HUST

ĐẠI HỌC BÁCH KHOA HÀ NỘI
HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY

ONE LOVE. ONE FUTURE.

DATA STRUCTURES AND ALGORITHMS

ĐẠI HỌC
BÁCH KHOA HÀ NỘI
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Week 6: Dynamic programming

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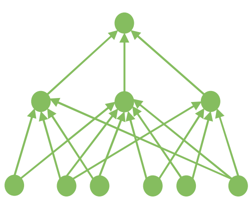
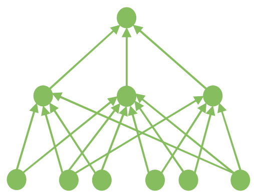
INTRODUCTION

- Dynamic programming was invented by Bellman during World War II. The first name of this algorithm was multi-stage decision process (decision making through many stages).
- The dynamic programming algorithm is a powerful technique for solving optimization problems by dividing them into smaller problems and solving the sub-problems only once.
- Dynamic programming algorithms have many similarities with backtracking and divide and conquer algorithms.



ALGORITHM DIAGRAM

- DIVIDE the starting problem into subproblems that are not necessarily independent of each other
- SOLVE sub-problems from small to large, the solutions are stored in memory (to ensure each problem is only solved correctly once)
  - The smallest subproblem must be solved in a direct, simple way
- COMBINE the solution of the larger problem from the existing solutions of the smaller sub-problems (need to use a recurrence formula)
  - The number of subproblems needed is bounded by a polynomial function of the input data size



ALGORITHM DIAGRAM

```
map<problem, value> Memory;

value DP(problem P) {
    if (is_base_case(P))
        return base_case_value(P);

    if (Memory.find(P) != Memory.end())
        return Memory[P];

    value result = some value;
    for (problem Q in subproblems(P))
        result = Combine(result, DP(Q));

    Memory[P] = result;
    return result;
}
```

Memory, mapping of programs and solutions

The smallest subproblem, corresponding to the basic step of the recursive algorithm

Always check the memory to see if the sub-problem has been solved. Once it have been solved it, don't solve it again. If it haven't been solved it, try solving it.

Solve a subproblem

Solve all subproblems and combine the solutions found to form the solution to the larger problem

Save the subproblem results in memory

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CHARACTERISTICS

- Compare to the divide and conquer algorithm, the dynamic programming algorithm also has 3 steps: Divide, Solve subproblems and Combine. However, in divide and conquer, the subproblems are independent; In dynamic programming, subproblems overlap or overlap.
- The biggest difficulty in proposing dynamic programming algorithms is the Recursive Formula (other names are Dynamic Programming Formula, Recursive Formula)
- There are 2 approaches: Top-Down and Bottom-Up, in which Top-Down is natural and easy to understand and easy to install.
- Memory design greatly affects the speed of the algorithm
- Memory is still used to trace and explicitly find the optimal solution

EXAMPLE: SUBSEQUENCE WITH THE LARGEST SUM

- **Problem**: Given a sequence of n integers $(a_1, a_2, \ldots, a_n)$, find a subsequence consisting of consecutive elements of the sequence such that the sum of the selected elements is maximized.
- **Example**: Given a sequence of 7 intergers:

-16	7	-3	0	-1	5	-4

Subsequence with sum of -12 — **Subsequence with sum of 0**

  - *Optimal solution: (subsequence with the largest sum of 8): 7, -3, 0, -1, 5*

- The exhaustive algorithm has complexity $O(n^2)$, can we do better with the divide and conquer algorithm?

EXAMPLE: SUBSEQUENCE WITH THE LARGEST SUM

- **Problem**: Given a sequence of n integers $(a_1, a_2, \ldots, a_n)$, find a subsequence consisting of consecutive elements of the sequence such that the sum of the selected elements is maximized.
- **Determine the subproblems**: P_i is the problem of finding a sub-segment consisting of consecutive elements with the largest sum whose last element is a_i, for all i = 1,..., n.

Solution of P_2 — *Solution of P_6*

-16	7	-3	0	-1	5	-4

Solution of P_4 — *Solution of P_7*

EXAMPLE: SUBSEQUENCE WITH THE LARGEST SUM

- **Problem**: Given a sequence of n integers $(a_1, a_2, \ldots, a_n)$, find a subsequence consisting of consecutive elements of the sequence such that the sum of the selected elements is maximized.
- **Determine the subproblems**: P_i is the problem of finding a sub-segment consisting of consecutive elements with the largest sum whose last element is a_i, for all i = 1,..., n.
- **Dynamic programming formula** (formula combining solutions to sub-problems to obtain solution to parent problem): Let S_i be the sum of elements of the solutions of $P_i, \forall\, i = 1, \ldots, n$.

We have: $S_1 = a_1$,

$$S_i = \begin{cases} s_{i-1} + a_i \ if \ s_{i-1} > 0 \\ a_i \ if \ s_{i-1} \le 0 \end{cases}$$

EXAMPLE: SUBSEQUENCE WITH THE LARGEST SUM

- **Example:**

-16	7	-3	0	-1	5	-4

$\boldsymbol{S_1 = -16, S_2 = a_2 = 7, S_3 = S_2 + a_3 = 4, S_4 = S_3 + 0 = 4,}$

$\boldsymbol{S_5 = S_4 + (-1) = 3, S_6 = S_5 + 5 = 8, S_7 = S_6 + (-4) = 4}$

EXAMPLE: SUBSEQUENCE WITH THE LARGEST SUM

- **Dynamic programming formula** (formula combining solutions to sub-problems to obtain solution to parent problem): Let S_i be the sum of elements of the solutions of $P_i, \forall\, i = 1, \dots, n$.

