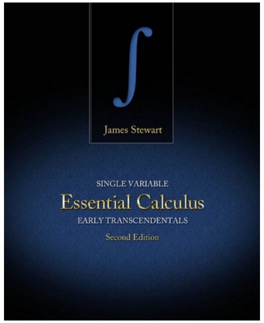
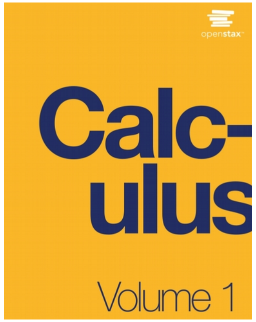
Chapter 3: Inverse Functions





3.1 Exponential Functions

3.2 Inverse Functions and Logarithms

3.3 Derivatives of Logarithmic and Exponential Functions

3.4 Exponential Growth and Decay

3.5 Inverse Trigonometric Functions

3.6 Hyperbolic Functions

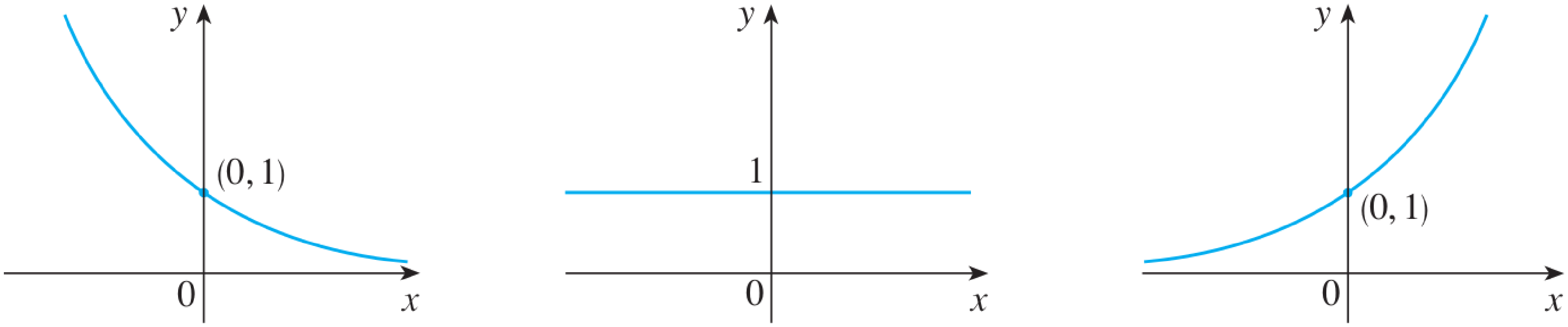
3.7 Indeterminate Forms and l'Hospital's Rule

The pictures are taken from the books:

[1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280
2) G. Strang and E. J. Herman, Calculus 1, https://openstax.org/details/books/calculus-volume-1]

3.1 Exponential Functions

$$f(x) = a^x\,, \qquad a \in \mathbb{R}^+ \backslash \{1\},\ x \in (0, \infty),\ f(x) \in (0, \infty)$$



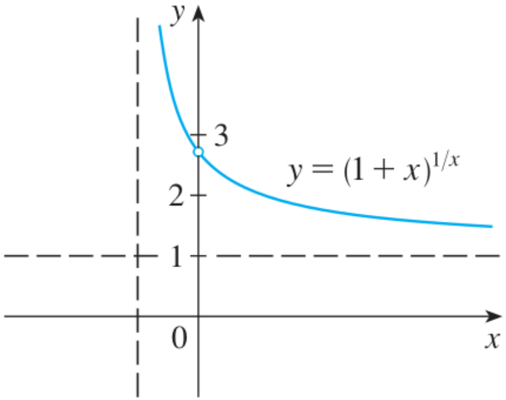
(a) $y = a^x,\ 0 < a < 1$ (b) $y = 1^x$ (c) $y = a^x,\ a > 1$

3.1 Properties

- If $0 < a < 1$, then $\lim_{x\to\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = \infty$
- If $a > 1$, then $\lim_{x\to\infty} a^x = \infty$ and $\lim_{x\to-\infty} a^x = 0$

The natural exponential function, $a = e$

$e = \lim_{x\to 0}(1+x)^{1/x} = 2.7182$ $\quad f(x) = e^x$, $\quad D = \mathbb{R}$, $\quad R = (0, \infty)$



$$f'(x) = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0}\frac{e^h - 1}{h} = e^x$$

$$f^{(n)}(x) = e^x$$

3.1 Examples

- Find the following limits

1. $\lim\limits_{x \to 0^-} e^{1/x}$

2. $\lim\limits_{x \to -\infty} \dfrac{3^{x+1} + 5e^x}{2 \cdot 3^x - e^x}$

3. $\lim\limits_{x \to -\infty} \ln^2(1 + e^x)$

4. $\lim\limits_{x \to \infty} \dfrac{e^{2x}}{e^{2x} + 1}$

5. $\lim\limits_{x \to \infty} \ln^3(1 + e^x)$

3.2 Inverse Functions and Logarithms

Motivation

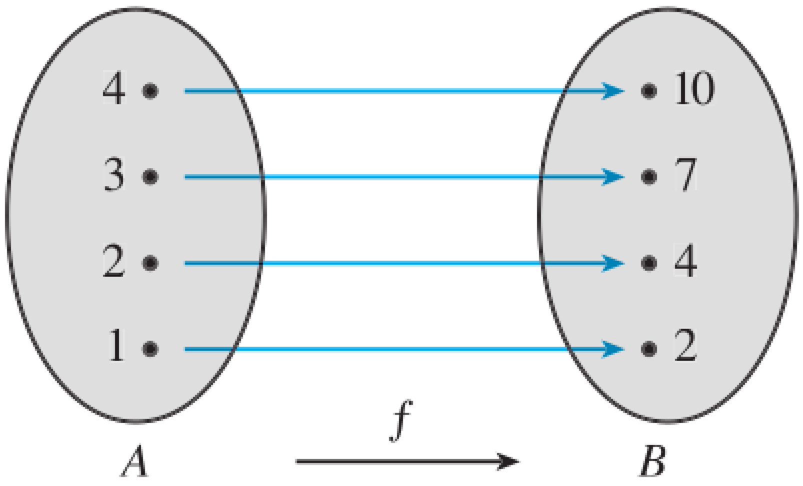
- Time t as a function of the position x: $\quad t(x) = \sqrt{\dfrac{2x}{g}}$ $\quad$ (Motion with constant acceleration)

