Introduction to Communications Engineering

IT4593E

Đỗ Công Thuần, Ph.D.

Dept. of CE, SoICT, HUST

Email: thuandc@soict.hust.edu.vn

ONE LOVE. ONE FUTURE.

Thông tin chung

- Tên học phần: **Nhập môn kỹ thuật truyền thông**
- Mã học phần: **IT4593E**
- Khối lượng: **2 TC (2-1-0-4)**
- Lý thuyết và bài tập: **10 buổi lý thuyết, 5 buổi bài tập**
- Đánh giá học phần:

  30% QT (kiểm tra + bài tập/project + chuyên cần-quiz)

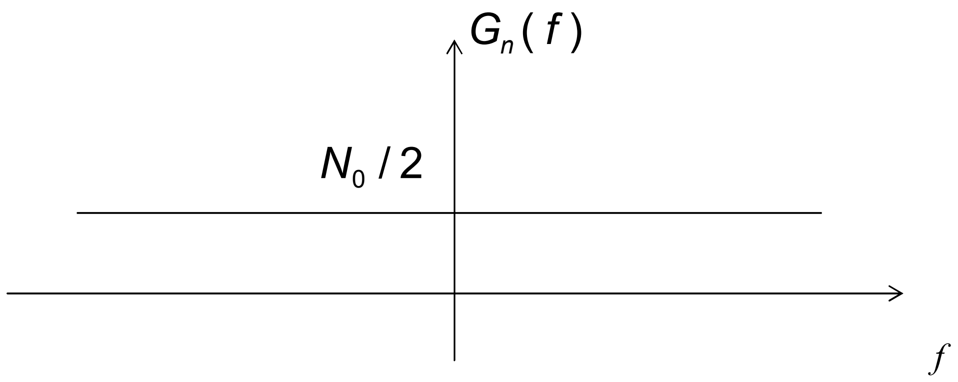
  70% CK (trắc nghiệm + tự luận)
- Tài liệu tham khảo:
  - Lecture slides
  - Lecture notes
  - Textbooks, ví dụ ***Communication Systems Engineering***, 2nd Edition, by John G. Proakis Masoud Salehi
  - Internet

Lec 04: Decision Theory
4.2 MAP and ML Criteria

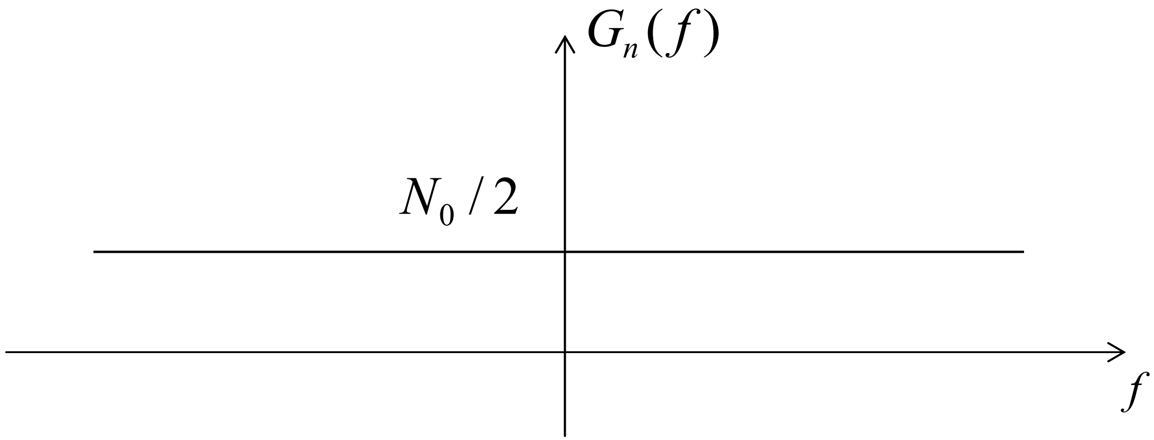
Channel Model

Additive White Gaussian Noise *n(t)*

- «Ergodic» random process
- Each random variable is a Gaussian random variable with zero mean
- Constant power spectral density: $G_n(f) = N_0/2$



