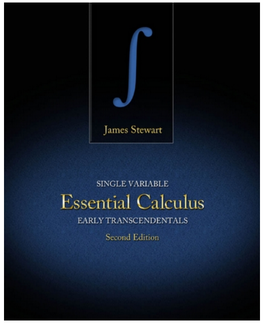
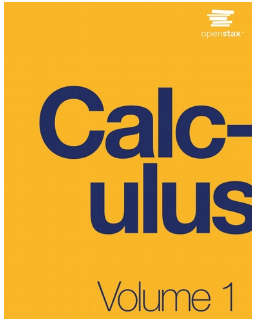
Chapter 4: Applications of Differentiation





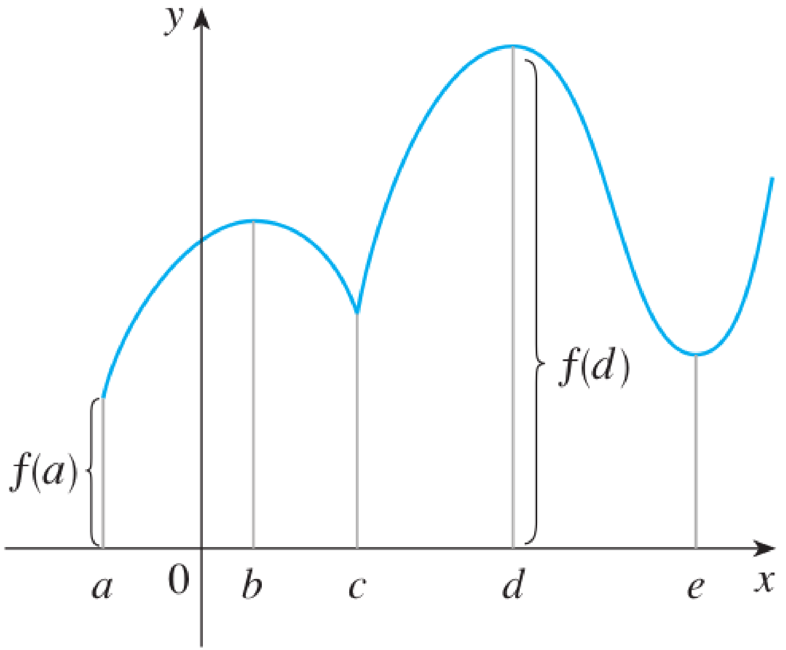
The pictures are taken from the books:

[1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280
2) G. Strang and E. J. Herman, Calculus 1, https://openstax.org/details/books/calculus-volume-1]

4.1 Minimum and Maximum Values

Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

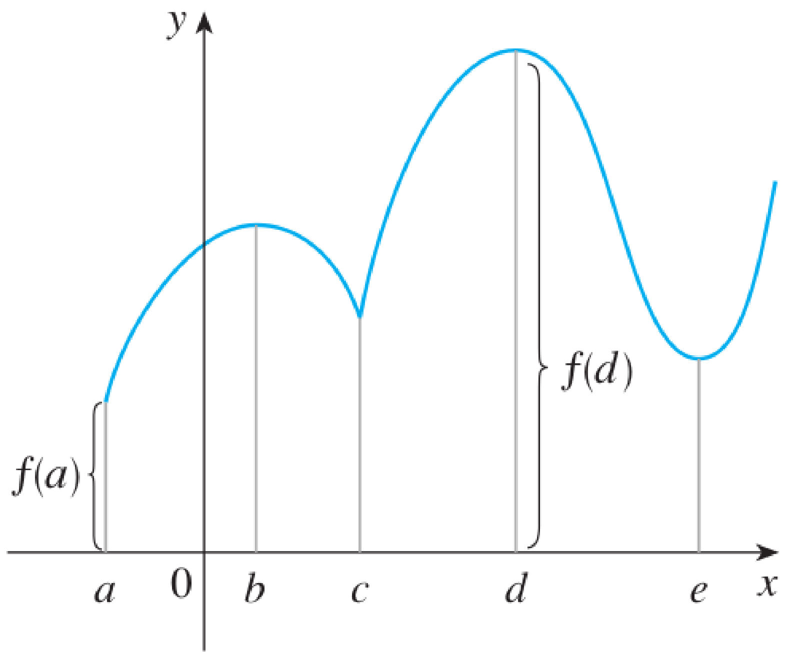
- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all $x \in D$.
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all $x \in D$.



4.1 Minimum and Maximum Values

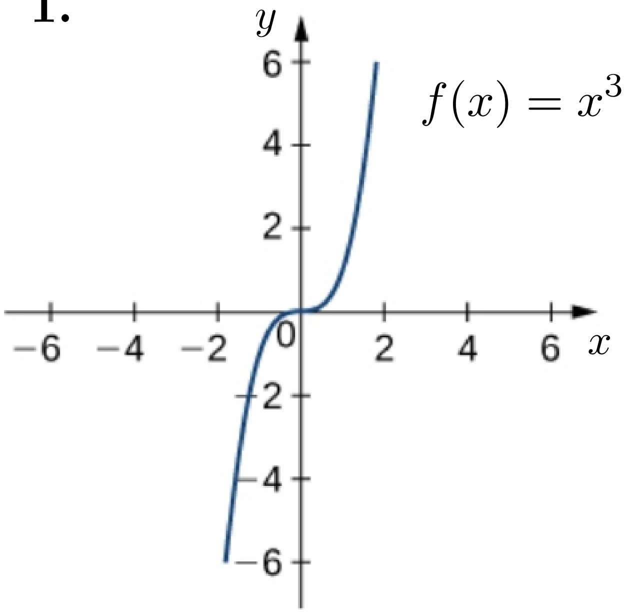
Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **local maximum** value of f on D if $f(c) \geq f(x)$ when x is near c .
- **local minimum** value of f on D if $f(c) \leq f(x)$ when x is near c.

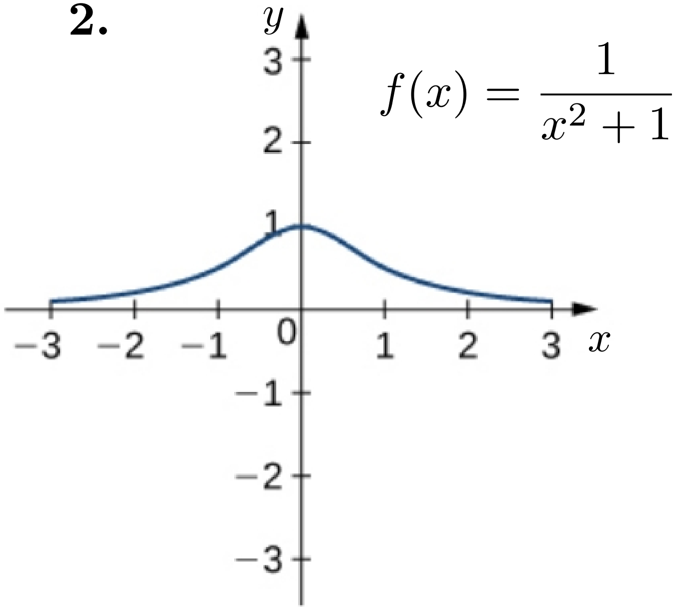


4.1 Examples

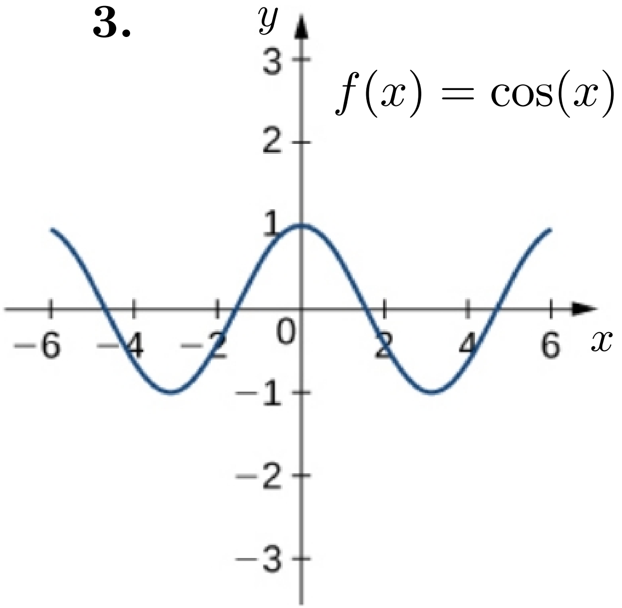
1.



2.

