Hanoi University of Science and Techonology

School of Applied Mathematics and Informatics

CALCULUS I

Course ID: MI 1114E

Chapter 1

Derivative and Differentiation of a function

1.1-1.4. Sequences; Functions

Exercise 1. Determine the domains of the following functions

a) $y = \sqrt{2\operatorname{arccot} x - \pi}$

b) $y = \arcsin \dfrac{2x}{1+x}$

c) $y = \dfrac{\sqrt{x}}{\sin \pi x}$

d) $y = \arccos(\sin x)$

Exercise 2. Prove the following identities

a) $\sinh(-x) = -\sinh x$

b) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

d) $\sinh 2x = 2 \sinh x \cosh x$

e) $\cosh^2 x - \sinh^2 x = 1$

f) $\cosh 2x = \cosh^2 x + \sinh^2 x$

Exercise 3. Determine the ranges of the following functions

a) $y = \log(1 - 2\cos x)$

b) $y = \arcsin\left(\log \dfrac{x}{10}\right)$

c) $y = \operatorname{arccot}(\sin x)$

d) $y = \arctan(e^x)$

Exercise 4. Find the function $f(x)$ such that

a) $f\left(x+\dfrac{1}{x}\right)=x^2+\dfrac{1}{x^2}$

b) $f\left(\dfrac{x}{1+x}\right)=x^2$

Exercise 5. Find the inverse functions of the following functions

a) $y=2\arcsin x$

b) $y=\dfrac{1-x}{1+x}$

c) $y=\dfrac{1}{2}\left(e^x-e^{-x}\right)$

Exercise 6. Determine whether the following functions are odd, even or neither.

a) $f(x)=a^x+a^{-x}$, $(a>0)$

b) $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$

c) $f(x)=\sin x+\cos x$

d) $f(x)=\arcsin(\tan x)$

Exercise 7. Prove that any function $f(x)$ defined on an open interval $(-a,a)$, for some $(a>0)$, can be expressed as a sum of one odd and one even function.

Exercise 8. Given two functions $f(x)$ and $g(x)$ on an interval $(-a,a)$, for some $(a>0)$. Prove that:

a) If both $f(x)$ and $g(x)$ are even functions then their sum and their product are also even functions.

b) If both $f(x)$ and $g(x)$ are odd functions then their sum is an odd function and their product is an even function.

c) If $f(x)$ is odd and $g(x)$ is even then their product is an odd function.

Week 2 **Exercise 9.** Analyze the periodicity and find the basic period (if exists) of the following functions

fundamental period

a) $f(x)=A\cos\lambda x+B\sin\lambda x$

b) $f(x)=\sin(x^2)$

c) $f(x)=\sin x+\dfrac{1}{2}\sin 2x+\dfrac{1}{3}\sin 3x$

d) $f(x)=\cos^2 x$

a') f(x) = 2 sin(4x) + 3 sin (6x) => A sin(mx) + B sin(nx) where m,n are natural numbers

Exercise 10. Find the limit of the following sequences (if exists)

a) $x_n=n-\sqrt{n^2-n}$

b) $x_n=\dfrac{1}{1.2}+\dfrac{1}{2.3}+\ldots+\dfrac{1}{(n-1)n}$

c) $x_n=\dfrac{\sin^2 n-\cos^3 n}{n}$

d) $x_n=\dfrac{\sqrt{n}\cos n}{n+1}$

Exercise 11. Find the limit of the following sequences (if exist)

a) $x_n = \sqrt[n]{n^2+2}$

b) $x_n = \dfrac{1}{2}\left(x_{n-1} + \dfrac{1}{x_{n-1}}\right)$, $x_0 > 0$

1.5-1.6. Limit of a function

Exercise 12. Calculate the followings

a) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\sqrt{1+x} - \dfrac{1}{x}\right)$

b) $\lim\limits_{x\to +\infty}\left(\sqrt[3]{x^3+x^2-1} - x\right)$

c) $\lim\limits_{x\to 1}\dfrac{x^{100}-2x+1}{x^{50}-2x+1}$

d) $\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+\alpha x} - \sqrt[n]{1+\beta x}}{x}$, $\quad (m, n \in \mathbb{N}^*)$

