Chapter 2: Multiple integrals

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Contents

2.1.1.Definition, geometric meaning and properties

Let D be a rectangle $[a, b] \times [c, d]$ and $f(x, y)$ is a function defined over D.
Split D into small rectangles by splitting $[a, b]$ and $[c, d]$:

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b,$$
$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d.$$

We have a partion P of D consisting of mn smaller rectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \quad (1 \leq i \leq m, 1 \leq j \leq n).$$

Rectangle R_{ij} has the area $\Delta S_{ij} = \Delta x_i \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1})$, with the diagonal $\mathrm{diam}(R_{ij}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2}$.
The value $||P|| = \max \mathrm{diam}(R_{ij})$ is called the norm of partion P.

In each rectangle R_{ij} we take one (x_{ij}^*, y_{ij}^*) and set the Riemann sum

$$R(f, P) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta S_{ij}.$$

Definition (Double integrals over rectangles)

If $||P|| \to 0$ and the sum $R(f, P)$ approaches to a limit I, not depending on P and (x_{ij}^*, y_{ij}^*) then I is called the integeral of $f(x, y)$ over D, denoted by

$$\iint\limits_D f(x, y) dS \text{ hay } \iint\limits_D f(x, y) dxdy.$$

In this case we say f is integral over D.
D: region, f: function, dS: area.

So $I = \iint\limits_D f(x,y)dS$ if and only if for all $\epsilon > 0$, there exists δ such that

$$|R(f,P) - I| < \epsilon,$$

for all partitions P of D satisfying $||P|| < \delta$ for all points (x_{ij}^*, y_{ij}^*).

Integrable functions

If f in continous in $D = [a, b] \times [c, d]$ then it is integrable in D.

Doube integrals over general regions

Let $f(x, y)$ be a function defined over a bounded closed region D. Take a rectangle $R = [a, b] \times [c, d]$ containing D and define function F over R as

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in R \setminus D. \end{cases}$$

If F is integrable over R then we say f is integrable over D and we let

$$\iint\limits_D f(x, y)dS = \iint\limits_R F(x, y)dS$$

Theorem

If f is continuous over a bounded closed region D then it is integrable over D.

Geometric meaning

- The area of D is $A(D) = \iint\limits_D 1dxdy = \iint\limits_D dxdy$.
- If $f(x, y)$ is continous and non-negative, in D then the volume of the cylinder with the lower base D and the upper base $z = f(x, y)$ is

$$V = \iint\limits_D f(x, y)dxdy.$$

Property

- $(a, b \in \mathbb{R})$

$$\iint\limits_{D} (af(x,y) + bg(x,y)]dxdy = a \iint\limits_{D} f(x,y)dxdy + b \iint\limits_{D} g(x,y)dxdy,$$

- If D is the union of D_1, D_2 without interior common point then

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy.$$

- If $f(x,y) \leq g(x,y) \forall (x,y) \in D$ then $\iint\limits_{D} f(x,y)dxdy \leq \iint\limits_{D} g(x,y)dxdy$.

Mean value theorem

Theorem

Let $f(x,y)$ be continous on a bounded, connected, closed region D. Then there exists a point $(\bar{x},\bar{y})$ in D such that

$$\iint\limits_D f(x,y)dxdy = f(\bar{x},\bar{y})S(D).$$

The value $f(\bar{x},\bar{y})$ is called the mean value of $f(x,y)$ in D:

$$f(\bar{x},\bar{y}) = \frac{1}{A(D)}\iint\limits_D f(x,y)dxdy$$

2.1.2. Calculating the double integral in the xy-plane

Let $D = [a, b] \times [c, d]$ be a rectangle.

Fubini's theorem

Let $f(x, y)$ be continuous in $D = [a, b] \times [c, d]$. Then

$$\iint\limits_D f(x, y)dxdy = \int\limits_a^b \left(\int\limits_c^d f(x, y)dy \right) dx = \int\limits_c^d \left(\int\limits_a^b f(x, y)dx \right) dy$$

Iterated integrals

In order the compute the interated integral $\int\limits_a^b \left(\int\limits_c^d f(x,y)dy \right) dx$, we compute the integral

$$I(x) = \int\limits_c^d f(x,y)dy,$$

(x is treated as a constant), then we compute $\int\limits_a^b I(x)dx$.

We often ignore the parentheses

$$\int\limits_a^b \left(\int\limits_c^d f(x,y)dy \right) dx = \int\limits_a^b \int\limits_c^d f(x,y)dydx = \int\limits_a^b dx \int\limits_c^d f(x,y)dy.$$

Special cases

If $f(x,y) = g(x)h(y)$ và $D = [a,b] \times [c,d]$ then

$$\iint\limits_D f(x,y)dxdy = \left(\int\limits_a^b g(x)dx\right)\left(\int\limits_c^d h(y)dy\right).$$

Example

Compute $\iint\limits_D (x - y^2)dxdy$, where $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

Answer: $1/6$.

$$I = \int_0^1 dx \int_0^1 (x - y^2)dy = \int_0^1 \left[xy - \frac{y^3}{3}\right]_{y=0}^{y=1} dx = \int_0^1 (x - \tfrac{1}{3})dx = \frac{1}{6}.$$

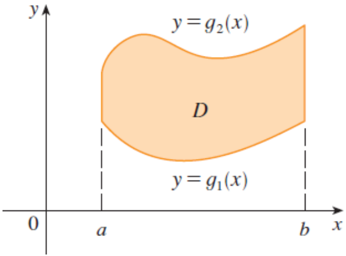
$$I = \int_0^1 dy \int_0^1 (x - y^2)dx = \int_0^1 \left[\frac{x^2}{2} - xy^2\right]_{x=0}^{x=1} dy = \int_0^1 (\tfrac{1}{2} - y^2)dy = \frac{1}{6}.$$

Integrals over general regions: Type I regions

Let D:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

with g_1 and g_2 are continuous in $[a, b]$.



