Contents

3.1.1. Definition

Consider a function $f : [a, b] \times [c, d] \to \mathbb{R}$.

- Assume that for each $t \in [c, d]$ the function $f(x, t)$ is integrable over $[a, b]$.
- The function $I : [a, b] \to \mathbb{R}$ is defined by

$$I(t) = \int_a^b f(x, t)dx.$$

- $I(t)$ is called an integral depending on the paramerter t.

3.1.2. Continuity, integrability, differentiability

Theorem (Continuity)

If $f(x,t)$ is continuous on $[a,b]\times[c,d]$, then $I(t)$ is continuous on $[c,d]$.

Remark: From the above theorem, one can deduce:
If $f(x,t)$ is continuous on $[a,b]\times D$, where D is an *open* interval, then $I(t)$ is continous on D.

$$\lim_{t\to t_0}\int_a^b f(x,t)dx = \int_a^b \lim_{t\to t_0} f(x,t)dx = \int_a^b f(x,t_0)dx.$$


*me
*Limit
*Integral
*Integral
*Limit

Sketch of proof

- Let $t_0 \in [c, d]$ and let $\epsilon > 0$.

$$|I(t) - I(t_0)| = \left| \int_a^b (f(x,t) - f(x,t_0)) \right| dx \leq \int_a^b |f(x,t) - f(x,t_0)| dx.$$

- Since $f(x,t)$ is continuous in $R = [a,b] \times [c,d]$, it is uniformly continuous in R.
- For $\epsilon > 0$, there exists $\delta > 0$ such that:

$$|f(x_1,t) - f(x_2,t_0)| < \frac{\epsilon}{b-a+1}, \quad \forall |x_1 - x_2| < \delta, |t - t_0| < \delta.$$

- With $|t - t_0| < \delta$,

$$|I(t) - I(t_0)| \leq \int_a^b |f(x,t) - f(x,t_0)| \leq (b-a)\frac{\epsilon}{b-a+1} < \epsilon.$$

- I is continuous at t_0.

Example (Midterm 20201)

Let $I(y) = \int_{-1}^{1} \sqrt{x^4 + x^2 + y^4}dx$. Consider the continuity of $I(y)$. Find $\lim\limits_{y \to 0} I(y)$.

- The function $f(x, y) = \sqrt{x^4 + x^2 + y^4}$ is continuous on $[-1, 1] \times [c, d]$.
- So $I(y)$ is continuous on $[c, d]$ for all $[c, d]$. Thus $I(y)$ is continuous in $\mathbb{R}$.
- $$\begin{aligned}\lim_{y \to 0} I(y) = I(0) &= \int_{-1}^{1} \sqrt{x^4 + x^2}dx = 2\int_{0}^{1} \sqrt{x^4 + x^2}dx \\ &= \int_{0}^{1} \sqrt{1 + x^2}d(1 + x^2) = \frac{2}{3}\left.(1 + x^2)^{3/2}\right|_{0}^{1} = \frac{2}{3}(2\sqrt{2} - 1).\end{aligned}$$

Integrability

Theorem

If $f(x,t)$ is continuous in $[a,b] \times [c,d]$, then $I(t)$ is integrable on $[c,d]$ and

$$\int_c^d I(t)dt = \int_c^d dt \int_a^b f(x,t)dx = \int_a^b dx \int_c^d f(x,t)dt.$$

Differentiability

Theorem

If $f(x,t)$ and $f_t'(x,t)$ are continuous on $[a,b]\times[c,d]$, then $I(t)$ is differentiable on $[c,d]$ and

$$I'(t)=\int_a^b f_t'(x,t)dx.$$

$$\frac{d}{dt}I(t)=\int_a^b \frac{\partial f}{\partial t}(x,t)dx$$

Sketch of proof

- Let $J(t) = \int_a^b f'_t(x,t)dx$. Then $J(t)$ is continuous on $[c,d]$
- For all $y \in [c,d]$, one has

$$\begin{aligned} LHS &= \int_c^y J(t)dt = \int_c^y \int_a^b f'_t(x,t)dxdt = \int_a^b \int_c^y f'_t(x,t)dtdx \\ &= \int_a^b \left(f(x,t)\big|_c^y \right) dx = \int_a^b f(x,y)dx - \int_a^b f(x,c)dx = RHS. \end{aligned}$$

- Take the derivatives of both sides with respect to y.
- The derivative of the left side is $J(y)$.
- The derivative of the right side is $I'(y)$.
- Thus $I'(y) = J(y)$.

