Introduction to Communications Engineering

IT4593E

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ONE LOVE. ONE FUTURE.

Thông tin chung

- Tên học phần: **Nhập môn kỹ thuật truyền thông**
- Mã học phần: **IT4593E**
- Khối lượng: **2 TC (2-1-0-4)**
- Lý thuyết và bài tập: **10 buổi lý thuyết, 5 buổi bài tập**
- Đánh giá học phần:

  30% QT (kiểm tra + bài tập/project + chuyên cần-quiz)

  70% CK (trắc nghiệm + tự luận)
- Tài liệu tham khảo:
  - Lecture slides
  - Lecture notes
  - Textbooks, ví dụ ***Communication Systems Engineering***, 2nd Edition, by John G. Proakis Masoud Salehi
  - Internet

Lec 04: Decision Theory

4.1 Signal Space Representation

4.1 Decision Theory: Signal Space Representation

Channel Transmission

Binary data sequence $\underline{\mathrm{u}}_T$

⇓

Waveform $s(t)$

⇓

Transmitted over the channel to the destination

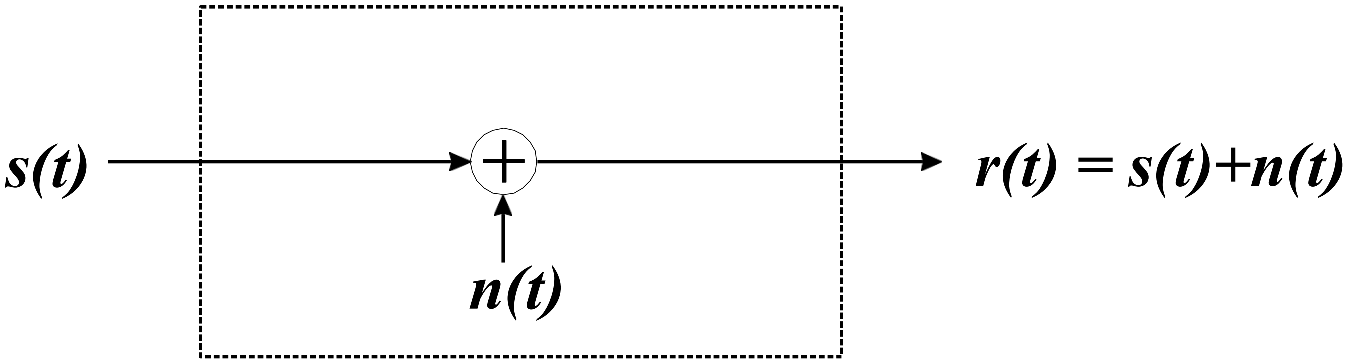
Channel Model

Additive White Gaussian Noise - AWGN

Channel Transmission

AWGN Channel Characteristics

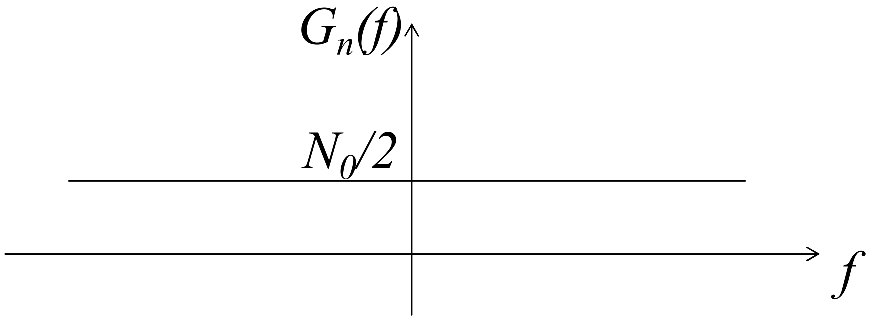
- Linear and time-invariant
- Ideal frequency response: $H(f)=1$
- Additive Gaussian noise: $n(t)$



