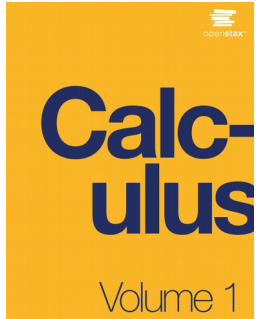
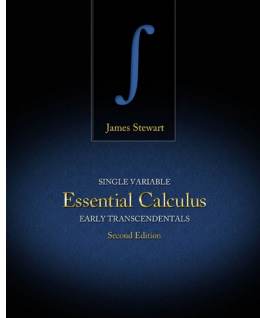


# Chapter 5: Integrals



5.1 Areas and Distances

5.2 The definite Integral

5.3 Evaluating Definite Integrals

5.4 The Fundamental Theorem of Calculus

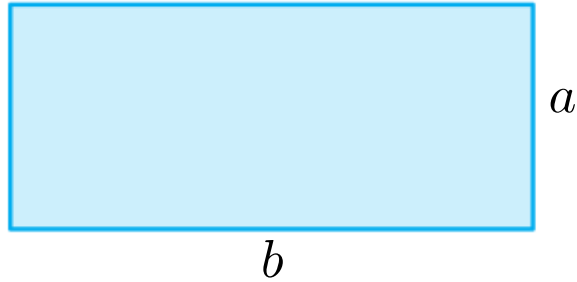
5.5 The Substitution Rule

**The pictures are taken from the books:**

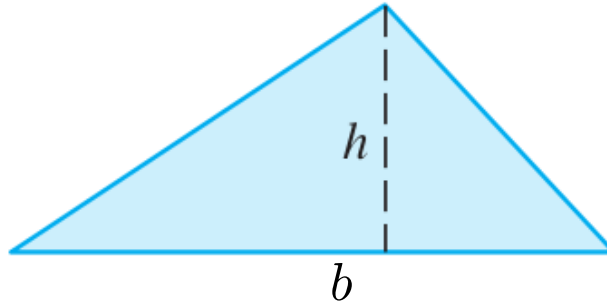
- [ 1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280  
2) G. Strang and E. J. Herman, Calculus 1, <https://openstax.org/details/books/calculus-volume-1> ]

## 5.1 Areas and Distances

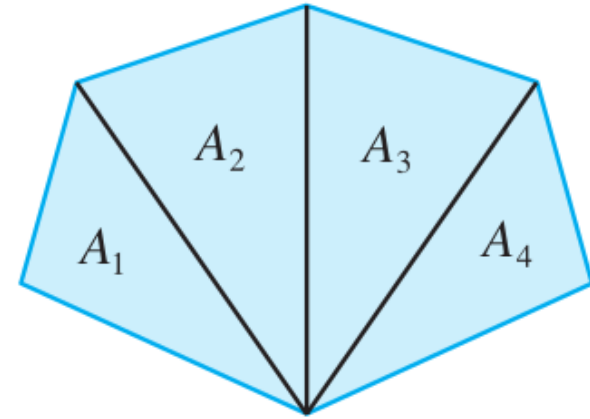
- Find the area of the geometric objects below.



$$A = ab$$



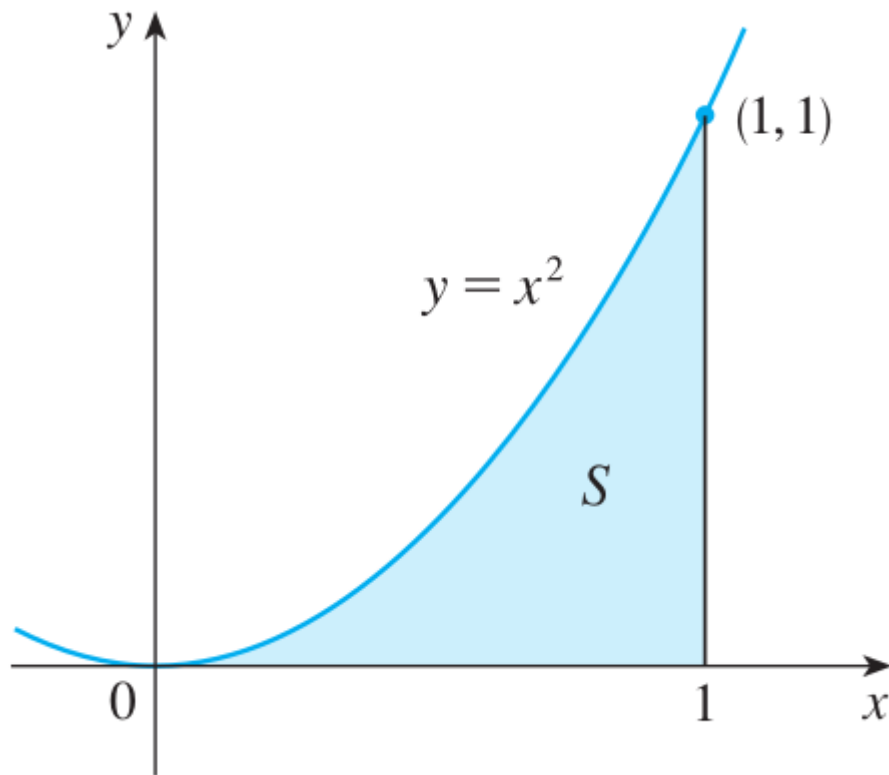
$$A = \frac{1}{2}ab$$



$$A = A_1 + A_2 + A_3 + A_4 + A_5$$

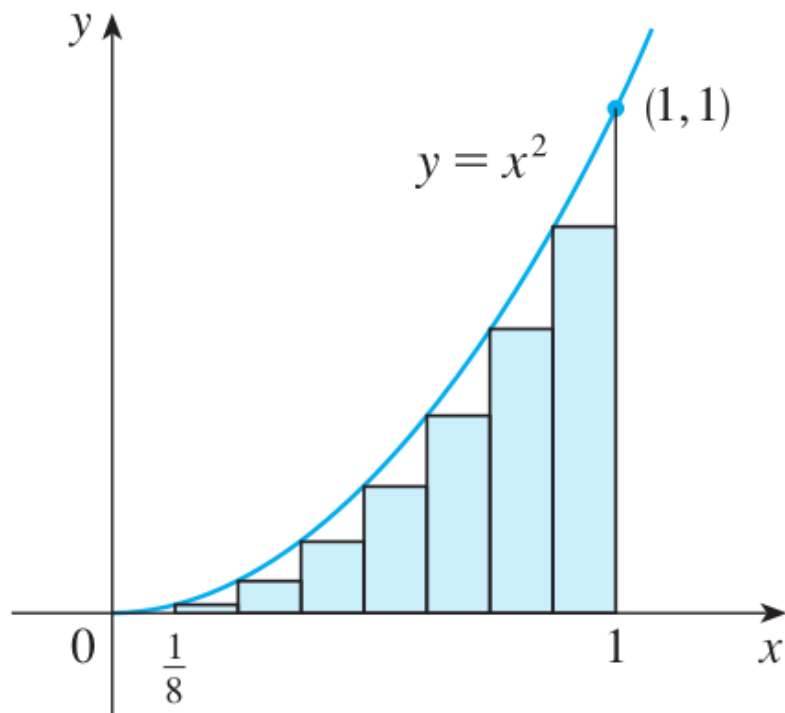
## 5.1 Areas and Distances

- Find the area of the region  $S$  that lies under the curve  $y = x^2$  from  $a = 0$  to  $b = 1$ .

