Chapter 6: Field theory

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Contents

6.1.1.Scalar fields

- A scalar field (in R^3) is a function $u\colon V \to \mathbb{R}$, mapping each point $M(x, y, z)$ to a real number $u(x, y, z)$, where $V \subset \mathbb{R}^3$.
- For each $c \in \mathbb{R}$, the set of point $M(x, y, z)$ such that $u(M) = C$ is called a level surface.

6.1.2. Directional derivative

Definition

Let $u(x, y, z)$ be a scalar field and $\vec{e} = (a, b, c)$ be a unit vector. Let $M(x_0, y_0, z_0)$ be a fixed point. If the following limit exists,

$$\lim_{t \to 0} \frac{u(x_0 + ta, y_0 + tb, z_0 + tc) - u(x_0, y_0, z_0)}{t},$$

then the limit is called the directional derivative of u at M in the direction of vector $\vec{e}$, and is denoted by $\frac{\partial u}{\partial \vec{e}}(M)$ or $D_{\vec{e}}\, u(M)$.

Remarks

- If $\vec{e} = \vec{i} = (1, 0, 0)$, then $\dfrac{\partial u}{\partial \vec{e}} = \dfrac{\partial u}{\partial x}$.
- If $\vec{e} = \vec{j} = (0, 1, 0)$, then $\dfrac{\partial u}{\partial \vec{e}} = \dfrac{\partial u}{\partial y}$.
- If $\vec{e} = \vec{k} = (0, 0, 1)$, then $\dfrac{\partial u}{\partial \vec{e}} = \dfrac{\partial u}{\partial z}$.
- Let $\alpha = (\vec{e}, Ox)$, $\beta = (\vec{e}, Oy)$, $\gamma = (\vec{e}, Oz)$. Then $\vec{e} = (\cos\alpha, \cos\beta, \cos\gamma)$.
- Let $\vec{v}$ be a vector $\neq \vec{0}$. The derivative of u at M in the direction of vector $\vec{v}$, denoted by $\dfrac{\partial u}{\partial \vec{v}}(M)$, is the derivative of u at M in the direction of the unit vector $\vec{e} = \dfrac{\vec{v}}{||v||}$.
- The directional derivative $\dfrac{\partial u}{\partial \vec{v}}(M)$ measures the rate of change of the function u in the direction of vector $\vec{v}$.

Theorem

If $u(x, y, z)$ is differentiable at $M(x, y, z)$ then it has directional derivative in the direction of any unit vector $\vec{e} = (a, b, c)$ and

$$\frac{\partial u}{\partial \vec{e}}(M) = \frac{\partial u}{\partial x}(M)a + \frac{\partial u}{\partial y}(M)b + \frac{\partial u}{\partial z}(M)c$$

If $\alpha = (\vec{e}, Ox)$, $\beta = (\vec{e}, Oy)$, $\gamma = (\vec{e}, Oz)$ then $\vec{e} = (\cos\alpha, \cos\beta, \cos\gamma)$ và

$$\frac{\partial u}{\partial \vec{e}}(M) = \frac{\partial u}{\partial x}(M)\cos\alpha + \frac{\partial u}{\partial y}(M)\cos\beta + \frac{\partial u}{\partial z}(M)\cos\gamma.$$

Example (CK20192)

Find the directional derivative in the direction of the vector $\vec{\ell}(1,2,-2)$ of $u(x,y,z) = e^x(y^2+z) - 2xyz^3$ at $A(0,1,2)$.

- $u'_x = e^x(y^2+z) - 2yz^3$, $u'_y = 2e^x y - 2xz^3$, $u'_z = e^x - 6xyz^2$.
- $u'_x(A) = -13$, $u'_y(A) = 2$, $u'_z(A) = 1$.
- $\dfrac{\vec{\ell}}{||\vec{\ell}||} = (\dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3})$.
- $\dfrac{\partial u}{\partial \vec{\ell}}(A) = -13 \times \dfrac{1}{3} + 2 \times \dfrac{2}{3} + 1 \times \dfrac{-2}{3} = -\dfrac{11}{3}$