Example

Example 1: Consider $I(t) = \int\limits_0^1 (x+t^2)^2\,dx$.

- One has $I(t) = \frac{1}{3}(x+t^2)^3\Big|_{x=0}^{x=1} = \frac{(1+t^2)^3 - t^6}{3} = \frac{1}{3} + t^2 + t^4$.
- Hence $I'(t) = 2t + 4t^3$.
- On the other hand,

$$\int_0^1 \frac{\partial}{\partial t}(x+t^2)^2\,dx = \int_0^1 2(x+t^2)(2t)\,dx = 2t\int_0^1 (x+t^2)^2\,dx = 2t((1+t^2)^2 - t^4) = 2t(1+t^2).$$

- Hence $\frac{d}{dt}\int\limits_0^1 (x+t^2)^2\,dx = \int\limits_0^1 \frac{\partial}{\partial t}(x+t^2)^2\,dx$.

Example 2:

$$\int_0^1 x^t\,dx = \frac{1}{t+1}, \quad (t \geq 1) \Rightarrow \frac{d}{dt}\int_0^1 x^t\,dx = \int_0^1 \frac{\partial}{\partial t}(x^t)\,dx \Rightarrow \int_0^1 x^t \ln x\,dx = -\frac{1}{(t+1)^2}.$$

"Fun" example: Feynman's trick

Putnam 2005 A6, Serret's integral

Evaluate $\int\limits_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$.

- Let $I(t) = \int_0^1 \frac{\ln(1+tx)}{1+x^2}\,dx$. Then $I(0) = 0$.
- $I'(t) = \int\limits_0^1 \frac{\partial}{\partial t}\left(\frac{\ln(1+tx)}{1+x^2}\right)dx = \int\limits_0^1 \frac{x}{(1+tx)(1+x^2)}dx =$ $\frac{1}{1+t^2}\int\limits_0^1 \left(\frac{t}{1+x^2} + \frac{x}{1+x^2} - \frac{t}{1+tx}\right)dx = \frac{1}{1+t^2}\left(t\frac{\pi}{4} + \frac{\ln 2}{2} - \ln(1+t)\right).$
- $I = I(1) = \int_0^1 I'(t)dt = \frac{\pi}{4}\int\limits_0^1 \frac{t}{1+t^2}dt + \frac{\ln 2}{2}\int\limits_0^1 \frac{1}{1+t^2}dt - \int\limits_0^1 \frac{\ln(1+t)}{1+t^2}dt = \frac{\pi \ln 2}{4} + \frac{\ln 2\pi}{4} - I.$
- Hence, $2I = \frac{\pi \ln 2}{4}$ and $I = \frac{\pi \ln 2}{8}$.

Example (Midterm 20162)

Let $f(y) = \int_0^{\pi/2} \ln(y^2 \sin^2 x + \cos^2 x)dx$. Find $f'(1)$.

- The function $F(x,y) = \ln(y^2 \sin^2 x + \cos^2 x)$ and its partial derivative $F'_y(x,y) = \dfrac{2y\sin^2 x}{y^2\sin^2 x + \cos^2 x}$ are continuous in $[0,\pi/2] \times [1/2, 2]$.
- So $f(y)$ is differentiable in $[1/2, 2]$ and

$$f'(y) = \int_0^{\pi/2} \frac{2y\sin^2 x}{y^2\sin^2 x + \cos^2 x}dx.$$

- $f'(1) = \int\limits_0^{\pi/2} \dfrac{2\sin^2 x}{\sin^2 x + \cos^2 x}dx = \int\limits_0^{\pi/2} (1-\cos(2x))dx = \pi/2.$

