Classical Mechanics

Lesson 1

Vector Algebra and Vector Calculus

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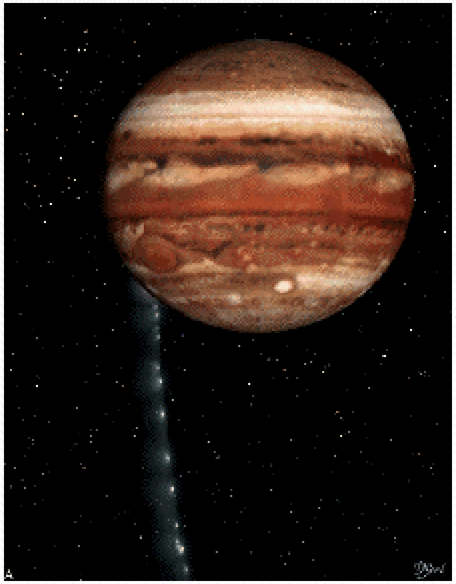
Outline

- **Introduction**
- **Vector Algebra**
  - **Addition/Subtraction**
  - **Multiplication on a scalar**
  - **Rotation**
  - **Scalar (or *dot*) product**
  - **Vector(or *cross*) product**
- **Vector Calculus**
  - **Derivatives**
  - **Gradient**
  - **Divergence**
  - **Rotor (curl)**
  - **Line integrals**
  - **Vector line integrals**
  - **Volume integrals**

Steps in Science

- One of the things that scientists do is make *predictions* - predictions based on their ***hypotheses***, ***laws***, and ***theories***. The test of a prediction is whether it works in the "real world" - do the results of experiments match the theoretical prediction? If the results don't match then the hypothesis that generated the prediction must be modified or abandoned. The ultimate authority in science is nature.
- A *hypothesis* is an "educated guess“ about what nature is going to do, or about why nature does what it does. A *scientific* hypothesis must be ***testable***, and ***falsifiable***.
- A scientific *law* (sometimes called a *principle*) is a powerful ***summary*** of many previously unrelated facts.
- A scientific *theory* is a synthesis of a large body of information that encompasses well-tested and verified hypotheses about certain aspects of the natural world. A physical theory is ***explanatory***, ***well tested***, and ***mathematical*** in nature.
- Carefully planned and executed ***experiments*** are designed to make observations. Experiment data is a *merit* of validity for any theory.

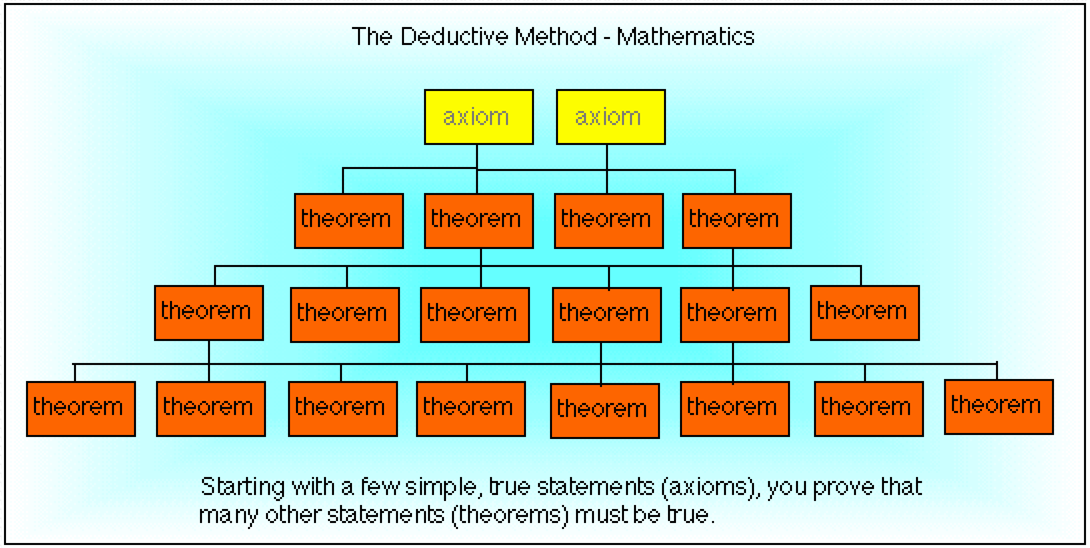


Comet Shoemaker-Levy hits Jupiter, as seen by the Hubble Space Telescope

Methods in Physics (and in other Science)

