Power series

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Definition

A power series (centered at x_0) is a **function series** of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + \ldots + a_n(x-x_0)^n + \ldots$$

where $a_n \in \mathbb{R}$ are the coefficients, x is the variable.

Set $X = x - x_0$, in the following we consider power series of the form $\sum\limits_{n=0}^{\infty} a_n x^n$.

Example

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}, \ |x| < 1.$$

Theorem (Abel Theorem)

If the series $\sum_{n=1}^{\infty} a_n x^n$ *converges at* $x_0 \neq 0$ *then the series converges absolutely at all* x *that* $|x| < |x_0|$.

Proof.

Assume that the series converges at $x_0 \neq 0$. Neccessary condition implies that $\lim_{n\to\infty} a_n x_0^n = 0 \Rightarrow \exists M > 0 : |a_n x_0^n| \leq M$ for all n.

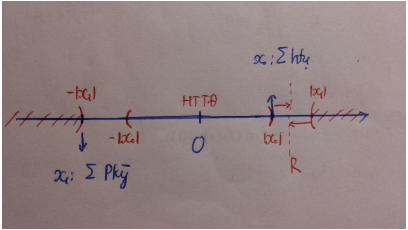
If $|x| < |x_0|$, we estimate $|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n$, the series $\sum M \left|\frac{x}{x_0}\right|^n$ converges. Hence, the series $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely. $\square$

Corollary

If the series $\sum_{n=1}^{\infty} a_n x^n$ diverges at $x_1 \neq 0$ then the series diverges at all x that $|x| > |x_1|$.

Assume that the series diverges at x_1 and converges at x_2 where $|x_2| > |x_1|$. The previous part implies that, the series converges at x_2 then converges absolutely at x where $|x| < |x_2|$, in particular, it converges at x_1 (contradiction).

The series $\sum_{n=1}^{\infty} a_n x^n$ always converges at $x = 0$.



$\exists\, R > 0$ such that the power series converges absolutely in $(-R, R)$ and diverges in $(-\infty, -R) \cup (R, \infty)$.
At the end points $x = \pm R$, the series may converge or diverge.

Definition

R is called the radius of convergence of the series.

$$D = \lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

$D < 1$, the series converges; $D > 1$ the series diverges.

Denote $\rho = \lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$.

Compare $|x|.\rho$ vs 1 or equivalently, compare $|x|$ vs $\dfrac{1}{\rho}$.

Similarly, if we use the root test, we will compare $|x|$ vs $\dfrac{1}{\bar{\rho}}$, where $\bar{\rho} = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$.

Theorem

The radius of convergence of the series $\sum_{n=1}^{\infty} a_n x^n$ *is determined by*

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \text{ or } R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}.$$

Example

Find the domain of convergence of the following power series

$$\text{a)} \sum_{n=1}^{\infty} \frac{(x-3)^n}{n+2} \qquad \text{b)} \sum_{n=1}^{\infty} n!x^n \qquad \text{c)} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

a) Power series with coefficients $a_n = \dfrac{1}{n+2}$, $X = x - 3$.

- $R = \lim\limits_{n\to\infty} \dfrac{n+3}{n+2} = 1.$
  $|X| < 1$: the series converges. $|X| > 1$: the series diverges.
- At $X = 1$: the series becomes $\sum \dfrac{1}{n+2}$, which diverges.
- At $X = -1$: the series becomes $\sum \dfrac{(-1)^n}{n+2}$, which converges.

The series is convergent $\Leftrightarrow -1 \leq x - 3 < 1 \Leftrightarrow 2 \leq x < 4$.

b) Power series with $a_n = n!$.

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \tfrac{1}{n+1} = 0.$$

Domain of convergence $\{0\}$.

c) Power series with $a_n = \dfrac{(-1)^n}{(2n)!}$, $X = x^2$.

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n}{(2n)!} \frac{(2n+2)!}{(-1)^{n+1}} \right| = \lim_{n\to\infty} (2n+1)(2n+2) = \infty.$$

Domain of convergence $\mathbb{R}$.