Some exercises

- (Midterm 20152) Find $\lim\limits_{y\to 0} \int\limits_{-1}^{1} \dfrac{x^{2015}\cos(xy)}{1+x^2+2y^2}dx$.
- (Midterm 20182) Find $\lim\limits_{x\to 0} \int\limits_{\pi/4}^{\pi/3} \dfrac{1}{x^4+\sin^2 y}dy$.
- (Midterm 20212) Let $I(y) = \int\limits_{0}^{1} \sqrt{x^3y+x^2+x^4}dx$. Consider the continuity of $I(y)$ and find $\lim\limits_{y\to 0} I(y)$.
- (Midterm20222) Let $I(y) = \int\limits_{0}^{\pi/2} \ln(1-y\cos^2 x)dx$. Show that $I(y)$ is differentiable with $y \leq 1$. Find $I'(-1)$.
- (Midterm20222) Let $I(y) = \int\limits_{0}^{1} \ln(x^2+y^2)dx$, where $y > 0$. Show that $I(y)$ is monotone on $(0,+\infty)$ and find $I(y)$.
- (Midterm 20193) Find $\lim\limits_{y\to 0} \int_0^1 (x+3y)\sqrt{x^2+y^3+1}dx$.

The case: the limits of integration depend on a prameter

Consider the integral

$$I(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)dx,$$

where

- $f(x,t)$ is defined in $[a,b] \times [c,d]$,
- and $\alpha(t), \beta(t)$ are defined in $[c,d]$ such that

$$a \leq \alpha(t) \leq b, \quad a \leq \beta(t) \leq b, \ \forall t \in [c,d].$$

Theorem (Continuity)

If $f(x,t)$ is continuous on $[a,b] \times [c,d]$, and $\alpha(t), \beta(t)$ are continuous on $[c,d]$ and whose values in $[a,b]$, then $I(t)$ is continuous on $[c,d]$.

Example (Midterm 20192)

Find the limit $\lim\limits_{y\to 0} \int\limits_{y}^{\pi/2} \sin(x^2y + 2x + y^2)dx$.

- The function $f(x, y) := \sin(x^2y + 2x + y^2)$ is continuous on $[-1, \pi/2] \times [-1, 1]$.
- The functions $\alpha(y) := y$ and $\beta(y) = \pi/2$ are continuous on $[-1, 1]$, and they takes values in $[-1, \pi/2]$ (for $y \in [-1, 1]$). (The integrand and the limits of integration are continuous functions.)
- So $I(y) = \int\limits_{y}^{\pi/2} \sin(x^2y + 2x + y^2)dx$ is continuous on $[-1, 1]$, hence it is continuous at $y = 0$.
- Thus $\lim\limits_{y\to 0} \int\limits_{y}^{\pi/2} \sin(x^2y + 2x + y^2)dx = I(0) = \int\limits_{0}^{\pi/2} \sin(2x)dx = -\left.\dfrac{\cos(2x)}{2}\right|_0^{\pi/2} = 1.$

Differentiability

Consider the integral

$$I(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)dx.$$

Theorem (Leibniz's Formula)

Suppose that the following conditions are satisfied.

- $f(x,t)$ and is partial derivative $f_t'(x,t)$ are continuous on $[a,b] \times [c,d]$,
- $\alpha(t)$ and $\beta(t)$ are differentiable on $[c,d]$.

Then $I(t)$ is differentiable on $[c,d]$ and

$$I'(t) = \int_{\alpha(t)}^{\beta(t)} f_t'(x,t)dx + f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t).$$

(Sketch of) Proof

- Consider a function with three variables: $I(t,u,v) = \int_u^v f(x,t)\,dx$.
- The following properties hold:

$$\frac{\partial I}{\partial t}(t,u,v) = \int_u^v \frac{\partial}{\partial t} f(x,t)\,dx.$$
$$\frac{\partial I}{\partial u}(t,u,v) = -f(u,t).$$
$$\frac{\partial I}{\partial v}(t,u,v) = f(v,t).$$

- By applying the chain rule, we obtain:

$$\begin{aligned}\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx &= \frac{d}{dt} I(t,\alpha(t),\beta(t)) \\ &= \frac{\partial I}{\partial t}(t,\alpha(t),\beta(t)) - f(\alpha(t),t)\alpha'(t) + f(\beta(t),t)\beta'(t).\end{aligned}$$

Example (Midterm 20192)

Let $I(y) = \int_y^1 \sin(x^2 + xy + y^2)dx$. Find $I'(0)$.

