Chapter 1: Sets -Maps - Complex numbers

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Contents

1.1.1. Sets

1. A set is an (unordered) collection of (distinct) objects, called elements or members of the set. A set is said to contain its elements.
   We write $x \in X$ to denote that x is an element of the set X.
   We write $x \notin X$ to denote that x is not an element of the set X.
2. Example: $A =$ the set of prime numbers, $1 \notin A$, $2 \in A$.
3. We usually use uppercase letters such as $A, B, \ldots, X, Y, \ldots$ to denote sets. Lowercase letters are usually used to denote elements of sets: $a, b, \ldots, x, y, \ldots$
4. The *empty set* is the set that has no elements. The empty set is denoted by $\emptyset$.

Some common sets

- $\mathbb{N}$ the set of all natural numbers.
- $\mathbb{Z}$ the set of all integers.
- $\mathbb{Q}$ the set of rational numbers.
- $\mathbb{R}$ the set of real numbers.

Desrcibe a set

- List all the elements, when this is possible. Example, $A = \{0, 1, 2, 3\}$.
- (Use set builder notation) State the property or properties that elements in the set must have. The general form
$$\{x \mid x \text{ has property } P\}.$$
We also usually use
$$\{x \in X \mid x \text{ has property } P\}$$
to describe the set of those elements of X that have property P.
Example, the set of even integers can be described as follows
$$\{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\}.$$

Relations between sets: Subsets

Let A and B be sets.

- If every element of A is also an element of B then we say that A is a *subset* of B, or A is contained in B, or B contains A, and write $A \subset B$ (or $A \subseteq B$). So

$$A \subseteq B \Leftrightarrow \forall x, (x \in A) \rightarrow (x \in B).$$

- If A is not a subset of B then we write $A \not\subset B$.
- Showing that A is a subset of B. To show that $A \subset B$, show that if x belongs to A then x also belongs to B.
- Showing that A is a subset of B. To show that $A \not\subset B$, find a single x belongs to A but x does not belong to B.

Relations between sets: Equal sets

Two sets A and B are equal, written $A = B$, if they have the same elements.
In other words, if every element of A is also an element of B and every element of B is also an element of A.
Therefore, $A = B$ if and only if $A \subset B$ and $B \subset A$.

$$A = B \Leftrightarrow \forall x, (x \in A) \leftrightarrow (x \in B).$$

Example: $A = \{0, 1, 2\}$, $B = \{0, 1, 2, 3\}$, $C = \{3, 2, 0, 1\}$. We have

$$A \subset B = C.$$

Remarks

Let A and B two sets.

$$\boxed{A \subset B \text{ if and only if} x \in A \Rightarrow x \in B.}$$

- To show $A \subset B$, we can use the *element method* as follows:
  1. Let x be an arbitrary element of A.
  2. Show that x is in B.
- To show that $A = B$, we may:
  1. Use the element method to show $A \subset B$ and $B \subset A$

     $$\boxed{A = B \text{ if and only if } x \in A \Leftrightarrow x \in B.}$$

  2. Use (some basic) set identities.

1.1.2. Set operations

Let A and B be sets.

- Union: The *union* of A and B, denoted by $A \cup B$, is the set defined by

$$\boxed{A \cup B = \{x \mid x \in A \text{ or } x \in B\}.}$$

- Intersection: The *intersection* of A and B, denoted by $A \cap B$, is the set defined by

$$\boxed{A \cap B = \{x \mid x \in A \text{ and } x \in B\}.}$$

- Difference: The *difference* of A and B, denoted by $A \setminus B$ (or $A - B$), is the set defined by

$$\boxed{A \setminus B = \{x \mid x \in A \text{ và } x \notin B\}.}$$

- Complement: If $A \subset X$ then $X \setminus A$ is said to be the *complement* of A in X.
  The complment of A in X is denoted by $\bar{A}$ or $C_X(A)$.

Example

Let $A = \{0, 1, 2, 3, 4\}$, $B = \{2, 3, 4, 5, 6\}$. Find

$$A \cup B, A \cap B, A \setminus B, B \setminus A.$$

- $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$.
- $A \cap B = \{2, 3, 4\}$.
- $A \setminus B = \{0, 1\}$.
- $B \setminus A = \{5, 6\}$.

Example (GK20161)

Let $f(x), g(x)$ be functions on $\mathbb{R}$. Let $A = \{x \in \mathbb{R} \mid f(x) = 0\}$, $B = \{x \in \mathbb{R} \mid g(x) = 0\}$. Use A, B to describe the set of the solutions of the equation: $\dfrac{f(x)}{f(x)+g(x)} = 0$.

Let S be the set of the solutions of the given equation. Then

$$x \in S \Leftrightarrow \frac{f(x)}{f(x)+g(x)} = 0 \Leftrightarrow \begin{cases} f(x) = 0 \\ f(x)+g(x) \neq 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} f(x) = 0 \\ g(x) \neq 0 \end{cases} \Leftrightarrow \begin{cases} x \in A \\ x \notin B \end{cases} \Leftrightarrow x \in A \setminus B.$$

Hence $S = A \setminus B$.

Some properties

1. Commutative laws: $$\begin{aligned} A \cup B &= B \cup A, \\ A \cap B &= B \cap A. \end{aligned}$$
2. Associative laws: $$\begin{aligned} (A \cup B) \cup C &= A \cup (B \cup C), \\ (A \cap B) \cap C &= A \cap (B \cap C). \end{aligned}$$
3. Distributive laws: $$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C). \end{aligned}$$
4. De Morgan's laws: $$\begin{aligned} X \setminus (A \cup B) &= (X \setminus A) \cap (X \setminus B), \\ X \setminus (A \cap B) &= (X \setminus A) \cup (X \setminus B) \end{aligned}$$ or $$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Union and intersection operators can be defined for several sets (not necessary two sets), and have similar properties as above.

