Chapter 4: Linear mappings

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4.1.1. Definitions, examples

Definition

Let V and W be vector spaces over a field K. A mapping $f: V \to W$ is called a *linear mapping* (or a linear map, or a linear transformation, or a vector space homomorphism) if it satisfies the following conditions:

a) $f(u+v) = f(u) + f(v), \quad , \forall u, v \in V;$
b) $f(cv) = cf(v), \quad \forall c \in K, \forall v \in V.$

A linear mapping from V to itself is called a linear endomorphism (or linear operator).

Example

Example

: The following mappings are linear:

1. $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 - 2x_2 + x_3)$.
2. The zero mapping $f\colon V \to W$, $f(v) = \mathbf{0}, \quad \forall v \in V$.
3. The identity mapping $\mathrm{id}_V\colon V \to V$, $\mathrm{id}_V(v) = v$, $\forall v \in V$.
4. For a given $a \in K$, the map $f\colon V \to V$ given by $f(v) = av$.
5. For a real matrix A of size $m \times n$, the map $f\colon \mathcal{M}_{n\times 1}(\mathbb{R}) \to \mathcal{M}_{m\times 1}(\mathbb{R})$ given by $f(X) = AX$.

Example: The following mappings are not linear:

1. $f\colon \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 - 2x_2 + x_3 + 1)$.
2. $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x^2 + y, 2x - y)$.

Properties

Let $f\colon V \to W$ be a linear mapping. Then

- $f(\mathbf{0}) = \mathbf{0}$;
- $f(-v) = -f(v), \quad \forall v \in V$;
- $f(c_1 v_1 + \cdots + c_m v_m) = c_1 f(v_1) + \cdots + c_m f(v_m), \quad , \forall c_i \in K, \forall v_i \in V.$

Example: Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear mapping such that

$$f(1,0,0) = (1,-1,2), f(0,1,0) = (2,3,1), f(0,0,1) = (-1,2,2).$$

Find $f(1,-2,3)$.

Solution.

- Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$, $v = (1,2,3)$. Then $v = e_1 - 2e_2 + 3e_3$.
- $f(v) = f(e_1 - 2e_2 + 3e_3) = f(e_1) - 2f(e_2) + f(e_3) = (1,-1,2) - 2(2,3,1) + 3(-1,2,2) = (-6,-1,6).$

Theorem

Let V and W be vector spaces over K. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V. Let $\{w_1, \ldots, w_n\}$ be n arbitrary vectors in W. Then there exists a unique linear mapping $f : V \to W$ such that

$$f(v_1) = w_1, f(v_2) = w_2, \ldots, f(v_n) = w_n.$$

(A linear map is completely determined by its values on a basis.

Operations

Definition

Let f and g be linear mappings from V to W.

- The sum of f and g is a mapping $f+g\colon V \to W$ which is given by
$$(f+g)(v) = f(v) + g(v), \quad v \in V.$$
- The product of a scalar $a \in K$ and a linear mapping f is a mapping $af\colon V \to W$ given by
$$(af)(v) = af(v), \quad v \in V.$$

Property

The mappings $f+g$ and af are linear.

Proposition

Let $f\colon V \to W$ và $g\colon W \to U$ be linear mappings. Then the mapping $g \circ f\colon V \to U$ is also linear.

4.1.2. Kernel and image

Definition

Let $f\colon V \to W$ be a linear mapping.

- The set $\ker(f) = \{v \in V \mid f(v) = \mathbf{0}\}$ is called the kernel of f.
- The set $\mathrm{im}(f) = \{f(v) \mid v \in V\}$ is called the image of f.

So, $\ker(f) = f^{-1}(\{\mathbf{0}\})$ and $\mathrm{im}(f) = f(V)$.

Property

- $\ker(f)$ is a vector subspace of V.
- $\mathrm{im}(f)$ is a vector subspace W.

Theorem (Rank-nullity theorem; Fundamental theorem of linear maps)

Let $f\colon V \to W$ be a linear mapping and $\dim V = n$. Then

$$\dim(\mathrm{im} f) + \dim(\ker f) = n.$$

Proposition

Let $f\colon V \to W$ be a linear mappings. Suppose that $S = \{v_1, \ldots, v_n\}$ is a spanning set of V. Then $\{f(v_1), \ldots, f(v_n)\}$ is a spanning set of $\mathrm{im} f$.

Thus,

$$\boxed{V = \mathrm{span}\{v_1, \ldots, v_n\} \Rightarrow f(V) = \mathrm{span}\{f(v_1), \ldots, f(v_n)\}}$$

In particular, if S is a basis for V then $f(S)$ is a spanning set of $f(V) = \mathrm{im} f$.

