:

$$\begin{cases}\dfrac{dR}{dt}=aR-bRF\\ \dfrac{dF}{dt}=cRF-dF\end{cases}$$

in which:

- R and F are the population of the rabbit and the fox respectively.
- a, b, c and c are parameters to determine the growth and the interaction rate.

Given $a=1$, $b=0.02$, $c=0.01$ and $d=0.4$.

a) Determine the initial population of rabbit and fox such that the total population of both species is unchanging with respect to time.

b) Determine the implicit relationship between the population of rabbits and the population of foxes.

Prob 5. (1 points) Solve the following problem $y''-9y=2\sin^2(x)$

Prob 6. (1 point) Find the Laplace Transform of $f(t)$

$$f(t)=e^t(\cos t+t^2\sin 2t)$$

Prob 7. (1 point) Using the Laplace Transform, solve the following problem:

$$y(t) = te^{-2t} + e^{-2t}\int_0^t y(u)e^{2u}du$$

Prob 8. (1 point) Using the Laplace Transform, prove that

$$\mathbf{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

in which $\mathbf{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ and $\Gamma(n) = \int_0^{+\infty} e^{-t}t^{n-1}dt$

Chúc các bạn hoàn thành tốt bài thi

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Prob 1.

a)

+) We can see that: $u_n = \left(\dfrac{2n^4+10}{3n^4+2}\right)^n > 0 \Rightarrow$ The given series is a positive series.

+) Consider $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = \lim\limits_{n\to\infty} \sqrt[n]{\left(\dfrac{2n^4+10}{3n^4+2}\right)^n} = \lim\limits_{n\to\infty} \dfrac{2n^4+10}{3n^4+2} = \dfrac{2}{3} < 1.$

+) The given series converges (According to Cauchy's test)

b)

We have: $\sum\limits_{n=2}^{\infty} \dfrac{\cos(n\pi)}{n+3} = \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n+3}$

+) $u_n = \dfrac{1}{n+3} > 0, \forall n \geq 2 \Rightarrow \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n+3}$ is an alternating series.

+) $u_n = \dfrac{1}{n+3} > \dfrac{1}{n+4} = u_{n+1}, \forall n \geq 2 \Rightarrow \{u_n\}$ is monotonously decreasing when $n \to \infty$

+) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{n+3} = 0$

$\Rightarrow \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n+3}$ converges by Lebnitz test .

$\Rightarrow \sum\limits_{n=2}^{\infty} \dfrac{\cos(n\pi)}{n+3}$ converges.

Prob 2.

Let $Y = \left(\dfrac{2x-1}{x}\right)^2 \geq 0$ the series becomes $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2-2} Y^n$.(*)

We have $a_n = \dfrac{n}{n^2-2}$. The radius of convergence is

$$R = \lim_{n\to+\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to+\infty} \left|\frac{\frac{n}{n^2-2}}{\frac{n+1}{(n+1)^2-2}}\right| = \lim_{n\to+\infty} \left|\frac{\frac{n}{n^2}}{\frac{n}{n^2}}\right| = 1$$

$\Rightarrow$ The interval of convergence (*) is: (-1,1)

Consider $Y = 1$,

The series (*) becomes $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2-2}$ is a positive series with $u_n = \dfrac{n}{n^2-2} > 0, \forall n \geq 2$

When $n \to \infty$ then $u_n \sim \dfrac{n}{n^2} = \dfrac{1}{n}$; nevertheless, $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges (since $\alpha = 1$)

$\Rightarrow \sum\limits_{n=1}^{\infty} \dfrac{n}{n^2-2}$ diverges according to limit-comparison test.

$\Rightarrow$ The series (*) diverges at $Y = 1$

Since $Y \geq 0$, there's no need to consider $Y = -1$

$\Rightarrow$ (*) converges when $\Leftrightarrow 0 \leq Y < 1 \Leftrightarrow 0 \leq \left(\dfrac{2x-1}{x}\right)^2 < 1 \Leftrightarrow -1 < \dfrac{2x-1}{x} < 1$

$\Rightarrow \dfrac{1}{3} < x < 1$

In conclusion, the domain of convergence of the given series is $\left(\dfrac{1}{3}, 1\right)$.

Prob 3.

f(x) = x, $\forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is a periodic function with the period $T = 2L = \pi$. It can be seen that $f(x)$ is an odd function; thus, the Fourier expansion of $f(x)$ has the form

$$f(x) = \sum_{n=0}^{+\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

We have

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin 2nx dx \\ &= \frac{(-1)^{n+1}}{n} \end{aligned}$$

In conclusion, the Fourier expansion of the given function is

$$f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \sin(2nx)$$

Prob 5.

$$\begin{cases} \dfrac{dR}{dt} = R - 0.02RF & (1) \\ \dfrac{dF}{dt} = 0.01RF - 0.4F & (2) \end{cases}$$

a) When the population of both species are unchanging, then $\dfrac{dR}{dt} = \dfrac{dF}{dt} = 0$. Then, we'll have a system of equations:

$$\begin{cases} R - 0.02RF = 0 \\ 0.01RF - 0.4F = 0 \end{cases}$$

Solving the system of equations, we can find that $F = 50$ and $R = 40$.

b) Divide equation 2 by equation 1, we have:

$$\frac{dF}{dR} = \frac{0.01RF - 0.4F}{R - 0.02RF} \tag{3}$$

Equation 3 is equivalent to:

$$\frac{1 - 0.02F}{F}dF = \frac{0.01R - 0.4}{R}dR \tag{4}$$

Integrate both sides of equation 4, we have:

$$\ln F - 0.02F = 0.01R - 0.4\ln R + C. \tag{5}$$

Problem solved.