e) $\lim\limits_{x\to +\infty} x\left(\sqrt{x^2+2x} - 2\sqrt{x^2+x} + x\right)$

f) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+4x}-1}{\ln(1+3\sin x)}$

Exercise 13. Calculate the following limits (if exist)

a) $\lim\limits_{x\to 0^+}\dfrac{\ln(x+\arccos^3 x) - \ln x}{x^2}$

b) $\lim\limits_{x\to +\infty}\left(\sin\sqrt{x+1} - \sin\sqrt{x}\right)$

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x} - \sqrt[3]{\cos x}}{\sin^2 x}$

d) $\lim\limits_{x\to 0}\dfrac{1-\cos x\cos 2x\cos 3x}{1-\cos x}$

Exercise 14. Calculate the following limits (if exist)

a) $\lim\limits_{x\to\infty}\left(\dfrac{x^2-1}{x^2+1}\right)^{\frac{x-1}{x+1}}$

b) $\lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{1}{x}}$

c) $\lim\limits_{n\to\infty} n^2\left(\sqrt[n]{x} - \sqrt[n+1]{x}\right), x > 0.$

d) $\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x} + \cos\dfrac{1}{x}\right)^x$

e) $\lim\limits_{x\to 1}(1+\sin\pi x)^{\cot\pi x}$

f) $\lim\limits_{x\to 0}[\ln(e+2x)]^{\frac{1}{\sin x}}$

tends to

Exercise 15. Compare the order of the following infinitesimal as x approaches 0.

a) $\alpha(x) = \sqrt{x+\sqrt{x}}$ and $\beta(x) = e^{\sin x} - \cos x$, for $x \to 0^+$

b) $\alpha(x) = \sqrt[3]{x} - \sqrt{x}$ and $\beta(x) = \cos x - 1$, for $x \to 0^+$

c) $\alpha(x) = x^3 + \sin^2 x$ and $\beta(x) = \ln\left(1 + 2\arctan(x^2)\right)$, for $x \to 0$

Week 3

1.7. Continuous function

Exercise 16. Find a such that the following functions are continuous at $x = 0$

a) $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & \text{if } x \neq 0, \\ a, & \text{if } x = 0. \end{cases}$

b) $g(x) = \begin{cases} ax^2 + bx + 1, & \text{if } x \geq 0, \\ a\cos x + b\sin x, & \text{if } x < 0. \end{cases}$

Exercise 17. At which points the following functions are continuous?

a) $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational}, \\ 1, & \text{if } x \text{ is irrational}. \end{cases}$

b) $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational}, \\ x, & \text{if } x \text{ is irrational}. \end{cases}$

Exercise 18. Find the type of discontinuity of the point $x = 0$, given the following functions

a) $y = \dfrac{8}{1 - 2^{\cot x}}$

b) $y = \dfrac{1}{x} \arcsin x$

c) $y = \dfrac{\sin \frac{1}{x}}{e^{\frac{1}{x}} + 1}$

d) $y = \dfrac{e^{ax} - e^{bx}}{x}$ $(a \neq b)$

Exercise 19. Are the following functions uniformly bounded on their domains?

a) $y = \dfrac{x}{4 - x^2}$; $-1 \leq x \leq 1$

b) $y = \ln x$; $0 < x < 1$

1.8. Derivatives and Differentiation of a function

Exercise 20. Calculate the derivatives of the following functions

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ (1-x)(2-x), & \text{if } 1 \leq x \leq 2, \\ x - 2, & \text{if } x > 2. \end{cases}$$

Exercise 21. Find $f'(x)$ given that $\dfrac{d}{dx}[f(2017x)] = x^2$.

Exercise 22. For which condition the function

$$f(x) = \begin{cases} x^n \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases} \qquad (n \in \mathbb{Z})$$

a) is continuous at $x = 0$

b) is differentiable at $x = 0$

c) has a first order derivative f' continuous at $x = 0$.

Exercise 23. Prove that the function $f(x) = |x - a|\varphi(x)$, where $\varphi(x)$ is a continuous function and $\varphi(a) \neq 0$, is not differentiable at $x = a$.

Exercise 24. Calculate the differentiation of the following functions

a) $y = \dfrac{1}{a}\arctan\dfrac{x}{a}, (a \neq 0)$

b) $y = \arcsin\dfrac{x}{a}, (a \neq 0)$

c) $y = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|, (a \neq 0)$

d) $y = \ln\left|x + \sqrt{x^2+a}\right|$.