Theorem

Let f be a continuous function over D. Let

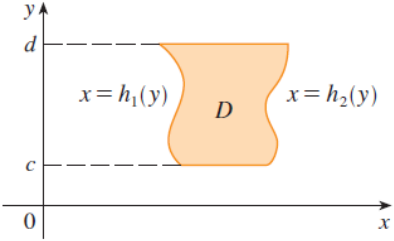
$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy \right) dx =: \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy.$$

Integrals over general regions: Type II regions

Let D:

$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\},$$

with h_1 and h_2 continuous functions over $[c,d]$.



Theorem

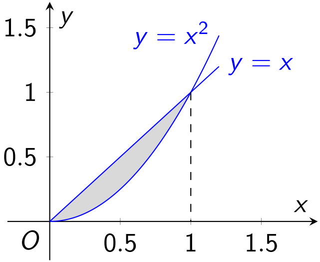
Let f be a continuous function on D. Then

$$\iint_D f(x,y)dxdy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)dx \right) dy =: \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x,y)dx.$$

Example(GK20201)

Calculate $\iint\limits_D (x^2+3y^2)dxdy$, where D is the region bounded by $y=x^2$ and $y=x$.

Solutions: (Sketch the region D.)



Domain $D=\{(x,y)\mid 0\le x\le 1, x^2\le y\le x\}$.

One has $\iint\limits_D (x^2+3y^2)dxdy = \int\limits_0^1 dx \int\limits_{x^2}^{x} (x^2+3y^2)dy = \int\limits_0^1 (2x^3-x^4-x^6)dx = 11/70$.

Changing the order of integrals

Suppose a region D is of both type I and type II

$$\begin{aligned} D &= \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\} \\ &= \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}. \end{aligned}$$

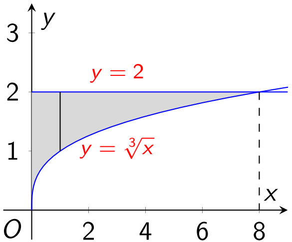
In this case, we have the formula

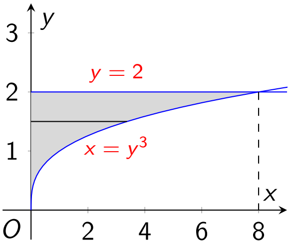
$$\int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)dy = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x,y)dx.$$

Example(GK20172)

Evaluate $\int\limits_0^8 dx \int\limits_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} dy$.

Solution: $\int\limits_0^8 dx \int\limits_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} dy = \iint\limits_D \frac{1}{y^4+1} dxdy$, where $D: \begin{cases} 0 \le x \le 8 \\ \sqrt[3]{x} \le y \le 2 \end{cases}$





$$D: \begin{cases} 0 \le x \le 8 \\ \sqrt[3]{x} \le y \le 2 \end{cases}$$

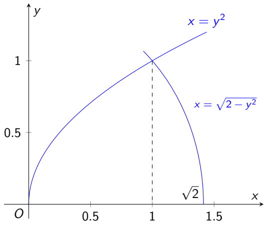
$$\Leftrightarrow \begin{cases} 0 \le y \le 2 \\ 0 \le x \le y^3. \end{cases}$$

The integral $\int\limits_0^8 dx \int\limits_{\sqrt[3]{x}}^2 \frac{1}{y^4+1} dy = \int\limits_0^2 dy \int\limits_0^{y^3} \frac{1}{y^4+1} dx = \int\limits_0^2 \frac{y^3}{y^4+1} dy = \frac{\ln 17}{4}$.

Example (GK20192)

Change the order of integration $\int\limits_0^1 dy \int\limits_{y^2}^{\sqrt{2-y^2}} f(x,y)dx$.

Solution: $\int\limits_0^1 dy \int\limits_{y^2}^{\sqrt{2-y^2}} f(x,y)dx = \iint\limits_D f(x,y)dxdy$, where $D: \begin{cases} 0 \le y \le 1 \\ y^2 \le\le \sqrt{2-y^2} \end{cases}$.



Decompose $D = D_1 \cup D_2$, where

$D_1 : \begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{x} \end{cases}$ and

$D_2 : \begin{cases} 1 \le x \le \sqrt{2} \\ 0 \le y \le \sqrt{2-x^2}. \end{cases}$

Hence $\int\limits_0^1 dy \int\limits_{y^2}^{\sqrt{2-y^2}} f(x,y)dx = \int\limits_0^1 dx \int\limits_0^{\sqrt{x}} f(x,y)dy + \int\limits_1^{\sqrt{2}} dx \int\limits_0^{\sqrt{2-x^2}} f(x,y)dy$.