- The function $f(x, y) = \sin(x^2 + xy + y^2)$ and is partial derivative f'_y are continuous. The limits of integrations are continuous functions. (For more a rigorous argument, choose appropriate intervals [a, b] and [c, d] as in the previous example.)
- So $I(y)$ is differentiable and

$$\begin{aligned} I'(y) &= \int_y^1 f'_y(x, y)dx - f(y, y) \\ &= \int_y^1 (x + 2y)\cos(x^2 + xy + y^2)dx - \sin(3y^2). \end{aligned}$$

- $I'(0) = \int\limits_0^1 x\cos(x^2)dx - \sin 0 = \dfrac{1}{2}\sin 1.$

Some exercises

- (Midterm 20181) Find $\lim\limits_{y\to 0} \int\limits_{\frac{1}{2}+y^2}^{\sin y} \dfrac{\arcsin(x+3y)}{\sqrt{1-x^2+3y^2}}dx$.
- (Midterm 20213) Let $I(y) = \int\limits_0^y \arctan \frac{x}{y} dx$. Find $I'(1)$.
- (Midterm 20212) Let $I(y) = \int\limits_{2y}^1 \sin(2x^2+4xy+y^2)dx$. Find $I'(0)$.

3.2.1. Definition

- Given a function $f\colon [a,+\infty)\times[c,d]\to\mathbb{R}$, assume that for each $t\in[c,d]$, the improper integral

$$I(t)=\int_a^{+\infty} f(x,t)dx$$

converges. The integral $I(t)$ is called an improper integral depending on the parameter t.
- We say the improper integral $I(t)$ is uniformly convergent in $[c,d]$ if for all $\epsilon>0$, there exists $A\geq a$ such that:

$$b\geq A\Rightarrow\left|I(t)-\int_a^b f(x,t)dx\right|=\left|\int_b^{+\infty} f(x,t)dx\right|<\epsilon,\quad \forall t\in[c,d].$$

Remark: (Suppose for each $t\in[c,d]$, the function $f(x,t)$ is integrable over $[a,b]$ for every $b>a$.) The improper integral $\int_a^{+\infty} f(x,t)dx$ is uniformly convergent in $[c,d]$ if and only if the improper integral $\int_u^{+\infty} f(x,t)dx$ is uniformly convergent in $[c,d]$ for all $u>a$.

3.2.2. Sufficient Conditions for Uniform Convergence of an Improper Integral Depending on a Parameter

The Weierstrass test

- Let $f(x,t)$ be defined in $R = [a, +\infty) \times [c, d]$ such that for each $t \in [c, d]$ the function $f(x,t)$ is integrable over $[a, b]$, $\forall b \geq a$.
- Assume that exists a function $\varphi(x)$ defined in $[a, +\infty)$ such that $|f(x,t)| \leq \varphi(x)$ for all $(x,t) \in R$ and $\int\limits_a^{+\infty} \varphi(x)dx < +\infty$.
- Then the integral $I(t) = \int_a^{+\infty} f(x,t)dx$ is uniformly convergent in $[c, d]$.

Example

Consider the uniform convergence of $I(t) = \int\limits_0^{+\infty} e^{-x} x^t dx$ in $[1, a]$, where $a > 1$.

- We have $|e^{-x} x^t| = e^{-x} x^t \leq e^{-x} x^a$ for all $x \geq 1$ and $t \in [1, a]$.
- The integral $\int\limits_1^{+\infty} e^{-x} x^a dx$ is convergent.
- By the Weierstrass test, the integral $\int\limits_1^{+\infty} e^{-x} e^t dx$ is uniformly convergent in $[1, a]$.

