<u>CHAPTER 2:</u> RANDOM VARIABLES and PROBABILITY DISTRIBUTIONS

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Problem 2.1. A civil engineer is studying a left-turn lane that is long enough to hold 7 cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x + 1)(8 - x)$ for $x = 0, 1, \ldots, 7$.
(a) Find the probability mass function (PMF) of X.
(b) Find the probability that X will be at least 5.
(c) Find $\mathbb{E}[X]$, Var(X), σ_X.
(d) Find the mode of X.
(e) Find the median of X.

Problem 2.2. A multiple choice test has 4 questions, each of which has 4 possible answers, only one of which is correct. If you just randomly guess on each of the 4 questions, what is the probability that you get exactly 2 questions correct? Assume that you answer all and you will get (+5) points for each correct question, (-2) points for each wrong question. Let X be the total points that you get. Find the probability mass function (PMF) of X and the expected value of X.

Problem 2.3. Suppose that a box contains seven red balls and three blue balls. If five of them are selected at random, without replacement, determine the PMF of the number of red balls that will be obtained.

Problem 2.4. There are two boxes, the first box consists of 7 red balls and 3 white balls, the second box consists of 5 red balls and 2 white balls. Draw randomly 2 balls from the first box and then put them into the second box. Then, continue to draw randomly 2 balls from the second one. Let X be the number of white balls out of these 2 balls. Find the probability mass distribution (PMF) of X and $\mathbb{E}[X]$.

Problem 2.5. Three balls are randomly chosen with replacement from an urn containing 5 blue, 4 red, and 2 yellow balls. Let X denote the number of red balls chosen. Find the probability mass distribution (PMF) of X and $\mathbb{E}[X]$.

Problem 2.6. Two balls are chosen randomly from an urn containing 8 white balls, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose $\$1$ for each white ball selected. Let X denote our winnings. Find the probability mass distribution (PMF) of X and $\mathbb{E}[X]$.

Problem 2.7. When someone presses "SEND" on a cellular phone, the phone attempts to set up a call by transmitting a "SETUP" message to a nearby base station. The phone waits for a response and if none arrives within 0.5 seconds, it tries again. If it doesnt get a response after $n = 6$ tries, the phone stops transmitting messages and generates a busy signal.
(a) Draw a tree diagram that describes the call setup procedure.
(b) If all transmissions are independent and the probability is p that a "SETUP" message will get through, what is the PMF of K, the number of messages transmitted in a call attempt?
(c) What is the probability that the phone will generate a busy signal?

Problem 2.8. Find the value of the constant c for which the given function, $p_X(x)$, is the probability mass function (PMF) of certain discrete random variable, where

$$p_X(x) = \begin{cases} c\left(\dfrac{2}{3}\right)^x & x = 1, 2, 3, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2.9. Let $p_X(n)$ be the probability mass function (PMF) of certain discrete random variable

$$p_X(n) = \begin{cases} k\dfrac{3^{n-1}}{n!} & n = 1, 2, 3, \dots \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant k.
(c) Find $\mathbb{E}[X]$ and $\text{Var}(X)$.

Problem 2.10. Let X be the number of insurance claims that a person makes in a year. Assume that X can take the values $0, 1, 2, 3, \dots$ with probabilities $\mathbb{P}(X = 0) = \dfrac{2}{3}$, $\mathbb{P}(X = 1) = \dfrac{2}{9}, \dots, \mathbb{P}(X = n) = \dfrac{2}{3^{n+1}}, \dots$. Find the expected number of claims this person makes in a year.

Problem 2.11. The random variable N has PMF

$$p_N(n) = \begin{cases} c\left(\dfrac{1}{2}\right)^n & n = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?
(b) What is $\mathbb{P}(N \leq 1)$?
(c) What is the cumulative distribution function (CDF) of N?
(d) Find $\mathbb{E}[N]$, Var(N).
(e) Find the mode of N.
(f) Find the median of N.

Problem 2.12. The random variable V has PMF

$$p_V(v) = \begin{cases} cv^2 & v = 1, 2, 3, 4 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?
(b) Find $\mathbb{P}\Big(V \in \{u^2 | u = 1, 2, 3, \ldots\}\Big)$
(c) Find the probability that V is an even number.
(d) Find $\mathbb{P}(V > 2)$.
(e) Find $\mathbb{E}[V]$, Var(V).

