Linear Algebra

Dr. Bui Xuan Dieu

School of Applied Mathematics and Informatics,
Hanoi University of Science and Technology

Some Information

About this course

1) 4 Credit points, 3 hours of lecture and 2 hours of exercises and discussion per week.
2) Grade:

$$\text{Progress Grades } 30\% \text{ and Final Exam Grade } 70\%, \text{ where}$$
$$\text{Progress Grades} = \text{Midterm Exam Grades} + \text{Attendance Grades}$$

Chapter 1: Sets, mapping and complex numbers

Propositions

Definition

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i) Any proposition has two possible truth values: $1 =$ true or $0=$ false.

ii) For notational simplicity, the symbol A may stand for the proposition A or its truth-value, depending on the situation.

Example

i) $A = 2017$ *is an odd number,* $V(A) = 1$.

ii) $B =$ *There exists life outside the earth,* $V(B) =?$

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5) Equivalence

A	B	$A \leftrightarrow B$
1	1	1
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Properties

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5) Property of the implication operator $A \rightarrow B \Leftrightarrow \overline{A} \vee B$

6) Property of the equivalence operator

$$A \leftrightarrow B \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$$

How to prove a proposition is a tautology?

Prove that the following proposition is tautology $\left[\overline{A} \wedge (A \vee C)\right] \to C$.

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A	C	$\overline{A}$	$A \vee C$	$\overline{A} \wedge (A \vee C)$	$[\overline{A} \wedge (A \vee C)] \to C$
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1	0	0	1	0	1
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0	0	1	0	0	1

2) **Logical Equivalence**

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2) **Logical Equivalence**

$$
\begin{aligned}
&[\overline{A} \wedge (A \vee C)] \rightarrow C \\
\Leftrightarrow &[(\overline{A} \wedge A) \vee (\overline{A} \wedge C)] \rightarrow C \\
\Leftrightarrow &[0 \vee (\overline{A} \wedge C)] \rightarrow C \\
\Leftrightarrow &[(\overline{A} \wedge C)] \rightarrow C \Leftrightarrow \overline{\overline{A} \wedge C} \vee C \\
\Leftrightarrow &A \vee \overline{C} \vee C \Leftrightarrow 1.
\end{aligned}
$$

3) **by contradiction**.

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\Leftrightarrow &[(\overline{A} \wedge C)] \to C \Leftrightarrow \overline{\overline{A} \wedge C} \vee C \\
\Leftrightarrow &A \vee \overline{C} \vee C \Leftrightarrow 1.
\end{aligned}
$$

3) **by contradiction**. Suppose that the propostion is false. Then,

i) $\overline{A} \wedge (A \vee C) = 1$ and $C = 0$.

ii) $\overline{A} \wedge (A \vee C) = \overline{A} \wedge (A \vee 0) = \overline{A} \wedge A = 0$.

Exercises

Exercise

Show that the following propositions are tautology

a) $[(A \to B) \wedge (B \to C)] \to (A \to C)$.

Exercise

Which of the following propositions are tautology, contradiction

a) $(p \vee q) \to (p \wedge q)$,

b) $(p \wedge q) \vee (p \to q)$,

c) $p \to (q \to p)$,

d) $(q \to (q \to p))$,

e) $(p \to q) \to q$,

f) $(p \wedge q) \leftrightarrow (q \updownarrow p)$.

