Chapter 2. Random Variables and Probability Distributions

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- Thus, each outcome in the sample space will be assigned a numerical value of 0, 1 or 2.
- These values are random quantities determined by the outcome of the experiment.
- They may be viewed as values assumed by the random variable X, the number of defective items when two electronic components are tested.

Definition of Random Variables

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- The sample space $S = \{DF, FD, DT, FF, FT, TF\}$, where D denotes one dollar coin, F denotes fifty cent coin and T denotes twenty cent coin.
- $Y : S \to S_Y \subset R$, where
  $DF \mapsto 1.5$; $FD \mapsto 1.5$; $DT \mapsto 1.2$; $FF \mapsto 1.0$; $FT \mapsto 0.7$; $TF \mapsto 0.7$

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Select at random an electronic component from a production line and let T be the life time in years of this electronic component.

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Select at random an electronic component from a production line and let T be the life time in years of this electronic component.

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Definition 3.2:

A probability distribution is a table, a formula (function), or a graph that describes the values of a random variable and the probability associated with each value or each interval of values.

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- The range of Y is $S_Y = \{0.7; 1.0; 1.2; 1.5\}$.
- The event $(Y = 0.7) = \{FT; TF\}$ so $P(Y = 0.7) = \frac{2}{6}$. Similarly, $P(Y = 1.0) = \frac{1}{6}; P(Y = 1.2) = \frac{1}{6}$ and $P(Y = 1.5) = \frac{2}{6}$

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- The distribution of Y is the following table:

Y	0.7	1.0	1.2	1.5
$f(x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

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- or the following function (probability mass function p.m.f):

$$f_Y(x) = P(Y = x) = \begin{cases} \frac{1}{6} \text{ if } x = 1.0 \text{ or } x = 1.2 \\ \frac{2}{6} \text{ if } x = 0.7 \text{ or } x = 1.5 \\ 0 \text{ otherwise.} \end{cases}$$

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- The probability distribution of T is given by a function (probability density function, p.d.f)

$$f_T(t) = \begin{cases} 0.2e^{-0.2t} \text{ if } t > 0 \\ 0 \text{ otherwise} \end{cases}$$

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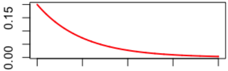

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Definition 3.3:

Let X be a discrete random variable where the range $S_X = \{x_1, x_2, ...\}$ is finte or countable. The probability distribution of X is defined by a probability distribution table

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or a probability mass fuction (pmf)

$$f_X(x) = \begin{cases} P(X = x) \text{ if } x \in S_X \\ 0 \text{ otherwise} \end{cases}$$

that satisfies the following two requirements:
$p = P(X = x) \geq 0, \forall x \in S_X$ and $\sum_{x \in S_X} P(X = x) = 1$.

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Suppose that the rate of defective component in a production line is 5%. Let X be the number of defective components when two electronic components are tested. Develop the probability distribution of X.

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- The range of X is $S_X = \{0; 1; 2\}$ and
  $P(X = 0) = P(NN) = 0.95 * 0.95 = 0.9025;$
  $P(X = 0) = P(DN) + P(ND) = 2 * 0.05 * 0.95 = 0.095; P(X = 2) = P(DD) = 0.05 * 0.05 = 0.0025$

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- The probability mass function (pmf) of X is:

$$f_X(x) = P(X = x) = \begin{cases} 0.9025 \text{ if } x = 0 \\ 0.095 \text{ if } x = 1 \\ 0.0025 \text{ if } x = 2 \\ 0 \text{ otherwise.} \end{cases}$$

Probability Distribution of Discrete Random Variables

Example 3.9:

The probability distribution of Y, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

Y	0	1	2	3	4
$f(x)$	0.41	0.37	c	0.05	0.01

- Find the constant c.
- Find the following probabilities:
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- $P(Y > 2) = P(Y = 3) + P(Y = 4) = 0.05 + 0.01 = 0.06$;
- $P(Y = 3.5) = 0$.

Cumulative distribution function

- There are many problems where we may wish to compute the probability that the observed value of a random variable X will be less than some real number t.

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Definition 3.4:

Let X be a discrete random variable. The cumulative distribution function (cdf) of X is denoted by $F_X(t)$ and defined as the following:

$$F_X(t) = P(X < t) = \sum_{x_i < t} P(X = x_i) \text{ for } t \in R.$$

Cumulative distribution function

Example 3.10:

The discrete random variable X (Example 3.8) has the following probability distribution table:

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The cdf of X is:

$$F_X(t) = P(X < t) = \begin{cases} 0 \text{ if } t \leq 0 \\ 0.9025 \text{ if } 0 < t \leq 1 \\ 0.9975 \text{ if } 1 < t \leq 2 \\ 1 \text{ if } t > 2. \end{cases}$$

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- $P(t_1 \leq X < t_2) = F_X(t_2) - F_X(t_1)$ for all $t_1 < t_2$.

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- $P(t_1 \leq X < t_2) = F_X(t_2) - F_X(t_1)$ for all $t_1 < t_2$.
- $P(X = t_0) = \lim_{t \to t_0^+} P(t_0 \leq X < t) = \lim_{t \to t_0^+} [F_X(t) - F_X(t_0)]$.

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- $P(t_1 \leq X < t_2) = F_X(t_2) - F_X(t_1)$ for all $t_1 < t_2$.
- $P(X = t_0) = \lim_{t \to t_0^+} P(t_0 \leq X < t) = \lim_{t \to t_0^+} [F_X(t) - F_X(t_0)]$.
- $F(+\infty) = \lim_{t \to +\infty} F_X(t) = 1$ and $F(-\infty) = \lim_{t \to -\infty} F_X(t) = 0$.

Cumulative distribution function

Example 3.11:

A discrete random variable X has the following cdf:

$$F_X(t) = \begin{cases} 0 \text{ if } t \leq 0 \\ 0.09 \text{ if } 0 < t \leq 1 \\ c \text{ if } 1 < t \leq 2 \\ 0.79 \text{ if } 2 < t \leq 3 \\ 1 \text{ if } t > 3. \end{cases}$$

- Find the constant c given that $P(X = 1) = 0.33$.
- Find the following probabilities: $P(1 \leq X < 3)$; $P(1 < X \leq 3)$; $P(1 \leq X \leq 3)$.

Cumulative distribution function

Solution of Example 3.11:

- We have

$$P(X = 1) = \lim_{t \to 1^+} P(1 \le X < t) = \lim_{t \to 1^+} [F_X(t) - F_X(1)] = c - 0.09$$

then $c - 0.09 = 0.33$ or $c = 0.42$.

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- By using the properties:

$$P(X = t_0) = \lim_{t \to t_0^+} P(t_0 \leq X < t) = \lim_{t \to t_0^+} [F_X(t) - F_X(t_0)]$$

we can find that $P(X = 0) = 0.09$; $P(X = 1) = 0.33$;
$P(X = 2) = 0.37$ and $P(X = 3) = 0.21$.

