Integrals depending on parameters

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1. Definite Integrals Depending on a Parameter
   - Continuity and taking limits under the integral sign
   - Differentiation under the Integral Sign
   - Integration under the Integral Sign

2. Improper Integrals depending on a parameter
   - Uniform Convergence of Improper Integrals
   - Continuity and taking limits under the integral sign
   - Differentiation under the Integral Sign
   - Integration under the Integral Sign

3. Euler Integral
   - The Gamma Function
   - The Beta Function

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Definite Integrals Depending on a Parameter

Definition

Suppose that $f(x, y)$ is a continuous function defined on $[a, b] \times [c, d]$
Then

$$I(y) = \int_a^b f(x, y)dx \tag{1}$$

is a function defined on $[c, d]$ and is called an *integral depending on a parameter* of the function $f(x, y)$.

Continuity and taking limits under the integral sign

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If function $f(x, y)$ is defined and continuous on the rectangle $[a, b] \times [c, d]$ then the integral $I(y)$ is continuous on $[c, d]$, i.e.,

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$$\lim_{y \to y_0} I(y) = \boxed{\lim_{y \to y_0} \int_a^b f(x, y)dx = \int_a^b \lim_{y \to y_0} f(x, y)dx} = \int_a^b f(x, y_0)dx = I(y_0).$$

Example

Compute $\lim\limits_{y \to 0} \int\limits_0^2 x^2 \cos xy dx$.

Differentiation under the Integral Sign

Leibniz's Theorem

Suppose that

i) $f(x, y)$ is continuous on $[a, b] \times [c, d]$,

ii) $f'_y(x, y)$ is continuous on $[a, b] \times [c, d]$.

Then the integral $I(y)$ is differentiable on (c, d) and

$$I'(y) = \left(\int_a^b f(x, y)dx \right)'_y$$

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$$I'(y) = \left(\int_a^b f(x, y)dx \right)'_y = \int_a^b f'_y(x, y)dx.$$

Differentiation under the Integral Sign

Example

Evaluate

a) $I(y) = \int\limits_0^1 \arctan \frac{x}{y} dx.$

b) $J(y) = \int\limits_0^1 \ln\left(x^2 + y^2\right) dx.$

Hint:

a) S1. Check the conditions of the Leibniz' Theorem.
   S2. Calculate $I'(y) = \frac{1}{2} \ln \frac{y^2}{1+y^2}$.
   S3. $I(y) = \arctan \frac{1}{y} + \frac{1}{2} y \ln \frac{y^2}{1+y^2}$.

b) S1. Check the conditions of the Leibniz' Theorem.
   S2. Calculate $I'(y) = 2 \arctan \frac{1}{y}$.
   S3. $I(y) = \ln(1+y^2) - 2 + 2y \arctan \frac{1}{y}$.

Integration under the Integral Sign

If $f(x,y)$ is defined and continuous on the rectangle $[a,b]\times[c,d]$, then

$$\int\limits_c^d I(y)dy =$$

Integration under the Integral Sign

If $f(x, y)$ is defined and continuous on the rectangle $[a, b] \times [c, d]$, then

$$\int_c^d I(y)dy = \boxed{\int_c^d dy \int_a^b f(x,y)dx = \int_a^b dx \int_c^d f(x,y)dy}.$$

Example

By integrating under the integral sign, compute the integral

$$\int_0^1 \frac{x^b - x^a}{\ln x},\ (0 < a < b).$$

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Improper Integrals depending on a parameter

Consider the integral $I(y) = \int_a^{+\infty} f(x,y)dx,\ y \in [c,d]$.

Definition

We say that the integral $I(y)$ is

i) convergent at $y_0 \in [c,d]$ if $\int_a^{\infty} f(x,y_0)dx$ is convergent, i.e.,

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i) convergent at $y_0 \in [c,d]$ if $\int\limits_a^{\infty} f(x,y_0)dx$ is convergent, i.e.,
$\forall \epsilon > 0, \exists b = b(\epsilon, y_0) > a$ (depending on ϵ and y_0) such that

$$\left| I(y_0) - \int_a^A f(x,y_0)dx \right| = \left| \int_A^{\infty} f(x,y_0)dx \right| < \epsilon \text{ for all } A > b.$$

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ii) convergent on $[c,d]$ if $I(y)$ is convergent at any $y \in [c,d]$,

Improper Integrals depending on a parameter

Consider the integral $I(y) = \int\limits_a^{+\infty} f(x, y)dx$, $y \in [c, d]$.

Definition

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i) convergent at $y_0 \in [c, d]$ if $\int\limits_a^{\infty} f(x, y_0)dx$ is convergent, i.e.,

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$$\left| I(y_0) - \int_a^A f(x, y_0)dx \right| = \left| \int_A^{\infty} f(x, y_0)dx \right| < \epsilon \text{ for all } A > b.$$

ii) convergent on $[c, d]$ if $I(y)$ is convergent at any $y \in [c, d]$,

iii) uniformly convergent on $[c, d]$ if

Improper Integrals depending on a parameter

Consider the integral $I(y) = \int\limits_a^{+\infty} f(x, y)dx,\ \mathrm{y} \in [c, d]$.