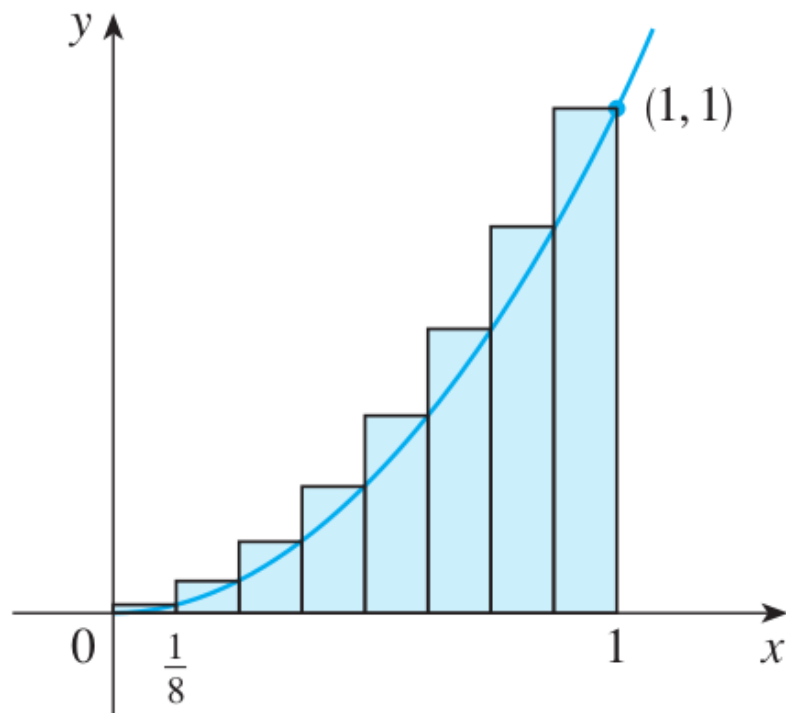


## 5.1 Areas and Distances

- Approximate  $S$  with  $n = 8$  **rectangles** of **equal width**:



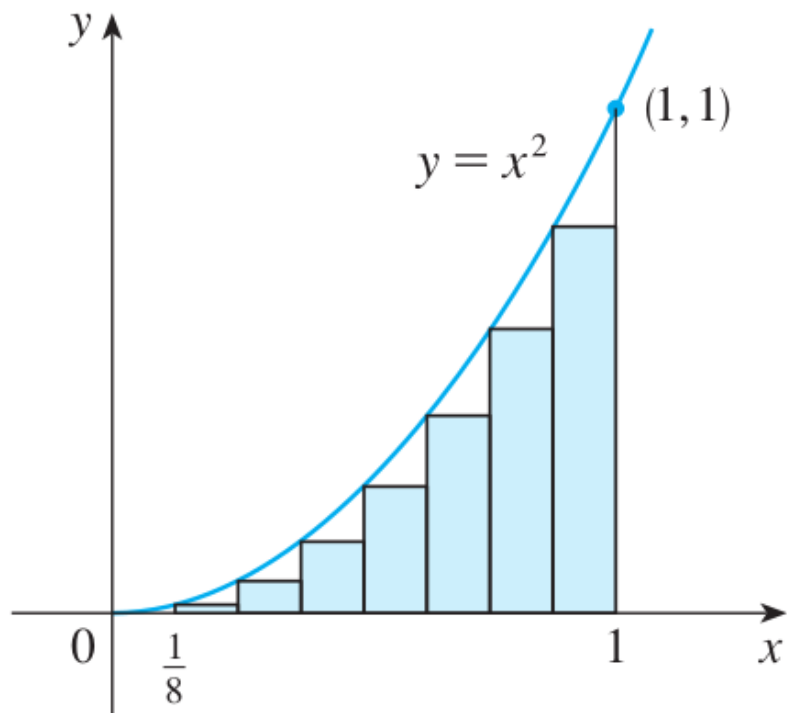
Left endpoints, area  $L_8$



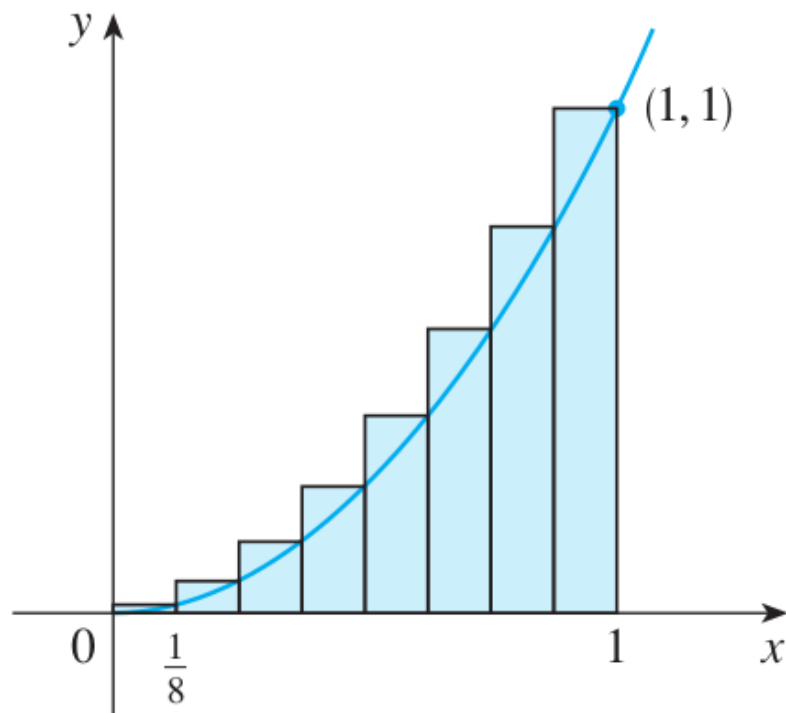
Right endpoints, area  $R_8$

## 5.1 Areas and Distances

$$L_8 = 0.2734375 < A < 0.3984375 = R_8$$



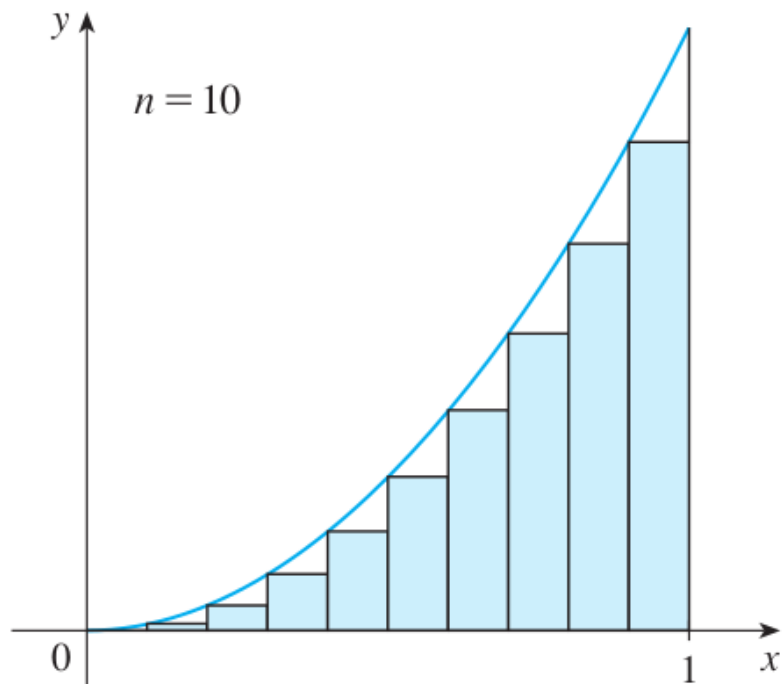
Left endpoints, area  $L_8$



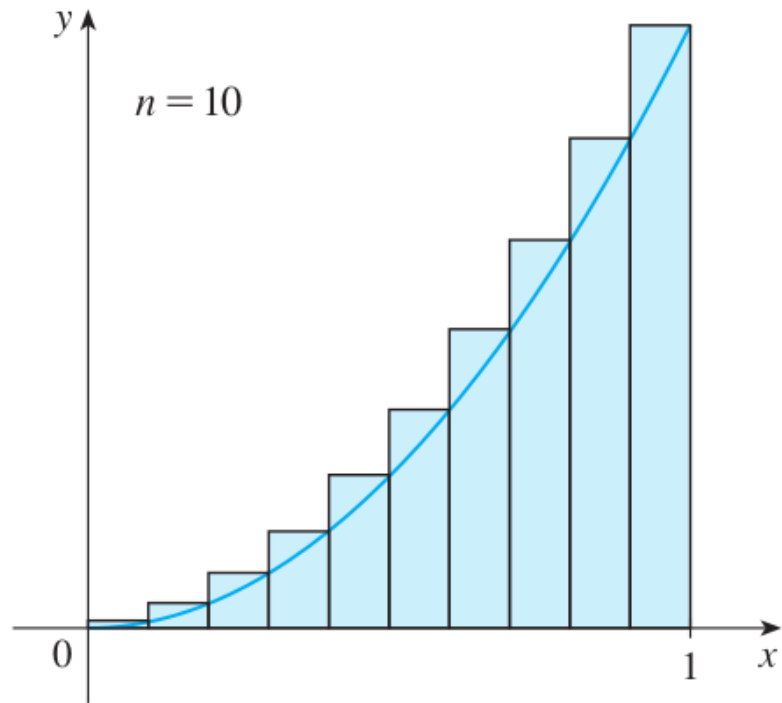
Right endpoints, area  $R_8$

## 5.1 Areas and Distances

$$L_{10} = 0.285 < A < 0.385 = R_{10}$$

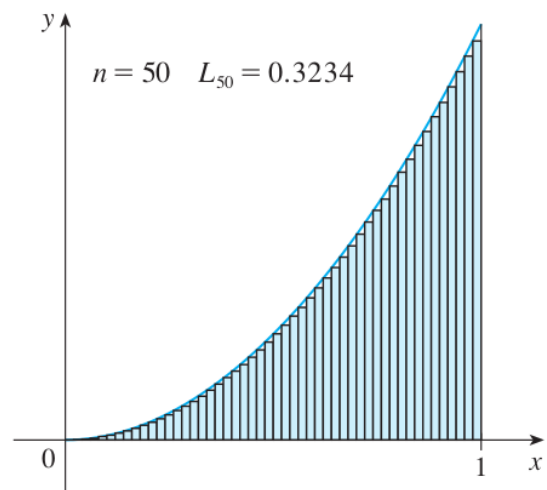
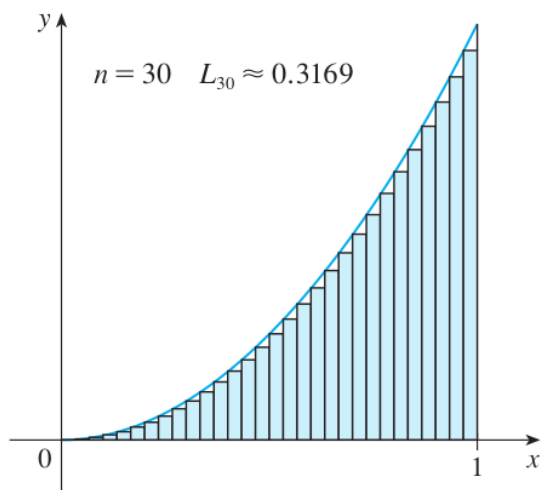
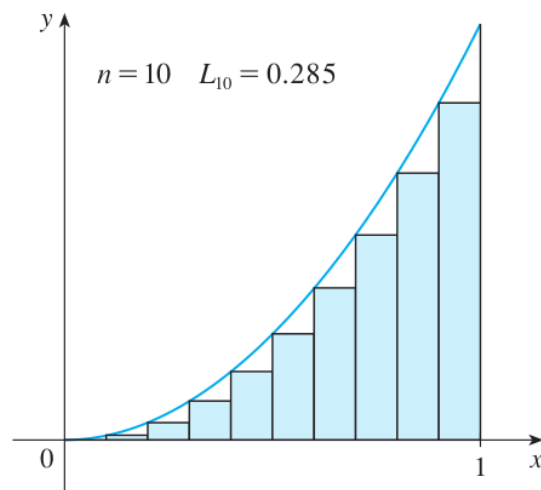
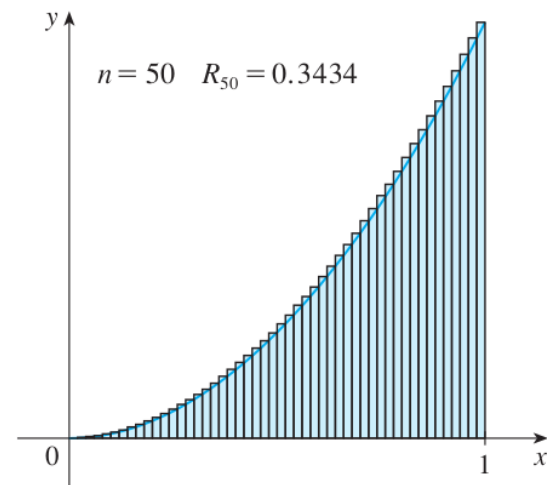
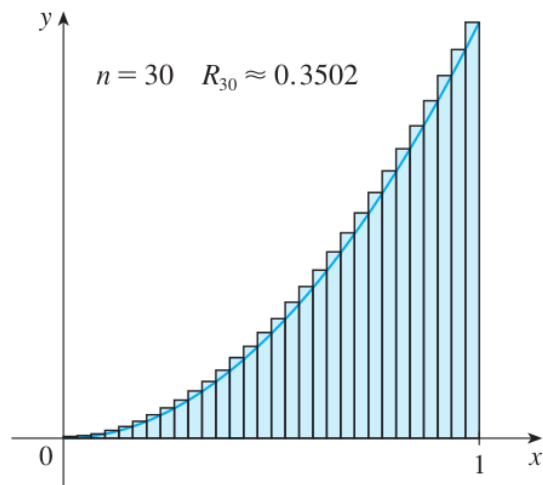
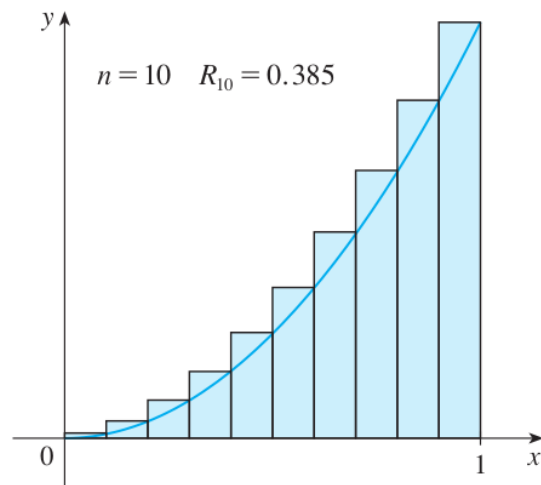


