UNIT 4

LINEAR TIME-INVARIANT DISCRETE-TIME SYSTEMS

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❑ Contents

❑ Learning Objectives

After completing this lesson, you will have grasped the following topics:

- Concept of the discrete-time signal processing system.
- Linearity and time invariance of discrete-time system.
- The impulse response of linear time-invariant (LTI) discrete-time system.
- Convolution operation and its properties.

1. Discrete-time signal processing system

$$x(n) \longrightarrow \boxed{T[\ ]} \longrightarrow y(n) = T[x(n)]$$

- x(n): input/impact signal
- y(n): output/response signal
- Example: image noisy filter

2. Linear system

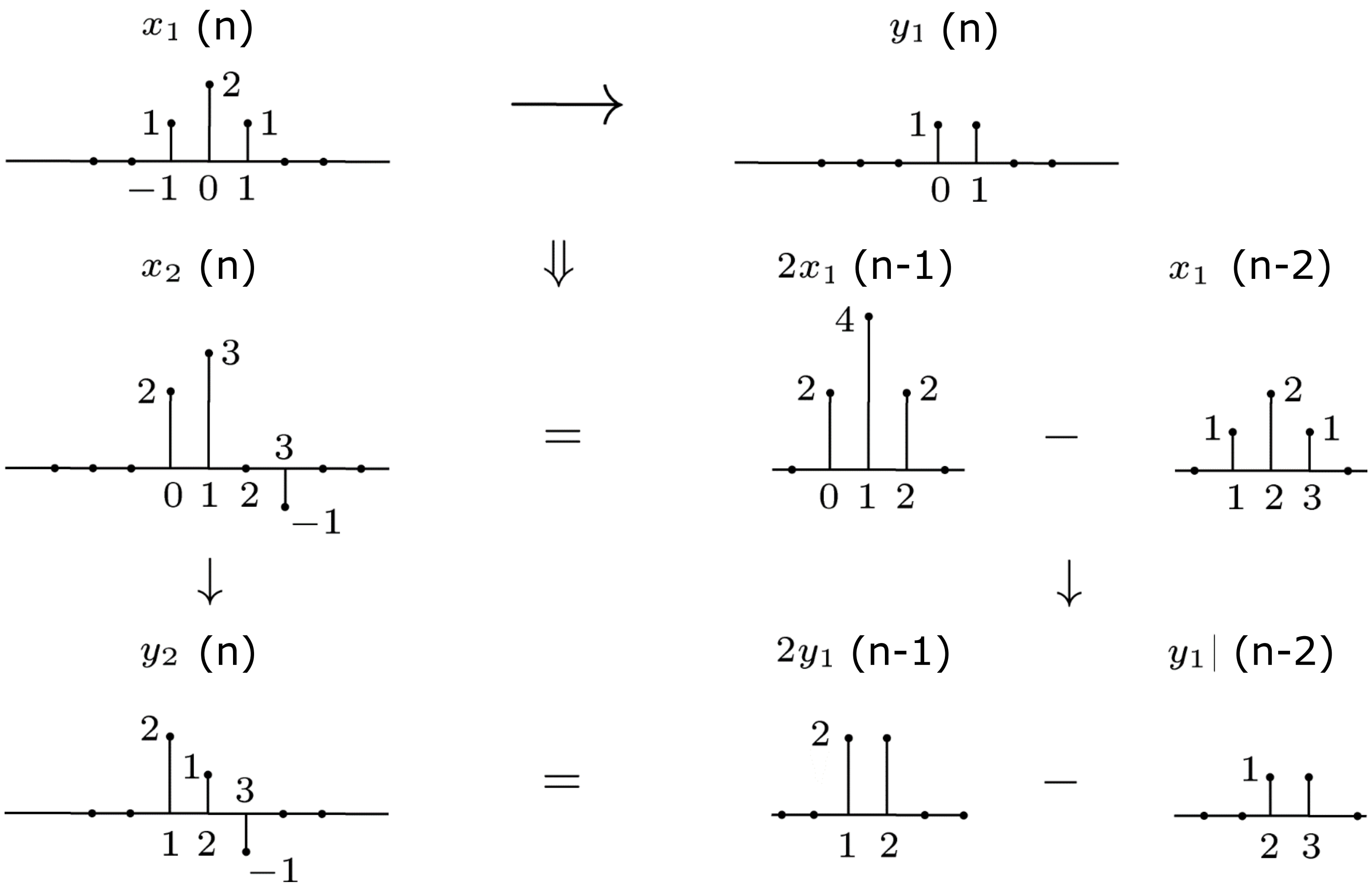
- Definition

$$x_1(n) \rightarrow y_1(n),\ \ x_2(n) \rightarrow y_2(n)$$
$$T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$
$$= ay_1(n) + by_2(n)$$

