GENERAL PHYSICS PH1110

Nam Le, Dr.

School of Engineering Physics

Hanoi University of Science and Technology

1. MECHANICS

1.5 DYNAMICS OF RIGID BODIES

1. Rigid body
2. Center of mass
3. Translation and rotation
4. Rotational energy and moment of inertia
5. Torque of a force
6. Angular momentum
7. Torque and angular acceleration
8. Torque and angular momentum

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  - Some results studied in this chapter can be applied for solid bodies which are not rigid.

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- For a rigid body, the general motion is the combination of the translational and rotational motions.

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- Therefore, in studying a pure translational motion, we only need to study the motion of the CM of the body.

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The vectors $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\beta}}$ lie on the z-axis; direction of $\vec{\boldsymbol{\omega}}$ depends on the sign of $\dot{\varphi}$; direction of $\vec{\boldsymbol{\beta}}$ depends on the sign of $\ddot{\varphi}$.

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- The unit vectors $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}})$ form a **right-handed triad**:

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$$\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\rho}} = 0, \qquad \hat{\boldsymbol{\varphi}} \times \hat{\boldsymbol{\varphi}} = 0, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$$

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$$\boxed{\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\rho}}} \qquad \boxed{\vec{\mathbf{a}}_t = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{\rho}}} \qquad \boxed{\vec{\mathbf{a}}_n = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}}}$$

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$$\boxed{\vec{\boldsymbol{\omega}} = \frac{\vec{\boldsymbol{\rho}} \times \vec{\mathbf{v}}}{\rho^2}} \qquad \boxed{\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{\rho}} \times \vec{\mathbf{a}}}{\rho^2}}$$

3. Translation and rotation

- **Rotation with constant acceleration**

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$$\overline{\omega} =$$

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A disc of radius of 10 cm rotates from rest with a constant angular acceleration. It requires 2 s for it to rotate through an angular displacement of 60°. Find:

(a) the angular acceleration of the disc,

(b) its angular velocity at 2 s and at 6 s,

(c) the linear speed at 2 s of a point that is at a distance of 7 cm from the center of the disc,

(d) the distance moved by this point in the first 2 s.

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A disc of radius of 10 cm rotates from rest with a constant angular acceleration. It requires 2 s for it to rotate through an angular displacement of 60°. Find:

(a) the angular acceleration of the disc, [$0.525\ \mathrm{rad\,s^{-2}}$]

(b) its angular velocity at 2 s and at 6 s,

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(a) the angular acceleration of the disc, [$0.525\ \mathrm{rad\,s^{-2}}$]

(b) its angular velocity at 2 s and at 6 s, [$1.05; 3.18\ \mathrm{rad\,s^{-1}}$]

(c) the linear speed at 2 s of a point that is at a distance of 7 cm from the center of the disc,

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1. MECHANICS

1.5 DYNAMICS OF RIGID BODIES

1. Rigid body
2. Center of mass
3. Translation and rotation
4. Rotational energy and moment of inertia
5. Torque of a force
6. Angular momentum
7. Torque and angular acceleration
8. Torque and angular momentum

4. Rotational energy and moment of inertia

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♣ **Rotation of a system about a fixed axis** (the z-axis).

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- Kinetic energy:

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$$E_{\mathrm{k}} = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \left(\rho_i \omega\right)^2$$

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$$\boxed{E_\mathrm{k} = \frac{1}{2} I \omega^2} \qquad \boxed{I = \sum m_i \rho_i^2}$$

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Here, ρ_i is the distance of m_i <u>from the axis</u>.

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- The quantity I is called the **moment of inertia** of the system.

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  ▹ For a continuous distribution of mass:

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Here, ρ_i is the distance of m_i from the axis.

