Chapter 4: Line integrals

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Content

4.1.1. Definition

- Let $f(x, y)$ be a function defined on a curve $\overset{\frown}{AB}$.
- Divide $\overset{\frown}{AB}$ into n smaller arcs by points $A = A_0$, A_1, ..., $A_n = B$. Let the length of $\overset{\frown}{A_{i-1}A_i}$ be Δs_i.
- In each arc $\overset{\frown}{A_{i-1}A_i}$, take a point $M_i(x_i^*, y_i^*)$ and define the sum

$$\sum_{i=1}^{n} f(M_i)\Delta s_i.$$

- If the sum $\sum_{i=1}^{n} f(M_i)\Delta S_i$ approaches to a limit as $\max \Delta s_i \to 0$, not depending on A_0, A_1,..., A_n and M_i, then the limit is called the line integral of $f(x, y)$ along the arc $\overset{\frown}{AB}$ and is denoted by

$$\int\limits_{\overset{\frown}{AB}} f(x, y) ds.$$

In this case, we say that f is integral along the arc $\overset{\frown}{(}AB)$.

Remark

- If $f(x, y)$ is piece-wise smooth in (smooth) $\overset{\frown}{AB}$, then f is integrable along $\overset{\frown}{AB}$.
- The integral does not depend on the direction of $\overset{\frown}{AB}$.
- The arc length $\overset{\frown}{AB}$ is $\ell = \int\limits_{\overset{\frown}{AB}} ds$.

4.1.2. How to compute

- If $\overset{\frown}{AB}$ is defined by $y = y(x)$, where $a \leq x \leq b$, then

$$\int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_a^b f(x, y(x))\sqrt{1 + (y'(x))^2}dx.$$

- If $\overset{\frown}{AB}$ is defined by $x = x(y)$, where $c \leq y \leq d$, then

$$\int\limits_{\overset{\frown}{AB}} f(x,y)ds = \int\limits_c^d f(x(y), y)\sqrt{1 + (x'(y))^2}dy.$$

- If $\overset{\frown}{AB}$ is defined by $x = x(t)$, $y = y(t)$, where $\alpha \leq t \leq \beta$, then

$$\int_{\overset{\frown}{AB}} f(x,y)ds = \int_{\alpha}^{\beta} f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

- Sometimes, we need to split AB into smaller arcs and then compute the integral in each smaller integral.

- The line integral of $f(x, y, z)$ along the arc $\widehat{AB}$ is three dimensional space is defined similarly.
- If $\widehat{AB}$ has parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$, where $\alpha \leq t \leq \beta$, then

$$\int\limits_{\widehat{AB}} f(x, y, z)ds = \int\limits_{\alpha}^{\beta} f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

Example (Final 20152)

Evaluate the line integral $\int\limits_C (x+y)ds$, where C is defined by $x = 2 + 2\cos t$, $y = 2\sin t$, where $0 \leq t \leq \pi$.

- $I = \int\limits_0^\pi (2 + 2\cos t + 2\sin t)\sqrt{(-2\sin t)^2 + (2\cos t)^2}dt = 4\int\limits_0^\pi (1 + \cos t + \sin t)dt = 4\pi + 8.$

Example (Final 20182)

Evaluate the line integral $\int_C (y^2+1)\,ds$, where C is the astroid curve $x^{2/3}+y^{2/3}=1$ in the first quadrant, connecting the two points $A(1,0)$ and $B(0,1)$.

- The astroid can be parametrized by: $x=\cos^3 t, \quad y=\sin^3 t, \quad t\in\left[0,\frac{\pi}{2}\right]$.
  At $t=0$, $(x,y)=(1,0)$, and at $t=\frac{\pi}{2}$, $(x,y)=(0,1)$, matching points A and B.
- We have
  $$\frac{dx}{dt}=-3\cos^2 t\sin t, \quad \frac{dy}{dt}=3\sin^2 t\cos t.$$
  Hence $ds=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt=3\cos t\sin t\sqrt{\cos^2 t+\sin^2 t}\,dt=3\cos t\sin t\,dt$.
- We have $\int_C (y^2+1)\,ds=\int_0^{\pi/2}(\sin^6 t+1)\cdot 3\cos t\sin t\,dt=$
  $3\left[\int_0^{\pi/2}\sin^7 t\cos t\,dt+\int_0^{\pi/2}\sin t\cos t\,dt\right]=\dfrac{3}{8}+\dfrac{3}{2}=\dfrac{15}{8}$.