Left endpoints, area  $L_{10}$

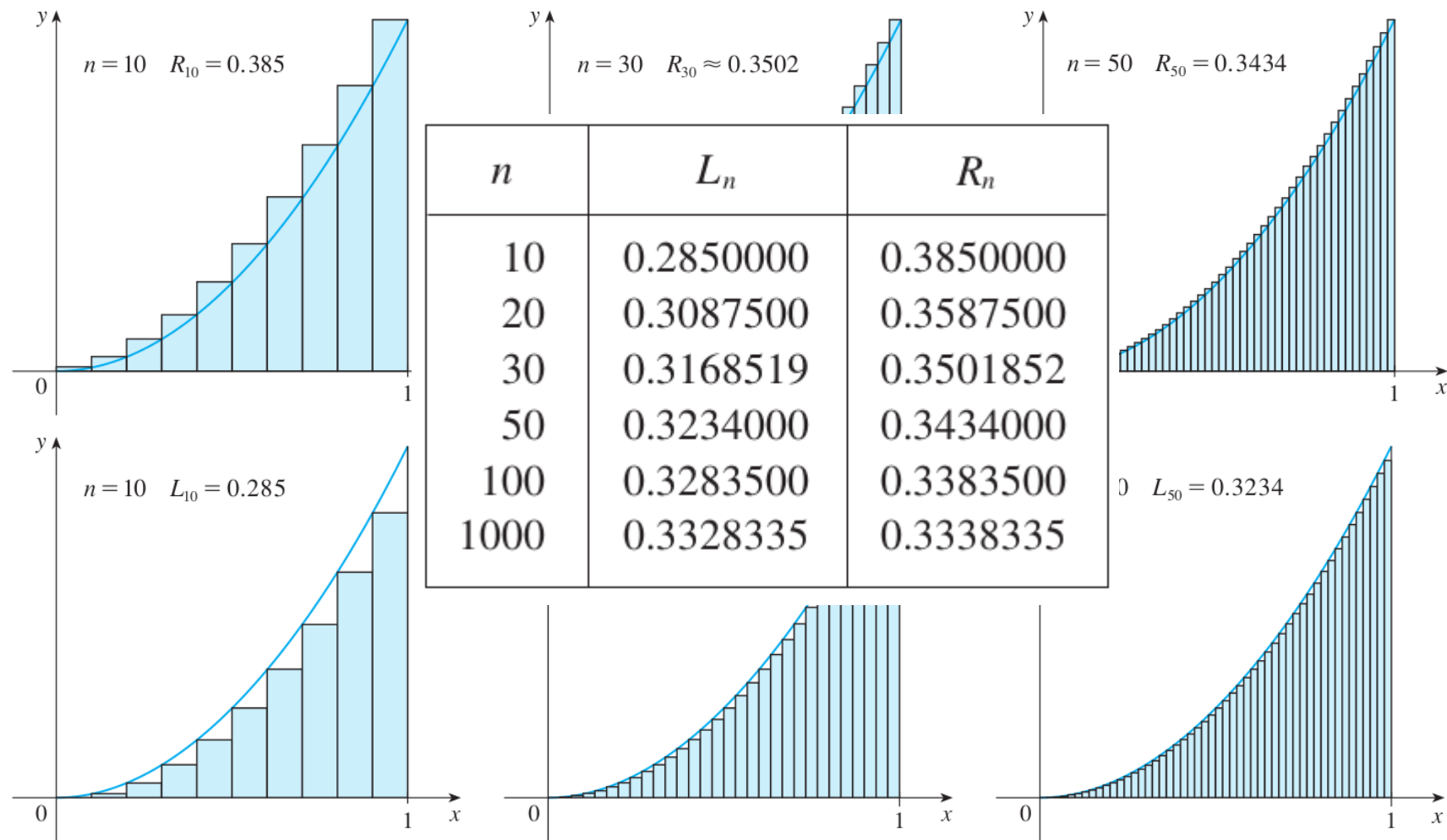


Right endpoints, area  $R_{10}$

## 5.1 Areas and Distances



## 5.1 Areas and Distances





## 5.1 Areas and Distances

- We identify

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n$$

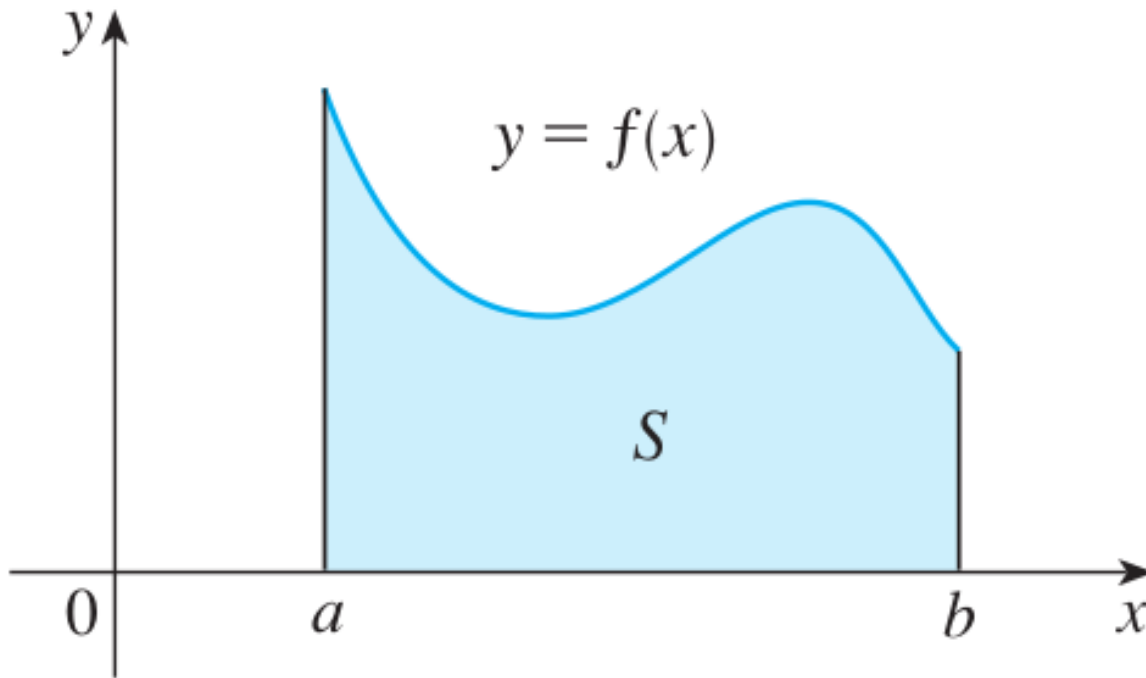
Example:

Calculate for  $y = x^2$ :

$$A = \lim_{n \rightarrow \infty} R_n$$

## 5.1 Areas and Distances

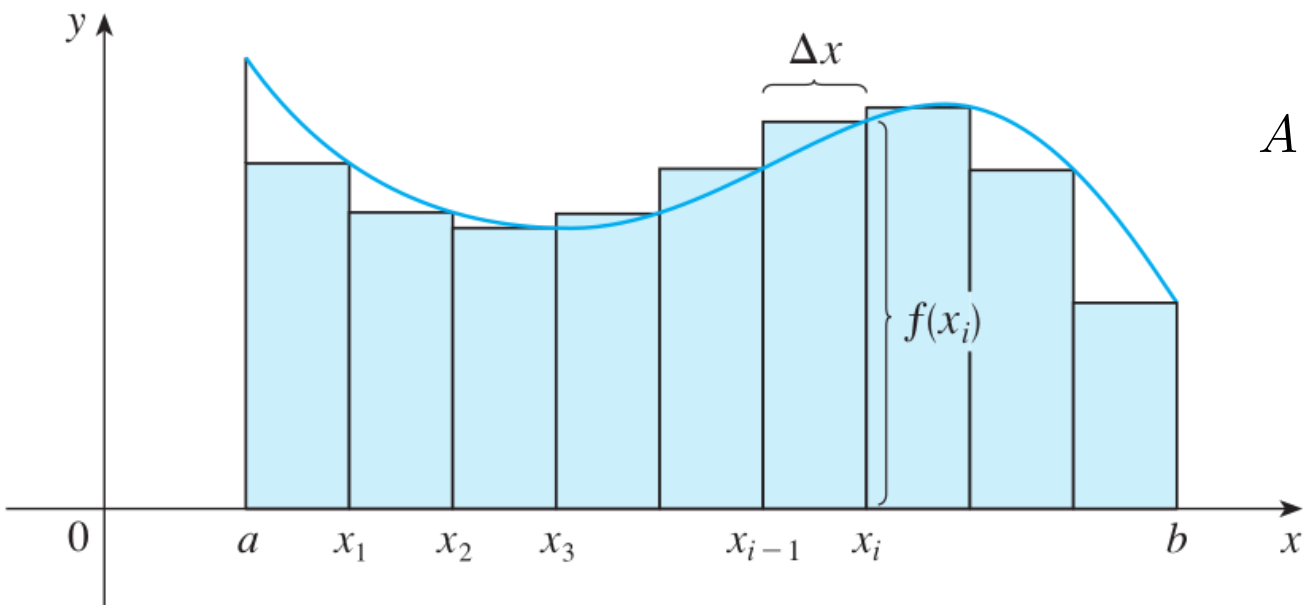
- Find the area of the region  $S$  that lies under the curve  $y = f(x)$  from  $a$  to  $b$ .



## 5.1 Areas and Distances

- Find the area of the region  $S$  that lies under the curve  $y = f(x)$  from  $a$  to  $b$ .

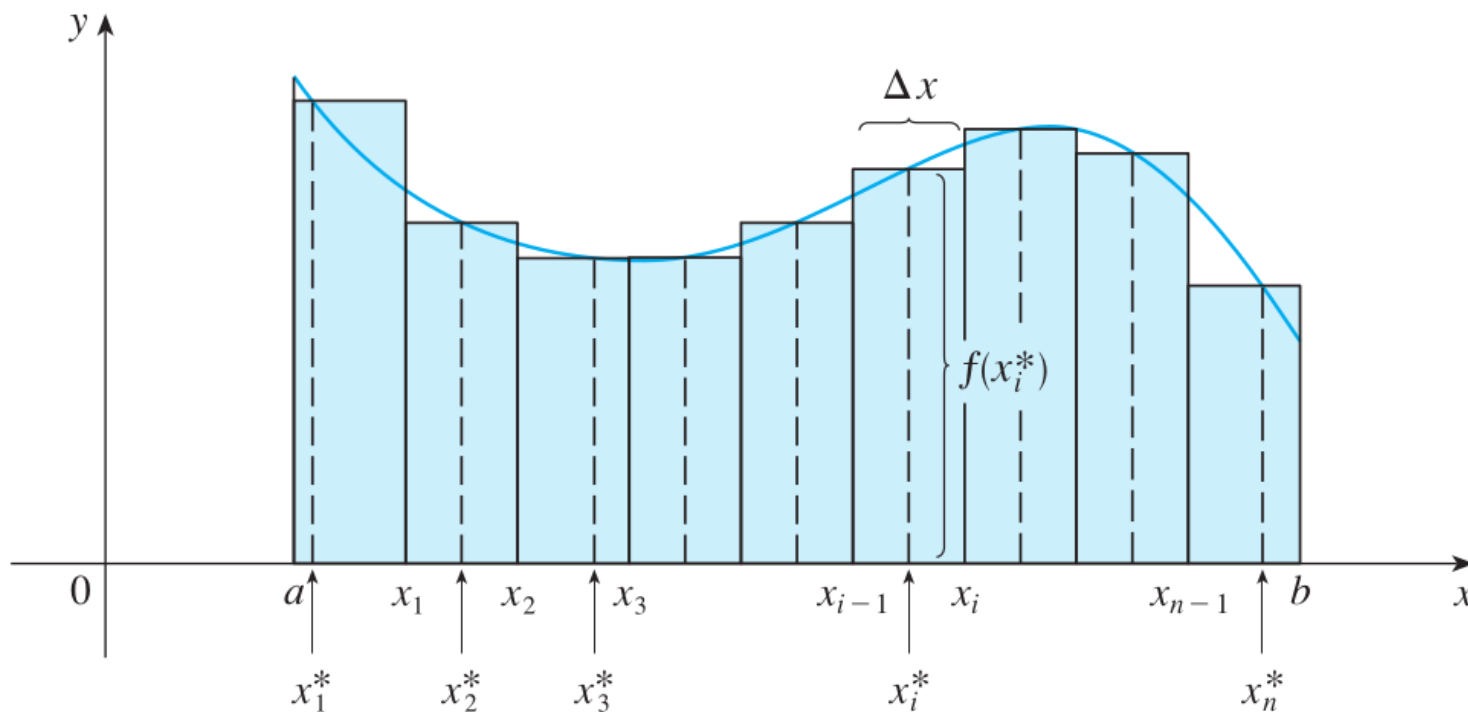
$$\Delta x = x_i - x_{i-1} = \frac{b - a}{n}$$



$$\begin{aligned} A &= \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \\ &= \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i-1}) \Delta x \end{aligned}$$

## 5.1 Areas and Distances

- A more general expression for the area of the region  $S$  is: 
$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