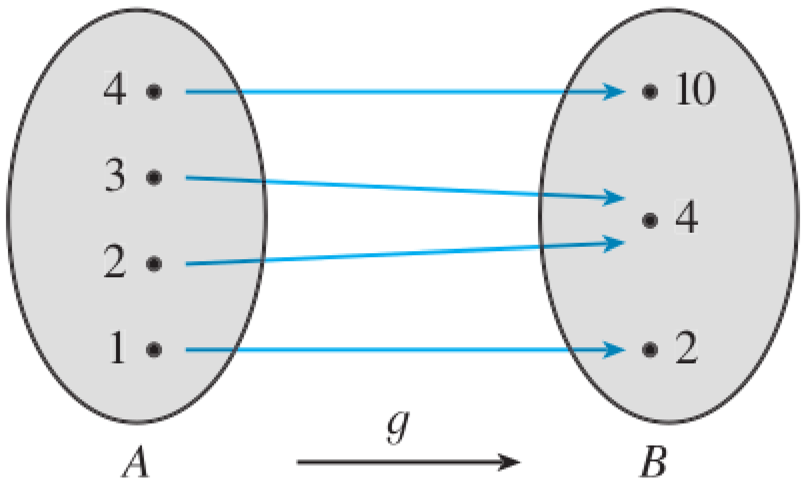
- Physically: can we obtain $x(t)$?

- Mathematically: can $t(x)$ be inverted to $x(t)$?

3.2 Inverse Functions

Definition A function $f : A \to B$ is called a **one-to-one** function if it never takes on the same value twice; that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.





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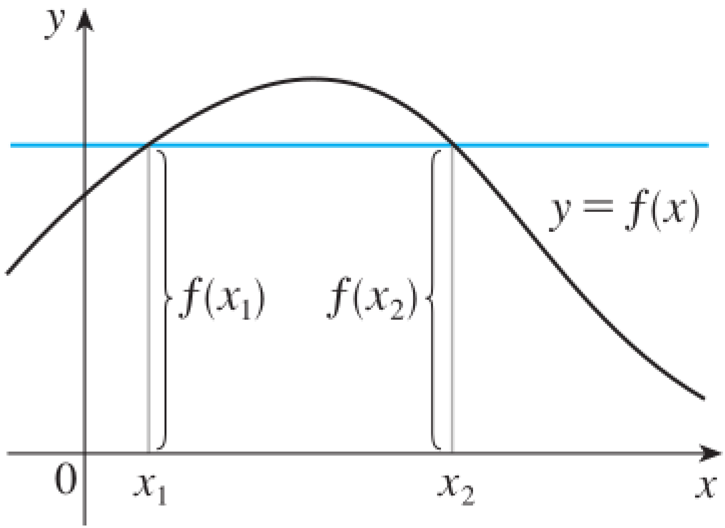
Horizontal Test Line A function is one-to-one if and only if no horizontal line intersects its graph more than once.



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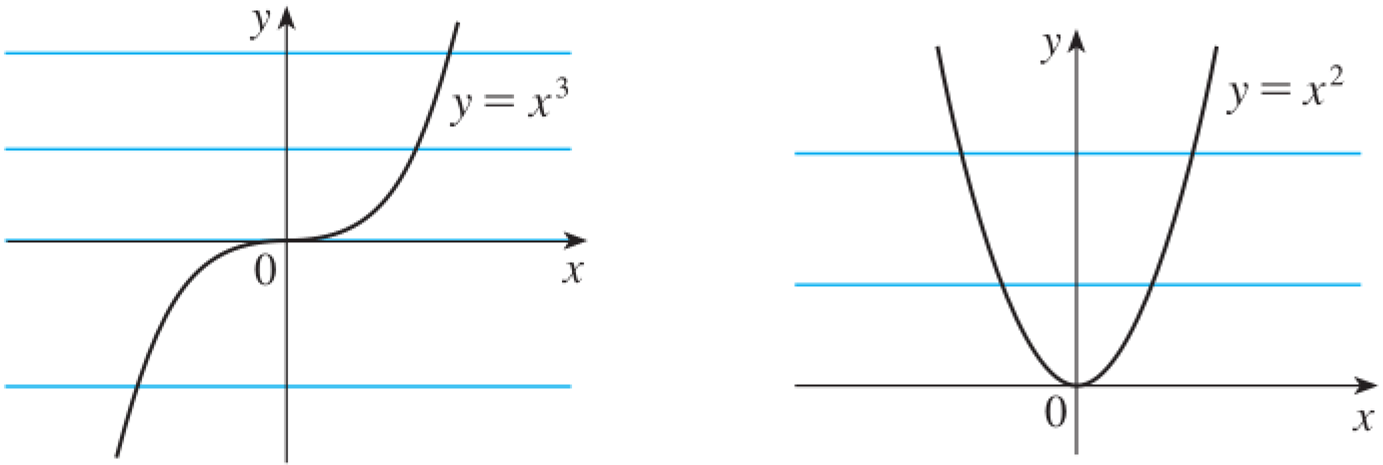
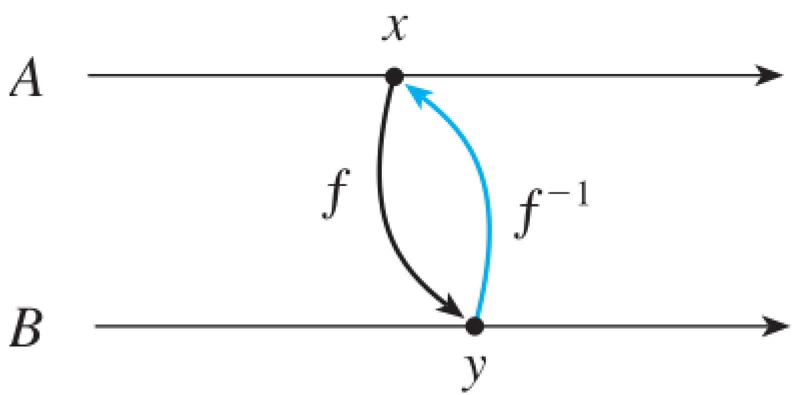


3.2 Inverse Functions

Definition Let f be a one-to-one function with domain A and range B . Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y$$

for any $y \in B$.