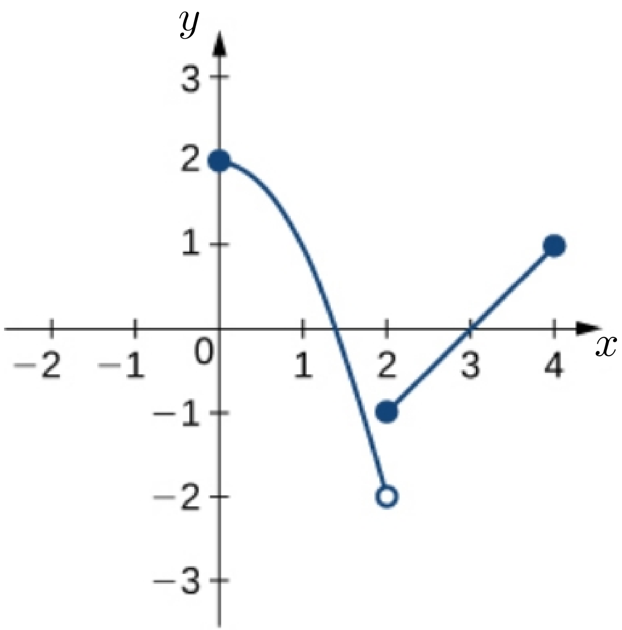


3.



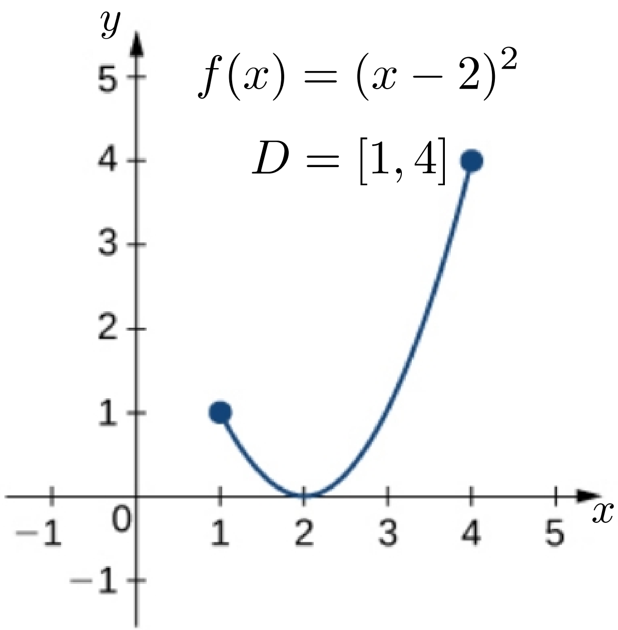
4.1 Examples

4.

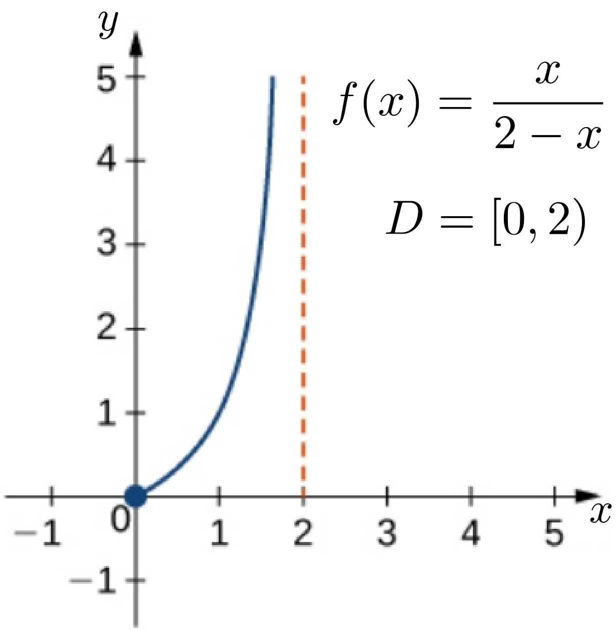


$$f(x) = \begin{cases} 2 - x^2\,, & 0 \le x < 2 \\ x - 3\,, & 2 \le x \le 4 \end{cases}$$

5.

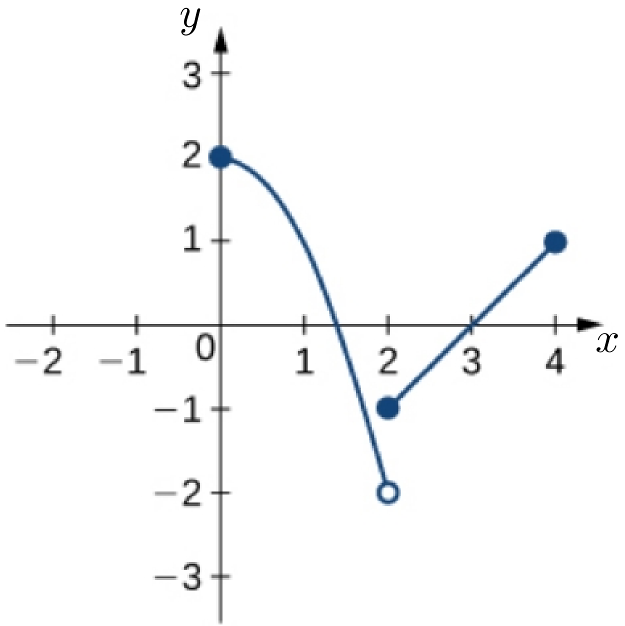


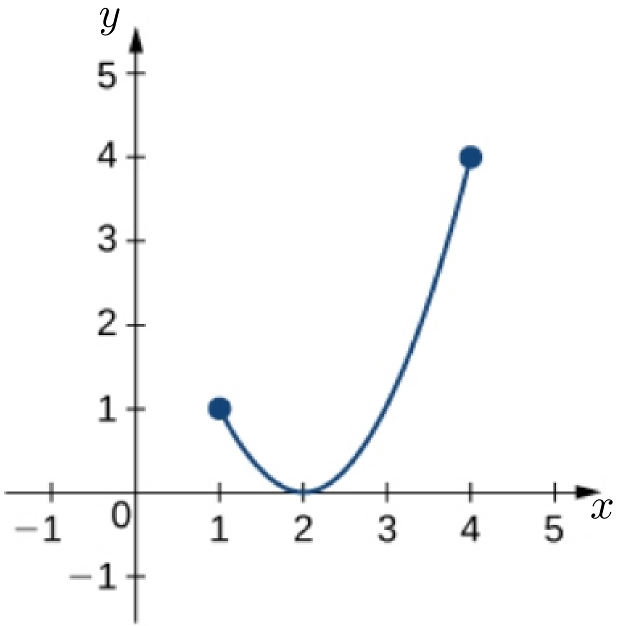
6.

