Chapter 3: Vector spaces

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Contents

3.1.1. Definitions, examples

Let K be the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Definition

Let V be a nonempty set, together with two operations:

- Vector addition:
$$\begin{aligned} +\colon V \times V &\to V \\ (u, v) &\mapsto u + v \end{aligned}$$
- Scalar multiplication:
$$\begin{aligned} \cdot\colon K \times V &\to V \\ (a, v) &\mapsto av \end{aligned}$$

The set V together with these two operations is called a *vector space* over K, or a K-*vector space* if the following conditions (axioms) are satisfied.

For every $u, v, w \in V$, and $a, b \in K$:

1. $(u+v)+w = u+(v+w)$,
2. $\exists \mathbf{0} \in V$: $v+\mathbf{0} = \mathbf{0}+v = v$,
3. $\forall v \in V$, $\exists v' \in V$: $v+v' = v'+v = \mathbf{0}$,
4. $u+v = v+u$,
5. $(a+b)v = av+bv$,
6. $a(u+v) = au+av$,
7. $a(bv) = (ab)v$,
8. $1v = v$.

- An element of a vector space are called a vector. A element in K is called a scalar.
- Conditions 1-4 say that V is a commutative group .
- The element $\mathbf{0}$ in Condition 2 is called the zero vector.
- Element v' in Condition 3 is called the opposite vector of v, denoted by $-v$.

Example

- Consider $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$.
- Consider the following operations:

$$\begin{aligned}(x_1, \ldots, x_n) + (y_1, \ldots, y_n) &= (x_1 + y_1, \ldots, x_n + y_n),\\ a(x_1, \ldots, x_n) &= (ax_1, \ldots, ax_n).\end{aligned}$$

- The set $\mathbb{R}^n$ together with these operations is a vector space over $\mathbb{R}$.
- The zero vector is $\mathbf{0} = (0, 0, \ldots, 0)$.
- The opposite vector of $v = (x_1, \ldots, x_n)$ is $-v = (-x_1, \ldots, -x_n)$.

Example

- The set $\mathcal{M}_{2,2}(\mathbb{R})$ of real matrices of size 2×2, with the operations of matrix addition and scalar multiplication is a vector space (over $\mathbb{R}$).
  - Commutativity, associativity, distribution properties,.... are clear;
  - The zero vector is the zero matrix $\mathcal{O}_2$;
  - The opposite vector of vector $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.
- The set $\mathcal{M}_{m,n}(\mathbb{R})$ of real matrices of size $m \times n$, with the operations of matrix addition and scalar multiplication is a vector space (over $\mathbb{R}$).

Example

- The set $P_2[x] = \{a_2x^2 + a_1x + a_0 | a_0, a_1, a_2 \in \mathbb{R}\}$ of real polynomials of degree less than or equal to *2* together with two operations:

$$\begin{aligned}(a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0) &= (a_2 + b_2)x^2 + (a_1 + b_1)x \\ &\quad + (a_0 + b_0) \\ c(a_2x^2 + a_1x + a_0) &= ca_2x^2 + ca_1x + ca_0\end{aligned}$$

  is a vector space over $\mathbb{R}$.
  - These two operations are closed.
  - Commutativity, associativity, distribution properties,.... are clear.
  - The zero vector is the zero polynomial $\mathbf{0} = 0 \cdot x^2 + +0 \cdot x + 0$.
  - The opposite (vector) of $a_2x^2 + a_1x + a_0$ is $-a_2x^2 - a_1x - a_0$.
- The set $P_n[x]$ of real polynomials of degree *not exceeding n* together with the usual polynomial addition and scalar multiplication, is a vector space over $\mathbb{R}$.

Example

The set $\mathcal{C}[a, b]$ of real continuous functions on $[a, b]$ is a vector space over $\mathbb{R}$, together with two natural operations,

$$(f + g)(x) = f(x) + g(x),$$
$$(cf)(x) = cf(x),$$

is a vector space.

- The operations are closed.
- Commutativity, associativity, distribution properties,.... are clear.
- The zero vector is $f_0 \equiv 0$
- The opposite of f is $-f$ which is defined by $(-f)(x) = -f(x)$, for all $x \in \mathbb{R}$.

Examples

- The set $\mathbb{Z}$ of integers together with the usual operations, is not a vector space over $\mathbb{R}$.
- The set $\mathbb{R}^+$ of real numbers together with the usual operations, is not a vector space over $\mathbb{R}$.
- The set of real polynomials of degree *exactly* n together with the polynomial addition and scalar multiplication, is not a vector space over $\mathbb{R}$.
- The set$\mathbb{R}^2$ together with the usual addition and the following scalar multiplication:

$$c(x_1, x_2) = (cx_1, 0)$$

  is not a vector spaces.