4.2.1. Definition

- Let $P(x, y)$, $Q(x, y)$ be two functions defined over the arc $\overparen{AB}$, with the direction from A to B.
- Divide $\overparen{AB}$ into n smaller arcs by points $A = A_0$, A_1, ..., $A_n = B$. Let $\overrightarrow{A_{i-1}A_i} = (\Delta x_i, \Delta y_i)$.
- In each arc $\overparen{A_{i-1}A_i}$, take a point $M_i(x_i^*, y_i^*)$ and define the sum $\sum\limits_{i=1}^{n} P(M_i)\Delta x_i + Q(M_i)\Delta y_i$.
- If $\max|\Delta x_i| \to 0$, $\max|\Delta y_i| \to 0$, and the sum $\sum_{i=1}^{n} P(M_i)\Delta x_i + Q(M_i)\Delta y_i$ approaches to a limit, not depending on A_i and M_i, then the limit is called the integral of $P(x, y)$, $Q(x, y)$ along the arc $\overparen{AB}$, and is denoted by

$$\int\limits_{\overparen{AB}} P(x, y)dx + Q(x, y)dy.$$

Remark

- Line integrals depend on the direction along the arc $\overset{\frown}{AB}$:

$$\int_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy = -\int_{\overset{\frown}{BA}} P(x,y)dx + Q(x,y)dy.$$

- (Physical meaning) Let an object M move a long curve L from A to B under the force $\vec{F} = \vec{F}(M)$, with $\vec{F}(M) = P(M)\vec{i} + Q(M)\vec{j}$. Then the work done by $\vec{F}$ along the curve from A to B is

$$\int_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy.$$

4.1.2. How to compute

- If $\overparen{AB}$ is defined by $y = y(x)$, where $a \leq x \leq b$, then

$$\int\limits_{\overparen{AB}} P(x,y)dx + Q(x,y)dy = \int\limits_a^b \left(P(x,y(x)) + Q(x,y(x))y'(x)\right) dx.$$

- If $\overparen{AB}$ is defined by $x = x(y)$, where $c \leq y \leq d$, then

$$\int\limits_{\overparen{AB}} P(x,y)dx + Q(x,y)dy = \int\limits_c^d \left(P(x(y),y)x'(y) + Q(x(y),x)\right) dy.$$

How to compute

- If $\overset{\frown}{AB}$ has parametric equations $x = x(t)$, $y = y(t)$, where $\alpha \leq t \leq \beta$, then

$$\int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy = \int\limits_{\alpha}^{\beta} \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right) dt.$$

- Sometimes it is useful to divide the arc AB into smaller arcs and compute the integral over each smaller arc.

Example (Final 20181)

Evaluate $\int\limits_C ydx - 2xdy$, where C is defined by $y = \sin x$, from $O(0,0)$ to $A(\pi,0)$.

$$I = \int\limits_0^\pi \sin xdx - 2x\cos xdx = -\cos x|_0^\pi - 2x\sin x|_0^\pi + \int_0^\pi 2\sin xdx = 6.$$

Example (Final 20212)

Evaluate

$$\int_C e^{x^2+5y}(2xy\,dx + (1+5y)\,dy),$$

where C is the part of the curve $y = x^3$ from $O(0,0)$ to $B(1,1)$.

$$\int_C e^{x^2+5y}(2xy\,dx + (1+5y)\,dy) = \int_0^1 e^{x^2+5x^3}\left(2x(x^3) + (1+5x^3)3x^2\right)dx$$

$$= \int_0^1 e^{x^2+5x^3}(3x^2 + 2x^4 + 15x^5)dx = \left(x^3 e^{x^2+5x^3}\right)\Big|_0^1 = e^6.$$

Example (Final 20182)

Evaluate the line integral $\int\limits_C \frac{dx+dy}{|x|+|y|}$ along the circle C $x^2+y^2=1$ with counterclockwise direction.

