Chapter 5: Euclidean spaces, quadratic forms

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5.1.1. Inner products

Let V be a vector space over $\mathbb{R}$.

Definition

An *inner product* on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following conditions: for all $u, v, w \in V$, and for all $k \in \mathbb{R}$, we have

1. $\langle u, v \rangle = \langle v, u \rangle$;
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$;
3. $\langle ku, v \rangle = k \langle u, v \rangle$;
4. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

A vector space together with an inner product is called an *inner product space*.
A finite dimensional vector space together with an inner product is called a *Euclidean space*.

Standard inner product on $\mathbb{R}^n$

On $\mathbb{R}^n$ we consider the following inner product defined as follows: if $u = (x_1, \dots, x_n)$ and $v = (y_1, \dots, y_n)$, then

$$\langle u, v \rangle := x_1 y_1 + \cdots + x_n y_n.$$

Can check that this is indeed an inner product on $\mathbb{R}^n$.
This inner product on $\mathbb{R}^n$ is called the *Euclidean* (or *standard*, or *canonical*, or *usual*) inner product, or the *dot product* on $\mathbb{R}^n$.

Another inner product on $\mathbb{R}^2$

On $\mathbb{R}^2$, consider

$$\langle u, v \rangle := x_1 y_1 + 2x_2 y_2.$$

This defines an inner product on $\mathbb{R}^2$.

Examples

On $\mathbb{R}^2$, consider

$$\langle u, v \rangle := x_1 y_1 - 2x_2 y_2.$$

This is not an inner product on $\mathbb{R}^2$.

Basic properties

Let V be an inner product space with the inner product $\langle\ ,\ \rangle$.

Properties

For all $u, v, w \in V$ and all $c \in \mathbb{R}$:

- $\langle 0, u\rangle = \langle u, 0\rangle = 0$.
- $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.
- $\langle u, cv\rangle = c\langle u, v\rangle$.

Length and distance

Definition

1. The *length* or *norm* of a vector u is $\|u\| = \sqrt{\langle u, u\rangle}$.
2. The *distance* between two vectors $u, v \in V$ is $d(u, v) = \|u - v\|$.

Example: In $\mathbb{R}^n$ with the Euclidean inner product (the dot product), the length of $u = (x_1, x_2, \ldots, x_n)$ is

$$\|u\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Theorem (Cauchy – Schwarz inequality)

For all $u, v \in V$, we have $|\langle u, v\rangle| \leq \|u\| \, \|v\|$.

The equality holds if and only if u and v are linearly dependent.

Example: Consider $\mathbb{R}^n$ with the dot product, the Cauchy–Schwarz inequality becomes

$$|x_1 y_1 + \cdots + x_n y_n| \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}.$$

Some properties of lengths and distances

Proposition

- $\|v\| \geq 0$, for all $v \in V$.
- $\|v\| = 0 \Leftrightarrow v = 0$.
- $\|av\| = |a| \ \|v\|$, $\forall a \in \mathbb{R}, v \in V$.
- $\|u + v\| \leq \|u\| + \|v\|$, $\forall u, v \in V$.

Proposition

- $d(u, v) \geq 0$, for all $u, v \in V$.
- $d(u, v) = 0 \Leftrightarrow u = v$.
- $d(u, v) = d(v, u)$, $\forall u, v \in V$.
- $d(u, v) \leq d(u, w) + d(w, v)$, $\forall u, v, w \in V$.

Angle

Definition

1. The *angle* of two non-zero vectors u, v is an angle θ $(0 \leq \theta \leq \pi)$ such that

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\| \; \|v\|}.$$

2. Two vectors u, v are said to be *orthogonal* (or *perpendicular*), denoted $u \perp v$, if $\langle u, v\rangle = 0$.

Remark: The zero vector 0 is orthogonal to every vector.

Orthogonal and orthonormal sets

Let $S = \{v_1, \dots, v_k\}$ be a set of vectors in V.

Definition

- S is said to be *orthogonal* if any two vectors in S are orthogonal, that means

$$\langle v_i, v_j \rangle = 0, \forall i \neq j;$$

- S is said to be *orthonormal* if S is orthognal and every vector in S has length 1, that means
$$\langle v_i, v_j \rangle = \begin{cases} 0 & \forall i \neq j \\ 1 & \forall i = j. \end{cases}$$

Proposition

Let $S = \{v_1, \dots, v_k\}$ be an orthogonal set of nonzero vectors in V. Then

1. S is linearly independent,
2. The set $\{\frac{v_1}{\|v_1\|}, \dots, \frac{v_k}{\|v_k\|}\}$ is orthonormal.

Orthogonal and orthonormal bases

Defintion

- An *orthogonal basis* for V is a basis for V that is also an orthogonal set.
- An *orthonormal basis* for V is a basis for V that is also an orthonormal set.