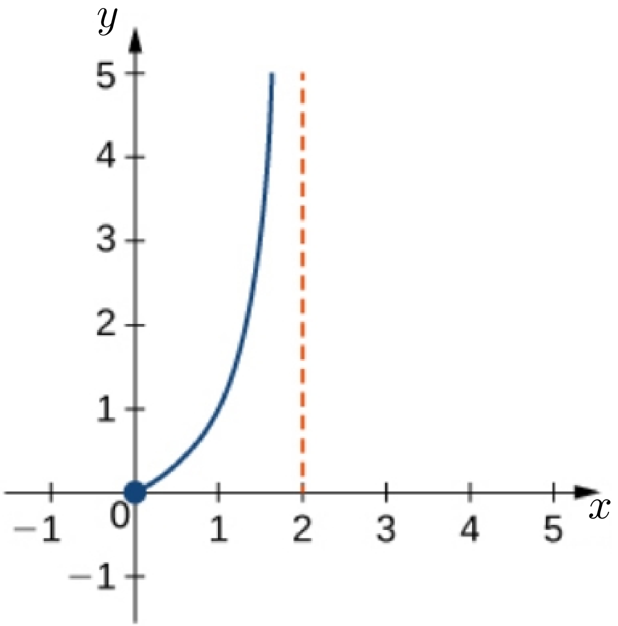


4.1 The Extreme Value Theorem

• If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

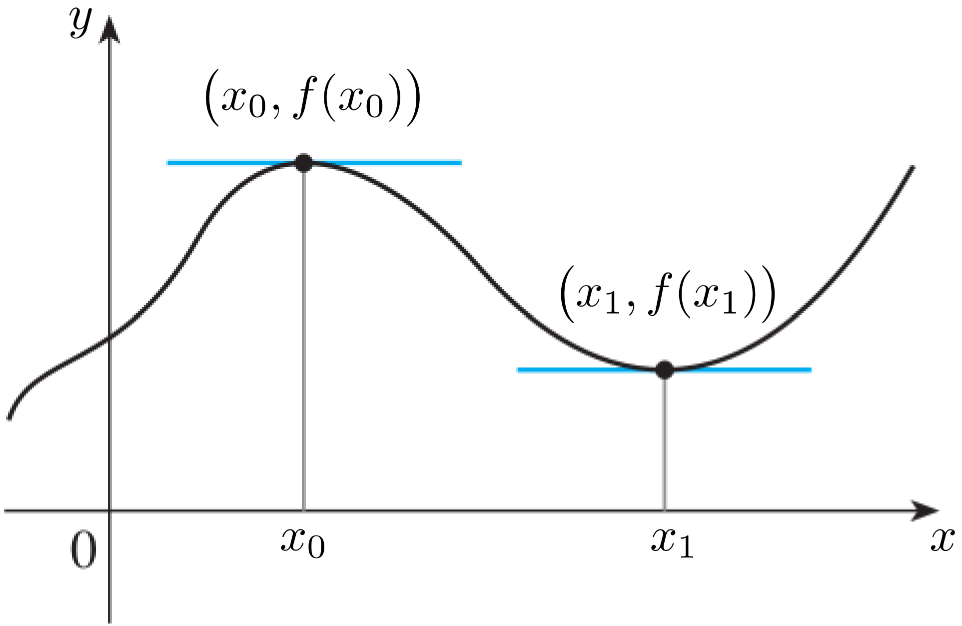






4.1 Fermat's Theorem

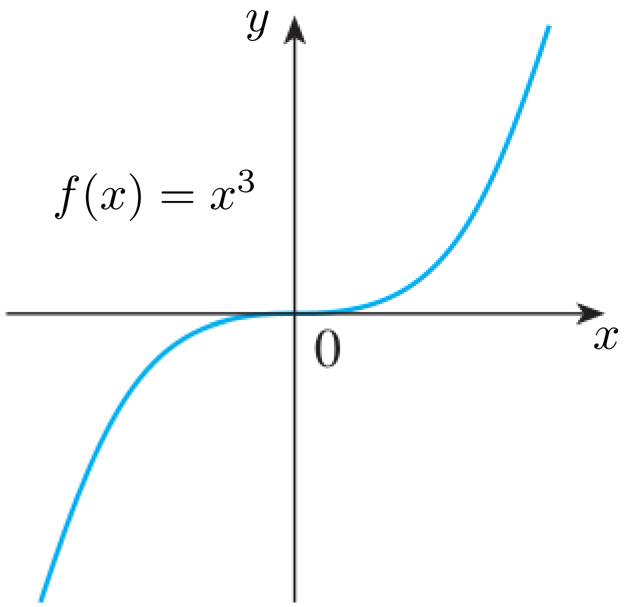
• **If** f has a local maximum or minimum at x_0, and **if** $f'(x_0)$ exists, **then** $f'(x_0) = 0$.

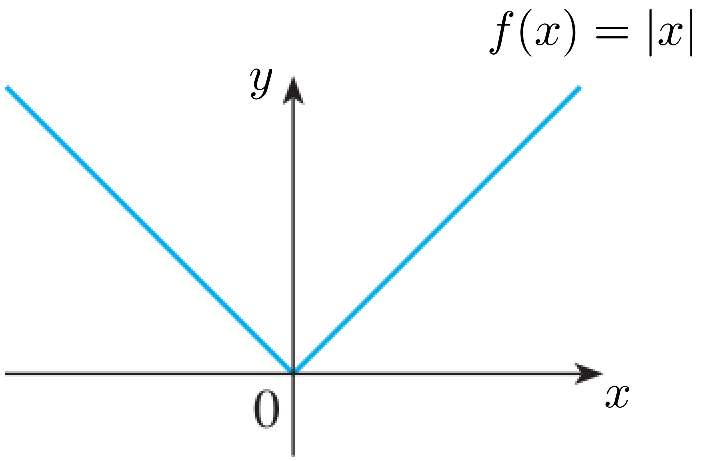


4.1 Fermat's Theorem

• **If** f has a local maximum or minimum at x_0, and **if** $f'(x_0)$ exists, **then** $f'(x_0) = 0$.

Caution The converse of Fermat's Theorem is false in general



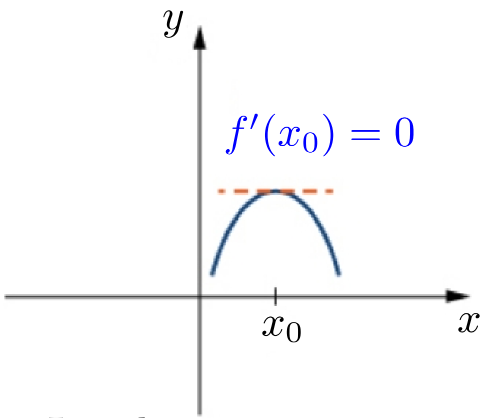


4.1 Critical Points

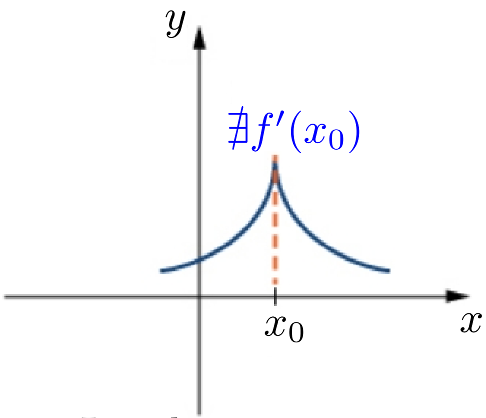
• A **critical point** of a function f is a number x_0 in the domain of f such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

If f has a local maximum or minimum at x_0, then x_0 is a critical point of f.