Exercise

Prove that

a) $A \leftrightarrow B$ *and* $(A \wedge B) \vee (\overline{A} \wedge \overline{B})$ *are logically equivalent.*

b) $(A \to B) \to C$ *and* $A \to (B \to C)$ *are not logically equivalent.*

Binary operators

1) Binary operator XOR

A	B	$A \updownarrow B$
1	1	0
1	0	1
0	1	1
0	0	0

$$(A \updownarrow B) = \overline{A \leftrightarrow B}$$

2) Binary operator NOR

A	B	$A \uparrow B$
1	1	0
1	0	0
0	1	0
0	0	1

$$(A \uparrow B) = \overline{A \vee B}$$

3) Binary Operator NAND

A	B	$A \downarrow B$
1	1	0
1	0	1
0	1	1
0	0	1

$$(A \downarrow B) = \overline{A \wedge B}$$

Propositions with quantifiers $\forall, \exists$

1) "Every element x of the set X satisfies property $\mathcal{P}(x)$"

$$\forall x \in X, \mathcal{P}(x).$$

2) "There exists at least one element x of the set X that satisfies properties $\mathcal{P}(x)$"

$$\exists x \in X, \mathcal{P}(x).$$

Relations

$$\overline{\forall x \in X, \mathcal{P}(x)} = \exists x \in X, \overline{\mathcal{P}(x)}$$

$$\overline{\exists x \in X, \mathcal{P}(x)} = \forall x \in X, \overline{\mathcal{P}(x)}$$

Propositions with quantifiers $\forall, \exists$

Remark

To receive the negation of a proposition containing qualifiers $\forall, \exists$ *and statement* $P(x_1, \ldots, x_n)$, *we*

1) *change* $\forall$ *by* $\exists$,
2) *change* $\exists$ *by* $\forall$,
3) *change* $P(x_1, \ldots, x_n)$ *by* $\bar{P}(x_1, \ldots, x_n)$.

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1) *change $\forall$ by $\exists$,*
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3) *change $P(x_1, \ldots, x_n)$ by $\bar{P}(x_1, \ldots, x_n)$.*

Exercise

Find the negation p if

a) $p =$ "$\forall \epsilon > 0, \exists \delta > 0 : \forall x, |x - x_0| < \delta, |f(x) - f(x_0)| < \epsilon.$"

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a) $p =$ "$\forall \epsilon > 0, \exists \delta > 0 : \forall x, |x - x_0| < \delta, |f(x) - f(x_0)| < \epsilon.$"
b) $p = \lim\limits_{n \to +\infty} x_n = \infty \Leftrightarrow \forall M > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n| > M.$
c) $p = \lim\limits_{n \to +\infty} x_n = L \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - L| < \epsilon.$

Chapter 1: Sets, mapping and complex numbers

Definitions and notations

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2) Let A be a set. If a is an element of A, then we denote by $a \in A$. Otherwise, $a \notin A$.
3) The set containing no any element is called the empty set and denoted by $\emptyset$.
4) Description of a set:
   i) Roster notation (or listing notation).
   ii) Set-builder notation.
   iii) Venn diagram.

Algebra of sets

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6) Complement
   If $A \subset X$, then $\overline{A} = X \setminus A$ is called the complement of A in X.

Algebra Sets

Example

Let

$$A = \{x \in \mathbb{R} | x^2 - 4x + 3 \leq 0\}, \quad B = \{x \in \mathbb{R} || x - 1| \leq 1\},$$

and

$$C = \{x \in \mathbb{R} | x^2 - 5x + 6 \leq 0\}.$$

Find $(A \cup B) \cap C$ and $(A \cap B) \cup C$.

Set Identities

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4) Property of the complement
If $A, B \subset X$, then $A \setminus B = A \cap \overline{B}$

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4) Property of the complement
   If $A, B \subset X$, then $A \setminus B = A \cap \overline{B}$
5) De Moorgan's Law
$$\overline{A \cap B} = \overline{A} \cup \overline{B}, \quad \overline{\cap A_i} = \cup \overline{A_i}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{\cup A_i} = \cap \overline{A_i}$$

Mathematical Logic and Sets

1) Negation $\overline{A}$
2) Conjunction $A \wedge B$
3) Disjunction $A \vee B$
4) Implication $A \Rightarrow B$
5) Equivalence $A \Leftrightarrow B$

1) Complement $\overline{A} = X \setminus A$
2) Intersection $A \cap B$
3) Union $A \cup B$
4) Inclusion $A \subset B$
5) Set equality $A = B$

Three possible methods to prove set equality

1) Double inclusion
2) Set identities
3) Membership tables.