Exercise 25. Calculate

a) $\dfrac{d}{d(x^2)}\left(\dfrac{\sin x}{x}\right)$

b) $\dfrac{d(\sin x)}{d(\cos x)}$

c) $\dfrac{d}{d(x^3)}\left(x^3 - 2x^6 - x^9\right)$.

Exercise 26. Approximate the followings

a) $\sqrt[3]{7,97}$

b) $\sqrt[7]{\dfrac{2-0,02}{2+0,02}}$

c) $\sqrt{3e^{0,04}+1,02^2}$

Exercise 27. If $C(x)$ is the production cost of x units of a certain item then the marginal cost is $C'(x)$ which indicates the cost that must be spent in order to increase the amount output by one unit. For a given function

$$C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3,$$

find the marginal cost function. Determine the marginal cost at $x = 100$. What is the meaning of that value?

Exercise 28. Calculate the following high-order derivatives.

a) Given $y = \dfrac{x^2}{1-x}$, calculate $y^{(8)}$

b) Given $y = \dfrac{1+x}{\sqrt{1-x}}$, calculate $y^{(100)}$

c) Given $y = \ln(2x - x^2)$, calculate $y^{(5)}$

d) Given $y = x^2\sin x$, calculate $y^{(50)}$

e) Given $y = e^{x^2}$, calculate $y^{(10)}(0)$

f) Given $y = x\ln(1+2x)$, calculate $y^{(10)}(0)$

Exercise 29. Calculate the $n-$ derivatives of the following functions

a) $y = \dfrac{x}{x^2-1}$

b) $y = \dfrac{1}{x^2-3x+2}$

c) $y = \dfrac{x}{\sqrt[3]{1+x}}$

d) $y = e^{ax}\sin(bx+c)$

e) $y = \sin^4 x + \cos^4 x$

f) $y = x^{n-1}e^{\frac{1}{x}}$

Exercise 30. Calculate the high-order differentiations of the following functions.

a) Given $y = (2x+1)\sin x$. calculate $d^{10}y(0)$

b) Given $y = e^x\cos x$. calculate $d^{20}y(0)$

c) Given $y = x^9\ln x$. calculate $d^{10}y(1)$

d) Given $y = x^2e^{ax}$. calculate $d^{20}y(0)$

Exercise 31. In one fish pond, fish in the lake are continuously born and exploited. The amount of fish in this lake, denoted by $P(t)$, satisfies the differential equation

$$P'(t) = r_0 \left(1 - \frac{P(t)}{P_c}\right) P(t) - \beta P(t),$$

where r_0 is the reproduction rate, P_c is the maximum number of fish the lake can maintain, and β is the exploitation rate. Given $P_c = 10000$, the production rate and the exploitation rate are 5% and 4%, respectively. Find a stable number of fish.

Applications of Derivatives and Differentials

Exercise 32. Prove that $\forall a, b, c \in \mathbb{R}$, the equation

$$a\cos x + b\cos 2x + c\cos 3x = 0$$

has a solution in $(0, \pi)$.

Exercise 33. Prove that the equation $x^n + px + q = 0$ for $n \in \mathbb{N}$, $n \geq 2$, could not have more than two roots if n is even, and no more than 3 roots if n is odd.

Exercise 34. Given three real numbers a, b, c that satisfy $a + b + c = 0$. Prove that equation $8ax^7 + 3bx^2 + c = 0$ has at least one solution in the interval $(0, 1)$.

Exercise 35. Explain why the Cauchy formula $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ could not be applied for the following functions $f(x) = x^2, \quad g(x) = x^3, \quad -1 \leq x \leq 1.$

Exercise 36. Prove the following inequalities

a) $|\sin x - \sin y| \leq |x - y|$

b) $\dfrac{a-b}{a} < \ln\dfrac{a}{b} < \dfrac{a-b}{b}, 0 < b < a.$

c) $\dfrac{b-a}{1+b^2} < \arctan b - \arctan a < \dfrac{b-a}{1+a^2},$
$0 < a < b$

Exercise 37. Whether there exists a function $f(x)$ such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x?