6.1.3. Gradient

Let $u(x, y, z)$ be a scalar vector field, the gradient of u at M, denoted by $\nabla u(M)$ or $\overrightarrow{\mathrm{grad}} u(M)$, is the vector

$$\nabla u(M) = \overrightarrow{\mathrm{grad}} u(M) = \left(\frac{\partial u}{\partial x}(M), \frac{\partial u}{\partial y}(M), \frac{\partial u}{\partial z}(M)\right) = \frac{\partial u}{\partial x}(M)\vec{i} + \frac{\partial u}{\partial y}(M)\vec{j} + \frac{\partial u}{\partial z}(M)\vec{k}.$$

Theorem

The directional derivative of $u(x, y, z)$ at M in the direction of a unit vector $\vec{e}$ is equal to the dot product (canonical inner product) of the gradient of u at M and $\vec{e}$:

$$\frac{\partial u}{\partial \vec{e}}(M) = \overrightarrow{\mathrm{grad}} u(M) \cdot \vec{e}.$$

Example (CK20173)

Consider $u = \ln(3x + 2y^2 - z^3)$ and two points $A(1, -1, 1)$, $B(0, 1, 3)$. Find $\dfrac{\partial u}{\partial \vec{\ell}}(A)$, where $\vec{\ell} = \overrightarrow{AB}$.

- $u'_x = \dfrac{3}{3x + 2y^2 - z^3}$, $u'_y = \dfrac{4y}{3x + 2y^2 - z^3}$, $u'_z = \dfrac{-3z^2}{3x + 2y^2 - z^3}$.
- $\overrightarrow{\mathrm{grad} u}(A) = (\dfrac{3}{4}, -1, \dfrac{-3}{4})$.
- $\vec{\ell} = \overrightarrow{AB} = (-1, 2, 2)$, $\vec{e} = \dfrac{\vec{\ell}}{||\vec{\ell}||} = (\dfrac{-1}{3}, \dfrac{2}{3}, \dfrac{2}{3})$.
- $\dfrac{\partial u}{\partial \vec{\ell}}(M) = \overrightarrow{\mathrm{grad} u}(A) \cdot \vec{e} = \dfrac{3}{4} \times \dfrac{-1}{3} + (-1) \times \dfrac{2}{3} + \dfrac{-3}{4} \times \dfrac{2}{3} = -\dfrac{17}{12}$.

Example (Final 20142)

Let $u(x,y,z)=x^3+2yz^2+3xyz$. Evaluate $\dfrac{\partial u}{\partial \vec{n}}(A)$, where $\vec{n}$ is the normal vector to the sphere $x^2+y^2+z^2=3$ at $A(1,1,1)$ with the outward direction.

- $u'_x=3x^2+3yz,\ u'_y=2z^2+3xz,\ u'_z=4yz+3xy$.
- $\overrightarrow{\mathrm{grad}\, u}(A)=(6,5,7)$.
- $\vec{n}=(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$.
- $\dfrac{\partial u}{\partial \vec{n}}(A)=\overrightarrow{\mathrm{grad}\, u}(A)\cdot\vec{n}=6\times\dfrac{1}{\sqrt{3}}+5\times\dfrac{1}{\sqrt{3}}+7\times\dfrac{1}{\sqrt{3}}=6\sqrt{3}$.

Some past exam problems

- (Final 20182) Let $u = \dfrac{x}{x^2+y^2+z^2}$ and $A(1,2,2)$, $B(-3,1,0)$. Find the angle between $\overrightarrow{\mathrm{grad}\,u}(A)$ and $\overrightarrow{\mathrm{grad}\,u}(B)$.
- (Final 20162) Find the derivative $u = x^3 + 2y^3 + 3z^2 + 2xyz$ in the direction $\vec{\ell} = (1,1,2)$ at $P(2,1,1)$.
- (Final 20152) Find the derivative of $u = x^3 + y^3 + z^2 + 2$ in the direction of $\vec{\ell} = (5,5,2)$ at $P(1,1,1)$.

Maximizing and minimizing the directional derivative

Question

Fix a function u and a point M. In which direction $\vec{e}$ does u change fastest and what is the maximum of rate of change? In other words, find directions $\vec{e}$ such that $\left|\frac{\partial u}{\partial \vec{e}}(M)\right|$ is maximal.

