LESSON 14

SPECTRUM ANALYSIS OF CONTINUOUS SIGNALS

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❑ CONTENT

1. Signal representation in the frequency domain.
2. Spectral analysis of a continuous cyclic signal.
3. Spectral analysis of a continuous non-periodic signal.

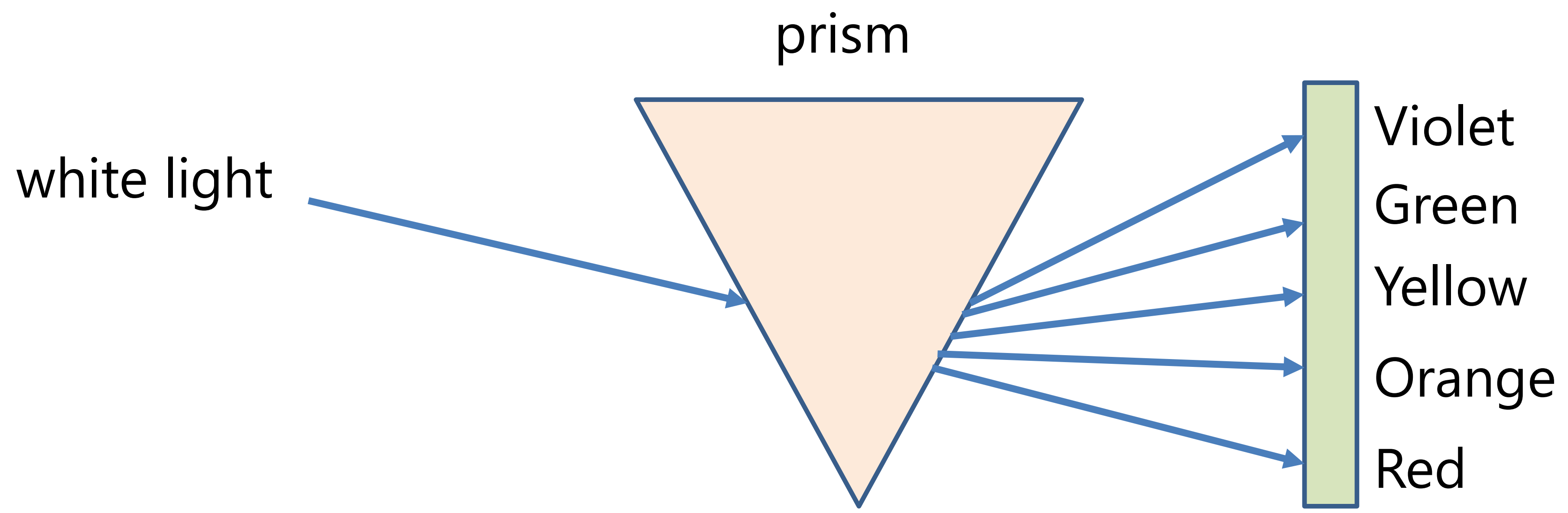
❑ Lesson Objectives

After completing this lesson, you will be able to understand the following topics:

- Signal representation in the frequency domain.
- Spectral analysis of a continuous periodic signal.
- Spectral analysis of a continuous non-periodic signal.

1. Signal representation in the frequency domain

- Analysis of white light (sunlight) using a prism:



- Prism is used to analyze white light into monochromatic light
- Color range created : *spectrum* [Isaac Newton]

The idea of signal analysis in the frequency domain

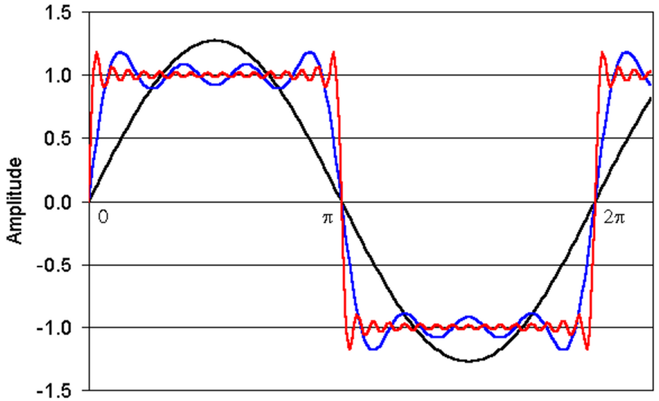
- To study the response of a linear system to any signal x(n):
  - First we need to decompose the signal x(n) into a linear combination of simple signals.