Exercise 38. Calculate the following limits (if exist)

a) $\displaystyle\lim_{x\to+\infty}\left(\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}\right)$

b) $\displaystyle\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

c) $\displaystyle\lim_{x\to\infty}\frac{e^{\frac{1}{x}} - \cos\frac{1}{x}}{1 - \sqrt{1 - \frac{1}{x^2}}}$

d) $\displaystyle\lim_{x\to 0}\frac{e^x \sin x - x(1+x)}{x^3}$

e) $\displaystyle\lim_{x\to 1}\tan\frac{\pi x}{2}\ln(2 - x)$

f) $\displaystyle\lim_{x\to 0}(1 - a\tan^2 x)^{\frac{1}{x\sin x}}$

g) $\displaystyle\lim_{x\to 1^-}\frac{\tan\frac{\pi}{2}x}{\ln(1-x)}$

h) $\displaystyle\lim_{x\to 0}(1 - \cos x)^{\tan x}$

i) $\displaystyle\lim_{x\to -\infty}(x^2 + 2^x)^{\frac{1}{x}}$

j) $\displaystyle\lim_{x\to +\infty}(x^3 + 3^x)^{\tan\frac{1}{x}}$

Exercise 39. Find a, b such that there exists a limit of the following function as $x \to 0$

$$f(x) = \frac{1}{\sin^3 x} - \frac{1}{x^3} - \frac{a}{x^2} - \frac{b}{x}.$$

Exercise 40. Given a real-valued, function f on $[a,b]$ and twice-differentiable on (a,b). Prove that for all $x \in (a,b)$ there exists at least one point $c \in (a,b)$ such that

$$f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) = \frac{(x-a)(x-b)}{2} f''(c).$$

Exercise 41. Use the Newton method, approximate $\sqrt[6]{2}$ to 8 decimal digits.

Exercise 42. Explain why the Newton method cannot be applied directly to the equation $x^3 - 2x + 2 = 0$ for an initial point $x_0 = 1$.

Exercise 43. Analyze the monotonicity of the following functions

a) $y = x^4 - 2x^3 + 2x - 1$

b) $y = 3\arctan x - \ln(1+x^2)$

c) $y = x + |\sin 2x|$, $x \in [0, \pi]$

Exercise 44. Prove the following inequalities

a) $2x \arctan x \geq \ln(1+x^2)$ for all $x \in \mathbb{R}$

b) $x - \dfrac{x^2}{2} \leq \ln(1+x) \leq x$ for all $x \geq 0$

c) $\cos x \leq 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24}$, $\forall x \in \left[0, \dfrac{\pi}{2}\right)$

Exercise 45. Find all extreme points of the following functions

a) $y = \dfrac{3x^2+4x+4}{x^2+x+1}$

b) $y = x - \ln(1+x)$

c) $y = \sqrt[3]{(1-x)(x-2)^2}$

d) $y = x^{\frac{2}{3}} + (x-2)^{\frac{2}{3}}$

Exercise 46. Given a convex function $f(x)$ on $[a,b]$. Prove that $\forall c \in (a,b)$ we have

$$\frac{f(c)-f(a)}{c-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(c)}{b-c}.$$

Exercise 47. Prove the following inequalities

a) $\tan \dfrac{x+y}{2} \leq \dfrac{\tan x + \tan y}{2}$, $\forall x, y \in \left(0, \dfrac{\pi}{2}\right)$

b) $x \ln x + y \ln y \geq (x+y) \ln \dfrac{x+y}{2}$, $\forall x, y > 0$

1.10. Curve sketching

Exercise 48. Find all the asymptotes of the graph of $y = f(x)$

a) $y=\sqrt[3]{1+x^3}$

b) $y=\ln(1+e^{-x})$

c) $y=\dfrac{x^3\operatorname{arccot}x}{1+x^2}$

d) $\begin{cases} x=2t-t^2 \\ y=\dfrac{2016t^2}{1-t^3} \end{cases}$

e) $\begin{cases} x=t \\ y=t+2\arctan t \end{cases}$

Exercise 49. Analyze and sketch the curve of the following functions (curves)

a) $y=e^{\frac{1}{x}-x}$

b) $y=\sqrt[3]{x^3-x^2-x+1}$

c) $y=\dfrac{x^3}{x^2+1}$

d) $y=\dfrac{x-2}{\sqrt{x^2+1}}$

e) $\begin{cases} x=\dfrac{2t}{1-t^2} \\ y=\dfrac{t^2}{1+t} \end{cases}$

f) $\begin{cases} x=2t-t^2 \\ y=3t-t^3 \end{cases}$

g) $r=a+b\cos\varphi,(0<a\le b)$

h) $r=a\sin 3\varphi,(a>0)$.