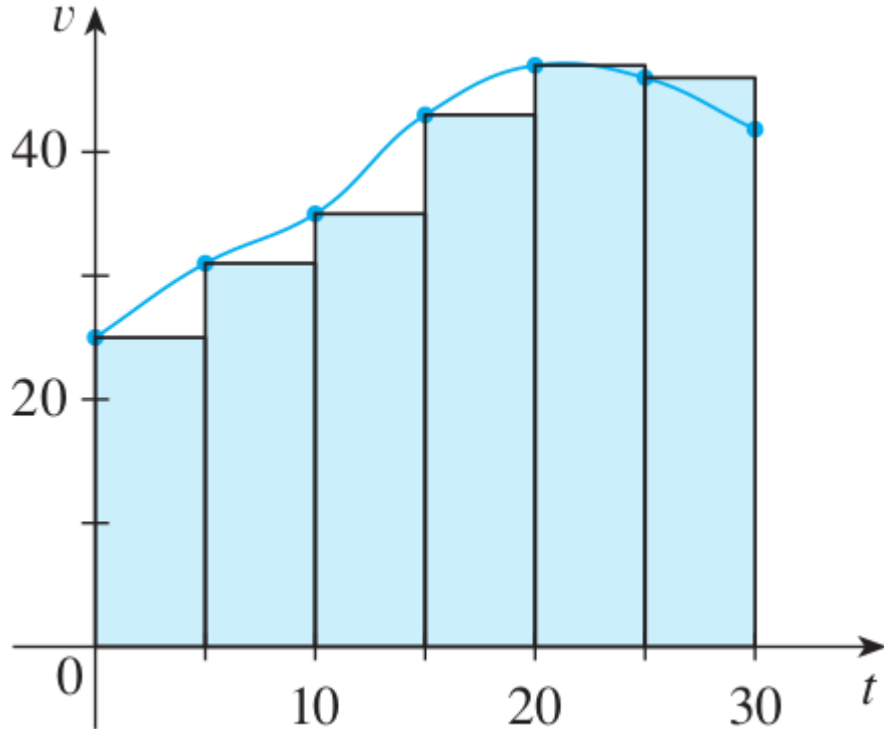


## 5.1 Example

- Let  $A$  be the area of the region that lies under the graph of  $f(x) = e^{-x}$  between  $x = 0$  and  $x = 2$ . **(a)** Using right endpoints, find an expression for  $A$  as a limit. Do not evaluate the limit. **(b)** Estimate the area by taking the sample points to be midpoints and using four sub intervals and then ten subintervals.

## 5.1 The Distance Problem

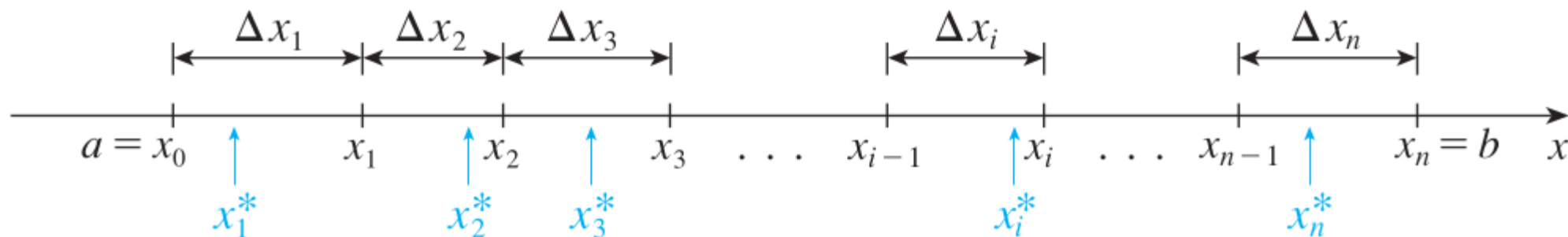
Area of a rectangle =  $vt = \frac{\text{m}}{\text{sec}} \text{sec} = \text{m} = \text{Distance}$



$$\text{Distance} = \lim_{n \rightarrow \infty} \sum_{i=1}^n v(t_i^*) \Delta t_i$$

## 5.2 The definite Integral

- **Generalization:** We consider limits similar in which  $f$  need not be positive or continuous and the subintervals don't necessarily have the same length.

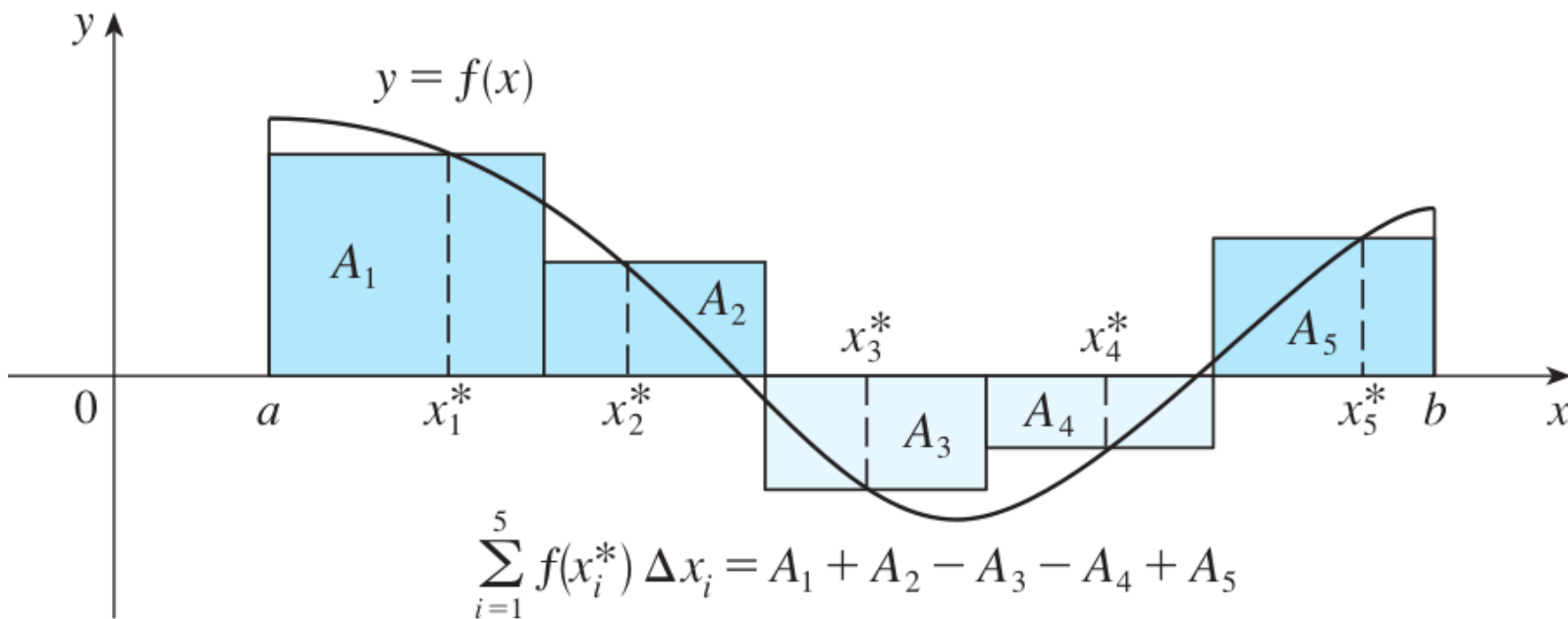


- A **partition**  $P$  of  $[a, b]$ :  $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$
- Choose **sample points (tags)**:  $x_1^*, x_2^*, \dots, x_n^*$  with  $x_i^* \in [x_{i-1}, x_i]$
- **Riemann Sum**:  $\sum_{i=1}^n f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \dots + f(x_n^*)\Delta x_n$

## 5.2 The definite Integral

- Geometric Interpretation of the **Riemann Sum**:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \cdots + f(x_n^*) \Delta x_n$$





## 5.2 The definite Integral

**Definition of a definite Integral:** If  $f$  is a function defined on  $[a, b]$ , the **definite integral** of  $f$  from  $a$  to  $b$  is the number

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

provided that this limit exists. If it does exist, we say that  $f$  is **integrable** on  $[a, b]$ .