Answer: From $\frac{\partial u}{\partial \vec{e}}(M) = \overrightarrow{\mathrm{grad}}\, u(M) \cdot \vec{e}$, we see that

$$\left|\frac{\partial u}{\partial \vec{e}}(M)\right| \leq ||\overrightarrow{\mathrm{grad}}\, u(M)||,$$

The equality holds when $\vec{e}$ has the same direction as $\overrightarrow{\mathrm{grad}}\, u(M)$. Hence

- $\max \frac{\partial u}{\partial \vec{e}}(M) = ||\overrightarrow{\mathrm{grad}}\, u(M)||$, it occurs when $\vec{e}$ has the same direction as $\overrightarrow{\mathrm{grad}}\, u(M)$. In other words, in the direction of $\overrightarrow{\mathrm{grad}}\, u(M)$ the function u increases fastest.
- $\min \frac{\partial u}{\partial \vec{e}}(M) = -||\overrightarrow{\mathrm{grad}}\, u(M)||$, it occurs when $\vec{e}$ has the opposite direction as $\overrightarrow{\mathrm{grad}}\, u(M)$. In other words, in the direction of $-\overrightarrow{\mathrm{grad}}\, u(M)$ the function u decreases fastest.

Example

Suppose that the temperature at a poin (x, y, z) in space is given by $T(x, y, z) = \dfrac{80}{1 + x^2 + 2y^2 + 3z^2}$, here is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$?

- $T'_x = -\dfrac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2}$, $T'_y = -\dfrac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2}$, $T'_x = -\dfrac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}$.
- $\overrightarrow{\mathrm{grad}}\, T(A) = (-\dfrac{5}{8}, -\dfrac{10}{8}, \dfrac{30}{8})$.
- In the direction of $\ell = -\vec{i} - 2\vec{j} + 6\vec{k}$ the temperature increases fastest.

Some properties

- (Linearity) $\overrightarrow{\text{grad}}(\alpha u + \beta v) = \alpha\overrightarrow{\text{grad}}u + \beta\overrightarrow{\text{grad}}v$, với $\alpha, \beta \in \mathbb{R}$.
- $\overrightarrow{\text{grad}}(uv) = u\overrightarrow{\text{grad}}v + v\overrightarrow{\text{grad}}u$.
- $\overrightarrow{\text{grad}}f(u) = f'(u)\overrightarrow{\text{grad}}u$.
- $\overrightarrow{\text{grad}}\left(\dfrac{u}{v}\right) = \dfrac{v\overrightarrow{\text{grad}}u - u\overrightarrow{\text{grad}}v}{v^2}$.

6.2.1. Vector fields

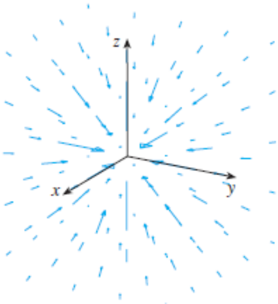
Let $V \subseteq \mathbb{R}^n$. A vector field in $\mathbb{R}^n$ is a function $\vec{F}$ mapping each M in V to a vector $\vec{F}(M) \in \mathbb{R}^n$. We mainly work witn vector fields when $n = 3$.

Example (Electric fields)

Suppose an electric charge q is located at the origin O. The *electric field* of q:

$$\vec{E} = \frac{q}{(x^2 + y^2 + z^2)^{3/2}}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{q}{r^3}\vec{r},$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.



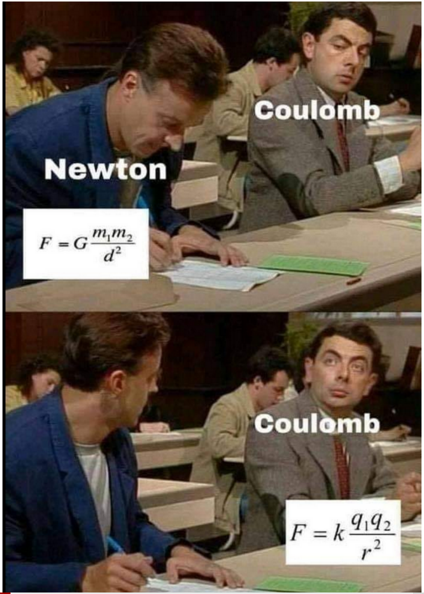
Example (Gravitational field)

Suppose an object with mass M locates at the origin O. Suppose another object with mass m locates at (x, y, z). Then the gravitational force exerted on the second object is

$$\vec{F}(x, y, z) = -\frac{mMG}{(x^2 + y^2 + z^2)^{3/2}}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{mMG}{r^3}\vec{r},$$

ở đây $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

JUST FOR FUN!