$$x(n) = a_1.x_1(n) + a_2.x_2(n) + \ldots$$

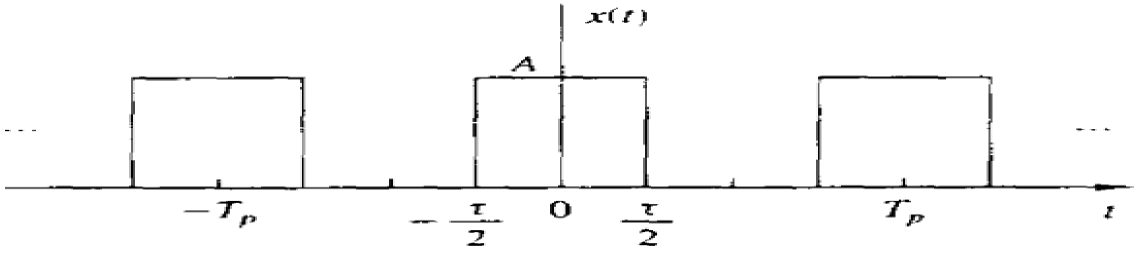
  - Simple signals $\delta(n)$; $\cos(\omega n + \varphi)$; $e^{j\omega n}$
- Frequency analysis of a signal is the breakdown of the signal into its frequency (sinusoidal) components.
- The role of the prism will be performed by analytical tools:
  - Fourier series
  - Fourier transform

Some terms

- *Spectrum: refers to the frequency content of the signal.*
- *Frequency analysis / spectrum analysis: is the process of obtaining the spectrum of a signal using mathematical tools.*
- *Spectral evaluation: is the process of determining the spectrum of a signal in practice, based on the actual measurement of the signal.*
- *Spectrum analyzer: is a hardware device or software program used to determine the signal spectrum*



2. Spectral analysis of cyclic continuous signal



- $x(t)$ cyclic with period T_p, frequency $F_0 = 1/T_p$, $\omega_0 = \frac{2\pi}{T_p}$
- Basic function: $e^{j\omega_k t} = e^{j2\pi k F_0 t}$ với $\omega_k = k\omega_0 = \frac{k2\pi}{T_p}$
- Fourier series for periodic signals:

Synthetic equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

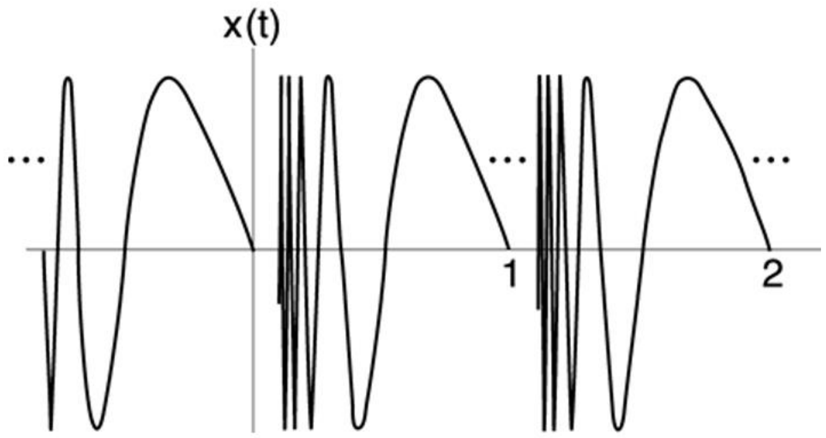
Analytical Equation

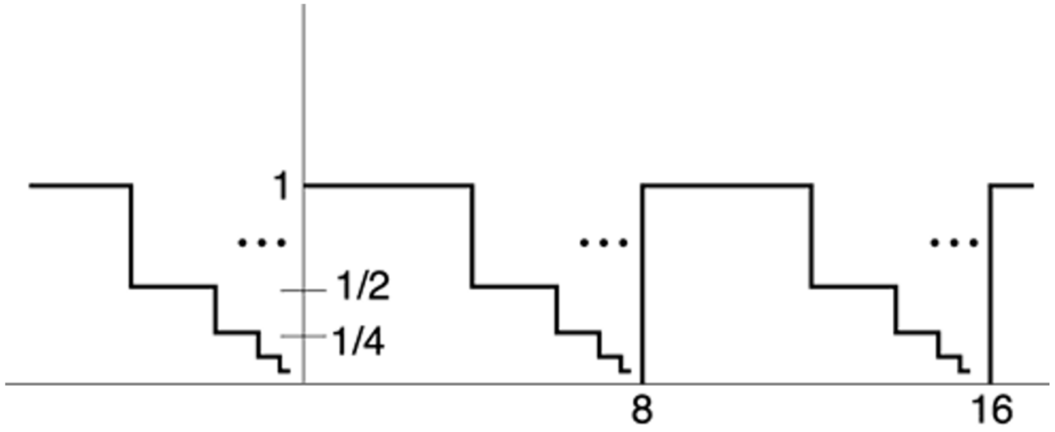
$$c_k = \frac{1}{T_p}\int_{T_p} x(t) e^{-j2\pi k F_0 t}\, dt$$

Dirichlet's conditions

- The signal x(t) must have an absolute integral in one period.
- The signal x(t) contains a finite number of maximums and minimums in a period.
- The signal x(t) has a finite number of discontinuities in one period.

$$\frac{1}{T_p}\int_{T_p}|x(t)|\,dt < \infty$$





Real cyclic signal

- c_k and c_{-k} are conjugate complex numbers : $c_k = |c_k|e^{j\theta_k}, c_{-k} = |c_k|e^{-j\theta_k}$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t} \quad \Rightarrow \quad x(t) = c_0 + 2\sum_{k=1}^{\infty} |c_k|\cos(2\pi kF_0 t + \theta_k)$$