Cumulative distribution function

Solution of Example 3.11:

- We have

$$P(X = 1) = \lim_{t \to 1^+} P(1 \le X < t) = \lim_{t \to 1^+} [F_X(t) - F_X(1)] = c - 0.09$$

then $c - 0.09 = 0.33$ or $c = 0.42$.

- By using the properties:

$$P(X = t_0) = \lim_{t \to t_0^+} P(t_0 \le X < t) = \lim_{t \to t_0^+} [F_X(t) - F_X(t_0)]$$

we can find that $P(X = 0) = 0.09$; $P(X = 1) = 0.33$;
$P(X = 2) = 0.37$ and $P(X = 3) = 0.21$.

- Then $P(1 \le X < 3) = P(X = 1) + P(X = 2) = 0.7$;
$P(1 < X \le 3) = P(X = 2) + P(X = 3) = 0.58$;
$P(1 \le X \le 3) = P(X = 1) + P(X = 2) + P(X = 3) = 0.91$.

Probability Density Function

- A continuous random variable has an uncountable range and has a probability of 0 of assuming exactly any of its values.

Probability Density Function

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Definition 3.5:

Let X be a continuous random variable. The probability density function (pdf) of X is a function $f_X(x)$ defined over the set of real numbers that satisfies the following requirements:

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Definition 3.5:

Let X be a continuous random variable. The probability density function (pdf) of X is a function $f_X(x)$ defined over the set of real numbers that satisfies the following requirements:

- $f_X(x) \geq 0$ for all $x \in R$.

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- $f_X(x) \geq 0$ for all $x \in R$.
- $\int_{-\infty}^{+\infty} f_X(x)dx = 1$.
- $P(a < X < b) = \int_a^b f_X(x)dx$.

Cumulative distribution function

Definition 3.6:

Let X be a continuous random variable with the probability density function (pdf) $f_X(x)$. The cumulative distribution function (cdf) of X is defined by:

$$F_X(t) = P(X < t) = \int_{-\infty}^{t} f_X(x)dx \text{ for } t \in R.$$

Cumulative distribution function

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Properties of cdf of a continuous random variable:

- $0 \leq F_X(t) \leq 1, \forall t \in R.$

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- $F_X(t)$ is a non-decreasing function, i.e., for all $t_1 < t_2$, we have $F_X(t_1) \leq F_X(t_2)$.

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- $F_X(t)$ is a non-decreasing function, i.e., for all $t_1 < t_2$, we have $F_X(t_1) \leq F_X(t_2)$.
- $F_X(t)$ is differentiable and $f_X(t) = F'_X(t)$.

Cumulative distribution function

Definition 3.6:

Let X be a continuous random variable with the probability density function (p.d.f) $f_X(x)$. The cumulative distribution function (cdf) of X is defined by:

$$F_X(t) = P(X < t) = \int_{-\infty}^{t} f_X(x)dx \text{ for } t \in R.$$

Properties of c.d.f of a continuous random variable:

- $F(+\infty) = \lim_{t\to+\infty} F_X(t) = 1$ and $F(-\infty) = \lim_{t\to-\infty} F_X(t) = 0$.

Cumulative distribution function

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Let X be a continuous random variable with the probability density function (p.d.f) $f_X(x)$. The cumulative distribution function (cdf) of X is defined by:

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Properties of c.d.f of a continuous random variable:

- $F(+\infty) = \lim_{t\to+\infty} F_X(t) = 1$ and $F(-\infty) = \lim_{t\to-\infty} F_X(t) = 0$.
- $P(X = c) = 0\ \forall c \in R$, then $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$.

Cumulative distribution function

Example 3.12:

The life time X (in years) of a type of machines has the following pdf:

$$f(x) = \begin{cases} A(x-2)(8-x) \text{ if } x \in [2;8], \\ 0 \text{ otherwise} \end{cases}$$

- Find the constsant A.
- Compute the probability that a machine has life time between 3 and 5 years.
- Find the cdf of X.

Cumulative distribution function

Solution of Example 3.12:

- The function $f(x)$ satisfies the following two requirements:
  - $f(x) \geq 0, \forall x \in R \Leftrightarrow A \geq 0.$

Cumulative distribution function

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  - $f(x) \geq 0, \forall x \in R \Leftrightarrow A \geq 0.$
  - $\int_{-\infty}^{+\infty} f(x)dx = A\int_2^8 (x-2)(8-x)dx = 36A = 1$ so $A = \frac{1}{36}$

Cumulative distribution function

Solution of Example 3.12:

- The function $f(x)$ satisfies the following two requirements:
  - $f(x) \geq 0, \forall x \in R \Leftrightarrow A \geq 0.$
  - $\int_{-\infty}^{+\infty} f(x)dx = A\int_{2}^{8}(x-2)(8-x)dx = 36A = 1$ so $A = \frac{1}{36}$
- The probability that a machine has life time between 3 and 5 years is: $P(3 \leq X \leq 5) = \int_{3}^{5} f(x)dx = \frac{1}{36}\int_{3}^{5}(x-2)(8-x)dx = \frac{23}{54}.$

Cumulative distribution function

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- The function $f(x)$ satisfies the following two requirements:
  - $f(x) \geq 0, \forall x \in R \Leftrightarrow A \geq 0.$
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- The cdf of X is $F(x) = \int_{-\infty}^{x} f(t)dt$

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- The cdf of X is $F(x) = \int_{-\infty}^{x} f(t)dt$
  - If $x \leq 2$ then $F(x) = \int_{-\infty}^{x} 0dt = 0$.

Cumulative distribution function

Solution of Example 3.12:

- The function $f(x)$ satisfies the following two requirements:
  - $f(x) \geq 0, \forall x \in R \Leftrightarrow A \geq 0.$
  - $\int_{-\infty}^{+\infty} f(x)dx = A\int_2^8 (x-2)(8-x)dx = 36A = 1$ so $A = \frac{1}{36}$
- The probability that a machine has life time between 3 and 5 years is: $P(3 \leq X \leq 5) = \int_3^5 f(x)dx = \frac{1}{36}\int_3^5 (x-2)(8-x)dx = \frac{23}{54}.$
- The cdf of X is $F(x) = \int_{-\infty}^{x} f(t)dt$
  - If $x \leq 2$ then $F(x) = \int_{-\infty}^{x} 0dt = 0.$
  - If $2 < x \leq 8$ then $F(x) = \int_{-\infty}^{2} 0dt + \frac{1}{36}\int_2^x (t-2)(8-t)dt = \frac{5x^2}{36} - \frac{x^3}{108} - \frac{4x}{9} + \frac{11}{27}.$

Cumulative distribution function

Solution of Example 3.12:

- The function $f(x)$ satisfies the following two requirements:
  - $f(x) \geq 0, \forall x \in R \Leftrightarrow A \geq 0.$
  - $\int_{-\infty}^{+\infty} f(x)dx = A\int_2^8 (x-2)(8-x)dx = 36A = 1$ so $A = \frac{1}{36}$
- The probability that a machine has life time between 3 and 5 years is: $P(3 \leq X \leq 5) = \int_3^5 f(x)dx = \frac{1}{36}\int_3^5 (x-2)(8-x)dx = \frac{23}{54}.$
- The cdf of X is $F(x) = \int_{-\infty}^{x} f(t)dt$
  - If $x \leq 2$ then $F(x) = \int_{-\infty}^{x} 0dt = 0.$
  - If $2 < x \leq 8$ then $F(x) = \int_{-\infty}^{2} 0dt + \frac{1}{36}\int_2^x (t-2)(8-t)dt = \frac{5x^2}{36} - \frac{x^3}{108} - \frac{4x}{9} + \frac{11}{27}.$
  - If $x > 8$ then $F(x) = \frac{1}{36}\int_2^8 (t-2)(8-t)dt = 1.$

Cumulative distribution function

Solution of Example 3.12:

- The cdf of X is

$$F(x) = \begin{cases} 0 \text{ if } x \leq 2, \\ \frac{5x^2}{36} - \frac{x^3}{108} - \frac{4x}{9} + \frac{11}{27} \text{ if } 2 < x \leq 8, \\ 1 \text{ if } x > 8. \end{cases}$$

Cumulative distribution function

Example 3.13:

A continuous random variable Y has the following cdf:

$$F(x) = a + b \arctan x \text{ for all } x \in R.$$

- Find the constsants a and b.
- Find the pdf of Y.
- Find the probability that out of 5 independent trials of observing the values of Y, there are exactly 3 times Y takes values in $[-1; 1]$.

Cumulative distribution function

Solution of Example 3.13:

- We have $F(-\infty) = a - b\frac{\pi}{2} = 0$ and $F(+\infty) = a + b\frac{\pi}{2} = 1$ then $a = \frac{1}{2}; b = \frac{1}{\pi}$.

Cumulative distribution function

Solution of Example 3.13:

- We have $F(-\infty) = a - b\frac{\pi}{2} = 0$ and $F(+\infty) = a + b\frac{\pi}{2} = 1$ then $a = \frac{1}{2}; b = \frac{1}{\pi}$.
- The pdf of Y is $f(x) = F'(x) = \frac{1}{\pi}.\frac{1}{1+x^2}$ for all $x \in R$.

Cumulative distribution function

Solution of Example 3.13:

- We have $F(-\infty) = a - b\frac{\pi}{2} = 0$ and $F(+\infty) = a + b\frac{\pi}{2} = 1$ then $a = \frac{1}{2}; b = \frac{1}{\pi}$.
- The pdf of Y is $f(x) = F'(x) = \frac{1}{\pi}.\frac{1}{1+x^2}$ for all $x \in R$.
- The probability that Y takes values in $[-1; 1]$ is: $P(-1 \leq Y \leq 1) = F(1) - F(-1) = (\frac{1}{2} + \frac{1}{\pi}\frac{\pi}{4}) - (\frac{1}{2} + \frac{1}{\pi}\frac{(-\pi)}{4}) = 0.5$

Cumulative distribution function

Solution of Example 3.13:

- We have $F(-\infty) = a - b\frac{\pi}{2} = 0$ and $F(+\infty) = a + b\frac{\pi}{2} = 1$ then $a = \frac{1}{2}; b = \frac{1}{\pi}$.
- The pdf of Y is $f(x) = F'(x) = \frac{1}{\pi}.\frac{1}{1+x^2}$ for all $x \in R$.
- The probability that Y takes values in $[-1; 1]$ is:
  $P(-1 \leq Y \leq 1) = F(1) - F(-1) = (\frac{1}{2} + \frac{1}{\pi}\frac{\pi}{4}) - (\frac{1}{2} + \frac{1}{\pi}\frac{(-\pi)}{4}) = 0.5$
  Then the probability that out of 5 independent trials, there are exactly 3 times Y takes values in $[-1; 1]$ is:
  $P_5(3) = C_5^3 0.5^3(1 - 0.5)^2 = 0.3125$

Expected Value

Example 3.14:

- The probability distribution of the number of heads when two fair coins were tossed is

X	0	1	2
$f(x)$	0.25	0.5	0.25

Expected Value

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- The probability distribution of the number of heads when two fair coins were tossed is

X	0	1	2
$f(x)$	0.25	0.5	0.25

- These probabilities are just the relative frequencies for the given events in the long run.

Expected Value

Example 3.14:

- The probability distribution of the number of heads when two fair coins were tossed is

X	0	1	2
$f(x)$	0.25	0.5	0.25

- These probabilities are just the relative frequencies for the given events in the long run.
- The number of heads, on the average, when two fair coins were tossed over and over again is $\mu = 0 * 0.25 + 1 * 0.5 + 2 * 0.25 = 1$. This value is called the mean, or the expected value of X.

Expected Value

Definition 3.7:

Let X be a random variable with probability distribution $f(x)$ (pmf or pdf). The mean, or expected value, of X is defined by:

$$\mu = E(X) = \sum_{x \in S_X} x f(x)$$

if X is discrete,

Expected Value

Definition 3.7:

Let X be a random variable with probability distribution $f(x)$ (pmf or pdf). The mean, or expected value, of X is defined by:

$$\mu = E(X) = \sum_{x \in S_X} x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

if X is continuous.

Expected Value

Definition 3.7:

Let X be a random variable with probability distribution $f(x)$ (pmf or pdf). The mean, or expected value, of X is defined by:

$$\mu = E(X) = \sum_{x \in S_X} x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

if X is continuous.

- The expected value of X is a measure of its central location and is calculated by using the probability distribution.

Expected Value

Theorem 3.1:

Let X be a random variable with probability distribution $f(x)$ (pmf or pdf) and the random variable $Y = g(X)$. Then

$$E(Y) = E[g(X)] = \sum_{x \in S_X} g(x)f(x)$$

if X is discrete,

Expected Value

Theorem 3.1:

Let X be a random variable with probability distribution $f(x)$ (pmf or pdf) and the random variable $Y = g(X)$. Then

$$E(Y) = E[g(X)] = \sum_{x \in S_X} g(x)f(x)$$

if X is discrete, and

$$E(Y) = E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

if X is continuous.

Expected Value

Example 3.15:

The number of times X that a sttudent entered a pizza store has the following distribution:

Y	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

- Find the expected value of X.
- Find the expected value of $Y = X^2$.

Expected Value

Example 3.15:

The number of times X that a sttudent entered a pizza store has the following distribution:

Y	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

- Find the expected value of X.
- Find the expected value of $Y = X^2$.