Example (GK20201-N2)

Let A, B, C be sets. Is the following inclusion true or not? Why?

$$[(A \cap B) \setminus C] \subset [A \cap (B \setminus C)].$$

- Pick an arbitrary element $a \in (A \cap B) \setminus C$.
- Then $a \in A$, $a \in B$ and $a \notin C$.
- Since $a \in B$ and $a \notin C$, $a \in B \setminus C$.
- Since $a \in A$ and $a \in B \setminus C$, $a \in A \cap (B \setminus C)$.
- Hence the above inclusion is true.

Example (GK20201-N1)

Let A, B, C be sets. Show that

$$(A \cap B) \setminus C = A \cap (B \setminus C).$$

Solution 1: Use element method.
Solution 2: Use set identities.
Suppose A, B, C are subset of certain X, and let $\bar{C}$ the complement of C in X. We have

$$(A \cap B) \setminus C = (A \cap B) \cap \bar{C} = A \cap (B \cap \bar{C}) = A \cap (B \setminus C).$$

Solution 2': Ta có

$$x \in (A \cap B) \setminus C \Leftrightarrow \begin{cases} x \in A \cap B \\ x \notin C \end{cases} \Leftrightarrow \begin{cases} x \in A \\ x \in B \\ x \notin C \end{cases} \Leftrightarrow \begin{cases} x \in A \\ x \in B \setminus C \end{cases}$$

$$\Leftrightarrow x \in A \cap (B \setminus C).$$

Remark: Suppose A, B are subsets of a set X. Let $\bar{B}$ be the complement of B in X. Then $A \setminus B = A \cap \bar{B}$.

Some exercises

- (CK20151) Let A, B, C be (arbitrary) sets. Show that $[(A \cup B) \setminus C] \subset [(A \setminus B) \cup (B \setminus C)]$.
- (GK20161) Let A, B, C be sets. Show that $(A \setminus B) \setminus C = A \setminus (B \cup C)$.
- (GK20161-No 5) Let $A = [a, a+1]$, $B = [b-1, b+1]$, where a, b are real numbers. Find conditions on a, b to ensure that $A \cap B = \emptyset$.
- (GK20171) Let A, B, C be sets. Show that $(A \setminus B) \cap C = (A \cap C) \setminus B$.
- (CK20181*) Consider the following subsets $\mathbb{R}$: $A = [1, 3]$, $B = (m, m+3)$. Find m such that $(A \setminus B) \subset (A \cap B)$.
- (GK20191-N3) Let A, B, C be non-empty sets. Show that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.
- (GK20201*) Let A, B be subsets of a set X and suppose that $(X \setminus B) \subset A$. Show that $(X \setminus A) \subset B$.

Cartesian product (Tích Descartes)

- Let A and B be set. The *Cartesian product* (the direct product) of A and B, denoted by $A \times B$, the set of all order pairs (a, b), where $a \in A$, $b \in B$:

$$\boxed{A \times B = \{(a, b) \mid a \in A, b \in B\}.}$$

- Consider n sets $A_1, A_2, \ldots, A_n$. The Cartesian product of these n sets is of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i = 1, \ldots n$. The Cartersian product of $A_1, A_2, \ldots, A_n$ is denoted $A_1 \times A_2 \times \cdots \times A_n$, or $\prod\limits_{k=1}^{n} A_k$. Hence

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \text{ với mọi } i = 1, 2, \ldots n\}.$$

- If $A_1 = A_2 = \cdots = A_n = A$, we denote the Cartesian of $A_1, \ldots, A_n$ by A^n.

Remarks

1. Consider two elements (a, b) and (c, d) in the Cartesian product $A \times B$. Then

$$(a, b) = (c, d) \Leftrightarrow \begin{cases} a = c \\ b = d. \end{cases}$$

2. Consider two elements$(a_1, a_2, \ldots, a_n)$ và $(b_1, b_2, \ldots, b_n)$in the Cartesian product $A_1 \times A_2 \cdots \times A_n$. Then

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \Leftrightarrow \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ \vdots \\ a_n = b_n \end{cases}.$$

Example

Let $A = \{a, b\}$, $B = \{0, 1, 2\}$.

- Find $A \times B$, $B \times A$, B^2.
- Which set does the element $(a, 1, 2, b)$ belong to?

- $A \times B = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\}$.
- $B \times A = \{(0, a), (1, a), (2, a), (0, b), (1, b), (2, b)\}$.
- $B^2 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.
- $(a, 1, 2, b)$ belongs to $A \times B \times B \times A$.

Example (CK20201-N3)

Let A, B, C, D be sets. Is the folling inclustion true or not? Why?

$$(A \times C) \cap (B \times D) \subset (A \cap B) \times (C \cap D).$$

- Let $(x, y) \in (A \times C) \cap (B \times D)$ be an arbitrary element.
- Then $(x, y) \in A \times C$ và $(x, y) \in B \times D$.
- Since $(x, y) \in A \times C$, this implies that $x \in A$ và $y \in C$.
- Since $(x, y) \in B \times D$, this implies that $x \in B$ và $y \in D$.
- Hence $x \in A \cap B$ and $y \in C \cap D$.
- Thus $(x, y) \in (A \cap B) \times (C \cap D)$.
- Therefore the inclusion is true.