- Advantages: It enables the determination of the response of complex input signals based on the known simple component responses.
- Check for the linearity of the system
  - Scaling: $T[ax_1(n)] = aT[x_1(n)] = ay_1(n)$
  - Combination: $T[x_1(n) + x_2(n)] = T[x_1(n)] + T[x_2(n)] = y_1(n) + y_2(n)$
- Ví dụ. Check for the linearity of the following systems?

$a. y(n) = 2x(n)$ $\quad\quad$ $b. y(n) = x^2(n)$

Example of a linear system



Linear system analysis techniques.

$$x(n) = \sum_{k=1}^{M-1} a_k x_k(n) \xrightarrow{T} y(n) = \sum_{k=1}^{M-1} a_k y_k(n)$$

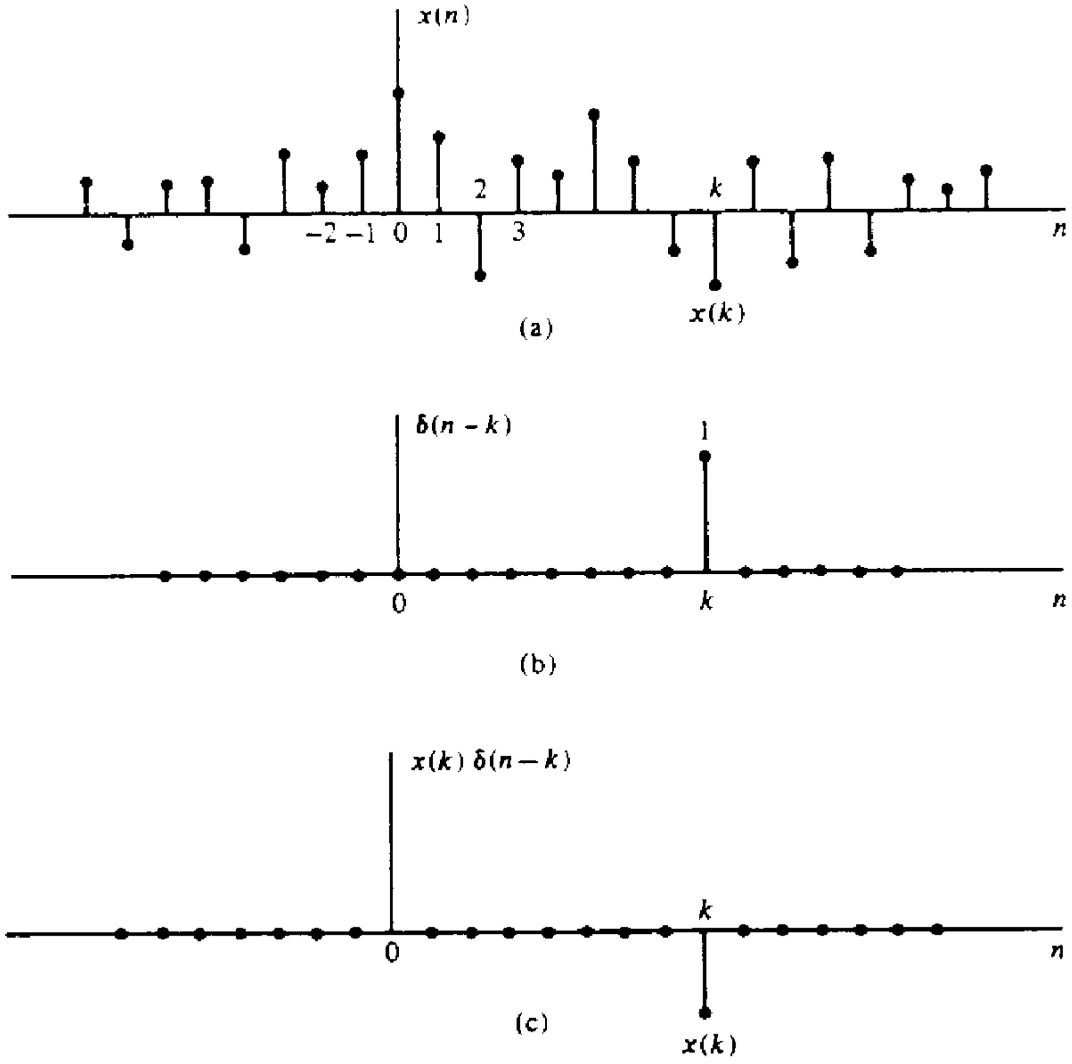
$$y_k(n) = T[x_k(n)] \qquad\qquad k = 1, 2, \dots, M-1$$