$$f : A \to B$$

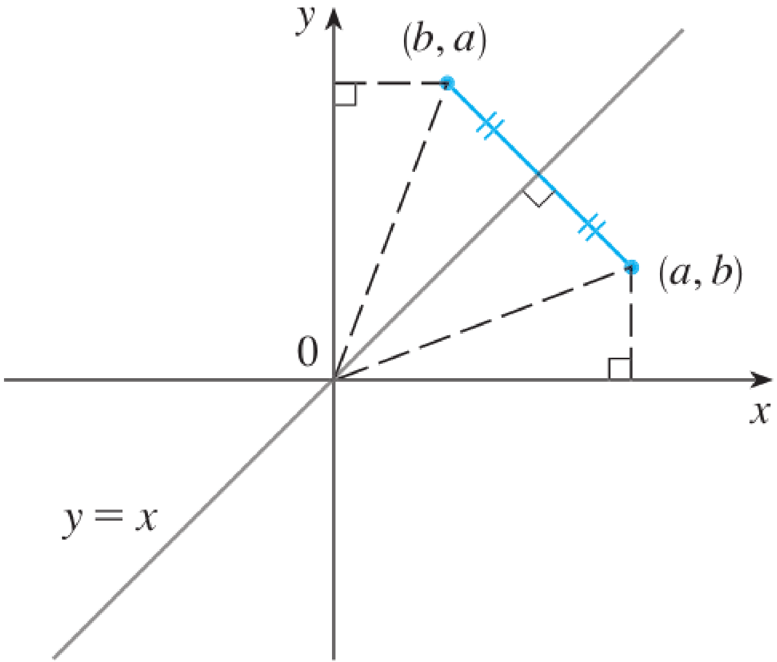
$$f^{-1} : B \to A$$

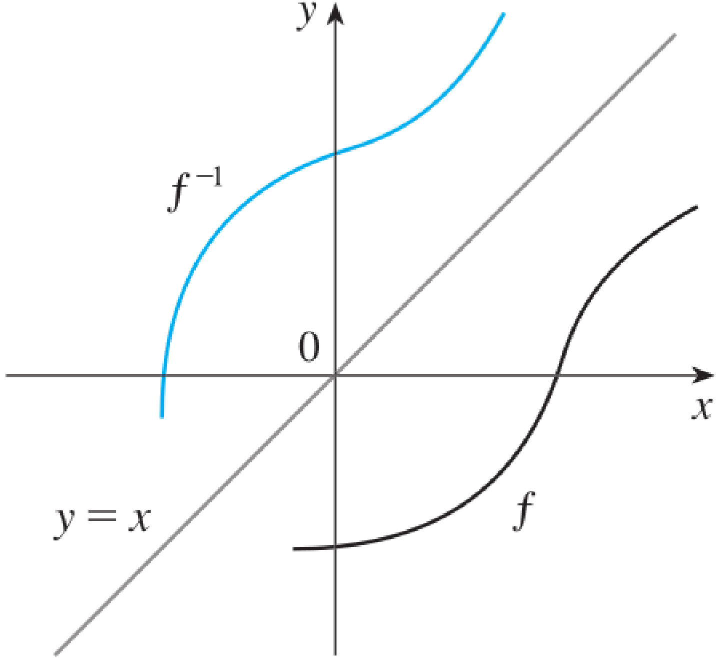
$$f^{-1}\big(f(x)\big) = x \quad \forall \in A$$

$$f\big(f^{-1}(x)\big) = x \quad \forall \in B$$

3.2 Inverse Functions

- The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.



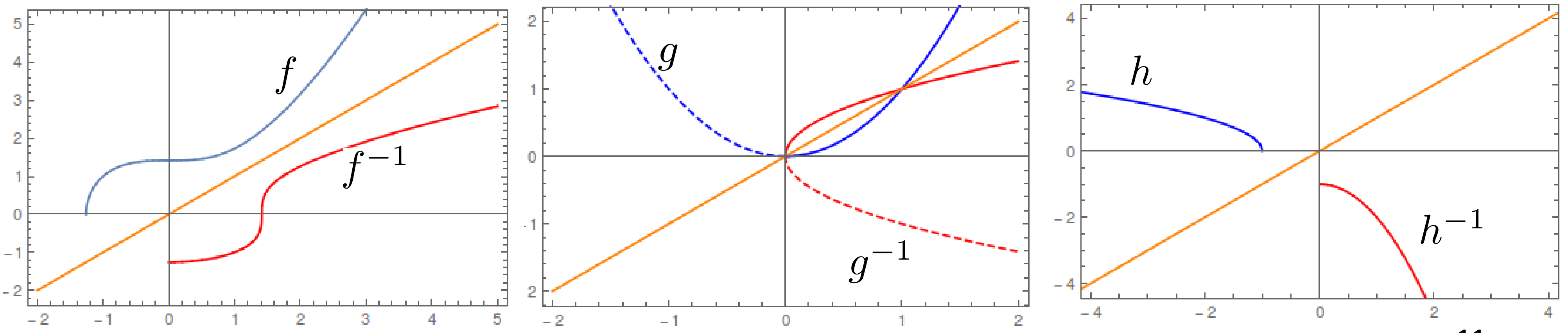


3.2 Examples

- Sketch the graphs of

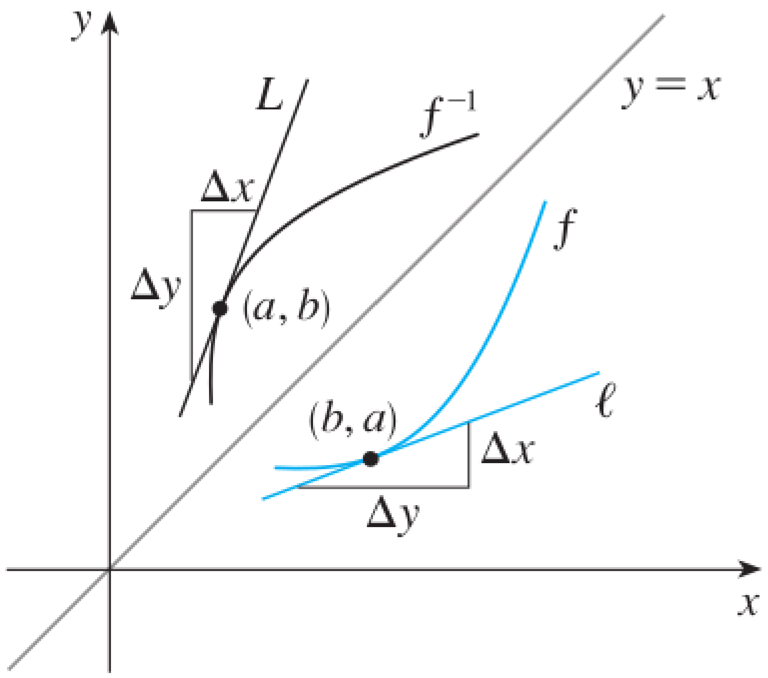
$$\textbf{1.}\quad f(x)=\sqrt{x^3+2}\qquad \textbf{2.}\quad g(x)=x^2\qquad \textbf{3.}\quad h(x)=\sqrt{-1-x}$$

and its inverse function using the same coordinate axes.