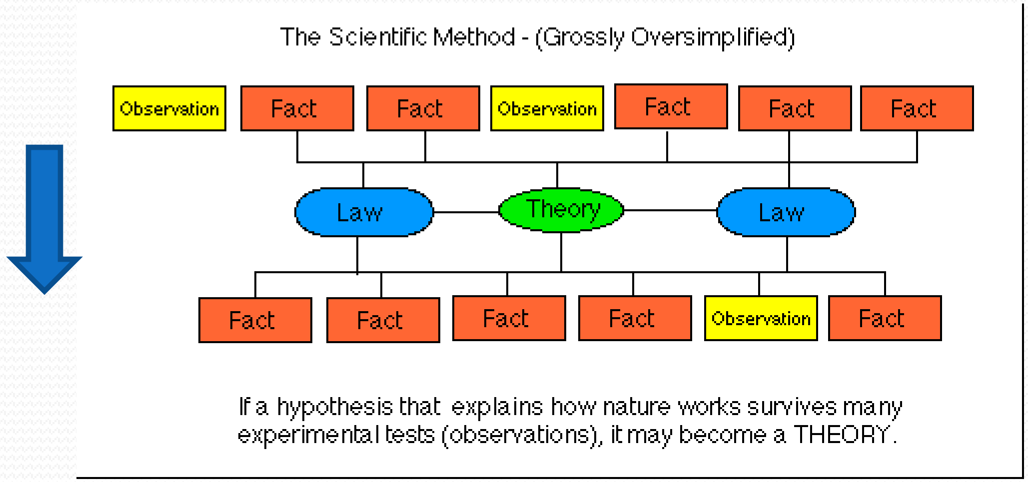
- **Deductive**

1. You have to start with **undefined terms** which must be very common and self-evident.
2. Once agreed on, you can use them to create **definitions**.
3. Next, you need to pick some simple, obviously true statements about the undefined terms and definitions. These statements are called **axioms** or **postulates**. The number of axioms must be limited minimum.
4. You can combine your axioms, definitions, and undefined terms with the rules of logic to **prove** that other statements must be true. These statements are called **theorems**.
5. Once a theorem is **proven,** you can use it, along with other proven theorems, axioms, definitions, and undefined terms to prove other theorems.



Methods in Physics (cont.)

- **Inductive** (scientific)
  1. The inductive method starts with many ***observations*** of nature, with the goal of finding a few, powerful statements about how nature works (*laws* and *theories*).
  2. In the scientific method, observation of nature is the authority. If an idea conflicts with what happens in nature, the idea must be changed or abandoned.



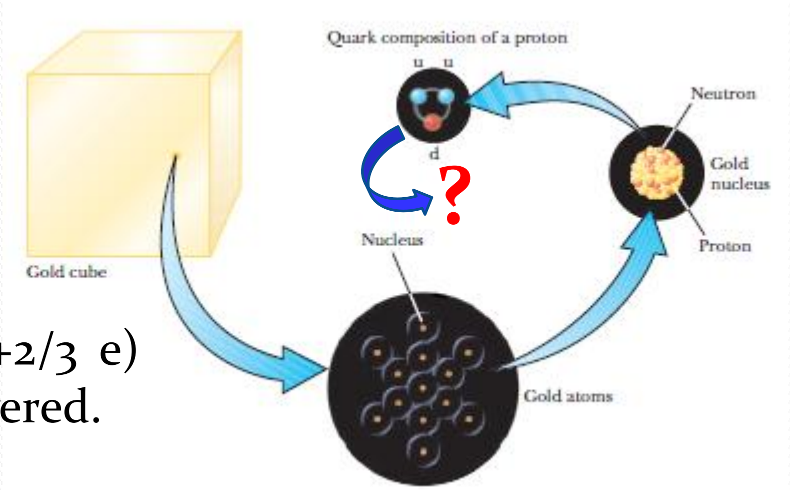
Why Physics – at first – Dynamics??

- "**Physics**" is the most fundamental science for understanding ***objects*** in surrounding world and ***relations*** between them.
- "**Dynamics**" is the study of the *motions* of the various objects in the world around us.
- Newton's theory (1687) is only *approximately* true, for *large* (compared to an atoms) and *slowly moving* (compared to light speed) objects in *Euclidean* space .

The building blocks of matter

A 1-kg cube of solid gold has a length of about 3.73 cm on a side. What would be if we could cut this cube smaller and smaller, infinitively?

- Two early Greek philosophers – *Leucippus* and *Democritus* – couldn't accept the idea that such cutting could go on forever. They speculated that the process ultimately would end when it produced a particle that could no longer be cut. This particle has the name *atomos* – by Greek means "not sliceable."
- Then Danish Niels Bohr has developed the *atom model* which is similar to the Solar system, with a dense, positively charged nucleus occupying the position of the Sun, with negatively charged electrons orbiting like planets. Bohr model successfully explained properties of simpler atoms , but failed to explain many fine details of atomic structure.
- In 1930 – 1932, one discovered that two basic entities—protons and neutrons—occupy the nucleus. The *proton* is of +1e-charge and number of protons determine what element is. Neutrons are no-charged and act as "glue" to keep nucleus stable.

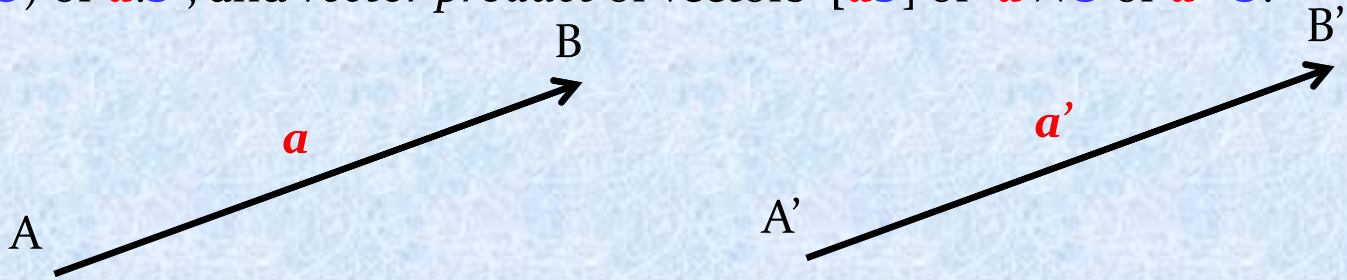


- The ***quarks*** named as *up*, *charm*, *top* (with +2/3 e) and *down*, *strange*, *bottom* (with -1/3 e) are discovered.
- **And ???**

For dealing with physical problems it is necessary to have mathematical tools. The next slides will provide enough knowledge on Vector Algebra and Vector Calculus.