3.1.2. Some basic properties

Properties

Let V be a vector space over K. For every $u, v \in V$ and $c \in K$, the following statements are true.

1. The zero vector $\mathbf{0}$ is unique.
2. The opposite vector $(-v)$ of vector v is unique.
3. $(-1)v = -v$.
4. $0v = \mathbf{0}$.
5. $c\mathbf{0} = \mathbf{0}$.
6. If $cv = \mathbf{0}$ then either $c = 0$ or $v = \mathbf{0}$.

3.2.1. Definition

Let V be a vector space over K.

Vector subspace

A nonempty subset W of V is a *vector subspace* (or simply called, a *subspace*) of V if W is closed under the operations inherited from V, that means if

$$\begin{cases} u + v \in W, & \forall u, v \in W \\ cv \in W, & \forall c \in K, v \in W, \end{cases}$$

and under the operations of V, W is a vector space over K.

Test for a subspace

A nonempty subset W of V is a vector subspace of V if and only if if W is closed under the operations inherited from V, that means

$$\begin{cases} u+v \in W, & \forall u, v \in W \\ cv \in W, & \forall c \in K, v \in W. \end{cases}$$

Examples: Let V be a vector space.

- $\{\mathbf{0}\}$ (the subset of V consisting of the zero vector) is a subspace of V.
- V itself is a subspace of V.

Remark: If W is a subspace of V then $\mathbf{0} \in W$.

Example

Consider the vector space $V = \mathbb{R}^3$ (with two usual operations).
The subset $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ is a subspace of $\mathbb{R}^3$.

- Because $\mathbf{0} = (0, 0, 0)$ is in W, $W \neq \emptyset$.
- Consider $u = (x_1, y_1, z_1) \in W$ and $v = (x_2, y_2, z_2) \in W$, and let $c \in \mathbb{R}$.
- Then $u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$ since

$$\begin{aligned}(x_1 + x_2) + 2(y_1 + y_2) + 3(z_1 + z_2) &= (x_1 + 2y_1 + 3z_1) + (x_2 + 2y_2 + 3z_2) \\ &= 0 + 0 = 0.\end{aligned}$$

- Hence W is closed under vector addition.
- We have $cv = (cx_2, cy_2, cz_2) \in W$ since

$$cx_2 + 2cy_2 + 3cz_2 = c(x_2 + 2y_2 + 3z_2) = c \cdot 0 = 0.$$

- Hence W is closed under scalar multiplication.
- Therefore W is a subspace of $\mathbb{R}^3$.

Solution spaces of homogenous linear systems

Consider the vector space $\mathbb{R}^n$ (with usual operations). The subset W of $\mathbb{R}^n$ consisting of all solutions of a homogenous linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = 0 \\ \dots\dots\dots\dots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = 0 \end{cases}$$

is a vector subspace of $\mathbb{R}^n$.

Intersection of subspaces

Tính chất

If U và W are subspaces of a vector space V then $U \cap W$ is a subspace of V.

Remark:

- More generally, the intersection of a finite number of subspaces is a subspace.
- The union of two subspaces is not (in general) a subspace.

3.2.2. Subspaces spanned by subsetss

Let V be a vector space over K. (If we write $V = \mathbb{R}^n$ then we assume implicitly that the operations are the usual (standard/canonical) ones on $\mathbb{R}^n$.)

Definition (linear combinations)

Let $v_1, v_2, \ldots, v_n$ be vectors in V. A vector $v \in V$ of the form

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, \text{ for } c_1, c_2, \ldots, c_n \in K,$$

is a *linear combination* of the vectors $v_1, v_2, \ldots, v_n$.

Example: $V = \mathbb{R}^2$, $v_1 = (3, 4)$, $v_2 = (1, 1)$ and $v = (1, 2)$. Then

$$v = (1, 2) = (3, 4) - 2(1, 1) = v_1 - 2v_2$$

and v is a linear combination of v_1 and v_2.

Example

$V = \mathbb{R}^3$, $v_1 = (1,2,3)$, $v_2 = (0,1,2)$, $v_3 = (-1,0,1)$ and $v = (1,1,1)$. Is v a linear combination of v_1, v_2, v_3?