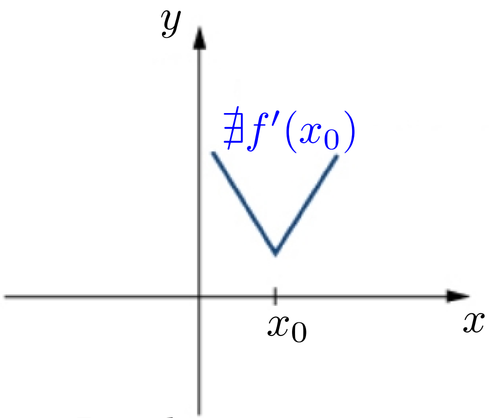
4.1 Critical Points



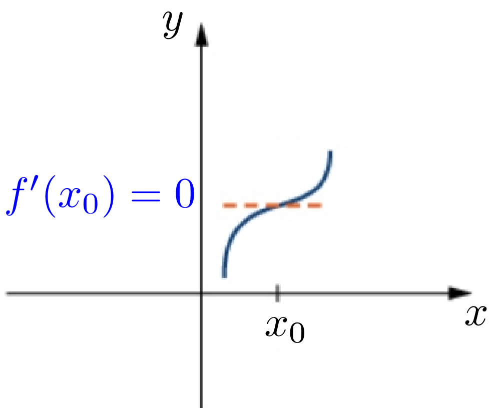
Local maximum at x_0



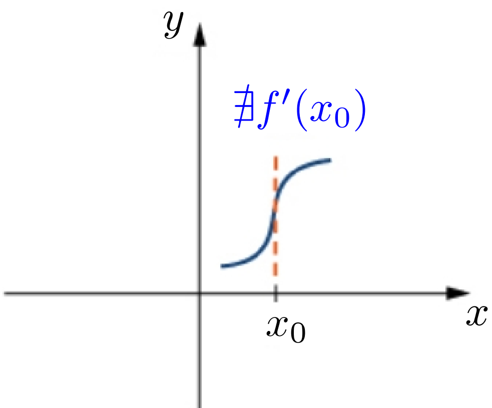
Local maximum at x_0



Local minimum at x_0



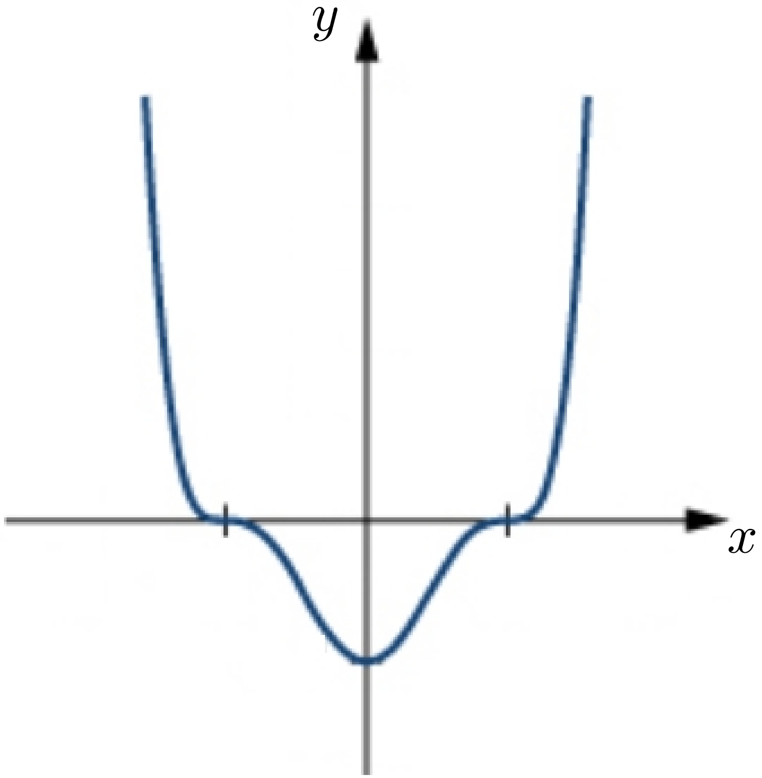
No local extremum at x_0



No local extremum at x_0

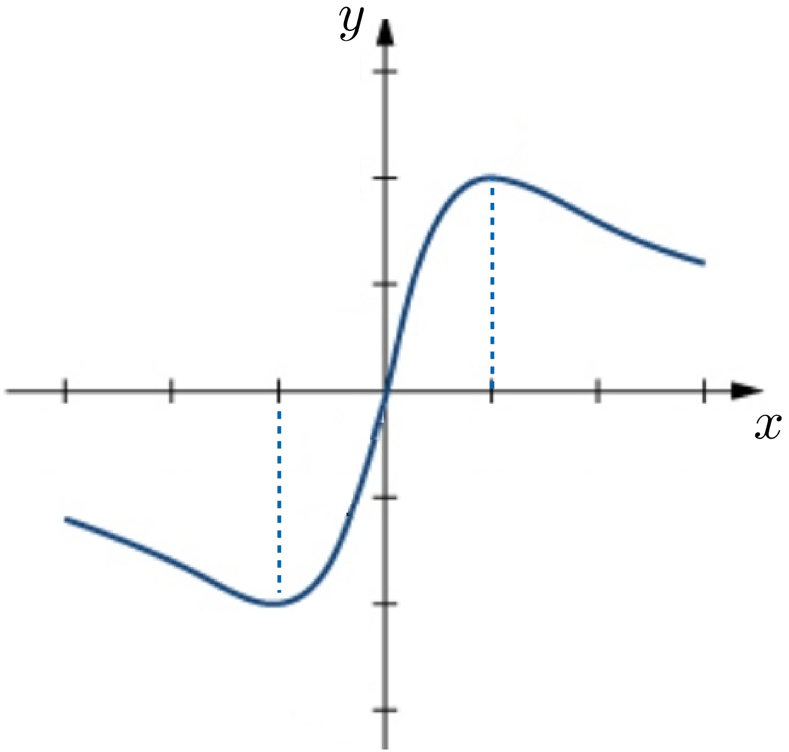
4.1 Examples

- Find all critical points for the function $f(x) = (x^2 - 1)^3$. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.



4.1 Examples

• Find all critical points for the function $f(x) = \dfrac{4x}{1+x^2}$. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.



Locating Extrema: The Big Picture

$f'(x) = 0$

$\{x_1, x_2, \ldots\}$

Critical points

Minima / Maxima

Inflection points

$\nexists f'(x)$

Minima/Maxima: Corner

Discontinuity

Vertical Tangent

4.1 Methodology: Locating Absolute Extrema in $[a, b]$

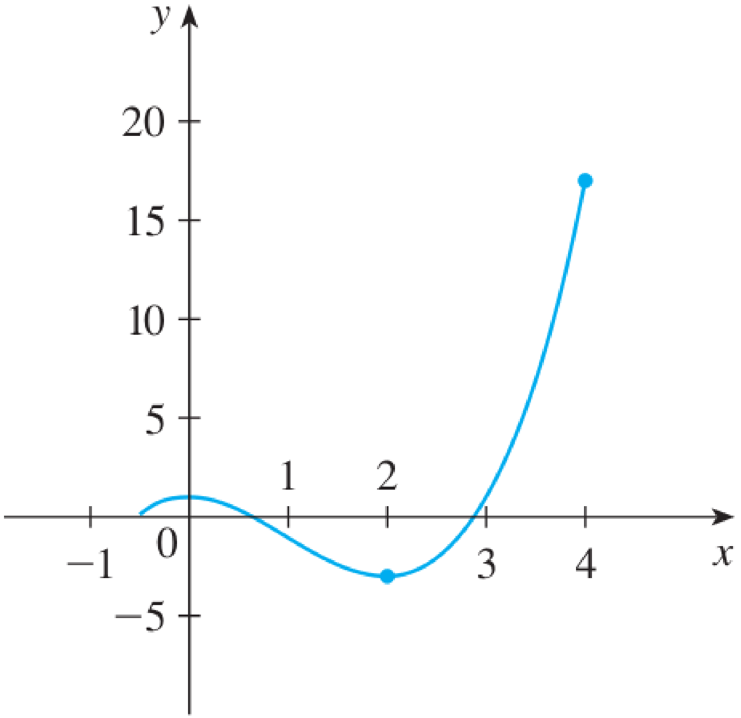
- Consider a continuous function f defined over the closed interval $[a, b]$.

1. Find the critical point of f in (a, b): $f'(x) = 0$.
2. Evaluate f at the end points of the interval, $x = a$ and $x = b$.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

4.1 Example

- Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1\,, \qquad -\frac{1}{2} \le x \le 4$$



4.2 Rolle's Theorem

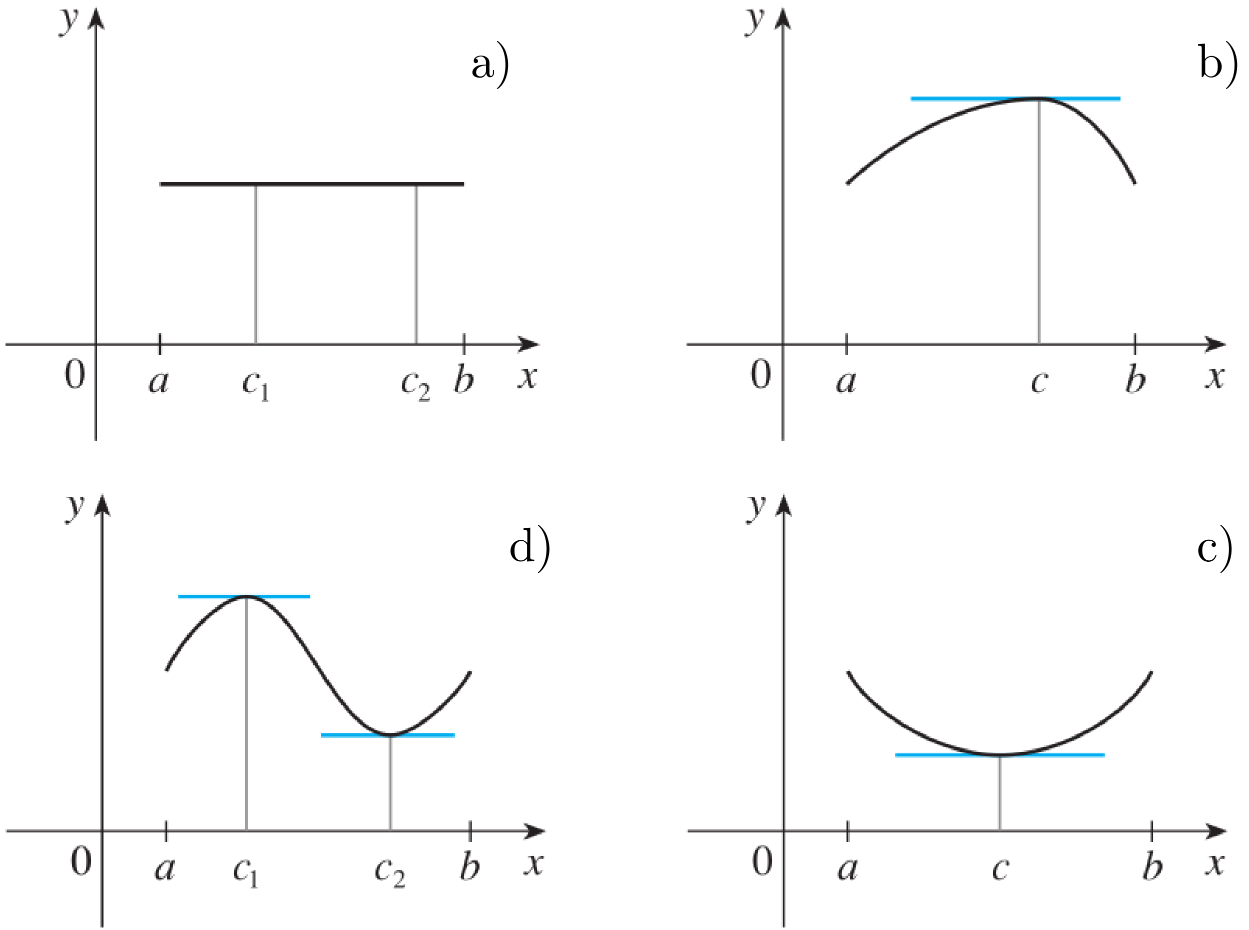
- Let f be a function that satisfies the following three hypotheses:

  1. f is continuous on the closed interval $[a, b]$.
  2. f is differentiable on the open interval (a, b).
  3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

4.2 Rolle's Theorem

$f : [a, b] \to R$
- Continuous
- Differentiable
- $f(a) = f(b)$



4.2 The Mean Value Theorem

- Let f be a function that satisfies the following hypotheses:

  1. f is continuous on the closed interval $[a, b]$.