## 5.2 The definite Integral

**Theorem** If  $f$  is continuous on  $[a, b]$ , or if  $f$  has only a finite number of jump discontinuities, then  $f$  is integrable on  $[a, b]$ ; that is, the definite integral  $\int_a^b f(x)dx$  exists.

**Theorem** If  $f$  is integrable on  $[a, b]$ , then

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x,$$

where  $\Delta x = \frac{b-a}{n}$  and  $x_i = a + i\Delta x$ .

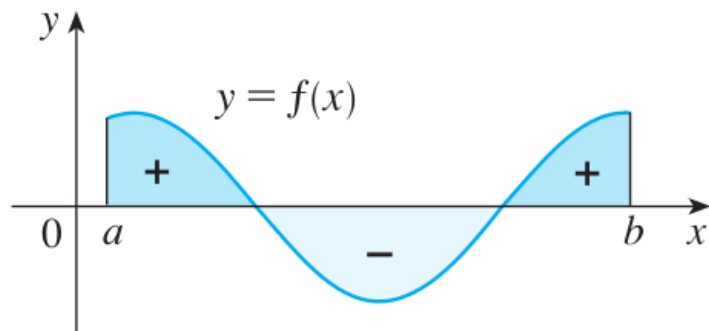
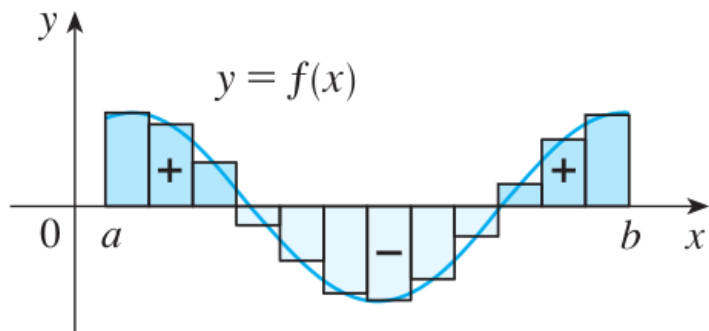
## 5.2 The definite Integral

**Note 1** The definite integral  $\int_a^b f(x)dx$  is a number, thus

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

**Note 2** A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)dx = (\text{Area above the } x\text{-axis}) - (\text{Area below the } x\text{-axis})$$



## 5.2 Examples

**1. (a)** Evaluate the Riemann sum for  $f(x) = x^3 - 6x$  taking the sample points to be right endpoints and  $a = 0$ ,  $b = 3$ , and  $n = 6$ . **(b)** Evaluate  $y = \int_0^3 (x^3 - 6x)dx$ .

**2.** Evaluate the following integrals by interpreting each in terms of areas:

**(a)**  $\int_0^1 \sqrt{1 - x^2}dx$ ,    **(b)**  $\int_0^3 (x - 1)dx$ .

## 5.2 Properties of the definite Integral

---

$$1. \quad \int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$2. \quad \int_a^a f(x)dx = 0$$

$$3. \quad \int_a^b c \, dx = c(b - a), \quad c \text{ constant}$$

$$4. \quad \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$5. \quad \int_a^b cf(x)dx = c \int_a^b f(x)dx$$

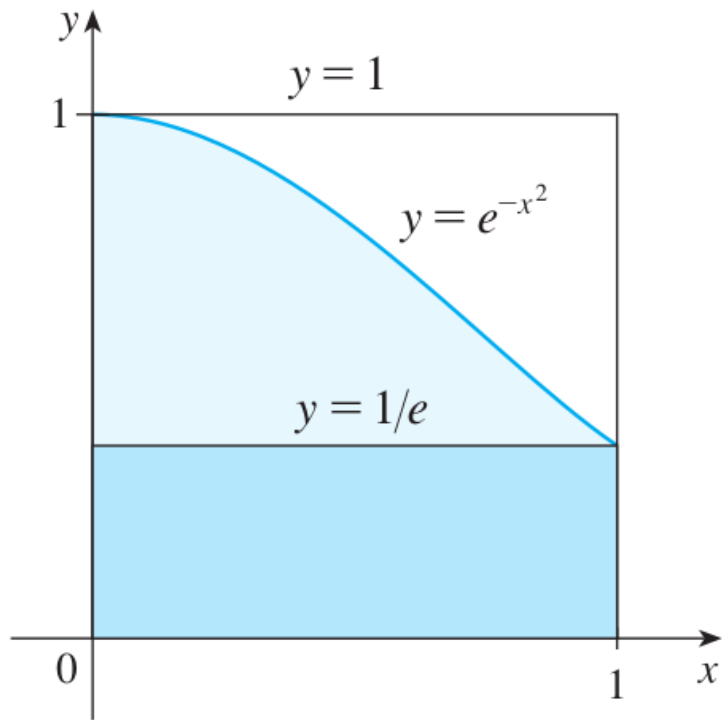
$$6. \quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad c \in (a, b)$$

## 5.2 Properties of the definite Integral

7. If  $f(x) \geq 0$  for  $x \in [a, b]$ , then  $\int_a^b f(x)dx \geq 0$
8. If  $f(x) \geq g(x)$  for  $x \in [a, b]$ , then  $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
9. If  $m \leq f(x) \leq M$  for  $x \in [a, b]$ , then  $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$

## 5.2 Example

- Use Property 9 (p. 22) to estimate  $\int_0^1 e^{-x^2} dx$ .



(whiteboard)

## 5.3 Evaluating definite Integrals

### Evaluation Theorem

If  $f$  is continuous on the interval  $[a, b]$ , then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where  $F$  is any antiderivative of  $f$ , that is,  $F' = f$ .



## 5.3 Indefinite Integrals

$$\boxed{\int f(x)dx = F(x) \quad \Rightarrow \quad \frac{dF(x)}{dx} = f(x)}$$

**Note** Distinguish between **definite** and **indefinite** integrals. A definite integral  $\int_a^b f(x)dx$  is a number, whereas an indefinite integral  $\int f(x)dx$  is a function (or family of functions).