Problem 2.13. The random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x \leq -1, \\ 0.2 & -1 < x \leq 0, \\ 0.7 & 0 < x \leq 1, \\ 1 & x > 1. \end{cases}$$

(a) Draw a graph of the CDF.
(b) Write the PMF of X and the probability distribution of X.
(c) Find $\mathbb{E}[X]$, Var(X).
(e) Find the mode of X.
(f) Find the median of X.

Problem 2.14. The random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x \leq -3, \\ c & -3 < x \leq 5, \\ 0.8 & 5 < x \leq 7, \\ 1 & x > 7. \end{cases}$$

(a) Find the constant c given that $\mathbb{P}(X = -3) = \mathbb{P}(X = 5)$.
(b) Draw a graph of the CDF.
(c) Write the PMF of X and the probability distribution (table) of X. Sketch the graph of the PMF of X.

(d) Find $\mathbb{E}[X]$, Var(X).
(e) Find the mode of X.
(f) Find the median of X.
(g) Define a new random variable by $Y = |X|$. Write the PMF and the CDF of Y. Find $\mathbb{E}[Y]$, Var(Y), the mode and the median of Y.

Problem 2.15. The number of buses that arrive at a bus stop in T minutes is a Poisson random variable B with expected value $T/5$.
(a) What is the PMF of B, the number of buses that arrive in T minutes?
(b) What is the probability that in a two-minute interval, three buses will arrive?
(c) What is the probability of no buses arriving in a 10-minute interval?
(d) How much time should you allow so that with probability 0.99 at least one bus arrives?

Problem 2.16. You roll a fair die repeatedly until a number larger than 4 is observed. Let N be the total number of times that you roll the die. Find the PMF of N.

Problem 2.17. The CDF of a continuous random variable X is

$$F_X(x) = \begin{cases} 0 & x \le -5, \\ c(x+5)^2 & -5 < x \le 7, \\ 1 & x > 7. \end{cases}$$

(a) What is c?
(b) Find $\mathbb{P}(X \ge 4)$, $\mathbb{P}(-3 \le X < 0)$, $\mathbb{P}(|X| > 6)$.
(c) Find the value of a such that $\mathbb{P}(X \ge a) = \dfrac{2}{3}$.
(d) Write the PDF of X.
(e) Find $\mathbb{E}[X]$, Var(X), σ_X.
(f) Find the mode of X.
(g) Find the median of X.

Problem 2.18. The probability density function (PDF) of a continuous random variable X is

$$f_X(x) = \begin{cases} \dfrac{k}{x+1} & x \in [0, e-1], \\ 0 & x \notin [0, e-1]. \end{cases}$$

(a) Find the constant k.
(b) Find $\mathbb{P}(0.7 \le X \le 1.7)$.
(c) Find $\mathbb{E}[X]$, Var(X).
(d) Find the CDF of X.
(e) Determine the CDF and PDF of the random variable $Y = \ln(1+X)$.
(f) Determine the CDF and PDF of the random variable $Z = -2\ln Y$.

Problem 2.19. The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} C\dfrac{\ln x}{x^3} & x \geq 1, \\ 0 & x < 1. \end{cases}$$

(a) Find the constant C.
(b) Find the CDF of X.

Problem 2.20. A continuous random variable X has the following PDF

$$f_X(x) = Ce^{-|x|}, \qquad x \in \mathbb{R}.$$

(a) Find the constant C.
(b) Find the CDF of X.
(c) Find $\mathbb{E}[X]$, Var(X).
(d) Find the probability that out of 5 independent trials of observing the value of X, there are exactly 3 times that X takes value in the interval $[2, 3]$.
(e) Find the conditional probability $\mathbb{P}(|X| \leq \ln 2 | X \leq \ln 2)$.
(f) Determine the CDF and PDF of the random variable $Y = X^2$.
(g) Determine the CDF and PDF of the random variable $Z = X^{2n}$ with $n \in \mathbb{N}^*$.
(h) Find $\mathbb{E}[Z]$.
(i) Determine the CDF and PDF of the random variable $U = e^X - 1$.

Problem 2.21. The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} C\,(20 - x) & x \in [0, 20], \\ 0 & x \notin [0, 20]. \end{cases}$$

(a) Find the normalizing constant C.
(b) Find $\mathbb{E}[\max\{10, X\}]$.
(c) Define $Y = \min\{10, X\}$. Determine the CDF of Y.