Solution:

- The expected value of X is:
  $\mu = E(X) = 0 * 0.41 + 1 * 0.37 + 2 * 0.16 + 3 * 0.05 + 4 * 0.01 = 0.88.$

Expected Value

Example 3.15:

The number of times X that a sttudent entered a pizza store has the following distribution:

Y	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

- Find the expected value of X.
- Find the expected value of $Y = X^2$.

Solution:

- The expected value of X is:
  $\mu = E(X) = 0 * 0.41 + 1 * 0.37 + 2 * 0.16 + 3 * 0.05 + 4 * 0.01 = 0.88.$
- The expected value of $Y = X^2$ is:
  $E(X^2) = 0^2 * 0.41 + 1^2 * 0.37 + 2^2 * 0.16 + 3^2 * 0.05 + 4^2 * 0.01 = 1.62$

Expected Value

Example 3.16:

The life time X (in years) of a type of machines has the following pdf:

$$f(x) = \begin{cases} \frac{1}{36}(x-2)(8-x) \text{ if } x \in [2;8], \\ 0 \text{ otherwise} \end{cases}$$

- Find the mean life time (in years) of all machines.
- Find the pdf of $Y = \sqrt{X}$ and the expected value of Y by two methods.

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The cdf of $Y = \sqrt{X}$ is $F_Y(y) = P(Y < y) = P(\sqrt{X} < y).$

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is
  $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The cdf of $Y = \sqrt{X}$ is $F_Y(y) = P(Y < y) = P(\sqrt{X} < y)$.
  - If $y \leq 0$ then $F_Y(y) = P(\emptyset) = 0$.

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is
  $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The cdf of $Y = \sqrt{X}$ is $F_Y(y) = P(Y < y) = P(\sqrt{X} < y)$.
  - If $y \leq 0$ then $F_Y(y) = P(\emptyset) = 0$.
  - If $y > 0$ then $F_Y(y) = P(X < y^2) = F_X(y^2)$.

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The cdf of $Y = \sqrt{X}$ is $F_Y(y) = P(Y < y) = P(\sqrt{X} < y)$.
  - If $y \leq 0$ then $F_Y(y) = P(\emptyset) = 0$.
  - If $y > 0$ then $F_Y(y) = P(X < y^2) = F_X(y^2)$.
    So if $y^2 \leq 2 \Leftrightarrow 0 < y \leq \sqrt{2}$, $F_Y(y) = 0$.

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The cdf of $Y = \sqrt{X}$ is $F_Y(y) = P(Y < y) = P(\sqrt{X} < y)$.
  - If $y \leq 0$ then $F_Y(y) = P(\emptyset) = 0$.
  - If $y > 0$ then $F_Y(y) = P(X < y^2) = F_X(y^2)$.
    So if $y^2 \leq 2 \Leftrightarrow 0 < y \leq \sqrt{2}$, $F_Y(y) = 0$.
    If $2 < y^2 \leq 8 \Leftrightarrow \sqrt{2} < y \leq \sqrt{8}$ then $F_Y(y) = \frac{5y^4}{36} - \frac{y^6}{108} - \frac{4y^2}{9} + \frac{11}{27}$

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The cdf of $Y = \sqrt{X}$ is $F_Y(y) = P(Y < y) = P(\sqrt{X} < y)$.
  - If $y \leq 0$ then $F_Y(y) = P(\emptyset) = 0$.
  - If $y > 0$ then $F_Y(y) = P(X < y^2) = F_X(y^2)$.
    So if $y^2 \leq 2 \Leftrightarrow 0 < y \leq \sqrt{2}$, $F_Y(y) = 0$.
    If $2 < y^2 \leq 8 \Leftrightarrow \sqrt{2} < y \leq \sqrt{8}$ then $F_Y(y) = \frac{5y^4}{36} - \frac{y^6}{108} - \frac{4y^2}{9} + \frac{11}{27}$
    If $y^2 > 8 \Leftrightarrow y > \sqrt{8}$ then $F_Y(y) = 1$.

Expected Value

Solution of Example 3.16:

- The mean life time (in years) of all machines is $\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The cdf of $Y = \sqrt{X}$ is $F_Y(y) = P(Y < y) = P(\sqrt{X} < y)$.
  - If $y \le 0$ then $F_Y(y) = P(\emptyset) = 0$.
  - If $y > 0$ then $F_Y(y) = P(X < y^2) = F_X(y^2)$.
    So if $y^2 \le 2 \Leftrightarrow 0 < y \le \sqrt{2}$, $F_Y(y) = 0$.
    If $2 < y^2 \le 8 \Leftrightarrow \sqrt{2} < y \le \sqrt{8}$ then $F_Y(y) = \frac{5y^4}{36} - \frac{y^6}{108} - \frac{4y^2}{9} + \frac{11}{27}$
    If $y^2 > 8 \Leftrightarrow y > \sqrt{8}$ then $F_Y(y) = 1$.
  - Then the pdf of Y is

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{5y^3}{9} - \frac{y^5}{18} - \frac{8y}{9} \text{ if } y \in [\sqrt{2}; \sqrt{8}], \\ 0 \text{ otherwise} \end{cases}$$

Expected Value

Solution of Example 3.16:

- The expected value of $Y = \sqrt{X}$ is:

$$E(Y) = \int_{-\infty}^{+\infty} \sqrt{x} f_X(x) dx = \frac{1}{36} \int_2^8 \sqrt{x}(x-2)(8-x) dx \approx 2.215$$

Expected Value

Solution of Example 3.16:

- The expected value of $Y = \sqrt{X}$ is:

$$E(Y) = \int_{-\infty}^{+\infty} \sqrt{x} f_X(x)dx = \frac{1}{36}\int_2^8 \sqrt{x}(x-2)(8-x)dx \approx 2.215$$

or

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y)dy = \int_{\sqrt{2}}^{\sqrt{8}} \left(\frac{5y^4}{9} - \frac{y^6}{18} - \frac{8y^2}{9}\right)dy \approx 2.215$$

Variance and Standard Deviation

Definition 3.8:

Let X be a random variable with probability distribution $f(x)$ (pmf or pdf). The variance of X is defined by:

$$\sigma^2 = V(X) = E[(X-\mu)^2] = E(X^2) - \mu^2$$

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- if X is continuous $V(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2$.

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- if X is continuous $V(X) = \int_{-\infty}^{+\infty} x^2 f(x) dx - \mu^2$.
- The variance and the standard deviation of X are measures of its variability (disperson).