3.2 The Calculus of Inverse Functions

THEOREM If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.



$$(f^{-1})'(a) = \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x/\Delta y} = \frac{1}{f'(b)}$$

3.2 The Calculus of Inverse Functions

THEOREM If f is a one-to-one differentiable function with inverse function f^{-1} and $f'\big(f^{-1}(a)\big) \neq 0$, then the inverse function is differentiable at a and

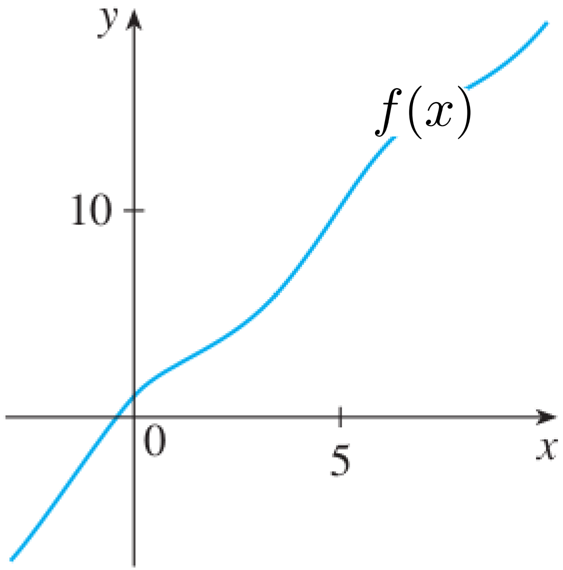
$$(f^{-1})'(a) = \frac{1}{f'\big(f^{-1}(a)\big)}$$

Note Replacing a by x, we get

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

3.2 Example

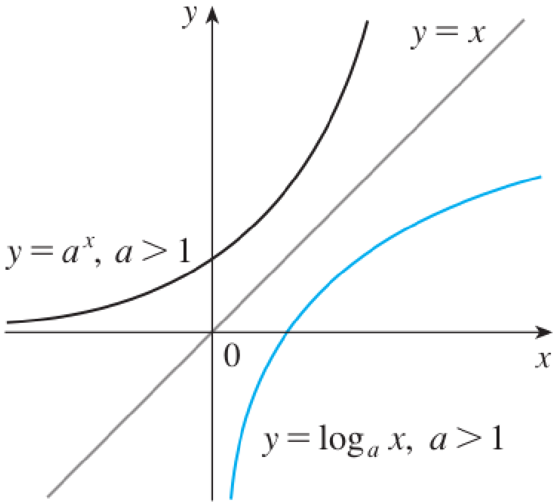
- If $f(x) = 2x + \cos(x)$, find $(f^{-1})'(1)$.

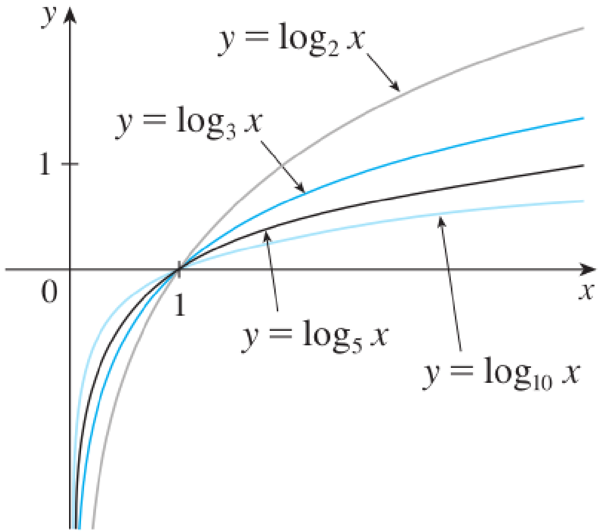


3.2 Inverse Exponential Functions: Logarithmic Functions

$$y = f(x) = a^x \qquad \Rightarrow \qquad f^{-1}(x) = \log_a(x), \qquad \text{such that}$$

$$\log_a(a^x) = x \qquad \forall x \in \mathbb{R}\,, \qquad\qquad a^{\log_a(x)} = x \qquad \forall x \in (0, \infty)$$





3.2 Properties of Logarithmic Functions

- If $x, y > 0$ and $r \in \mathbb{R}$, then

$$\begin{aligned}
\textbf{1}. \quad & \log_a(xy) = \log_a(x) + \log_a(y) \\
\textbf{2}. \quad & \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y) \\
\textbf{3}. \quad & \log_a(x^r) = r\log_a(x)
\end{aligned}$$

- If $a > 1$, then

$$\lim_{x\to\infty} \log_a(x) = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \log_a(x) = -\infty$$

3.2 Natural Logarithms

- Special notation: $\log_e(x) = \ln(x)$. Then $\log_e(e) = \ln(e) = 1$ and

$$f(x) = e^x\,, \qquad\qquad f : \mathbb{R} \to (0, \infty)\,,$$
$$f^{-1}(x) = \ln(x)\,, \qquad\qquad f^{-1} : (0, \infty) \to \mathbb{R}\,,$$

- Change of base formula

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

- Limits

$$\lim_{x\to\infty} \ln(x) \stackrel{?}{=} \quad , \qquad \lim_{x\to 0^+} \ln(x) \stackrel{?}{=}$$