Some problems

- (GK20212) Change the order of integration $\int\limits_0^1 dx \int\limits_{x^2}^{\sqrt{2-x^2}} f(x,y)dy$.
- (GK20192) Find $\iint\limits_D 4ydxdy$, D is bounded by $x^2+y^2 \le 1$, $x+y \ge 1$.
- (GK20181) Change the order of integration $\int\limits_{-2}^1 dx \int\limits_{x^2}^{2-x} f(x,y)dy$.
- (GK20181) Find $\iint\limits_D x^2ydxdy$, D is bounded by $x=-1, x=0, y=-1, y=x^2$.
- (GK20182) Find $\iint\limits_D (2y-x)dxdy$, D is bounded by $y=x^2$ and Ox.
- (GK20182) Find $\int\limits_1^2 dx \int\limits_{\sqrt{x-1}}^1 \frac{1-\cos 2\pi y}{y^2}dy$.
- (GK2016) Find $\iint_D (x^2+y)dxdy$, D is bounded by $y^2=x, y=x^2$.

2.1.3. Change of variables in double integrals

Let $f(x, y)$ be a (continuous) function definded on $D \subseteq \mathbb{R}^2$. Suppose $x = x(u, v)$, $y = y(u, v)$. We assume that

- $(x(u, v)$ and $y(u, v))$ define a bijection $T(u, v) = (x(u, v), y(u, v))$ from D' (a region in $O'uv$) onto D (a region in Oxy).
- $x'_u, x'_v, y'_u, y'v$ are continuous D' the Jacobian

$$J = \frac{D(x, y)}{D(u, v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} \neq 0$$

  in D'.

Then

$$\boxed{\iint\limits_{D} f(x, y)dxdy = \iint\limits_{D'} f(x(u, v), y(u, v))|J|dudv.}$$

Remarks

- The goal of the substitution is to simplify the integration:
- The transformation will map the boundary of D' onto the boundary of D.
- We can compute J by computing $\dfrac{1}{J} = \dfrac{D(u,v)}{D(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix}$.
- For a proof see [Puhg, Section 8, pages 306-312]: C. C. Pugh, "Real Mathematical Analysis", Undergraduate Texts in Mathematics (2002).

Example (GK20172)

Evaluate $I = \iint_D (x^2 + xy - y^2)dxdy$, where D is the region bounded by $y = -2x + 1$, $y = -2x + 3$, $y = x - 2$, $y = x$.

Let $u = y + 2x$, $v = y - x$. Then $x = (u - v)/3$, $y = (u + 2v)/3$.
D': $1 \leq u \leq 3$, $-2 \leq v \leq 0$.
$J = \dfrac{D(x,y)}{D(u,v)} = \begin{vmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{vmatrix} = 1/3$. So

$$
\begin{aligned}
I &= \iint\limits_{D'} \left(\frac{(u-v)^2}{9} + \frac{(u-v)(u+2v)}{9} - \frac{(u+2v)^2}{9} \right) \frac{1}{3} dudv \\
&= \frac{1}{27} \int_1^3 du \int_{-2}^0 (u^2 - 5uv - 5v^2) dv = \frac{1}{27} \int_1^3 (2u^2 + 10u - \frac{40}{3}) du \\
&= \frac{1}{27} \left(\frac{52}{3} + 40 - \frac{80}{3} \right) = \frac{92}{81}.
\end{aligned}
$$

Example(GK20172)

Evaluate $I = \iint_D (3x + 2xy)dxdy$, where D: $1 \le xy \le 9$, $y \le x \le 4y$.

Let $u = xy$, $v = x/y$. Then $x = \sqrt{uv}$, $y = \sqrt{u/v}$. D': $1 \le u \le 9$, $1 \le v \le 4$.

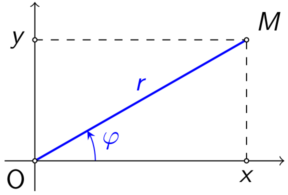
$J = \left(\dfrac{D(u,v)}{D(x,y)}\right)^{-1} = \begin{vmatrix} y & x \\ 1/y & -x/y^2 \end{vmatrix}^{-1} = -\left(\dfrac{2x}{y}\right)^{-1} = -\dfrac{1}{2v}$.

So

$$I = \iint\limits_{D'} \left(3\sqrt{uv} + 2u\right) \frac{1}{2v} dudv = \int_1^4 dv \int_1^9 (\frac{3}{2}\frac{\sqrt{u}}{\sqrt{v}} + \frac{u}{v})du$$

$$= \int_1^4 (\frac{26}{\sqrt{v}} + \frac{40}{v})dv = 26 \cdot 2v^{1/2}\Big|_1^4 + 40 \ln v\Big|_1^4 = 52 + 40 \ln 4.$$

Polar coordinate substitution



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

Remark: Some textbooks use θ instead of φ.

- The Jacobian $J = \dfrac{D(x,y)}{D(r,\varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r(\neq 0)$.
- The formula is

$$\boxed{\iint\limits_{D} f(x,y)dxdy = \iint\limits_{D'} f(r\cos\varphi, r\sin\varphi)rdrd\varphi.}$$

Example (GK20201)

Evaluate $\iint_D \cos(x^2+y^2)dxdy$, where D is defined by $x^2+y^2 \leq 4$, $x \geq 0$.

