UNIT 5

CAUSALITY AND STABILITY OF THE DISCRETE-TIME SYSTEM

PhD. Nguyễn Hồng Quang
Assoc Prof. Trịnh Văn Loan
PhD. Đoàn Phong Tùng
Department of Computer Engineering

❑ Contents

1. Causality of the discrete-time system

2. Stability of the discrete-time system

❑ Learning Objectives

After studying this lesson, students will be able to understand:

- The concept of causal system.
- Methods for studying the causality of a system and how to calculate the response of a causal system.
- The concept of stability and methods for studying the stability of a system.

1. Memoryless system

- The output signal depends only on the input signal at the same time instant:

$$y(n) = T[x(n), n]$$

- Static systems
- Example: $y(n) = A.x(n)$

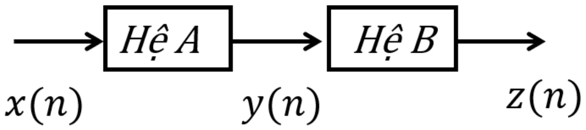
System with memory

- The output signal depends on the input signal at multiple time instants.
- Dynamic systems
- Example: $y(n) = x(n) - x(n-1)$
- A system with memory of duration N
  - $0 < N < \infty$: the system has a finite memory
  - $N = \infty$: the system has an infinite memory

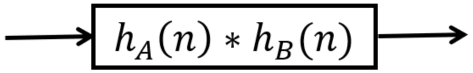
$$y(n) = \sum_{k=0}^{\infty} x(n-k)$$

The concepts of invertibility and inverse system

- A homogeneous system is a system in which the output signal is equal to the input signal: $y(n) = x(n)$
- System A is the inverse of system B if connecting these two systems in series results in a homogeneous system.



$$x(n) = z(n)$$



$$h(n) = h_A(n) * h_B(n) = \delta(n)$$

- Exercise: Given a system with input-output equations as follows

$$y(n) = 2x(n) + 3x(n-1).$$

Find the inverse system of this system.

Causal system

- The output signal in a causal system only depends on the input signal in the present and past.
  - Without any input signal, there is no output response
  - The output response cannot occur before the input signal
- Non-causal system: the output signal depends on the input signal in the present, past, and future.
- Anti-causal system: the output signal only depends on the input signal in the future.
- Only causal systems can be implemented in practice.
- Example: $y(n) = 2x(n) + 3x(n-1)$

$$y(n) = 4x(n+1) + 5x(n+2)$$

$$y(n) = 4x(n+1) + 2x(n) + 3x(n-1)$$

Impulse response of a linear time-invariant causal system

- If $x(n) = 0$ for $n < n_0$, then $y(n) = 0$ for $n < n_0$
- If the system is causal, then $y(n)$ does not depend on $x(k)$ for $k > n$

$$y(n) = F[x(n), x(n-1), x(n-2), \dots]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- $h(n-k) = 0$ for $k > n$ means that $h(n) = 0$ với $n < 0$

$$y(n_0) = \sum_{k=0}^{\infty} h(k)x(n_0 - k) + \sum_{k=-\infty}^{-1} h(k)x(n_0 - k)$$

$$= [h(0)x(n_0) + h(1)x(n_0 - 1) + \cdots] + [[h(0)x(n_0 + 1) + h(1)x(n_0 + 2) + \cdots]]$$

Causal system

- Impulse response of causal system $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

  $h(n) = 0$ for $n < 0$: $y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k)$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

- Causal signal : $x(n) = 0$ với $n < 0$

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$\Rightarrow \quad y(n) = \sum_{k=0}^{n} h(k)x(n-k)$$

2. Stability

- Definition: A system is stable if and only if for a bounded input signal, the output signal is also bounded

$$|x(n)| \leq M_x < \infty \;\Rightarrow\; |y(n)| \leq M_y < \infty$$

- An unstable system
  - Causes overflow
  - Leads to abnormal operation

Impulse response of a linear time-invariant stable system

- Assume $|x(n)| < B$

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right|$$

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$$

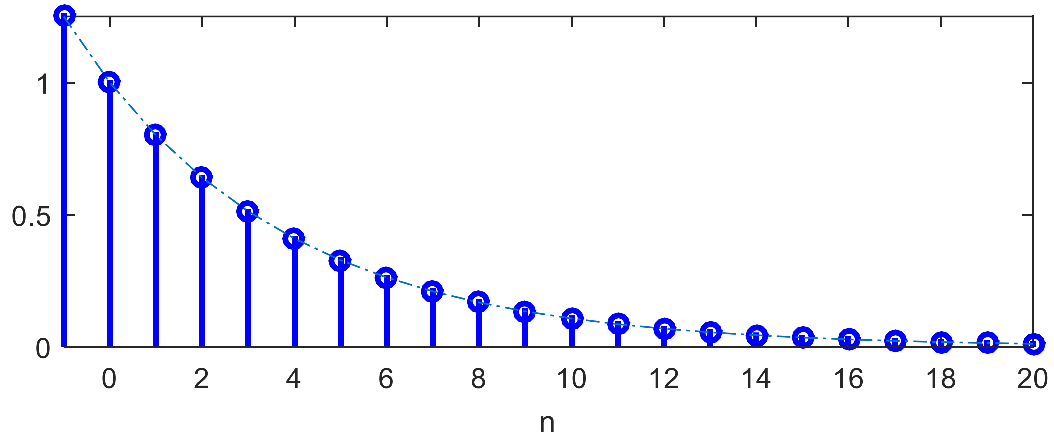
$$|y(n)| \le B\sum_{k=-\infty}^{\infty} |h(k)|$$