Some exercises

- (CK20181-N3 = GK20191). Let A, B, C be sets. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
- (GK20181-N2) In $\mathbb{R}^2$, given the following subsets $A = \{(x, y) \in \mathbb{R}^2 \mid x + y = 4\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y = 8\}$. Determine $A \cap B$.

1.2.1. Definitions

Let X and Y be nonempty sets.

Definition

A *map* f from X to Y is an assigment which assigns to each element of X a unique element of Y. The unique elemen y corresponding to x is called the image of x (via f) and we write $y = f(x)$. Element x is called an preimage of y.

We usually denote the function f from X to Y that sends each $x \in X$ to element $y = f(x) \in Y$ by $f : X \to Y$, or

$$\begin{aligned} f : X &\to Y \\ x &\mapsto y = f(x). \end{aligned}$$

X is called the domain (tập nguồn) of f; Y is called the codomain (tập đích) of f.

Image and preimages

Let $f: X \to Y$ be a mapping.

- Let $A \subset X$. The set

$$\boxed{f(A) = \{f(x) \mid x \in A\} = \{y \in Y \mid \exists x \in A, f(x) = y\}}$$

  is called the *image* of A via f. In the particular case when $A = X$, the set $f(X)$ is called the image of f, denoted $\mathrm{Im}(f)$.

- Let $B \subset Y$. The set

$$\boxed{f^{-1}(B) = \{x \in X \mid f(x) \in B\}}$$

  is called the *preimiage* of B via f. Hence, for v $x \in X$,

$$x \in f^{-1}(B) \Leftrightarrow f(x) \in B.$$

  In the particular case when $B = \{y\}$ a singleton consiting of only one element y, we simply write $f^{-1}(y)$ for $f^{-1}(\{y\})$.

Example (GK20161-No 5)

Consider the mapping (function) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 3x - 4$, and $A = \{0, -6\}$. Find $f(A)$ and $f^{-1}(A)$.

- $f(A) = \{f(x) \mid x \in A\} = \{f(0), f(-6)\} = \{-4, 50\}$.
- $x \in f^{-1}(A) \Leftrightarrow f(x) \in A \Leftrightarrow \left[\begin{array}{l} f(x) = 0 \\ f(x) = -6 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = -1, x = 4 \\ x = 1, x = 2 \end{array}\right. \Leftrightarrow x \in \{-1, 1, 2, 4\}$.
- $f^{-1}(A) = \{-1, 1, 2, 4\}$.

Example (*GK20201)

Let $f: X \to Y$ be a mapping and let $A, B \subset X$. Is the following inclustion true or not? Why?

$$f(A) \setminus f(B) \subset f(A \setminus B).$$

- Let $y \in f(A) \setminus f(B)$ be an aribitary element.
- Then $y \in f(A)$ và $y \notin f(B)$.
- Since $y \in f(A)$, there exists $x \in A$ such that $f(x) = y$.
- Since $y = f(x) \notin f(B)$, $x \notin B$.
- From $x \in A$ và $x \notin B$, we deduce that $x \in A \setminus B$.
- Hence $y = f(x) \in f(A \setminus B)$.
- Therefore the inclusion is (always) true.

Some exercises

- (GK20171) Consider the functions $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 3x + 2$. Find $f^{-1}((0,2])$.
- (GK20171*) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be mapping defined by $f(x,y) = (x-y, x+y)$. Find $f(A)$, where $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
- (GK20161) Consider the mapping $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 4x$. Determine a, b such that $f^{-1}(\{a\}) = \{0, 2, b\}$.
- (GK20161) Consider the mapping $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f(x) = \dfrac{x+1}{x-1}$. Find $f^{-1}((0,1])$.
- (GK20191*) Consider the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2 - 2x + 4y - 1$, and $A = [-1,1] \times [0,2]$. Find $f(A)$.
- (GK20191-N2) Consider the mapping $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 3x$ and the set $A = \{x \in \mathbb{R} : \dfrac{x-1}{2-x} \geq 0\}$. Find $f(A)$.
- (*GK20191-N3) Consider the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (x+2y, 3x-y)$ and the set $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Determine $f(A)$.
- (GK20181-N2) Consider the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x+2y, 2x-y)$. Let $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$. Find $f(A)$.
- (GK20201) Consider the mapping $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x$ and the set $A = (0,3)$. Find $f(A)$ and $f^{-1}(f(A))$.
- (GK20191) Let $f\colon E \to F$ be a mapping and $\neq B \subset F$. Show that $f^{-1}(F \setminus B) = E \setminus f^{-1}(B)$.

1.2.2. Injective, surjective and bijective mappings

Let $f : X \to Y$ be a mapping.

1. f is said to be *injective* (or *one-to-one*) if for $x_1, x_2 \in X$,

$$\boxed{x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).}$$

   In other words, for $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
   In other words, for every $y \in Y$, the "equation" $f(x) = y$ has at most one solution $x \in X$.
2. f is said to be *surjective* (or *onto*) if for every $y \in Y$, there exists at least one element $x \in X$ such that $f(x) = y$.
   In other words, for every $y \in Y$, the "equation" $f(x) = y$ has at leat one solution $x \in X$.
3. f is said to be *bijective* if f is both injective and surjective.
   In other words, for every $y \in Y$, the "equation" $f(x) = y$ always has a unique solution $x \in X$.

Example (GK20181)

Is the mapping $f: \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x^2 - 4, x^3 + 1)$ injective? Explain your answer?