3.2.3. Continuity, integrability, differentiability of a parameter-dependent improper integral

Theorem (Continuity)

If f is continuous on $[a, +\infty) \times [c, d]$, and the improper integral

$$I(t) = \int_a^{+\infty} f(x,t)dx \text{ is uniformly convergent on } [c, d],$$

then $I(t)$ is continuous on $[c, d]$.

Theorem (Integrability)

If $f(x,t)$ is continuous on $[a,+\infty)\times[c,d]$ and

$$I(t)=\int\limits_a^{+\infty} f(x,t)dx \text{is uniformly convergent on } [c,d],$$

then $I(t)$ is integrable on $[c,d]$ and

$$\int_c^d I(t)dt=\int_c^d dt\int_a^{+\infty} f(x,t)dx=\int_a^{+\infty} dx\int_c^d f(x,t)dt.$$

Theorem (Differentiability)

Let $f(x,t)$ and its partial derivative $f'_t(x,t)$ be continuous on $[a,+\infty) \times [c,d]$. Assume that

$$I(t) = \int\limits_a^{+\infty} f(x,t)dx \text{ converges and } J(t) = \int\limits_a^{+\infty} f'_t(x,t)dx \text{ uniformly converges on } [c,d].$$

Then $I(t)$ is differentiable in $[c,d]$ and

$$I'(t) = \int_a^{+\infty} f'_t(x,t)dx.$$

$$\frac{d}{dt}I(t) = \int_a^{+\infty} \frac{\partial f}{\partial t}(x,t)dx$$

Example (Midterm-20193)

Show that the function $I(y) = \int\limits_0^{+\infty} \frac{\sin(x^6 + 3y + 2)}{1 + x^6 + y^2} dx$ is continuous and differentiable in $\mathbb{R}$.

- We only need to show that $I(y)$ is continuous and differentiable in each closed interval $[c, d]$.
- The function $f(x, y) = \frac{\sin(x^6 + 3y + 2)}{1 + x^6 + y^2}$ is continuous on $[0, +\infty) \times [c, d]$.
- $\left| \frac{\sin(x^6 + 3y + 2)}{1 + x^6 + y^2} \right| \leq \frac{1}{1 + x^6}$, for all $x \geq 0, c \leq y \leq d$.
- The integral $\int\limits_0^{+\infty} \frac{1}{1 + x^6} dx$ converges.
- So $I(t)$ uniformly converges on $[c, d]$.
- And $I(t)$ is continuous on $[c, d]$.

- The function $f'_y(x,y) = \dfrac{3\cos(x^6+3y+2)}{1+x^6+y^2} - \dfrac{2y\sin(x^6+3y+2)}{1+x^6+y^2}$ is continuous on $[0,+\infty) \times [c,d]$.
- $\left|f'_y(x,y)\right| \leq \dfrac{3}{1+x^6} + \dfrac{M}{1+x^6} := \varphi(x)$, for all $x \geq 0, c \leq y \leq d$, where $M = 2\max\{|c|,|d|\}$.
- The integral $\int\limits_0^{+\infty} \varphi(x)$ converges.
- So $\int\limits_0^{+\infty} f'_y(x,y)dx$ uniformly converges.
- Thus $I(t)$ is differentiable.

Example (Final-20172)

Evaluate $\int\limits_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx$, where $a, b > 0$.

- $\int\limits_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx = \int\limits_0^{+\infty} \left(\int\limits_a^b xe^{-x^2 y} dy \right) dx.$
- The integral $I(y) = \int\limits_0^{+\infty} xe^{-x^2 y} dx$ uniformly converges for each y in $[a, b]$:
- $xe^{-x^2 y} \leq xe^{-x^2 a}$ for all $x \geq 0$ and $y \in [a, b]$, and $\int\limits_0^{+\infty} xe^{-x^2 a} dx$ converges.
- We can change the order of integration:

$$\int\limits_0^{+\infty} \left(\int\limits_a^b xe^{-x^2 y} dy \right) dx = \int\limits_a^b \int\limits_0^{+\infty} xe^{-x^2 y} dx dy = \int\limits_a^b \frac{1}{2y} dy = \frac{1}{2} (\ln b - \ln a).$$

3.3.1. The gamma Function

- For each positive integer n, $n! = 1 \cdot 2 \cdots n$.
- Euler showed that $n! = \int\limits_0^{+\infty} x^n e^{-x} dx$ (around 1730).