Channel Transmission

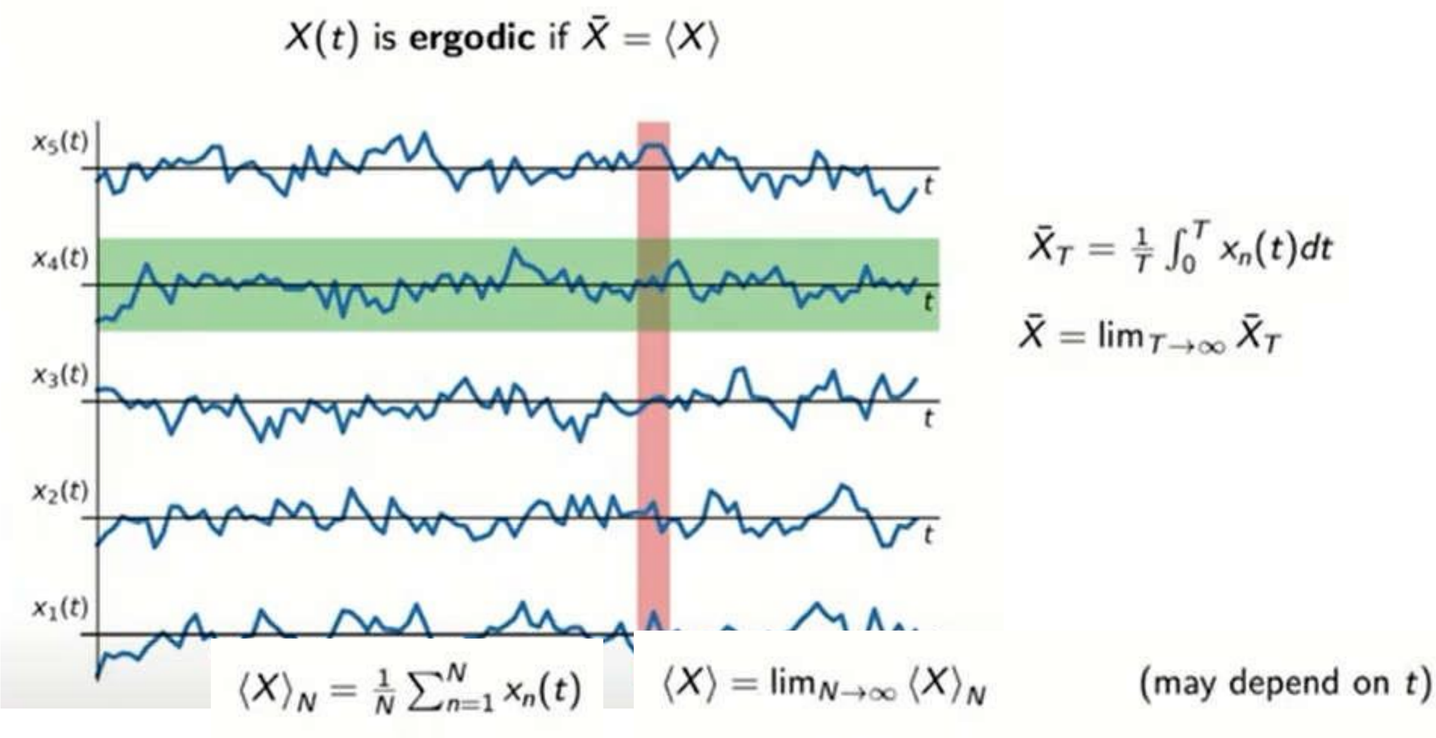
Additive White Gaussian Noise $n(t)$

- **Ergodic random process**
- **Each random variable follows a Gaussian distribution with zero mean**
- **Power spectral density (PSD) is constant:** $G_n(f)=N_0/2$



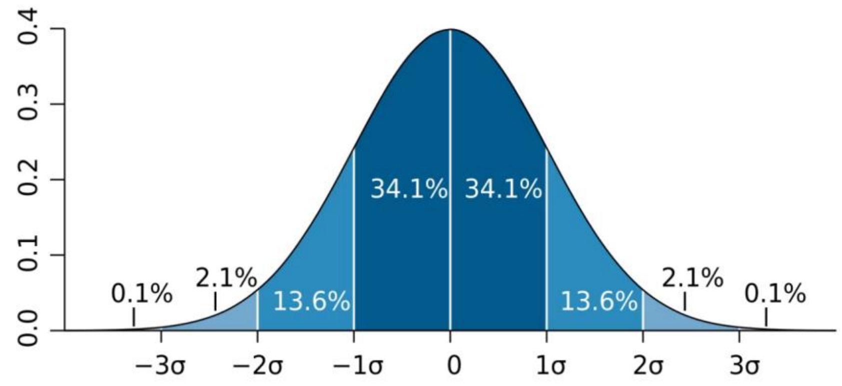
Ergodic Random Process

- A **random process** is called **ergodic** if its statistical characteristics can be inferred from **a sufficiently long sequence** of its samples.



Why is Noise Gaussian Distributed?

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



G → Gaussian

- Why Gaussian
- ***Central limit theorem*** → sum of ***independent and identically distributed (i.i.d)*** random variables approaches normal distribution as sample size N → ∞
- pdf of summation to two random variables is the convolution of their pdf's

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n: number of uniformly distributed random variables, Xi