Remark: If $\dim V = n$ then any orthonormal set of n vectors of V is automatically a(n orthonormal) basis for V.

Example: Consider $\mathbb{R}^2$ with the standard inner product.

- $\{(1,1),(1,-1)\}$ is an orthogonal basis,
- $\{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})\}$ is an orthonormal basis,
- $\{(1,0),(0,1)\}$ is an orthonormal basis.

Coordinates of a vector relative to an orthogonal basis

Theorem

Let $S = \{v_1, \ldots, v_n\}$ be an orthogonal basis for an Eulcidean vector space V and v a vector in V. Then

$$v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \cdots + \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} v_n.$$

In particular, if S is orthonormal if

$$v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n.$$

Example: In $\mathbb{R}^2$, find the coordinates of $v = (1,1)$ relative to the basis $S = \{(1,2),(2,-1)\}$.

Consider $\mathbb{R}^2$ with the standard inner product. Then S is an orthogonal basis. The coordinate vector of v relative to S is $(v)_S = (c_1, c_2)$, where

$$c_1 = \frac{\langle (1,1),(1,2) \rangle}{||(1,2)||^2} = \frac{3}{5}, \quad c_2 = \frac{\langle (1,1),(2,-1) \rangle}{||(2,-1)||^2} = \frac{1}{5}.$$

Thus, $(v)_S = (\frac{3}{5}, \frac{1}{5})$.

5.1.2. Gram-Schmidt orthonormalization process

- Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for a Euclidean vector space V.
- Construct $\mathcal{B}' = \{w_1, w_2, \ldots, w_n\}$ as follows:

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \\
\ldots \\
w_n &= v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}.
\end{aligned}
$$

Theorem (Gram-Schmidt orthonormalization)

- Then $\mathcal{B}'$ is an orthogonal basis for V.
- Set $u_i = \dfrac{w_i}{\|w_i\|}$, $1 \leq i \leq n$.
  Then $\mathcal{B}'' = \{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis for V.
- Morever, $\mathrm{span}\{v_1, \ldots, v_k\} = \mathrm{span}\{u_1, \ldots, u_k\}$, for every $1 \leq k \leq n$.

The process of calculating the orthonormal basis $\{u_1, \ldots, u_n\}$ from a basis $\{v_1, \ldots, v_n\}$ as above is called the Gram-Schmidt orthonormalization (process).

Example: In $\mathbb{R}^3$, with the standard inner product, consider a basis $\mathcal{B} = \{v_1, v_2, v_3\} = \{(1,1,0), (1,2,0), (0,1,2)\}$.

- We have

$$\begin{aligned} w_1 &= v_1 = (1,1,0) \\ w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = \left(-\frac{1}{2}, \frac{1}{2}, 0\right) \\ w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 = (0,0,2) \end{aligned}$$

  Then $B' = \{w_1, w_2, w_3\}$ in an orthogonal basis for $\mathbb{R}^3$.

- Normalize vectors in $\mathcal{B}'$:

$$u_1 = \frac{w_1}{\|w_1\|} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$u_3 = \frac{w_3}{\|w_3\|} = (0,0,1)$$

  $\mathcal{B}'' = \{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

Example

In $\mathbb{R}^3$ with the standard inner product, consider $v_1 = (0,1,0)\,, v_2 = (1,1,1)$. Find an orthonormal basis for $\mathrm{span}\{v_1, v_2\}$.

- Orthogonalize $\{v_1, v_2\}$ we obtain

$$\begin{aligned} w_1 &= v_1 = (0,1,0) \\ w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = (1,1,1) - (0,1,0) = (1,0,1). \end{aligned}$$

- Normalize w_1, w_2 we obtain an orthonormal basis for $\mathrm{span}\{v_1, v_2\}$

$$\left\{(0,1,0), \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\right\}.$$

Example

In $\mathbb{R}^3$, consider the followin inner product

$$\langle u, v\rangle = 2x_1y_1 + 3x_2y_2 + 2x_3y_3 - 2x_1y_2 - 2x_2y_1 + x_1y_3 + x_3y_1 - x_2y_3 - x_3y_2,$$

với $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$. Find an orthonormal basis for $\mathbb{R}^3$ relative to this inner product by using the Gram-Schmidt orthonormalization process on the standard basis for $\mathbb{R}^3$.