Proving set equality

Example

Prove that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

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Double inclusion

$\boxed{\Rightarrow}$ Suppose that $x \in A \cap (B \setminus C)$

Need to prove $x \in (A \cap B) \setminus (A \cap C)$.

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Double inclusion

$\boxed{\Rightarrow}$ Suppose that $x \in A \cap (B \setminus C)$
Need to prove $x \in (A \cap B) \setminus (A \cap C)$.

$\boxed{\Leftarrow}$ Suppose $x \in (A \cap B) \setminus (A \cap C)$
Need to prove $x \in A \cap (B \setminus C)$.

Proving set equality

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Set identities

$$
\begin{aligned}
&(A \cap B) \setminus (A \cap C) \\
=&(A \cap B) \cap (\overline{A} \cup \overline{C}) \\
=&[(A \cap B) \cap \overline{A}] \cup [A \cap B \cap \overline{C}] \\
=&A \cap (B \setminus C).
\end{aligned} \tag{1}
$$

Proving set equality

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Membership tables

A	B	C	$B \setminus C$	$A \cap (B \setminus C)$	$A \cap B$	$A \cap C$	$(A \cap B) \setminus (A \cap C)$
1	1	1	0	0	1	1	0
1	1	0	1	1	1	0	1
1	0	1	0	0	0	1	0
1	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

1) "1" =membership, "0" =non-membership.
2) Two sets are equal, iff they have identical columns.

Proving set equality

Example

Let A, B, C be arbitrary sets. Prove that

a) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

b) $A \cup (B \setminus A) = A \cup B$.

c) If $(A \cap C) \subset (A \cap B)$ and $(A \cup C) \subset (A \cup B)$, then $C \subset B$.

d) $A \setminus (A \setminus B) = A \cap B$.

e) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

f) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

g) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

h) Is it true that $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$. If not, give a counterexample.

The cartesian product: Let A, B be sets.

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}, \quad (a, b) = (c, d) \Leftrightarrow a = c, b = d.$$

Chapter 1: Sets, mapping and complex numbers

Maps

Definition

$$f : X \to Y,$$
$$x \mapsto y \in Y(unique)$$

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Image, Preimage

Let $f : X \to Y$ and $A \subset X, C \subset Y$.

i) **The Image**
$f(A) = \{y \in Y | \exists x \in A, f(x) = y\}$.

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$\boxed{y \in f(A) \Leftrightarrow \exists x \in A : y = f(x)}$ $\boxed{x \in f^{-1}(C) \Leftrightarrow f(x) \in C}$

Image, preimage

Properties

Let $f : X \to Y$ and $A, B \subset X, \quad C, D \subset Y$.

a) $f(A \cup B) = f(A) \cup f(B)$,

b) $f(A \cap B) \subset f(A) \cap f(B)$,

c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$,

d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

i) $y \in f(A) \Leftrightarrow \exists x \in A : y = f(x)$,

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Image, preimage

Example

Let $f: X \to Y$ and $A, B \subset X, \quad C, D \subset Y$. Prove that

e) $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$,

f) $A \subset f^{-1}(f(A))$,

g) $C \supset f(f^{-1}(C))$.

Image, preimage

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Example

Let $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (2x, 2y)$ and

$$A = \{(x, y) \in \mathbb{R}^2 | (x - 4)^2 + y^2 = 4\}.$$

Find $f(A), f^{-1}(A)$.

Maps

Injective,surjective, bijective mappings

Let $f : X \to Y$ be a map.

a) **Injective**

i) $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, or

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b) **Surjective**

i) $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$, or

ii) $\forall y \in Y$, the eq. $f(x) = y$ has **at least** one solution.

c) **Bijective** $=$ injective $+$ surjective.

