Vector Calculus

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Surface Integrals

1. Scalar Fields
2. Vector Fields

Scalar Fields

Definition

Let Ω be an open domain of $\mathbb{R}^3$ (or $\mathbb{R}^2$, too). A function

$$\begin{aligned} u : \Omega &\to \mathbb{R} \\ (x, y, z) &\mapsto u = f(x, y, z) \end{aligned}$$

is called a scalar field defined on Ω.

Let c be a constant, then the surface $S = \{(x, y, z) \in \Omega | f(x, y, z) = c\}$ is called the level surface corresponding to c.

Directional Derivatives

Recall:
$f'_x = \lim\limits_{h\to 0} \frac{f(x_0+h,y_0,z_0)-f(x_0,y_0,z_0)}{h}$ represents the rates of change of f in the direction of $\vec{i}$.

Problem:
Find the rate of change of f at (x_0, y_0, z_0) in the direction of an arbitrary unit vector $u = (a, b, c)$.

Definition

The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $u = (a, b, c)$ is

$$\frac{\partial f}{\partial u}(x_0, y_0, z_0) := \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

- If $u = \vec{i}$, then $\frac{\partial f}{\partial u}(x_0, y_0, z_0) = \frac{\partial f}{\partial x}(x_0, y_0, z_0)$.
- $\frac{\partial f}{\partial u}(x_0, y_0, z_0)$ represents the rate of change of f at (x_0, y_0, z_0) in the direction of u.

Directional Derivatives vs Partial Derivatives

Theorem

If $f(x, y, z)$ is differentiable at $M_0(x_0, y_0, z_0)$, then f has a directional derivative at M_0 in the direction of any unit vector u and

$$\frac{\partial f}{\partial \vec{u}}(M_0) = \frac{\partial f}{\partial x}(M_0)\cos\alpha + \frac{\partial f}{\partial y}(M_0)\cos\beta + \frac{\partial f}{\partial z}(M_0)\cos\gamma,$$

where $u = (\cos\alpha, \cos\beta, \cos\gamma)$.

Example

Find the directional derivative of the function $f(x, y, z) = x^2y^3z^4$ at the point $M(1, 1, 1)$ in the direction of the vector $\vec{l} = (1, 1, 1)$.

The Gradient Vector

If f is a function of three variables x, y and z, then the gradient of f is the vector function ∇f defined by

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

Example

Find ∇u, *where* $u = r^2 + \frac{1}{r} + \ln r$ *and* $r = \sqrt{x^2 + y^2 + z^2}$.

Directional Derivatives vs The Gradient Vector

If the function $f(x, y, z)$ is differentiable at M_0, then

$$\frac{\partial f}{\partial u}(M_0) = \nabla f \cdot u.$$

The Gradient Vector

- Suppose we have a function of two or three variables f.
- In what direction does f change fastest and what is the maximum rate of change?

Significance of the Gradient Vector

$\dfrac{\partial f}{\partial u}(M_0)$ represents the rates of change of f at M_0 in the direction of $\vec{u}$.
From the formula $\dfrac{\partial f}{\partial u}(M_0) = \nabla f \cdot u$ we have that $\left|\dfrac{\partial f}{\partial u}(M_0)\right|$ attains the maximum value $|\nabla f|$ if $u // \nabla f$.

- The function f increases fastest at M_0 if $u \upuparrows \nabla f$.
- The function f decreases fastest at M_0 if $u \uparrow\downarrow \nabla f$.

Example

In what direction from $O(0,0)$ does $f = x\sin z - y\cos z$ have the maximum rate of change.