- Vector v is a linear combination of v_1, v_2, v_3 if and only if there exists real numbers c_1, c_2, c_3 such that $c_1 v_1 + c_2 v_2 + c_3 v_3 = v$.
- This is equivalent to the condition that the following system of linear equations has a solution (where c_1, c_2, c_3 are variables)

$$\begin{cases} c_1 \qquad\quad - c_3 = 1 \\ 2c_1 + \ \ c_2 \qquad = 1 \\ 3c_1 + 2c_2 + c_3 = 1 \end{cases}$$

- Solve this system we find that this system has infinitely many solutions: $c_1 = 1 + t, c_2 = -1 - 2t, c_3 = t$ $(t \in \mathbb{R})$. In particular, v is a linear combination of v_1, v_2, v_3.
- For example, choose $t = 1$, we obtain a representation of v as a linear combination of v_1, v_2, v_3:

$$v = 2v_1 - 3v_2 + v_3 \,.$$

Example

$V = \mathbb{R}^3$, $v_1 = (1, 2, 3)$, $v_2 = (0, 1, 2)$, $v_3 = (-1, 0, 1)$ and $v = (1, -2, 2)$. Is v a linear combination of v_1, v_2, v_3?

- Vector v is a linear combination of v_1, v_2, v_3 if and only if there exists real numbers c_1, c_2, c_3 such that $c_1 v_1 + c_2 v_2 + c_3 v_3 = v$.
- This is equivalent to the condition that the following system of linear equations has a solution (where c_1, c_2, c_3 are variables)

$$\begin{cases} c_1 - c_3 = 1 \\ 2c_1 + c_2 = -2 \\ 3c_1 + 2c_2 + c_3 = 2 \end{cases}$$

- This system has no solution. Hence v is not a linear combination of v_1, v_2, v_3.

Let $S = \{v_1, v_2, \dots, v_n\}$ be a set of vectors in a K-vector space V.

Definition

The set of all linear combinations of vectors in S is called the *(linear) span* of S and denoted by $\operatorname{span}(S)$ or $\operatorname{span}\{v_1, v_2, \dots, v_n\}$:

$$\operatorname{span}(S) = \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \mid c_1, c_2, \dots, c_n \in K\} \subset V.$$

Theorem

The subset $\operatorname{span}(S)$ is a subspace of V. Moreover, $\operatorname{span}(S)$ is the smallest subspace of V that contains S.

Remark

$\operatorname{span}(S)$ is also called the subspace spanned (generated) by S.

Example: $V = \mathbb{R}^2$, $v_1 = (1, 1)$, $v_2 = (1, 2)$. Then $\operatorname{span}\{v_1\} = \{c(1, 1) \mid c \in \mathbb{R}\} = \{(c, c) \mid c \in \mathbb{R}\}$.
$\operatorname{span}\{v_1, v_2\} = \{a(1, 1) + b(1, 2) \mid a, b \in \mathbb{R}\} = \{(a + b, a + 2b) \mid a, b \in \mathbb{R}\} = \mathbb{R}^2$.

Spanning sets

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a se of vectors in a K-vector space V.

Định nghĩa

If $\mathrm{span}(S) = V$ then we say that S is a *spanning set* (or a *generating set*) of V. In this case we also say that V *is spanned by* S, or S spans V.

Hence, $S = \{v_1, v_2, \ldots, v_n\}$ is a spanning set of V if and only if every vector $v \in V$ can be expressed a linear combination of vectors in S, that means there exist $c_1, \ldots, c_n \in K$ such that

$$v = c_1 v_1 + \cdots + c_n v_n.$$

Example

- $\{(1,1),(1,2)\}$ is a spanning set of $\mathbb{R}^2$.
- $\{(1,1),(1,2),(1,3)\}$ is a spanning set of $\mathbb{R}^2$.
- $\{(1,1)\}$ is not a spanning set of $\mathbb{R}^2$.
- $\{(1,0),(0,1)\}$ is a spanning set of $\mathbb{R}^2$.
- $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a spanning set of $\mathbb{R}^3$.
- $\{1,x,x^2\}$ is a spanning set of $P_2[x]$.
- The set of vectors consisting of

$$e_1=(1,0,0,\dots,0), e_2=(0,1,0,\dots,0),\dots, e_n=(0,0,\dots,0,1)$$

  is a spanning set of $\mathbb{R}^n$.
- $\{1,x,\dots,x^n\}$ is a spanning set of $P_n[x]$.

Example

Let $v_1 = (1,2,3)$, $v_2 = (0,1,2)$, $v_3 = (-1,1,1)$. Is $S = \{v_1, v_2, v_3\}$ a spanning set of $\mathbb{R}^3$?