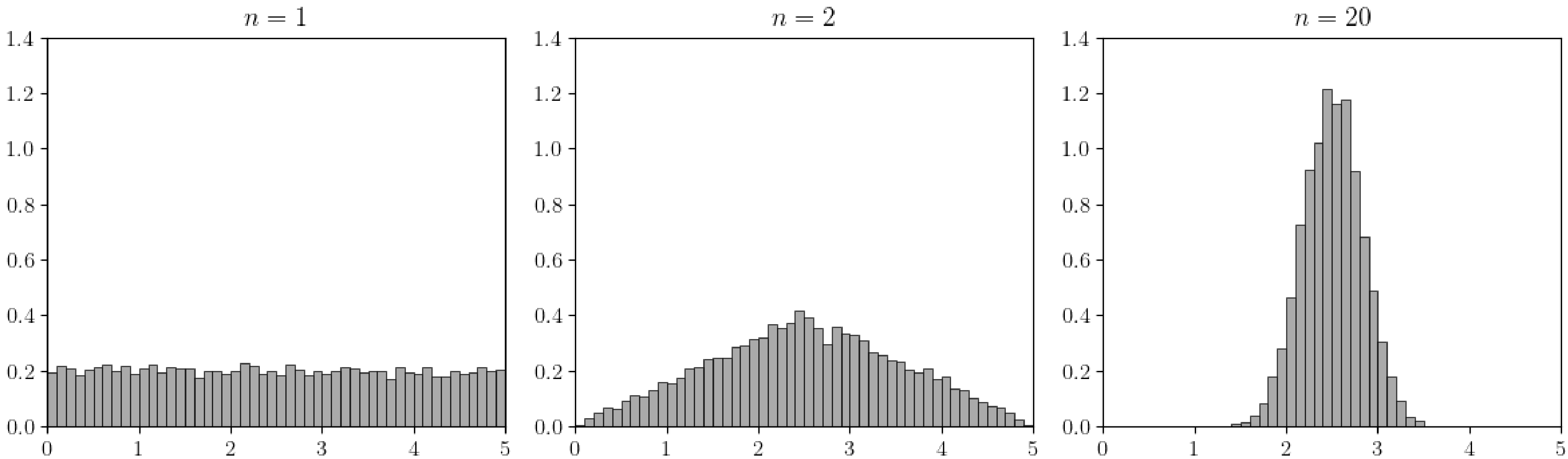
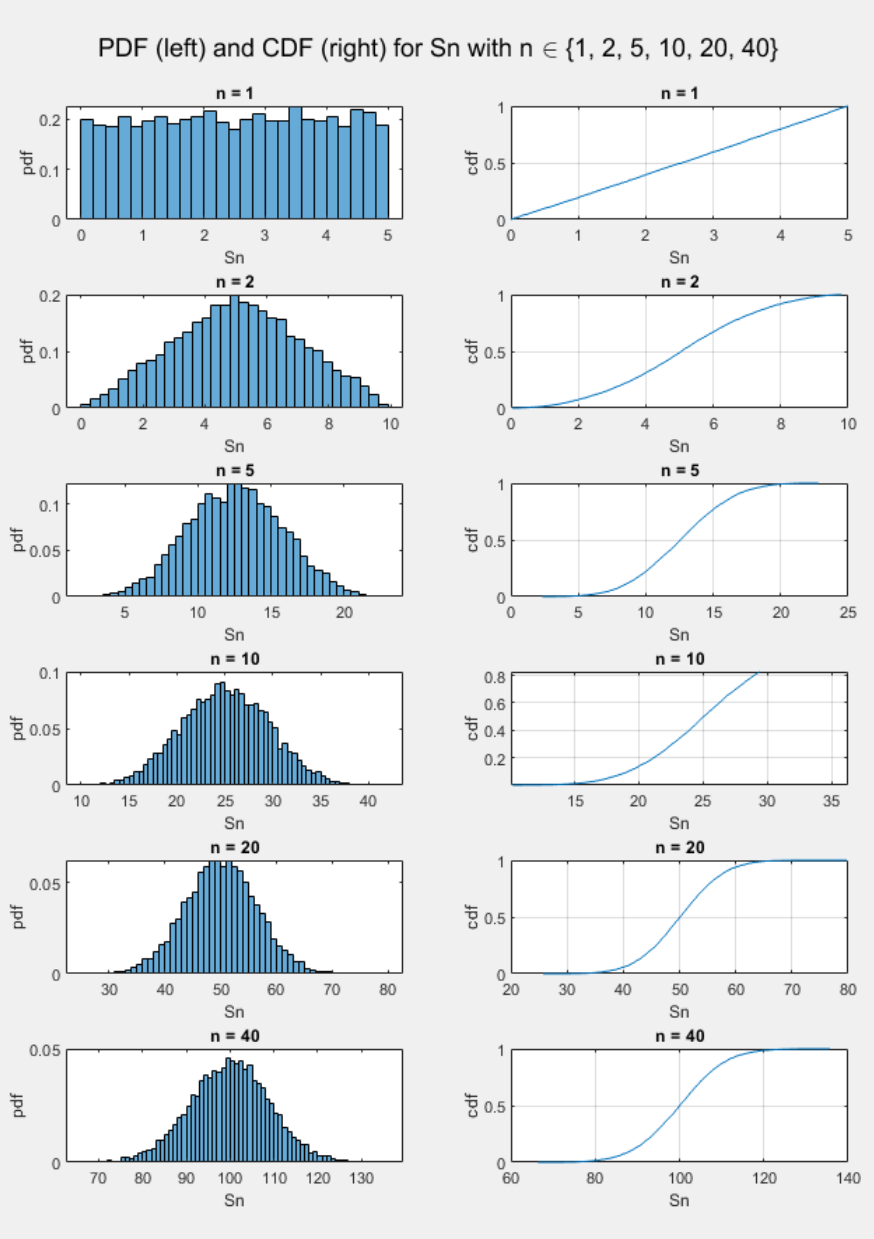


Figure 1. For each n, we draw a uniformly distributed random variable $X_i \sim \mathcal{U}(0, 5)$ and compute the sum $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. We sample a new S_n ten thousand times for each n and then compute the histogram of the variables S_n.