Flow lines (or streamlines) of a vector fields

- Consider a vector field $\vec{F} = \vec{F}(M) = P(M)\vec{i} + Q(M)\vec{j} + R(M)\vec{j}$.
- The flow lines (or streamlines) of the vector field $\vec{F}$ are the curves C such at each point M in C the tangent line of C at M is parallel to the vector $\vec{F}(M)$.
- Suppose a flow line C has parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$. Then we have

$$\frac{x'(t)}{P} = \frac{y'(t)}{Q} = \frac{z'(t)}{R}$$

- The above system of differential equations is called differential equations of follow lines of $\vec{F}$.
- Example: the flow lines of a electric field is straight lines passing the origin O.

6.2.2. Flux, divergence, incompressible fields

- Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector defined on a oriented surface S with unit normal vector $\vec{n}$. The flux of $\vec{F}$ across S is

$$\Phi = \iint\limits_S \vec{F} \cdot \vec{n}\, dS = \iint\limits_S Pdydz + Qdzdx + Rdxdy.$$

- If, for instance, $\vec{F}$ is a velocity field describing the flow of a fluid with density 1, then the flux of $\vec{F}$ across S is the rate of flow through S (in units of mass per unit time) .

Example (CK20152)

Find the flux of vector field $\vec{F} = (x^3 - z)\vec{i} - y\vec{j} + (3y^2 z + 2y)\vec{k}$ through the surface $S : x^2 + y^2 + z^2 = 1$, with outward orientation.

- The flux $\Phi = \iint\limits_S (x^3 - z)dydz - ydzdx + (3y^2 z + 2y)dxdy$.
- By Ostrogradsky formula $\Phi = \iiint\limits_V (3x^2 - 1 + 3y^2)dxdydz$, where V is the sphere $x^2 + y^2 + z^2 \leq 1$.
- Change of variables $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$, $|J| = r^2\sin\theta$, $0 \leq r \leq 1, 0 \leq \varphi \leq 2\pi$, $0 \leq \theta \leq \pi$.
- $\Phi = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 dr \int\limits_0^{\pi} 3r^2\sin^2\theta r^2\sin\theta - \frac{4\pi}{3} = 6\pi\int\limits_0^1 r^4 dr \int\limits_0^{\pi} \sin^3\theta d\theta - \frac{4\pi}{3} = \frac{8\pi}{5} - \frac{4\pi}{3} = \frac{4\pi}{15}$.

Example (CK20182)

Let $\vec{F} = (x^2 - y)\vec{i} + (x + 2y)\vec{j} + (x + y + z)\vec{k}$. Find the flux $\vec{F}$ across the flux $|x - y| + |x + 2y| + |x + y + z| = 1$, with outward orientation.

- The flux $\Phi = \iint\limits_S (x^2 - y)dydz + (x + 2y)dzdx + (x + y + z)dxdy$.
- By Ostrogradsky formula $\Phi = \iiint\limits_V (2x + 3)dxdydz$, where $V\!:\!|x - y| + |x + 2y| + |x + y + z| \leq 1$.
- Change of variables $u = x - y$, $v = x + 2y$, $w = x + y + z$, $1/J = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 3$. Miền

  $V'\!: |u| + |v| + |w| = 1$.
- $\Phi = \iiint\limits_{V'} (2 \cdot \dfrac{2u + v}{3} + 3)\dfrac{1}{3}dudvdw = \iiint\limits_{V'} dudvdw = Vol(V') = \dfrac{4}{3}$ (Since V' is symmetrical with respect to the plane $u = 0$, the functions u and v are odd functions).