3.3 Derivatives of Logarithmic Functions

- The function $f(x) = \log_a(x)$ is differentiable and

$$f'(x) = \frac{1}{x}\log_a(e) = \frac{1}{x}\ln(a)$$

- The derivative of the natural logarithmic function $f(x) = \ln(x)$,

$$f'(x) = \frac{1}{x}$$

- The derivative of the natural logarithm of a function $f(x) = \ln(g(x))$,

$$f'(x) = \frac{g'(x)}{g(x)}$$

3.3 Examples

Find $f'(x)$

1. $y = \ln(1 + x^3)$

2. $y = \ln(\sin(x))$

3. $y = \sqrt{\ln(x)}$

4. $y = \ln(|x|)$

5. $y = \ln\left(\dfrac{x+1}{\sqrt{x-2}}\right)$

6. $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

3.3 Derivatives of Exponential Functions

- The function $f(x) = a^x$, $a > 0$, is differentiable and

$$f'(x) = a^x \ln(a) = \frac{1}{x} \ln(a)$$

- The derivative of the natural exponential function $f(x) = e^x$,

$$f'(x) = e^x$$

- The derivative of the natural exponential of a function $f(x) = e^{g(x)}$,

$$f'(x) = e^{g(x)} g'(x)$$

3.3 Examples

- Differentiate the following functions

1. $y = e^{\tan(x)}$

2. $y = e^{-4x}\sin(5x)$

3. $y = x^{\sqrt{x}}$

4. $y = \sqrt{1 + xe^{-2x}}$

5. $y = \sin\left(e^{\sin^2(t)}\right)$

6. $y = e^{\sin^2(z)+\cos^2(z)}$

3.4 Exponential Growth and Decay

- Considering the rate of change

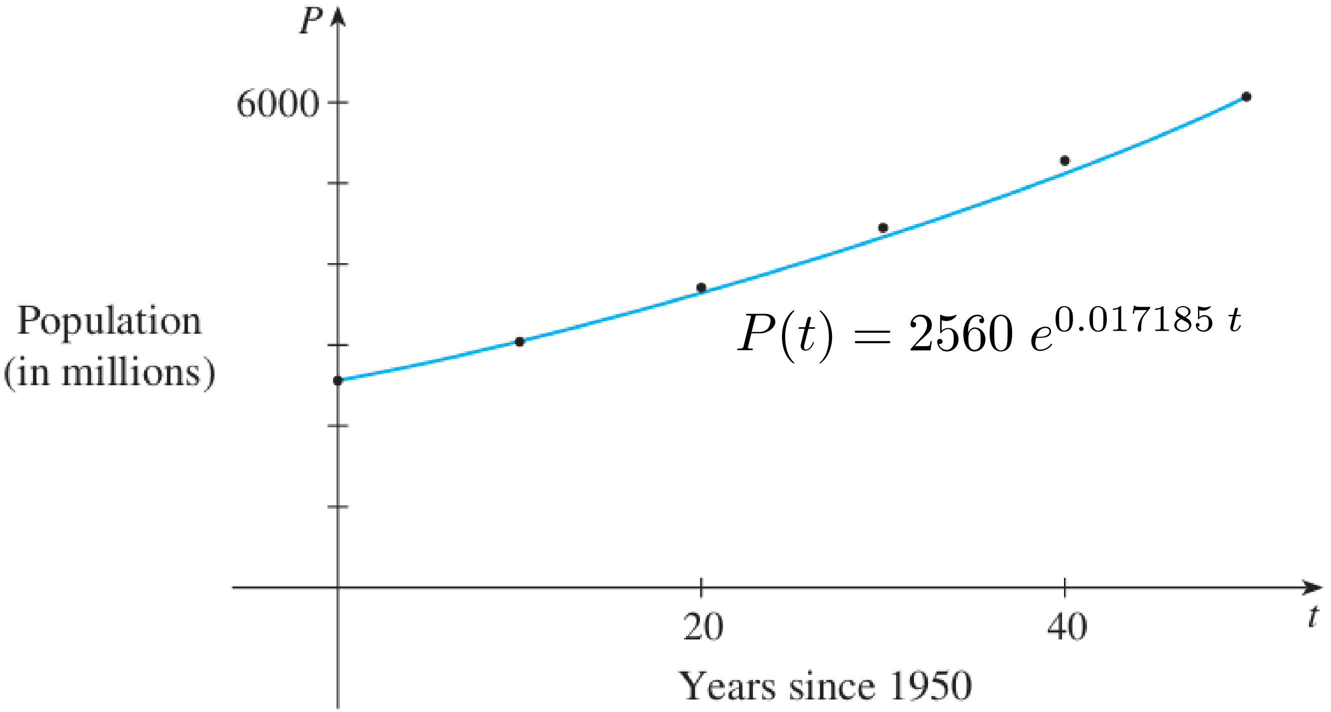
$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = ky(t)\,,$$

where k is the **relative growth/decay rate**. The only solutions are exponential functions

$$y(t) = y(0)e^{kt}$$

3.4 Example

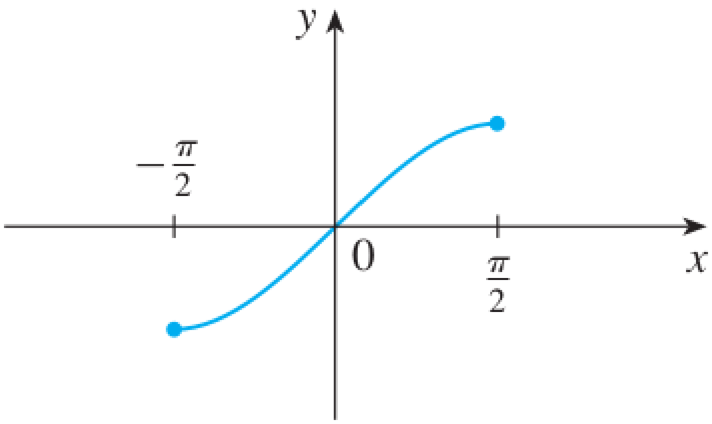
- Population Growth: A model for world population growth in the second half of the 20th century

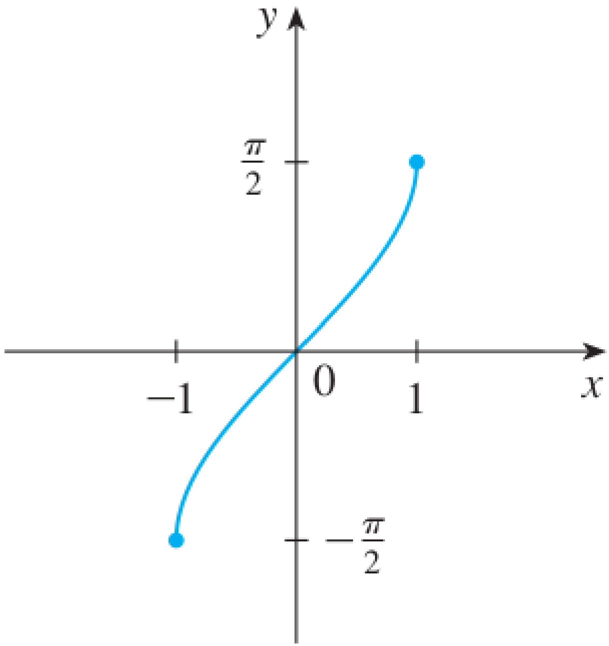


3.5 Inverse Trigonometric Functions

$$f(x) = \sin(x)\,, \qquad f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$$