$\forall y \in Y$, the eq. $f(x) = y$ has **a unique solution**.

Injective,surjective, bijective mappings

Example

Which of the following maps are injective, surjective, bijective?

a) $f:\mathbb{R}\to\mathbb{R}, f(x)=3-2x$,

b) $f:(-\infty,0]\to[4,+\infty), f(x)=x^2+4$,

c) $f:(1,+\infty)\to(-1,+\infty), f(x)=x^2-2x$,

d) $f:\mathbb{R}\setminus\{1\}\to\mathbb{R}\setminus\{3\}, f(x)=\frac{3x+1}{x-1}$,

e) $f:[4,9]\to[21,96], f(x)=x^2+2x-3$,

f) $f:\mathbb{R}\to\mathbb{R}, f(x)=3x-2|x|$,

g) $f:(-1,1)\to\mathbb{R}, f(x)=\ln\frac{1+x}{1-x}$,

h) $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}, f(x)=\frac{1}{x}$,

i) $f:\mathbb{R}\to\mathbb{R}, g(x)=\frac{2x}{1+x^2}$.

Composition of maps, inverse maps

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Let $f: X \to Y, \quad g: Y \to Z$.

i) Composition $(g \circ f)(x) = g(f(x))$.

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Composition of maps, inverse maps

Example

Let $f : X \to Y, \quad g : Y \to Z$. Prove that

a) f surjective and $g \circ f$ injective $\Rightarrow g$ injective,

b) give an example to show that $g \circ f$ is injective, but g is not,

c) g injective and $g \circ f$ surjective $\Rightarrow f$ surjective,

d) give an example to show that $g \circ f$ is surjective but f is not.

Restriction, characteristic functions

Restriction

i) Let $f : X \to Y$ and $A \subset X$. The restriction $f_A : A \to Y$ given by $f_A(x) = f(x)\ \forall x \in A$.

ii) g is the restriction of $f \Rightarrow f$ is an extension of g.

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Characteristic functions

Let $A \subset X$, the map $f : A \to \{0, 1\}$ given by $f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases}$

is called the characteristic function.

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The canonical projection

Let $X = X_1 \times X_2$. The map $p_1 : X \to X_1$ given by $p(x_1, x_2) = x_1$ is called the canonical projection on X_1.

Substitutions

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A bijection $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$ is called a substitution (or permutation).

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Example

a) Let $|X| = m, |Y| = n$. Find the number of maps from X to Y.
b) Let $|X| = m, |Y| = n$, where $m < n$. Find the number of injection from X to Y.

Substitutions

Cycle

i) cycle of length k

$$(i_1, i_2, \ldots, i_k) \Leftrightarrow \begin{cases} f(i_1) &= i_2, \\ f(i_2) &= i_3, \\ &\vdots \\ f(i_k) &= i_1 \end{cases} \text{ and } f(j) = j \text{ otherwise.}$$

ii) A cycle of length 2 is called a transposition.

Substitutions

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ii) A cycle of length 2 is called a transposition.

Theorem

i) Any substitution is a product of cycles.

ii) Any substitution is a product of transpositons.

Substitutions

Example

Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 2 & 1 & 5 & 7 & 6 & 9 & 10 & 8 \end{pmatrix},$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 1 & 2 & 5 & 7 & 6 & 9 & 8 & 10 \end{pmatrix}$$

i) Compute σ^{-1} and $\tau \circ \sigma$.

ii) Write σ, τ as a product of disjoint cycles.

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Example

Let $|X| = n$ and f be a bijection from X to X. Prove that there exists $k \in \mathbb{N}$ such that $f^k = \mathrm{Id}_X$, where $f^k = f \circ f \cdots \circ f$ (k-times).

Substitutions

Parity of a permutation

Let $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$ be a permutatiton.

i) (i, j) is called an inversion if $i < j$ and $f(i) > f(j)$.