Surface Integrals

Vector Fields

Let Ω be an open domain in $\mathbb{R}^3$. A vector field on Ω is the function

$$\begin{aligned} F : \Omega &\to \mathbb{R}^3 \\ M &\mapsto \overrightarrow{F} = \overrightarrow{F}(M), \end{aligned}$$

where

$$F = P(M)\overrightarrow{i} + Q(M)\overrightarrow{j} + R(M)\overrightarrow{k}.$$

Flux

Let S be an oriented surface and F be a vector field. The quatity

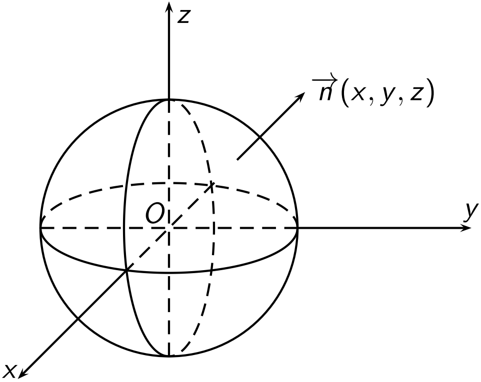
$$\phi = \iint_S F_x dydz + F_y dzdx + F_z dxdy \tag{1}$$

is called the flux of F across S.

Flux

Example

Let $F = xz^2\overrightarrow{i} + yx^2\overrightarrow{j} + zy^2\overrightarrow{k}$. *Find the flux of* F *across the surface* $S: x^2 + y^2 + z^2 = 1$ *with the outward direction.*



Vector Fields

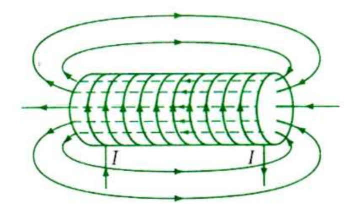
The Divergence

If $F = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is a vector field on Ω and $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}$ then the divergence of F is the function of three variables defined by $\operatorname{dive} F := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ i.e.,

$$\operatorname{dive} F = \nabla \cdot F.$$

Solenoidal Vector Field

- A solenoidal vector field (also known as an incompressible vector field, a divergence-free vector field, or a transverse vector field) is a vector field with $\operatorname{dive} F(M) = 0 \ \forall M \in \Omega$.
- The flux going into a region equals the flux coming out, i.e., nothing is lost.



Vector Fields

Circulation

Let $\mathbb{C}$ be a closed path in $\mathbb{R}^3$. The quatity

$$\int\limits_{\mathbb{C}} Pdx + Qdy + Rdz \tag{2}$$

is called the circulation of F across C.

Example

Let $F = x(y+z)\overrightarrow{i} + y(z+x)\overrightarrow{j} + z(x+y)\overrightarrow{k}$ *and* L *is the intersection between the quatity* $x^2 + y^2 + y = 0$ *and a half of the sphere* $x^2 + y^2 + z^2 = 2, z \geq 0$. *Prove that the circulation of* F *across* L *is equal to* 0.

Conservative Vector Fields and Potential Functions

Curl (Rot) Vector

The vector

$$\operatorname{curl} F := \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = \nabla \times F$$

is called the curl of F.

Conservative Vector Fields

A vector field F is called a conservative vector field if it is the gradient of some scalar function, that is, if there exists a function f such that $F = \nabla f$. In this situation f is called a potential function for F.

Theorem

F is a conservative vector field on Ω iff $\operatorname{curl} F(M) = 0 \ \forall M \in \Omega$.

Conservative Vector Fields and Potential Functions

Potential Functions

If $\overrightarrow{F}$ is a conservative vector field, then the its potential function is calculated by

$$u = \int_{x_0}^{x} F_x(x, y_0, z_0)dx + \int_{y_0}^{y} F_y(x, y, z_0)dy + \int_{z_0}^{z} F_z(x, y, z)dz + C. \quad (3)$$

Example

Which of the following fields are conservative and find their potential functions.

a. $F = 5(x^2 - 4xy)\overrightarrow{i} + (3x^2 - 2y)\overrightarrow{j} + \overrightarrow{k}$.

b. $G = yz\overrightarrow{i} + xz\overrightarrow{j} + xy\overrightarrow{k}$.

c. $H = (x + y)\overrightarrow{i} + (x + z)\overrightarrow{j} + (z + x)\overrightarrow{k}$.