Variance and Standard Deviation

Example 3.17:

A discrete random variable X has the following distribution:

Y	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

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  $\mu = E(X) = 0 * 0.41 + 1 * 0.37 + 2 * 0.16 + 3 * 0.05 + 4 * 0.01 = 0.88$

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- The variance of X is: $\sigma^2 = V(X) = \sum_{x \in S_X} x^2 f(x) - \mu^2$
  $= 0^2 * 0.41 + 1^2 * 0.37 + 2^2 * 0.16 + 3^2 * 0.05 + 4^2 * 0.01 - 0.88^2 = 0.8456$

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- The standard deviation of X is $\sigma = \sqrt{0.8456} \approx 0.92$

Variance and Standard Deviation

Example 3.18:

The life time X (in years) of a type of machines has the following pdf:

$$f(x) = \begin{cases} \frac{1}{36}(x-2)(8-x) \text{ if } x \in [2;8], \\ 0 \text{ otherwise} \end{cases}$$

Variance and Standard Deviation

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  $\mu = E(X) = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$

Variance and Standard Deviation

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- The expected value of X is: $\mu = E(X) = \frac{1}{36}\int_2^8 x(x-2)(8-x)dx = 5.$
- The variance of X is: $\sigma^2 = V(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2$ $= \frac{1}{36}\int_2^8 x^2(x-2)(8-x)dx - 5^2 = 1.8.$

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  $= \frac{1}{36}\int_2^8 x^2(x-2)(8-x)dx - 5^2 = 1.8.$
- The standard deviation of X is $\sigma = \sqrt{1.8} \approx 1.34.$

Laws of Expectation and Variance

Laws of Expectation and Variance:

For any random variables X, Y and any constants a, b, C:

- $E(C) = C; V(C) = 0$

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- $E(aX+b) = aE(X)+b; V(aX+b) = a^2V(X)$.

Laws of Expectation and Variance

Example 3.19:

The monthly sales at a computer store have a mean of 25000$ and a standard deviation of 4000$. Profits are 30% of the sales less fixed costs of 6000$. Find the mean and standard deviation of the monthly profit.

Laws of Expectation and Variance

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Solution:

Let X be the monthly sales of the computer store and let Y be its monthly profit.

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Let X be the monthly sales of the computer store and let Y be its monthly profit.

- We have $E(X) = 25000; \sigma(X) = 4000$ and $Y = 0.3X - 6000$.
- The mean of the monthly profit is:
  $E(Y) = E(0.3X - 6000) = 0.3E(X) - 6000 = 1500$

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- The mean of the monthly profit is:
  $E(Y) = E(0.3X - 6000) = 0.3E(X) - 6000 = 1500$
- The variance of the monthly profit is:
  $V(Y) = V(0.3X - 6000) = 0.3^2 V(X)$
  Then the standard deviation of the monthly profit is:
  $\sigma(Y) = 0.3\sigma(X) = 0.3 * 4000 = 1200.$

Discrete Uniform Distribution

Definition 3.9:

A discrete random variable X with the range $S_X = \{x_1, x_2, ..., x_n\}$ follows uniform distribution if X has the following probability distribution table

X	x_1	x_2	...	x_n
$f(x)$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

or the pmf of X is given by

$$f(x) = P(X = x) = \begin{cases} \frac{1}{n} \text{ if } x \in S_X, \\ 0 \text{ otherwise} \end{cases}$$

Discrete Uniform Distribution

Example 3.20:

Each odd number from 1 to 99 is written on an individual tile and one is chosen at random. The random variable T represents the number on the chosen tile. Find $E(T)$ and $V(T)$.

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Solution:

- Let X be the discrete uniform distribution on the range $S_X = \{1, 2, ..., 50\}$. A formula linking X and T is $T = 2X - 1$.

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Solution:

- Let X be the discrete uniform distribution on the range $S_X = \{1, 2, ..., 50\}$. A formula linking X and T is $T = 2X - 1$.
- Then $E(T) = 2E(X) - 1$ and $V(T) = 2^2 V(X) = 4V(X)$.

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- Let X be the discrete uniform distribution on the range $S_X = \{1, 2, ..., 50\}$. A formula linking X and T is $T = 2X - 1$.
- Then $E(T) = 2E(X) - 1$ and $V(T) = 2^2 V(X) = 4V(X)$.
- The mean of X is $E(X) = \sum_{i=1}^{50} i * \frac{1}{50} = \frac{50(50+1)}{2}\frac{1}{50} = 25.5$, so the mean of T is $E(T) = 2 * 25.5 - 1 = 50$.

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Each odd number from 1 to 99 is written on an individual tile and one is chosen at random. The random variable T represents the number on the chosen tile. Find $E(T)$ and $V(T)$.

Solution:

- Let X be the discrete uniform distribution on the range $S_X = \{1, 2, ..., 50\}$. A formula linking X and T is $T = 2X - 1$.
- Then $E(T) = 2E(X) - 1$ and $V(T) = 2^2 V(X) = 4V(X)$.
- The mean of X is $E(X) = \sum_{i=1}^{50} i * \frac{1}{50} = \frac{50(50+1)}{2}\frac{1}{50} = 25.5$, so the mean of T is $E(T) = 2 * 25.5 - 1 = 50$.
- The variance of X is $V(X) = \sum_{i=1}^{50} i^2 * \frac{1}{50} - 25.5^2 = \frac{50(50+1)(2*50+1)}{6}\frac{1}{50} - 25.5^2 = 208.25$, so the variance of T is $V(T) = 4 * 208.25 = 833$.

Hyper-Geometric Distribution

Definition 3.11:

- A hypergeometric experiment, that is, one that possesses the following two properties:
  1. A random sample of size n is selected without replacement from N items.
  2. Of the N items, k may be classified as successes and $N - k$ are classified as failures.

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  2. Of the N items, k may be classified as successes and $N-k$ are classified as failures.
- The number X of successes of a hypergeometric experiment is called a hyper-geometric random variable.
- The probability distribution of the hypergeometric variable is called the hypergeometric distribution and is given as follows:

$$p(x; N, n, k) = \frac{C_k^x C_{N-k}^{n-x}}{C_N^n} \text{ if } \max\{0, n-(N-k)\} \leq x \leq \min\{n, k\}.$$

Hyper-Geometric Distribution

Theorem 3.2:

The mean and variance of the hypergeometric distribution are

$$\mu = \frac{nk}{N} \text{ and } \sigma^2 = \frac{N-n}{N-1}.n.\frac{k}{N}.\left(1 - \frac{k}{N}\right)$$

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Example 3.22:

Lots of 40 components each are deemed unacceptable if they contain 3 or more defectives. The procedure for sampling a lot is to select 5 components at random and to reject the lot if a defective is found.

- What is the probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot?
- Find the mean and variance of the number of defective items found in the sample if there are 3 defectives in the entire lot.