- Let $v = (a, b, c)$ be any vector $\mathbb{R}^3$. Consider the condition $v = c_1 v_1 + c_2 v_2 + c_3 v_3$. This produces a system of linear equation

$$(a,b,c) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,1,1) \Leftrightarrow \begin{cases} c_1 - c_3 = a \\ 2c_1 + c_2 + c_3 = b \\ 3c_1 + 2c_2 + c_3 = c \end{cases}$$

- This system (with variable c_1, c_2, c_3) has the coefficient matrix with nonzero determinant $\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$. Hence the system always has a solution for every a, b, c.
- Thus for every $v \in \mathbb{R}^3$, we always can find c_1, c_2, c_3 such that $v = c_1 v_1 + c_2 v_2 + c_3 v_3$.
- Therefore $S = \{v_1, v_2, v_3\}$ is a spanning set of $\mathbb{R}^3$.

Example

Let $v_1 = (1, 2, 3)$, $v_2 = (0, 1, 2)$, $v_3 = (-1, 0, 1)$. Is $S = \{v_1, v_2, v_3\}$ a spanning set of $\mathbb{R}^3$?

NO

3.3.1. Linear dependence, linear independence

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a K-vector space V.

Definition

- The set S is said to be *linearly dependent* if there are numbers c_1, c_2, ..., c_n, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \mathbf{0}.$$

- The set S is said to be *linearly independent* if it is not linearly dependent. Thus, S is linearly independent if the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \mathbf{0} \quad (\text{với } c_1, c_2, \ldots, c_n \in K)$$

has only one solution $c_1 = c_2 = \cdots = c_n = 0$.

- $S = \{(1,1),(2,2)\} \subset \mathbb{R}^2$ is linearly dependent because
$$2 \cdot (1,1) + (-1) \cdot (2,2) = \mathbf{0} = (0,0).$$
- $S = \{(1,0),(0,1),(-2,4)\} \subset \mathbb{R}^2$ is linearly dependent because
$$2(1,0) - 4(0,1) + (-2,4) = (0,0).$$
- $S = \{(1,1),(1,2)\} \subset \mathbb{R}^2$ is linearly independent.
In fact, equation $c_1(1,1) + c_2(1,2) = \mathbf{0}$ is equivalent to
$$(c_1 + c_2, c_1 + 2c_2) = (0,0) \Leftrightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 + 2c_2 = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$
- $S = \{(1,1),(1,2),(1,3)\} \subset \mathbb{R}^2$ is linearly dependent. In fact, equation $c_1(1,1) + c_2(1,2) + c_3(1,3) = \mathbf{0}$ is equivalent to
$$(c_1 + c_2 + c_3, c_1 + 2c_2 + 3c_3) = (0,0) \Leftrightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 + 3c_3 = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases}$$
This system has a nontrivial solution, for example $c_1 = 1$, $c_2 = -2$, $c_3 = 1$. Hence S is linearly dependent.

- In $\mathbb{R}^n$, the set of vectors consisting of

$$e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$$

  is linearly independent.
- In $P_n[x]$, the set $\{1, x, \ldots, x^n\}$ is linearly independent.

Example

In $\mathbb{R}^3$, determine whether the following set of vectors is linearly independent of dependent:
$S = \{v_1, v_2, v_3\}$, where $v_1 = (1, 2, 3)$, $v_2 = (1, 1, -2)$, $v_3 = (2, 3, 2)$.

- The relation $c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0}$ is equivalent to

$$c_1(1, 2, 3) + c_2(1, 1, -2) + c_3(2, 3, 2) = (0, 0, 0) \Leftrightarrow \begin{cases} c_1 + c_2 + 2c_3 = 0 \\ 2c_1 + c_2 + 3c_3 = 0 \\ 3c_1 - 2c_2 + 2c_3 = 0 \end{cases}$$

- The coefficient matrix of this homogeneous linear system has non-zero determinant:
$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & -2 & 2 \end{vmatrix} \neq 0.$
- Hence the system only has the trivial solution $(c_1, c_2, c_3) = (0, 0, 0)$.
- Hence, S is linearly independent.

Example

In $\mathbb{R}^3$, determine whether the following set of vectors is linearly independent of dependent:
$S = \{v_1, v_2, v_3\}$, where $v_1 = (1, 2, 3)$, $v_2 = (1, 1, -2)$, $v_3 = (2, 3, 1)$.

- The relation $c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0}$ is equivalent to

$$c_1(1, 2, 3) + c_2(1, 1, -2) + c_3(2, 3, 1) = (0, 0, 0) \Leftrightarrow \begin{cases} c_1 + c_2 + 2c_3 = 0 \\ 2c_1 + c_2 + 3c_3 = 0 \\ 3c_1 - 2c_2 + 1c_3 = 0 \end{cases}$$

- The coefficient matrix of this homogeneous linear system has zero determinant: $\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & -2 & 1 \end{vmatrix} = 0.$
- Hence the system only has a nontrival solution $(c_1, c_2, c_3) \neq (0, 0, 0)$. (For example, $c_1 = 1, c_2 = 1, c_3 = -1$.)
- Hence, S is linearly dependent.