Let $x = r\cos\varphi$, $y = r\sin\varphi$.
Then D': $0 \leq r \leq 2$, $-\pi/2 \leq \varphi \leq \pi/2$.
$J = r$. So

$$\iint\limits_D \cos(x^2+y^2)dxdy = \iint\limits_{D'} \cos(r^2)rdrd\varphi = \int\limits_{-\pi/2}^{\pi/2} d\varphi \int\limits_0^2 \cos(r^2)rdr$$

$$= \int\limits_{-\pi/2}^{\pi/2} d\varphi \, \frac{1}{2}\sin(r^2)\Big|_0^2 = \int\limits_{-\pi/2}^{\pi/2} \frac{1}{2}\sin 4 d\varphi = \frac{\pi}{2}\sin 4.$$

Example (GK20192)

Evaluate $\iint_D (4x^2+1)dxdy$, where D is defined by $(x-1)^2+y^2 \leq 1$.

Let $x = 1 + r\cos\varphi$, $y = r\sin\varphi$.
Then D': $0 \leq r \leq 1$, $0 \leq \varphi \leq 2\pi$. $J = r$. Vậy

$$\iint\limits_D (4x^2+1)dxdy = \iint\limits_{D'} (4(1+r\cos\varphi)^2+1)rdrd\varphi$$

$$= \int_0^{2\pi} d\varphi \int_0^1 (5r + 8r^2\cos\varphi + 4r^3\cos^2\varphi)dr$$

$$= \int_0^{2\pi} d\varphi \left(\frac{5}{2}r + \frac{8}{3}r^3\cos\varphi + r^3\cos^2\varphi\right)\Big|_0^1 = \int_0^{2\pi} \left(\frac{5}{2} + \frac{8}{3}\cos\varphi + \cos^2\varphi\right) d\varphi$$

$$= \int_0^{2\pi} \left(3 + \frac{8}{3}\cos\varphi + \frac{1}{2}\cos(2\varphi)\right) d\varphi = 6\pi.$$

Example (GK20192)

Evaluate $\iint_D (4x^2+1)dxdy$, where D is defined by $(x-1)^2+y^2 \le 1$.

Let $x = r\cos\varphi$, $y = r\sin\varphi$.
Then D': $-\pi/2 \le \varphi \le \pi/2$, $0 \le r \le 2\cos\varphi$. $J = r$. So

$$\iint\limits_D (4x^2+1)dxdy = \iint\limits_D 4x^2 dxdy + S(D) = \pi + \iint\limits_{D'} 4(r\cos\varphi)^2 rdrd\varphi$$

$$= \pi + \int\limits_{-\pi/2}^{\pi/2} d\varphi \int\limits_0^{2\cos\varphi} 4r^3\cos^2\varphi dr = \pi + \int\limits_{-\pi/2}^{\pi/2} d\varphi \left(r^4\cos^2\varphi\right)\Big|_0^{2\cos\varphi} = \pi + 32\int\limits_0^{\pi/2} \cos^6\varphi d\varphi$$

$$= \pi + 32\cdot\frac{5\pi}{32} = 6\pi.$$

Integration over symmetric regions

Theorem

Let D be a region with the x-axis as a symmetrical axis.

- If $f(x,y)$ is an odd function with respect to y then $\iint\limits_D f(x,y)dxdy = 0$.
- If $f(x,y)$ is an even function with respect to y then $\iint\limits_D f(x,y)dxdy = 2\iint\limits_{D'} f(x,y)dxdy$, where D' is the sub-region of D lying above the x-axis.

We have similar results for the y-axis.

Theorem

If D is a region with O as the center of symmetric and a function $f(x,y)$ satisfies $f(-x,-y) = -f(x,y)$ $(\forall(x,y) \in D)$, then $\iint\limits_D f(x,y)dxdy = 0$.

Example

Evaluate $\iint\limits_D (2 + x^2y^3 - y^2 \sin x)dxdy$, where $D = \{(x, y) \mid |x| + |y| \leq 1\}$.

- We have $I := \iint\limits_D (2 + x^2y^3 - y^2 \sin x)dxdy = \iint\limits_D 2dxdy + \iint\limits_D x^2y^3dxdy - \iint\limits_D y^2 \sin xdxdy$.
- Since the function x^2y^3 is odd with respect to (w.r.t) y and D is symmetric w.r.t the line $y = 0$, we have $\iint\limits_D x^2y^3dxdy = 0$.
- Similarly, $\iint\limits_D y^2 \sin xdxdy = 0$.
- Hence $I = \iint\limits_D 2dxdy = 2\mathrm{Area}(D) = 4$.

Example (GK20192)

Evaluate $\iint_D (4x^2+1)dxdy$, where D is defined by $(x-1)^2+y^2 \le 1$.