Prob 5.

The initial equation is equal to:

$$y^{''} - 9y = 1 - \cos(2x) \quad (1)$$

Consider the homogeneous differential equation: $y^{''} - 9y = 0 \quad (2)$

The characteristic equation (2) : $\lambda^2 - 9 = 0 \Leftrightarrow \lambda = \pm 3$

The general solution of (2) is: $\overline{y} = C_1e^{3x} + C_2e^{-3x}$

Since $\lambda_1 = 0$ và $\lambda_2 = \pm 2i$ aren't solutions of the characteristic equation (2); therefore, a particular solution of (1) has the form:

$$Y = A + B\cos(2x) + C\sin(2x)$$

Ta có:

$Y^{'} = -2B\sin(2x) + 2C\cos(2x)$

$Y^{''} = -4B\cos(2x) - 4C\sin(2x)$

Replace into equation (1), we have:

$-4B\cos(2x) - 4C\sin(2x) - 9[A + B\cos(2x) + C\sin(2x)] = 1 - \cos(2x)$

$\Leftrightarrow -9A - 13B\cos(2x) - 13C\sin(2x) = 1 - \cos(2x) \quad \forall x$

$\Leftrightarrow A = \frac{-1}{9}, B = \frac{1}{13}, C = 0$

$\Rightarrow$ Particular solution (1) is $Y = \frac{-1}{9} + \frac{1}{13}\cos(2x)$

In conclusion, the general solution of the given equation is

$$y = \overline{y} + Y = C_1e^{3x} + C_2e^{-3x} - \frac{1}{9} + \frac{1}{13}\cos(2x)$$

Prob 6.

+) $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$
+) $\mathcal{L}\{t^2 \sin 2t\} = (\dfrac{2}{s^2+4})^{(2)} = \dfrac{12s^2-16}{(s^2+4)^3}$
+) $\mathcal{L}\{\cos t + t^2 \sin 2t\} = \dfrac{s}{s^2+1} + \dfrac{12s^2-16}{(s^2+4)^3}$
+) $\mathcal{L}\{e^t(\cos t + t^2 \sin 2t)\} = \dfrac{s-1}{(s-1)^2+1} + \dfrac{12(s-1)^2-16}{((s-1)^2+4)^3}$

Prob 7.

Laplace both sides, we have

$$\mathcal{L}\{y(t)\}(s) = \mathcal{L}\{te^{-2t}\}(s) + \mathcal{L}\left\{e^{-2t}\int_0^t y(u)e^{2u}du\right\}(s)$$

$$\begin{aligned}
\mathcal{L}\{y(t)\}(s) &= F(s) \\
\mathcal{L}\{te^{-2t}\}(s) &= \mathcal{L}\{t\}(s+2) \\
&= \frac{1}{(s+2)^2} \\
\mathcal{L}\left\{e^{-2t}\int_0^t y(u)e^{2u}du\right\}(s) &= \mathcal{L}\left\{\int_0^t y(u)e^{2u}du\right\}(s+2) \\
&= \frac{\mathcal{L}\{y(t)e^{2t}\}(s+2)}{(s+2)} \\
&= \frac{\mathcal{L}\{y(t)\}(s)}{(s+2)} \\
&= \frac{F(s)}{s+2}
\end{aligned}$$

From there, we can solve $F(s) = \dfrac{1}{(s+1)(s+2)} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$
Inverse Laplace both sides, we can gain $y(t) = e^{-t} - e^{-2t}$

Prob 8.

Firstly, we recall the convolution operation. Given two functions $f(x)$ and $g(x)$, the convolution of f and g is defined as:

$$f(x) * g(x) = \int_0^x f(y)g(x-y)dy \tag{6}$$

Let $f(x) = x^{m-1}$ and $g(x) = x^{n-1}$, equation 6 becomes:

$$f(x) * g(x) = \int_0^x y^{m-1}(x-y)^{n-1}dy \tag{7}$$

Let $u = \dfrac{y}{x}$, then $du = \dfrac{1}{x}dy$, the integral 7 becomes:

$$\begin{aligned} f(x) * g(x) &= \int_0^x y^{m-1}(x-y)^{n-1}dy && (8) \\ &= \int_o^1 x^{m+n-1}u^{m-1}(1-u)^{n-1}du && (9) \\ &= x^{m+n-1}\mathbf{B}(m,n) && (10) \end{aligned}$$

Now, we will perform Laplace transformation on both sides of equation 10, we have:

$$\begin{aligned} \mathcal{L}\{f(x) * g(x)\}(s) &= \mathbf{B}(m,n)\mathcal{L}\{x^{m+n-1}\}(s) && (11) \\ \iff \mathcal{L}\{x^{m-1}\}(s)\mathcal{L}\{x^{n-1}\}(s) &= \mathbf{B}(m,n)\mathcal{L}\{x^{m+n-1}\}(s) && (12) \end{aligned}$$

Recall that:

$$\mathcal{L}\{x^{\alpha-1}\}(s) = s^{\alpha}\Gamma(\alpha) \tag{13}$$

Replace 13 into equation 12, we have:

$$s^m\Gamma(m)s^n\Gamma(n) = \mathbf{B}(m,n)s^{m+n}\Gamma(m+n) \tag{14}$$

Simplify the equation 14, we will have:

$$\mathbf{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof completes.