Vector Algebra and Vector Calculus

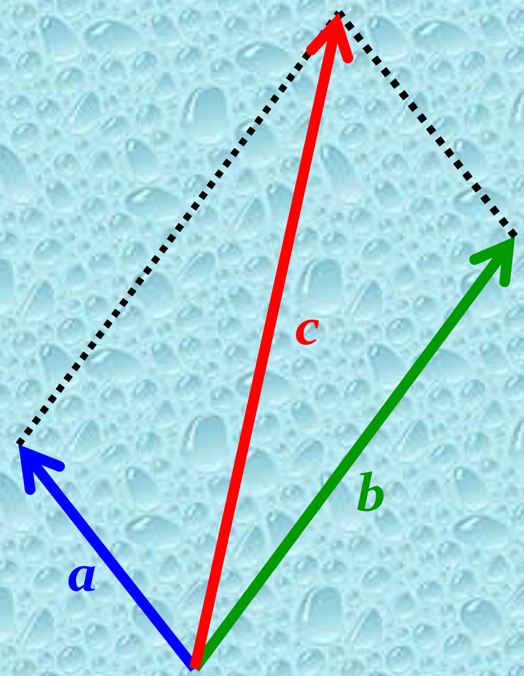
- **Physical quantities** are distincted by two classes of objects: *scalars*, represented by *real numbers;* and *vectors*, represented by *directed line elements* in space.
- While scalar have only *magnitude,* vectors possess a *magnitude* and a *direction*, but are *movable*. $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are considered as a same vector. The bold letters may be used for naming vectors, *e.g*., ***a*** and ***a’***.
- In the coordinate approach, a vector is denoted as the row matrix of its components : $\boldsymbol{a} \equiv (a_x, a_y, a_z)$ or $\boldsymbol{a} = a_x\boldsymbol{e}_x + a_y\boldsymbol{e}_y + a_z\boldsymbol{e}_z$, with $\boldsymbol{e}_i$ – the unit vector along *i*-th Cartesian coordinate axis.
- Algebraic operations on vectors: *addition/subtraction* (*i.e.*, addition with inversed vector) $\boldsymbol{a} \pm \boldsymbol{b}$, *rotation*, *multiplication/division* a vector on a scalar n$\boldsymbol{a}$, *scalar product* ($\boldsymbol{ab}$) or $\boldsymbol{a}.\boldsymbol{b}$, and *vector product* of vectors [$\boldsymbol{ab}$] or $\boldsymbol{a} \wedge \boldsymbol{b}$ or $\boldsymbol{a} \times \boldsymbol{b}$.

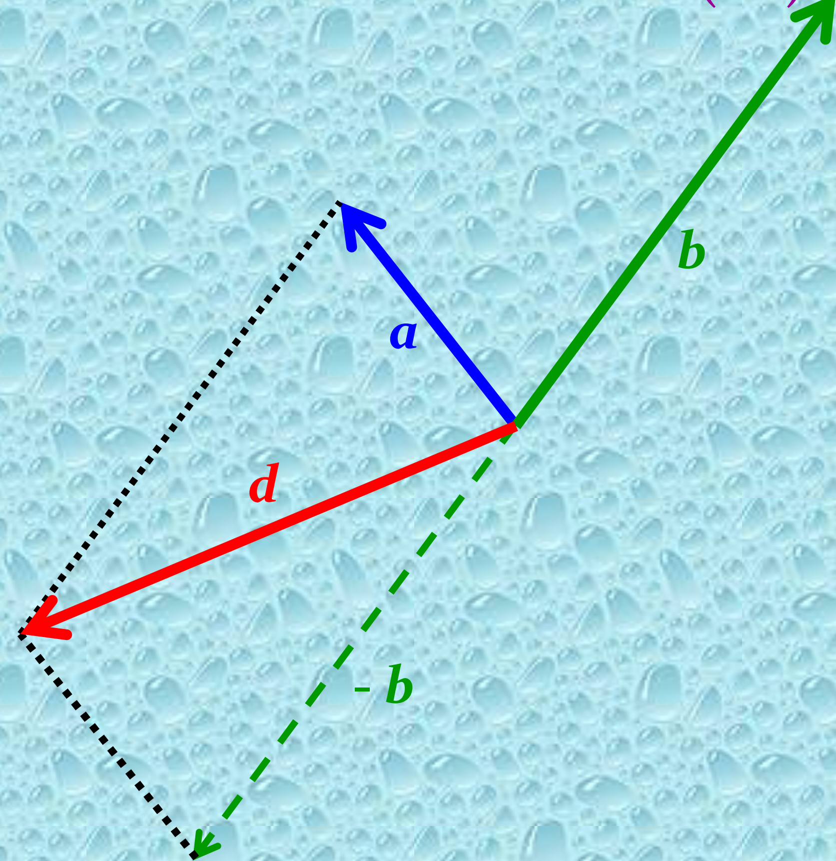


Vector Algebra

Vector Addition/Subtraction

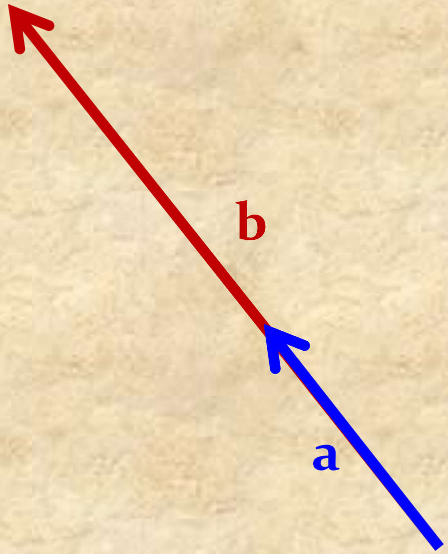
Addition: $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ Subtraction: $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}) = \boldsymbol{d}$