- The quantity I is called the **moment of inertia** of the system.
  - ▹ For a continuous distribution of mass: $\boxed{I = \int \rho^2 \, \mathrm{d}m}$

4. Rotational energy and moment of inertia

♣ **Parallel-axis theorem**

4. Rotational energy and moment of inertia

♣ **Parallel-axis theorem** (**Huygens-Steiner theorem**)

4. Rotational energy and moment of inertia

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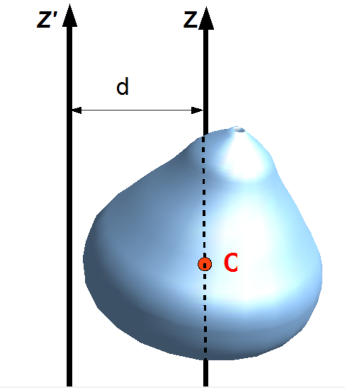
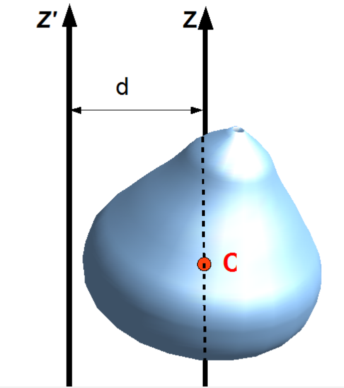


4. Rotational energy and moment of inertia

♣ **Parallel-axis theorem** (**Huygens-Steiner theorem**)

The moment of inertia I of a system about a given axis is calculated as

$$I = I_{\mathrm{cm}} + md^2$$



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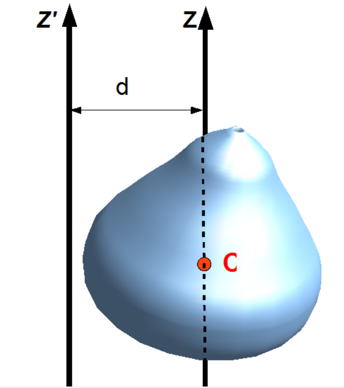
where I_{cm} is the moment of inertia of the system about the axis which passes through the center of mass of the system,



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The moment of inertia I of a system about a given axis is calculated as

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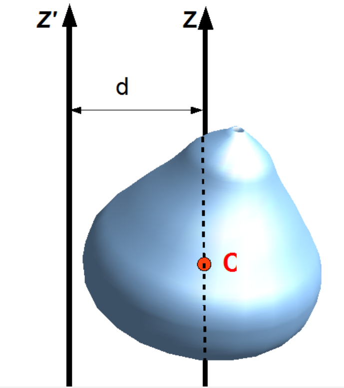
where I_{cm} is the moment of inertia of the system about the axis which passes through the center of mass of the system, is parallel to and is at distance d from the given axis.



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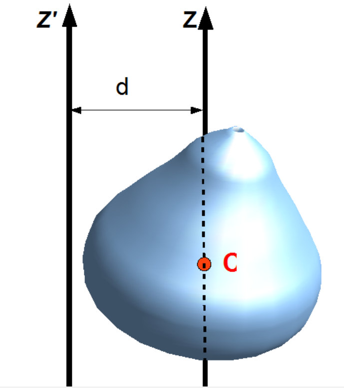
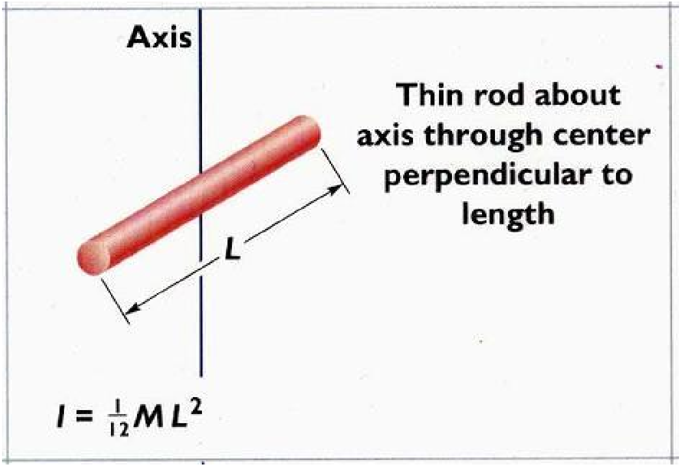


- The proof will be given as a homework.