Definition

Let $f\colon V \to W$ be a linear mapping, the *rank* of f, denoted by $\mathrm{rank}(f)$ is defined to be the dimension of the image of f:

$$\mathrm{rank}(f) = \dim(\mathrm{im}(f)).$$

Example

Consider the linear mapping $f\colon \mathbb{R}^4 \to \mathbb{R}^3$,
$f(x_1, x_2, x_3, x_4) = (x_1 - x_2 + 2x_3 + x_4, 2x_1 - 2x_2 + 3x_3 + 4x_4, x_1 - x_2 + x_3 + 3x_4)$. Find a basis for $\ker(f)$ and a basis for $\mathrm{im}(f)$.

- $v = (x_1, x_2, x_3, x_4) \in \ker(f) \Leftrightarrow f(v) = \mathbf{0} \Leftrightarrow (x_1, x_2, x_3, x_4)$ is a solution of the system
$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 & = 0 \\ 2x_1 - 2x_2 + 3x_3 + 4x_4 & = 0 \\ x_1 - x_2 + x_3 + 3x_4 & = 0 \end{cases}.$$
- Solve the homogeneous linear system: $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & -2 & 3 & 4 \\ 1 & -1 & 1 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
- This system has infinitely many solutions: $x_1 = a - 5b$, $x_2 = a$, $x_3 = 2b$, $x_4 = b$. Và $v = (a - 5b, a, 2b, b) = (a, a, 0, 0) + (-5b, 0, 2b, b) = a(1, 1, 0, 0) + b(-5, 0, 2, 1)$.
- $S = \{(1, 1, 0, 0), (-5, 0, 1, 1)\}$ is a spanning set of $\ker(f)$.
- Can check that S is linearly independent. Hence S is a basis for $\ker(f)$.

- Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $\mathbb{R}^4$.
- $\mathrm{im}(f) = \mathrm{span}\{f(e_1), f(e_2), f(e_3), f(e_4)\} = \mathrm{span}\{(1,2,1), (-1,-2,-1), (2,3,1), (1,4,3)\}$.
- $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 2 & 3 & 1 \\ 1 & 4 & 3 \end{bmatrix} \to \cdots \to C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- A basis for $\mathrm{im}(f)$ is $\{(1,2,1), (0,-1,-1)\}$.
- (Another basis for $\mathrm{im}(f)$ is $\{(1,2,1), (2,3,1)\}$.)

4.1.3. Injective, surjective, bijective linear mappings

Proposition

Let $f: V \to W$ be a linear mapping.

- f is injective $\Leftrightarrow \ker(f) = \{\mathbf{0}\}$.
- f is surjective $\Leftrightarrow \mathrm{rank}(f) = \dim(W)$. [Nhắc lại $\mathrm{rank}(f) = \dim(\mathrm{im}(f))$.]
- If f is bijective then it inverse $f^{-1}: W \to V$ is linear and bijective.

A bijective linear mapping is also called an isomorphism.

Theorem

Let $f\colon V \to W$ be a linear mapping. Suppose that $\dim V = \dim W = n$. The following statements are equivalent.

- f is injective.
- f is surjective.
- f is bijective.

Exercise: (CK20171-No3) Let $P_2[x]$ the vector space of all real polynomials of degree less than or equal and let $\varphi\colon P_2[x] \to \mathbb{R}^3$ be a mapping given by $\varphi(p(x)) = (p(0), p(1), p(-1))$. Is φ an isomorphism? Explain your answer?

Isomorphic vector spaces

Định nghĩa

We say that a vector space V is *is isomorphic to* a vector space W if there is an isomorphism $f : V \to W$. In this case, we also say that V and W are *isomorphic*.

Proposition

Let V and W be finite dimensional vector spaces. Then

$$V \text{ and } W \text{ isomorphic} \Leftrightarrow \dim V = \dim W.$$

Corollary

Every real vector space of dimension n is isomorphic to $\mathbb{R}^n$.

4.2.1. Matrix of a linear mapping

Problem

Let $f\colon V \to W$ be a linear map. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for V, $\mathcal{B}' = \{w_1, \dots, w_m\}$ a basis for W. Find a relation between $[f(v)]_{\mathcal{B}'}$ and $[v]_{\mathcal{B}}$.