- We have

$$\begin{aligned} w_1 &= e_1 = (1,0,0) \\ w_2 &= e_2 - \frac{\langle e_2, w_1\rangle}{\|w_1\|^2} w_1 = (0,1,0) - \frac{-2}{2}(1,0,0) = (1,1,0) \\ w_3 &= e_3 - \frac{\langle e_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle e_3, w_2\rangle}{\|w_2\|^2} w_2 = (0,0,1) - \frac{1}{2}(1,0,0) - \frac{0}{1}(1,1,0) \\ &= (-1/2, 0, 1). \end{aligned}$$

- $\mathcal{B}' = \{w_1, w_2, w_3\}$ is an orthogonal basis for $\mathbb{R}^3$.
- Normalize vectors in $\mathcal{B}'$:

$$u_1 = \frac{w_1}{\|w_1\|} = (\frac{1}{\sqrt{2}}, 0, 0)$$
$$u_2 = \frac{w_2}{\|w_2\|} = (1, 1, 0)$$
$$u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{6}}(-1, 0, 2).$$

- $\mathcal{B}'' = \{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{R}^3$ with respect to $\langle \cdot, \cdot \rangle$.

Remark

Normalizing of vectors can be done right after orthogonalizing each vector as follows:

$$u_1 = \frac{w_1}{\|w_1\|} \text{ where } w_1 = v_1$$

$$u_2 = \frac{w_2}{\|w_2\|} \text{ where} w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$$u_3 = \frac{w_3}{\|w_3\|} \text{ where} w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$$

$$\dots$$

$$u_n = \frac{w_n}{\|w_n\|} \text{ where } w_n = v_n - \langle v_n, u_1 \rangle u_1 - \cdots - \langle v_n, u_{n-1} \rangle u_{n-1}$$

Corollary

Every Euclidean vector space has an orthonormal basis.

Proposition

Let V be an Euclidean vector space with an orthonormal basis $\mathcal{B}$. Let $u, v \in V$. Let $(u)_{\mathcal{B}} = (x_1, x_2, \ldots, x_n)$ and $(v)_{\mathcal{B}} = (y_1, y_2, \ldots, y_n)$. Then

- $\langle u, v \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$;
- $\|u\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$;
- $d(u, v) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$.

Orthogonal subspaces

Consider a Euclidean vector space V with inner product $\langle \quad , \quad \rangle$ and $\dim(V) = n$.

Definition

Giả sử U là một không gian con của V và $v \in V$. We say that v is *orthogonal* (or *perpendicular*) to U, and write $v \perp U$, if $v \perp w$, $\forall w \in U$, i.e., $\langle v, w \rangle = 0$, $\forall w \in U$.

Remark: $v \perp U \Leftrightarrow v$ is orthognal to every vector in a basis (of a spanning set) for U.

Definition

Two vector subspaces U, W of V are said to be *orthogonal* (or *perpendicular*), written $U \perp W$, if for every $u \in U$ and every $w \in W$, we have that u and v are orthogonal.

Remark: If $U \perp W$ then $U \cap W = \{0\}$.

Example: In $\mathbb{R}^3$ (with the standard inner product), consider $U = \mathrm{span}\,\{(0,1,-1),(1,1,0)\}$, $w = (-1,1,1)$, $W = \mathrm{span}\{w\}$.
We have $w \perp U$ và $U \perp W$.

Orthogonal complements

Definition

The *orthogonal complement* of a vector subspace U (of V), denoted by $U^{\perp}$, is defined as

$$U^{\perp} = \{v \in V \mid v \perp U\} = \{v \in V \mid \langle v, u\rangle = 0, \forall u \in U\} .$$

Example: In $\mathbb{R}^4$ (with the standard inner product), find the orthogonal complement of $U = \operatorname{span}\{v_1 = (1, 2, 1, 0), v_2 = (0, 0, 0, 1)\}$.

We have $v = (x_1, x_2, x_3, x_4) \in U^{\perp} \Leftrightarrow \begin{cases} \langle v, v_1\rangle = 0 \\ \langle v, v_2\rangle = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_4 = 0 \end{cases}$

Solve this system we obtain $U^{\perp} = \operatorname{span}\{u_1, u_2\}$ where $u_1 = (-2, 1, 0, 0)$, $u_2 = (-1, 0, 1, 0)$.