Select $x_k(n)$ as the impulse signal δ(n-k)

Signal analysis as a combination of elementary impulses

- Principles:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$



- Example:

$$x(n) = \{2, 4, 0, 3\}$$
$$\uparrow$$

$$x(n) = 2\delta(n+1) + 4\delta(n) + 3\delta(n-2)$$

Response of the linear system

$$y(n) = T[x(n)]$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)] = \sum_{k=-\infty}^{\infty} x(k)h(n,k)$$

$$h(n,k) = h_k(n) = T[\delta(n-k)]$$

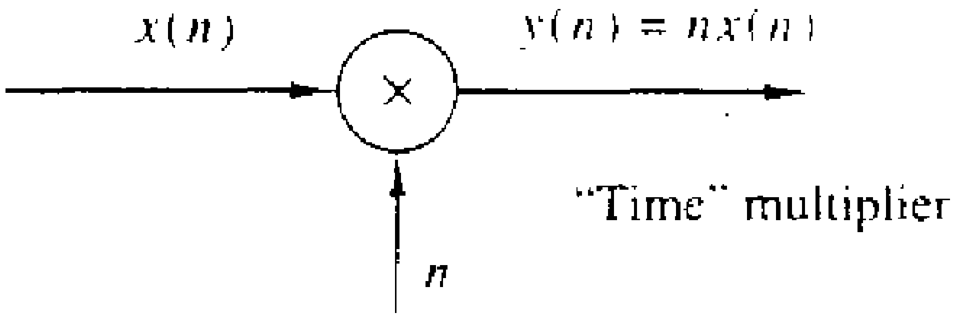
- It is necessary to specify the sequence $h(n,k) = h_k(n)$ for all $-\infty \leq k \leq \infty$
- It is necessary to reduce the number of functions $h_k(n)$

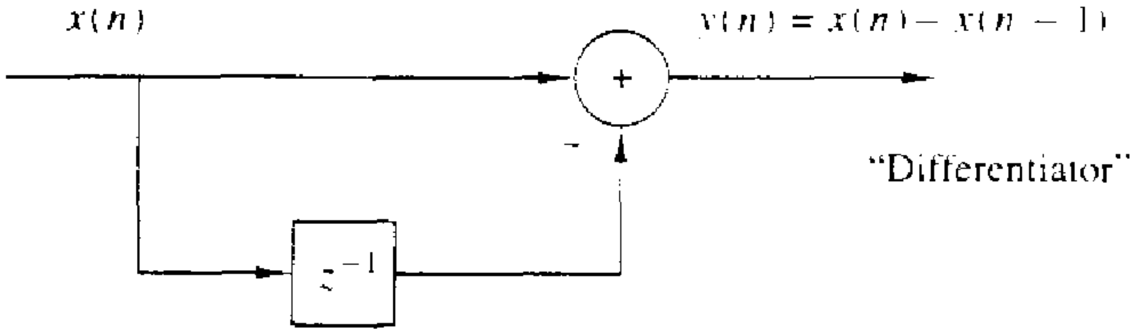
Time invariant system

$$x(n) \xrightarrow{T} y(n)$$

$$\Rightarrow \quad x(n-k) \xrightarrow{T} y(n-k) \qquad \forall\, x(n) \text{ và } \forall\, k$$

- Example:





Linear time-invariant system

- If the system is linear with respect to time
  - Impact signal $\delta(n)$ induces the response $h(n)$
  - Impact signal $\delta(n-k)$ induces the response $h(n-k)$
- For a linear time-invariant system (LTI)