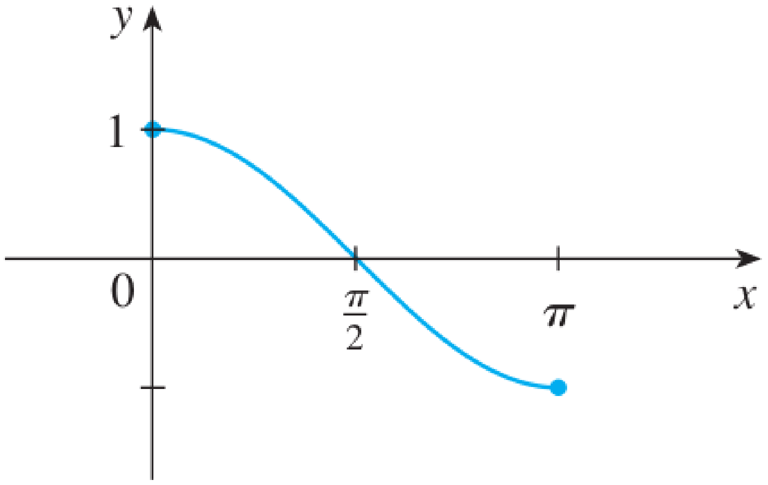
$$f^{-1}(x) = \arcsin(x)\,, \qquad f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

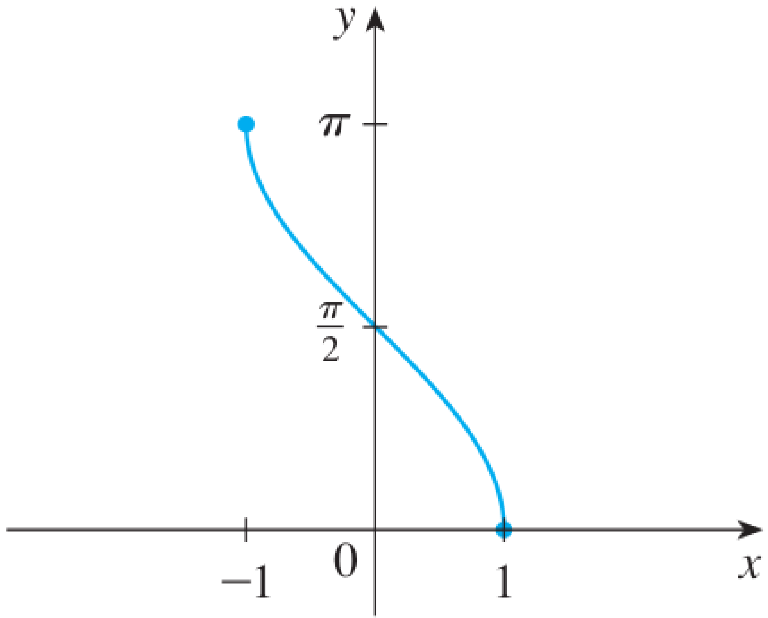




3.5 Inverse Trigonometric Functions

$$f(x) = \cos(x)\,, \qquad f : [0, \pi] \to [-1, 1]$$
$$f^{-1}(x) = \arccos(x)\,, \qquad f^{-1} : [-1, 1] \to [0, \pi]$$

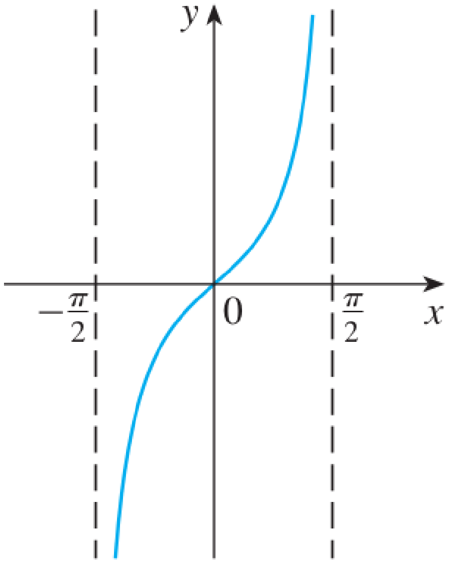


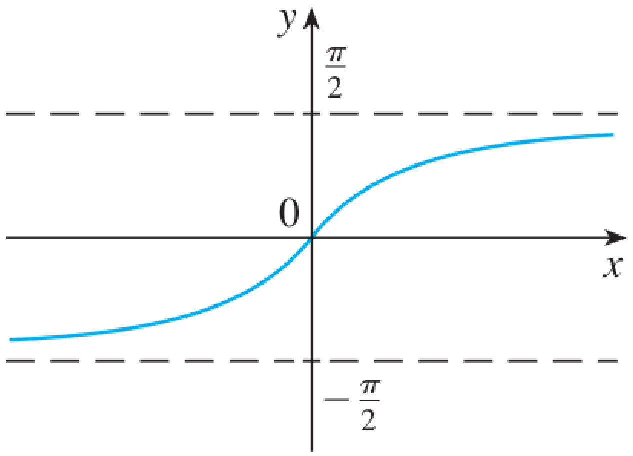


3.5 Inverse Trigonometric Functions

$$f(x) = \tan(x)\,, \qquad f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$$

$$f^{-1}(x) = \arctan(x)\,, \qquad f^{-1} : \mathbb{R} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$





$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$$

3.5 Inverse Trigonometric Functions

- Derivatives

$$y = \arcsin(x) \quad \Rightarrow \quad \sin(y) = x \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 - \sin^2(y)}} \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 - x^2}}}$$

- $\dfrac{\mathrm{d}}{\mathrm{d}x} \arcsin(x) = \dfrac{1}{\sqrt{1 - x^2}}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \arccos(x) = -\dfrac{1}{\sqrt{1 - x^2}}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \arctan(x) = \dfrac{1}{1 + x^2}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{arccsc}(x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{arcsec}(x) = \dfrac{1}{x\sqrt{x^2 - 1}}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{arccot}(x) = -\dfrac{1}{1 + x^2}$