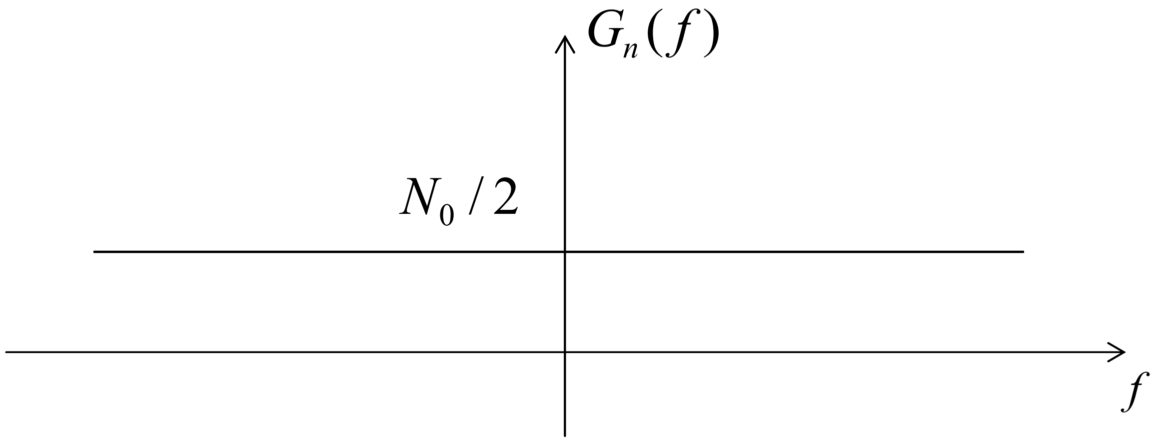
AWGN



$$G_n(f) = N_0/2 \quad \Longleftrightarrow \quad R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

(autocorrelation function)

AWGN



$$R_n(\tau) = \frac{N_0}{2}\delta(\tau) \quad \Leftrightarrow \quad E\left[n(t_1)n(t_1+\tau)\right] = \frac{N_0}{2}\delta(\tau)$$

n(t) is an «ergodic» process
(Time properties = Statistical properties)

AWGN

Consider two time instants t_1 and t_2

Corresponding to two random variables:

$$t_1 \longrightarrow n(t_1)$$

$$t_2 \longrightarrow n(t_2)$$

hey are Gaussian random variables with property:

$$E[n(t_1)n(t_2)] = \frac{N_0}{2}\delta(t_1 - t_2)$$

Statistically independent

The Problem at the Receiver

$$\underline{\mathrm{u}}_{\mathrm{T}} \longrightarrow s(t) \longrightarrow r(t) = s(t) + n(t)$$

Problem: $\boxed{\text{given } r(t) \rightarrow \text{recover } s(t)}$

Split $r(t)$ into segments corresponding to time intervals T:

$$r(t) = (\underbrace{r[0](t)}_{T} \;|\; \underbrace{r[1](t)}_{T} \;|\; \ldots \;|\; \underbrace{r[n](t)}_{T} \;|\; \ldots$$

Question: Is it possible to analyze the received signal in any given time interval independently?

$$\mathrm{r}(\mathrm{t}) = (\underbrace{\mathrm{r}[0](\mathrm{t})}_{T} \mid \underbrace{\mathrm{r}[1](\mathrm{t})}_{T} \mid \ldots \mid \underbrace{\mathrm{r}[\mathrm{n}](\mathrm{t})}_{T} \mid \ldots$$

$$s(t) = (s[0](t) \mid s[1](t) \mid \ldots \mid s[n](t) \mid \ldots$$

$$\mathrm{n}(t) = (\mathrm{n}[0](t) \mid \mathrm{n}[1](t) \mid \ldots \mid \mathrm{n}[n](t) \mid \ldots$$

We have:

$$r(t) = s(t) + n(t)$$

Consider the n-th time interval: $\boxed{nT \le t < (n+1)T}$

$$r[n](t) = s[n](t) + \mathrm{n}[n](t)$$

Each $r[n](t)$ depends entirely on:

- The transmitted signal: $s[n](t)$
- The noise: $n[n](t)$

which are random variables existing in the time interval:

$$nT \le t < (n+1)T$$

SOICT

$$s(t) = (\underbrace{s[0](t)}_{T} \mid \underbrace{s[1](t)}_{T} \mid \ldots \mid \underbrace{s[m](t)}_{T} \mid \ldots\ldots \mid \underbrace{s[n](t)}_{T} \mid \ldots$$

Each signal $s[n](t)$

- exists in the time interval T
- is statistically independent of signals in other time intervals $s[m](t),\ m \neq n$

→ $\boxed{r[n](t) \text{ is independent of } s[m](t),\ m \neq n}$

SOICT

$$\mathrm{n}(t) = (\underbrace{\mathrm{n}[0](t)}_{T} \mid \underbrace{\mathrm{n}[1](t)}_{T} \mid \ldots \mid \underbrace{\mathrm{n}[m](t)}_{T} \mid \ldots\ldots \mid \underbrace{\mathrm{n}[n](t)}_{T} \mid \ldots$$

Each noise $\mathrm{n}(t_i)$ is also statistically independent

→ $\boxed{r[n](t) \text{ independent of } n[m](t),\ m \neq n}$

The Problem at the Receiver

Consider time interval n: $nT \le t < (n+1)T$

Received signal: $r[n](t) = s[n](t) + \mathrm{n}[n](t)$

Depends **only** on:

- Transmitted signal $s[n](t)$
- Noise $\mathrm{n}[n](t)$ in the time interval $nT \le t < (n+1)T$

Each time interval can be analyzed independently

NO INTERSYMBOL INTERFERENCE (ISI)

$$r(t) = (\underbrace{r[0](t)}_{T} \mid \underbrace{r[1](t)}_{T} \mid \ldots \mid \underbrace{r[n](t)}_{T} \mid \ldots$$

Each time interval is analyzed independently:

Assume considering the original time interval, with $0 \leq t < T$

$$r(t) = (\underbrace{r[0](t)}_{T} \mid r[1](t) \mid \ldots \mid r[n](t) \mid \ldots$$

Consider the original time interval $0 \leq \mathrm{t} < \mathrm{T}$

$$s[0](t) \longrightarrow r[0](t) = s[0](t) + n[0](t)$$

For simplicity, we can omit the index [0]

$$s(t) \longrightarrow r(t) = s(t) + n(t)$$

Problem: $\boxed{\text{given } r(t) \rightarrow \text{recover } s(t)}$

The transmitted signal s(t) certainly belongs to the signal space S.