Hyper-Geometric Distribution

Solution of Example 3.22:

Using the hypergeometric distribution with $n = 5, N = 40, k = 3$:

- The probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot is

$$p(1; 40, 5, 3) = \frac{C_3^1 C_{37}^4}{C_{40}^5} = 0.3011$$

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Solution of Example 3.22:

Using the hypergeometric distribution with $n = 5, N = 40, k = 3$:

- The probability that exactly 1 defective is found in the sample if there are 3 defectives in the entire lot is

$$p(1; 40, 5, 3) = \frac{C_3^1 C_{37}^4}{C_{40}^5} = 0.3011$$

- The mean and variance are

$$\mu = \frac{nk}{N} = \frac{5 * 3}{40} = 0.375$$

and

$$\sigma^2 = \frac{N-n}{N-1}.n.\frac{k}{N}.\left(1 - \frac{k}{N}\right) = \frac{40-5}{40-1}.5.\frac{3}{40}.\left(1 - \frac{3}{40}\right) = 0.3113$$

Binomial Distribution

Definition 3.12:

A Bernoulli process is a sequence of trials that satisfies the following properties:

1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

Binomial Distribution

Example 3.23:

Some examples of Bernoulli process:

1. Testing 10 electronic components from a production line consisting of 5% defective items.

Binomial Distribution

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Some examples of Bernoulli process:

1. Testing 10 electronic components from a production line consisting of 5% defective items.
2. A salesperson calls to 20 customers where the probability that she closes a sale is 0.6.

Binomial Distribution

Example 3.23:

Some examples of Bernoulli process:

1. Testing 10 electronic components from a production line consisting of 5% defective items.
2. A salesperson calls to 20 customers where the probability that she closes a sale is 0.6.
3. The probability that a patient recovers from a rare blood disease is 0.4. Consider 15 people that are known to have contracted this disease and check the number of patients who will survive.

Binomial Distribution

Definition 3.13:

Consider a Bernoulli process of n trials where the probability of success in each trial is p.

- The number X of successes in n Bernoulli trials is called a binomial random variable.

Binomial Distribution

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Consider a Bernoulli process of n trials where the probability of success in each trial is p.

- The number X of successes in n Bernoulli trials is called a binomial random variable.
- The probability distribution of this discrete random variable is called the binomial distribution and given as follows:

$$p(x; n, p) = P(X = x) = C_n^x p^x (1 - p)^{n-x} \text{ for } x = 0, 1, ..., n.$$

Binomial Distribution

Example 3.24:

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

- exactly 5 survive?
- at least 10 survive?
- from 3 to 8 survive?

Binomial Distribution

Solution of Example 3.24:

Let X be the number of patients who survive out of 15 people. Then $X \sim B(15, 0.4)$.

Binomial Distribution

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Let X be the number of patients who survive out of 15 people. Then $X \sim B(15, 0.4)$.

- The probability that exactly exactly 5 survive is:
  $P(X = 5) = C_{15}^{5} 0.4^5 0.6^{10} = 0.186$

Binomial Distribution

Solution of Example 3.24:

Let X be the number of patients who survive out of 15 people. Then $X \sim B(15, 0.4)$.

- The probability that exactly exactly 5 survive is:
  $P(X = 5) = C_{15}^{5} 0.4^{5} 0.6^{10} = 0.186$
- The probability that at least 10 survive is
  $P(X \geq 10) = P(X = 10) + ... + P(X = 15) = C_{15}^{10} 0.4^{10} 0.6^{5} + ... + C_{15}^{15} 0.4^{15} 0.6^{0} = 0.034$

Binomial Distribution

Solution of Example 3.24:

Let X be the number of patients who survive out of 15 people. Then $X \sim B(15, 0.4)$.

- The probability that exactly exactly 5 survive is: $P(X = 5) = C_{15}^{5} 0.4^5 0.6^{10} = 0.186$
- The probability that at least 10 survive is $P(X \geq 10) = P(X = 10) + ... + P(X = 15) = C_{15}^{10} 0.4^{10} 0.6^5 + ... + C_{15}^{15} 0.4^{15} 0.6^0 = 0.034$
- The probability that from 3 to 8 survive is $P(3 \leq X \leq 8) = P(X = 3) + ... + P(X = 8) = C_{15}^{3} 0.4^3 0.6^{12} + ... + C_{15}^{8} 0.4^8 0.6^7 = 0.878$

Binomial Distribution

Theorem 3.3:

- The mean and variance of the Binomial distribution $B(n, p)$ are

$$\mu = np \text{ and } \sigma^2 = np(1-p) = npq.$$

- The mode of X is the value that occurs the most frequently (or the value with the largest probability) and equals to:

$$m = \begin{cases} np - q \text{ or } np - q + 1 & \text{if } np - q \in Z, \\ [np - q] + 1 & \text{if } np - q \notin Z, \end{cases}$$

Binomial Distribution

Example 3.25:

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the

- average number of of patients who survive?
- variance of the number of patients who survive?
- mode of the number of patients who survive?

Binomial Distribution

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Binomial Distribution

Solution of Example 3.25:

Let X be the number of patients who survive out of 15 people. Then $X \sim B(15, 0.4)$.

- The average number of of patients who survive is $\mu = np = 15 * 0.4 = 6$.

Binomial Distribution

Solution of Example 3.25:

Let X be the number of patients who survive out of 15 people. Then $X \sim B(15, 0.4)$.

- The average number of of patients who survive is $\mu = np = 15 * 0.4 = 6$.
- The variance of number of of patients who survive is $\sigma^2 = npq = 15 * 0.4 * 0.6 = 3.6$.

Binomial Distribution

Solution of Example 3.25:

Let X be the number of patients who survive out of 15 people. Then $X \sim B(15, 0.4)$.

- The average number of of patients who survive is $\mu = np = 15 * 0.4 = 6$.
- The variance of number of of patients who survive is $\sigma^2 = npq = 15 * 0.4 * 0.6 = 3.6$.
- Since $np - q = 15 * 0.4 - 0.6 = 5.4 \notin Z$ then $mode(X) = [np - q] + 1 = 6$. So there are 6 patients who survive with the most likely (lagest probability).

Poisson Distribution

Definition 3.14:

- Experiments yielding numerical values of a random variable X, the number of outcomes occurring during a given time interval or in a specified region, are called Poisson experiments.

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- The parameter of Poisson distribution is it mean μ, the average number of outcomes occurring during the given time interval.

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- The number X of outcomes occurring during a Poisson experiment is called a Poisson random variable, and its probability distribution is called the Poisson distribution.
- The parameter of Poisson distribution is it mean μ, the average number of outcomes occurring during the given time interval.
- The range of X is $S_X = \{0, 1, 2, 3, ...\}$ and the probability associated with each value is

$$p(x; \mu) = P(X = x) = e^{-\mu} \frac{\mu^x}{x!} \text{ for } x = 0, 1, 2, 3, ...$$

Poisson Distribution

Theorem 3.4:

The mean and variance of the Poisson distribution $P(\lambda)$ is $\mu = \sigma^2 = \lambda$.

Poisson Distribution

Theorem 3.4:

The mean and variance of the Poisson distribution $P(\lambda)$ is $\mu = \sigma^2 = \lambda$.