3.5 Examples

1. Evaluate **a.** $\arcsin(1/2)$, **b.** $\tan(\arcsin(1/3))$

2. Differentiate $f(x) = x \arctan(\sqrt{x})$

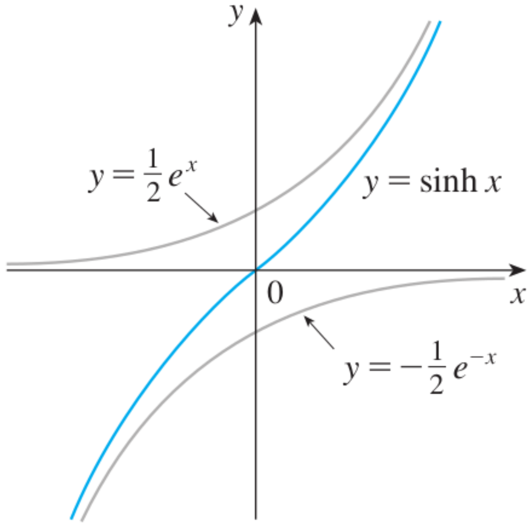
3.6 Hyperbolic Functions

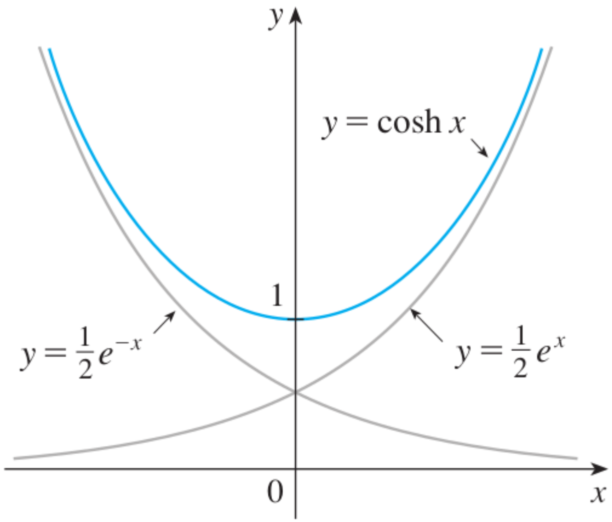
- $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$,
- $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$,
- $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$,

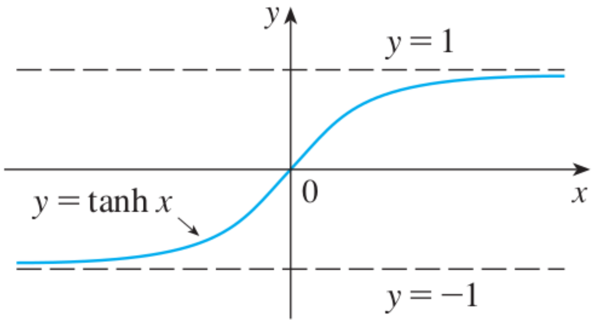
- $\text{csch(x)} = \dfrac{1}{\sinh(x)}$
- $\text{sech(x)} = \dfrac{1}{\cosh(x)}$
- $\text{coth(x)} = \dfrac{1}{\tanh(x)}$

3.6 Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$







3.6 Hyperbolic Identities

- $\sinh(-x) = -\sinh(x)\,,$
- $\cosh^2(x) - \sinh^2(x) = 1\,,$
- $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \sinh(y)\cosh(x)\,,$
- $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)\,,$

- $\cosh(-x) = \cosh(x)$
- $1 - \tanh^2(x) =^2 (x)$

3.6 Derivatives

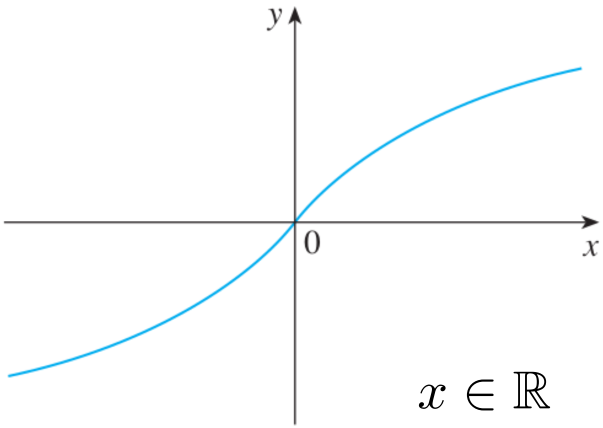
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \sinh(x) = \cosh(x)\,,$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \cosh(x) = \sinh(x)\,,$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \tanh(x) = \mathrm{sech}^2(x)\,,$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \mathrm{csch}(x) = -\mathrm{csch}(x)\coth(x)$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \mathrm{sech}(x) = -\mathrm{sech}(x)\tanh(x)$
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \coth(x) = -\mathrm{csch}^2(x)$

3.6 Inverse Hyperbolic Functions

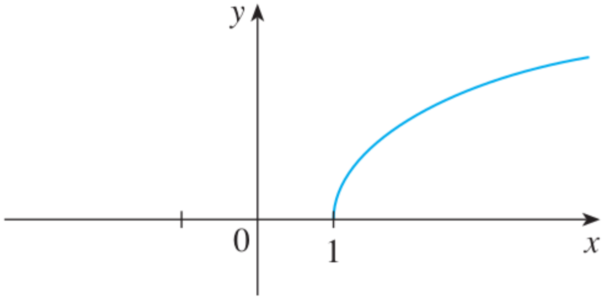
$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$$

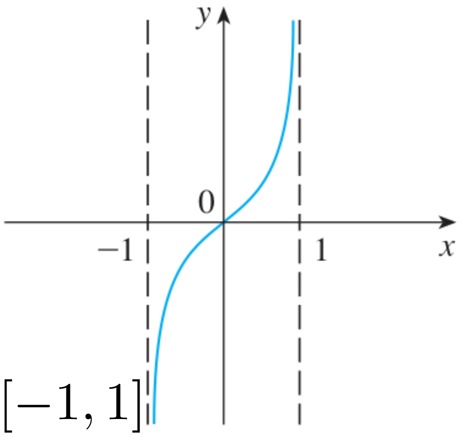
$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$