4. Rotational energy and moment of inertia



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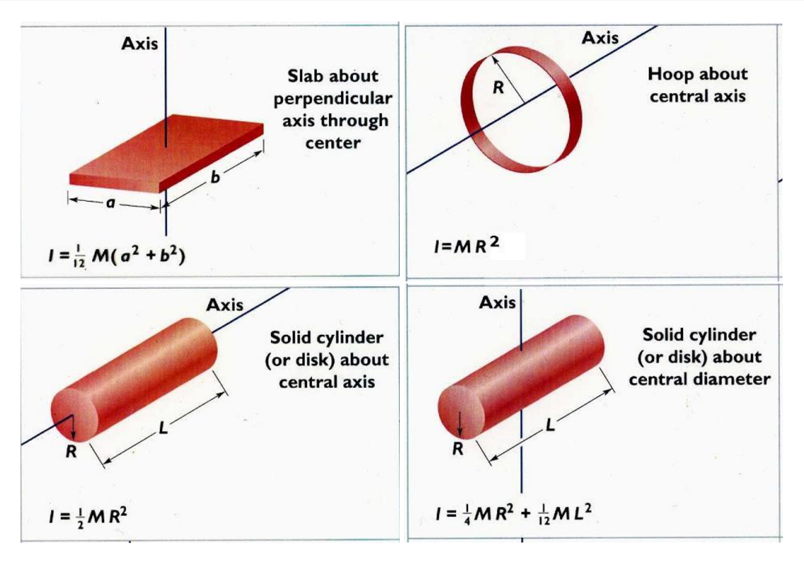
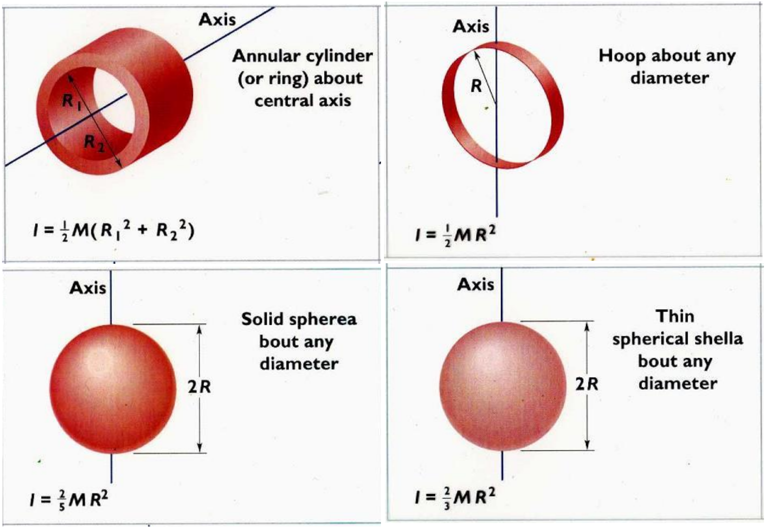