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i) (i, j) is called an inversion if $i < j$ and $f(i) > f(j)$.

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The sign of a permutation

Example

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iii) Compute $\text{sign}(\sigma), \text{sign}(\tau)$.

Chapter 1: Sets, mapping and complex numbers

Binary operators

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Let G be a set. A binary operator on G is a map

$$\begin{aligned} * : G \times G &\to G, \\ (x, y) &\mapsto x * y. \end{aligned}$$

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iv) $x' \in X$ is called the inverse element of x if $x * x' = x' * x = e$.

Binary operators

i) commutative $a * b = b * a$.

ii) associative $(a * b) * c = a * (b * c)$.

iii) identity $a * e = e * a = a$.

iv) the inverse $x * x' = x' * x = e$.

Example

Consider the commutativity, associativity and find the identity element, the inverse element.

a) $x * y := xy + 1$,

b) $x * y := \frac{1}{2}xy$,

c) $(x_1, x_2) \circ (y_1, y_2) := \left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right)$.

Groups

Definition

A group is a pair $(G, *)$ *satisfies*

G1) *Associativity:*

$$(x * y) * z = x * (y * z), \quad \forall x, y, z \in G,$$

G2) $\exists$ *the identity element* e

$$x * e = e * x = x, \quad \forall x \in G,$$

G3) $\exists$ *the inverse for any* $x \in G$

$$x * x' = x' * x = e.$$

Groups

Definition

A group is a pair $(G, *)$ *satisfies*

G1) *Associativity:*

$$(x * y) * z = x * (y * z), \quad \forall x, y, z \in G,$$

G2) $\exists$ *the identity element* e

$$x * e = e * x = x, \quad \forall x \in G,$$

G3) $\exists$ *the inverse for any* $x \in G$

$$x * x' = x' * x = e.$$

G is called *commutative* or abelian if $x * y = y * x, \ \forall x, y \in G$.

Groups

	Logics		Sets		Maps	$\mathbb{N}$		$\mathbb{Z}$		$\mathbb{Q}$		$\mathbb{R}$	
	$\vee$	$\wedge$	$\cup$	$\cap$	$\circ$	$+$	$\times$	$+$	$\times$	$+$	$\times$	$+$	$\times$
Asso.	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Identity	F	T	$\emptyset$	X	Id	0	1	0	1	0	1	0	1
Inverse	x	x	x	x	f^{-1}	✓	x	✓	x	✓	x	✓	x
Commu.	✓	✓	✓	✓	x	✓	✓	✓	✓	✓	✓	✓	✓

i) Logics $=$ the set of propositions,

ii) Sets $= P(X)$ the collection of subsets of X,

iii) Maps $= B(X)$ the set of bijection from X to X.

Open your mind

$$1+1=1$$

$$1+2=2 \qquad 2+1=2$$

$$2+2 = 1$$

Open your mind

$$1+1=1$$

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wtf? No, I'm not kidding!

Open your mind

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wtf? No, I'm not kidding!

Prove that $(\{1,2\},+)$ is a group.

Groups

Properties

1) the identity element e is unique.
2) the inverse element x' of x is unique.

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Notationally,

i) $"+"$ the identity $e := 0$ and the inverse element of x is $-x$.

ii) $"\times"$ the identity $e =: 1$ and the inverse element of x is x^{-1}.

Groups

Example

Let X be arbitrary set and $x * y = x, \forall x, y \in X$. Prove that $(X, *)$ is a semigroup.