Chapter 2

Integral

2.1 Indefinite integrals

Exercise 50. Evaluate the following integrals

a) $\int e^{\sin^2 x} \sin 2x dx$

b) $\int (x+2) \ln x dx$

c) $\int |x^2 - 3x + 2| dx$

d) $\int \frac{x dx}{(x+2)(x+5)}$

e) $\int \frac{(x^2+2)dx}{x^3+1}$

f) $\int \frac{dx}{(x+a)^2(x+b)^2}$

g) $\int \sin 5x \cos 3x dx$

h) $\int \tan^3 x dx$

i) $\int \frac{dx}{3\sin x - 4\cos x}$

j) $\int \frac{(3-2x)dx}{\sqrt{1-x^2}}$

k) $\int \frac{dx}{1+\sqrt{x^2+4x+5}}$

l) $\int \frac{(x+1)dx}{\sqrt{x^2-2x-1}}$

Exercise 51. Evaluate the following integrals

a) $\int \frac{x^4 dx}{x^{10}-1}$

b) $\int x\sqrt{-x^2+3x-2} dx$

c) $\int \frac{dx}{(x^2+2x+5)^2}$

d) $\int \sin^{n-1} x \sin(n+1)x dx, n \in \mathbb{N}^*$

e) $\int e^{-2x} \cos 3x dx$

f) $\int \arcsin^2 x dx$

Exercise 52. Construct the recurrence formula to evaluate I_n, $n \in \mathbb{N}$

a) $I_n = \int x^n e^x dx$

b) $I_n = \int \sin^n x dx$

c) $I_n = \int \frac{dx}{\cos^n x}$

2.2 Definite integral

Exercise 53. Evaluate the following derivatives

a) $\dfrac{d}{dx}\int\limits_x^y e^{t^2}dt$ b) $\dfrac{d}{dy}\int\limits_x^y e^{t^2}dt$ c) $\dfrac{d}{dx}\int\limits_{x^2}^{x^3} \dfrac{dt}{\sqrt{1+t^4}}$

Exercise 54. Use definition and the method to calculate definite integral, evaluate

a) $\lim\limits_{n\to\infty}\left[\dfrac{1}{n\alpha}+\dfrac{1}{n\alpha+\beta}+\dfrac{1}{n\alpha+2\beta}+\cdots+\dfrac{1}{n\alpha+(n-1)\beta}\right], (\alpha,\beta>0)$

b) $\lim\limits_{n\to\infty}\dfrac{1}{n}\left(\sqrt{1+\dfrac{1}{n}}+\sqrt{1+\dfrac{2}{n}}+\cdots+\sqrt{1+\dfrac{n}{n}}\right)$

Exercise 55. Calculate the following limits (if exist)

a) $\lim\limits_{x\to 0^+}\dfrac{\int\limits_0^{\sin x}\sqrt{\tan t}dt}{\int\limits_0^{\tan x}\sqrt{\sin t}dt}$ b) $\lim\limits_{x\to+\infty}\dfrac{\int\limits_0^x(\arctan t)^2dt}{\sqrt{x^2+1}}$ c) $\lim\limits_{x\to+\infty}\dfrac{\left(\int\limits_0^x e^{t^2}dt\right)^2}{\int\limits_0^x e^{2t^2}dt}$

Exercise 56. Evaluate the following integrals

a) $\int\limits_{1/e}^{e}|\ln x|\,(x+1)\,dx$

b) $\int\limits_1^e (x\ln x)^2dx$

c) $\int\limits_0^{3\pi/2}\dfrac{dx}{2+\cos x}$

d) $\int\limits_0^1 \dfrac{\sin^2 x\cos x}{(1+\tan^2 x)^2}dx$

e) $\int\limits_0^3 \arcsin\sqrt{\dfrac{x}{1+x}}dx$

f) $\int\limits_0^{\pi/2}\cos^n x\cos nx dx, n\in\mathbb{N}^*$

Exercise 57. Prove that if $f(x)$ is continuous on $[0,1]$ then

a) $\int\limits_0^{\pi/2} f(\sin x)dx=\int\limits_0^{\pi/2} f(\cos x)dx$ b) $\int\limits_0^{\pi} xf(\sin x)dx=\int\limits_0^{\pi}\dfrac{\pi}{2}f(\sin x)dx$