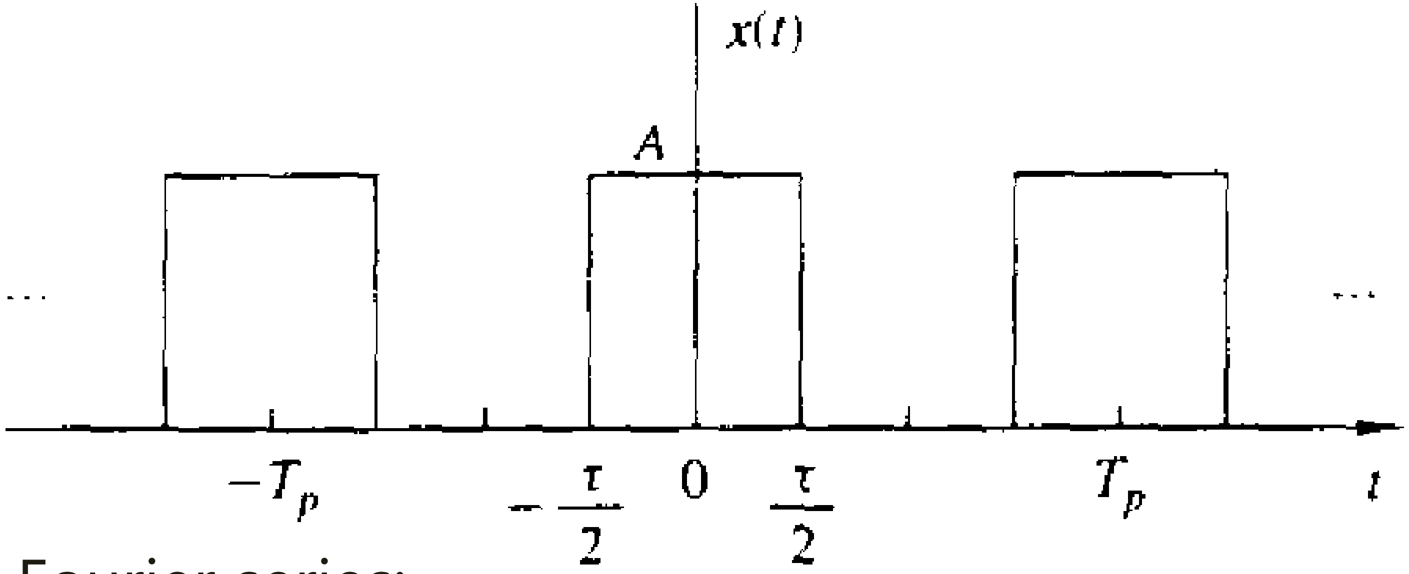
$$a_o = c_0$$

$$a_k = 2|c_k|\cos\theta_k \quad \Rightarrow \quad x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi kF_0\, t - b_k \sin 2\pi k\, F_0 t)$$

$$b_k = 2|c_k|\sin\theta_k$$

Example: spectrum analysis of a square pulse signal

- Square pulse signal continuously cyclic period $\mathrm{T_p}$, pulse width τ:



- Determine the Fourier series:
  - The frequencies ω_k
  - Amplitude $\mathrm{A_k}$ and phase angle φ_k corresponding to frequency ω_k
- Plot amplitude and phase spectrum

Solution

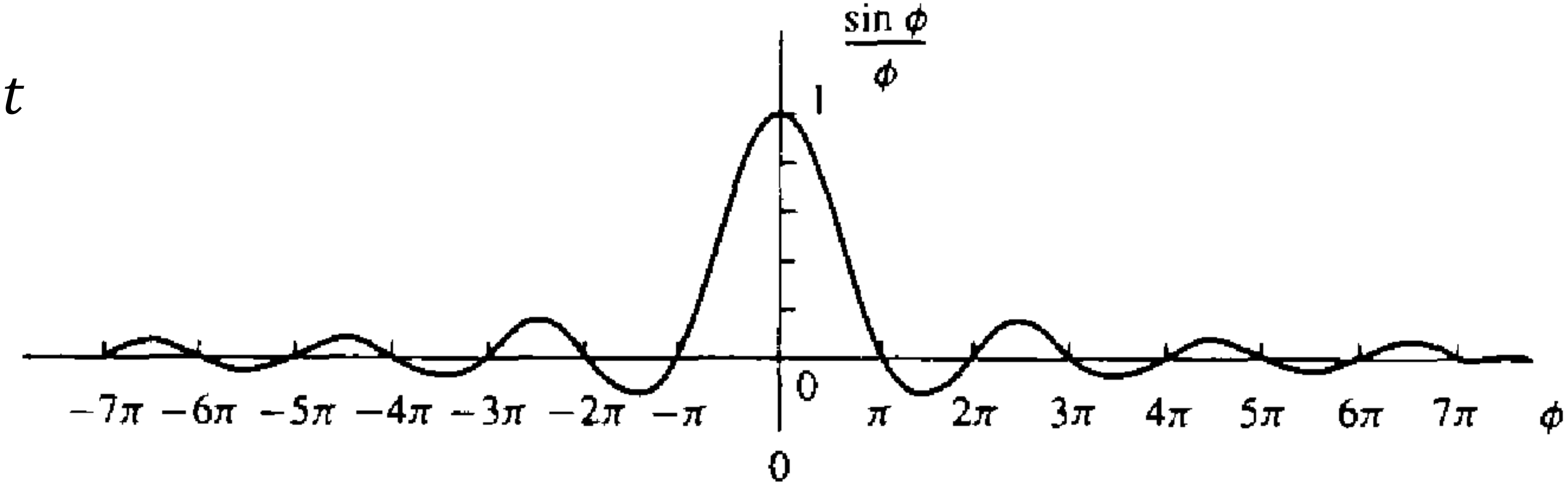
$$\omega_k = k\omega_0 = \frac{k2\pi}{T_p}$$