- Let $x=\cos t$, $y=\sin t$, where $0\le t\le 2\pi$.
- $I=\int\limits_0^{2\pi}\frac{-\sin t+\cos t}{|\sin t|+|\cos t|}dt=\int\limits_0^{\pi/2}+\int\limits_{\pi/2}^{\pi}+\int\limits_{\pi}^{3\pi/2}+\int\limits_{3\pi/2}^{2\pi}$
- $I_2+I_4=\int\limits_{\pi/2}^{\pi}\frac{-\sin t+\cos t}{\sin t-\cos t}dt+\int\limits_{3\pi/2}^{2\pi}\frac{-\sin t+\cos t}{-\sin t+\cos t}dt=0.$
- $I_1+I_3=\int\limits_0^{\pi/2}\frac{-\sin t+\cos t}{\sin t+\cos t}dt+\int\limits_{\pi}^{3\pi/2}\frac{-\sin t+\cos t}{-\sin t-\cos t}dt=$

$$\int\limits_0^{\pi/2}\frac{-\sin t+\cos t}{\sin t+\cos t}dt+\int\limits_0^{\pi/2}\frac{-\sin(t+\pi)+\cos(t+\pi)}{-\sin(t+\pi)-\cos(t+\pi)}dt=0.$$

Theorem ("Fundamental of line integrals")

If $Pdx + Qdy$ is the differential of $u(x, y)$, then

$$\int\limits_{\overset{\frown}{AB}} Pdx + Qdy = u(B) - u(A).$$

- $P = \dfrac{\partial u}{\partial x}$, $Q = \dfrac{\partial u}{\partial y}$
- Assume that $\overset{\frown}{AB}$ has parametric equations $x = x(t)$, $y = y(t)$, where $\alpha \le t \le \beta$. Then
- $\int\limits_{\overset{\frown}{AB}} Pdx + Qdy = \int\limits_{\alpha}^{\beta} \left(\dfrac{\partial u}{\partial x}(x(t), y(t)) \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}(x(t), y(t)) \dfrac{dy}{dt} \right) dt = \int\limits_{\alpha}^{\beta} \dfrac{du(x(t), y(t))}{dt} dt =$
$u(x(\beta), y(\beta)) - u(x(\alpha), y(\alpha)) = u(B) - u(A).$

Example (Final 20212)

Evaluate

$$\int_C e^{x^2+5y}(2xy\,dx + (1+5y)\,dy),$$

where C is the part of the curve $y = x^3$ from $O(0,0)$ to $B(1,1)$.

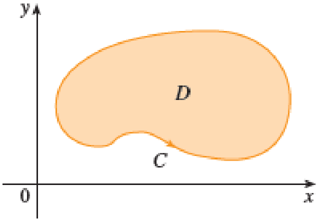
- Let $u(x,y) = ye^{x^2+5y}$. Then $u'_x = 2xye^{x^2+5y}$ and $u'_y = (1+5y)e^{x^2+5y}$.
- Hence $du = e^{x^2+5y}(2xy\,dx + (1+5y)\,dy)$.
- Therefore

$$\int_C e^{x^2+5y}(2xy\,dx + (1+5y)\,dy) = u(B) - u(O) = e^6.$$

Line integrals in space

- The line integral
$$\int\limits_{(\overset{\frown}{AB})} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$
is defined similarly as the line integral $\int\limits_{\overset{\frown}{AB}} P(x,y)dx + Q(x,y)dy$.
- If the curve $\overset{\frown}{AB}$ has parametric equations $x = x(t)$, $y = y(t)$, $z = x(t)$, where $\alpha \leq t \leq \beta$.
$\int\limits_{\overset{\frown}{AB}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$
$\int\limits_{\alpha}^{\beta} P(x(t),y(t),z(t))x'(t)dt + \int\limits_{\alpha}^{\beta} Q(x(t),y(t),z(t))y'(t)dt + \int\limits_{\alpha}^{\beta} R(x(t),y(t),z(t))z'(t)dt.$