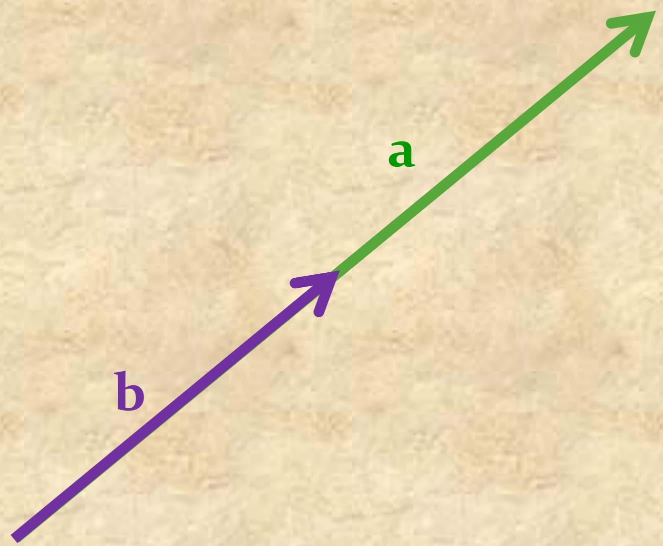




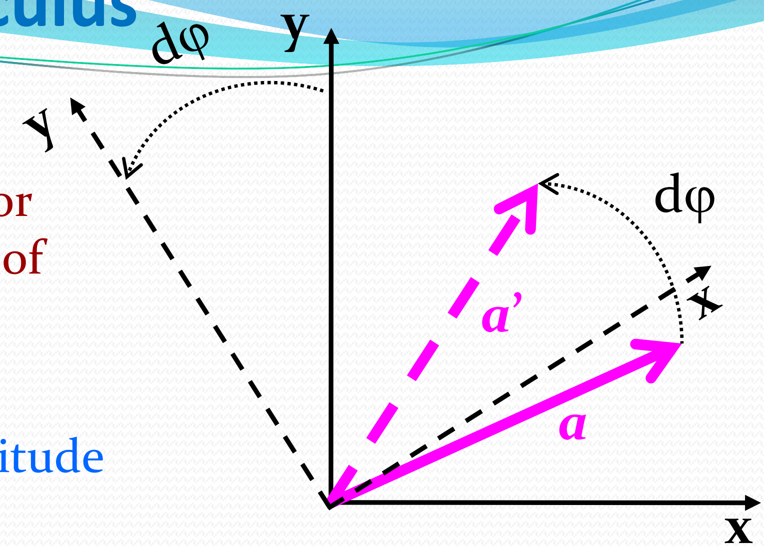
Multiplication a vector on a scalar: b = na

n > 1 | n < 1



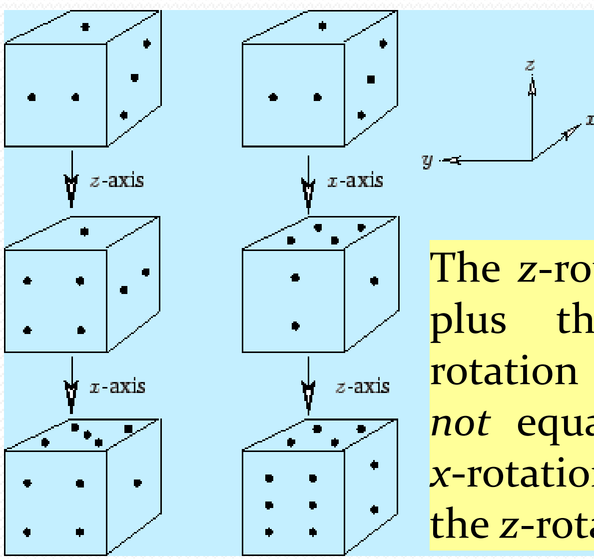


Vector rotations



Combination of axes x, y is as a ***basis*** for vector ***a*** in x-y plan. Rotation relatively to the vector of basis around z-axis is equivalent to the vector rotation in relation to the basis.

Although rotations have a well-defined magnitude and direction, they are ***not*** vector quantities.