$$c_0 = \frac{A\tau}{T_p} \qquad c_k = \frac{A_\tau}{T_p}\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau} \qquad k = \pm 1, \pm 2, \ldots$$

$$c_k = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi k F_0 t}\, dt$$

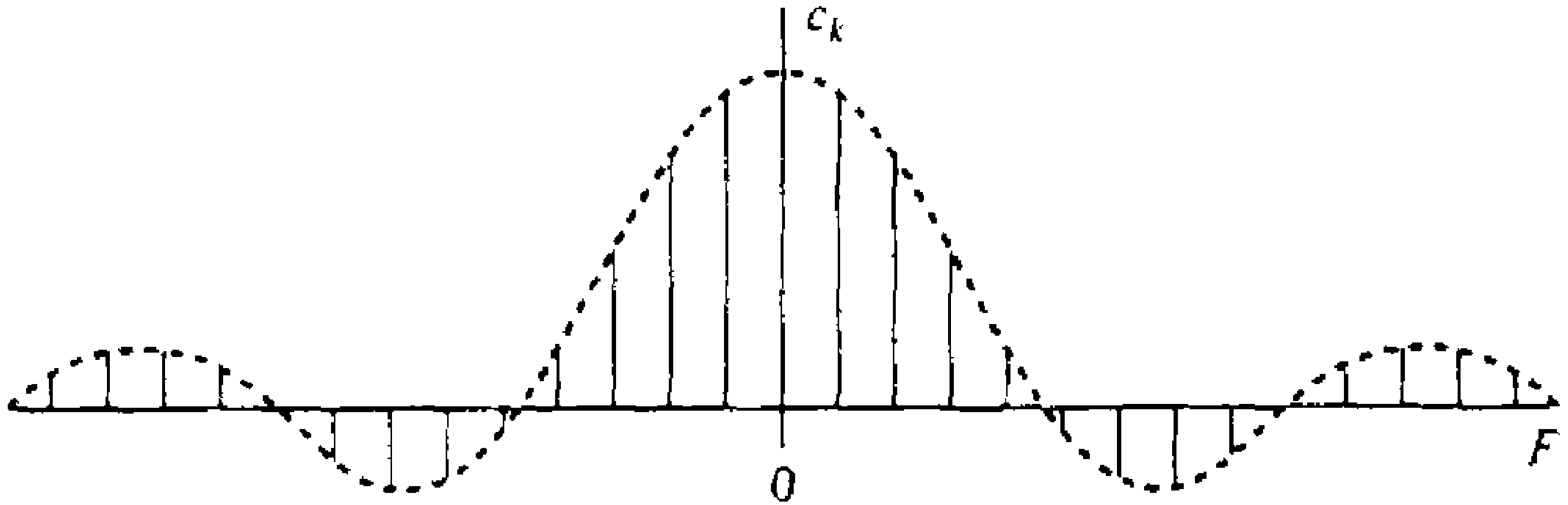
$$= \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} e^{-j2\pi k F_0 t} dt$$

$$= \frac{1}{T_p} \cdot \frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0} \Big|_{-\tau/2}^{\tau/2}$$



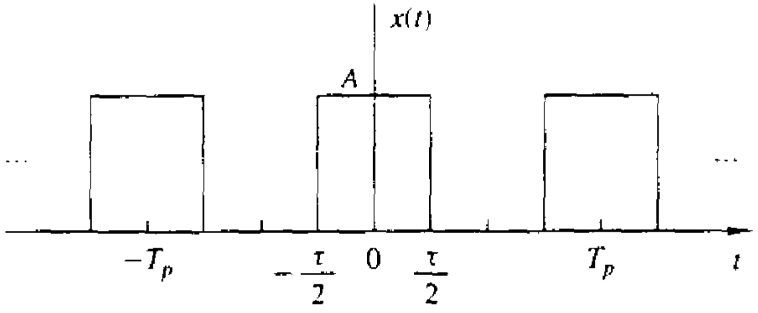
c_k **are samples of the function** $\frac{\sin \phi}{\phi}$