Some exercises

- (CK20142) Find the flux of $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ across the surface S which is the boundary of the region V: $0 \leq z \leq \sqrt{1-y^2}$, $0 \leq x \leq 2$, with outward orientation.
- (CK20161) Find the flux of $\vec{F} = (x+y)\vec{i} + 2y\vec{j} + (3y+z)\vec{k}$ across the hemisphere $S : x^2 + y^2 + z^2 = 1$, $z \geq 0$, with outward orientation.
- *(CK20181) Show that flux Φ of the vector field $\vec{F} = \dfrac{1}{3}(3x)^n\vec{i} + \dfrac{1}{2}(2y)^n\vec{j} - 2z^n\vec{k}$ across the surface $9x^2 + 4y^2 + z^2 = 1$ with outward orientation, is always equal to 0 for every positive integer n.

Divergence (div)

- If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field, then the divergence of $\vec{F}$ is

$$\mathrm{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

- $\mathrm{div}\vec{F}$ is scalar field.
- Some properties
  - $\mathrm{div}(\vec{F} + \vec{G}) = \mathrm{div}(\vec{F}) + \mathrm{div}(\vec{F})$.
  - $\mathrm{div}(f\vec{F}) = f\mathrm{div}(\vec{F}) + \vec{F} \cdot \overrightarrow{\mathrm{grad}}f$.
  - $\mathrm{div}(\overrightarrow{\mathrm{grad}}f \wedge \overrightarrow{\mathrm{grad}}g) = 0$.

- Ostrogradsky's formula (sometimes called the divergence theorem) can be rewritten in vector form as

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_V \mathrm{div} \vec{F} dV.$$

- For a fixed $M(x_0, y_0, z_0)$, let B_a be the sphere with center M and radius a. Then

$$\mathrm{div} \vec{F}(M) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_S \vec{F} \cdot \vec{n}\, dS.$$

Example

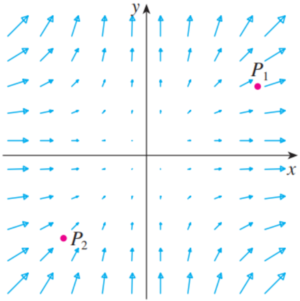
Find the divergence of a electric field $\vec{E} = \dfrac{q}{r^3}\vec{r}$.

- $P = \dfrac{qx}{r^3}$, $Q = \dfrac{qy}{r^3}$, $R = \dfrac{qz}{r^3}$.
- $P'_x = \dfrac{q}{r^5}(r^2 - 3x^2)$, $Q'_y = \dfrac{q}{r^5}(r^2 - 3y^2)$, $R'_z = \dfrac{q}{r^5}(r^2 - 3z^2)$.
- $\mathrm{div}\vec{E} = P'_x + Q_y + R'_z = 0$.

Incompressible vector fields

Suppose $\mathrm{div}\vec{F}$ is a continous function on (a open set containing) V.

- Vector field $\vec{F}$ (defined over V) is said to be incompressible if $\mathrm{div}\vec{F} = 0$ at every point $M \in V$.
- If $\mathrm{div}\vec{F}(M) > 0$ then M is called a source (điểm nguồn) of $\vec{F}$.
- If $\mathrm{div}\vec{F}(M) < 0$ then M is called a sink (điểm rò) of $\vec{F}$.



(Vector field $\vec{F} = x^2\vec{i} + y^2\vec{j}$)

Example

Find differentiable function $f(r)$ such that the following vector field is incompressible: $\vec{F} = f(r)\vec{r}$, $\vec{r} = x\vec{i} + y\vec{i} + z\vec{k}$.

- $P = f(r)x$, $Q = f(r)y$, $R = f(r)z$.
- $P'_x = f(r) + \dfrac{f'(r)}{r}x^2$, $Q'_y = f(r) + \dfrac{f'(r)}{r}y^2$, $R'_z = f(r) + \dfrac{f'(r)}{r}z^2$.
- $\vec{F}$ is incompressible $\Leftrightarrow 0 = \mathrm{div}\vec{F} = 3f(r) + f'(r)r$.
- Hence $f(r) = \dfrac{c}{r^3}$.

6.2.3. Circulation and curl

We have

$$\int_L Pdx + Qdy + Rdz = \int_L \vec{F} \cdot \vec{\mathcal{T}} ds,$$

where $\mathcal{T}$ is unit tangent vector (field) on L (in the direction of L).