Example 3.26:

During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4.

- What is the probability that 6 particles enter the counter in a given millisecond?
- What is the probability that 10 particles enter the counter in three given milliseconds?

Binomial Distribution

Solution of Example 3.26:

- Let X be the number of particles enter the counter in a given millisecond. Then $X \sim P(4)$, so the probability that 6 particles enter the counter in a given millisecond is

$$P(X = 6) = e^{-4}\frac{4^6}{6!} = 0.1042.$$

Binomial Distribution

Solution of Example 3.26:

- Let X be the number of particles enter the counter in a given millisecond. Then $X \sim P(4)$, so the probability that 6 particles enter the counter in a given millisecond is

$$P(X=6) = e^{-4}\frac{4^6}{6!} = 0.1042.$$

- Let Y be the number of particles enter the counter in three given milliseconds. Then $Y \sim P(12)$, so the probability that 10 particles enter the counter in three given milliseconds is

$$P(Y=10) = e^{-12}\frac{12^{10}}{10!} = 0.105$$

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Approximation of Binomial Distribution by a Poisson Distribution

Approximation of Binomial Distribution by a Poisson Distribution:

When n is vary large and p is very small, then the binomial distribution $B(n; p)$ can be approximated by the Poisson distribution $\approx P(np)$, it means that

$$P(X = k) = C_n^k p^k (1-p)^{n-k} \approx e^{-np}\frac{(np)^k}{k!}.$$

Approximation of Binomial Distribution by a Poisson Distribution

Example 3.27:

In a certain industrial facility, accidents occur infrequently. It is known that the probability of at least one accident on any given day is 0.005 and accidents are independent of each other.

- What is the probability that in any given period of 400 days there will be at least one accident on one day?
- What is the probability that there are at most three days with at least one accident?

Approximation of Binomial Distribution by a Poisson Distribution

Solution of Example 3.27:

Let X be the number of days having at least one accident out of 400 days observed. Then $X \sim B(400; 0.005)$, since $n = 400$ is large and $p = 0.005$ is small so $B(400; 0.005) \approx P(2)$.

Approximation of Binomial Distribution by a Poisson Distribution

Solution of Example 3.27:

Let X be the number of days having at least one accident out of 400 days observed. Then $X \sim B(400; 0.005)$, since $n = 400$ is large and $p = 0.005$ is small so $B(400; 0.005) \approx P(2)$.

- The probability that in any given period of 400 days there will be at least one accident on one day is
  $P(X = 1) \approx e^{-2}\frac{2^1}{1!} = 0.271$

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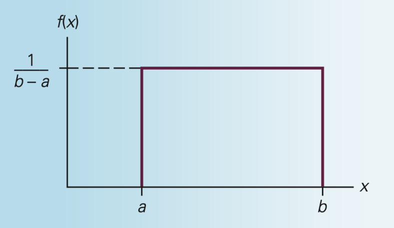
- The probability that in any given period of 400 days there will be at least one accident on one day is
  $P(X = 1) \approx e^{-2}\frac{2^1}{1!} = 0.271$
- The probability that there are at most three days with at least one accident is:
  $P(X \leq 3) \approx e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = 0.857.$

Continuous Uniform Distribution

Definition 3.15:

The density function of the continuous uniform random variable X on the interval $[a, b]$ is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a; b], \\ 0 & \text{otherwise.} \end{cases}$$



Continuous Uniform Distribution

Theorem 3.5:

The mean and variance of the continuous uniform distribution in $[a; b]$ are $\mu = \frac{a+b}{2}$ and $\sigma^2 = \frac{(b-a)^2}{12}$.

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Example 3.28:

The amount of gasoline sold daily at a service station is uniformly distributed with a minimum of 2000 gallons and a maximum of 5000 gallons.

- What is the probability density function?
- Find the probability that daily sales will fall between 2500 and 3000 gallons.
- What are the average amount of gasoline sold daily and the variance of the amount of gasoline sold daily?

Continuous Uniform Distribution

Solution of Example 3.28:

Let X be the amount of gasoline sold daily. Then $X \sim U([a; b])$.

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Let X be the amount of gasoline sold daily. Then $X \sim U([a;b])$.

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$$f(x) = \begin{cases} \frac{1}{5000-2000} = \frac{1}{3000} \text{ if } x \in [2000; 5000], \\ 0 \quad \text{otherwise.} \end{cases}$$

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  $P(2500 \le X \le 3000) = (3000 - 2500)\frac{1}{3000} = \frac{1}{6}$.

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$$f(x) = \begin{cases} \frac{1}{5000-2000} = \frac{1}{3000} \text{ if } x \in [2000; 5000], \\ 0 \ \ \text{otherwise.} \end{cases}$$

- The probability that daily sales will fall between 2500 and 3000 gallons is:
  $P(2500 \leq X \leq 3000) = (3000 - 2500)\frac{1}{3000} = \frac{1}{6}$.
- The average amount of gasoline sold daily is $\mu = \frac{2000+5000}{2} = 3500$ and the variance of the amount of gasoline sold daily is $\sigma^2 = \frac{(5000-2000)^2}{12} = 750000$.

Normal Distribution

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- Many random variables can be properly modeled as normally distributed.
- Many distributions can be approximated by a normal distribution.
- Normal distribution is the cornerstone distribution of statistical inference.

Normal Distribution

Definition 3.16:

X is called a normal random variable if the density function of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in R.$$

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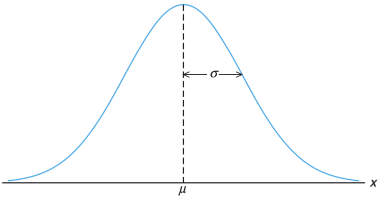
X is called a normal random variable if the density function of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in R.$$

We denote by $X \sim N(\mu, \sigma^2)$. The density function of X has a bell shape.



Normal Distribution

The normal distribution is fully defined by two parameters μ and σ^2, where:

- μ is the mean, median and mode of X:
  $\mu = E(X) = \text{med}(X) = \text{mode}(X)$.

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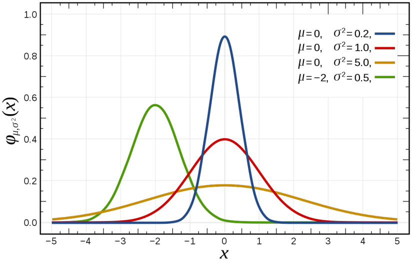
- σ^2 is the variance of X: $\sigma^2 = V(X)$.
- σ is the standard deviation of X: $\sigma = \sigma(X)$.



Standard Normal Distribution

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The distribution of a normal random variable Z with mean 0 and variance 1 is called a standard normal distribution $N(0, 1)$.