  2. f is differentiable on the open interval (a, b).

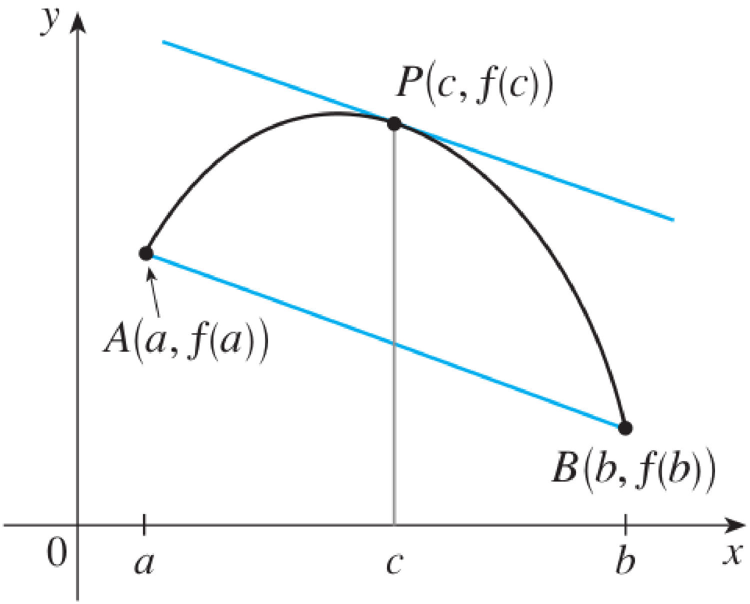
Then there is a number c in (a, b) such that

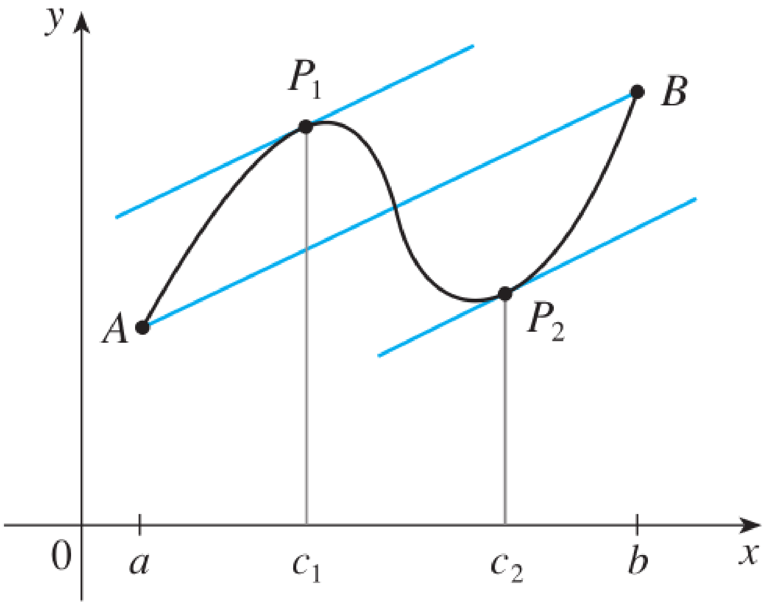
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

4.2 The Mean Value Theorem

$f : [a, b] \to R$

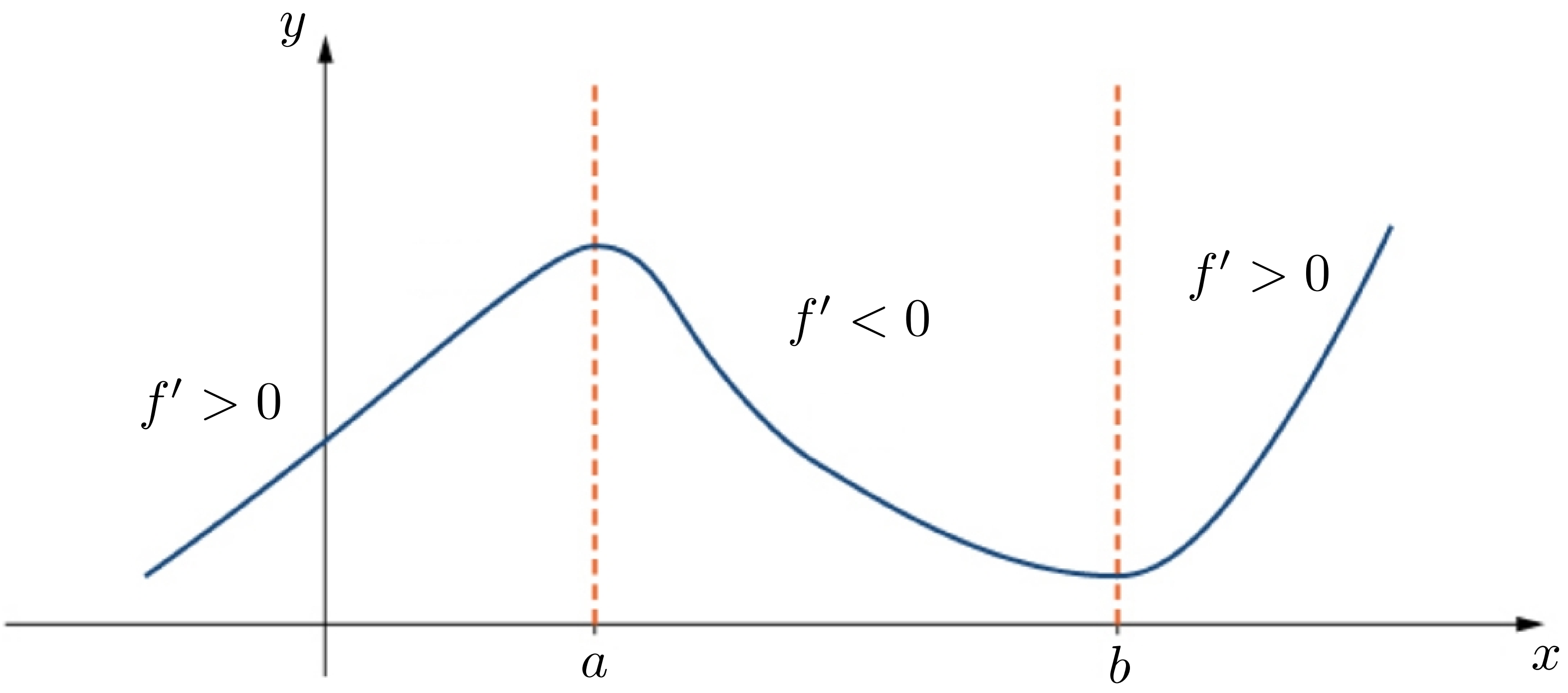
- Continuous
- Differentiable





4.3 Derivatives and the Shape of Graphs

- What does f' say about f?



4.3 Derivatives and the Shape of Graphs

• What does f' say about f?

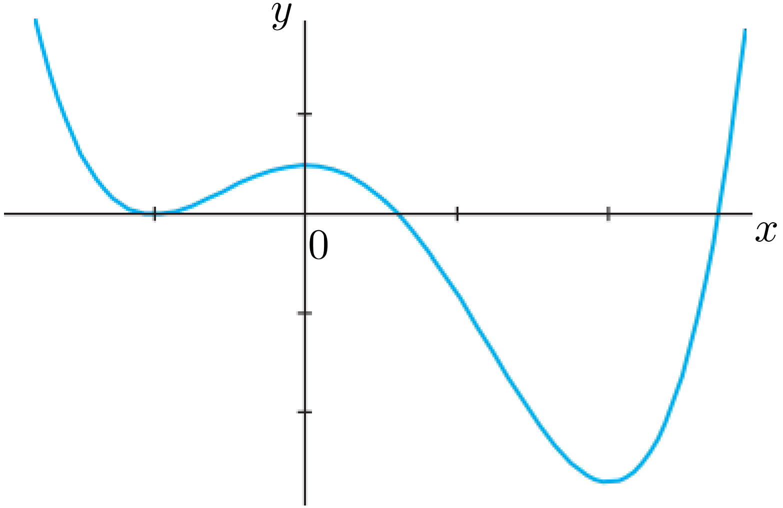
• Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.