Definition

The gamma function is an improper integral depending on a parameter t:

$$\Gamma(t) = \int\limits_0^{+\infty} x^{t-1} e^{-x} dx.$$

Fix $t > 0$.

- $\int_a^1 e^{-x}x^{t-1}dx < \int_a^1 x^{t-1}dx = \frac{1}{t} - \frac{a^t}{t} < \frac{1}{t}$.
- So there exists $\int_0^1 e^{-x}x^{t-1}dx = \lim_{a\to 0}\int_a^1 e^{-x}x^{t-1}dx$.
- $e^x > \frac{x^n}{n!}$, for each n, take $n > t+1$.
- $e^{-x}x^{t-1} < \frac{n!}{x^{n+1-t}}$.
- $\int_1^b e^{-x}x^{t-1}dx < \int_1^b \frac{n!}{x^{n+1-t}}dx = n!\left(\frac{1}{n-t} - \frac{1}{b^{n-t}}\right) < \frac{n!}{n-t}$.

  So there exists $\int_1^\infty e^{-x}x^{t-1}dx = \lim_{b\to\infty}\int_1^b e^{-x}x^{t-1}dx$.

Properties

- The gamma function is infinitely differentiable
- $\Gamma(1) = 1.$
- $\Gamma(t+1) = t\Gamma(t)$, $t > 0.$
- $\Gamma(n+1) = n!.$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$
- $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{n!2^{2n}}\sqrt{\pi} = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}.$
- $\boxed{\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}}$, $(0 < p < 1)$. (Euler's reflection formula.)

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

"A mathematician is one to whom *that* is as obvious as that twice two makes four is to you. Liouville was a mathematician."

Bohr-Mollerup Theorem

Bohr-Mollerup Theorem (1922)

The gamma function $\Gamma(x)$ is the only function that is defined in $(0, +\infty)$ and satisfies the following conditions:

- $f(1) = 1$,
- $f(x+1) = xf(x)$.
- $\log f(x)$ is convex.

3.3.2. Beta function

Definition

The Beta function is the improper integral depending parameters p, q:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt.$$

This is the first form of the Beta function

Properties

- The gamma function $B(p,q)$ is defined and infinitely differentiable for all $p>0$, $q>0$.
- $B(p,q) = B(q,p)$.
- $B(p+1,q) = \dfrac{p}{p+q} B(p,q)$.
- $B(p,q) = \displaystyle\int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}}\,dt$.
- $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.
- $\Gamma(p)\Gamma(1-p) = B(p,1-p) = \displaystyle\int_0^{+\infty} \frac{t^{p-1}}{1+t}\,dt = \frac{\pi}{\sin(p\pi)}$.

Example (Final 20171)

Evaluate $I = \int\limits_0^{+\infty} x^4 e^{-x^2} dx$.

- Let $t = x^2$. Then $dt = 2xdx \Rightarrow dx = \dfrac{dt}{2\sqrt{t}}$.
- $I = \int\limits_0^{+\infty} t^2 e^{-t} \dfrac{1}{2\sqrt{t}} dt = \dfrac{1}{2} \int\limits_0^{+\infty} e^{-t} t^{3/2} dt = \dfrac{1}{2}\Gamma(5/2) = \dfrac{1}{2} \cdot \dfrac{3}{2}\Gamma(3/2) = \dfrac{1}{2} \cdot \dfrac{3}{2} \cdot \dfrac{1}{2}\Gamma(1/2) = \dfrac{3\sqrt{\pi}}{8}$.

Example (Final 20192)

Evaluate $I = \int\limits_1^{+\infty} \frac{(\ln x)^{3/2}}{x^4} dx$.