Properties

1. Any (non empty) subset of a linearly independent set is linearly independent.
2. Any set containing a linearly dependent set is linearly dependent.
3. A set $\{v\}$ consisting of one vector is linearly independent if and only if $v \neq \mathbf{0}$.
4. A set consisting two vectors are linearly dependent if one vector is a scalar multiple of the other.

Remark: A set containing the zero $\mathbf{0}$ is always linearly dependent.

Proposition

A set $S = \{v_1, \ldots, v_k\}$, $k \geq 2$, linearly dependent if and only if at least one of the vector v_j can be written as a linear combination of the other vectors in S.

Theorem

Let V be a vector space. Suppose $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in V and $\{w_1, \ldots, w_m\}$ is a spanning set of V. Then $n \leq m$.

3.3.2. Basis, dimension

Definition (basis)

A set $\mathcal{B} = \{v_1, \ldots, v_n\}$ of vectors in a vector space V is called a *basis* for V if the following conditions are true:

1. $\mathcal{B}$ is linearly independent;
2. $\mathcal{B}$ is a spanning set of V.

Example:

- $\{(1,1),(1,2)\}$ is a basis for $\mathbb{R}^2$.
- $\{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$.
- $\{(1,1),(1,2),(1,3)\}$ is not a basis for $\mathbb{R}^2$.

- Set $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$ consisting of the following vectors

$$e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1),$$

  is a basis for $\mathbb{R}^n$. This basis is called the *standard basis* (or the canonical basis) for $\mathbb{R}^n$.
- Set $\{1, x, x^2, \ldots, x^n\}$ is basis for $P_n[x]$. This basis is called the *standard basis* (or the canonical basis) for $P_n[x]$.
- A basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$ is the following set, which consist of the following vectors

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Theorem

If V has one basis with n vector, then every basis for V has n vector.

Definition (dimension)

If V has a basis consisting of n vector then the number n is called the *dimension* of V, denoted by $\dim V = n$, and we say that V is an n-dimensional vector space.

Remark:

- If $V = \{\mathbf{0}\}$, then we make a convention that $\dim V = 0$, and $\emptyset$ is a basis for V.
- If $V \neq \{\mathbf{0}\}$ and V does not have a finite basis then we say that V is infinitely dimensional, and write $\dim V = \infty$.

If $\dim V = n$ or $\dim V = 0$ then we say that V is finite dimensional.

Example

- $\dim \mathbb{R}^n = n$.
- $\dim P_n[x] = n + 1$.
- Let $P[x] := \mathbb{R}[x]$ be the vector of all real polynomials (with standard operations). Then $\dim P[x] = \infty$.
- $\dim \mathcal{M}_{m\times n}(\mathbb{R}) = mn$.

Example

Find the dimension of the following subspace of $\mathbb{R}^3$:

$$W = \{(a, a + b, b) \mid a, b \in \mathbb{R}\}.$$

- Let $v = (a, a + b, b) \in W$ be any vector in W. We have $(a, a + b, b) = (a, a, 0) + (0, b, b) = a(1, 1, 0) + b(0, 1, 1)$.
- Hence W is spanned by $S = \{(1, 1, 0), (0, 1, 1)\}$.
- We can show that S is linearly independent.
- Hence S is a basis for W and $\dim W = 2$.

Theorem (Dimension of the solution space of a homogeneous system)

Let V be the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$, where n=number of variables=number of columns of A. Then

$$\dim V = n - \mathrm{rank}(A).$$

Example (CK20151)

Find m such that the solution space of the following system has dimension 2:

$$\begin{cases} 2x_1 + x_2 - x_3 + 3x_4 - 2x_5 & = 0 \\ x_1 - 2x_2 + 3x_3 + mx_4 + x_5 & = 0 \\ 3x_1 - x_2 + 2x_3 + 4x_4 - x_5 & = 0 \end{cases}.$$

Properties

Let V be a vector space of dimension n.

- Every set having fewer than n vectors is not a spanning set of V.
- Every set having more than n vectors is linearly dependent.

In particular, for any set S of vectors in V, if S has m vectors and $m \neq n$, then S is not a basis for V.

Theorem

Let V be a vector space of dimension n.

- Every linearly indenpendent set having exactly n vectors is a basis for V.
- Every spanning set having exactly n vectors is a basis for V.

Some extra properties

Let V be a vector space of dimension n.