The z-rotation plus the x-rotation *does not* equal the x-rotation plus the z-rotation

Rotation of vector ***a*** about the z-axis by a ***small*** angle $\delta\theta_z$ leads to a new vector ***a***'

$$\boldsymbol{a}' \approx \boldsymbol{a} + \delta\theta_z \mathbf{e}_z \times \boldsymbol{a} \approx \boldsymbol{a} + \delta\boldsymbol{\theta} \times \boldsymbol{a}$$

so, we can write small rotation angles as vector:

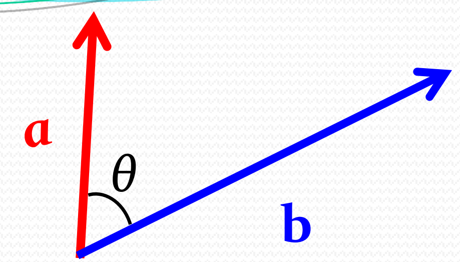
$$\delta\boldsymbol{\theta} = \delta\theta_x \mathbf{e}_x + \delta\theta_y \mathbf{e}_y + \delta\theta_z \mathbf{e}_z$$

For example, angular velocity is $\vec{\omega} = \lim\limits_{\delta t \to 0} \dfrac{\delta\vec{\theta}}{\delta t}$

and equation of motion of a vector precessing about the origin

$$\frac{d\vec{a}}{dt} = \lim_{\delta t \to 0} \frac{\delta\left(\vec{a}' - \vec{a}\right)}{\delta t} \lim_{\delta t \to 0} \frac{\delta\left[\vec{a}\left(t + \delta t\right) - \vec{a}\right]}{\delta t} = \lim_{\delta t \to 0} \frac{\delta\vec{\theta} \times \vec{a}}{\delta t} = \vec{\omega} \times \vec{a}$$

Scalar product of vectors



- Scalar product of vectors results a **scalar number**

  $\mathbf{a}.\mathbf{b} = a_x b_x + a_y b_y + a_z b_z = \text{scalar}$

- Scalar (or *dot*) product is ***invariant*** about axes, *e.g.*, *z*-axis, therefore, about coordinate

$$\vec{a}\cdot\vec{b} = \left(a_x\cos\theta + a_y\sin\theta\right)\left(b_x\cos\theta + b_y\sin\theta\right) + \\ + \left(-a_x\sin\theta + a_y\cos\theta\right)\left(-b_x\sin\theta + b_y\cos\theta\right) + a_z b_z \\ = a_x b_x + a_y b_y + a_z b_z$$

- Scalar product is ***commutative***: $\mathbf{a}.\mathbf{b} = \mathbf{b}.\,\mathbf{a}$ and ***distributive***: $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\,\mathbf{c}$
- From the "cosine rule" of trigonometry

$$\left.\begin{array}{l} \left|\vec{b}-\vec{a}\right|^2 = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta \\ \left(\vec{b}-\vec{a}\right)\left(\vec{b}-\vec{a}\right) = \left|\vec{a}\right|^2 + \left|\vec{b}\right|^2 - 2\vec{a}\cdot\vec{b} \end{array}\right\rangle \Rightarrow \vec{a}\cdot\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$$

Triple scalar product:
- is ***invariant*** under any *cyclic permutation* :

$\mathbf{a}.\mathbf{b}\times\mathbf{c} = \mathbf{b}.\mathbf{c}\times\mathbf{a} = \mathbf{c}.\mathbf{a}\times\mathbf{b}$

- **changes the sign** with any *anti-cyclic permutation*:

$\mathbf{a}.\mathbf{b}\times\mathbf{c} = -\,\mathbf{b}.\mathbf{a}\times\mathbf{c}$

Vector product of vectors

- Vector (or *cross*) product of vectors is a **vector**

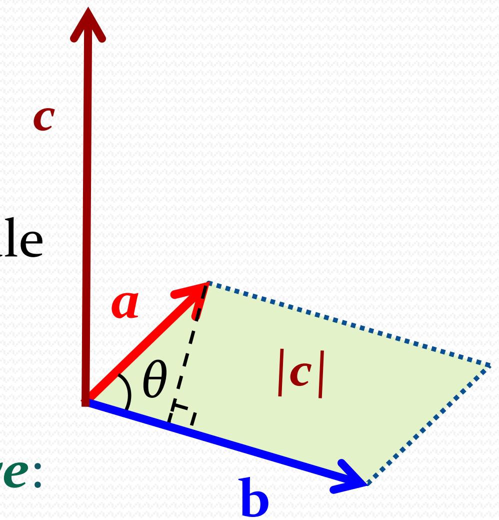
$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x) = \boldsymbol{c} = (c_x, c_y, c_z)$

or

$$\vec{a} \wedge \vec{b} = \vec{c} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

by from the right-hand rule

$\boldsymbol{c} \perp \boldsymbol{a}$;
$\boldsymbol{c} \perp \boldsymbol{b}$



- Vector product is ***anticommutative*** and ***distributive***:

$\boldsymbol{a} \times \boldsymbol{b} = -\,\boldsymbol{b} \times \boldsymbol{a}$; $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$

- Vector product is **not *associative***: $\boldsymbol{a} \times [\boldsymbol{b} \times \boldsymbol{c}] \neq [\boldsymbol{a} \times \boldsymbol{b}] \times \boldsymbol{c}$

- Magnitude

$$\left[\vec{a} \wedge \vec{b}\right]^2 = \left(a_y b_z - a_y b_z\right)^2 + \left(a_z b_x - a_x b_z\right)^2 + \left(a_x b_y - a_y b_x\right)^2$$

$$= \left(a_x^2 + a_y^2 + a_z^2\right)\left(b_x^2 + b_y^2 + b_z^2\right) - \left(a_x b_x + a_y b_y + a_z b_z\right)^2$$

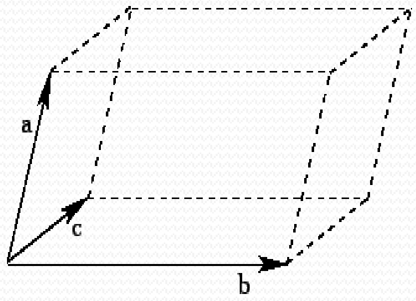
$$= \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 - \left(\vec{a} \cdot \vec{b}\right)^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 \left(1 - \cos^2\theta\right) = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2 \sin^2\theta$$

$$\Rightarrow \left|\vec{a} \wedge \vec{b}\right| = \left|\vec{a}\right| \left|\vec{b}\right| \sin\theta$$

Vector product of vectors (cont.)