- $y(n)$ is bounded if : $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

Impulse response of a system

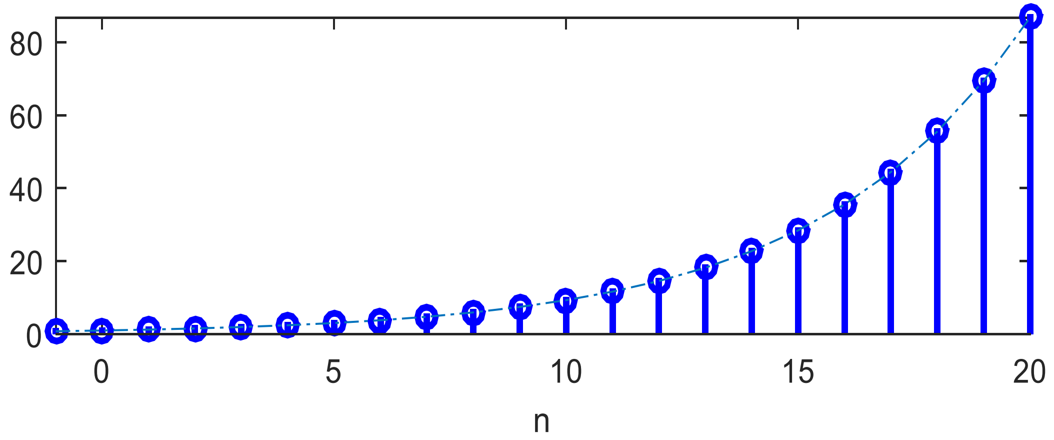
- Impulse response of causal system

h(n)



- Impulse response of non-causal system

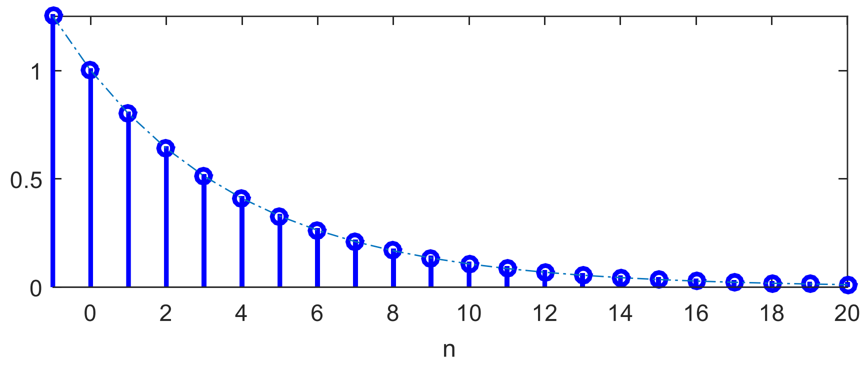
h(n)



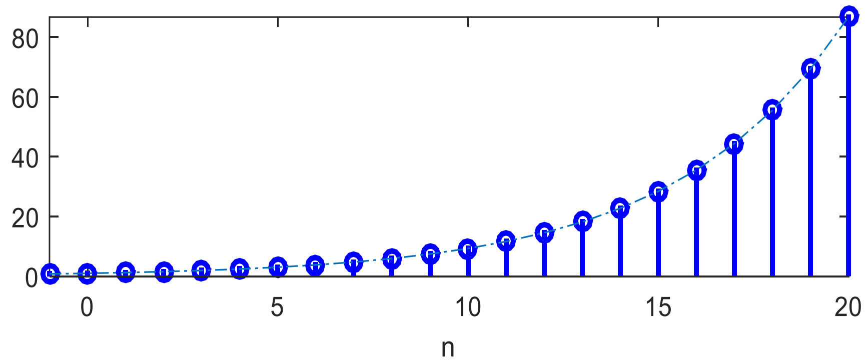
Example

- Study causality and stability of a system with impulse response given as

  $h(n) = \alpha^n u(n)$



- This is a causal system because $h(n) = 0$ với $n < 0$
- For the stability:
  - $\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a|^n$
  - This is a geometric series that
    - Converge if $|\alpha| < 1$
    - Diverge if $|\alpha| > 1$
  - $\Longrightarrow$ The system is stable if $|\alpha| < 1$



4. Summary

- A system is causal if the output signal only depends on the input signal in the present and past. Only causal systems can be implemented in practice. A system is causal if $h(n) = 0$ với $n < 0$.
- A system is stable if the output signal is bounded for any bounded input signal. A system is stable if its impulse response $h(n)$ is absolutely summable.

5. Assignment

- Assignment 1
  - ❑ Compute impulse response $h(n)$ and investigate causality of the following systems
    a. $y(n) = 2x(n) + 3x(n-1)$
    b. $y(n) = 4x(n+1) + 5x(n+2)$
    c. $y(n) = 4x(n+1) + 2x(n) + 3x(n-1)$
    d. $y(n) - ay(n-1) = x(n)$

Homework

- Assignment 2
  a. Find the conditions of a under which the system is stable

$$y(n) - ay(n-1) = x(n)$$

  b. Check the conditions of a and b for the system to be stable :

$$h(n) = \begin{cases} a^n & n \geq 0 \\ b^n & n < 0 \end{cases}$$

The next unit 6

IMPLEMENTATION OF DISCRETE-TIME SYSTEMS

References:

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wishing you all the best in your studies!