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We say that the integral $I(y)$ is

i) *convergent at $y_0 \in [c, d]$ if $\int\limits_a^{\infty} f(x, y_0)dx$ is convergent, i.e.,*
$\forall \epsilon > 0, \exists b = b(\epsilon, y_0) > a$ (depending on ϵ and y_0) such that

$$\left| I(y_0) - \int_a^A f(x, y_0)dx \right| = \left| \int_A^{\infty} f(x, y_0)dx \right| < \epsilon \text{ for all } A > b.$$

ii) *convergent on $[c, d]$ if $I(y)$ is convergent at any $y \in [c, d]$,*

iii) *uniformly convergent on $[c, d]$ if $\forall \epsilon > 0, \exists b = b(\epsilon) > a$ such that*

$$\left| \int_A^{\infty} f(x, y)dx \right| < \epsilon \text{ for all } A > b \text{ and } y \in [c, d].$$

Improper Integrals depending on a parameter

Example

Show that the integral $I(y) = \int\limits_1^{\infty} \sin(yx)dx$ is convergent if $y = 0$ and is divergent if $y \neq 0$.

Improper Integrals depending on a parameter

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Show that the integral $I(y) = \int\limits_1^{\infty} \sin(yx)dx$ is convergent if $y = 0$ and is divergent if $y \neq 0$.

Example

a) Evaluate $I(y) = \int\limits_0^{+\infty} ye^{-yx}dx \quad (y > 0)$.

b) Prove that $I(y)$ converges to 1 uniformly on $[y_0, +\infty)$ for all $y_0 > 0$.

c) Explain why $I(y)$ is not uniformly convergent on $(0, +\infty)$.

Sufficient Conditions for Uniform Convergence

Theorem (Weierstrass Criterion)

If

i) $|f(x,y)| \leqslant g(x) \, \forall (x,y) \in [a,+\infty) \times [c,d]$,

ii) The improper integral $\int_a^{+\infty} g(x)\, dx$ *is convergent,*

then $I(y) = \int_a^{+\infty} f(x,y) dx$ *is uniformly convergent on* $[c,d]$.

Example

Prove that

a) $I(y) = \int\limits_0^{\infty} \frac{\cos \alpha x}{x^2+1}$ is uniformly convergent on $\mathbb{R}$.

Continuity and taking limits under the integral sign

Example

Prove that $\lim\limits_{y\to 0^+}\left(\int\limits_0^{+\infty} ye^{-yx}\,dx\right) \neq \int\limits_0^{+\infty}\left(\lim\limits_{y\to 0^+} ye^{-yx}\right)dx$

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Prove that $\lim\limits_{y\to 0^+}\left(\int\limits_0^{+\infty} ye^{-yx}dx\right) \neq \int\limits_0^{+\infty}\left(\lim\limits_{y\to 0^+} ye^{-yx}\right)dx$

Theorem (Continuity and taking limits under the integral sign)

If

i) $f(x,y)$ *is continuous on* $[a,+\infty)\times[c,d]$,

ii) $I(y)=\int_a^{+\infty} f(x,y)dx$ *is uniformly convergent on* $[c,d]$,

then $I(y)$ *is continuous on* $[c,d]$, *i.e.,*

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i) $f(x,y)$ is continuous on $[a,+\infty)\times[c,d]$,

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$$\lim_{y\to y_0} I(y) = \boxed{\lim_{y\to y_0}\int\limits_a^{+\infty} f(x,y)dx = \int_a^{+\infty}\lim_{y\to y_0} f(x,y)dx} = \int_a^{+\infty} f(x,y_0)dx.$$

Differentiation under the Integral Sign

Example

Evaluate $\int\limits_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x}, \quad (\alpha, \beta > 0).$

Differentiation under the Integral Sign

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Theorem (Differentiation under the Integral Sign)

If

i) $f(x,y)$ *and* $f'_y(x,y)$ *are continuous on* $[a,+\infty) \times [c,d]$,

ii) $I(y) = \int\limits_a^{+\infty} f(x,y)dx$ *is convergent on* $[c,d]$,

iii) $\int\limits_a^{+\infty} f'_y(x,y)dx$ *is uniformly convergent on* $[c,d]$,

then $I(y)$ *is differentiable on* $[c,d]$ *and* $I'(y) = \int\limits_a^{+\infty} f'_y(x,y)\,dx$.

Integration under the Integral Sign

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Theorem (Integration under the Integral Sign)

If

i) $f(x, y)$ *is continuous on* $[a, +\infty) \times [c, d]$,

ii) $I(y) = \int_a^{+\infty} f(x, y) dx$ *is uniformly continuous on* $[c, d]$,

then $I(y)$ *is integrable on* $[c, d]$ *and*

$$\int_c^d I(y)\, dy := \int_c^d \left(\int_a^{+\infty} f(x, y)\, dx \right) dy = \int_a^{+\infty} \left(\int_c^d f(x, y)\, dy \right) dx.$$

Integration Techniques

Evaluate $I(y) = \int\limits_a^{+\infty} f(x,y)dx$.