- Suppose $f(x_1) = f(x_2)$.
- We have $f(x_1) = f(x_2) \Leftrightarrow (x_1^2 - 4, x_1^3 + 1) = (x_2^2 - 4, x_2^3 + 1) \Leftrightarrow \begin{cases} x_1^2 - 4 = x_2^2 - 4 \\ x_1^3 + 1 = x_2^3 + 1 \end{cases} \Leftrightarrow$
  $\begin{cases} x_1^2 = x_2^2 \\ x_1^3 = x_2^3 \end{cases} \Rightarrow x_1 = x_2.$
- Thus f is injective. (f is an injective map, f is an injection.)

Example (GK20181)

Is the mapping $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (2x+1, x-3)$ surjective? Explain your answer?

- Consider $(0,0) \in \mathbb{R}^2$ and consider the equation $f(x) = (0,0)$.
- We have $f(x) = (0,0) \Leftrightarrow (2x+1, x-3) = (0,0) \Leftrightarrow \begin{cases} 2x+1 = 0 \\ x-3 = 0 \end{cases}$.
- This system of equations has no solutions.
- Thus f is not surjective.

Example (GK20161)

Is the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x + y^2, y^3)$ a bijection? Explain your answer?

- Pick an arbitrary element $(a, b) \in \mathbb{R}^2$ and consider the equation $f(x, y) = (a, b)$ (*).
- We have $f(x, y) = (a, b) \Leftrightarrow (2x + y^2, y^3) = (a, b) \Leftrightarrow \begin{cases} 2x + y^2 & = a \\ y^3 & = b \end{cases} \Leftrightarrow \begin{cases} x & = (a - \sqrt[3]{b^2})/2 \\ y & = \sqrt[3]{b} \end{cases}$.
- Thus (*) always has a unique solution.
- Therefore f is a bijection. (f is a bijective mapping).

Some excercises

- (GK20171) Consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^2 - y, x + y)$. Is f injective, surjective? Why?
- (*GK20171) Consider the function $f : [m, 2] \to \mathbb{R}$, $f(x) = x^3 - 3x^2 - 9x + 1$. Find m suc that f is an injection.
- (GK20191-N2) Find the largest integer m such that the mapping $f : [m, 2] \to [0, 4]$, $f(x) = x^2$, is surjecive but not injective.
- (GK20201-N2) Consider the mapping $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x - 2, x^2 - 2x)$.
    1. Is f surjective? Why?
    2. Find $f^{-1}(A)$, where $A = [0, 1] \times (-\infty, 3)$.

1.2.3. Compositions and inverses of mappings

Let $f: X \to Y$ and $g: Y \to Z$ be mapping.
The composition (or *product*) of g and f is a mapping $h: X \to Z$ defined by

$$h(x) = g(f(x)), \quad \forall x \in X.$$

The composition of g and f is denoted by $g \circ f$ or gf.

Example

Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x + 1$. Determine $g \circ f$ and $f \circ g$.

- $g \circ f(x) = g(f(x)) = g(x^2) = x^2 + 1.$
- $f \circ g(x) = f(x + 1) = (x + 1)^2.$

Some properties

The identity mapping

Let X be a nonempty set. The mapping from X to X that sends each element $x \in X$ to x, is called the identity mapping on X, and denoted by id_X.

Some properties

Let $f\colon X \to Y$, $g\colon Y \to Z$, $h\colon Z \to W$ be mappings. Then

1. $(h \circ g) \circ f = h \circ (g \circ f)$,
2. $\mathrm{id}_Y \circ f = f$,
3. $f \circ \mathrm{id}_X = f$.

Inverse mappings

Let $f\colon X \to Y$ be a bijective mapping.

- Since f is bijective, for each $y \in Y$ there exists a unique element $x \in X$ such that $f(x) = y$. We denote the element x by $x = f^{-1}(y)$.
- The assigment that assigns to ech element $y \in Y$ the element $x = f^{-1}(y)$, is a mapping. This mapping is called the *inverse mapping* of f, and denoted by $f^{-1}\colon Y \to X$.
- Remember: $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

Example: The mapping $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$, is bijective and its inverse $f^{-1}\colon \mathbb{R} \to \mathbb{R}$ is given by $f^{-1}(x) = x - 1$.

Properties

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et $f : X \to Y$ be a bijection. We have the following statements.

- f^{-1} is also a bijection and $(f^{-1})^{-1} = f$,
- $f \circ f^{-1} = \mathrm{id}_Y$, $f^{-1} \circ f = \mathrm{id}_X$.

Let $f : X \to Y$ and $g : Y \to Z$ be bijective mappings. Then $g \circ f$ is bijective and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Example (GK20201-N3)

Consider the mapping $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$. Show that $f^2 = f \circ f$ is bijective. Find the inverse mapping of f^2.

- $f^2(x) = (f \circ f)(x) = f(f(x)) = f(2x-3) = 2(2x-3) - 3 = 4x - 9.$
- Let $y \in \mathbb{R}$ be an arbitrary element. Consider the equation $f^2(x) = y$ (*). We have $f(x) = y \Leftrightarrow 4x - 9 = y \Leftrightarrow x = \dfrac{y+9}{4}$.
- $(*)$ always has a unique solution $x \in \mathbb{R}$, for every $y \in \mathbb{R}$.
- Hence f^2 is bijective. Moreover, $(f^2)^{-1}(y) = \dfrac{y+9}{4}$.
- Thus $(f^2)^{-1}(x) = \dfrac{x+9}{4}$.