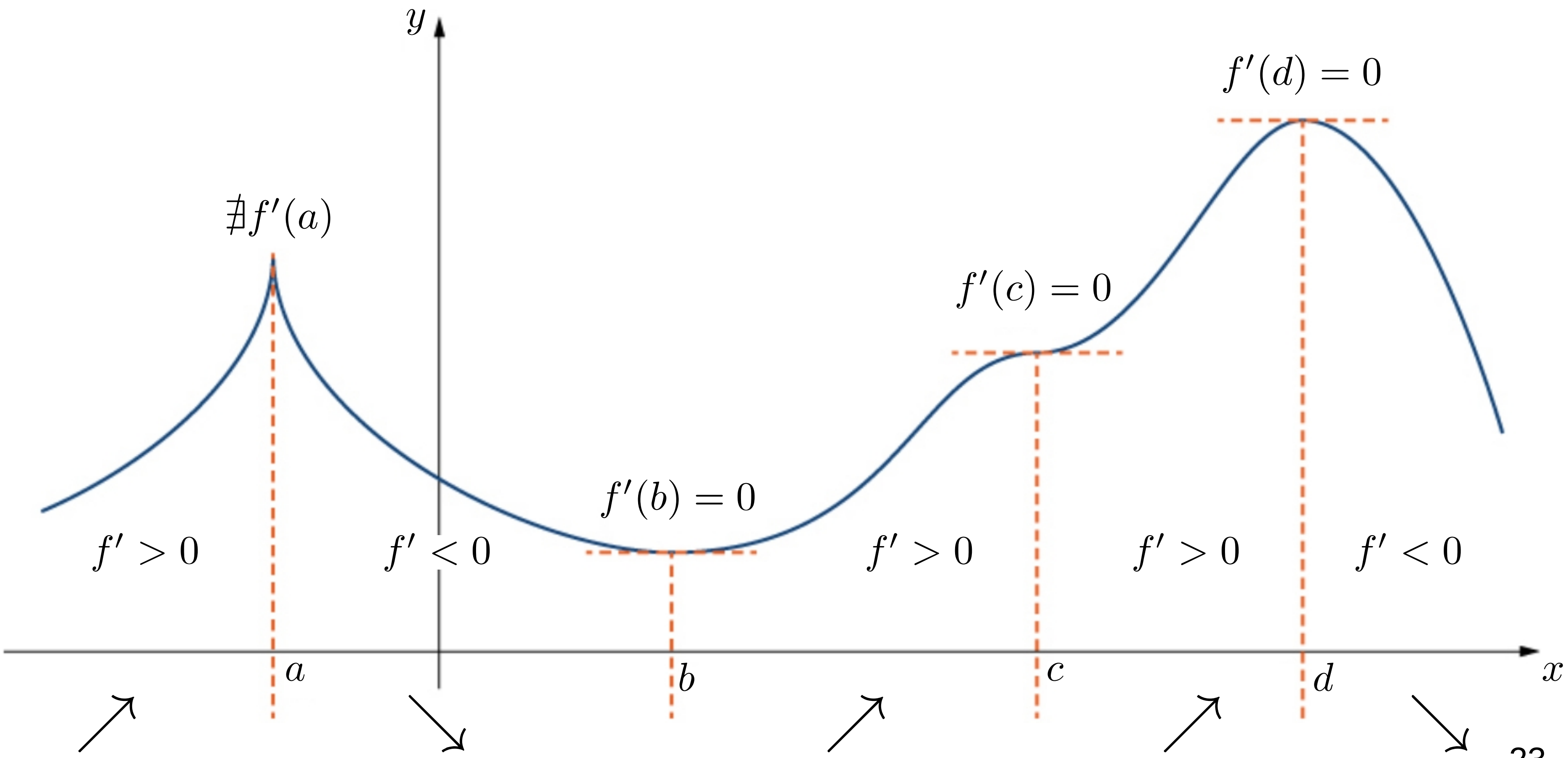
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

4.3 Example

- Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.



4.3 Observation on f' sign



4.3 The First Derivative Test

- Suppose that c is a critical number of a continuous function f.

  (a) If f' changes from **positive to negative** at c, then f has a **local maximum** at c.

  (b) If f' changes from **negative to positive** at c, then f has a **local minimum** at c.

  (c) If f' **does not change sign** at c, then f has **no local extremum** at c.

Locating Extrema: The Big Picture

$f'(x) = 0$

$\{x_1, x_2, \ldots\}$
Critical points

$\boxed{\text{Minima / Maxima} \qquad f'(\{x_1, \ldots, x_n\} \pm \delta) \gtrless 0}$

Inflection points $\qquad f'(\{x_1, \ldots, x_n\} \pm \delta) > 0$ or < 0

$\nexists f'(x)$

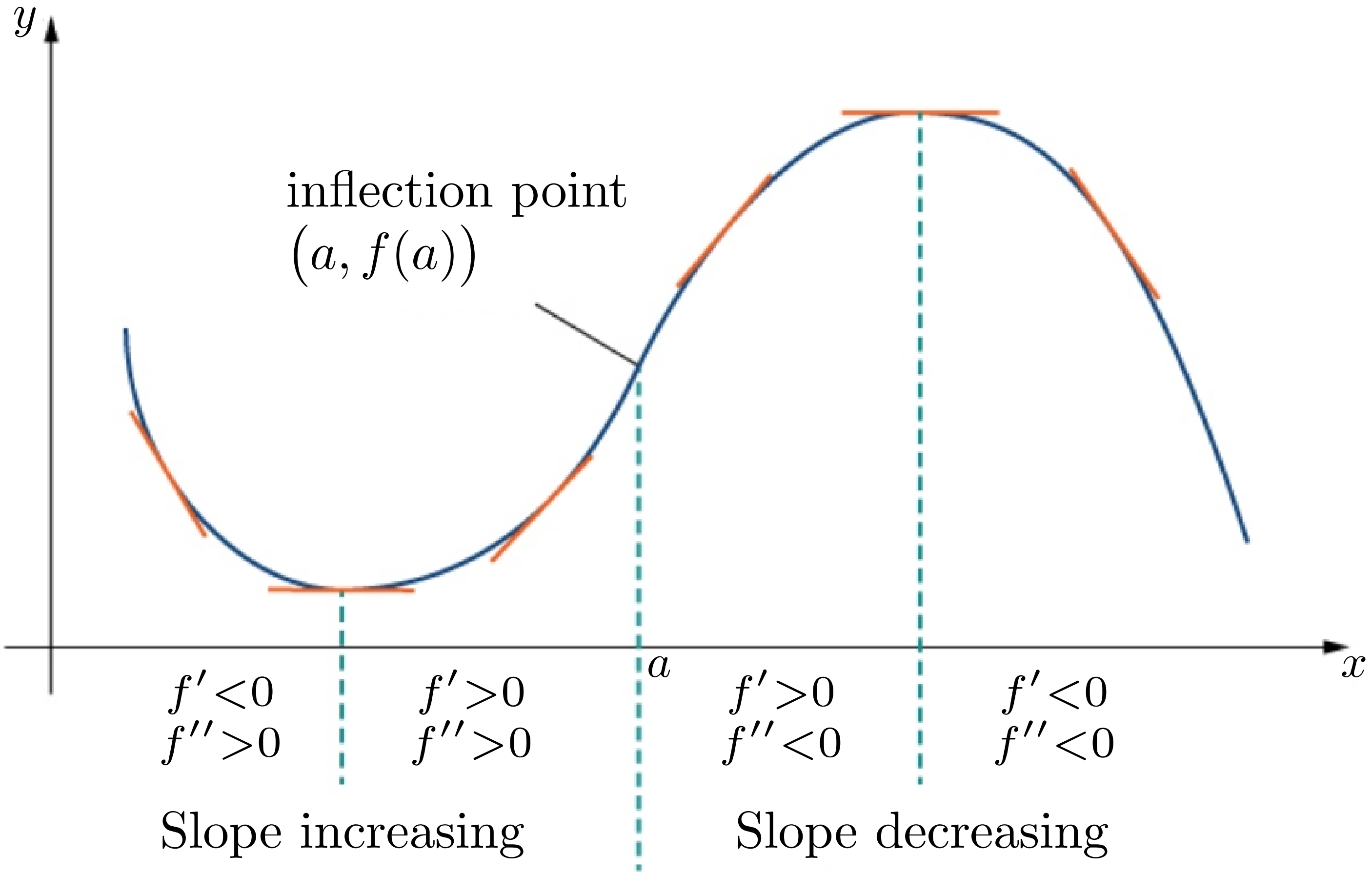
$\boxed{\text{Minima/Maxima: Corner}}$

Discontinuity

Vertical Tangent

4.3 Derivatives and the Shape of Graphs

- What does f'' say about f?



4.3 Derivatives and the Shape of Graphs

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Concavity test

(a) If $f''(x) > 0$ for all $x \in I$, then the graph of f is **concave upward** on I.

(b) If $f''(x) < 0$ for all $x \in I$, then the graph of f is **concave downward** on I.