Exercise (CK20183-N2)

In $\mathbb{R}^4$ with the standard inner product, let $u_1 = (-1, -2, 1, 0)$, $u_2 = (1, -1, 2, 3)$, $u_3 = (-3, -2, 0, 1)$. The vectors in $\mathbb{R}^4$ that are orthognal to these there vectors form a vector subspace U of $\mathbb{R}^4$. Find a basis for U.

Orthogonal decomposition

Theorem

Cho U là một không gian véc tơ con của V và $\dim V = n$. Khi đó

- $U^{\perp}$ là một không gian véc tơ con của V.
- $U + U^{\perp} = V$ và $\dim U + \dim U^{\perp} = n$.
- $(U^{\perp})^{\perp} = U$.

Remark: V is a direct sum of U and $U^{\perp}$. That means every vector $v \in V$ has a unique representation $v = u + w$, where $u \in U$, $w \in U^{\perp}$.

Orthogonal projection

Definition

Let U be a subspace of V and $v \in V$. Let

$$v = u + w, \text{với } u \in U, w \in U^{\perp},$$

be the unique representation of v as a sum of a vector in U and a vecttor in $U^{\perp}$.
We say that u is the *orthogonal projection* of v onto the subspace U, and denoted by $\mathrm{pr}_U(v) = u$.

Remark:

- The orthogonal projection $\mathrm{pr}_U(v)$ of v onto the subspace U is the unique vector u such that $u \in U$ and $v - u \perp U$.
- If $v = u + w$, where $u \in U$ and $w \in U^{\perp}$, then $w = \mathrm{pr}_{U^{\perp}}(v)$. In other words, $v - \mathrm{pr}_U(v)$ is the orthogonal projection of v onto the subspace $U^{\perp}$.

$$v = \mathrm{pr}_U(v) + \mathrm{pr}_{U^{\perp}}(v).$$

Theorem

Let U be a subspace of a Euclidean space V and $v \in V$. Then

$$\|v - \mathrm{pr}_U(v)\| \leq \|v - w\|, \quad \forall w \in U,$$

and the "=" holds $\Leftrightarrow$ $w = \mathrm{pr}_U(v)$.

Among all the vector in the subspace U, the vector $\mathrm{pr}_U(v)$ is the closet vector to v.

Finding orthogonal projection

Định lý

If $S = \{u_1, u_2, \ldots, u_k\}$ is an orthogonal basis for the subspace U of V then for every $v \in V$, we have

$$\mathrm{pr}_U(v) = \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} u_1 + \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} u_2 + \cdots + \frac{\langle v, u_k\rangle}{\langle u_k, u_k\rangle} u_k \,.$$

In particular, if S is an orthonormal basis for U then

$$\mathrm{pr}_U(v) = \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \cdots + \langle v, u_k\rangle u_k \,.$$

Particular case

If $U = \mathrm{span}\{u\}$ where $u \neq 0$, then we also call the orthogonal projection of v onto U as theorthogonal projection of v onto the vector u, and we have

$$\mathrm{pr}_u(v) := \mathrm{pr}_U(v) = \frac{\langle v, u\rangle}{\langle u, u\rangle} u$$

Example

In $\mathbb{R}^3$ with the standard inner product, consider $U = \mathrm{span}\{v_1 = (1,2,2), v_2 = (1,1,0)\}$ and $v = (1,1,3)$. Find $\mathrm{pr}_U(v)$.

Using Gram-Schmidt process on the basis $\{w_1, w_2\}$ for U, we find an orthonormal basis for U:

$$u_1 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$
$$u_2 = (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$$

Hence

$$\mathrm{pr}_U(v) = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 = (\frac{1}{3}, \frac{5}{3}, \frac{8}{3}).$$

Solution 2:

- Let $u = \mathrm{pr}_U(v)$. Then $u \in U$ and $v - u \perp U$.
- Since $u \in U$, one has $u = c_1 v_1 + c_2 v_2$, for some $c_1, c_2 \in \mathbb{R}$.
- We have $v - u \perp U \Leftrightarrow \begin{cases} v - u \perp v_1 \\ v - u \perp v_2 \end{cases} \Leftrightarrow \begin{cases} \langle v - u, v_1 \rangle = 0 \\ \langle v - u, v_2 \rangle = 0 \end{cases} \Leftrightarrow \begin{cases} \langle v, v_1 \rangle = \langle u, v_1 \rangle \\ \langle v, v_2 \rangle = \langle u, v_2 \rangle \end{cases} \Leftrightarrow$
  $\begin{cases} c_1 \langle v_1, v_1 \rangle + c_2 \langle v_2, v_1 \rangle = \langle v, v_1 \rangle \\ c_1 \langle v_1, v_2 \rangle + c_2 \langle v_2, v_2 \rangle = \langle v, v_2 \rangle \end{cases} \Leftrightarrow \begin{cases} 9c_1 + 3c_2 = 9 \\ 3c_1 + 2c_2 = 2 \end{cases} \Leftrightarrow \begin{cases} c_1 = 4/3 \\ c_2 = -1 \end{cases}$
- Thus $u = \frac{4}{3}(1, 2, 2) + (-1)(1, 1, 0) = (\frac{1}{3}, \frac{5}{3}, \frac{8}{3})$.

Another way to find orthogonal projections (optional)

- Let V be a Euclidean space. Suppose that $U = \mathrm{span}\{v_1, \ldots, v_k\}$ is a subspace of V.
- Let $v \in V$. Set $u = \mathrm{pr}_U(v)$. Write $u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$.
- Then $(c_1, c_2, \ldots, c_k)$ is a solution of the system

$$\begin{cases} \langle v_1, v_1\rangle c_1 + \langle v_1, v_2\rangle c_2 + \cdots + \langle v_1, v_k\rangle c_k & = \langle v, v_1\rangle \\ \langle v_2, v_1\rangle c_1 + \langle v_2, v_2\rangle c_2 + \cdots + \langle v_2, v_k\rangle c_k & = \langle v, v_2\rangle \\ \vdots & \\ \langle v_k, v_1\rangle c_1 + \langle v_k, v_2\rangle c_2 + \cdots + \langle v_k, v_k\rangle c_k & = \langle v, v_k\rangle \end{cases}$$

Another way to find orthogonal projections (matrix form, optional)

Use the same notation as in the previous slide.
Furthermore, we fix an orthonormal basis $\mathcal{E}$ for V. For simplicity, we use $[u]$ to denote the column vector $[u]_{\mathcal{E}}$ (the coordinate of $u \in V$ relative $\mathcal{E}$.

- Let$A$ be the column coordinate matrix of the set $S = \{v_1, \ldots, v_k\}$ relative to $\mathcal{E}$.
- Set $[u] := x = [c_1 \ \cdots c_k]^T$. Then x satisfies the system

$$A^T Ax = A^T [v].$$

- Solve x from the above system, and we find $u = c_1 v_1 + \cdots + c_k v_k$.
- In the case that S is a basis for U then $A^T A$ is invertible and

$$[u] = A(A^T A)^{-1} A^T [v].$$

Remark: When $V = \mathbb{R}^n$ with the standard inner product, we usually use the standard basis $\mathcal{E}$.
Example: In $\mathbb{R}^3$ with the standard inner product, consider $U = \mathrm{span}\{v_1 = (1, 2, 2), v_2 = (1, 1, 0)\}$ and $v = (1, 1, 3)$. Find $\mathrm{pr}_U(v)$.

$A^T = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix}$. The system $A^T Ax = A^T [v] \Leftrightarrow \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1 \end{bmatrix}$.

Some exercises

- (CK20181) Consider the system $\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ 2x_1 - x_2 + 3x_3 - 2x_4 = 0 \\ -x_1 + (m-3)x_2 - 3x_3 + 7x_4 = m \end{cases}$, ($m$ is a parameter).
  - c) When $m = 0$, the set of solutions of the system is a subspace U of $\mathbb{R}^4$. Find the dimension and a basis of U.
  - d) In $\mathbb{R}^4$ with the standard inner product, find the orthogonal projection of $v = (4, 5, -6, -9)$ ontho the subspace U in part c.
- (CK20181-N2) In $\mathbb{R}^4$ with the standard inner product, consider the vectors $v_1 = (1, 1, 2, -1)$, $v_2 = (1, 2, 1, 1)$, $v_3 = (3, 4, 5, -1)$. Let $V = \mathrm{span}\{v_1, v_2, v_3\}$.
  - a) Find the dimension and a basis of V.
  - b) Find the orthogonal projection of $v = (4, 1, 0, 4)$ onto V.
- (CK20181-N3) In $\mathbb{R}^5$ with the standard inner product, consider the vectors $v_1 = (-1, 1, 1, -1, -1)$, $v_2 = (2, 1, 4, -4, 2)$, $v_3 = (5, -4, -3, 7, 1)$. Let V be the subspace spanned by v_1, v_2, v_3.
  - a) Find an orthogonal basis for V.
  - b) Find the orthogonal projection of $v = (1, 2, 3, 4, 5)$ onto V.