An answer: $\exists!$: $[f(v)]_{\mathcal{B}'} = A[v]_{\mathcal{B}}$, $\forall v \in V$.
For each $v_j \in \mathcal{B}$, express $f(v_j)$ as a linear combination of vectors in $\mathcal{B}'$:

$$\begin{aligned}
f(v_1) &= a_{11}w_1 +_{21} w_2 + \cdots + a_{m1}w_m \\
f(v_2) &= a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\
&\dots \\
f(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m.
\end{aligned}$$

In other words,

$$[f(v_1)]_{\mathcal{B}'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, [f(v_2)]_{\mathcal{B}'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, [f(v_n)]_{\mathcal{B}'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Definition

Matrix $A = [f]_{\mathcal{B},\mathcal{B}'} = [[f(v_1)]_{\mathcal{B}'}\ [f(v_2)]_{\mathcal{B}'} \cdots [f(v_n)]_{\mathcal{B}'}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ is called the *(representation) matrix of f with respect to (relative to) the bases $\mathcal{B}$ and $\mathcal{B}'$.*

Theorem

$$[f(v)]_{\mathcal{B}'} = A[v]_{\mathcal{B}}, \quad \forall v \in V.$$

Moreover, if B is a matrix such that $[f(v)]_{\mathcal{B}'} = B[v]_{\mathcal{B}}, \forall v \in V$, then $B = A$.

Proposition (Rank of a linear map and rank of its matrix)

$$\mathrm{rank}(f) = \mathrm{rank}(A).$$

Example

Consider the linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (x - y + z, x + 2y - z)$.
Find the matrix of f relative to the standard bases.

- $f(e_1) = f(1, 0, 0) = (1, 1)$ and $[f(e_1)] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- $f(e_2) = f(0, 1, 0) = (-1, 2)$ and $[f(e_2)] = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
- $f(e_3) = f(0, 0, 1) = (1, -1)$ and $[f(e_3)] = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- The matrix of f relative to the standard bases. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$.

Ví dụ

Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (x - y + z, x + 2y - z)$.
Find the matrix of f relative to the bases $\mathcal{B} = \{v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1)\}$ and $\mathcal{B}' = \{w_1 = (1, 0), w_2 = (1, 1)\}$.

- $f(v_1) = f(1, 0, 0) = (1, 1)$ and $[f(v_1)]_{\mathcal{B}'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- $f(v_2) = f(1, 1, 0) = (0, 3)$ and $[f(v_2)]_{\mathcal{B}'} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$.
- $f(v_3) = f(1, 1, 1) = (1, 2)$ and $[f(v_3)]_{\mathcal{B}'} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
- The matrix of f relative to the bases $\mathcal{B}$ and $\mathcal{B}'$ is $\begin{bmatrix} 0 & -3 & -1 \\ 1 & 3 & 2 \end{bmatrix}$.

Example

Suppose that a linear map $f\colon \mathbb{R}^3 \to P_2[x]$ has the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$ relative to the bases $\mathcal{B} = \{(1,1,1),(1,1,0),(0,1,1)\}$ và $\mathcal{B}' = \{1, 1+x, 1+x^2\}$. Find $f(2,3,2)$.

- Set $v_1 = (1,1,1), v_2 = (1,1,0), v_3 = (0,1,1)$.
- $[f(v_1)]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow f(v_1) = 1\cdot 1 + 1\cdot(1+x) + 2\cdot(1+x^2) = 4 + x + 2x^2$.
- $[f(v_2)]_{\mathcal{B}'} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow f(v_2) = 2\cdot 1 + (-1)\cdot(1+x) + 1\cdot(1+x^2) = 2 - x + x^2$.
- $[f(v_3)]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \Rightarrow f(v_3) = 1\cdot 1 + 2\cdot(1+x) + (-1)\cdot(1+x^2) = 2 + 2x - x^2$.
- We have $v = v_1 + v_2 + v_3$ and

$$\begin{aligned} f(v) &= f(v_1+v_2+v_3) = f(v_1) + f(v_2) + f(v_3) \\ &= (4+x+2x^2) + (2-x+x^2) + (2+2x-x^2) = 8 + 2x + 2x^2 \end{aligned}$$

Example

Suppose that a linea map $f : \mathbb{R}^3 \to P_2[x]$ has the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$ relative to the bases $\mathcal{B} = \{(1,1,1),(1,1,0),(0,1,1)\}$ *and* $\mathcal{B}' = \{1, 1+x, 1+x^2\}$. Find $f(2,3,2)$.

- Set $v = (2,3,2)$. We have $[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ và

$$[f(v)]_{\mathcal{B}'} = A[v]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

- $f(v) = 4 \cdot 1 + 2 \cdot (1+x) + 2 \cdot (1+x^2) = 8 + 2x + 2x^2$.