Then apply to evaluate the following integrals

1. $\int\limits_0^{\pi}\dfrac{x\sin xdx}{1+\cos^2 x}$ 2. $\int\limits_0^{\pi/2}\dfrac{\sqrt{\sin x}dx}{\sqrt{\sin x}+\sqrt{\cos x}}$

Exercise 58. Given two integrable $f(x), g(x)$ functions $[a,b]$. Prove the inequality below (for $a<b$)

$$\left(\int\limits_a^b f(x)g(x)dx\right)^2\le\left(\int\limits_a^b f^2(x)dx\right)\left(\int\limits_a^b g^2(x)dx\right)$$

(Cauchy-Schwartz inequality for integrals).

2.3 Improper integrals

Exercise 59. Determine whether each integral below is convergent or divergent. Calculate the convergent integrals.

a) $\int\limits_{-\infty}^{0} xe^x dx$

b) $\int\limits_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2}$

c) $\int\limits_{0}^{1} \frac{dx}{\sqrt{x(1-x)}}$

d) $\int\limits_{2}^{+\infty} \frac{dx}{x\ln x}$

e) $\int\limits_{0}^{+\infty} \frac{dx}{x^2+3x+2}$

f) $\int\limits_{0}^{+\infty} \frac{x^2+1}{x^4+1}dx$

Exercise 60. Determine whether each integral below is convergent or divergent.

a) $\int\limits_{1}^{+\infty} \frac{\ln(1+x)\,dx}{x^2}$

b) $\int\limits_{1}^{+\infty} \frac{dx}{\sqrt{x+x^3}}$

c) $\int\limits_{2}^{+\infty} \frac{xdx}{\ln^3 x}$

d) $\int\limits_{0}^{1} \frac{dx}{\tan x - x}$

e) $\int\limits_{0}^{1} \frac{\sqrt{x}dx}{\sqrt{1-x^4}}$

f) $\int\limits_{0}^{\pi} \frac{dx}{\sqrt[3]{\sin x}}$

g) $\int\limits_{0}^{+\infty} \frac{\ln(1+3x)}{x\sqrt{x}}dx$

h) $\int\limits_{0}^{+\infty} \frac{x-\sin x}{\sqrt{x^7}}dx$

i) $\int\limits_{0}^{+\infty} \frac{\arctan x dx}{\sqrt{x^3}}$

j) $\int\limits_{0}^{+\infty} \frac{\sin 2x}{x}dx$

Exercise 61. Provided that $\int\limits_{0}^{+\infty} f(x)dx$ converges, can we deduce that $\lim\limits_{x\to+\infty} f(x) = 0$? Discuss the example $\int\limits_{0}^{+\infty} \sin(x^2)\,dx$.

Exercise 62. Given a continuous function $f(x)$ on $[a, +\infty)$ and $\lim\limits_{x\to+\infty} f(x) = A \neq 0$. Does the integral $\int\limits_{a}^{+\infty} f(x)dx$ converges?

2.4 Application of definite integrals

Exercise 63. Calculate the area of the region enclosed by the curve

a) The parabola $y = x^2 + 4$ and the straight line $x - y + 4 = 0$.

b) The curve $y = x^3$ and the straight lines $y = x, y = 4x, (x \geq 0)$.

c) The circle $x^2 + y^2 = 2x$ and the parabola $y^2 = x, (y^2 \leq x)$

d) The curve $y^2 = x^2 - x^4$

Exercise 64. Calculate the volume of the solid generated by the common part of the two cylinders $x^2 + y^2 \leq a^2$ and $y^2 + z^2 \leq a^2$, $(a > 0)$.

Exercise 65. Calculate the volume of an object limited by the curved surface $z = 4 - y^2$, the coordinate planes $x = 0, z = 0$ and the plane $x = a$ $(a \neq 0)$.