4.2.3. Green's Formula



Let C be a closed simple piecewise-smooth curve with the positive orientation. Let D be the region bounded by C. If P and Q are two functions with continuous derivatives in an open region containing D, then

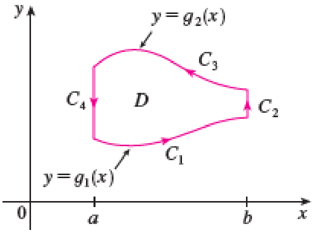
$$\boxed{\int\limits_C P dx + Q dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy}. \tag{1}$$

Proof when D is a trapezium.

$$\int_{C_1} P dx = \int_a^b P(x, g_1(x)) dx$$

$$\int_{C_3} P dx = -\int_a^b P(x, g_2(x)) dx$$

$$\int_{C_2} P dx = 0 = \int_{C_4} P dx$$



- $$\int_C P dx = \int_a^b P(x, g_1(x)) dx - \int_a^b P(x, g_2(x)) dx$$

- $$\iint_D \frac{\partial P}{\partial y} dx dy = \int_a^b dx \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))] dx$$

- So

$$\int\limits_C P dx = -\iint\limits_D \frac{\partial P}{\partial y} dxdy \tag{2}$$

- Similarly

$$\int\limits_C Q dy = \iint\limits_D \frac{\partial Q}{\partial x} dxdy$$

Example(Final 20161)

Evaluate $\int\limits_L (xy + x + y)dx + (2x + 3)dy$, where $L = ABCA$ with $A(0,0)$, $B(1,1)$, and $C(0,2)$.

- By Green's formula: $I = \iint_D (2 - 1 - x)dxdy$, với $D : 0 \leq x \leq 1$, $x \leq y \leq 2 - x$.
- $I = \int\limits_0^1 dx \int\limits_x^{2-x} (1 - x)dy = 2\int\limits_0^1 (1 - x)^2 dx = \dfrac{2}{3}$.

Example (Final 20192)

Evaluate $\int\limits_C (2e^x + y^2)dx + (x^4 + e^y)dy$, where C is the curve $y = \sqrt[4]{1 - x^2}$ from $A(-1, 0)$ to $B(1, 0)$.

- $P = 2e^x + y^2$, $Q = x^4 + e^y$. Adding the segment BA. Green's formula gives

$$J = \int\limits_{BA \cup C} Pdx + Qdy = -\iint\limits_D (4x^3 - 2y)dxdy,$$

  với $D : -1 \leq x \leq 1,\ 0 \leq y \leq \sqrt[4]{1 - x^2}$.
- D has Oy as a symmetrical axis and x^3 is an odd function in x. So $J = 2\iint\limits_D ydxdy = 2\int\limits_{-1}^{1} dx \int_0^{\sqrt[4]{1-x^2}} ydy = \int\limits_{-1}^{1} \sqrt{1 - x^2}dx = \pi/2.$
- $\int\limits_{AB} Pdx + Qdy = \int\limits_{-1}^{1} 2e^x dx = 2e - \dfrac{2}{e}.$
- $I = \pi/2 + 2e - 2/e.$

Corollary

Area S of a region D with boundary C with positive direction is

$$S = \int_C xdy = -\int_C ydx = \frac{1}{2}\int_C xdy - ydx.$$

Some problems

- (Final 20192) Evaluate $\int\limits_C (e^x + y^2)dx + x^2 e^y dy$, where C is the boundary of the region bounded by $y = 1 - x^2$ and $y = 0$ with positive direction.
- (Final 20181) Evaluate $\int\limits_{ABCDA} \dfrac{2ydx - xdy}{|x| + |y|}$, with $A(1,0)$, $B(0,1)$, $C(-1,0)$, $D(0,-1)$.
- (Final 20171) Evaluate $\int\limits_C (\arctan\dfrac{x}{y})(xdx + ydy)$, where C is the curve parameterized by $x = 3 + \sqrt{2}\cos t$, $y = 3 + \sqrt{2}\sin t$ with $t \in [-\dfrac{3\pi}{4}, \dfrac{\pi}{4}]$.