We do not consider $X = x$, $a_{2n} = \dfrac{(-1)^n}{(2n)!}, a_{2n-1} = 0$, so $\dfrac{a_{2n}}{a_{2n-1}}$ is undefined, we need more general formula for R.

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Proposition

Assume that $\sum\limits_{n=0}^{\infty} a_n x^n = S(x)$ *has the radius of convergence* $R \neq 0$. *Then*

1. $\sum\limits_{n=0}^{\infty} a_n x^n$ *converges uniformly on* $[a; b] \subset (-R; R)$.
2. $S(x)$ *is continuous on* $(-R, R)$.
3. $S(x)$ *is integrable on* $[a, b] \subset (-R, R)$.
$$\int \Big(\sum_{n=0}^{\infty} a_n x^n\Big) dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + C.$$
4. $S(x)$ *is differentiable on* $(a, b) \subset (-R, R)$.
$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Remark

These series have the same radius of convergence R. But their domains of convergence might be different, because of the convergence at the endpoints $x = \pm R$.

Example

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ DoC } =[-1,1).$$

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \text{ DoC } =[-1,1].$$

$$\sum_{n=1}^{\infty} x^{n-1} \text{ DoC } =(-1,1).$$

although they have the same radius of convergence $R = 1$.

Example

1. Find the sum $\sum_{n=1}^{\infty}(3n+1)x^{3n}$, $x \in (-1, 1)$.
2. (K60) Find the sum $\sum_{n=1}^{\infty}\dfrac{(-1)^n(3n+1)}{8^n}$.

a)

- Domain of convergence $(-1, 1)$. Set $S(x) = \sum_{n=1}^{\infty}(3n+1)x^{3n}$.
- $S(x) = \sum_{n=1}^{\infty}[x^{3n+1}]' = \left[\sum_{n=1}^{\infty}x^{3n+1}\right]' = \left[x\sum_{n=1}^{\infty}(x^3)^n\right]'$, so

$$S(x) = \left[\frac{x^4}{1-x^3}\right]' = \frac{4x^3 - x^6}{(1-x^3)^2}, x \in (-1, 1).$$

b) Obviously, $\sum_{n=1}^{\infty}\dfrac{(-1)^n(3n+1)}{8^n} = S\left(-\dfrac{1}{2}\right) = ?$

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}) \text{ as } x \to 0.$$

Now

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots$$

If $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, formally we obtain

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Taylor series

Definition

Let $f(x)$ be an infinitely differentiable function at x_0.
The Taylor series of $f(x)$ at x_0 is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

If $x_0 = 0$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

is called the Maclaurin series of $f(x)$.

Example

Consider $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

$f(x)$ has derivatives of all orders and $f^{(n)}(0) = 0$, the Maclaurin series of $f(x)$ is 0.

Remark

The Taylor series of $f(x)$ at x_0 may converge or diverge.
In case it converges, the sum may not equal $f(x)$.

Theorem

Let $f(x)$ have the derivatives of all orders in $I = (x_0 - R; x_0 + R)$. If there is $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in I$, $n \in \mathbb{N}$. Then the Taylor series $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ converges to $f(x)$ in $(x_0 - R; x_0 + R)$.

Ways to expand a function to Taylor series

1. Using definition: calculate $f^{(n)}(x_0)$ and substitute in the series.
2. Using fundamental expansion.
3. Via differentiation or integration.