$x \in \mathbb{R}$



$x \in [1, \infty)$



$x \in [-1, 1]$

3.6 Derivatives

- $\dfrac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}(x) = \dfrac{1}{\sqrt{x^2+1}}\,,$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}\,,$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(x) = \dfrac{1}{1-x^2}\,,$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\operatorname{csch}^{-1}(x) = -\dfrac{1}{|x|\sqrt{1+x^2}}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\operatorname{sech}^{-1}(x) = -\dfrac{1}{x\sqrt{1-x^2}}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}\coth^{-1}(x) = \dfrac{1}{1-x^2}$

3.7 Indeterminate Forms and L'Hospital's Rule

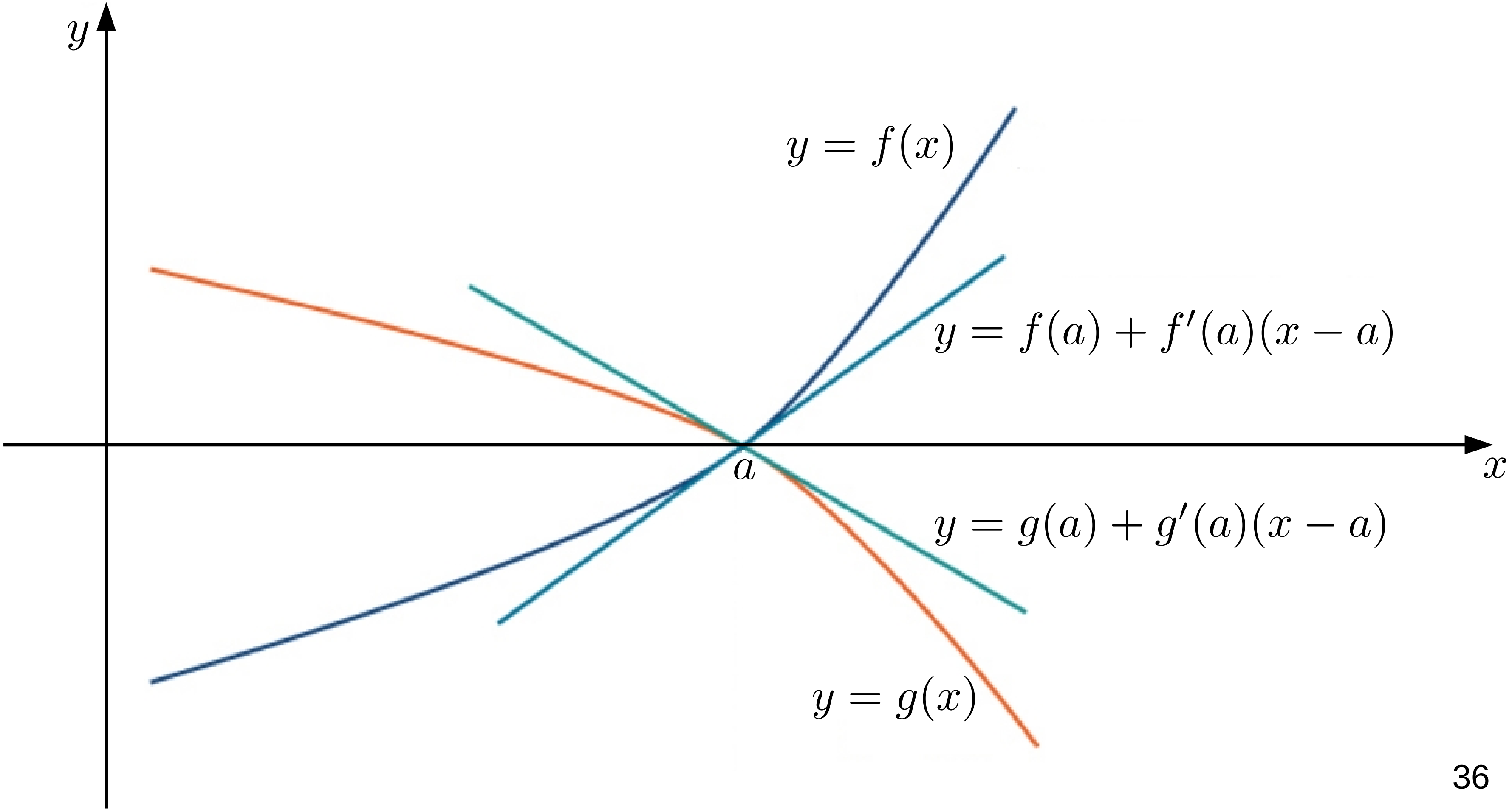
• Indeterminate forms:
(Rational functions)

$$\lim_{x\to 1} \frac{x^2 - x}{x^2 - 1} = \frac{0}{0}, \qquad \lim_{x\to\infty} \frac{x^2 - x}{x^2 - 1} = \frac{\infty}{\infty}$$

• Indeterminate forms:
(Non-rational functions)

$$\lim_{x\to 1} \frac{\ln(x)}{x - 1} = \frac{0}{0}, \qquad \lim_{x\to\infty} \frac{\ln(x)}{x - 1} = \frac{\infty}{\infty}$$

3.7 Indeterminate Forms and L'Hospital's Rule



3.7 Indeterminate Forms and L'Hospital's Rule

Theorem

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ and $-\infty$.

3.7 Examples

1. $\displaystyle\lim_{x\to 1}\frac{\ln(x)}{x-1}=1$

2. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}=\infty$

3. $\displaystyle\lim_{x\to\infty}\frac{\ln(x)}{x^{1/3}}=0$

4. $\displaystyle\lim_{x\to 1}\frac{\sin(\pi x)}{\ln(x)}=-\pi$

5. $\displaystyle\lim_{x\to\pi^+}\frac{\sin(x)}{1+\cos(x)}=-\infty$

6. $\displaystyle\lim_{x\to 0}\frac{\tan(x)-x}{x^3}=\frac{1}{3}$

3.7 Other Indeterminate Forms

$$0 \cdot (\pm\infty)\,, \qquad \infty - \infty\,, \qquad 0^0\,, \qquad \infty^0\,, \qquad 1^\infty \qquad \Big| \qquad 0^\infty$$

Examples:

1. $\displaystyle\lim_{x\to 0^+} (1+\sin(4x))^{\cot(x)} = e^4$

2. $\displaystyle\lim_{x\to 0^+} x^x = 1$

3. $\displaystyle\lim_{x\to 0^+} x\ln(x) = 0$

4. $\displaystyle\lim_{x\to (\pi/2)^-} (\sec(x) - \tan(x)) = 0$