4.2.4. Path independence for line integrals

Theorem

Let P, Q be two continuous functions with continuous first derivatives in an open region D. Then the following claims are equivalent:

1. $\int\limits_{\overset{\frown}{AB}} Pdx + Qdy$ only depends on A and B and does not depend on any smooth curve that lies inside D and connects A to B.
2. $\oint\limits_{C} Pdx + Qdy = 0$, for all closed simpled and piecewise-smooth curves C in D.
3. $Q'_x(M) = P'_y(M)$ for all $M \in D$.
4. The expression is $Pdx + Qdy$ the differential of a function $u(x, y)$ defined over D.

- Proof: (4) $\overset{Schwarz}{\Rightarrow}$ (3) $\overset{Green}{\Rightarrow}$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4).
- $1 \Rightarrow (4)$: $u(x,y) = \int\limits_A^M Pdx + Qdy + C$, ở đây $A(x_0, y_0)$ Fix $M(x,y)$ in D.

Corollary

If $D = \mathbb{R}^2$ then $Pdx + Qdy$ is the differential of the function $u(x,y)$ given by

$$u(x,y) = \int\limits_{x_0}^{x} P(t, y_0)dt + \int\limits_{y_0}^{y} Q(x,t)dt + C,$$

or

$$u(x,y) = \int\limits_{y_0}^{y} P(x_0, t)dt + \int\limits_{x_0}^{x} Q(t,y)dt + C,$$

Example (Final 20152)

Evaluate $\int\limits_C e^{2x+y^2}[(1+2x)dx + 2xydy]$, where C is the curve $x = y^3$ from $O(0,0)$ to $N(1,1)$.

- $P = (1+2x)e^{2x+y^2}$, $Q = 2xye^{2x+y^2}$.
- $Q'_x = 2ye^{2x+y^2} + 2xy \cdot 2e^{2x+y^2}$.
- $P'_y = (1+2x)2ye^{2x+y^2} = Q'_x$.
- The integral is path independent. Choose the path OAN with $A = (1,0)$.
- $I = \int\limits_{OA} + \int\limits_{AN} = \int\limits_0^1 (1+2x)e^{2x}dx + \int\limits_0^1 e^{2+y^2} 2ydy = e^2 + (e^3 - e^2) = e^3$.

Example (Final 20171)

Find a such that

$$\left(y^3+\frac{y}{1+x^2y^2}\right)dx+\left(axy^2+\frac{x}{1+x^2y^2}\right)dy$$

is the differential of a function $u(x,y)$. Find $u(x,y)$.

- $P=y^3+\dfrac{y}{1+x^2y^2}$, $Q=axy^2+\dfrac{x}{1+x^2y^2}$.
- $P'_y=3y^2+\dfrac{(1+x^2y^2)-2x^2y^2}{1+x^2y^2}$, $Q'_x=ay^2+\dfrac{(1+x^2y^2)-2x^2y^2}{1+x^2y^2}$
- $P'_y=Q'_x \Leftrightarrow a=3$.
- $u=\int\limits_0^x P(x,0)dx+\int\limits_0^y Q(x,y)dy+C=0+\int\limits_0^y(3xy^2+\dfrac{x}{1+x^2y^2})dy+C=xy^3+\arctan(xy)+C$

Example (Final20162)

Evaluate the integral $\int\limits_C (y^2 - e^y \sin x)dx + (x^2 + 2xy + e^y \cos x)dy$, where C is the semicircle $x = \sqrt{2y - y^2}$, from $O(0,0)$ to $P(0,2)$.

- $P = y^2 - e^y \sin x$, $Q = x^2 + 2xy + e^y \cos x$.
- $P'_y = 2y - e^y \sin x$, $Q'_x = 2x + 2y - e^y \sin x$.
- $I = I_1 + \int\limits_C x^2 dy$, where $I_1 = \int\limits_C Pdx + Q_1 dy$ is path independent, with $Q_1 = 2xy + e^y \cos x$.
- Evaluate I_1: choose the path OP. Then $I_1 = \int\limits_0^2 e^y dy = e^2 - 1$.
- Evaluate $\int\limits_C x^2 dy = \int\limits_0^2 (2y - y^2)dy = 4/3$.
- $I = e^2 + 1/3$.