## 5.3 Table of definite Integrals

$$\bullet \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\bullet \int e^x dx = e^x + C$$

$$\bullet \int \sin(x) dx = -\cos(x) + C$$

$$\bullet \int \sinh(x) dx = \cosh(x) + C$$

$$\bullet \int \frac{1}{x} dx = \ln(|x|) + C$$

$$\bullet \int a^x dx = \frac{a^x}{\ln(a)} + C$$

$$\bullet \int \cos(x) dx = \sin(x) + C$$

$$\bullet \int \cosh(x) dx = \sinh(x) + C$$

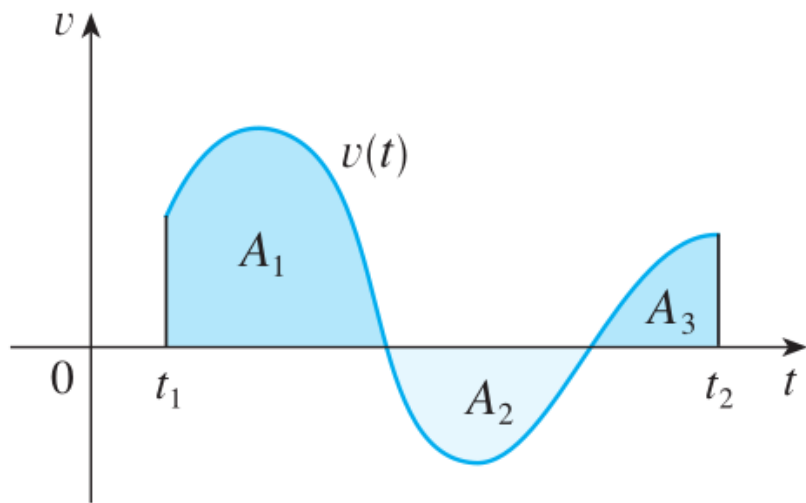
## 5.3 Example

- Find  $\int_0^2 \left( 2x^3 - 6x + \frac{3}{x^2 + 1} \right) dx$  and interpret the result in terms of areas.

## 5.3 Applications

### Net Change Theorem

The integral of a rate of change is the net change:  $\int_a^b F'(x)dx = F(b) - F(a)$



$$\text{Displacement} = \int_{t_1}^{t_2} v(t)dt = A_1 - A_2 + A_3$$

$$\text{Distance} = \int_{t_1}^{t_2} |v(t)|dt = A_1 + A_2 + A_3$$

## 5.3 Example

- A particle moves along a line so that its velocity at time  $t$  is  $v(t) = t^2 - t - 6$  (measured in meters per second).
  - (a) Find the displacement of the particle during the time period  $1 \leq t \leq 4$ .
  - (b) Find the distance traveled during this time period.

## 5.4 The Fundamental Theorem of Calculus

- If  $f$  is continuous on  $[a, b]$ , then the function  $t$  defined by

$$g(x) = \int_a^x f(t)dt, \quad x \in [a, b]$$

is an antiderivative of  $f$ , that is,

$$g'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x), \quad x \in (a, b).$$

## 5.4 Examples

1. Find the derivative of the function  $g(x) = \int_0^x \sqrt{1+t^2} dt$

2. Find the derivative of the function  $h(x) = \int_1^{x^4} \sec(t) dt$

## 5.4 Average Value of a Function

---

- The **average value** of  $f$  on the interval  $[a, b]$  is given by

$$\langle f \rangle = \frac{1}{b-a} \int_a^b f(x) dx$$

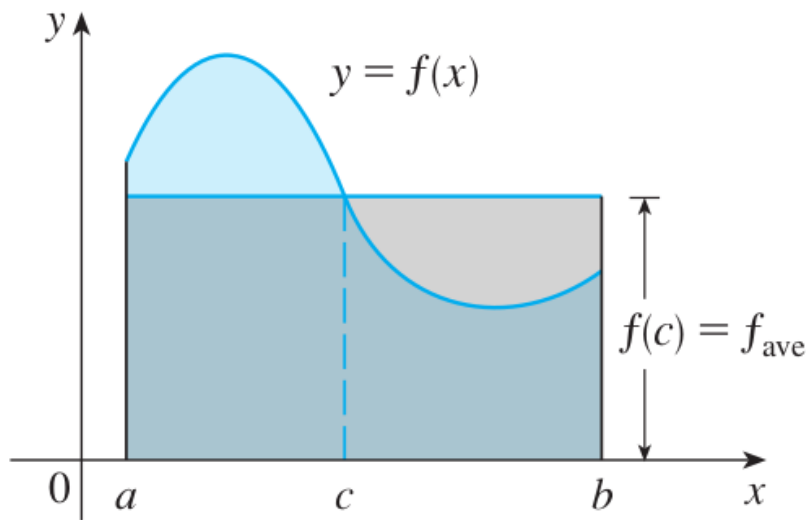
(proof  
whiteboard)



## 5.4 Average Value of a Function

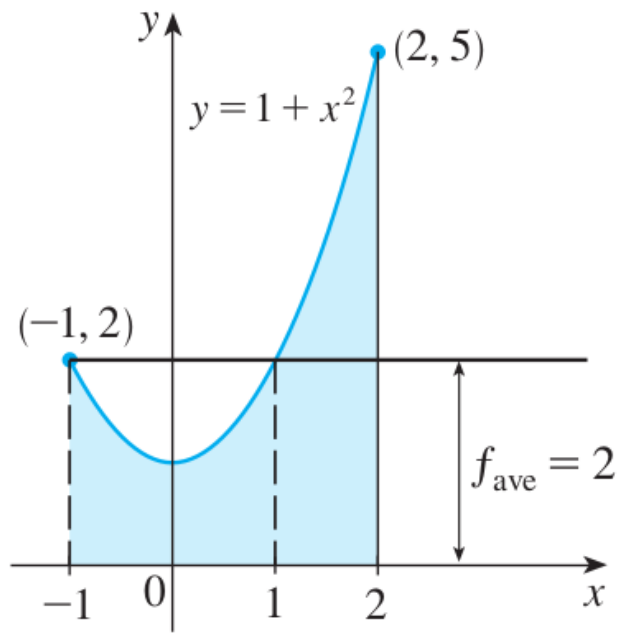
**The Mean Value Theorem for Integrals** If  $f$  is continuous on  $[a, b]$ , then there exists a number  $c \in [a, b]$  such that

$$f(c) = \langle f \rangle = \frac{1}{b-a} \int_a^b f(x) dx \quad \Rightarrow \quad \int_a^b f(x) dx = f(c)(b-a)$$



## 5.4 Example

Find the average value  $\langle f \rangle$  of the function  $f(x) = 1 + x^2$  on the interval  $[-1, 2]$  and the value  $c$  for which  $f(c) = \langle f \rangle$ .



## 5.5 The Substitution Rule (indefinite Integrals)

- If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Examples:

Find (a)  $\int x^3 \cos(x^4 + 2)dx$ ,    (b)  $\int \tan(x)dx$ ,    (c)  $\int \frac{x}{\sqrt{1 - 4x^2}}dx$

## 5.5 The Substitution Rule (definite Integrals)

- If  $g'(x)$  is continuous on  $[a, b]$  and  $f$  is continuous on the range of  $u = g(x)$ , then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example:

Find

(a)  $\int_1^2 \frac{1}{(3-5x)^2} dx,$       (b)  $\int_1^e \frac{\ln(x)}{x} dx$

## 5.5 Symmetry

- Suppose  $f$  is continuous on  $[-a, a]$ .

(a) If  $f$  is even [ $f(-x) = f(x)$ ], then  $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$

(b) If  $f$  is odd [ $f(-x) = -f(x)$ ], then  $\int_{-a}^a f(x)dx = 0$