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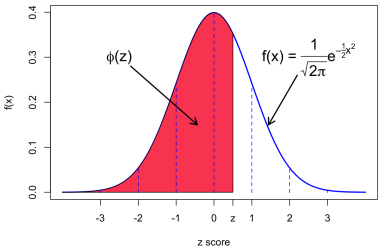
- The pdf of Z is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, x \in R$.
- The cdf of Z is $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, z \in R$.

The probability density function of the standard normal distribution



Standard Normal Distribution

Remark:

- The values of function $\Phi(x)$ is given in the Table A.3.
- $\Phi(-z) = 1 - \Phi(z)$.

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

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Example:

- $\Phi(0.68) = 0.7517$ and $\Phi(-0.68) = 1 - \Phi(0.68) = 0.2483$.

Standardization Rule

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Let $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$.

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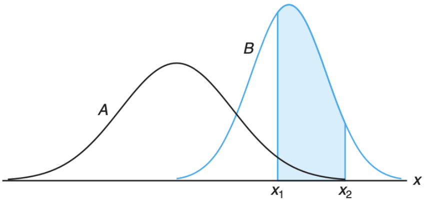
- $P(x_1 < X < x_2) = P(X < x_2) - P(X < x_1) = \Phi(\frac{x_2-\mu}{\sigma}) - \Phi(\frac{x_1-\mu}{\sigma})$.



Normal Distribution

Example 3.29:

A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that

- a given battery will last less than 2.3 years.
- a given battery will last between 2.5 and 3.5 years.

Normal Distribution

Solution of Example 3.29:

Let X be the battery life, then $X \sim N(3, 0.5^2)$. By the standardization rule, $Z = \frac{X-3}{0.5} \sim N(0, 1)$.

Normal Distribution

Solution of Example 3.29:

Let X be the battery life, then $X \sim N(3, 0.5^2)$. By the standardization rule, $Z = \frac{X-3}{0.5} \sim N(0, 1)$.

- The probability that a given battery will last less than 2.3 years is

$$P(X < 2.3) = P(Z < \frac{2.3 - 3}{0.5}) = P(Z < -1.4) = \Phi(-1.4) = 0.081$$

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- The probability that a given battery will last between 2.5 and 3.5 years is

$$P(2.5 < X < 3.5) = P(\frac{2.5-3}{0.5} < Z < \frac{3.5-3}{0.5})$$

$$= P(-1 < Z < 1) = 2\Phi(1) - 1 = 2 * 0.8413 - 1 = 0.6826$$

Normal Distribution

Example 3.30:

The average grade for an exam is 7.4, and the standard deviation is 0.7. If 12% of the class is given As, and the grades are curved to follow a normal distribution, what is the lowest possible A?

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- Let x be the lowest possible A. Then

$$P(X \geq x) = 0.12 \Leftrightarrow 1 - P(Z < \frac{x - 7.4}{0.7}) = 0.12$$

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- Let x be the lowest possible A. Then

$$P(X \geq x) = 0.12 \Leftrightarrow 1 - P(Z < \frac{x-7.4}{0.7}) = 0.12$$

$$\Leftrightarrow \Phi(\frac{x-7.4}{0.7}) = 0.88 = \Phi(1.18) \Leftrightarrow x = 7.4 + 0.7 * 1.18 = 8.2$$

Normal Approximation for the Binomial Distribution

Let X follow a Binomial distribution $B(n, p)$, where p is closed to 0.5 or where n is large and p is far from 0 and 1. Then X approximately follows a normal distribution $N(np; npq)$.

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- $P(k_1 \leq X \leq k_2) = P(k_1 - 0.5 < X < k + 0.5) \approx \Phi\left(\frac{k_2+0.5-np}{\sqrt{npq}}\right) - \Phi\left(\frac{k_1-0.5-np}{\sqrt{npq}}\right)$.

Normal Approximation for the Binomial Distribution

Example 3.31:

The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that

- fewer than 30 survive?
- between 35 and 45 survive?

Normal Approximation for the Binomial Distribution

Solution of Example 3.31:

Let X present the number of patients who recover from the rare blood disease out of 100 patients observed. Then $X \sim B(100; 0.4) \approx N(40; 24)$.

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- The probability that fewer than 30 survive is:

$$P(X < 30) = P(X < 29.5) \approx \Phi\left(\frac{29.5 - 40}{\sqrt{24}}\right) = \Phi(-2.14) = 0.016$$

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$$P(35 \leq X \leq 45) = P(34.5 < X < 45.5)$$

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$$\approx \Phi(\frac{45.5 - 40}{\sqrt{24}}) - \Phi(\frac{34.5 - 40}{\sqrt{24}})$$

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Exponential Distribution

Definition 3.18:

The random variable X follows an exponential distribution if the density function of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} \text{ if } x > 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

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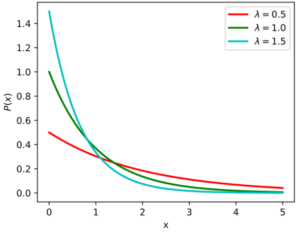
The random variable X follows an exponential distribution if the density function of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $X \sim E(\lambda)$.



Exponential Distribution

Theorem 3.6:

Let $X \sim E(\lambda)$. Then

- $E(X) = \frac{1}{\lambda}$ and $V(X) = \frac{1}{\lambda^2}$.

Exponential Distribution

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Let $X \sim E(\lambda)$. Then

- $E(X) = \frac{1}{\lambda}$ and $V(X) = \frac{1}{\lambda^2}$.
- The cdf of X is

$$F(x) = \begin{cases} 1 - e^{-\lambda x} \text{ if } x > 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

Exponential Distribution

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$$F(x) = \begin{cases} 1 - e^{-\lambda x} \text{ if } x > 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

- The Memoryless Property: $P(X > t_0 + t | X > t_0) = P(X > t)$.

Exponential Distribution

Example 3.31:

Suppose that a system contains a certain type of component whose time, in years, to failure is given by T . The random variable T is modeled nicely by the exponential distribution with $\lambda = 0.2$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Exponential Distribution

Solution of Example 3.30:

- The probability that a given component is still functioning after 8 years is given by

$$P(T > 8) = 1 - F(8) = 1 - (1 - e^{-0.2*8}) = e^{-1.6} = 0.2$$

Exponential Distribution

Solution of Example 3.30:

- The probability that a given component is still functioning after 8 years is given by

$$P(T > 8) = 1 - F(8) = 1 - (1 - e^{-0.2*8}) = e^{-1.6} = 0.2$$

- Let X represent the number of components functioning after 8 years. We have $X \sim B(5; 0.2)$, then

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - C_5^0 0.2^0 0.8^5 - C_5^1 0.2^1 0.8^4 = 0.2627$$

Student's t- Distribution

Definition 3.19:

Student's t-distribution has the probability density function (pdf) given by

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}, x \in R,$$

where ν the number of degrees of freedom and Γ is the gamma function. When ν is large, the t-distribution T_ν is closed to $N(0;1)$.