Relations between linear maps and matrices

- Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V, $\dim V = n$.
- Let $\mathcal{B}' = \{w_1, \ldots, w_m\}$ be a basis for W, $\dim V = m$.

Then

- For any linear map $f : V \to W$, $[f]_{\mathcal{B},\mathcal{B}'}$ is a matrix of size $m \times n$.
- Conversely, for any matrix A of size $m \times n$, the exists a unique linear map $f : V \to W$ such that $[f]_{\mathcal{B},\mathcal{B}'} = A$.

Thus, there is a bijection (1-1 correspondence) between the set of linear maps form V to W and the set of matrices of size $m \times n$.

Addition and và scalar multiplication

- Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V.
- Let $\mathcal{B}' = \{w_1, \ldots, w_m\}$ be a basis for W.
- Let f and g be linear maps from V to W.
- Let A and B be the matrix of f and g (respectively) with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$.

Then

- $A + B$ is the matrix of $f + g$ relative to the bases $\mathcal{B}$ and $\mathcal{B}'$;
- For $c \in K$, cA is the matrix of cf relative to the bases $\mathcal{B}$ and $\mathcal{B}'$.

Composition of linear maps and matrix multiplication

- Suppose V, W and U are vector spaces with basis $\mathcal{B}$, $\mathcal{B}'$ và $\mathcal{B}''$.
- Let $f : V \to W$ be a linear map and let A be the matrix of f relative to the bases $\mathcal{B}$ và $\mathcal{B}'$.
- Let $g : W \to U$ be a linear map and let B be the matrix of g relative to the bases $\mathcal{B}'$ và $\mathcal{B}''$.
- Then BA is the matrix of $g \circ f : V \to U$ relative to the bases $\mathcal{B}$ and $\mathcal{B}''$.

Corollary

Let $f\colon V \to W$ be a linear map. Let A be a matrix of f relative to the bases $\mathcal{B}$ và $\mathcal{B}'$. The following statements are equivalent.

- f is an isomorphism.
- A is invertible.

In this case, A^{-1} is the matrix of f^{-1} relative $\mathcal{B}'$ and $\mathcal{B}$.

4.2.2. Matrix of a linear endomorphism relative to a basis

- Let $f\colon V \to V$ be a linear endomorphismand $\mathcal{B}$ a basis for V.
- The matrix A of f relative the pair of matrices $\mathcal{B}$ and $\mathcal{B}' = \mathcal{B}$ is simply called the matrix f relative to the basis $\mathcal{B}$.
- Thus, if $\mathcal{B} = \{v_1, \dots, v_n\}$ then

$$A = [f]_{\mathcal{B}} = [[f(v_1)]_{\mathcal{B}} \cdots [f(v_n)]_{\mathcal{B}}].$$

Property

$$[f(v)]_{\mathcal{B}} = A[v]_{\mathcal{B}}, \quad \forall v \in V.$$

Moreover, if B is a matrix such that $[f(v)]_{\mathcal{B}} = B[v]_{\mathcal{B}}, \forall v \in V$, then $B = A$.

Change of basis

- Let $f \colon V \to V$ be a linear endomorphism.
- Let A be a matrix of f relative to a basis $\mathcal{B}$ for V.
- Let Bbe a matrix of f relative to a basis $\mathcal{B}'$ for V.
- Let P be the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Theorem

$$B = P^{-1}AP.$$

Proof: For any $v \in V$, we have $[f(v)]_{\mathcal{B}} = A[v]_{\mathcal{B}}$, $[f(v)]_{\mathcal{B}'} = B[v]_{\mathcal{B}'}$, $[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}$. Hence

$$[f(v)]_{\mathcal{B}} = A[v]_{B} = AP[v]_{\mathcal{B}'},$$

$$[f(v)]_{\mathcal{B}} = P[f(v)]_{\mathcal{B}'} = PB[v]_{\mathcal{B}'}.$$

Thus $AP[v]_{\mathcal{B}} = PB[v]_{\mathcal{B}'}$, for every $v \in V$. This implies that $AP = PB$. Hence $B = P^{-1}AP$.

Example

Consider the linear map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (2x-y, x+y)$. Find the matrix of f relative to the basis $\mathcal{B} = \{(1,0),(1,1)\}$.