4. Rotational energy and moment of inertia

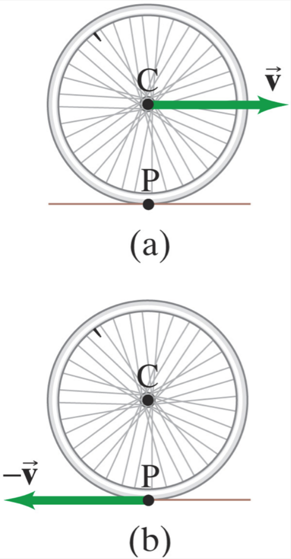


4. Rotational energy and moment of inertia

- **Rolling without slipping**

4. Rotational energy and moment of inertia

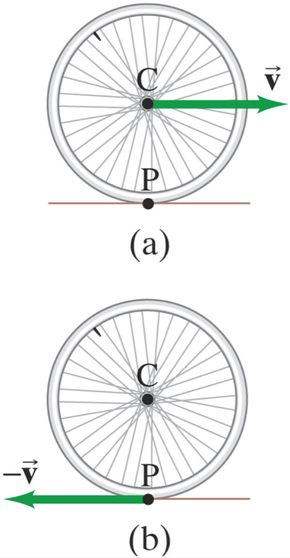
♣ **Rolling without slipping**



4. Rotational energy and moment of inertia

♣ Rolling without slipping

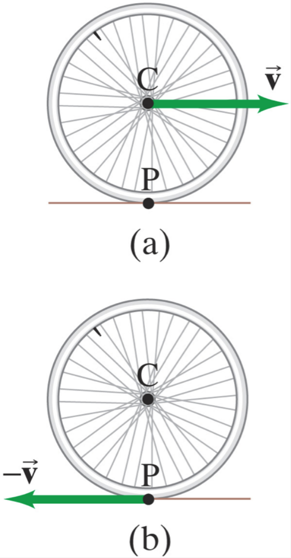
(a) A wheel is rolling without slipping.



4. Rotational energy and moment of inertia

♣ Rolling without slipping

(a) A wheel is rolling without slipping. Point P is instantaneously at rest and point C moves with velocity $\vec{\mathbf{v}}$.

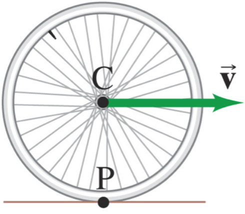


4. Rotational energy and moment of inertia

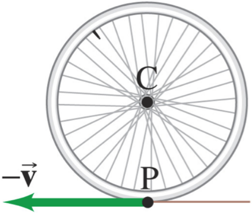
♣ Rolling without slipping

(a) A wheel is rolling without slipping. Point P is instantaneously at rest and point C moves with velocity $\vec{\mathbf{v}}$.

(b) If the wheel is seen from a reference frame where C is at rest



(a)



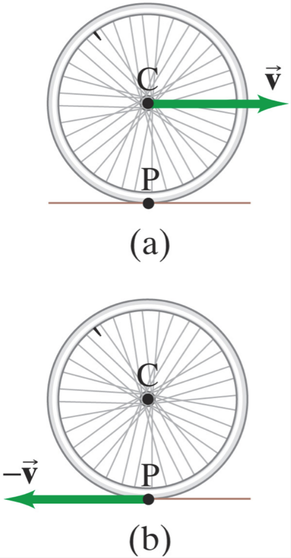
(b)

4. Rotational energy and moment of inertia

♣ Rolling without slipping

(a) A wheel is rolling without slipping. Point P is instantaneously at rest and point C moves with velocity $\vec{\mathbf{v}}$.

(b) If the wheel is seen from a reference frame where C is at rest then point P is moving with velocity



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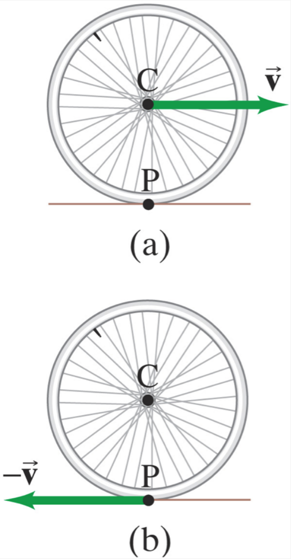
(b) If the wheel is seen from a reference frame where C is at rest then point P is moving with velocity $-\vec{\mathbf{v}}$.



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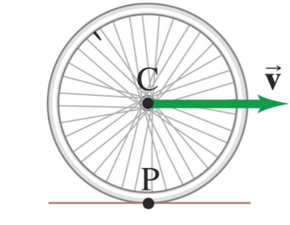
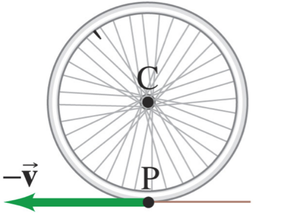
(a) A wheel is rolling without slipping. Point P is instantaneously at rest and point C moves with velocity $\vec{\mathbf{v}}$.

(b) If the wheel is seen from a reference frame where C is at rest then point P is moving with velocity $-\vec{\mathbf{v}}$.