Some important Maclaurin expansions

Example (using definition)

- $f(x) = e^x$, $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1.$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}.$$

- $f(x) = \sin x$, $f^{(k)}(x) = \sin\left(x + \frac{k\pi}{2}\right)$

$$\Rightarrow f^{(k)}(0) = \sin\frac{k\pi}{2} = \begin{cases} 0 & \text{if } k = 2n, \\ (-1)^n & \text{if } k = 2n+1 \end{cases}$$

$$\sin x = x - \frac{x^3}{3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, x \in \mathbb{R}.$$

Some important Maclaurin expansions

- $\cos x = 1 - \dfrac{x^2}{2!} + \ldots + \dfrac{(-1)^n x^{2n}}{(2n)!} + \ldots = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$, $x \in \mathbb{R}$.
- $(1+x)^\alpha = 1 + \sum\limits_{n=1}^{\infty} \dfrac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n, |x| < 1.$

In particular,

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + \ldots, |x| < 1$$
$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots, |x| < 1.$$

Example (using fundamental expansion)

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}, X \in \mathbb{R} \Rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, e^{2x} = \sum_{n=0}^{\infty} \frac{2^n.x^n}{n!}$$

Example (via integration)

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + \ldots$$

Integrating both sides, we obtain

$$\ln(1+x) + C = x - \frac{x^2}{2} + \ldots + (-1)^n \frac{x^{n+1}}{n+1} + \ldots, |x| < 1.$$

Evaluating at $x = 0$ yields that $C = 0$.

$$\ln(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^n \frac{x^{n+1}}{n+1} + \ldots, |x| < 1.$$

Example

Expand the following functions into Maclaurin series

$$a) f(x) = \frac{1}{x^2 - 3x + 2} \qquad b) f(x) = \frac{1}{(1-x)^2}$$

a) **1st way**:

$$f(x) = \frac{1}{x-2} - \frac{1}{x-1} \Rightarrow f^{(n)}(x) = \frac{(-1)^n.n!}{(x-2)^{n+1}} - \frac{(-1)^n.n!}{(x-1)^{n+1}}.$$

Hence,

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{(-1)^n.n!}{(-2)^{n+1}} - \frac{(-1)^n.n!}{(-1)^{n+1}} \right) x^n = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}} \right) x^n.$$

2nd way: $f(x) = \dfrac{1}{1-x} - \dfrac{1}{2}.\dfrac{1}{1-\frac{x}{2}}$.

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} x^n - \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}} \right) x^n, \ |x| < 1.$$

b) **1st way**

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1+\sum_{n=1}^{\infty} \frac{(-2)(-3)\dots(-2-n+1)}{n!}(-x)^n, |x| < 1.$$

2nd way $\dfrac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots, |x| < 1.$

$$\Rightarrow \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = 1 + 2x + \dots + nx^{n-1} + \dots,\ |x| < 1.$$

Example

Expand the function $f(x) = \sqrt{x}$ into Taylor series at $x = 4$.

1st way:

$$f(x) = x^{\frac{1}{2}} \Rightarrow f^{(n)}(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-n+1\right)x^{\frac{1}{2}-n}.$$ We get

$$f(x) = 2 + \sum_{n\geq 1} \frac{1}{n!}\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-n+1\right)4^{\frac{1}{2}-n}(x-4)^n.$$

$$f(x) = 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}(2n-3)!!}{n!.2^{3n-1}}(x-4)^n, |x-4| < 1.$$

2nd way: set $X = x - 4$, we need to write the Maclaurin expansion of the function $g(X) = \sqrt{X+4} = 2\left(1+\frac{X}{4}\right)^{\frac{1}{2}}$.

Applications of power series

We aim at estimating the value $f(x)$ at certain x in U_{x_0} with an indicated error. We assume that, in U_{x_0}, $f(x)$ is expanded into power series as follows

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots$$

If we approximate

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

then the error is

$$|R_n(x)| = \left| \frac{f^{(n+1)}(\bar{x})}{(n+1)!}(x - x_0)^{n+1} \right|$$

Estimating a definite integral

Example (GK20191)

Estimate $\int_1^2 \frac{\sin x}{x} dx$ by approximating $\sin x$ by the fifth partial sum (up to x^9) in its Maclaurin series.

We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

So

$$\int_1^2 \frac{\sin x}{x} dx \approx \int_1^2 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}\right) dx$$
$$\approx \left(x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \frac{x^9}{3265920}\right) \Big|_1^2 \approx 0,6593344689.$$