Exercise 66. Calculate the volume of a solid obtained by rotating the region bounded by the curves $y = 2x - x^2$ and $y = 0$

a) about the $0x$ axis once

b) about the $0y$ axis once

Exercise 67. Calculate the length of the curves

a) $y = \ln \dfrac{e^x + 1}{e^x - 1}$ for x varies from 1 to 2

b) $\begin{cases} x = a\left(\cos t + \ln \tan \dfrac{t}{2}\right) \\ y = a \sin t \end{cases}$ for t varies from $\dfrac{\pi}{3}$ to $\dfrac{\pi}{2}$, $(a > 0)$

Exercise 68. Calculate the volume of a solid obtained by rotating the curves

a) $y = \sin x, 0 \leq x \leq \dfrac{\pi}{2}$ about the $0x$ axis.

b) $y = \dfrac{1}{3}(1 - x)^3, 0 \leq x \leq 1$ about the $0x$ axis.

Chapter 3

Functions of several variables

3.1 Basic definitions

Exercise 69. Find the domains of the following functions

a) $z = \dfrac{1}{\sqrt{x^2+y^2-1}}$

b) $z = \sqrt{(x^2+y^2-1)(4-x^2-y^2)}$

c) $z = \arcsin \dfrac{y-1}{x}$

d) $z = \sqrt{x \sin y}$

Exercise 70. Calculate the limits (if exist)

a) $f(x,y) = \dfrac{xy}{x^2+y^2}, \quad (x \to 0, y \to 0)$

b) $f(x,y) = \dfrac{y^2}{x^2+3xy}, \quad (x \to \infty, y \to \infty)$

c) $f(x,y) = \dfrac{(x-1)^3-(y-2)^3}{(x-1)^2+(y-2)^2}, \quad (x \to 1, y \to 2)$

d) $f(x,y) = \dfrac{1-\cos\sqrt{x^2+y^2}}{x^2+y^2}, \quad (x \to 0, y \to 0)$

e) $f(x,y) = \dfrac{x(e^y-1)-y(e^x-1)}{x^2+y^2}, \quad (x \to 0, y \to 0)$

f) $f(x,y) = \dfrac{xy^2}{x^2+y^4}, \quad (x \to 0, y \to 0)$

Exercise 71. Calculate the following limits (if exists)

a) $\lim\limits_{x\to 0}\lim\limits_{y\to 0} \frac{x^2}{x^2+y^2}$,

b) $\lim\limits_{y\to 0}\lim\limits_{x\to 0} \frac{x^2}{x^2+y^2}$

c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

3.2 Partial derivatives and differentials

Exercise 72. Evaluate the following partial derivatives

a) $z = \ln\left(x + \sqrt{x^2 + y^2}\right)$

b) $z = y^2 \sin \dfrac{x}{y}$

c) $z = \arctan \sqrt{\dfrac{x^2 - y^2}{x^2 + y^2}}$

d) $z = x^{y^3}, (x > 0)$

e) $u = x^{y^z}, (x, y, z > 0)$

f) $u = e^{\frac{1}{x^2+y^2+z^2}}$.

Exercise 73. Analyze the continuity of the following functions and the existence of their partial derivatives

a) $f(x, y) = \begin{cases} x \arctan\left(\dfrac{y}{x}\right)^2, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$

b) $f(x, y) = \begin{cases} \dfrac{x \sin y - y \sin x}{x^2 + y^2}, & \text{if } (x, y) \neq (0; 0), \\ 0, & \text{if } (x, y) = (0; 0). \end{cases}$

Exercise 74. Given a function $z = yf(x^2 - y^2)$, where f is differentiable. Prove that

$$\frac{1}{x}{z_x}' + \frac{1}{y}{z_y}' = \frac{z}{y^2}.$$

Exercise 75. Evaluate the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

a) $z = e^{u^2 - 2v^2}, u = \cos x, v = \sqrt{x^2 + y^2}$

b) $z = \ln\left(u^2 + v^2\right), u = xy, v = \dfrac{x}{y}$

c) $z = \arcsin\left(x - y\right), x = 3t, y = 4t^3$

Exercise 76. Given a twice-differentiable function f on $\mathbb{R}$. Prove that the function $\omega(x, t) = f(x - 3t)$ satisfies the wave equation $\dfrac{\partial^2 \omega}{\partial t^2} = 9\dfrac{\partial^2 \omega}{\partial x^2}$.