Let $u = x-1$, $v = y$. Then D': $u^2+v^2 \le 1$, $J = 1$. Hence

$$I = \iint\limits_{D} (4x^2+1)dxdy = \iint\limits_{D'} (4(u+1)^2+1)dudv = \iint\limits_{D'} (4u^2+8u+5)dudv$$

$$= 5S(D') + \iint\limits_{D'} 4u^2 dudv + \iint\limits_{D'} 8ududv = 5\pi + \iint\limits_{D'} 2(u^2+v^2)dudv$$

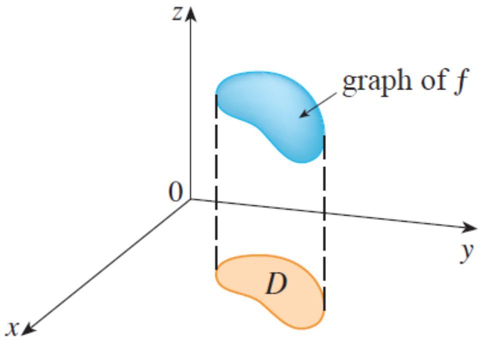
Let $u = r\cos\varphi$, $v = r\sin\varphi$.
Then D'': $0 \le r \le 1$, $0 \le \varphi \le 2\pi$, $J = r$. So

$$I = 5\pi + \int\limits_0^{2\pi} d\varphi \int\limits_0^1 2r^2 r dr = 5\pi + 2\pi \cdot \frac{1}{2} = 6\pi.$$

Some exercises

- (GK20212) Evaluate $\iint_D (xy+y^2)dxdy$, where D is bounded by $x+y=1$, $x+y=-1$, $x-2y=1$ và $x-2y=-1$.
- (GK20182) Evaluate $\iint\limits_D \sqrt{x^2+y^2}dxdy$, where $D: 1 \leq x^2+y^2 \leq 4, x+y \geq 0$.
- (CK20182) Evaluate $\iint\limits_D \sqrt{y^2-x^2}dxdy$, where D is defined by $0 \leq 2y \leq x^2+y^2 \leq 2x$.
- (GK20172) Evaluate $\iint\limits_D x\sqrt{x^2+y^2}dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq x\}$.
- (GK20162) Evaluate $\iint\limits_D \sin\sqrt{x^2+y^2}dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : \pi^2 \leq x^2+y^2 \leq 4\pi^2, x \geq 0, y \geq 0\}$.
- (GK20152) Evaluate $\iint\limits_D \sqrt{x^2+y^2}dxdy$, where $D: x^2+y^2 \leq 2y, |x| \leq y$.

2.1.4. Applications



Volume of a cylinder is

$$V = \iint\limits_{D} f(x, y)dxdy.$$

Example (CK20142)

Find the volume of the region bounded by $0 \leq z \leq 2 - x^2 - y^2$, $0 \leq y \leq \sqrt{3}x$.

Example (CK20142)

Find the volume of the region bounded by $0 \leq z \leq 2 - x^2 - y^2$, $0 \leq y \leq \sqrt{3}x$.

$V = \iint\limits_D (2 - x^2 - y^2)dxdy$, với $D : x^2 + y^2 \leq 2$, $0 \leq y \leq \sqrt{3}x$.

Substitution: $x = r\cos\varphi$, $y = r\sin\varphi$, $J = r$, $D' : 0 \leq r \leq \sqrt{2}$, $0 \leq \varphi \leq \pi/3$.

$$V = \int\limits_0^{\pi/3} d\varphi \int\limits_0^{\sqrt{2}} (2 - r^2) r dr = \frac{\pi}{3}.$$

Area

The suface area $S(D)$ of a region D is

$$S(D) = \iint\limits_D dxdy.$$

Example (GK20201)

Find the area of the region bounded by $2y \leq x^2 + y^2 \leq 4y$, $0 \leq x \leq y$.

The area is $S = \iint\limits_D dxdy$, where $D\colon 2y \leq x^2 + y^2 \leq 4y, 0 \leq x \leq y$.

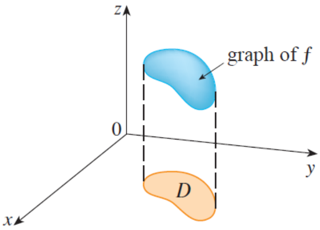
Let $x = r\cos\varphi$, $y = r\sin\varphi$, $J = r$,

D' : $2\sin\varphi \leq r \leq 4\sin\varphi$, $\pi/4 \leq \varphi \leq \pi/2$.

$$S = \int\limits_{\pi/4}^{\pi/2} d\varphi \int_{2\sin\varphi}^{4\sin\varphi} rdr = \int\limits_{\pi/4}^{\pi/2} 6\sin^2\varphi d\varphi = 3\int_{\pi/4}^{\pi/2}(1-\cos 2\varphi)d\varphi = \frac{3\pi}{4} + \frac{3}{2}.$$

Surface area

Let S be a suface defined by the equation $z = f(x, y)$, where (x, y) is in a closed and bounded region D in the Oxy plane. (D is the projection of S on Oxy.)



Then the area of σ of S is

$$\boxed{\sigma = \iint\limits_D \sqrt{1 + {z'_x}^2 + {z'_y}^2}\,dxdy.}$$

Example (GK20192)

Find the surface area of $z = x^2 + y^2 + 1$ inside the cylinder $x^2 + y^2 = 4$.