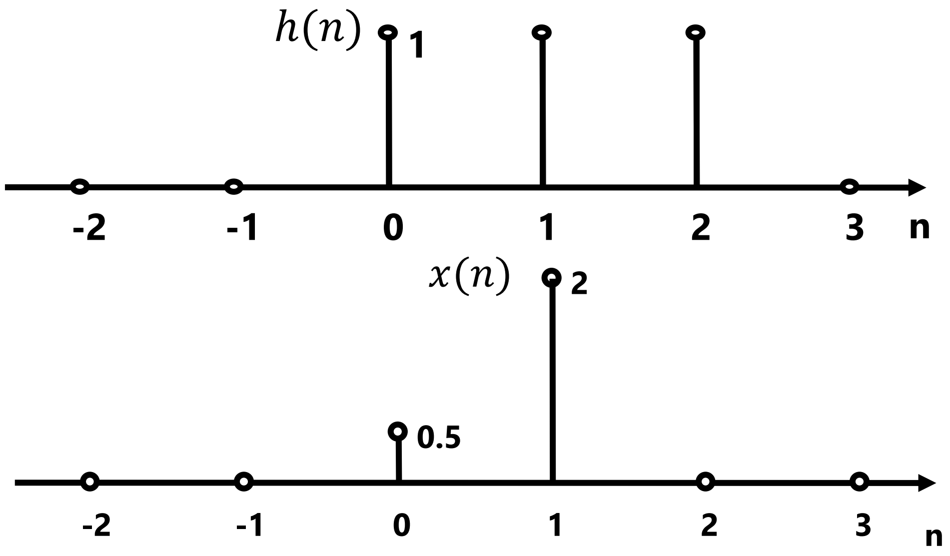
$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

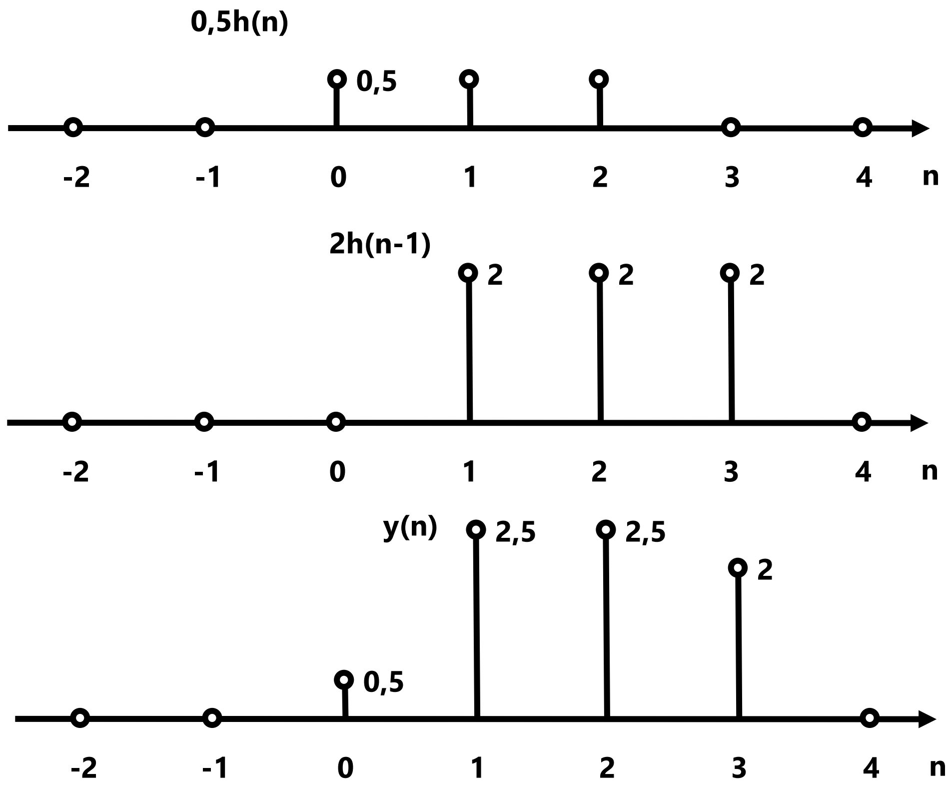
$$= \sum_{k=-\infty}^{\infty} x(k)\, T[\delta(n-k)] \Rightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- $h(n)$ is the impulse response of the system
- $y(n) = x(n) * h(n)$, where $*$ is the convolution operation

Example of convolution

- The input signal and impulse response of a LTI system is given in the following plots. Determine the corresponding output signal.