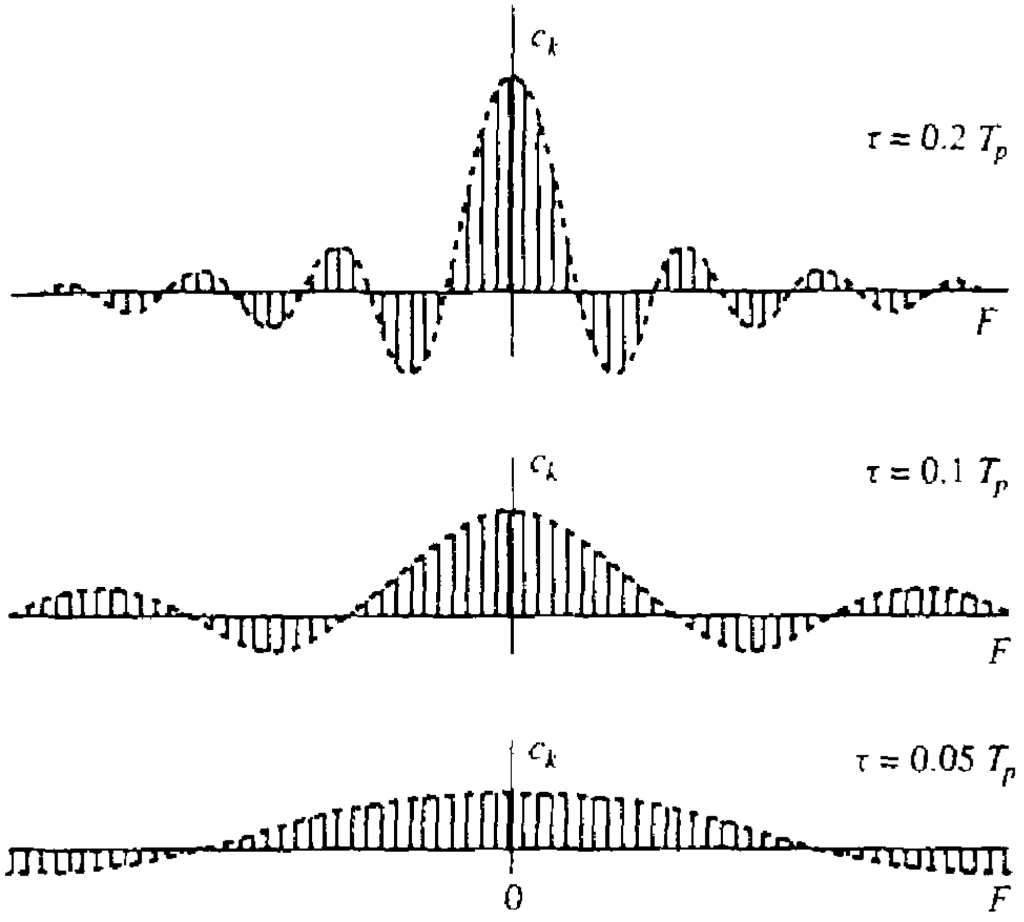
Comment



- Comment:
  - Line spectrum
  - x(t) even $\longrightarrow$ c_k are the real values $\longrightarrow$ phase spectrum is zero or equal to π
- So instead of plotting the amplitude and phase spectra separately, just plot c_k on a graph