1.3.1. Binary operations

Binary operation

Let X be a nonempty set. A *binary operation* on X is a mapping

$$*\colon X \times X \to X.$$

The image of (x, y) via $*$, denoted by $x * y$, is called the *product* of x and y.

We also use $\circ, \cdot, +, \times$,... to denote binary operations.

Algebraic structure

An algebraic structure consists of a nonempty set X, a collection of operations on X and a finite set of identities (axioms), that these operations must satisfy.

Some definitions

Let X be a nonempty set together with a binary operation $*$. We say that

- $*$ is *associative* if
$$(x*y)*z = x*(y*z), \quad \forall x,y,z \in X.$$
- $*$ is *commutative* if
$$x*y = y*x, \quad \forall x,y \in X.$$
- $*$ has an *identity* element (or *neutral*) if there is $e \in X$ such that
$$x*e = e*x = x, \quad \forall x \in X.$$
In this case, such an element e is called the *identity* (or *neutral*) element of (or with respect to) $*$.
- Suppose $*$ has an identity element $e \in X$. An element $x \in X$ is said to be *invertible* if there exists an element $y \in X$ such that
$$x*y = y*x = e.$$
In this case, y is called an inverse of x.

1.3.2. Groups, rings and fields

Let G be a nonempty set, equipped with a binary operation $*$. The set G with operation $*$ is called a *group* if the following conditions are satisfied.

1. $*$ is associative: $(x * y) * z = x * (y * z), \quad \forall x, y, z \in G.$
2. There exists an element $e \in G$, called a *neutral* (or *identity*) element such that $x * e = e * x = x$, for all $x \in G$.
3. For any element $x \in G$, there exists $x' \in G$, called an *opposition* or *inverse* of x, such that $x * x' = x' * x = e$.

A group G with binary operation $*$ is *commutative* (or *abelian*) if the operation $*$ is commutative:

$$x * y = y * x, \quad \forall x, y \in X.$$

Some properties and examples

Properties

Suppose G with a binary operation $*$ is a group.

- The identity element e is unique.
- The inverse of a given element x is unique. The inverse of x is denoted by x^{-1}
- Cancellation law: $a * b = a * c \Rightarrow b = c$, $b * a = c * a \Rightarrow b = c$.

- $\mathbb{Z}$ together with the usual addition is a (commutative) group.
- $\mathbb{R} \setminus \{0\}$ together with the usual multiplication is a (commutative) group.
- The set of bijections from X vào X together the operation of composition of functions, is a group. If X has more than two elements then this group is not commutative.
- The set $\mathbb{N}$ together with the usual addition is not a group.

Remarks

- Usually, the operation in an abelian group is denoted by "$+$", and called addition (though the underlying set might not be a set of numbers). In this case the identity element is denoted by 0. The inverse of x is denoted by $-x$ (the negative of x).
- In the general case, the operation $*$ in a group usually is denoted by "$\cdot$". Product $x \cdot y$ of x and y, is denoted by xy. The identity element is denoted by 1. The inverse of x is denoted by x^{-1}.

Example (GK20201)

Is the set $G = \{z \in \mathbb{C} \mid z^7 = 1\}$ with the multiplication of complex numbers a group? Why?

To determine a set G together with an operation $*$ is a group or not, we should first check that the operation is indeed a binary operation, that means, we should check that the operation is closed (if $a, b \in G$ then $a * b$ is also in G). After that, we check three group axioms.

- For all $z_1, z_2 \in G$, we have $(z_1 z_2)^7 = z_1^7 z_2^7 = 1$. Hence $z_1 z_2 \in G$, and the operation is closed.
- For all $z_1, z_2, z_3 \in G$, we have $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
- The element $e = 1 \in G$ is the element of the multiplication.
- For all $z \in G$, $\dfrac{1}{z} \in G$ is the inverse of z.
- Thus G with the multiplication of complex numbers, is a group.

Some exercises

- (GK20191) Is the set $G = \{z \in \mathbb{C} \colon |z| = 1\}$ with the multiplication of complex numbers? Why?
- (GK20191) Is the set $G = \{z = m + ni \in \mathbb{C} \mid m, n \in \mathbb{Q}, m^2 + n^2 \neq 0\}$ is a group under the multiplication of complex numbers? Why?
- (GK20173*) Let $G \neq \emptyset$ together with a binary operation be a group such that $x * x = e$, $\forall x \in G$, where e is the identity element of G. Is $(G, *)$ is a commutative group? Why?
- (GK20171) Is the set $W = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}\}$ is a group under the operation of matrix addittion? Why?
- (GK20201) Let X be the set of square matrices of order 2 whose determents are 0 or 1. Is X togethe with matrix multiplication a group? Why?

Rings

Let R be a nonempty set, equipped two binary operations, one operation denoted by $+$ (called addition), and one operation denoted by $\cdot$ (called multiplicaion). Then R together with these operations is a *ring* if the following conditions (axioms) are satisfied.

1. $(R, +)$ is an abelian group.
2. $\cdot$ is associative: $(xy)z = x(yz)$, với mọi $x, y, z \in R$.
3. The distributive laws holds in R:

$$(x+y)z = xz + yz, z(x+y) = zx + zy, \quad , \forall x, y, z \in R.$$

Examples

- $\mathbb{Z}$ under the usual operations of addition and muliplication is a ring.
- $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ vunder the usual operations of addition and muliplication is a ring.

- Ring R is called a *commutative* ring if the multiplication is commutative: $xy = yx, \quad \forall x, y \in R$.
- Ring R is said to have an identity (or R is said be unital) if the multiplication has an identity elements, i.e., there exists an element, denoted by $1 \in R$, such that $x \cdot 1 = 1 \cdot x = x$, for all $x \in R$.