Definition (circulation, hoàn lưu, lưu số)

The circulation of $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ around the (closed) curve L is

$$C = \int_L Pdx + Qdy + Rdz = \int_L \vec{F} \cdot \vec{\mathcal{T}} ds.$$

If $\vec{F}$ is a force field on L then the circulation of $\vec{F}$ around L is the work done by F on L.

In $\mathbb{R}^2$, we have

$$\int\limits_L Pdx + Qdy = \int\limits_L \vec{F} \cdot \vec{\mathcal{T}} ds,$$

where $\mathcal{T}$ is unit tangent vector (field) on L (in the direction of L).

- Suppose L is given in parametric form as $x = x(t)$, $y = y(t)$, where the initial point corresponds to $t = \alpha$ and the end point corresponds to $t = \beta$. Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$.
- Tangent vector at $(x(t), y(t))$: $\vec{r}'(t) = x'(t)\vec{i} + y'(t)\vec{j}$ and hence $\vec{\mathcal{T}}(x(t), y(t)) = \dfrac{\vec{r}(t)}{||\vec{r}(t)||}$.
- $(\vec{F} \cdot \vec{\mathcal{T}})(x(t), y(t)) = \dfrac{P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}$.
- $\int\limits_L \vec{F} \cdot \vec{\mathcal{T}} ds = \int\limits_\alpha^\beta (\vec{F} \cdot \vec{\mathcal{T}})(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2} dt =$
$\int\limits_\alpha^\beta [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)] dt = \int\limits_L Pdx + Qdy.$

Example (CK20171)

Find the circulation of $\vec{F} = (x^3 + y^3 + z^3)(\vec{i} + \vec{j} + \vec{k})$ around the curve of intersection of two surfaces $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$.

- Choose surface S to be the the surface $x + y + z = 1$ and $x^2 + y^2 + z^2 \leq 1$, in the direction of $\vec{n} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, choose the positive direction (with respect to $\vec{n}$) on the curve of intersection L.
- By Stokes' formula, the circulation

$$C = \int\limits_L (x^3 + y^3 + z^3)dx + (x^3 + y^3 + z^3)dy + (x^3 + y^3 + z^3)dz$$

$$= \iint\limits_S 3(y^2 - z^2)dydz + 3(z^2 - x^2)dzdx + 3(x^2 - y^2)dxdy$$

$$= \iint\limits_S [3(y^2 - z^2)\frac{1}{\sqrt{3}} + (z^2 - x^2)\frac{1}{\sqrt{3}} + (x^2 - y^2)\frac{1}{\sqrt{3}}]dS = 0.$$

Curl

Definition

The curl at a point M of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is the vector

$$\mathrm{curl}\vec{F}(M) = \overrightarrow{\mathrm{rot}}\vec{F}(M) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

Hence, $\mathrm{curl}\ \vec{F}$ is a vector field. Stokes' formula can be written in vector form

$$\oint\limits_{L} \vec{F}\cdot\vec{T}ds = \iint\limits_{S} \mathrm{curl}\vec{F}\cdot\vec{n}dS.$$

Circulation of $\vec{F}$ around a closed curve L is equal to the flux of $\mathrm{curl}\ \vec{F}$ across a surface S with boundary L.

- For a fixed point $M(x_0, y_0, z_0)$, let S_a be a disk with center M and radius a with orientation given by $\vec{n}(M)$, and boundary C_a. Then

$$\mathrm{curl}\vec{F}(M) \cdot \vec{n}(M) = \lim_{a \to 0} \frac{1}{\pi a^2} \int\limits_{C_a} \vec{F} \cdot \vec{T}\, ds.$$

- If $\mathrm{curl}\vec{F}(M) \neq 0$ then $\vec{F}$ is called rotational at M.
- If $\mathrm{curl}\vec{F}(M) = 0$ then $\vec{F}$ is called irrotational at M.