- Condition for rolling without slipping is



(a)



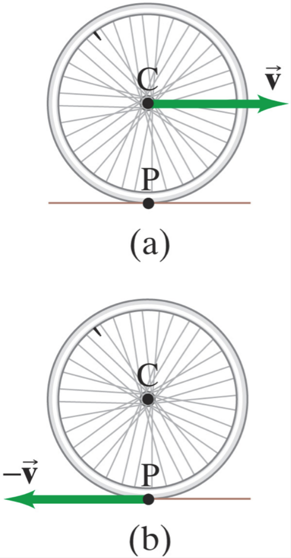
(b)

4. Rotational energy and moment of inertia

♣ Rolling without slipping

(a) A wheel is rolling without slipping. Point P is instantaneously at rest and point C moves with velocity $\vec{\mathbf{v}}$.

(b) If the wheel is seen from a reference frame where C is at rest then point P is moving with velocity $-\vec{\mathbf{v}}$.



- Condition for rolling without slipping is $v = R\omega$ $\quad$ (R = radius of the wheel).

4. Rotational energy and moment of inertia

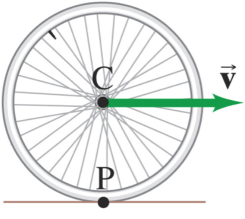
♣ Rolling without slipping

(a) A wheel is rolling without slipping. Point P is instantaneously at rest and point C moves with velocity $\vec{\mathbf{v}}$.

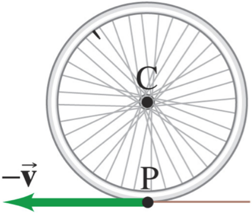
(b) If the wheel is seen from a reference frame where C is at rest then point P is moving with velocity $-\vec{\mathbf{v}}$.

- Condition for rolling without slipping is $v = R\omega$ (R = radius of the wheel).

$E_k =$



(a)



(b)

4. Rotational energy and moment of inertia

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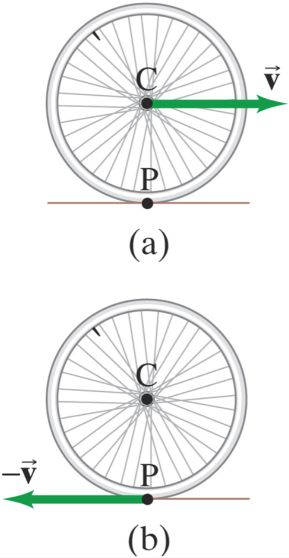
$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$



4. Rotational energy and moment of inertia

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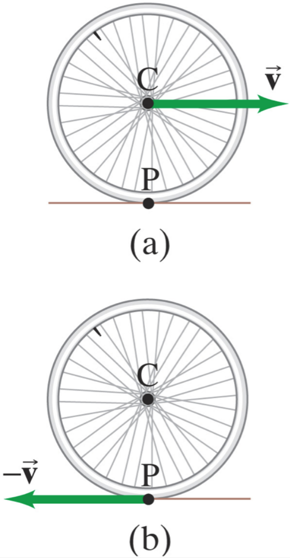
$$E_{\mathrm{k}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2$$



(a)
(b)

1. MECHANICS

1.5 DYNAMICS OF RIGID BODIES

1. Rigid body
2. Center of mass
3. Translation and rotation
4. Rotational energy and moment of inertia
5. Torque of a force
6. Angular momentum
7. Torque and angular acceleration
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5. Torque of a force

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- ♣ For a body rotating about point O,

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- ▷ $(\vec{\mathbf{r}}, \vec{\mathbf{F}}, \vec{\boldsymbol{\tau}})$ is a right hand triad,
- ▷ $\tau = rF\sin\alpha$, where α is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$.