- Let $t = \ln x$. Then $x = e^t$ và $dx = e^t dt$.
- $I = \int\limits_0^{+\infty} t^{3/2} e^{-4t} e^t dt = \int\limits_0^{+\infty} t^{3/2} e^{-3t} dt$.
- Let $u = 3t \Leftrightarrow t = u/3$.
- $I = \int\limits_0^{+\infty} \frac{1}{3^{3/2}} u^{3/2} e^{-u} \frac{1}{3} du = \frac{1}{9\sqrt{3}} \int\limits_0^{+\infty} u^{3/2} e^{-u} du = \frac{1}{9\sqrt{3}} \Gamma(5/2) = \frac{1}{9\sqrt{3}} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{12\sqrt{3}}$.

Some past exam problems

- (Final 20161) Evaluate $\int\limits_0^1 x^5(\ln x)^{10}dx$.
- (Final 20182) Evaluate $\int\limits_0^{+\infty} x^6 3^{-x^2}dx$.
- (Final 20152) Evaluate $\int\limits_0^{+\infty} x^{25}e^{-x^2}dx$.
- (Final 20152) Evaluate $\int\limits_0^{+\infty} x^6 e^{-\sqrt{x}}dx$.
- (Final 20142) Evaluate $\int\limits_0^{+\infty} x^9 e^{-x^4}dx$.

Example (Final 20152)

Evaluate $\int\limits_{-\infty}^{0} e^{2x}\sqrt[3]{1-e^{3x}}\,dx$.

- Let $t = e^{3x}$, $t\colon 0 \to 1$. Then $x = \dfrac{1}{3}\ln t$ and $dx = \dfrac{1}{3t}dt$.
- $I = \int\limits_0^1 t^{2/3}(1-t)^{1/3}\dfrac{1}{3t}dt = \dfrac{1}{3}B(2/3,4/3) = \dfrac{1}{3}\dfrac{\Gamma(2/3)\Gamma(4/3)}{\Gamma(2)} = \dfrac{1}{3}\Gamma(2/3)\dfrac{1}{3}\Gamma(1/3) =$
  $\dfrac{1}{9}\Gamma(2/3)\Gamma(1/3) = \dfrac{1}{9}\dfrac{\pi}{\sin(\pi/3)} = \dfrac{2\pi}{9\sqrt{3}}$.

Example (Final 20182)

Evaluate $\int\limits_0^{+\infty} \frac{x^2}{(1+x^4)^4}dx$.

- Let $t = x^4 \to x = t^{1/4} \to dx = \frac{t^{-3/4}}{4}dt$.
- $\int\limits_0^{+\infty} \frac{t^{1/2}}{(1+t)^4}\frac{t^{-3/4}}{4}dt = \frac{1}{4}\int\limits_0^{+\infty} \frac{t^{-1/4}}{(1+t)^4} = \frac{1}{4}B(\frac{3}{4}, \frac{13}{4}) = \frac{1}{4}\frac{\Gamma(3/4)\Gamma(13/4)}{\Gamma(4)} =$
$\frac{1}{4}\frac{1}{3!}\Gamma(3/4)\frac{9}{4}\frac{5}{4}\frac{1}{4}\Gamma(1/4) = \frac{15}{512}\Gamma(1/4)\Gamma(3/4) = \frac{15}{512}\frac{\pi}{\sin(\pi/4)} = \frac{15\pi}{256\sqrt{2}}$.

Some past exam problems

- (Final 20152) Evaluate $\int\limits_0^1 \sqrt[4]{\dfrac{x^3}{(1-\sqrt{x})^2}}\,dx$.
- (Final 20162) Evaluate $\int\limits_0^{+\infty} \dfrac{1}{(1+x^2)\sqrt[5]{x^4}}\,dx$.
- (Final 20171) Evaluate $\int\limits_0^{+\infty} \dfrac{1}{(1+4x^4)^2}\,dx$.
- (Final 20192) Evaluate $\int\limits_{-\infty}^{+\infty} \dfrac{e^{x/4}}{(1+e^x)^2}\,dx$.