While scalar triple product $\boldsymbol{a}.\boldsymbol{b}\times\boldsymbol{c}$ is the **volume** of the parallelepiped defined by vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, forming a right-handed set, vector triple product $\boldsymbol{a} \times [\boldsymbol{b} \times \boldsymbol{c}]$ give the **vector** as a result.

In vector triple products the brackets are important, because $\boldsymbol{a} \times [\boldsymbol{b} \times \boldsymbol{c}] \neq [\boldsymbol{a} \times \boldsymbol{b}] \times \boldsymbol{c}$



Frequently used formulas:

$$\boldsymbol{a} \times [\boldsymbol{b} \times \boldsymbol{c}] \equiv \boldsymbol{b}\,(\boldsymbol{a}\;\boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a}\;\boldsymbol{b})$$

and

$$[\boldsymbol{a} \times \boldsymbol{b}] \times \boldsymbol{c} \equiv \boldsymbol{b}\,(\boldsymbol{a}\;\boldsymbol{c}) - \boldsymbol{a}\,(\boldsymbol{b}\;\boldsymbol{c})$$

Useful additional formulae for products of vectors

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c})$$

$$[\mathbf{a} \times \mathbf{b}] \times [\mathbf{c} \times \mathbf{d}] = \mathbf{c}\,\{\mathbf{a} \cdot [\mathbf{b} \times \mathbf{d}]\} - \mathbf{d}\,\{\mathbf{a} \cdot [\mathbf{b} \times \mathbf{c}]\}$$

$$[\mathbf{a} \times \mathbf{b}] \times [\mathbf{c} \times \mathbf{d}] = \mathbf{b}\,\{\mathbf{a} \cdot [\mathbf{c} \times \mathbf{d}]\} - \mathbf{a}\,\{\mathbf{b} \cdot [\mathbf{c} \times \mathbf{d}]\}$$

Vector Calculus

Derivatives

- The laws of vector differentiation are **analogous** to those in conventional calculus.
- Assume vector ***a*** varies with time, so that ***a*** = ***a***(t). Time derivative is

$$\frac{d\vec{a}}{dt} = \lim_{\delta t \to 0} \frac{a(t+\delta t) - a(t)}{\delta t} = \dot{\vec{a}} \quad \text{or} \quad \frac{d\vec{a}}{dt} = \left(\frac{da_x}{dt}, \frac{da_y}{dt}, \frac{da_z}{dt} \right) = \left(\dot{a}_x, \dot{a}_y, \dot{a}_z \right)$$

- Assume vector ***a*** is a product of another vector ***b***(t) with a scalar $\phi(t)$

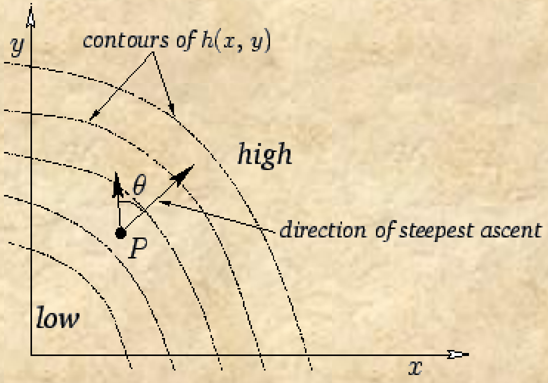
$$\dot{\vec{a}} = \frac{d\vec{a}}{dt} = \frac{d}{dt}\left[\phi(t)\vec{b}(t) \right] = \frac{d\phi(t)}{dt}\vec{b}(t) + \phi(t)\frac{d\vec{b}(t)}{dt}$$

- For the scalar and vector products of vectors ***a*** and ***b***

$$\frac{d}{dt}\left(\vec{a} \cdot \vec{b} \right) = \frac{d\vec{a}}{dt} \cdot \vec{b} + \vec{a} \cdot \frac{d\vec{b}}{dt} \qquad \frac{d}{dt}\left[\vec{a} \times \vec{b} \right] = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$$

Gradients

- Gradient is defined as the slope of the tangent to the curve at the variable(s). Gradient is *vector* and argument for gradient is *scalar*: **grad** $a = \nabla a = \mathbf{b}$
- Vector operator $\text{grad} = \nabla \equiv \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \right) \Leftrightarrow \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$
- For example, gradient of 2-D scalar field $h(x, y)$
- Frequently used formulae



$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(f/g) = (1/g)\nabla f - \left(f/g^2\right)\nabla g$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$$

$$\nabla\cdot(f\mathbf{A}) = (\nabla f)\cdot\mathbf{A} + f(\nabla\cdot\mathbf{A})$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$(\nabla\cdot\nabla)f = \nabla^2 f$$

$$\nabla\times(\nabla f) = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(f\mathbf{A}) = (\nabla f)\times\mathbf{A} + f(\nabla\times\mathbf{A})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B}$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla^2\mathbf{A} = \nabla\cdot(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A})$$