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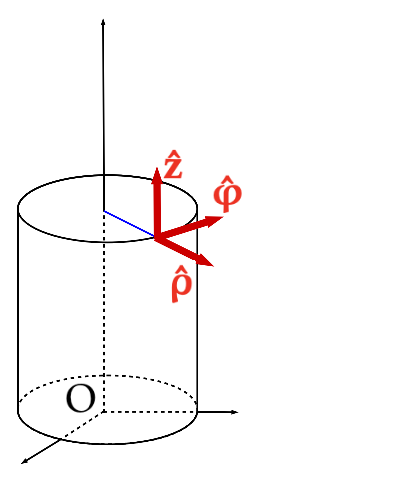

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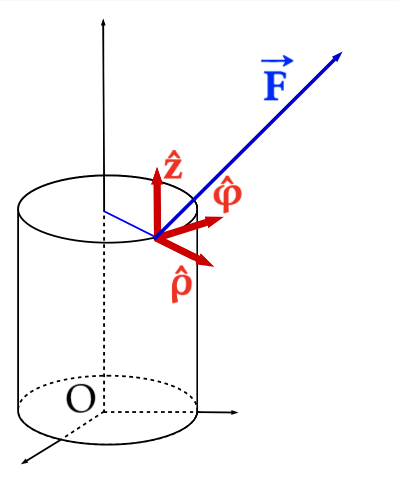

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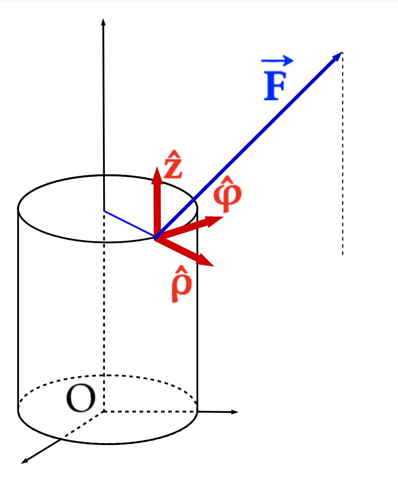

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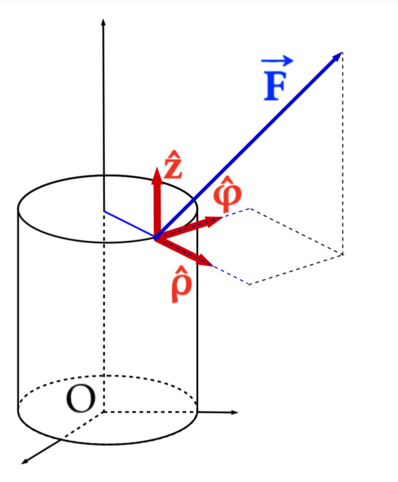

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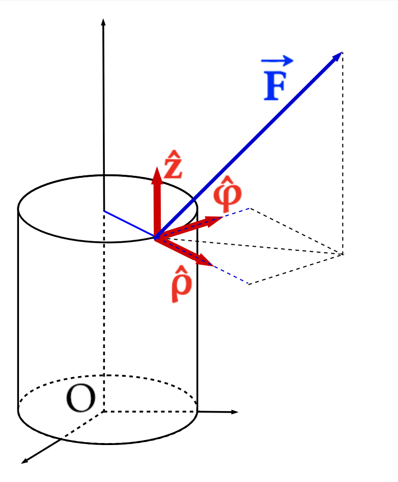

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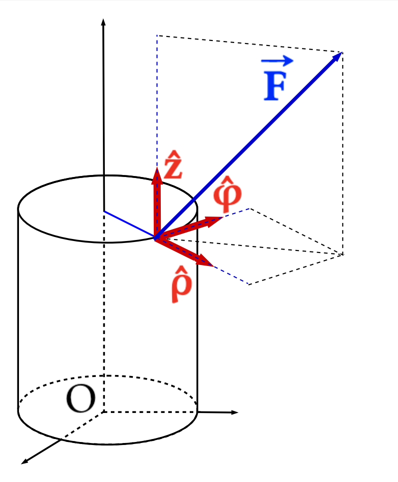