Example, $\mathbb{Z}$ under the usual operations of addition and muliplication is a commutative and unital ring.

Fields

Let K be a nonempty set, equipped two binary operations, one operation denoted by $+$ (called addition), and one operation denoted by $\cdot$ (called multiplicaion). Then R together with these operations is a *ring* if the following conditions (axioms) are satisfied.
Then K together with these operations is a *field* if the following conditions (axioms) are satisfied.

- K together with these two operations is a commutative ring which has the identity $1 \neq 0$ (recalled that 1 is the identity element of the multiplication, 0 is the identity element of addition).
- Every $x \neq 0$ in K is *invertible*, that means there exists $x' \in K$ such that $xx' = x'x = 1$.

Ví dụ

- $\mathbb{R}$ under the usual operations of addition and muliplication is a field.
- $\mathbb{Z}$ under the usual operations of addition and muliplication is not a field.
- $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ under the usual operations of addition and muliplication is a field.

1.3.3. Complex numbers

- An ordered pair (a, b) of real numebers is called a complex number. The set of complex numbers is denoted by $\mathbb{C}$. So $\mathbb{C} = \mathbb{R} \times \mathbb{R}$.
- We definition two operations: the addition $+$ and multiplication $\times$ on $\mathbb{C}$ as follows.

$$\boxed{(a, b) + (c, d) = (a + c, b + d); \quad (a, b)(c, d) = (ac - bd, ad + bc).}$$

Proposition

The set $\mathbb{C}$ together with two operations defined above, is a field.

- On $\mathbb{C}$: $(a, b) = (c, d) \Leftrightarrow a = c, b = d$.
- The identity element of the addition is $(0, 0)$.
- The identity element of the multiplication is (1,0).
- The inverse of $(a, b) \neq (0, 0)$ is $(a, b)^{-1} = (\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2})$.

Standard form of a complex number

Let F be the set of complex number of the form $(a, 0)$, where $a \in \mathbb{R}$. The mapping

$$f : \mathbb{R} \to F, a \mapsto (a, 0)$$

is bijective and $f(a + b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$. We will identify a real number a with the complex number $(a, 0)$. Then the set $\mathbb{R}$ of real number is identified with the set F. Via this identification, $\mathbb{C}$ contains $\mathbb{R}$.

- Set $i = (0, 1)$. Then $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$. The number i is called the *imaginary unit*.
- A complex number $z = (a, b)$ can be written as

  $$z = (a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1) = a + bi.$$

  Form $z = a + bi$ is called the *standard form* (or *algebraic form*) of z.
- $a = \mathrm{Re} z$ is the real part of z, $b = \mathrm{Im} z$ is the imaginary part of z.

Angela Suba Natarajan, From My Pen, On Life

"Life is a complex with real and imaginary parts"

Operations with complex numbers in standard forms

- $(a+bi)+(c+di)=(a+c)+(b+d)i$
- $(a+bi)-(c+di)=(a-c)+(b-d)i$
- $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$
- $\dfrac{a+bi}{c+di}=\dfrac{(a+bi)(c-di)}{c^2+d^2}.$

Conjugate of a complex number

Let $z = a + bi$ be a complex number.

- The complex number $\bar{z} = a - bi$ is called the (complex) *conjugate* of z.
- The nonnegative real number $|z| = \sqrt{a^2 + b^2}$ is called the modulus of z.

Properties

- $|z| = 0 \Leftrightarrow z = 0$.
- $z\bar{z} = |z|^2$.
- $|z_1 + z_2| \leq |z_1| + |z_2|$.
- $|z_1 z_2| = |z_1||z_2|, \quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.
- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$.
- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$.

Examples (GK20191)

Let z_1, z_2 be two nonzero complex numbers. Show that $\left|\frac{\bar{z}_1}{z_2} + \frac{\bar{z}_2}{z_1}\right| \geq 2.$

- $\left|\frac{\bar{z}_1}{z_2} + \frac{\bar{z}_2}{z_1}\right| = \left|\frac{\bar{z}_1 z_1 + \bar{z}_2 z_2}{z_1 z_2}\right| = \left|\frac{|z_1|^2 + |z_2|^2}{z_1 z_2}\right| = \frac{|z_1|^2 + |z_2|^2}{|z_1||z_2|}.$
- By Cauchy's inequality, we have $\frac{|z_1|^2 + |z_2|^2}{|z_1||z_2|} \geq 2.$
- We are done.

1.3.4. Polar form of a complex number

- Each complex number $z = a + bi$ can be represented by a point $M(a, b)$ on the complex plane Oxy. Conversely, each point $M(a, b)$ represents a complex number $z = a + bi$.
- The x-axis represents real numbers. This axis is called the real axis. The y-axis represents pure imaginary numbers. This axis is called the imaginary axis.

Polar (trigonometric) form of a complex number

Consider a complex number $z = a + bi \neq 0$ represented by a point $M(a, b)$. Set $r = OM = |z| = \sqrt{a^2 + b^2}$. Set $\varphi = (Ox, \vec{OM})$ (góc lượng giác), (an) argument of z. (Argument of z is determined upto $2k\pi$, $k \in \mathbb{Z}$.)

Then $\cos\varphi = \dfrac{a}{\sqrt{a^2 + b^2}}$, $\sin\varphi = \dfrac{b}{\sqrt{a^2 + b^2}}$ and

$$z = r(\cos\varphi + i\sin\varphi).$$

Multiplication, division and taking powers in polar forms

Proposition

Consider two nonzero complex numbers: $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$. Then

1. $z_1 z_2 = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$,
2. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$.