Exercise 77. Evaluate the total differentiation of the following functions

a) $z = \sin(x^2 + y^2)$

b) $z = \ln \tan \dfrac{y}{x}$

c) $z = \arctan \dfrac{x + y}{x - y}$

d) $u = x^{y^2 z}$

Exercise 78. Using differentiation to approximate the following

a) $A = \sqrt[3]{(1,02)^2 + (0,05)^2}$

b) $B = \ln\left(\sqrt[3]{1,03} + \sqrt[4]{0,98} - 1\right)$

c) $C = \sqrt{(2,02)^3 + e^{0,03}}$

d) $D = (1,02)^{1,01}$

Exercise 79. Given a function $z = f(x, y)$ determined via the equation $z - ye^{\frac{z}{x}} = 0$. Approximate $f(0,99; 0,02)$.

Exercise 80. Evaluate the partial derivatives of the functions determined via the following equations

a) $x^3y - y^3x = a^4$, calculate y'

b) $x + y + z = e^z$, calculate $z_x{}', z_y{}'$

c) $\arctan\dfrac{x+y}{a} = \dfrac{y}{a}$, calculate y'

d) $x^3+y^3+z^3-3xyz = 0$, calculate $z_x{}', z_y{}'$.

Exercise 81. Given a function $z = z(x, y)$ that satisfies the equation $2x^2y+4y^2+x^2z+z^3 = 3$. Calculate $\dfrac{\partial z}{\partial x}(0; 1), \dfrac{\partial z}{\partial y}(0; 1)$.

Exercise 82. Given z be a function of two variables x, y that satisfies the equation $ze^z = xe^x + ye^y$, and let $u = \dfrac{x+z}{y+z}$, calculate $u_x{}', u_y{}'$.

Exercise 83. Calculate the derivatives of functions $y(x), z(x)$ defined by the system

$$\begin{cases} x + y + z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

Exercise 84. Given a function $z = z(x, y)$ that satisfies the equation $z^2 + \dfrac{2}{x} = \sqrt{y^2 - z^2}$. Prove that

$$x^2 z_x{}' + \frac{1}{y} z_y{}' = \frac{1}{z}.$$

Exercise 85. Evaluate the second partial derivatives of the following functions

a) $z = \dfrac{1}{3}\sqrt{(x^2+y^2)^3}$

b) $z = x^2 \ln(x + y)$

c) $z = \arctan\dfrac{y}{x}$

d) $z = \sin(x^3 + y^2)$

Exercise 86. Evaluate the second partial derivatives of the following functions

a) $z = xy^3 - x^2y$

b) $z = e^{2x}(x + y^2)$

c) $z = \ln(x^3 + y^2)$

Exercise 87. a) Express the function $f(x, y) = x^2 + 3y^2 - 2xy + 6x + 2y - 4$ as the Taylor series in a neighborhood of the point $(-2, 1)$.

b) Express the function $f(x, y) = e^x \sin y$ as a Maclaurin series to the third order of x and y.

3.3 Extreme values of functions of several variables

Exercise 88. Find all extreme values of the following functions

a) $z = 4x^3 + 6x^2 - 4xy - y^2 - 8x + 2$

b) $z = 2x^2 + 3y^2 - e^{-(x^2+y^2)}$

c) $z = 4xy - x^4 - 2y^2$

d) $z = \dfrac{4}{x} + \dfrac{3}{y} - \dfrac{xy}{12}$

e) $z = e^{2x}(4x^2 - 2xy + y^2)$

f) $z = x^3 + y^3 - (x + y)^2$

Exercise 89. Find all extreme values of the following functions subject to given constraints.

a) $z = xy$ given that $x + y = 1$

b) $z = x^2 + y^2$ given that $3x - 4y = 5$

c) $z = \dfrac{1}{x} + \dfrac{1}{y}$ given that $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{a^2}$

Exercise 90. Find a point in the ellipse $4x^2 + y^2 = 4$ such that the distance to the point $A(1; 0)$ is longest.

Exercise 91. Find the (global) maximum and minimum values of the following functions

a) $z = x^2 + y^2 + xy - 7x - 8y$ in the triangle restricted by the straight lines $x = 0$, $y = 0$, and $x + y = 6$

b) $z = 4x^2 - 9y^2$ in the bounded region restricted by the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.