- Noise (total noise) is the aggregation of noise from many different sources.
- Example: A Bluetooth speaker receives a signal from your laptop, with the following noise sources:
  - Microwave oven with similar radio frequency, sensor error due to overheating, physical noise when you pick up the speaker, etc.
  - **How does the total noise follow a Gaussian distribution???**




```
N = [1:5 10 20 40];      % values of n we are interested in
LB = 0;                  % lowerbound for X ~ Uniform(LB,UB)
UB = 5;                  % upperbound for X ~ Uniform(LB,UB)
n = 10000;               % Number of copies (samples) for each random variable
% Generate random variates
X = LB + (UB - LB)*rand(max(N),n);              % X ~ Uniform(LB,UB) (i.i.d.)
Sn = cumsum(X);
figure
s(11) = subplot(6,2,1)  % n = 1
    histogram(Sn(1,:),'Normalization','pdf')
    title(s(11),'n = 1')
s(12) = subplot(6,2,2)
    cdfplot(Sn(1,:))
    title(s(12),'n = 1')
s(21) = subplot(6,2,3)   % n = 2
    histogram(Sn(2,:),'Normalization','pdf')
    title(s(21),'n = 2')
s(22) = subplot(6,2,4)
    cdfplot(Sn(2,:))
    title(s(22),'n = 2')
s(31) = subplot(6,2,5)  % n = 5
    histogram(Sn(5,:),'Normalization','pdf')
    title(s(31),'n = 5')
s(32) = subplot(6,2,6)
    cdfplot(Sn(5,:))
    title(s(32),'n = 5')
s(41) = subplot(6,2,7)  % n = 10
    histogram(Sn(10,:),'Normalization','pdf')
    title(s(41),'n = 10')
s(42) = subplot(6,2,8)
    cdfplot(Sn(10,:))
    title(s(42),'n = 10')
s(51) = subplot(6,2,9)   % n = 20
    histogram(Sn(20,:),'Normalization','pdf')
    title(s(51),'n = 20')
s(52) = subplot(6,2,10)
    cdfplot(Sn(20,:))
    title(s(52),'n = 20')
s(61) = subplot(6,2,11)   % n = 40
    histogram(Sn(40,:),'Normalization','pdf')
    title(s(61),'n = 40')
s(62) = subplot(6,2,12)
    cdfplot(Sn(40,:))
    title(s(62),'n = 40')
sgtitle({'PDF (left) and CDF (right) for Sn with n \in \{1, 2, 5, 10, 20, 40\}'})

for tgt = [11:10:61 12:10:62]
    xlabel(s(tgt),'Sn')
    if rem(tgt,2) == 1
        ylabel(s(tgt),'pdf')
    else                              %  rem(tgt,2) == 0
        ylabel(s(tgt),'cdf')
    end
end
```

Channel Transmission

Binary Data Sequence $\underline{\mathrm{u}}_{\mathrm{T}}$

⇩

Transmitted Waveform $s(t)$

⇩

AWGN Channel

⇩

Received Waveform $r(t) = s(t) + n(t)$

$$\boxed{\underline{\mathrm{u}}_{\mathrm{T}} \longrightarrow \boldsymbol{s}(t) \longrightarrow \boldsymbol{r}(t) = \boldsymbol{s}(t) + \boldsymbol{n}(t)}$$

The Problem at the Receiver

$$\underline{u}_T \longrightarrow \boldsymbol{s}(t) \longrightarrow \boldsymbol{r}(t) = \boldsymbol{s}(t) + \boldsymbol{n}(t)$$

Problem: $\boxed{\text{receive } \boldsymbol{r}(t) \rightarrow \text{recover } \underline{u}_T}$

$$\underline{u}_T \longrightarrow \boldsymbol{s}(t) \longrightarrow \boldsymbol{r}(t) = \boldsymbol{s}(t) + \boldsymbol{n}(t)$$

Problem: $\boxed{\text{receive } \boldsymbol{r}(t) \rightarrow \text{recover } \underline{u}_T}$

Divided into 2 steps:

1. Receive $r(t)$, recover $s(t)$: ***(difficult problem)***
2. Receive $s(t)$, recover $\underline{u}_T$: ***(easy problem:** labeling is a 1-1 mapping**)***

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$$\underline{u}_T \longrightarrow \boldsymbol{s}(t) \longrightarrow \boldsymbol{r}(t) = \boldsymbol{s}(t) + \boldsymbol{n}(t)$$

Problem: $\boxed{\text{receive } \boldsymbol{r}(t) \rightarrow \text{recover } \underline{u}_T}$

Instead of processing the actual waveform

$$\Downarrow$$

Easier to process on **VECTORS**

Given a signal set $M = \{s_1(t), \dots, s_i(t), \dots, s_m(t)\}$

1. Construct an orthonormal basis *B*

2. Process in the signal space *S* spanned by *B*

3. Each signal in *S* can be represented as a linear combination of the basis components → each signal of *S* corresponds to a real vector (= the coefficients of that linear combination)