Corollary (Moivre's formula)

Let $z = r(\cos\varphi + i\sin\varphi)$. Then, for all $n \in \mathbb{Z}$

$$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi)).$$

If $r = 1$, then Moivre's formula becomes

$$(\cos(\varphi) + i\sin(\varphi))^n = \cos(n\varphi) + i\sin(n\varphi).$$

Examplle (GK20191)

Let $z = \dfrac{\sqrt{2} - i\sqrt{2}}{2}$. Find $z^{2019} + (\bar{z})^{2019}$.

- $z = \dfrac{\sqrt{2} - i\sqrt{2}}{2} = \cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})$.
- $z^{2019} = \left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)^{2019} = \cos(-\frac{2019\pi}{4}) + i\sin(-\frac{2019\pi}{4}) = \cos(-\frac{3\pi}{4}) + i\sin(-\frac{3\pi}{4})$
- $(\bar{z})^{2019} = \overline{z^{2019}} = \cos(-\frac{3\pi}{4}) - i\sin(-\frac{3\pi}{4})$
- $z^{2019} + (\bar{z})^{2019} = 2\cos(-\frac{3\pi}{4}) = -\sqrt{2}$.

The nth roots of a complex number

Let z be a complex numbe and let $n \geq 2$ be a natural number. A complex number u satisfying $u^n = z$, is called an nth root of z.
If $z = 0$, then there is only one nth root of 0, which is 0.

nth roots of a complex number

Consider $z = r(\cos\varphi + i\sin\varphi) \neq 0$. Then there are exactly nth roots of z, they are

$$z_k = \sqrt[n]{r}(\cos(\frac{\varphi + 2k\pi}{n}) + i\sin(\frac{\varphi + 2k\pi}{n})), \text{ với } k = 0, 1, 2, \ldots, n-1.$$

Example (CK20181)

Find all complex numbers z satisfying that $z^3 = 4\sqrt{3} - 4i$, where i is the imaginary unit.

- $z^3 = 4\sqrt{3} - 4i = 8(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6}))$
- All complex numbers z satisfying the equation are

$$
\begin{aligned}
z &= 2(\cos(\frac{-\frac{\pi}{6} + 2k\pi}{3}) + i\sin((\frac{-\frac{\pi}{6} + 2k\pi}{3})) \\
&= 2(\cos(-\frac{\pi}{18} + \frac{2k\pi}{3}) + i\sin(-\frac{\pi}{18} + \frac{2k\pi}{3}), \text{ với } k = 0, 1, 2.
\end{aligned}
$$

Should avoid: $z^3 = 4\sqrt{3} - 4i = 8(\cos(\frac{\pi}{6}) - i\sin(\frac{\pi}{6}))$, hence $z = 2(\cos(\frac{\pi}{18} + \frac{2k\pi}{3}) - i\sin(\frac{\pi}{18} + \frac{2k\pi}{3}),$ với $k = 0, 1, 2.$

Some exercises

- (GK20201) Solve the following equation in $\mathbb{C}$: $(1+i\sqrt{3})^{11}z^3 = (\sqrt{3}+i)^{20}$.
- (GK20201-N2) Find all complex numbers z satisfying that $z^6(1+i)^4 = (2-i\sqrt{12})^6$.
- (GK20201-N3) Determine the real part and the imaginary part of $z = (1+i)^8(2-i\sqrt{12})^{2020}$.
- (GK20191) Find all complex numbers z satisfying that $(z+i)^{10} - (z-i)^{10} = 0$.
- (GK20191-N3) Solve the following equation in $\mathbb{C}$: $(z-2i)^3(1+i\sqrt{3}) = -16i$.
- (*GK20181) Let $z_n = \left(\frac{1+i\sqrt{3}}{\sqrt{3}+i}\right)^n$. Find the smallest natural number n such that $\mathrm{Re}(z_n) = 0$.
- (GK20181) Consider the mapping $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^5 + \sqrt{3}$. Find $f^{-1}(\{i\})$.
- (GK20181-N2) Solve the following equation in $\mathbb{C}$: $(z+i)^4 = (2z-i)^4$.
- (GK20171) Find all complex numbers z such that $z^3 + 2i|z|^2 = 0$.
- (GK20171-N2) Solve the following equation in $\mathbb{C}$: $(3z+4)^9 = 1+i$.

Some exercises*

- (GK20201-N3) Show that for every natural number $n > 1$, all (complex) roots of the equation $\left(\frac{z+i}{z-i}\right)^n = 1$ are real numbers.
- (GK20181) Evaluate the sum $S = C_{2018}^0 - 3C_{2018}^2 + 3^2 C_{2018}^4 - 3^3 C_{2018}^6 + \cdots - 3^{1009} C_{2018}^{2018}$.
- (GK20171) Let z_1, z_2 be two complex root of the equation $z^2 - z + ai = 0$, where a is a real number and i is the imaginary unit. Find a such that $|z_1^2 - z_2^2| = 1$.
- (GK20161) Let $\epsilon_k = \cos(\frac{k2\pi}{2016}) + i\sin(\frac{k2\pi}{2016})$, $k = 0, 1, \ldots, 2015$. Evaluate $S = \sum_{k=0}^{2015} \epsilon_k^{2015}$.
- (GK20161) Let z_1, z_2 be two complex root of the equation $iz^2 + (2 - i)z + 5 = 0$. Find $\left|\frac{z_1}{z_2} - \frac{z_2}{z_1}\right|$.