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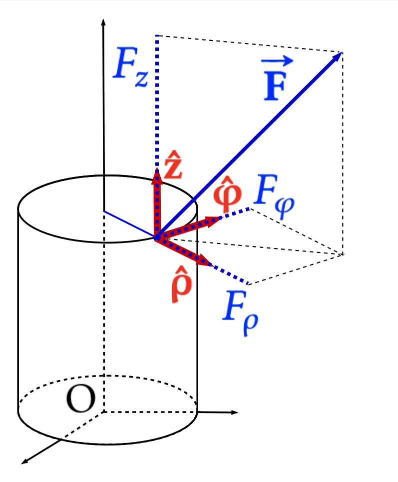

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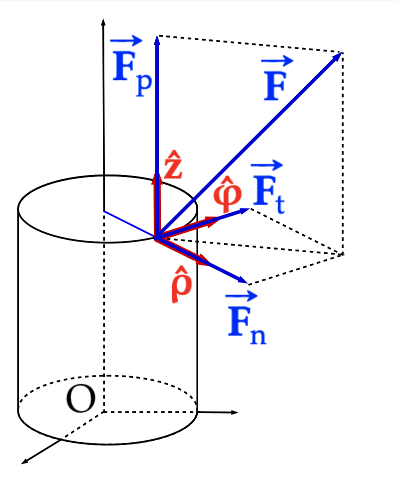

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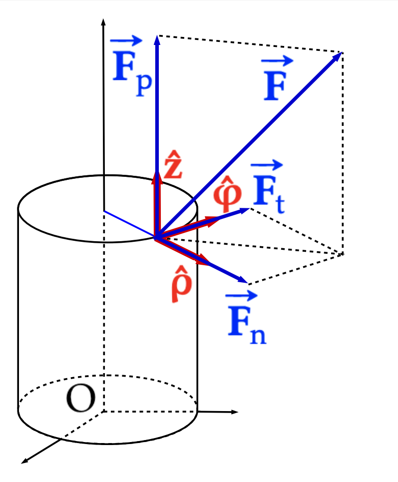
$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_n + \vec{\mathbf{F}}_t + \vec{\mathbf{F}}_p$$



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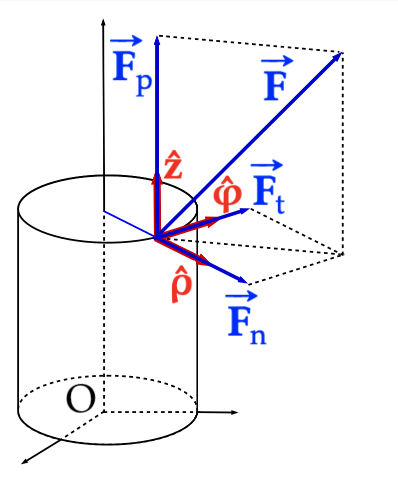
- For a body rotating about a fixed axis, chosen as the z-axis (in cylindrical coordinates):

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_{\mathrm{n}} + \vec{\mathbf{F}}_{\mathrm{t}} + \vec{\mathbf{F}}_{\mathrm{p}}$$

$$\equiv F_\rho\,\hat{\boldsymbol{\rho}} + F_\varphi\,\hat{\boldsymbol{\varphi}} + F_z\,\hat{\mathbf{z}}$$



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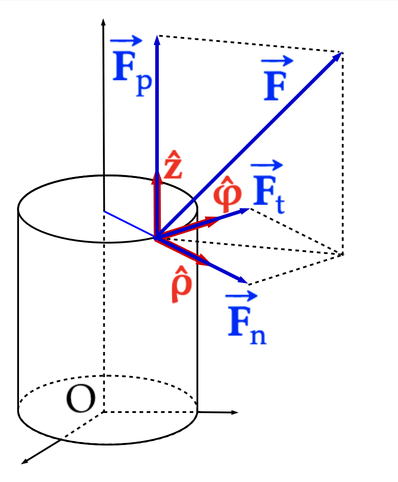