Some properties

- $\overrightarrow{\mathrm{rot}}(\alpha\vec{F} + \beta\vec{G}) = \alpha\overrightarrow{\mathrm{rot}}\vec{F} + \beta\overrightarrow{\mathrm{rot}}\vec{G}$.
- $\overrightarrow{\mathrm{rot}}(u\vec{C}) = \overrightarrow{\mathrm{grad}}u \wedge \vec{C}$, where $\vec{C} = \mathrm{const}$ vector, u is a function.
- $\overrightarrow{\mathrm{rot}}(u\vec{F}) = u\overrightarrow{\mathrm{rot}}\vec{F} + \overrightarrow{\mathrm{grad}}u \wedge \vec{F}$.
- $\mathrm{div}(\vec{F} \wedge \vec{G}) = \vec{G} \cdot \overrightarrow{\mathrm{rot}}\vec{F} - \vec{F} \cdot \overrightarrow{\mathrm{rot}}\vec{G}$.
- $\mathrm{div}\left(\overrightarrow{\mathrm{rot}}\vec{F}\right) = 0$. $[\mathrm{div}(\mathrm{curl}\vec{F}) = 0.]$
- $\overrightarrow{\mathrm{rot}}(\overrightarrow{\mathrm{grad}}u) = \vec{0}$. $[\mathrm{curl}(\nabla\vec{F}) = \vec{0}.]$

Example

Find the curl $\operatorname{curl}\vec{E}$ of electric field $\vec{E} = \dfrac{q}{r^3}\vec{r}$.

- $P = \dfrac{qx}{r^3}$, $Q = \dfrac{qy}{r^3}$, $R = \dfrac{qz}{r^3}$.
- $Q'_x = -qy\dfrac{3}{r^3}\dfrac{x}{r} = -\dfrac{3qxy}{r^5}$.
- $P'_y = -qx\dfrac{3}{r^3}\dfrac{y}{r} = -\dfrac{3qxy}{r^5}$.
- $P'_x - Q'_y = 0$. Similarly, $R'_y - Q_z = 0$, $P'_z - R'_x = 0$.
- $\operatorname{curl}\vec{E} = 0$.

6.2.4. Conservative vector fields

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field.

- We say that $\vec{F}$ (on V) is a conservative vector field (trường thế) if there exists a function u (scalar field) such that $\nabla u = \vec{F}$, or equivalently, $du = Pdx + Qdy + Rdz$. In this case, such a function u is called a potential function (hàm thế vị) of $\vec{F}$.
- Example: Gravitational field $\vec{F} = -\dfrac{mMG}{r^3}\vec{r}$ is conservative, with a potential function $u = \dfrac{mMG}{r}$.
- Example: Electric field $\vec{F} = -\dfrac{mMG}{r^3}\vec{r}$ is conservative, with a potential function $u = -\dfrac{q}{r}$.

Conservative vector fields

- $\boxed{\vec{F} \text{ is conservative} \Leftrightarrow \operatorname{curl} \vec{F} = \vec{0}}$ (over V simply connected, for instance $V = \mathbb{R}^3$, or $V = \mathbb{R}^3 \setminus \{0\}$).
- If $V = \mathbb{R}^3$, a potential function for a conservative vector field $\vec{F}$ is

$$u(x, y, z) = \int_{x_0}^{x} P(x, y_0, z_0)dx + \int_{y_0}^{y} Q(x, y, z_0)dy + \int_{z_0}^{z} R(x, y, z)dz + C.$$

- Vector field $\vec{F}$ is harmonic if it is both incompressible $(\operatorname{div}\vec{F} = 0)$ and conservative $(\operatorname{curl} \vec{F} = \vec{0})$.

Recall: Path independence in space

Theorem

Let P, Q, R be three continuous functions with continuous first derivatives in an open simply connected region V in space. Then the following claims are equivalent:

1. $\int\limits_{\overset{\frown}{AB}} Pdx + Qdy + Rdz$ only depends on A and B and does not depend on any smooth curve that lies inside V and connects A to B.
2. $\oint\limits_{C} Pdx + Qdy + Rdz = 0$, for all simple closed and piecewise-smooth curves C in V.
3. $R'_y(M) = Q'_z(M)$, $P'_z(M) = R'_x(M)$, $Q'_x(M) = P'_y(M)$ for all $M \in V$.
4. The expression is $Pdx + Qdy + Rdz$ the total differential of a function $u(x, y, z)$ defined over V.