Differentiation under the Integral Sign

S1. Evaluate $I'(y)$ by differentiating $I'(y) = \int\limits_a^{+\infty} f'_y(x,y)\,dx$.

S2. Evaluate $I(y)$ by integrating $I(y) = \int I'(y)\,dy + C$.

S3. Evaluate $I(y_0)$ to find C.

Remark: Remember to check the conditions.

Example

Evaluate $(a, b, \alpha, \beta > 0)$:

a) $\int_0^1 \frac{x^b - x^a}{\ln x} dx$.

b) $\int_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx$.

c) $\int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} dx$.

d) $\int_0^{+\infty} \frac{dx}{(x^2+y)^{n+1}}$.

Integration Techniques

Evaluate $I(y) = \int_a^{+\infty} f(x,y)dx$.

Integration under the Integral Sign

S1. Express $f(x,y) = \int_c^d F(x,y)\,dy$.

S2. Change the order of integration:

$$\int_a^{+\infty} f(x,y)dx = \int_a^{+\infty} \left(\int_c^d F(x,y)dy \right) dx = \int_c^d \left(\int_a^{+\infty} F(x,y)dx \right) dy.$$

Remark: Remember to check the conditions.

Example

Evaluate $(a, b, \alpha, \beta > 0)$

a) $\int_0^1 \frac{x^b - x^a}{\ln x} dx$.

b) $\int_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx$.

c) $\int_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2} dx$.

d) $\int_0^{+\infty} e^{-ax} \frac{\sin bx - \sin cx}{x}$.

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The Gamma Function

$\Gamma(p) = \int\limits_0^{+\infty} x^{p-1}e^{-x}dx$ defined on $(0, +\infty)$.

Example

Evaluate $\Gamma(1)$, $\Gamma\left(\frac{1}{2}\right)$.

Properties

a) $\Gamma(p+1) = p\Gamma(p)$.

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Evaluate $\Gamma(1)$, $\Gamma\left(\frac{1}{2}\right)$.

Properties

a) $\Gamma(p+1) = p\Gamma(p)$.
If $\alpha \in (n, n+1]$, then $\Gamma(\alpha) = (\alpha-1)(\alpha-2)\ldots(\alpha-n)\Gamma(\alpha-n)$.
Specially, $\begin{cases} \Gamma(1) = 1, \\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \end{cases}$ therefore $\begin{cases} \Gamma(n) = (n-1)! \\ \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}. \end{cases}$

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b) Derivative of the Gamma function:
   $\Gamma^{(k)}(p) = \int_0^{+\infty} x^{p-1} (\ln x)^k e^{-x} dx$.

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c) $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$ $\forall 0 < p < 1$.

The Beta Function

Form 1: $\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx.$

Form 2: $\mathrm{B}(p,q) = \int_0^{+\infty} \frac{x^{p-1}}{(1+x)^{p+q}}dx$ (change of variable $x = \frac{t}{t+1}$).

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a) Symmetry: $\mathrm{B}(p,q) = \mathrm{B}(q,p)$.

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Properties

a) Symmetry: $\mathrm{B}(p,q) = \mathrm{B}(q,p)$.

b) $\begin{cases} \mathrm{B}(p,q) = \frac{p-1}{p+q-1}\mathrm{B}(p-1,q), & \text{if } p > 1 \\ \mathrm{B}(p,q) = \frac{q-1}{p+q-1}\mathrm{B}(p,q-1), & \text{if } q > 1. \end{cases}$

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Specially, $\mathrm{B}(1,1) = 1$ therefore $\mathrm{B}(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$, $\forall m,n \in \mathbb{N}$.

Example

Express $\int_0^{\frac{\pi}{2}} \sin^m t \cos^n t dt$ as the Beta function.

Hint: Let $\sin t = \sqrt{x}$ to conclude $\int_0^{\frac{\pi}{2}} \sin^m t \cos^n t dt = \frac{1}{2}\mathrm{B}\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$.

Euler Integral

The Trigonometric Form of the Beta Function

$$\mathrm{B}\left(p,q\right)=2\int\limits_0^{\frac{\pi}{2}}\sin^{2p-1}t\cos^{2q-1}t\,dt.$$

Euler Integral

The Trigonometric Form of the Beta Function

$$\mathrm{B}(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1} t \cos^{2q-1} t dt.$$

Relation between the Gamma and Beta functions

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Example

a) $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x dx.$

b) $\int_0^a x^{2n}\sqrt{a^2 - x^2} dx \quad (a > 0).$

c) $\int_0^{+\infty} x^{10} e^{-x^2} dx.$

d) $\int_0^{+\infty} \frac{\sqrt{x}}{(1+x^2)^2} dx.$

e) $\int_0^{+\infty} \frac{1}{1+x^3} dx.$

f) $\int_0^1 \frac{1}{\sqrt[n]{1-x^n}} dx, \quad n \in \mathbb{N}^*.$