Fixed T_p, changed τ

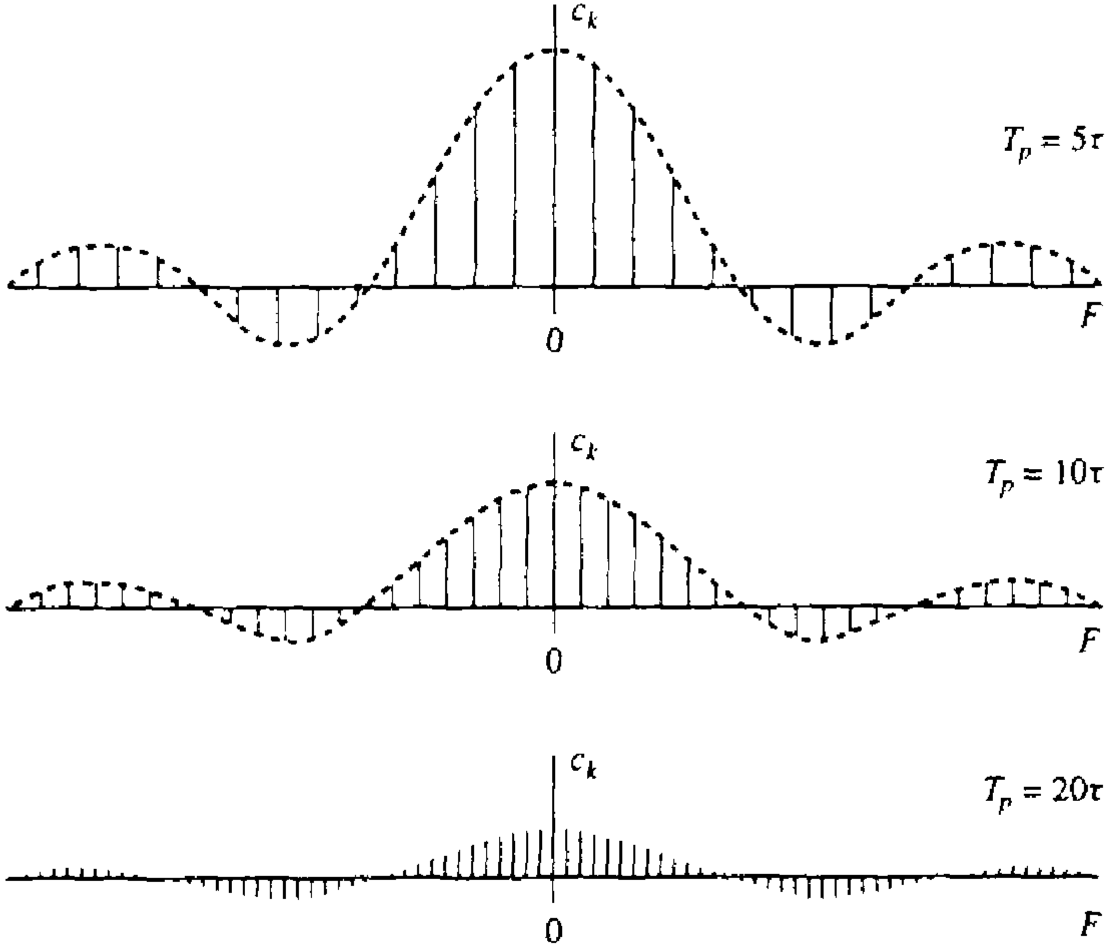




- The effect of reducing τ : spread the signal power over the frequency range
- The distance between two adjacent spectral lines is $F_0 = {}^{1}/_{T_p}$ Hz , independent of the value of pulse width τ.

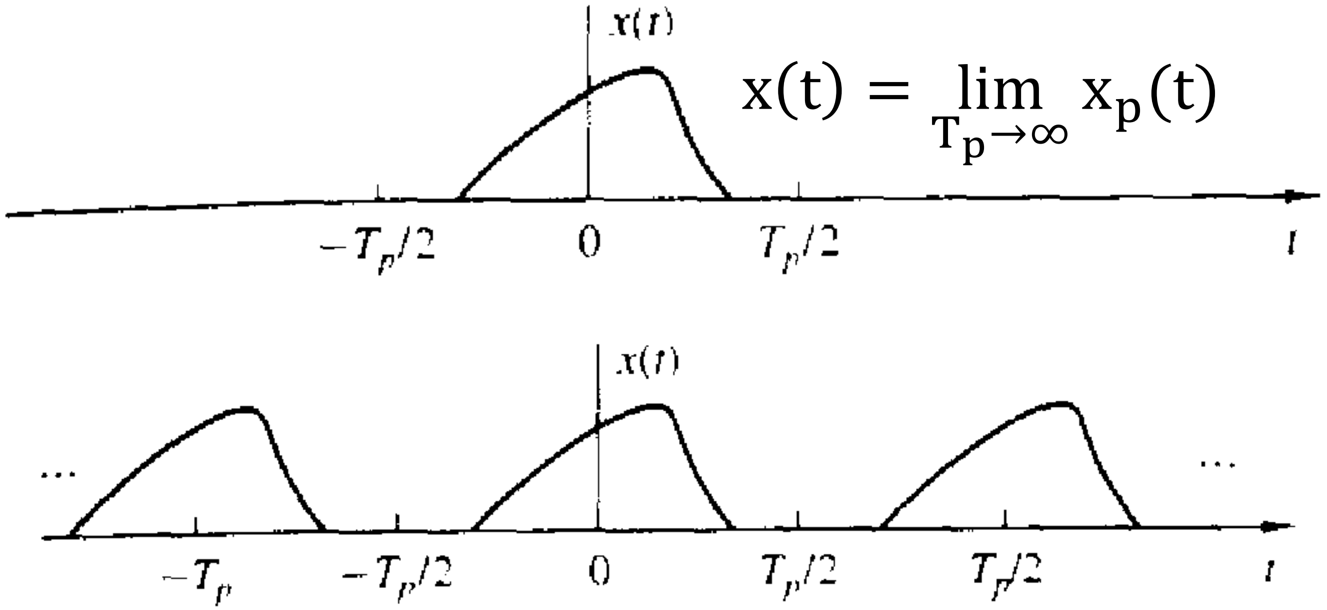
Fix τ and change the period T_p

- The distance between spectral lines decreases with increasing T_p.
- The amplitude of the spectral lines decrease:
- When $T_p \to \infty$:
  - The signal becomes acyclic
  - The distance between the spectral li gradually approaches 0, so the spectr becomes a continuous function



The spectrum of a non-periodic signal is the envelope of the spectral lines of the corresponding periodic signal

3. Spectral analysis of a non-periodic continuous signal



$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

Determine the spectrum of $x(t)$ from the spectrum of $x_p(t)$ by calculating the limit $T_p \longrightarrow \infty$

- Synthetic Equation (Inverse Fourier Transform)