- (CK20171) Consider the linear map $f\colon \mathbb{R}^4 \to \mathbb{R}^3$ defined by $f(x,y,z,s) = (x+2y+z-3s, 2x+5y+4z-5s, x+4y+5z-s)$.
  - a) Find the dimension and a basis of $\ker f$.
  - b) In $\mathbb{R}^4$ with the standard inner product, consder $u = (1,0,1,0)$. Find $w \in \ker f$ such that $\|u-w\| \le \|u-v\|$, for every $v \in \ker f$.
  - c) Add more vectors into the basis for $\ker f$ founded in part (a) so that the new set is a basis for $\mathbb{R}^4$.
- (CK20171) Trong không gian véc tơ $\mathbb{R}^4$ trang bị tích vô hướng chính tắc, cho $V_1 = \mathrm{span}\{v_1 = (1,2,3,1), v_2 = (1,3,3,2)\}$, $V_2 = \mathrm{span}\{v_3 = (1,2,5,3), v_4 = (1,3,4,3)\}$. Hãy tìm một cơ sở trực chuẩn của $V_1 + V_2$. Tìm hình chiếu của véc tơ $w = (1,1,2,0)$ lên $V_1 + V_2$.
- (CK20171-N2) In $\mathbb{R}^4$ with the standar inner product, consider vectors $v_1 = (1,0,-1,0)$, $v_2 = (1,-2m,m,1)$, $v_3 = (1,1,1,0)$.
  - a) Find m such that v_1, v_2 are orthogonal, and with such m prove that $\{v_1, v_2, v_3\}$ is linearly independent.
  - b) For m founded in part (a), find the orthogonal projection of $u = (0,2,1,-1)$ onto the subspace $\mathrm{span}\{v_1, v_2, v_3\}$.

- (CK20161) In $\mathbb{R}^3$ with the standar inner product, consider the vectors $u_1 = (1, 1, 0)$, $u_2 = (1, 2, 1)$, $u_3 = (3, 4, 1)$, $v = (2, 2, 3)$ and let $H = \operatorname{span}\{u_1, u_2, u_3\}$.
  a) Find an orthonormal basis for H.
  b) Find the orthogonal projection of v onto H.
- (CK20161-No7) In $\mathbb{R}^3$ with the standar inner product, consider the vectors $u = (1, 2, -1)$, $v = (-5, -2, 3)$, $u_3 = (3, 4, 1)$ and let $H = \{z \in \mathbb{R}^3 \mid z \perp u\}$.
  a) Find an orthonormal basis for H.
  b) Find the orthogonal projection of v onto H.
- (CK20193) In$\mathbb{R}^3$ with the standar inner product, consider $W = \operatorname{span}\{(0, 1, 2), (3, 4, 5), (6, 7, 8)\}$.
  a) Find an orthonormal basis fo$W$.
  b) Find the orthogonal projection of $u = (3, 1, 5)$ onto W.

Orthogonal matrix

Definition

Matrix $A \in \mathcal{M}_n(\mathbb{R})$ is said to be *orthogonal* if $A^T A = I_n$.

Example : $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ is an orthogonal matrix.

Proposition

Let $A \in \mathcal{M}_n(\mathbb{R})$. The following conditions are equivalent.

1. A is an orthogonal matrix.
2. A is invertible and $A^{-1} = A^T$.
3. The columns of A form an orthonormal basis for $M_{n\times 1}(\mathbb{R})$ (with the standard inner product).
4. The rows of A form an orthonormal basis for $M_{1\times n}(\mathbb{R})$ (with the standard inner product).

Proposition

Let V be a Euclidean space. Let P be the transition matrix from an orthonormal basis $\mathcal{B}$ to a basis $\mathcal{B}'$ for V. Then P is an orthogonal matrix if and only if $\mathcal{B}'$ is an orthonormal basis.