We have: $S_1 = a_1$,

$$S_i = \begin{cases} s_{i-1} + a_i \text{ if } s_{i-1} > 0 \\ a_i \text{ if } s_{i-1} \le 0 \end{cases}$$

- **Solution:** The sum of the elements of the sub-segment including consecutive elements of the sequence with the largest sum of selected elements is:

$$\boldsymbol{max(S_1, S_2, \dots, S_n)}$$

- **Complexity:** O(n)

EXAMPLE: SUBSEQUENCE WITH THE LARGEST SUM

- The sum of the elements of the sub-segment including consecutive elements of the sequence with the largest sum of selected elements is:

$$\boldsymbol{max(S_1, S_2, \dots, S_n)}$$

Example:

-16	7	-3	0	-1	5	-4

$\boldsymbol{S_1 = -16, S_2 = a_2 = 7, S_3 = S_2 + a_3 = 4, S_4 = S_3 + 0 = 4,}$

$\boldsymbol{S_5 = S_4 + (-1) = 3, S_6 = S_5 + 5 = 8, S_7 = S_6 + (-4) = 4}$

Solution: $\boldsymbol{max(S_1, S_2, \dots, S_7) = 8}$

EXERCISE: LONGEST INCREASING SUBSEQUENCE

- **Problem:** Given a list of integers $A = (a_1, a_2, \dots, a_n)$ satisfy the condition that the elements are pairwise different $(a_i \neq a_j, \forall i \neq j)$. A subsequence of A is a sequence obtained by deleting some elements in A. A subsequence $B = (b_1, \dots, b_k)$ is called tight increase when $b_i < b_{i+1}, \forall i \in \{1, \dots, k-1\}$. Find the length of the strongly increasing subsequence of A that has the greatest length.
- **Homework:** Submit code to the online system

EXAMPLE: LONGEST COMMON SUBSEQUENCE

- **Problem**: Given 2 lists of characters $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, \ldots, y_m\}$. A subsequence of A is a sequence obtained by deleting some elements in A. Find the length of the longest common subsequence of X and Y.
- **Example**:
  - $X = "abcb"$ and $Y = "bdcab"$
  - The longest common subsequence is "bcb" with length of 3
- Comment: The exhaustive algorithm, comparing all subsequences of X and Y will have complexity O(2^n×2^m×max(m,n)). Can we solve this problem faster with a dynamic programming algorithm?

EXAMPLE: LONGEST COMMON SUBSEQUENCE

- **Problem**: Given 2 lists of characters $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, \ldots, y_m\}$. A subsequence of A is a sequence obtained by deleting some elements from A. Find the length of the longest common subsequence of X and Y.
- **Determine the subproblem:** Let S(i,j) be the length of the longest common subsequence of the two sequences, subsequences of X are $X_i = (x_1, \ldots, x_i)$ with $i \in \{1, \ldots, n\}$ and subsequences of Y are $Y_j = (y_1, \ldots, y_j)$ with $j \in \{1, \ldots, m\}$.
- **Base problems (smallest subproblems):**

$$S(i,0) = 0, \forall i \in \{1, \ldots, n\}$$
$$S(0,j) = 0, \forall j \in \{1, \ldots, m\}$$

EXAMPLE: LONGEST COMMON SUBSEQUENCE

- **Determine the subproblem:** Let S(i,j) be the length of the longest common subsequence of the two sequences, subsequences of X are $X_i = (x_1, \ldots, x_i)$ with $i \in \{1, \ldots, n\}$ and subsequences of Y are $Y_j = (y_1, \ldots, y_j)$ with $j \in \{1, \ldots, m\}$.
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- **Dynamic programming formula:**

$$S(i,j) = max \begin{cases} S(i-1,j-1)\ if\ \ x_i = y_j \\ S(i-1,j) \\ S(i,j-1) \end{cases}$$

EXAMPLE: LONGEST COMMON SUBSEQUENCE

- Dynamic programming formula

$$S(i,j) = max \begin{cases} S(i-1,j-1)\ if\ \ x_i = y_j \\ S(i-1,j) \\ S(i,j-1) \end{cases}$$

X	3	7	2	5	1	4	9			
Y	4	3	2	3	6	1	5	4	9	7

	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	1	1	1	1	1
2	0	1	1	1	1	1	1	1	1	2
3	0	1	2	2	2	2	2	2	2	2
4	0	1	2	2	2	2	3	3	3	3
5	0	1	2	2	2	3	3	3	3	3
6	1	1	2	2	2	3	3	4	4	4
7	1	1	2	2	2	3	3	4	5	5

EXAMPLE: LONGEST COMMON SUBSEQUENCE

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- **Base problems (smallest subproblems):**

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- **Dynamic programming formula:**

$$S(i,j) = max \begin{cases} S(i-1,j-1)\ if\ \ x_i = y_j \\ S(i-1,j) \\ S(i,j-1) \end{cases}$$

- **Complexity:** $O(n \times m)$

EXERCISE: LONGEST INCREASING SUBSEQUENCE

- **Problem**: Given a sequence $A = (a_1, a_2, \dots, an)$. A subsequence of the sequence A is a sequence obtained by deleting some elements from A. Find the length of the subsequence of A that is a progression with a step of 1 and has the largest length.
- **Homework**: Submit code to the online system

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THANK YOU !

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