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Example

1) $(\mathbb{N}, +)$,

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1) $(\mathbb{N}, +), (\mathbb{N}, \times), (\mathbb{Z}, +), (\mathbb{Z}, \times), (\mathbb{Q}, +), (\mathbb{Q}, \times), (\mathbb{R}, +), (\mathbb{R}, \times)$,
2) $(\mathbb{Z}/5, +)$,

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3) $(2\mathbb{Z}, +)$,

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Group homomorphism

Group homomorphism

Let G and G' be groups. The map $\varphi : G \to G'$ is called a group homomorphism if

$$\varphi(xy) = \varphi(x)\varphi(y),\ \forall x, y \in G.$$

Remark: $\varphi(e) = e', \quad \varphi(x^{-1}) = [\varphi(x)]^{-1}.$

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1) *Group homomorphism + injective = Monomorphism,*
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3) *Group homomorphism + bijective = Isomorphism, write* $G \cong G'$.

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Example

i) $i : \mathbb{Z} \to \mathbb{Q}, n \mapsto n$ is a Monomorphism.

ii) The projection $p : \mathbb{Z} \to \mathbb{Z}/n$ is an Epimorphism.

iii) $\exp : \mathbb{R} \to \mathbb{R}^+, x \mapsto e^x$ is an Isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \times)$.

Rings

Definition

A ring is a tripple $(R, +, \times)$ *satisfies*

R1) $(R, +)$ *is a commutative group.*

R2) $\times$ *is associative:*

$$(xy)z = x(yz), \quad \forall x, y, z \in R$$

R3) *distributive*

$$\begin{aligned}(x+y)z &= xz + yz \\ z(x+y) &= zx + zy, \quad \forall x, y, z \in R\end{aligned}$$

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i) *commutative* or *abelian* $xy = yx, \quad \forall x, y \in R.$

ii) *Ring with identity* $\exists e$ such that $ex = xe = x, \quad \forall x \in R.$

Rings

	$\mathbb{N}$		$\mathbb{Z}$		$\mathbb{Q}$		$\mathbb{R}$	
	$+$	$\times$	$+$	$\times$	$+$	$\times$	$+$	$\times$
R1	x		✓		✓		✓	
R2	✓		✓		✓		✓	
R3	✓		✓		✓		✓	
commutative		✓		✓		✓		✓
with identity		✓		✓		✓		✓

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R3	✓		✓		✓		✓	
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with identity		✓		✓		✓		✓

Example

Let $X = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$ and $Y = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$.

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2},$$

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2},$$

Fields

Definition

F is a field if

i) F is a commutative ring with identity $1 \neq 0$.

ii) $\exists$ *the inverse element* x^{-1} *for every* $x \neq 0$.

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The characteristic of a field

Definition

Let R be a ring with identity 1. The smallest natural number n such that $0 = 1 + 1 + \cdots + 1$ (n terms) is called the characteristic of the ring R and is denoted by $\mathrm{Char}(R)$. If there is no such natural number, then the characteristic is zero.

Example

a) $\mathrm{Char}(\mathbb{Z}) = \mathrm{Char}(\mathbb{Q}) = 0$,

b) $\mathrm{Char}(\mathbb{Z}/n) = n$.

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Example

a) $\mathrm{Char}(\mathbb{Z}) = \mathrm{Char}(\mathbb{Q}) = 0$,

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Proposition

If R is a field, then $\mathrm{Char}(R)$ is either 0 or a prime number.

Ring of integers

Definition

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$$n = km, \text{ for some } k \in \mathbb{Z}.$$

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If $a, d \in \mathbb{Z}$ *and* $d > 0$ *then there exist unique integers* q *and* r *such that*

$$a = qd + r, 0 \leq r < d.$$

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The number q is called the quotient and r is called the remainder.

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Two integers a, b are said to be congruent modulo n if $n|(a-b)$.

Proposition

The congruence modulo m relation is an equivalence relation.

The set of congruence class modulo m is denoted by

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Proposition

$\mathbb{Z}_m$ together with the addition and multiplication defined as follow

$$\bar{a} + \bar{b} = \overline{a+b}, \quad \bar{a}.\bar{b} = \overline{ab}$$

is a ring.