$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=0}^{1} x(k)h(n-k)$$

$$y(n) = x(0)\,h(n-0) + x(1)\,h(n-1) = 0.5\,h(n-0) + 2\,h(n-1)$$

3. Properties of convolution

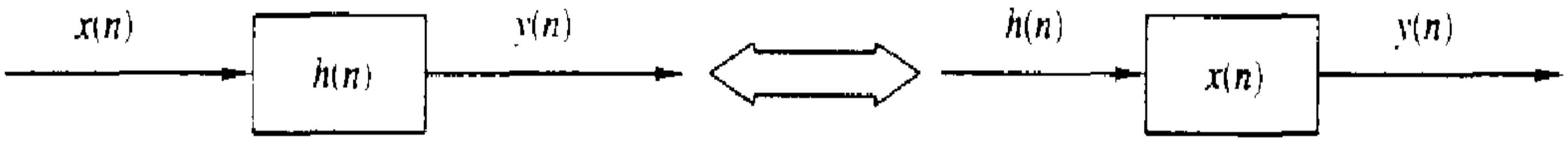
- Convolution

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- Commutative law

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

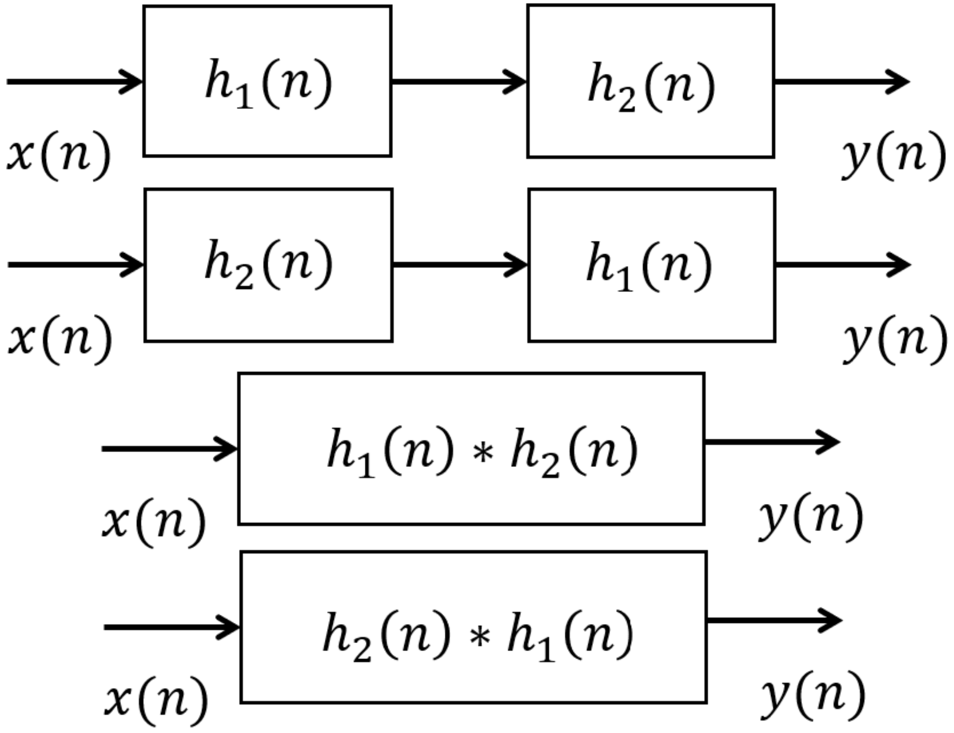


Distributive law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

$$[x(n) * h_2(n)] * h_1(n) = x(n) * [h_2(n) * h_1(n)]$$

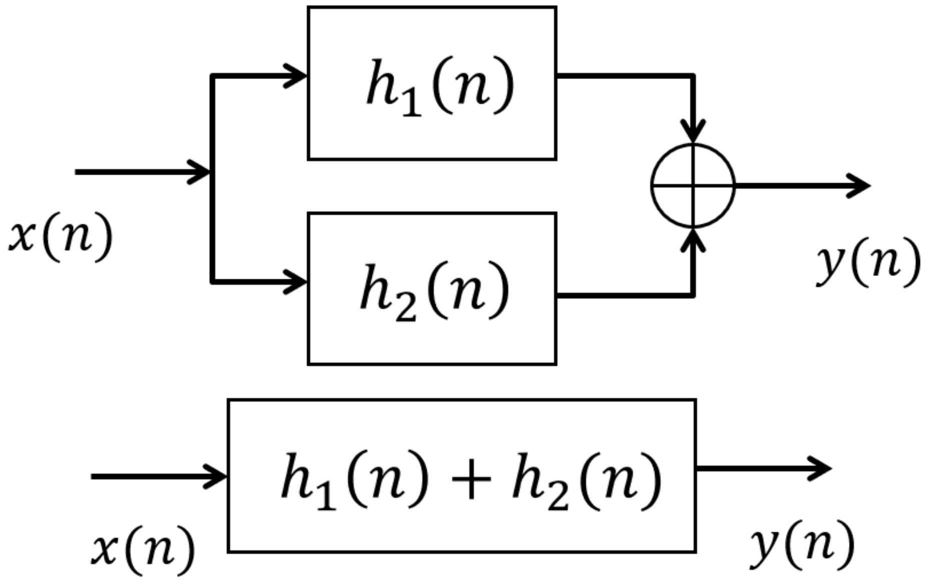
- Equivalent systems



Distributive law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

- Equivalent systems



4. Summary

- A discrete-time system receives a discrete-time signal as input, processes it, and generates a desired discrete-time signal at the output.
- A linear time-invariant system is a system that satisfies both linearity and time-invariance properties.
- Convolution allows the determination of the response of a LTI system to a given input signal before knowing the impulse response of the system.
- The convolution operation is commutative, associative, and distributive..

5. Assignment

- Assignment 1: Compute the output of the following system with the input $x(n)$