Theorem (Extending an independent set)

Let $S = \{v_1, \ldots, v_k\} \subset V$ be a linearly independent set of k vectors in V. Then $k \leq n$. Moreover, if $k < n$ then we can find $n - k$ vectors $v_{k+1}, \ldots, v_n$ such that the set $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ is a basis for V.

Theorem (Extracting a spanning set)

Let $S = \{v_1, \ldots, v_m\} \subset V$ be a spanning set of m vectors in V. Then $m \geq n$. Moreover, if $m \geq n$, we can remove $m - n$ vectors from S such that the subset set of n remaining vectors is a basis for V.

Theorem (Dimension of a subspace)

If W is a vector subspace of V, then $\dim W \leq \dim V$ with equality precisely when $W = V$.

3.3.3. Coordinates

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis for K-vector space V.

Theorem - Definition

Let $v \in V$. Then there is a unique n-tuple $(c_1, c_2, \dots, c_n) \in K^n$, such that

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n.$$

The numbers $c_1, c_2, \dots, c_n$ are called the coordinates of v with respect to (relative to) the basis $\mathcal{B}$. The n-tuple $(c_1, c_2, \dots, c_n)$ is called the (row) coordinate vector of v relative to $\mathcal{B}$, and denoted by

$$(v)_{\mathcal{B}} = (c_1, c_2, \dots, c_n).$$

We also denote $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, and call it the coordinate vector (or coordinate matrix)of v relative to $\mathcal{B}$.

Remarks: Coordinates of v depend on the order of the vectors in a basis.

Example

In $\mathbb{R}^2$, find the coordinates of $v = (1,4)$ relative to the basis $\mathcal{B} = \{v_1 = (1,1,), v_2 = (-1,2)\}$.

- $(v)_{\mathcal{B}} = (c_1, c_2) \Leftrightarrow v = c_1 v_1 + c_2 v_2 \Leftrightarrow (1,4) = c_1(1,1) + c_2(-1,2) \Leftrightarrow \begin{cases} c_1 - c_2 = 1 \\ c_1 + 2c_2 = 4 \end{cases} \Leftrightarrow$
$\begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}$
- Hence $(v)_{\mathcal{B}} = (2,1)$,or $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Change of basis

Problem

Le V be a K-vector space. Suppose we have two bases for V: $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{B}' = \{v_1', v_2', \ldots, v_n'\}$. We want to find a relation between $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}'}$.

For each j, we write v_j' as a linear combination of vectors in basis $\mathcal{B}$:

$$
\begin{aligned}
v_1' &= p_{11}v_1 + p_{21}v_2 + \cdots + p_{n1}v_n \\
v_2' &= p_{12}v_1 + p_{22}v_2 + \cdots + p_{n2}v_n \\
\cdots \\
v_n' &= p_{1n}v_1 + p_{2n}v_2 + \cdots + p_{nn}v_n.
\end{aligned}
$$

In other words,

$$
[v_1']_{\mathcal{B}} = \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix}, [v_2']_{\mathcal{B}} = \begin{bmatrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{bmatrix}, \ldots, [v_n']_{\mathcal{B}} = \begin{bmatrix} p_{1n} \\ p_{2n} \\ \vdots \\ p_{nn} \end{bmatrix}.
$$

Definition (Change of basis matrix)

Matrix $P = [[v_1']_{\mathcal{B}}\, [v_2']_{\mathcal{B}} \cdots [v_n']_{\mathcal{B}}] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$ is called the *transition matrix* (or the *change of basis*) from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$.

Theorem

We have

$$[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}, \quad \forall v \in V.$$

Moreover, if Q is a matrix such that $[v]_{\mathcal{B}} = Q[v]_{\mathcal{B}'}, \forall v \in V$, then $Q = P$.

Property

If P is the transition matrix from a basis $\mathcal{B}$ to a basis $\mathcal{B}'$ then P is invertible and P^{-1} is the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$, and

$$[v]_{\mathcal{B}'} = P^{-1}[v]_{\mathcal{B}}, \quad \forall v \in V.$$

Example

Let $\mathcal{B} = \{(1,1),(-1,2)\}$ and $\mathcal{B}' = \{(1,4),(-2,1)\}$ be two bases for $\mathbb{R}^2$. Find the transition matrix P from $\mathcal{B}$ to $\mathcal{B}'$.
For $v = (-1,5)$, compare $[v]_{\mathcal{B}}$ and $P[v]_{\mathcal{B}'}$.