Example (GK20192)

Find the surface area of $z = x^2 + y^2 + 1$ inside the cylinder $x^2 + y^2 = 4$.

The area is $\sigma = \iint_D \sqrt{1 + 4x^2 + 4y^2}dxdy$, where $D : x^2 + y^2 \leq 4$.
Let $x = r\cos\varphi$, $y = r\sin\varphi$, $J = r$,
$D' : 0 < r \leq 2$, $0 \leq \varphi < 2\pi$.

$$\sigma = \int\limits_0^{2\pi} d\varphi \int\limits_0^2 \sqrt{1 + 4r^2} rdr = 2\pi \cdot \frac{1}{8} \int\limits_1^{\sqrt{17}} \sqrt{u} du = \frac{\pi}{4} \cdot \frac{2}{3} \, u^{3/2}\Big|_1^{\sqrt{17}} = \frac{\pi}{6}(17\sqrt{17} - 1).$$

Some exercises

- (GK20212) Find the area of the surface $z = \sqrt{x^2 + y^2}$ inside $x^2 + y^2 = 2x$.
- (GK20192) Find the volume of the region bounded by Oxy and $z = x^2 + y^2 - 4$.
- (GK20192) Find the area of the region bounded by $(x^2 + y^2)^2 = 4xy$.
- (GK20182) Find the area of the region $x^2 + y^2 = 2x$ lying outside $x^2 + y^2 = 1$.
- (GK20181) The the area of the region bounded by $x = 2y^2$, $x = 5y^2$, $y = x^2$, $y = 4x^2$.

2.2.1. Definition, geometric meanings, properties

Triple integrals are similarly defined as double integrals.
Let $f(x, y, z)$ be a function defined on a rectangular box $B = [a, b] \times [c, d] \times [s, t]$ và.
Divide rectangular box B into smalles rectangular boxes:

$$a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b, c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d,$$
$$s = z_0 < z_1 < \cdots < z_{p-1} < z_p = t.$$

We get a partition P of B including mnp smaller rectangular boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k] \; (1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p).$$

The volume of the box B_{ij} is $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$, and the diagonal $\mathrm{diam}(R_{ijk}) = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2 + (\Delta z_k)^2}$.
The quantity $||P|| = \max \mathrm{diam}(B_{ijk})$ is called the norm of the partition P.

In each box B_{ijk}, we take one point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ and define the Riemann sum

$$R(f, P) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V_{ijk}.$$

Definition (Triple integrals over rectangular boxes)

When $||P|| \to 0$, the sum $R(f, P)$ has a limit I, not depending on P and take $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ then the limit I is called the integral of $f(x, y, z)$ over B,
denoted by

$$\iiint\limits_B f(x, y, z) dV \text{ or } \iiint\limits_B f(x, y, z) dxdydz.$$

In this case, f is called integrable over B.

So $I = \iiint\limits_B f(x, y, z)dV$, if and only if for all $\epsilon > 0$, there exists δ such that

$$|R(f, P) - I| < \epsilon,$$

for all partitions P of B such that $||P|| < \delta$ and for all points $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$.

Triple integrals over a general region

Let $f(x, y, z)$ be a function with a closed and bounded domain V.
Take a box $B = [a, b] \times [c, d] \times [s, t]$ containing V and define a function F on B as the following

$$F(x, y, z) = \begin{cases} f(x, y, z) & \text{nếu } (x, y, z) \in V \\ 0 & \text{nếu } (x, y) \in B \setminus V. \end{cases}$$

If F is integrable over B then we say f is integrable over V and we define the integration of f over V by:

$$\iiint\limits_V f(x, y, z) dV = \iiint\limits_B F(x, y, z) dV$$

Theorem

If f is continuous over a closed and bounded domain V then f is intebgrable over V.

Meaning and properties

- The volume of V is $\mathrm{Vol}(V) = \iiint\limits_V 1dxdydz = \iiint\limits_V dxdydz$.
- If $f(x, y, z)$ is the density of an object S then the mass of S is

$$\iiint\limits_V f(x, y, z)dxdydz.$$

Triple integrals have similar properties to double integrals.

- Linearity
- Additive
- Midpoint value theorem

2.2.2. Triple integrals in Cartesian coordinates

Fubini's theorem

If $B = [a, b] \times [c, d] \times [s, t]$ be a rectangular box, then

$$\iiint\limits_B f(x, y, z)dxdydz = \int\limits_a^b dx \int\limits_c^d dy \int\limits_s^t f(x, y, z)dz = \cdots = \int\limits_s^t dz \int\limits_c^d dy \int\limits_a^b f(x, y, z)dx.$$

Special case

If $f(x, y, z) = g(x)h(y)k(z)$ and $B = [a, b] \times [c, d] \times [s, t]$ be a rectangular box, then

$$\iiint\limits_B f(x, y, z)dxdydz = \left(\int\limits_a^b g(x)dx\right)\left(\int\limits_c^d h(y)dy\right)\left(\int\limits_s^t k(z)dz\right).$$

Triple integrals over more general solids

Region V is bounded by surfaces $z = z_1(x, y)$, $z = z_2(x, y)$, where z_1, z_2 are continuous functions on D with D is the projection of V on Oxy:

$$V = \{(x, y, z) \mid (x, y) \in D, z_1(x, y) \le z \le z_2(x, y)\}.$$

Then

$$\iiint\limits_V f(x, y, z)dxdydz = \iint\limits_D dxdy \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz.$$