Solution 1:

- $f(1,0) = (2,1) \Rightarrow [f(1,0)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- $f(1,1) = (1,2) \Rightarrow [f(1,1)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
- Matrix of f relative to $\mathcal{B}$ is $\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

Solution 2:

- Let A be matrix of f relative the standard basis $\Rightarrow A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.
- Let P be the transition matrix from the standard basis to the basis$\mathcal{B} \Rightarrow P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- The matrix of f relative to the basis $\mathcal{B}$ is

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

Example (CK20181-N2)

Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that $f(1,1,0) = (3,3,9)$, $f(2,-1,1) = (-1,3,1)$, $f(0,1,1) = (1,1,3)$.

a) Find the matrix of f relative to the standard basis for $\mathbb{R}^3$. [b)] Find $f(3,4,5)$.

c) Find the dimension and a basis of $\ker(f)$.

- Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.
  [We want to find $f(e_1)$, $f(e_2)$, $f(e_3)$.]
- $f(1,1,0) = f(e_1 + e_2) = f(e_1) + f(e_2) = (3,3,9) = v_1$.
- $f(2,-1,1) = f(2e_1 - e_2 + e_3) = 2f(e_1) - f(e_2) + f(e_3) = (-1,3,1) = v_2$.
- $f(0,1,1) = f(e_2 + e_3) = f(e_2) + f(e_3) = (1,1,3) = v_3$.
- We obtain a system $\begin{cases} f(e_1) + f(e_2) & = v_1 \\ 2f(e_1) - f(e_2) + f(e_3) & = v_2 \\ f(e_2) + f(e_3) & = v_3 \end{cases} \Leftrightarrow \begin{cases} f(e_1) & = (1,2,4) \\ f(e_2) & = (2,1,5) \\ f(e_3) & = (-1,0,-2) \end{cases}$
- The matrix of f relative to the standard basis of $\mathbb{R}^3$ is $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & -2 \end{bmatrix}$.

Example (CK20181-N2)

Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map such that $f(1,1,0) = (3,3,9)$, $f(2,-1,1) = (-1,3,1)$, $f(0,1,1) = (1,1,3)$.

a) Find the matrix of f relative to the standard basis for $\mathbb{R}^3$.

Solution 2:

- Let A be the matrix of f relative to the standard basis for $\mathbb{R}^3$. Then, for every $v \in \mathbb{R}^3$, one has
$$[f(v)] = A[v].$$
- $A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\3\\9\end{bmatrix}, \quad A\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}-1\\3\\1\end{bmatrix}, \quad A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\3\end{bmatrix}.$
- $A\begin{bmatrix}1 & 2 & 0\\1 & -1 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}3 & -1 & 1\\3 & 3 & 1\\9 & 1 & 3\end{bmatrix} \Leftrightarrow A = \begin{bmatrix}1 & 2 & -1\\2 & 1 & 0\\4 & 5 & -2\end{bmatrix}.$

Some exercises

- (CK20183) A linear map $f\colon P_2[x] \to P_2[x]$ has representation matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 3 \end{bmatrix}$ with respect to the basis $B = \{v_1, v_2, v_3\}$ với $v_1 = 1$, $v_2 = 1 + x$, $v_3 = 2 - x + x^2$.
  - Find the matrix of f relative the standard basis $E = \{1, x, x^2\}$. Find $f(4 + 3x + 2x^2)$.
  - Find the dimension and a basis of $\ker(f)$.
- (CK20193) Consider a linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 + 2x_3, 6x_1 - 2x_2 + 5x_3)$.
  - Find the matrix of f relative to the standard basis for $\mathbb{R}^3$.
  - Find $\dim \mathrm{im}(f)$ and $\dim \ker(f)$.
  - Is the vector $u = (1, 2, 3)$ in $\mathrm{im} f$? Why?
- (CK20193-N2) Let $f\colon P_2[x] \to P_3[x]$ be a linear map which is given by $f(p) = xp + 2p$. Find the matrix of f with respective to the standard bases of $P_2[x], P_3[x]$.
- (CK20161) Let $f\colon P_2[x] \to P_2[x]$ be a linear map such that $f(1 + x^2) = 2 + 5x + 3x^2$, $f(-1 + 2x + 3x^2) = 7(x + x^2)$, $f(x + x^2) = 3(x + x^2)$.
  - Find the matrices of f and $f^2 = f \circ f$ relative the standard basis $\{1, x, x^2\}$ of $P_2[x]$.
  - Determine the value of m such that the vector $v = 2 + mx + 5x^2$ is in $\mathrm{Im} f$.

4.2.3. Matrix similarity

4.3.1. Eigenvalues and eigenvectors of matrices

4.3.2. Eigenvalues and eigenvectors of linear endomorphisms

4.3.2. Matrix diagonalization