Locating Extrema: The Big Picture

$f'(x) = 0$

$\{x_1, x_2, \ldots\}$

Critical points

Minima / Maxima $\quad f'(\{x_1, \ldots, x_n\} \pm \delta) \gtrless 0$

$f''(\{x_1, \ldots, x_n\}) \gtrless 0$

Inflection points $\quad f'(\{x_1, \ldots, x_n\} \pm \delta) > 0$ or < 0

$f''(\{x_1, \ldots, x_n\} \pm \delta) \gtrless 0$ or $\lessgtr 0$

$\nexists f'(x)$

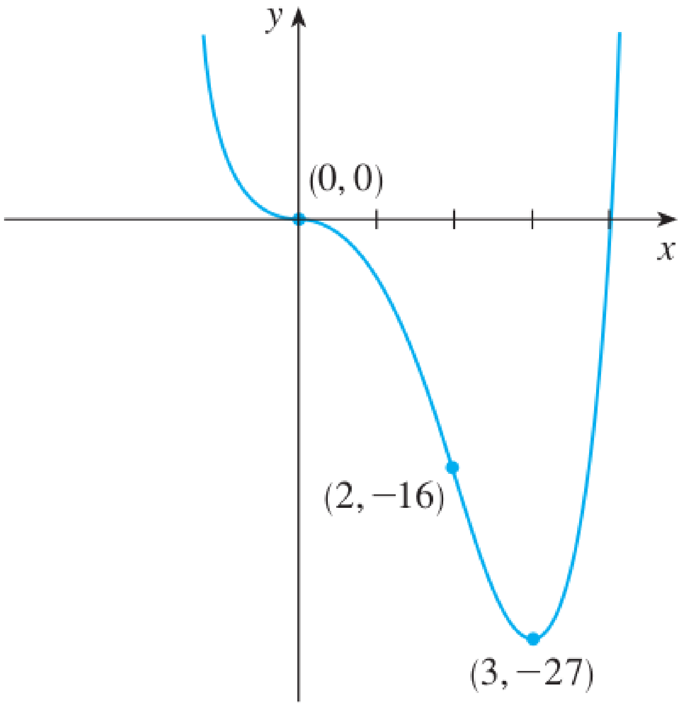
Minima/Maxima: Corner

Discontinuity

Vertical Tangent

4.3 Examples

• Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.



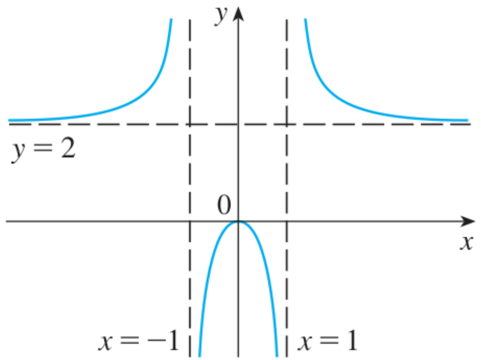
4.4 Curve Sketching

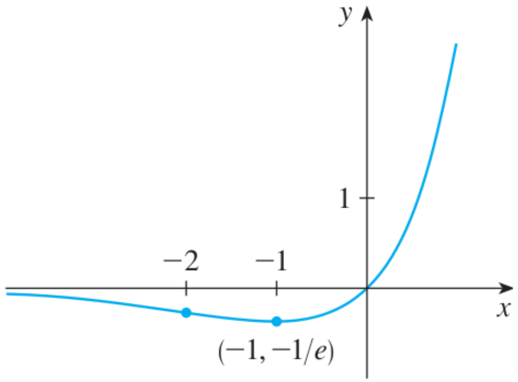
A. Domain
B. Intercepts
C. Symmetry
D. Asymptotes
E. Intervals of Increase or Decrease
F. Local Minima and Maxima
G. Concavity and Points of Inflection
H. Sketch the Curve

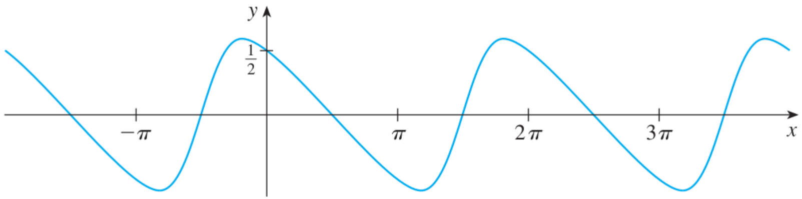
4.4 Example

- Use the guidelines to sketch the curve

1. $f(x) = \dfrac{2x^2}{x^2 - 1}$, **2.** $g(x) = x\,e^x$, **3.** $h(x) = \dfrac{\cos(x)}{2 + \sin(x)}$,

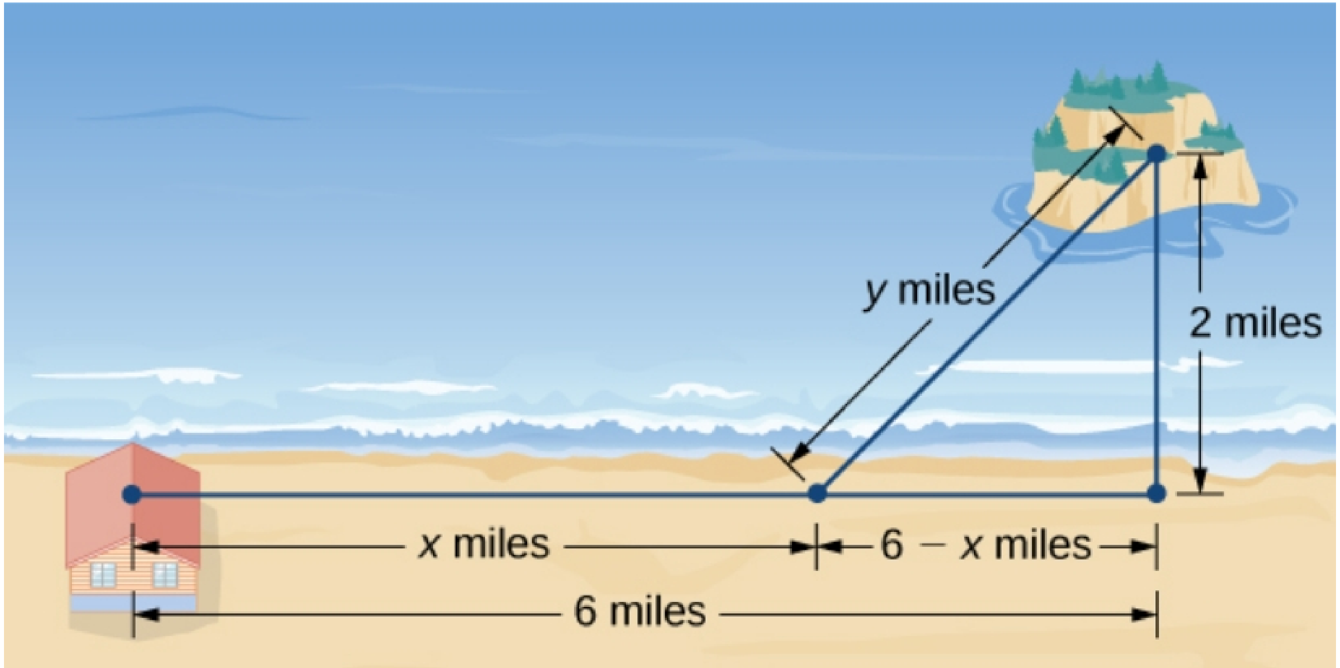






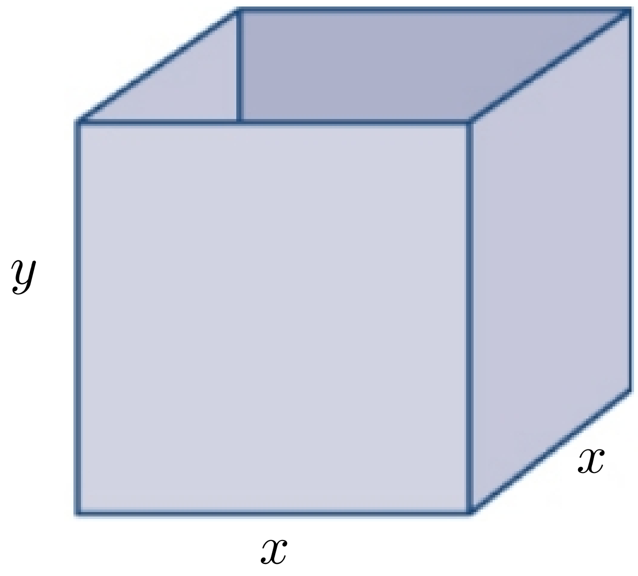
4.5 Optimization Problems

Minimizing Travel Time The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8mph and swims at a rate of 3mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