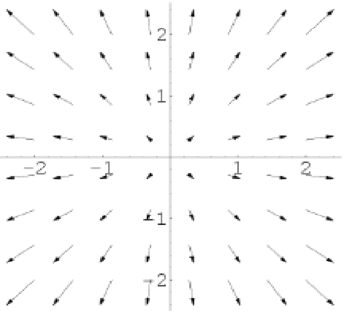
$$\nabla(1/r) = -\hat{\mathbf{r}}/r^2$$

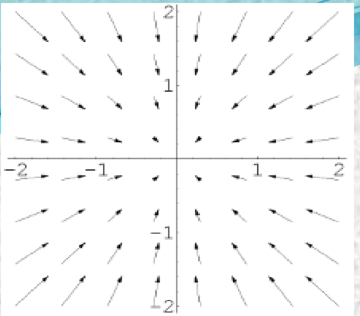
Gradient (cont.)

- In cylindrical (r, φ, z) and spherical (r, θ, φ) coordinate systems:

$$\text{grad} = \nabla = \frac{\partial}{\partial r}\vec{e}_r + \frac{\partial}{r\partial\varphi}\vec{e} + \frac{\partial}{\partial z}\vec{e}_z$$

$$\text{grad} = \nabla = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e}_\varphi$$





Divergence

- A *vector field* is defined as a set of vectors associated with each point in space.
- ***Divergence*** is *scalar product* of the vector operator ∇ applied to the vector field $\boldsymbol{a}(x, y, z)$. The divergence of a vector field is a scalar field.

$$\mathrm{div}\vec{a} = \nabla \cdot \vec{a} = \left(\frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \right) \cdot \left(a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z \right) = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right)$$

- Divergence measures expansion (when positive) or compression (when negative) of a vector field (left and right pictures at the slide's right-upper corner, respectively), *i.e.*, shows *density of the vector flow*.
- In cylindrical (r, φ, z) and spherical (r, θ, φ) coordinate systems :

$$\mathrm{div}\vec{a} = \nabla \cdot \vec{a} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r \right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(a_\theta \sin\theta \right) + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \varphi}$$

$$\mathrm{div}\vec{a} = \nabla \cdot \vec{a} = \frac{1}{r}\frac{\partial}{\partial r}\left(r a_r \right) + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{\partial a_z}{\partial z}$$

Divergence (cont.)

Several frequently used formulae with divergence

$$\mathrm{div}\left(f\vec{a} \right) = \nabla \cdot \left(f\vec{a} \right) = \nabla f \cdot \vec{a} + f\left(\nabla \cdot \vec{a} \right)$$

$$\mathrm{div}\left(\mathrm{rot}\vec{a} \right) = \nabla \cdot \left(\nabla \times \vec{a} \right) \equiv 0$$

$$\mathrm{div}\left[\vec{a} \times \vec{b} \right] = \vec{b} \cdot \left(\nabla \times \vec{a} \right) - \vec{a} \cdot \left(\nabla \times \vec{b} \right) = \vec{b} \cdot \mathrm{rot}\vec{a} - \vec{a} \cdot \mathrm{rot}\vec{b}$$

$$\Delta\vec{a} = \nabla^2\vec{a} = \nabla\left(\nabla \cdot \vec{a} \right) - \nabla \times \left(\nabla \times \vec{a} \right) = \mathrm{graddiv}\vec{a} - \mathrm{rotrot}\vec{a}$$

Divergence theorem

$$\oint\limits_{\text{closed surface } S} \vec{a}\,\mathrm{d}\vec{S} = \int\limits_{\substack{\text{volume } V \\ \text{limitted by } S}} \left(\mathrm{div}\vec{a} \right)\mathrm{dV}$$

Rotor (or curl)

- ***Rotor*** (curl) is the vector product of the vector operator ∇ applied to the vector field ***a***(x, y, z). The curl of a vector field is a vector field.

$$\mathrm{rot}\vec{a} = \nabla\times\vec{a} \equiv \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z$$

- Rotor ∇× ***a*** measures of the local "swirliness" of the current ***a***(x, y, z). In physics, ∇× ***a*** estimates the degree of *circulation* for vector field ***a***(x, y, z).
- In cylindrical (r, φ, z) and spherical (r, θ, φ) coordinate systems :

$$\mathrm{rot}\vec{a} = \nabla\times\vec{a} \equiv \left(\frac{1}{r}\frac{\partial a_z}{\partial\varphi} - \frac{\partial a_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\varphi + \frac{1}{r}\left(\frac{\partial\left(ra_\varphi\right)}{\partial r} - \frac{\partial a_r}{\partial\varphi}\right)\vec{e}_z$$

$$\mathrm{rot}\vec{a} \equiv \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(a_\varphi\sin\theta\right) - \frac{\partial a_\theta}{\partial\varphi}\right]\vec{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(ra_\varphi\right)\right]\vec{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(ra_\theta\right) - \frac{\partial a_r}{\partial\theta}\right]\vec{e}_\varphi$$

Rotor (cont.)