- $\vec{\mathbf{F}}_{\mathrm{n}}$ is perpendicular to the axis

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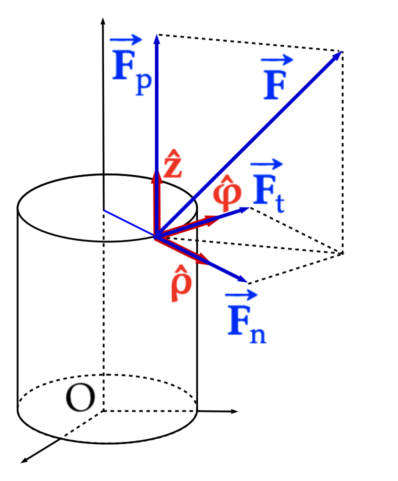

- $\vec{\mathbf{F}}_{\mathrm{n}}$ is perpendicular to the axis and $\vec{\mathbf{F}}_{\mathrm{p}}$ is parallel to the axis.

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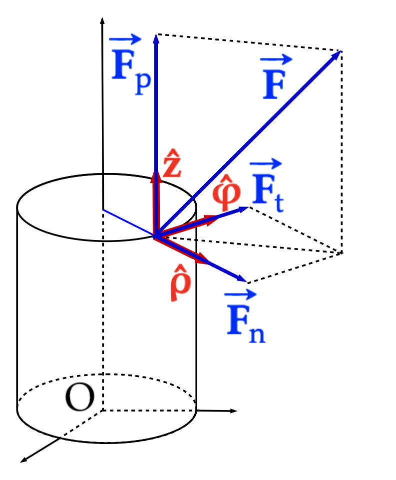

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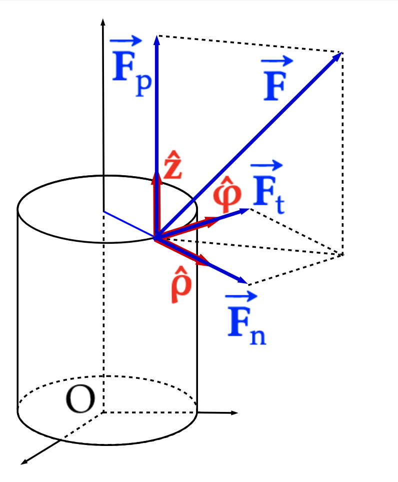

- $\vec{\mathbf{F}}_{\mathrm{n}}$ is perpendicular to the axis and $\vec{\mathbf{F}}_{\mathrm{p}}$ is parallel to the axis.

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  They are cancelled out by the reaction forces from the axis.

5. Torque of a force

- The tangential component $\vec{\mathbf{F}}_t$

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- The torque is along the axis of rotation.
- The torque is equal to the product of the perpendicular distance and the tangential component of the force.

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$\boxed{\tau_z = \rho F_\varphi}$ (same as the torque of $\vec{\mathbf{F}}$ about z-axis)

The torque of a force about an axis is equal to the component on the axis of the torque of that force about a point.

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$$\vec{\mathbf{L}} = \sum \vec{\mathbf{L}}_i = \sum \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i = \sum m_i(\vec{\mathbf{r}}_i \times \vec{\mathbf{v}}_i)$$

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$$\vec{\mathbf{L}} = \left(\sum m_i\rho_i^2\right)\omega\,\hat{\mathbf{z}} - \left(\sum m_i z_i\rho_i\,\hat{\boldsymbol{\rho}}\right)\omega$$

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- ▹ Recall that $\vec{\boldsymbol{\omega}} = \omega\,\hat{\mathbf{z}} \;\rightarrow\; \vec{\mathbf{L}}$ lies on the axis of rotation.
- ▹ In the next parts, we will relate $\vec{\boldsymbol{\tau}}$ with $\vec{\boldsymbol{\beta}}$ and $\vec{\mathbf{L}}$.

1. MECHANICS

1.5 DYNAMICS OF RIGID BODIES

1. Rigid body
2. Center of mass
3. Translation and rotation
4. Rotational energy and moment of inertia
5. Torque of a force
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7. Torque and angular acceleration
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▹ This is applied to any rigid object, symmetric or not.

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