- $v_1' = (1,4)$, $[v_1']_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
- $v_2' = (-2,1)$, $[v_2']_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
- The transition matrix from $\mathcal{B}$ to $\mathcal{B}'$:
$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$
- $[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $[v]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and
$$P[v]_{\mathcal{B}'} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [v]_{\mathcal{B}}.$$

3.3.4. Rank of a set of vectors

Let $S = \{v_1, v_2, \ldots, v_m\}$ be a set of vectors in a vector space V.

Định nghĩa

A subset T of S is said to be *maximal linearly independent* (in S) if the following conditions are true.

1. T is linearly independent.
2. If $T \subset T' \subset S$ and $T \not= T'$ then T is linearly dependent. (Equivalently, if we add any vector v in $S \setminus T$ to T then the set $T \cup \{v\}$ is linearly dependent.)

Example

Consider $S = \{v_1, v_2, v_3, v_3\}$ where $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, $v_3 = (2, 3, 4)$, $v_4 = (0, 1, 2)$. Then $T = \{v_1, v_2\}$ is a maximal linearly independent subset of S. Because

- $\{v_1, v_2\}$ is linearly independent, and
- If we add $v_3 \in S \setminus T$ to T we obtain the subset $\{v_1, v_2, v_3\}$, which is linearly dependent, and
- If we add $v_4 \in S \setminus T$ to T we obtain the subset $\{v_1, v_2, v_4\}$, which is linearly dependent.

Similarly, $\{v_1, v_3\}$ is also a maximal linearly independent subset of S.

Proposition

If T is a maximal linearly independent subset of S then T is a basis for $\mathrm{span}(S)$.

Corollary

Two maximal linearly independent subsets of S have the same number of vectors.

Definition (Rank of a set of vectors)

The number of vectors in a maximal linearly independent subset of $S = \{v_1, \ldots, v_m\}$ is called the rank of S, denoted by

$$\mathrm{rank}(S) = \mathrm{rank}\{v_1, \ldots, v_m\}.$$

Remarks:

- $$\dim \mathrm{span}(S) = \mathrm{rank}(S).$$
- $\mathrm{rank}(S) = \mathrm{rank}\{v_1, \ldots, v_m\} = r$ means that we can find r linearly indepedent vectors in S, and tính, every subset of $r + 1$ vectors in S, if any, is linearly dependent.
- $\mathrm{rank}\{v_1, \ldots, v_m\} = m \Leftrightarrow \{v_1, \ldots, v_m\}$ is linearly independent.

The row (column) coordinate matrix of a set of vectors

- Let $\mathcal{B}$ be a basis for V, $\dim V = n$, and $S = \{v_1, \dots, v_m\}$ a set of vectors in V.
- Suppose

$$\begin{aligned}(v_1)_{\mathcal{B}} &= (a_{11}, a_{12}, \dots, a_{1n})\\ (v_2)_{\mathcal{B}} &= (a_{21}, a_{22}, \dots, a_{2n})\\ &\dots\\ (v_m)_{\mathcal{B}} &= (a_{m1}, a_{m2}, \dots, a_{mn})\end{aligned}$$

- Then $(S)_{\mathcal{B}} := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} =$ *the row coordinate matrix* of S relative to $\mathcal{B}$.
- $[S]_{\mathcal{B}} := \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} =$ *the (column) coordinate matrix* of S relative to $\mathcal{B}$

Let V be an n-dimensional vector space, $\mathcal{B}$ a basis for V.

Proposition (Finding the rank of a set of vectors)

Let $S = \{v_1, \ldots, v_m\}$ be a set of vectors in V. Let A be the row coordinate matrix of S relative to $\mathcal{B}$. Then

$$\mathrm{rank}(S) = \mathrm{rank}(A).$$

Corollary

Let V be an n-dimensional vector space, $\mathcal{B}$ a basis for V, and $S = \{v_1, \ldots, v_n\}$ a set of n vectors in V. Let A be the row coordinate matrix of S relative to $\mathcal{B}$. Then

$$S \text{ is a basis for } V \Leftrightarrow \det(A) \neq 0.$$

Finding a basis for $\mathrm{span}(S)$

Let S be a set of vectors in V. To find a basis for $\mathrm{span}(S)$ we do the following.

- Find the row coordinate matrix A of S relative to some basis $\mathcal{B}$.
- Use **row** elementary operations to bring A to a row echelon matrix A'. Matrix A' has r nonzero rows $u_1, \ldots, u_r$.
- Let $v_1', \ldots, v_r'$ be vectors in V such that $(v_i')_{\mathcal{B}} = u_i$, for $i = 1, \ldots, r$.
- Then $\{v_1', \ldots, v_r'\}$ is a basis for $\mathrm{span}(S)$.