Problem 2.22. A continuous random variable X has the following PDF

$$f_X(x) = \begin{cases} Cxe^{-3x^2} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(a) Find the normalizing constant C.
(b) Find the CDF of X.
(c) Find $\mathbb{P}(0 < X < 1)$.

Problem 2.23. The CDF of a continuous random variable X is

$$F_X(x) = \begin{cases} 0 & x \leq 0, \\ \dfrac{1}{2} - k\cos x & 0 < x \leq \pi, \\ 1 & x > \pi. \end{cases}$$

(a) What is k?
(b) Find $\mathbb{P}\left(0 < X < \dfrac{\pi}{2}\right)$.
(c) Find $\mathbb{E}[X]$, $\mathrm{Var}(X)$.

Problem 2.24. The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} k\sin 3x & x \in \left(0, \dfrac{\pi}{3}\right), \\ 0 & x \notin \left(0, \dfrac{\pi}{3}\right). \end{cases}$$

(a) Find the constant k.
(b) Find $\mathbb{P}\left(\dfrac{\pi}{6} \leq X \leq \dfrac{\pi}{3}\right)$.
(c) Find $\mathbb{E}[X]$, $\mathrm{Var}(X)$.
(d) Find the CDF of X.

Problem 2.25. The CDF of a continuous random variable X is

$$F_X(x) = \begin{cases} 0 & x \leq -a, \\ A + B\arcsin\dfrac{x}{a} & -a < x \leq a, \\ 1 & x > a, \end{cases}$$

where a is a given positive constant.
(a) What are A and B?
(b) Find the PDF of X.

Problem 2.26. The CDF of a continuous random variable X is

$$F_X(x) = a + b\arctan x, \qquad x \in \mathbb{R}.$$

(a) What are a and b?
(b) Find the PDF of X.
(c) Find $\mathbb{P}(-1 < X < 1)$.
(d) Find the probability that out of 3 independent trials of observing the value of X, there are exactly 2 times (twice) that X takes value in the interval $(-1, 1)$.

Problem 2.27. A continuous random variable X has the CDF

$$f_X(x) = \frac{C}{e^x + e^{-x}}, \qquad x \in \mathbb{R}.$$

(a) Find the constant C.
(b) Find $\mathbb{E}[X]$.

Problem 2.28. The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} Ce^{-2\alpha x}\left(1-e^{-\alpha x}\right) & x \geq 0, \\ 0 & x < 0, \end{cases}$$

where α is a given positive constant.
(a) Find the normalizing constant C.
(b) Determine the CDF of X.
(c) Let $\alpha = 1$. Find $\mathbb{P}(-1 \leq X \leq 2.5)$ and $\mathbb{P}(X > 6)$.
(d) Define a new random variable by $Y = e^{-\alpha X}$. Determine the CDF and PDF of Y and then calculate $\mathbb{P}(|Y - 0.5| \geq 0.1)$.

Problem 2.29. The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} Ce^{-3x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(a) Find the normalizing constant C.
(b) Determine the CDF of X.
(c) Define a new discrete random variable by $Y = [X]$, where recall that $[x]$ denotes the interger part of the real number x, that is, $[x] = 0$ if $0 \leq x < 1$; $[x] = 1$ if $1 \leq x < 2$; $\ldots$; $[x] = k$ if $k \leq x < k+1$. Determine the PMF of Y and then calculate $\mathbb{E}[Y]$.

Problem 2.30. The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(a) Find $\mathbb{P}(X \geq 7)$.
(b) Determine the CDF of the random variable $Y = -2X + 5$.

Problem 2.31. The PDF of a continuous random variable X is

$$f_X(x) = \begin{cases} kx^2(1-x) & x \in [0,1], \\ 0 & x \notin [0,1]. \end{cases}$$

(a) Find k.
(b) Find the probability that out of 3 independent trials of observing the value of X, there is exactly one time (once) that X takes value in the interval $(0, \frac{1}{2})$.

Problem 2.32. The PDF of a continuous random variable X is

$$f_X(x) = Ae^{-(1+x+0.25x^2)}, \qquad x \in \mathbb{R}.$$

a) Find A.
(b) Find $\mathbb{P}(X > \mathbb{E}[X])$.