Basis B

Given a signal set:

$$M = \{s_1(t), \dots, s_i(t), \dots, s_m(t)\}$$

We must find an orthonormal basis:

$$B = \{b_1(t), \dots, b_j(t), \dots, b_d(t)\} \qquad (d \leq m)$$

B = set of signals

1. Mutually orthogonal $\Rightarrow$ $$\boxed{\int_0^T b_j(t) b_i(t) dt = 0 \quad khi \;\; j \neq i}$$

Given a signal set:

$$M = \{s_1(t), \dots, s_i(t), \dots, s_m(t)\}$$

We must find an orthonormal basis:

$$B = \{b_1(t), \dots, b_j(t), \dots, b_d(t)\} \qquad (d \le m)$$

B = set of signals

2. With unit energy $\Rightarrow$ $\boxed{\int_0^T b_j^2(t)dt = 1}$

Given a signal set:

$$M = \{s_1(t), \ldots, s_i(t), \ldots, s_m(t)\}$$

We must find an orthonormal basis:

$$B = \{b_1(t), \ldots, b_j(t), \ldots, b_d(t)\} \qquad (d \le m)$$

B = set of signals

3. The number of basis elements (*d*) is the minimum sufficient to represent each signal of M as a linear combination.

$$\Rightarrow \boxed{s_i(t) = \sum_{j=1}^{d} s_{ij} b_j(t) \qquad s_{ij} \in R}$$

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Basis B

Given a signal set: $M = \{s_1(t), \dots, s_i(t), \dots, s_m(t)\}$

We must find an orthonormal basis:

$$B = \{b_1(t), \dots, b_j(t), \dots, b_d(t)\} \quad (d \le m)$$

B = set of signals

1. Mutually orthogonal ⇒ $\boxed{\int_0^T b_j(t) b_i(t)\,dt = 0 \quad when j \ne i}$

2. With unit energy: ⇒ $\boxed{\int_0^T b_j^2(t)\,dt = 1}$

3. The number of basis elements (d) is the minimum sufficient to represent each signal of M as a linear combination

⇒ $\boxed{s_i(t) = \sum_{j=1}^{d} s_{ij} b_j(t) \quad s_{ij} \in R}$

Constructing the Basis B

Given *M*, how to construct *B*?

For simple signal sets, it is not difficult to construct **B** directly.

In the general case, we can use the following algorithm to construct **B** from **M**:

Gram-Schmidt Algorithm

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Gram-Schmidt Algorithm

$$M = \{s_1(t), \ldots, s_i(t), \ldots, s_m(t)\}$$

Step 1

For $s_1(t)$ → calculate the first vector.

Define:

$$\boxed{b_1^*(t) = s_1(t)}$$

Calculate:

$$\boxed{b_1(t) = \frac{b_1^*(t)}{\sqrt{E(b_1^*)}}}$$

(If $b_1^*(t) = 0$ → $b_1(t) = 0$)

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For $s_2(t)$, find the second vector.

Step 2

Calculate the projection onto the first vector:

$$s_{21} = \int_0^T s_2(t) b_1(t) dt$$

Define: $\boxed{b_2^*(t) = s_2(t) - s_{21} b_1(t)}$

Calculate: $\boxed{b_2(t) = \frac{b_2^*(t)}{\sqrt{E(b_2^*)}}}$ (If $b_2^*(t) = 0 \rightarrow b_2(t) = 0$)

$$s_{21} = \int_0^T s_2(t) b_1(t) dt \qquad b_2^*(t) = s_2(t) - s_{21} b_1(t)$$

Note:

- If $b_2^*(t) = 0$ ($s_2(t)$ is proportional to $b_1(t)$)

→ $b_2(t) = 0$ and no new vector is added. **Why?**

- If $b_2^*(t) \neq 0$ ($s_2(t)$ is not proportional to $b_1(t)$)

→ $b_2(t) \neq 0$ and a new vector is found.

For $s_i(t) \quad 3 \leq i < m$

Calculate the projection onto previous vectors:

Step i

$$s_{ij} = \int_0^T s_i(t) b_j(t) dt \qquad 1 \leq j \leq i-1$$

Define:

$$\boxed{b_i^*(\mathrm{t}) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} b_j(t)}$$

Calculate:

$$\boxed{b_i(t) = \frac{b_i^*(t)}{\sqrt{E(b_i^*)}}}$$

If $b_i^*(t) = 0 \rightarrow b_i(t) = 0$

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$$s_{ij} = \int_0^T s_i(t) b_j(t) dt \qquad b_i^*(\mathrm{t}) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} b_j(t)$$

Note:

- If $b_i^*(t) = 0$ $\quad (s_i(t)$ is a linear combination of the non-zero vectors: $b_0(t), \dots, b_{i-1}(t)$.)

→ $b_i(t) = 0$ and no new vector is added.

- If $b_i^*(t) \neq 0$ $\quad (s_i(t)$ is not a linear combination.)

→ $b_2(t) \neq 0$ and a new vector is found.