Example (CK20181-N2)

In $P_2[x]$, consider the following vectors: $v_1 = 1 + x + x^2$, $v_2 = 2 + mx - x^2$, $v_3 = 4 + 5x + x^2$, $v = 10 + 11x - 5x^2$.

a) Find m such that $B = \{v_1, v_2, v_3\}$ is linearly dependent.

b) For $m = 2$, show that B is basis for $P_2[x]$. Find coordinates of v relative to B.

- The row coordinate matrix of B relative to the standard basis $\{1, x, x^2\}$ is $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & m & -1 \\ 4 & 5 & 1 \end{bmatrix}$.
- B is linearly independent $\Leftrightarrow \mathrm{rank}(B) < 3 \Leftrightarrow \mathrm{rank}(A) < 3 \Leftrightarrow \det(A) = 0$.
- $\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & m & -1 \\ 4 & 5 & 1 \end{vmatrix} = 9 - 3m$.
- B is linearly independent $\Leftrightarrow 9 - 3m = 0 \Leftrightarrow m = 3$.

- When $m = 2$, $\det(A) \neq 0$, hence B is a basis for sở của $P_2[x]$.
- Let $(v)_B = (c_1, c_2, c_3)$ be the row coordinate vector of v relative to the basis B.
- We have $v = c_1 v_1 + c_2 v_2 + c_3 v_3 \Leftrightarrow \begin{cases} c_1 + 2c_2 + 4c_3 & = 10 \\ c_1 + 2c_2 + 5c_3 & = 11 \\ c_1 - c_2 + c_3 & = -5 \end{cases} \Leftrightarrow \begin{cases} c_1 = -2 \\ c_2 = 4 \\ c_3 = 1 \end{cases}$.
- Thus $(v)_B = (-2, 4, 1)$.

Sum and intersection of subspaces

Let V_1 and V_2 be two subspaces of a vector space V.
The subset $V_1 + V_2 = \{v = v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\}$ is a subspace of V, and it is called the *sum* of V_1 and V_2.

Theorem

If V_1 and V_2 are finite dimensional then $V_1 + V_2$ is finite dimensional and

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

Example (CK20181)

In $P_3[x]$, consider the vectors $u_1 = 1 + 2x - x^3$, $u_2 = 2 - x + x^2 + 2x^3$, $u_3 = -1 + x - x^2 + x^3$, $u_4 = 4 + 2x^2$ and let $V_1 = \mathrm{span}\{u_1, u_2\}$, $V_2 = \mathrm{span}\{u_3, u_4\}$. Find the dimension and a basis of the subspace $V_1 + V_2$ and of the subspace $V_1 \cap V_2$.

Some exercises

- (CK20151) In $\mathbb{R}^4$, consider the vectors $u_1 = (1, 3, -2, 1)$, $u_2 = (-2, 3, 1, 1)$, $u_3 = (2, 1, 0, 1)$, $u = (1, -1, -3, m)$. Find m such that $u \in \mathrm{span}\{u_1, u_2, u_3\}$.
- (CK20151-No.7) In $\mathbb{R}^4$, consider the vectors $u_1 = (1, 0, 1, 1)$, $u_2 = (-3, 2, 1, -1)$, $u_3 = (2, 1, 0, 2)$, $u = (1, 2, 1, m)$. Find m such that $S = \{u_1, u_2, u_3, u_4\}$ is linearly dependent.
- (CK20151) In $P_3[x]$, consider the vectors $v_1 = 1 + x + x^2$, $v_2 = x - x^2 + x^3$, $v_3 = 1 + 2x + x^2 + x^3$, $v_4 = 2 + 2x + 4x^2$, $V_1 = \mathrm{span}\{v_1, v_2\}$, $V_2 = \mathrm{span}\{v_3, v_4\}$. Find the dimension and a basis of $V_1 + V_2$.
- (CK20181-N3) Consider the vectors $v_1 = (2, 1, 5, 8), v_2 = (1, -1, 3, 5), v_3 = (0, 2, 1, 6), v_4 = (-3, 5, 2, 1)$
  a) Show that v_1, v_2, v_3, v_4 form a basis for $\mathbb{R}^4$.
  b) Find the coordinates of $v = (-5, 15, 15, 13)$ relative to the basis in part (a).
- (CK20193-N2) Consider the vectors $u_1 = (2, -1, 3, 0, 2)$, $u_2 = (1, -4, 2, 5, -1)$, $u_3 = (3, 2, 4, 6, 0)$, $u_4 = (7, 0, 10, 6, 4)$ in $\mathbb{R}^5$. Find a basis and the dimension of the subpace spanned by these vectors.