Additional formulae with rotor

$$\mathrm{rot}\left(\mathrm{grad}f\right) = \nabla \times \left(\nabla f\right) = 0$$

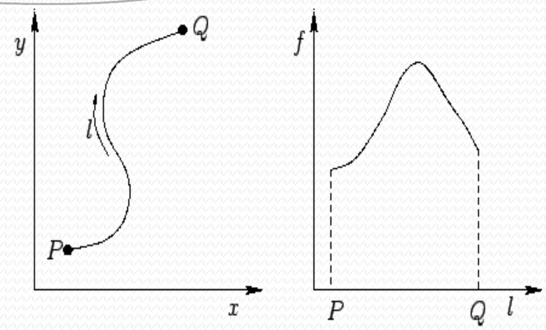
$$\mathrm{rot}\left(f\vec{a}\right) = \mathrm{grad}f \times \vec{a} + f\mathrm{rot}\vec{a} = \left(\nabla f\right) \times \vec{a} + f\left[\nabla \times \vec{a}\right]$$

$$\mathrm{rot}\left[\vec{a} \times \vec{b}\right] = \nabla \times \left[\vec{a} \times \vec{b}\right] = \left(\vec{a} \cdot \nabla\right)\vec{b} - \left(\vec{b} \cdot \nabla\right)\vec{a}$$

Stoke's theorem

$$\oint_C \vec{a}\mathrm{d}\vec{l} = \int_S \left[\nabla \times \vec{a}\right]\mathrm{d}\vec{S}$$

Line integrals



- Consider a two-dimensional function $f(x, y)$ which is defined for all x and y. The integral of f along a given curve l from P to Q in the x-y plane gives the *area under the curve*.
- Most common case, when the contribution from dx and dy can be evaluated separately.

$$\int_P^Q f(x, y)dl = \text{Area under the curve}$$

$$\int_P^Q \left[f(x, y)dx + g(x, y)dy \right]$$

There are *two distinct types of line integral*. Those which depend only on their endpoints and not on the path of integration, and those which depend both on their endpoints and the integration path. We shall distinguish between these two types in relation with two classes of physical quantities.

Vector line integrals

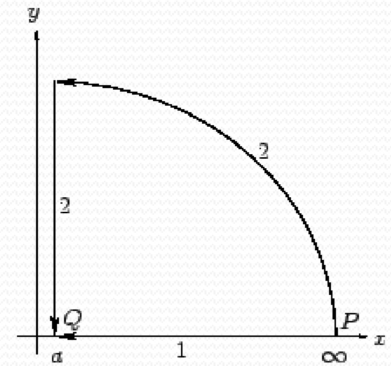
- Consider a general vector field, for simplicity, in terms of Cartesian coordinates : $\boldsymbol{A}(\mathbf{r}) = (A_x, A_y, A_z)$. Vector line integral often arises as:

$$\int_P^Q \vec{A}(\vec{r})d\vec{r} = \int_P^Q \left[A_x dx + A_y dy + A_z dz \right]$$

- Example: work done by a repulsive inverse-square central field $\boldsymbol{F} = -\mathbf{r}/r^3$ by two route from $P(\infty,0,0)$ to $Q(a,0,0)$: 1) along x-axis; 2) around the circle (r = const), and then parallel to y-axis.

$$1)\quad W = \int_P^Q F_x dx + \int_P^Q F_y dy = \int_\infty^a \left(-\frac{1}{x^2} \right) dx + 0 = \frac{1}{x}\Bigg|_\infty^a = \frac{1}{a}$$

$$2)\quad W = \int_P^Q F_x dx + \int_P^Q F_y dy = 0 + \int_\infty^0 \left(-\frac{y}{\left(a^2+y^2\right)^{3/2}} \right) = \frac{1}{\left(y^2+a^2\right)^{1/2}}\Bigg|_\infty^0 = \frac{1}{a}$$



- But ***not all vector line integrals are path independent***.

Volume integrals

- A volume integral takes the form $\iiint_V f(x,y,z)dV$ with $dV = dxdydz = d^3\vec{r}$
- There are two calculation methods
  - The **disk method** finds the volume of a revolution about an axis by summing cylinders of infinitesmal width about the axis. $\pi\int_a^b [R(x)]^2\,dx$ or $\pi\int_a^b [R(y)]^2\,dy$
  - The **shell method** finds the volume of a revolution about an axis , which is not zero, by summing the shells of infinitesmal width about the axis $2\pi\int_a^b xF(x)dx$ or $2\pi\int_a^b yF(y)dy$

One next slide: the examples of volume integral calculation

Volume integrals (cont.)

- The disk method

Find the volume generated by the rotation of the region bounded by x = (y+1)(y-3) and x = 0 about the y axis.
Instead of having the integral with respect to dx, we change the infinitesmal element to dy.
The bounds for this integral are now along the y axis, instead of the x axis.
First we need to find the bounds. In other words, when does x = (y+1)(y-3) intersect x = 0? It intersects when y = -1 or 3.

$$\pi\int_{-1}^{3}\left[(y+1)(y-3)\right]^2 dt = \frac{512}{15}\pi$$

- The shell method

Find the volume generated by the rotation of the region bounded by y = x ½ and y = 0 about the y axis on the interval 0 to 4.
Remembering what separates this from the disk method is that we don't square the radius here, and we include a variable that represents the distance from the axis. Also, the constant is different.

$$2\pi\int_{0}^{4} x\left(\sqrt{x}\right)dx = \frac{128}{5}\pi$$

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Thank you for attention!

Any suggestion or comment, please, send to:

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