Last step

- Remove all $b_i(t) = 0$
- Re-index the remaining non-zero $b_i(t)$ vectors
- We have the basis B:

$$\boxed{B = \{b_1(t), \ldots, b_j(t), \ldots, b_d(t)\} \quad (d \leq m)}$$

Exercise

Given the signal set:

$$M = \{s_1(t) = +P_T(t), s_2(t) = -P_T(t)\}$$

Construct the orthonormal basis *B*?

Constructing the Basis

As mentioned, for simple signal sets, B can be constructed directly without applying Gram Schmidt.

Just find ***d*** signals that satisfy the conditions of an orthonormal basis:

1. Orthogonal
2. Unit energy
3. The number of elements ***d*** is the minimum and sufficient to represent each signal of ***M*** as a linear combination

Exercise

Given the signal set:

$$M = \{s_1(t) = 0, s_2(t) = +P_T(t)\}$$

Construct the orthonormal basis *B*?

Exercise

Given the signal set:

$$M = \{s_1(t) = +P_T(t)\cos(2\pi f_0 t), s_2(t) = -P_T(t)\cos(2\pi f_0 t)\}$$

Construct the orthonormal basis *B*?

Signal Space S

With the orthonormal basis:

$$B = \{b_1(t), \ldots, b_j(t), \ldots, b_d(t)\} \qquad (d \leq m)$$

The space *S* represented by *B* is:

$$S = \left\{ a(t) = \sum_{j=1}^{d} a_j b_j(t) \qquad a_j \in R \right\}$$

(the set of all signals that can be represented as linear combinations of the basis signals)

Exercise

Given the basis B

$$B = \left\{ b_1(t) = +\frac{1}{\sqrt{T}} P_T(t) \right\}$$

What is the signal space S?

Exercise

Given the basis *B*

$$B = \left\{ b_1(t) = +\sqrt{\frac{2}{T}} P_T(t)\cos(2\pi f_0 t) \right\}$$

What is the signal space *S*?

Vector Representation

Given *B*, for each signal $a(t) \in S$ we have

$$a(t) = \sum_{j=1}^{d} a_j b_j(t)$$

The signal $a(t)$ corresponds to a real vector with d components (the coefficients a_j of the linear combination), and vice versa:

$$a(t) \equiv \underline{a} = (a_1, \dots, a_j, \dots, a_d)$$

Vector Representation

1. From vector $\underline{a}$ to signal $a(t)$

$$\underline{a} = (a_1, \dots, a_j, \dots, a_d)$$

$$a(t) = \sum_{j=1}^{d} a_j b_j(t)$$

---

2. From signal $a(t)$ to vector $\underline{a}$

$$\boxed{\text{Projection onto vector } b_j(t)}$$

$$a(t) \Rightarrow a_j = \int_0^T a(t) b_j(t) dt$$

$$\Downarrow$$

$$\underline{a} = (a_1, \dots, a_j, \dots, a_d)$$

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Vector Representation of the Signal Set

We have $$M \subseteq S$$

Each signal $s_i(t) \in S$ corresponds to a real vector with d components and vice versa:

$$s_i(t) \equiv \underline{s_i} = (s_{i1}, \dots, s_{ij}, \dots, s_{id})$$

The signal set M is a set of signals $M = \{s_1(t), \dots, s_1(t), \dots, s_m(t)\}$

The signal set M is a set of vectors $M = \{s_1, \dots, s_1, \dots, s_m\}$

1. From vector $\underline{s_i}$ to signal $s_i(t)$

$$\underline{s_i} = (s_{i1}, \dots, s_{ij}, \dots, s_{id})$$

$$s_i(t) = \sum_{j=1}^{d} s_{ij} b_j(t)$$

---

2. From signal $s_i(t)$ to vector $\underline{s_i}$

Projection onto vector $b_j(t)$

$$s_i(t) \Rightarrow s_{ij} = \int_0^T s_i(t) b_j(t) dt$$

$$\Downarrow$$

$$\underline{s_i} = (s_{i1}, \dots, s_{ij}, \dots, s_{id})$$

Note: for simple signal sets, the vector components can be deduced directly instead of calculating the projection.

We write:

$$s_i(t) = s_{i1}b_1(t) + \cdots s_{ij}b_j(t) + \cdots s_{id}b_d(t)$$

The basis signals $b_j(t)$ are known.

We find the set of coefficients s_{ij} that satisfy the above equation.
The solution is unique.

The signal space S is isomorphic to the Euclidean space R^d

(with the set of all vectors with *d* real components we can visualize in Cartesian space)

If d=1, S ≈ R and can be visualized as a 1-D line

If d=2, S ≈ R^2 and can be visualized as a 2-D plane

If d=3, S ≈ R^3 and can be visualized as a 3-D space

We will write:

$$M \subseteq R^d$$

(A signal set is a set of *m* points in the Euclidean space R^d)