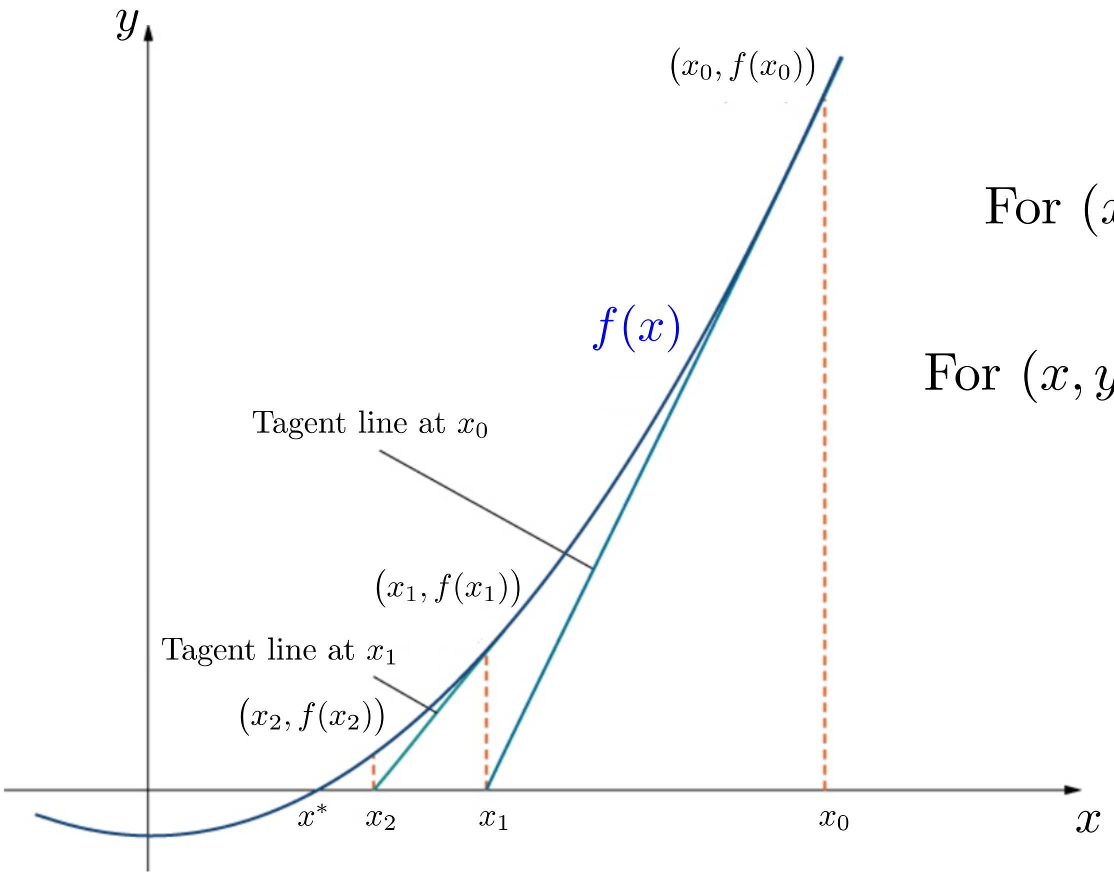


4.5 Optimization Problems

Minimizing Surface Area A rectangular box with a square base, an open top, and a volume of 216in.^{3} is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?



4.6 Newton's Approximation Method



Linear Approximation at x_0:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

For $(x, y) = (x_1, 0)$: $\quad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

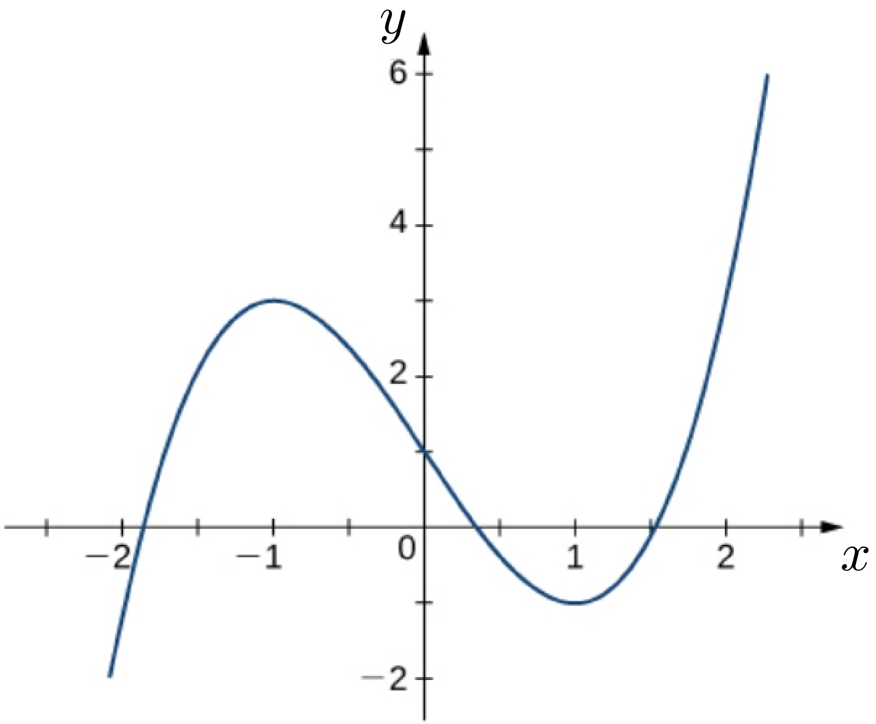
$$\vdots$$

For $(x, y) = (x_{n-1}, 0)$: $\quad x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$

$$\boxed{x^* = \lim_{n \to \infty} x_n}$$

4.6 Example

• Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find x_1, x_2, x_3, x_4, and x_5.

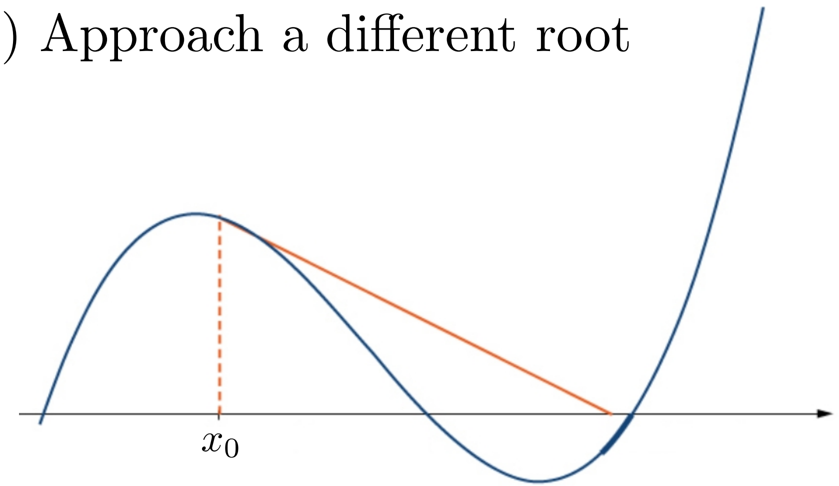


$$
\begin{aligned}
x_0 &= 2 \\
x_1 &\approx 1.666666667 \\
x_2 &\approx 1.548611111 \\
x_3 &\approx 1.532390162 \\
x_4 &\approx 1.532088989 \\
x_5 &\approx 1.532088886 \\
x_6 &\approx 1.532088886
\end{aligned}
$$

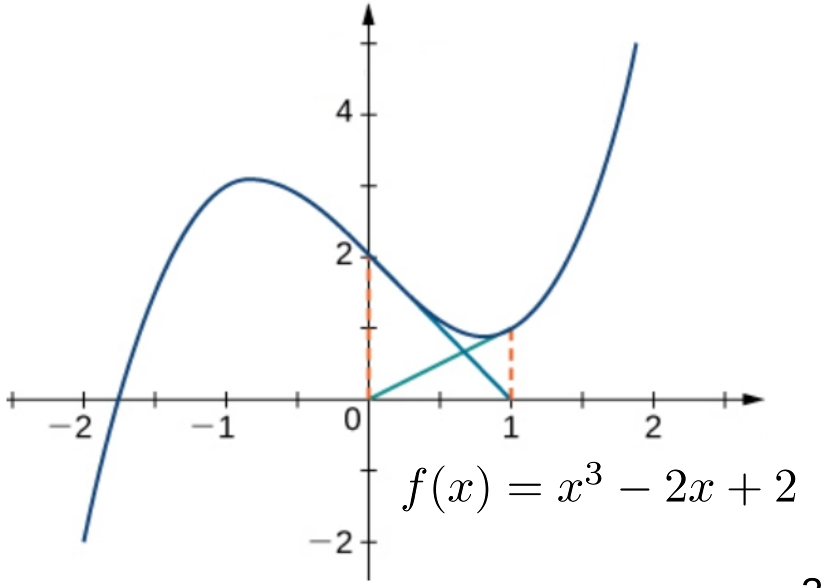
4.6 Failures of Newton's Method

(1) $f'(x_0) = 0$

(2) Approach a different root



(3) Approximations alternate back and forth



4.7 Antiderivatives

Definition A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Theorem If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C\,,$$

where C is an arbitrary constant.

4.7 Antiderivatives

Theorem Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\mathrm{d}x\,,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\mathrm{d}x = F(x) + C\,.$$

The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

4.7 Example

- Find the family of antiderivatives of $2x$.

$$\int 2x\mathrm{d}x = x^2 + C$$