Euclidean Algorithm

Definition

Given two natural integers a, b.

i) $\mathrm{GCD}(a, b) = \max\{d \text{ such that } d|a, d|b\}$.

ii) $\mathrm{LCD}(a, b) = \min\{d \text{ such that } a|d, b|d\}$.

If $\mathrm{GCD}(a, b) = 1$ then a, b are said to be coprime.

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For natural numbers a, b we have $ab = \text{GCD}(a, b).\text{LCD}(a, b)$.

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Proposition

Suppose that natural numbers a, b, q, r *satisfy*

$$a = bq + r,$$

then $\mathrm{GCD}(a, b) = \mathrm{GCD}(b, r)$.

The greatest common divisor

Euclidean Algorithm

1) Express $a = bq_1 + r_1$,
2) $b = r_1 q_2 + r_2$,
3) $r_1 = r_2 q_3 + r_3$,
4) $\cdots$
5) $r_{k-2} = r_{k-1} q_k + r_k$,
6) The last step $r_{n-1} = r_n q_{n+1}$.

Then $r_n = \text{GCD}(a, b)$.

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Example

Find $\mathrm{GCD}(3195, 630), \mathrm{GCD}(1243, 3124), \mathrm{GCD}(123456789, 987654321)$

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Example

Find integers a, b such that $1243a + 3124b = 11$.

Presentation of integers

Definition

Given a positive integer b. If

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0, \quad 0 \leq a_j < b, a_k \neq 0,$$

then the above presentation is said to be the expansion of n by base b and we denote $n = (a_k a_{k-1} \cdots a_1 a_0)_b$.

If $b = 2$ then we have the binary expansion of n.

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Algorithm for expantion of n by base b.

1) Write $n = bq_0 + a_0$,
2) $q_0 = bq_1 + a_1$,
3) $\cdots$
4) The last step $q_{m-1} = bq_m + a_m$ if $q_m = 0$.

Then $n = (a_m a_{m-1} \ldots a_1 a_0)_b$.

Presentation ofintegers

Example

Presentation the following numbers by the base 6:

a) 2011, b) 3125.

Presentation ofintegers

Example

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a) 2011,

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Example

a) $3145_{(7)} + 5436_{(7)}$,

b) $6145_{(7)} - 5451_{(7)}$,

c) $3142_{(7)} : 6_{(7)}$,

d) $3142_{(7)} \times 54_{(7)}$.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	11	13	15
3	3	6	12	15	21	24
4	4	11	15	22	26	33
5	5	13	21	26	34	42
6	6	15	24	33	42	51

Field of Complex Numbers

Introduction

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ii) The eq. $X^2 + 1 = 0$ has no real solution $\Rightarrow$ construct new numbers.

iii) Let i be a "formal notation" that satisfies $i^2 = -1$. Then, we have numbers of the form $a + bi, a, b \in \mathbb{R}$.

Field of Complex Numbers

Introduction

i) The eq. $X^2 - 2 = 0$ has no rational solution $\Rightarrow$ construct the field of real numbers,

ii) The eq. $X^2 + 1 = 0$ has no real solution $\Rightarrow$ construct new numbers.

iii) Let i be a "formal notation" that satisfies $i^2 = -1$. Then, we have numbers of the form $a + bi, a, b \in \mathbb{R}$.

Let $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. No structure is provided.

The field of complex numbers

We define

$$(a, b) + (c, d) = (a + c, b + d), \quad (a, c)(c, d) = (ac - bd, ad + bc).$$

Proposition

$(\mathbb{C}, +, \times)$ *is a field, called the field of complex numbers.*

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i) The additive identity is

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The field of complex numbers

Real numbers

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Each $(a,0)\in\mathbb{C}$ behaves like $a\in\mathbb{R}$.

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Each $(a,0)\in\mathbb{C}$ behaves like $a\in\mathbb{R}$.