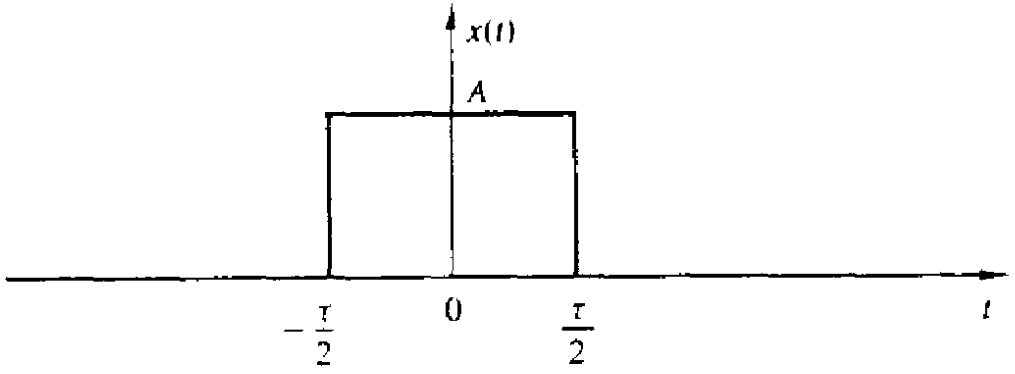
$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

- Analytical Equation (Forward Fourier Transform)

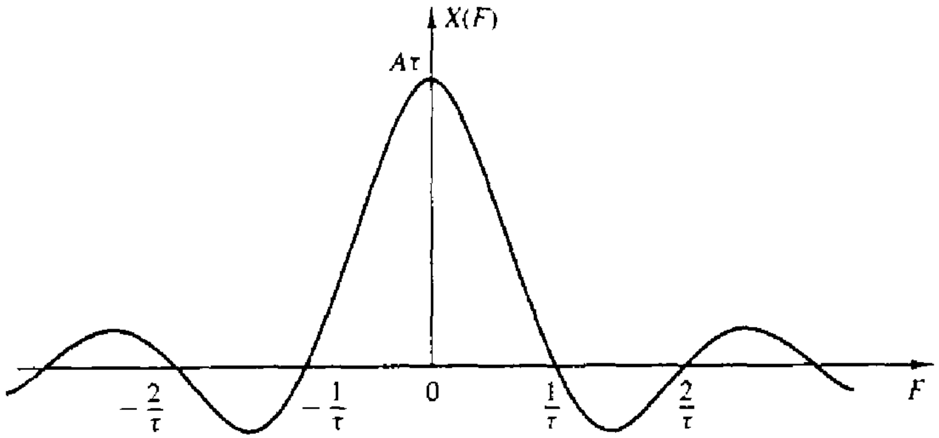
$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

Example

$$x(t) = \begin{cases} A, & |t| \le \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$



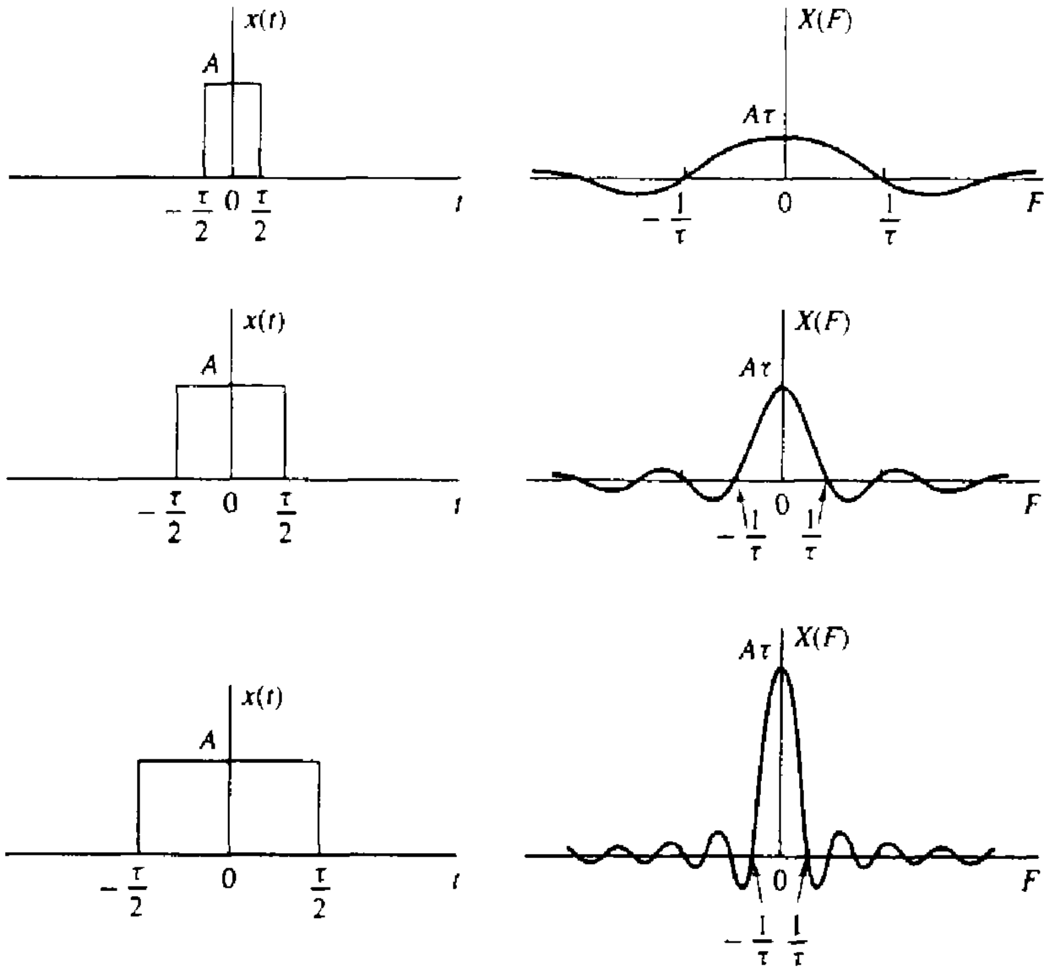
$$X(F) = \int_{-\tau/2}^{\tau/2} Ae^{-j2\pi Ft} dt = A\tau \frac{\sin \pi F\tau}{\pi F\tau}$$