$$x(n) = \begin{cases} |n|, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

a) $y(n) = x(n)$

b) $y(n) = x(n-1)$

c) $y(n) = x(n+1)$

d) $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

e) $y(n) = \max[x(n+1), x(n), x(n-1)]$

f) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots$

Assignment 2

- Given systems represented by the following input-output equations:

  (a) $y(n) = nx(n)$ (b) $y(n) = x(n^2)$ (c) $y(n) = x^2(n)$

  (d) $y(n) = Ax(n) + B$ (e) $y(n) = e^{x(n)}$

- Check their linearity.

Assignment 3

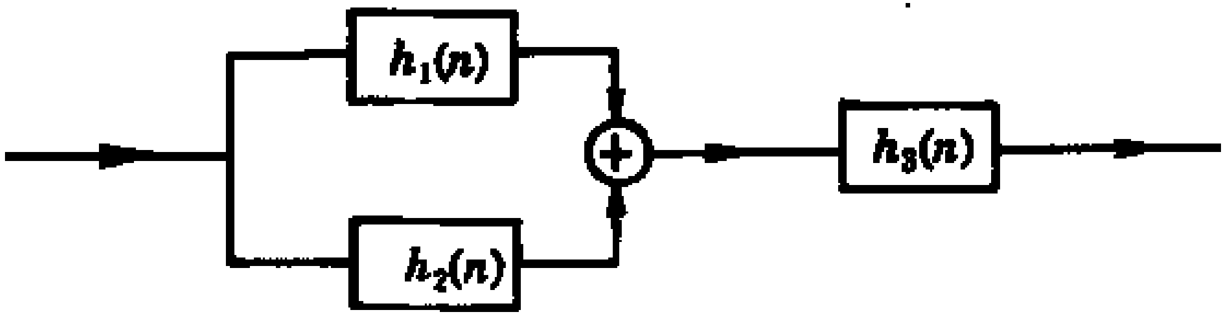
- Given impulse response of a linear time-invariant system as follows

$$h(n) = \begin{cases} 1 - \dfrac{n}{4} & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$$

- Let the input $x(n) = rect_3(n)$, determine the output $y(n)$?

Assignment 4

- Determine the impulse response $h(n)$ of the following system



$$h_1(n) = \begin{cases} 1 - \frac{n}{2} & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$h_2(n) = \frac{1}{2}\delta(n-1) + u(n-2) - u(n-4)$$

$$h_3(n) = \mathrm{rect}_3(n)$$

The next unit 5

CAUSALITY AND STABILITY OF A DISCRETE SYSTEM

References:

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wishing you all the best in your studies!