Quadratic equations

Consider an equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{C}$, $a \neq 0$. We solve this equation as follows.

- Calculate $\Delta = b^2 - 4ac$.
- Let δ be a complex square root of lΔ.
- All roots of the equation are

$$z_{1,2} = \frac{-b \pm \delta}{2a}.$$

Example (GK20191-N2)

Solve the following equation in $\mathbb{C}$: $z^2 - (3-i)z + 4 - 3i = 0$.

- $\Delta = (3-i)^2 - 4(4-3i) = -8 + 6i = (1+3i)^2$.
- Two roots are $z_1 = \dfrac{(3-i)+(1+3i)}{2} = 2 + i$ và $z_2 = \dfrac{(3-i)-(1+3i)}{2} = 1 - 2i$.

Polynomials

Let F be a field (for example, $F = \mathbb{C}$, $F = \mathbb{R}$).

- A polynomial in one variable x over F is a formal expression

$$p(x) = a_n x^n + \cdots + a_1 x + a_0,$$

  where $a_0, a_1, \ldots, a_n \in F$. The numbers $a_0, a_1, \ldots, a_n$ are called the coefficients of the polynomial $p(x)$.
- If $a_n \neq 0$ then we say that $p(x)$ has degree n, and write $\deg p(x) = n$.
- If $a_n = \cdots = a_1 = a_0 = 0$, then $p(x)$ is called the zero polynomial, and we make a convention that the degree of the zero polynomial is $-\infty$.
- We can define the addition and multiplication of two polynomials.
- The set of polynomials in variable x with coefficients over F is denoted by $F[x]$.
- An element $\alpha \in F$ is called a root (or zero) of $p(x)$ if $p(\alpha) = a_n \alpha^n + \cdots + a_1 \alpha + a_0 = 0$.

Polynomial division

Let $p(x)$ and $q(x) \neq 0$ be polynomials in $F[x]$. Then there exist unique polynomials $a(x)$ and $r(x) \in F[x]$ such that

$$p(x) = a(x)q(x) + r(x),$$

and $\deg(r(x)) < \deg(q(x))$.

If $r(x) = 0$ then we say that $p(x)$ is divisible by $q(x)$ (or $q(x)$ divides $p(x)$).

Fact (Bezout) A number $\alpha \in F$ is a root of $p(x)$ if and only if $p(x)$ is divisible by $x - \alpha$.

1.3.5. Fundamental theorem of algebra

Consider a polynomial $p(x)$ with complex coefficients and of degree $n \geq 1$:

$$p(x) = a_n x^n + \cdots + a_1 x + a_0 \quad (a_i \in \mathbb{C}, i = 0, \ldots, n, a_n \neq 0).$$

- (D'Alembert) $p(x)$ always has at least one complex root.
- (Định lý cơ bản của đại số) Polynomials $p(x)$ of degree n has exactly n complex roots (counted with multiplicity), and we have a factorization

$$p(x) = a_n(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n),$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

Some exercises

- (GK20201) Solve the following equation in $\mathbb{C}$: $1 + z + z^2 + z^3 + z^4 = 0$.
- (GK20191) Find all complex solutions to the equation: $z^{10} + z^5 + 1 = 0$.
- (GK20181) Find all $z \in \mathbb{C}$ such that $1 + (z + 2i) + (z + 2i)^2 + (z + 2i)^3 + (z + 2i)^4 = 0$.
- (CK20181) Let $f : \mathbb{C} \to \mathbb{C}$ be a map defined by $f(z) = 2z^3 - 1$. Is f injective?có phải là đơn ánh không vì sao? Find the product of the modulus of the complex numbers in the set $f^{-1}(\{5 + 2i\})$.
- (GK20171-N3) Consider the map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^6(m - i\sqrt{3})^{12}$, $m \in \mathbb{R}$.
  - Find m sucht that f is surjective.
  - When $m = 1$, find $f^{-1}(\{(\sqrt{3} + i)^6\})$.

Polynomials with real coefficients

Consider the following polynomial with real coefficients and of degree n:

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad (a_i \in \mathbb{R}, i = 0, \ldots, n).$$

Then

- If $\alpha \in \mathbb{C}$ is a complex root of $p(x)$ then $\bar{\alpha}$ is a root of $p(x)$.
- Polynomial $p(x)$ can be factorized as a product of degree one factors and degree two factors with negative discriminant:

  $$p_n(x) = a_n(x - b_1)\cdots(x - b_r)(x^2 + c_1 x + d_1)\cdots(x^2 + c_s x + d_s),$$

  where $b_1, \ldots, b_r, c_1, \ldots, c_s, d_1, \cdots, d_s$ are real numbers and $c_k^2 - 4d_k < 0$, for $k = 1, \ldots, s$.

Example (GK20161)

Factorize $p(x) = x^4 + 2x^3 + 7x^2 + 8x + 12$ as a product of two polynomials with real coefficients and of degree 2, provided that $p(2i) = 0$.

- Since $2i$ is a root of $p(x)$, $-2i$ is also of a root of $p(x)$ làm nghiệm.
- Polynomial $p(x)$ is divisible by $(x - 2i)(x + 2i) = x^2 + 4$.
- $p(x) = (x^2 + 4)(x^2 + 2x + 3)$.

Example (GK20201)

Factorize $f(x) = (x^2 - 4x + 5)^2 + (x + 1)^2$ as a product of two polynomials with real coefficients and of degree 2, provided that $f(1 + i) = 0$.