Remark:

- Condition (3) $\Leftrightarrow \mathrm{curl}\,\vec{F} = \vec{0}$ (from definition).
- Condition (4) $\Leftrightarrow \overrightarrow{\mathrm{grad}} u = \vec{F} \Leftrightarrow \vec{F}$ is conservative (from definition)

Example (CK20192)

Show that the vector field $\vec{F} = (e^x y^2 + e^{2z} - 2xy^3)\vec{i} + (2ye^x - 3x^2y^2)\vec{j} + (2xe^{2z} + 5)\vec{k}$ is a conservative vector field. Find a potential function for $\vec{F}$.

- $P = e^x y^2 + e^{2z} - 2xy^3$, $Q = 2ye^x - 3x^2y^2$, $R = 2xe^{2z} + 5$, $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$.
- $P'_y = 2e^x y - 6xy^2 = Q'_x$, $P'_z = 2e^{2z} = R'_x$, $Q'_z = 0 = R'_y$.
- $\mathrm{curl}\vec{F} = \vec{0}$ and hence $\vec{F}$ is conservative.
- Potential function
$u(x,y,z) = \int\limits_0^x P(x,0,0)dx + \int\limits_0^y Q(x,y,0)dy + \int\limits_0^z R(x,y,z)dz + C = \int\limits_0^x 1dx + \int\limits_0^y (2ye^x - 3x^2y^2)dy + \int\limits_0^z (2xe^{2z} + 5)dz + C = x + e^x y^2 - x^2y^3 + xe^{2z} - x + 5z + C = e^x y^2 - x^2 y^3 + xe^{2z} + 5z + C$.

Example (CK20192-N2)

Consider a conservative vector field $\vec{F} = e^y\vec{i} + xe^y\vec{j} + (z+1)e^z\vec{k}$. Find a potential function for $\vec{F}$ and evaluate $\int\limits_{(1,2,3)}^{(4,5,6)} e^y dx + xe^y dy + (z+1)e^z dz$.

- $P = e^y,\ Q = xe^y,\ R = (z+1)e^z$.
- Potential function $u = \int\limits_0^x P(x,0,0)dx + \int\limits_0^y Q(x,y,0)dy + \int\limits_0^z R(x,y,z)dz + C =$
  $\int\limits_0^x 1dx + \int\limits_0^y xe^y dy + \int\limits_0^z (z+1)e^z dz + C = xe^y + ze^z + C.$
- $\int\limits_{(1,2,3)}^{(4,5,6)} e^y dx + xe^y dy + (z+1)e^z dz = u(4,5,6) - u(1,2,3) = (4e^5 + 6e^6) - (e^2 + 3e^3).$

Some exercises

- (CK20182) Show that $\vec{F} = \left(\dfrac{y}{1+xy} - z\cos x\right)\vec{i} + \dfrac{x}{1+xy}\vec{j} - \sin x\vec{k}$ is a conservative vector field and find a potential function for $\vec{F}$.
- (CK20162) Show that $\vec{F} = (3x^2 + yz)\vec{i} + (6y^2 + xz)\vec{j} + (z^2 + xy + e^z)\vec{k}$ is a conservative vector field. Find a potential function for $\vec{F}$.
- (CK20152) Show that $\vec{F} = (2xe^z + y^2)\vec{i} + (2xy + 3z^2)\vec{j} + (x^2e^z + 6yz)\vec{k}$ is a conservative vector field. Find a potential function for $\vec{F}$.
- (CK20142) Show that $\vec{F} = (3x^2y + 2z^2)\vec{i} + (x^3 + 6y)\vec{j} + (4xz + e^z)\vec{k}$ is a conservative vector field. Find a potential function for $\vec{F}$.

Definition

- Laplace operator: $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.
- Hamilton operator: $\nabla = \vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}$.

- $\nabla u = \overrightarrow{\mathrm{grad}} u$.
- $\nabla \cdot \vec{F} = \mathrm{div}\vec{F}$.
- $\nabla \wedge \vec{F} = \overrightarrow{\mathrm{rot}}\vec{F}$.
- $\mathrm{div}\left(\overrightarrow{\mathrm{grad}} u\right) = \nabla \cdot \nabla u = \Delta u$.
- $\overrightarrow{\mathrm{rot}}\left(\overrightarrow{\mathrm{grad}} u\right) = 0$.
- $\mathrm{div}\left(\overrightarrow{\mathrm{rot}}\vec{F}\right) = 0$.

If people do not believe that mathematics is simple,
it is only because they do not realize how complicated life is.

John von Neumann