Canonical form

Let $i=(0,1)$, then $i^2=(0,1)(0,1)=(-1,0)\equiv -1$.

$$z=(a,b)=a(1,0)+b(0,1)=a+bi$$

What is i

i is nothing but $(0,1)$

The canonical form of complex numbers

Definition

$z = a + bi$, *where* $a, b \in \mathbb{R}$ *and* $i^2 = -1$, *is called the canonical form of* z.

i) $a = \mathrm{Re}(z)$ *the real part,*

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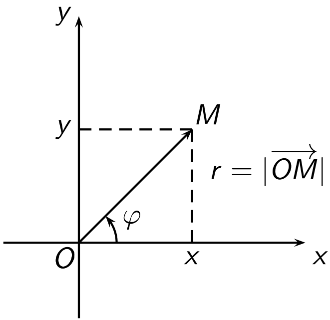
Polar form of complex numbers

Let $\begin{cases} r = |\overrightarrow{OM}| \\ \varphi = (Ox, \overrightarrow{OM}) \end{cases}$ then $z = r(\cos\varphi + i\sin\varphi)$ (the polar form).

i) Modulus $|z| = \sqrt{a^2 + b^2}$,

ii) Argument $\operatorname{Arg} z = \varphi$.



Polar form of complex numbers

Let $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1), \quad z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2).$

Operations in polar form

1) Multiplication

$$z_1 z_2 =$$

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$$z_1 z_2 = r_1 r_2.[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)]$$

Hence, $|z_1 z_2| = |z_1||z_2|, \quad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$

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2) Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}.\left(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)\right)$$

$$\text{Hence } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$

Polar form of complex numbers

Let $z = r(\cos\varphi + i\sin\varphi) \neq 0$.

Operations in polar form

3) Power (Moirve's formula)

$$z^n =$$

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$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right], k = \overline{0, n-1}.$$

Each nonzero complex number has exactly n different n-roots.

The field of complex numbers

Example

Find the canonical form of

a) $(1+i\sqrt{3})^9$,

b) $\frac{(1+i)^{21}}{(1-i)^{13}}$,

c) $(2+i\sqrt{12})^5(\sqrt{3}-i)^{11}$.

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Example

Solve the following equations

a) $z^2 + z + 1 = 0$,

b) $z^2 + 2iz - 5 = 0$,

c) $z^4 - 3iz^2 + 4 = 0$,

d) $z^6 - 7z^3 - 8 = 0$,

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f) $z^8(\sqrt{3} + i) = 1 - i$.

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Example

Prove that if $z+\frac{1}{z}=2\cos\varphi$, then $z^n+\frac{1}{z^n}=2\cos n\varphi, \forall n\in\mathbb{N}$.

The field of complex numbers

Example

a) Find the sum of n-roots of the complex number 1.

b) Find the sum of n-roots of an arbitrary complex number $z \neq 0$.

c) Let $\epsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, k = 0, 1, \ldots, n-1$. Compute $S = \sum_{k=0}^{n-1} \epsilon_k^m, \ (m \in \mathbb{N})$.

Example

Consider the equation $\frac{(z+1)^9 - 1}{z} = 0$.

a) Solve the above equation.

b) Compute the moduli of the solutions.

c) Compute the product of its solutions and $\prod_{k=1}^{8} \sin \frac{k\pi}{9}$.

Conjugation of complex numbers

Let $z = a + bi$.

i) $\overline{z} = a - bi$ is called the conjugate of z.

ii) In polar form, $z = r(\cos\varphi + i\sin\varphi) \Rightarrow \overline{z} = r(\cos\varphi - i\sin\varphi)$.

Properties

1) $\overline{\overline{z}} = z$

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8) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

The field of complex numbers

Example

Solve the following equation

a) $\overline{z^7} = \frac{1}{z^3}$,

b) $z^4 = z + \overline{z}$.

Example

Let x, y, z be complex numbers that satisfy $|x| = |y| = |z| = 1$. Compare the modulus of $x + y + z$ and $xy + yz + zx$.