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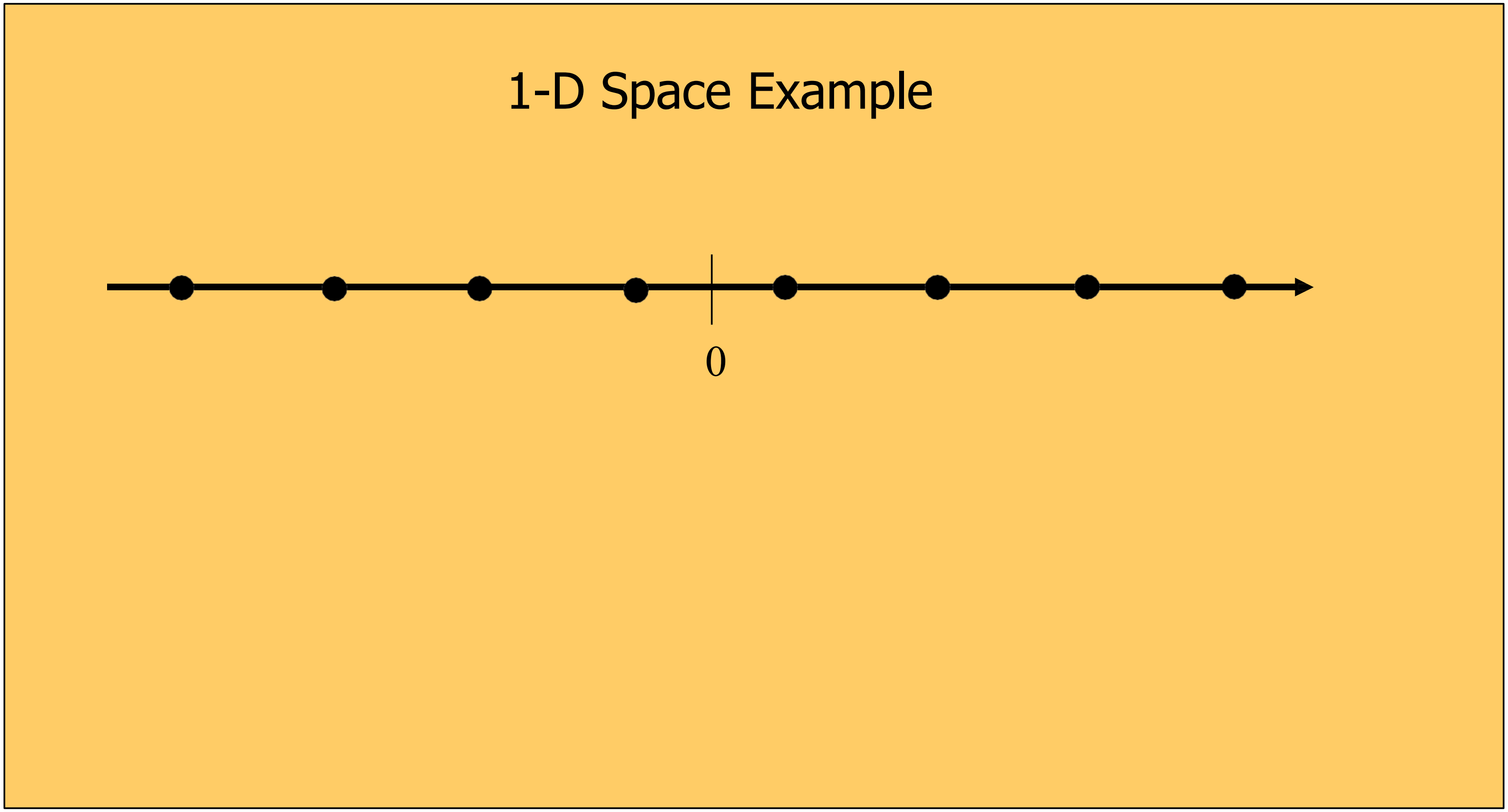
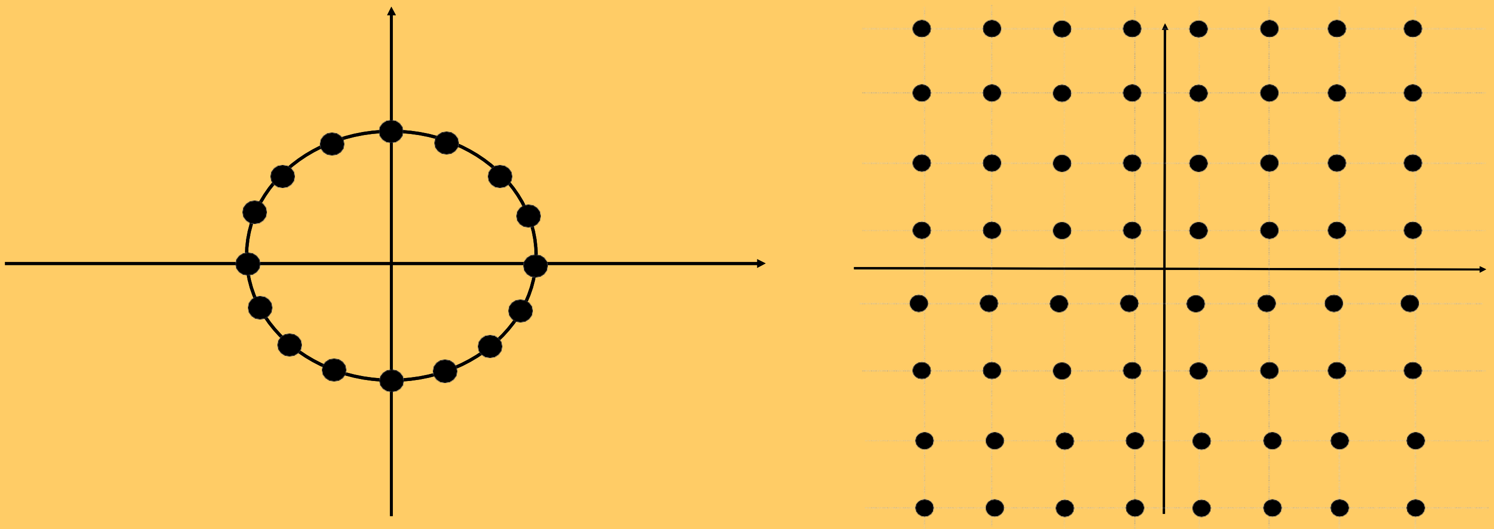
Example