In case D is a trapezium (type I region)

$$D = \{(x, y) \mid a \le x \le b, y_1(x) \le y \le y_2(x)\}.$$

Then

$$\iiint\limits_V f(x, y, z)dxdydz = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz.$$

Similar formulae for

$$V = \{(x, y, z) \mid (y, z) \in D, x_1(y, z) \leq x \leq x_2(y, z)\}$$

or

$$V = \{(x, y, z) \mid (x, z) \in D, y_1(x, z) \leq y \leq y_2(x, z)\}.$$

Example (GK20192)

Evaluate the triple integral $\iiint\limits_V x^2 e^z \, dxdydz$, where

$$V: 0 \le y \le 1, y \le x \le 1, 0 \le z \le xy + 1.$$

$$\begin{aligned}
\iiint\limits_V x^2 e^z \, dxdydz &= \int_0^1 dy \int_y^1 dx \int_0^{xy+1} x^2 e^z \, dz = \int_0^1 dy \int_y^1 (x^2 e^{xy+1} - x^2) dx \\
&= \int_0^1 dx \int_0^x (x^2 e^{xy+1} - x^2) dy = \int_0^1 (xe^{x^2+1} - ex - x^3) dx \\
&= \frac{1}{2}(e^2 - e) - \frac{1}{2}e - \frac{1}{4} = \frac{e^2}{2} - e - \frac{1}{4}.
\end{aligned}$$

2.2.3. Change of variables

Let $f(x, y, z)$ be contineous on $V \subseteq \mathbb{R}^3$. Let $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$. The Jacobian

$$J = \frac{D(x, y, z)}{D(u, v, w)} = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix} \neq 0.$$

Then

$$\iiint\limits_{V} f(x, y, z) dxdydz = \iiint\limits_{V'} f(x(u, v, w), y(u, v, w), z(u, v, w))|J| dudvdw.$$

Example (GK20182)

Evaluate the triple integral $\iiint_V (x+y+2z)dxdydz$, where V is the region bounded by $x-y=0$, $x-y=2$, $x+y=0$, $x+y=1$, $z=0$, $z=1$.

Let $u=x-y$, $v=x+y$, $w=z \Rightarrow x=(u+v)/2$, $y=(v-u)/2$, $z=w$.

$$\frac{D(u,v,w)}{D(x,y,z)} = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \Rightarrow J = \frac{D(x,y,z)}{D(u,v,w)} = 1/2.$$

V' is bounded by $u=0$, $u=2$, $v=0$, $v=1$,$w=0$, $w=1$.

$$\iiint_V (x+y+2z)dxdydz = \int_0^2 du \int_0^1 dv \int_0^1 (v+2w)\frac{1}{2}dw$$
$$= \left(\frac{1}{2}\int_0^2 du\right)\int_0^1 dv \int_0^1 (v+2w)dw = \int_0^1 (v+1)dv$$
$$= \frac{3}{2}.$$

Cylindrical coordinates

Then

$$x = r\cos\varphi, y = r\sin\varphi, z = z.$$

The Jacobian $J = \dfrac{D(x,y,z)}{D(r,\varphi,z)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$. We have the formulae

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f(r\cos\varphi, r\sin\varphi, z)rdrd\varphi dz.$$

Example (GK20162)

Evaluate the triple integral $\iiint_V zdxdydz$, where V is bounded by $z^2 = 4(x^2 + y^2)$, $z = 2$.

Let $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$.
Then $J = r$. The region V' is bounded by $z^2 = 4r^2$, $z = 2$. $V'\colon 0 \le \varphi \le 2\pi, 0 \le r \le 1, 2r \le z \le 2$.

$$
\begin{aligned}
&\iiint_V zdxdydz = \iiint_{V'} rzdrd\varphi dz = \int_0^{2\pi}\int_0^1\int_{2r}^2 rzd\varphi drdz \\
&= \int_0^{2\pi} d\varphi \int_0^1 rdr \int_{2r}^2 zdz = 2\pi \int_0^1 r(2 - 2r^2)dr \\
&= 2\pi \cdot (1 - \frac{1}{2}) = \pi.
\end{aligned}
$$

Spherical coordinates

- The spherical coordinate of the point $M(x, y, z)$ is the triple (r, θ, φ), where $r = OM$, θ is the angle between arrays Oz and OM, and φ is the angle between arrays Ox and $\overrightarrow{OM'}$, where M' is the projection of M on Oxy. Note that $0 \leq r < +\infty, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$.
- and $x = r\cos\varphi\sin\theta, y = r\sin\varphi\sin\theta, z = r\cos\theta$.
- The Jacobian $J = \dfrac{D(x,y,z)}{D(r,\theta,\varphi)} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$.
- The change of variable formula:

$$\iiint\limits_V f(x,y,z)dxdydz = \iiint\limits_{V'} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)r^2\sin\theta drd\theta d\varphi.$$

Example (GK20162)

Evaluate the triple integrals $\iiint_V xyzdxdydz$, where
$V = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.