In particular: the transition matrix from an orthonormal basis $\mathcal{B}$ to an orthonormal basis is an orthogonal matrix.

Orthogonal diagonalization

Problem

Let A be a square matrix. Does there exist an orthogonal matrix P such that $P^{-1}AP(= P^TAP) = D$ is a digonal matrix?

If such an orthogonal matrix P exists then we say that A is *orthogonally diagonalizable*. We also say that P orthogonally diagonalizes A. The process of finding an orthogonal matrix P and a diagonal matrix D satisfying that $P^{-1}AP = D$ is called the orthogonal diagonalization of matrix A.
s

Theorem (Condition for orthogonal diagonalization)

A square matrix A of size $n \times n$ is orthogonally diagonalizable if and only if A has n linearly independent eigenvectors which form an orthonormal basis for $M_{n\times 1}(\mathbb{R})$ (with the standard inner product).

Symmetric matrices - orthogonally diagonalizable matrices

Theorem

Let A be a real symmetric matrix of order n. The following statements are true.

- A has n real eigenvalue (counted with multiplicities).
- Two eigenvectors x_1 and x_2 with respect to distinct eigenvalues λ_1 and λ_2 are orthgonal (with the standard inner product).
- If λ_i is an eigenvalue with multiplicity m_i (viewed as a root of the characteristic polynomial of A) then $\dim E_{\lambda_i} = m_i$.

Theorem

Matrix A is orthogonally diagonalizable if and only if A is a (real) symmetric matrix.

Steps for orthogonally diagonalizing a symmetric matrix

Let A be a real symmetric $n \times n$ n.

1. Solve the equation $\det(A - \lambda I) = 0$ (*) to find the eigenvalues of A together with their multiplicities.
2. For each eigenvalue λ find an orthonormal basis for the eigenspace E_λ.
3. Take the union of all bases for eigenspaces founded in the previous step, we obtain an orthonormal basis consisting of n eigenvectors.
   Let $u_1, u_2, \ldots, u_n$ be these n linearly independent eigenvectors with corresponding eigenvalues $\alpha_1, \ldots, \alpha_n$.
4. Let $P = [u_1 u_2 \cdots u_n]$ be the $n \times n$ matrix whose columns are $u_1, \ldots, u_n$. Then P orthogonally diagonalizes A and

$$P^{-1}AP = \mathrm{diag}[\alpha_1, \alpha_2, \ldots, \alpha_n] = D.$$

Example

Orthogonally diagonalize the matrix $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$.

- The characteristic equation: $0 = \det(A - \lambda I_3) = -(\lambda - 3)^2(\lambda + 6)$.
- Eigenvalues $\lambda = -6$ (with multiplicty 1) and $\lambda = 3$ (with multiplicity 2).
- For $\lambda = -6$: Solve the system $(A + 6I_3)x = 0$, and we find an eigenvector $v_1 = [1\ -2\ 2]^T$. Normalize this vector, we obtain a vector of length 1: $u_1 = [\frac{1}{3}\ \frac{-2}{3}\ \frac{2}{3}]^T$ and $\{u_1\}$ is an orthonormal basis for the eigenspace $E_{\lambda=-6}$.
- For $\lambda = 3$: Solve the system $(A - 3I_3)x = 0$, and we find a basis for $E_{\lambda=3}$ consisting of two linearly independent eigevectors $v_2 = [2\ 1\ 0]^T$, $v_3 = [-2\ 0\ 1]^T$. The set $\{v_2, v_3\}$ is not orthonormal. We apply Gram-Schmidt process on this set:

$$w_2 = v_2 = [2\ 1\ 0]^T$$

$$w_3 = v_3 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = [-\frac{2}{5}\ \frac{4}{5}\ 1]^T$$

$$u_2 = \frac{w_2}{||w_2||} = [\frac{2}{\sqrt{5}}\ \frac{1}{\sqrt{5}}\ 0]^T, \quad u_3 = \frac{w_3}{||w_3||} = [\frac{-2}{3\sqrt{5}}\ \frac{4}{3\sqrt{5}}\ \frac{5}{3\sqrt{5}}]^T.$$

We obtain an orthonormal basis $\{u_2, u_3\}$ for $E_{\lambda=3}$.
Use u_1, u_2, u_3 to form the orthogonal matrix P, we have

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}, \text{ and } P^{-1}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

5.2.1. Quadratic forms

5.2.2. Reduction of quadratic forms