2-D Space Example



Signal Energy

For a signal $a(t) \in S$

Its energy is: $$E(a) = \int_0^T a^2(t)dt$$

If its vector representation is:

$$a(t) \equiv (a_1, \dots, a_j, \dots, a_d)$$

Then the energy is calculated as follows: (Parseval's identity)

$$E(a) = \sum_{j=1}^{d} a_j^2$$

In practice, since $$a(t) = \sum_{j=1}^{d} a_j b_j(t)$$

$$E(a) = \int_0^T a^2(t)dt = \int_0^T \left[\sum_{j=0}^{d-1} a_j b_j(t)\right]^2 dt = \sum_{j=0}^{d-1} a_j^2 \int_0^T b_j^2(t)dt = \sum_{j=0}^{d-1} a_j^2$$

Where the orthogonality property has been used:

$$\int_0^T b_j(t) b_i(t) dt = 0\,, với\ i\ \neq j$$

Signal Set Energy

Given the signal set $M = \{s_1, \dots, s_i, \dots, s_d\} \subseteq R_d$

With $\underline{s_i} = (s_{i1}, \dots, s_{ij}, \dots, s_{id})$

We have:

$$E(s_i) = \sum_{j=1}^{d} s_{ij}^2$$

Signal set energy (average):

$$E_s = \sum_{i=1}^{m} P(s_i)E(s_i)$$

Where $P(s_i)$ is the probability of transmitting s_i

Signal Set Energy

Binary data sequences: ideally random

Binary vectors $\underline{v} \in H_k$ have equal probability.

Labeling is a 1-1 mapping $e : H_k \leftrightarrow M$

The signals in the set $\underline{s_i} \in M$ have equal probability.

$$P(s_i) = \frac{1}{m}$$

The signal set has energy:

$$\boxed{E_s = \frac{1}{m}\sum_{i=1}^{m} E(s_i)}$$

Energy per Bit

Energy required to transmit one bit via *M*

$$\boxed{E_b = \frac{E_s}{k}}$$

Exercise

Given a bipolar NRZ signal set:

$$M = \{s_1(t) = +VP_T(t), s_2(t) = -VP_T(t)\}$$

- Construct the orthonormal basis.
- Represent the signal set in vector form.
- Plot in Euclidean space.
- Determine the signal space S?
- Calculate E_s and E_b.

Exercise

Given a unipolar NRZ signal set:

$$M = \{s_1(t) = +VP_T(t), s_2(t) = 0\}$$

- Construct the orthonormal basis.
- Represent the signal set in vector form.
- Plot in Euclidean space.
- Determine the signal space S?
- Calculate E_s and E_b.

Exercise

Given a 2-PSK signal set:

$$M = \{s_1(t) = +AP_T(t)\cos(2\pi f_0 t), s_2(t) = -AP_T(t)\cos(2\pi f_0 t)\}$$

- Construct the orthonormal basis.
- Represent the signal set in vector form.
- Plot in Euclidean space.
- Determine the signal space S?
- Calculate E_s and E_b.

Exercise

Given a 4-PSK signal set:

$$M = \{s_1(t) = +AP_T(t)\cos(2\pi f_0 t), s_2(t) = +AP_T(t)\sin(2\pi f_0 t),$$
$$s_3(t) = -AP_T(t)\cos(2\pi f_0 t), s_4(t) = -AP_T(t)\sin(2\pi f_0 t)\}$$

- Construct the orthonormal basis.
- Represent the signal set in vector form.
- Plot in Euclidean space.
- Determine the signal space S?
- Calculate E_s and E_b.

Hint: $A\cos(2\pi f_0 t - \vartheta) = \left(A\cos\vartheta\right)\cos(2\pi f_0 t) + \left(A\sin\vartheta\right)\sin(2\pi f_0 t)$

Exercise

Repeat for all the following signal sets:

- NRZ (bipolar and unipolar)
- RZ (bipolar and unipolar)
- 4-PAM
- 4-ASK
- 2-PSK
- 4-PSK
- 2-FSK