- The spectrum of a rectangular signal is the envelope of the line spectrum (Fourier coefficients) of the periodic square pulse signal.
- The zero crossings of X(F) occur at an integer multiple of ${}^{1}/_{\tau}$

Effect of rectangular pulse width τ

- As the pulse width τ increases, the frequency domain representation is compressed.
- And vice versa, when the pulse width τ decreases, the representation on the frequency will be stretched, the energy will gradually shift to high frequencies.



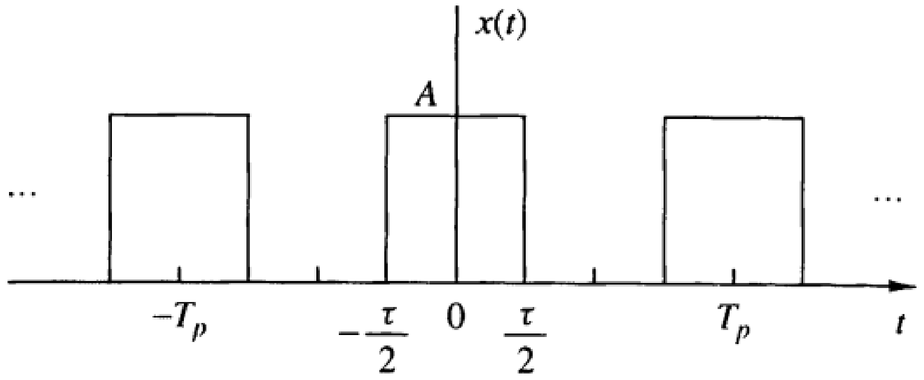
4. Summary

- Signals can be analyzed into or synthesized from frequency components using Fourier's analysis tools.
- The spectrum of a continuous cyclic signal is a discrete spectrum (line spectrum), while a non-periodic continuous signal has a continuous spectrum.
- The Fourier synthesis and analysis equation for a non-periodic continuous signal can be derived from a periodic signal with period T_p when considering $T_p \to \infty$

5. Exercise

- Exercise 1

  - Given the signal $x_a(t)$ as shown below :

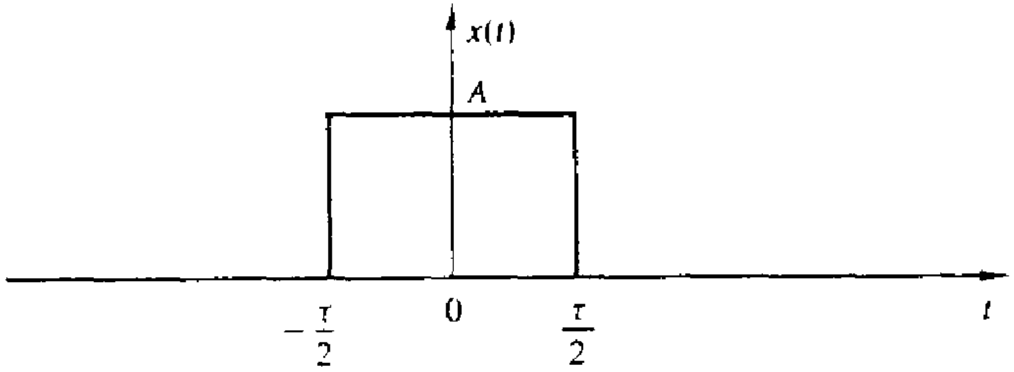


a. Determine the Fourier series c_k of this signal knowing $T_p = 0.25$ seconds, $\tau = 0.2T_p$

b. Draw the function c_k in the cases $\tau = 0.2T_p, \tau = 0.1T_p, \tau = 0.05T_p$, thereby commenting on the change of the signal spectrum shape when reducing the rectangular pulse width τ.

Homework

- Exercise 2



a) Determine and plot the spectrum of the rectangular pulse with $\tau = 0.25$ seconds.

b) Determine and plot the spectrum of the rectangular pulse with $\tau = 0.125$ seconds. From there, comment on the change of signal spectrum shape when reducing rectangular pulse width τ.

Next lesson. Lesson 15

SPECTRUM ANALYSIS OF DISCRETE SIGNALS

References :

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wish you all good study!