Let $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$.
We have $|J| = r^2\sin\theta$.
The region V' : $0 \leq r \leq 1$, $0 \leq \varphi \leq \pi/2$, $0 \leq \theta \leq \pi/2$.

$$\iiint_V xyzdxdydz = \iiint_{V'} r^3\cos\varphi\sin\varphi\sin^2\theta\cos\theta \cdot r^2\sin\theta drd\varphi d\theta$$
$$= \int_0^1 r^5 dr \int_0^{\pi/2} \cos\varphi\sin\varphi d\varphi \int_0^{\pi/2} \sin^3\theta\cos\theta d\theta$$
$$= \frac{1}{6} \cdot \left(\frac{1}{2}\sin^2\varphi\Big|_0^{\pi/2}\right) \cdot \left(\frac{1}{4}\sin^4\varphi\Big|_0^{\pi/2}\right) = \frac{1}{48}.$$

Symmetrical regions

Theorem: Let V be a region with a symmetrical plane Oxy.

- If $f(x,y,-z) = -f(x,y,z)$ for all $(x,y,z) \in V$ then $\iiint\limits_V f(x,y,z)dxdydz = 0$.
- If $f(x,y,-z) = f(x,y,z,)$ for all $(x,y,z) \in V$ then $\iiint\limits_V f(x,y,z)dxdydz = 2\iiint\limits_{V'} f(x,y,z)dxdydz$, where V' is the sub region of V lying above Oxy.

We have similar results for Oyz and Oxz .

Theorem

If the origin is the center of symmetry $f(-x,-y,-z) = -f(x,y,z)$ $(\forall(x,y,z) \in V)$ then

$$\iiint\limits_V f(x,y,z)dxdydz = 0.$$

Example (GK20181)

Evaluate $\iiint\limits_V (x+2y+3z+4)dxdydz$, where V is defined by

$$x^2+y^2+z^2+xy+yz+zx \leq 2.$$

Let $u=x+y$, $v=y+z$, $w=z+x$. Then $J=\left|\dfrac{D(u,v,w)}{D(x,y,z)}\right|^{-1}=1/2$ and V': $u^2+v^2+w^2 \leq 4$.

$$\iiint\limits_V (x+2y+3z+4)dxdydz = \iiint\limits_{V'} (2v+w+4)\cdot\frac{1}{2}dudvdw$$

$$= \iiint\limits_{V'} vdudvdw + \frac{1}{2}\iiint\limits_{V'} wdudvdw + 2\iiint\limits_{V'} dudvdw$$

$$= 2\mathrm{Vol}(V') = 2\frac{4\pi}{3}2^3 = \frac{64\pi}{3}.$$

Practice problems

- (GK20212) Evaluate $\iiint_V (x^2+y^2)dxdydz$, where V is bounded by $x^2+y^2+z^2=2x+4y$.
- (GK20212) Evaluate $\iiint_V (x+y)^2(x-y)^3z^2dxdydz$, where $V: 0 \leq x \leq 1, 0 \leq y \leq 1$, $0 \leq z \leq x^2+y^2$.
- (GK20201) Evaluate $\iiint_V (x^2+y^2)dxdydz$, where V is bounded by $z=x^2+y^2$ and $z=1$.
- (GK20201) Evaluate $\iiint_V \sqrt{x^2+y^2+z^2}dxdydz$, where V is bounded by $x^2+y^2+z^2 \leq 2x$.
- (GK20182) Evaluate $\iiint_V \dfrac{x^2}{\sqrt{4y-y^2-z^2}}dxdydz$, where V is bounded by $x^2+y^2+z^2 \leq 4y$, $x \leq 0$.
- (GK20172) Evaluate $\iiint_V (x^2+y^2+z^2)dxdydz$, where V is bounded by $x=y^2+4z^2$, $x \leq 4$.

2.2.4. Applications: Finding the volume

The volume of the object V in $\mathbb{R}^3$ is $\iiint\limits_V dxdydz$.

Example (GK20192)

Find the volume of the region bounded by $y = x^2$, $x = y^2$, $z = y^2$ and the Oxy plane.

- The volume is $I = \iiint_V dxdydz$, where V is bounded by $y = x^2$, $x = y^2$, $z = y^2$ and the Oxy plane.
- $$I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{y^2} dz = \int_0^1 dx \int_{x^2}^{\sqrt{x}} y^2 dy = \frac{1}{3}\int_0^1 (x^{3/2} - x^6)dx$$
  $$= \frac{1}{3}\left(\frac{2}{5} - \frac{1}{7}\right) = \frac{3}{35}.$$

Practice problems

- (GK20212) Find the volume of the solid bounded by $z \geq x^2 + y^2$ and $2x^2 + 2y^2 + z^2 \leq 3$.
- (GK20181) Find the volume of the solid bounded by $z = 2 - x^2 - y^2$, $z = x^2 + y^2$.
- (GK20172) Find the volume of the solid bounded by $z = x^2 + 3y^2$ and $z = 4 - 3x^2 - y^2$.
- (GK20162) Find the volume of the solid bounded by $x + y + z = 3$, $3x + y = 3$, $\frac{3}{2}x + y = 3$, $y = 0$, $z = 0$.

I my free time I do differential and integral calculus.

KARL MARX