Does the received signal *r(t)* belong to *S*?

$$r(t) = s(t) + n(t)$$

This depends on *n(t)*.

In general, *n(t)* is a signal not in *S:* $n(t) \notin S$

Therefore, $\boxed{r(t) \notin S}$

Random variables n_j

We know that $n(t) \notin S$

roject this noise signal onto the orthonormal basis.

$$B = \left(b_j(t)\right)_{j=1}^{d}$$

The *j-th* projection component is:

$$n_j = \int_0^T n(t) b_j(t) dt$$

$$n_j = \int_0^T n(t) b_j(t) dt$$

We can prove that this component n_j này is

a Gaussian random variable:

- Mean $E[n_j]=0$
- Variance $\sigma^2=N_0/2$
- Statistically independent

$$n_j = \int_0^T n(t) b_j(t) dt$$

are Gaussian random variables,

which are achieved through linear transformation of a Gaussian process

$$n_j = \int_0^T n(t) b_j(t) dt$$

- Mean: $E[n_j] = 0$

$$E\left[n_j\right] = E\left[\int_0^T n(t) b_j(t) dt\right] = \int_0^T E\left[n(t)\right] b_j(t) dt = 0$$

$$n_j = \int_0^T n(t) b_j(t) dt$$

- Variance: $\sigma^2 = N_0/2$
- Linearly independent

$$E\left[n_j n_i\right] = E\left[\int_0^T n(t) b_j(t) dt \int_0^T n(x) b_i(x) dx\right] = E\left[\int_0^T \int_0^T n(t) n(x) b_j(t) b_i(x) dt dx\right] =$$

$$= \int_0^T \int_0^T E\left[n(t) n(x)\right] b_j(t) b_i(x) dt dx = \int_0^T \int_0^T \frac{N_0}{2} \delta(t-x) b_j(t) b_i(x) dt dx =$$

$$= \frac{N_0}{2} \int_0^T b_j(t) b_i(t) dt = \begin{cases} N_0/2 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

SOICT

Note: 2 Gaussian RVs have the property,
***E[n(t₁)n(t₂)]*= (N₀/2).δ(t₁-t₂)**

Random noise in the signal space

Given $n(t)$ we have the projection components onto the orthonormal basis:

$$n_j = \int_0^T n(t) b_j(t) dt$$

Let

$$\boxed{n_S(t) = \sum_j n_j b_j(t)}$$

Clearly, $n_S(t) \in$ S: it is the part of $n(t)$ belonging to S

In general:

$$n(t) \neq n_S(t)$$

SOICT

We have

$$\boxed{n(t) = n_S(t) + e(t)}$$

$e(t)$ = is the part of $n(t)$ **not** in S

Choose a time instant $t = t^*$

$\Downarrow$

$n_S(t^*)$ and $e(t^*)$
are statistically independent

Need to prove

$$E\left[n_S(t^*)e(t^*)\right] = 0 = E\left[n_S(t^*)\right]E\left[e(t^*)\right]$$

$$\Downarrow$$

$n_S(t^*)$ and $e(t^*)$

are statistically independent

$$\Downarrow$$

The noise component outside the space S is statistically independent.

Received signal in the signal space

We have proven $r(t) \notin S$

Project $r(t)$ onto the orthonormal basis:

$$B = \left(b_j(t)\right)_{j=1}^{d}$$

The j-th component is:

$$r_j = \int_0^T r(t) b_j(t) dt$$

Define $\boxed{\mathrm{r_S}(\mathrm{t}) = \sum_j \mathrm{r_j} \mathrm{b_j}(\mathrm{t})}$ we have $r_S(t) \in S$

In general $r(t) \neq r_S(t)$

But $r(t) = s(t) + n(t) = \underbrace{s(t) + n_S(t)}_{\in S} + \underbrace{e(t)}_{\notin S}$

Therefore, $r(t) = r_S(t) + e(t)$ with $r_S(t) = s(t) + n_S(t)$

SOICT

Decision problem in the signal space

$P1$

Original basic problem:
cho $r(t)=s(t)+n(t)$ → khôi phục $s(t)$

$P2$

Equivalent problem:
cho $r_S(t) = s(t)+n_S(t)$ → khôi phục $s(t)$

The only difference is the existence of $e(t)$:
The noise component not in S, and it is statistically independent of both $s(t)$ and $n_S(t)$

- $r_S(t)$ is a sufficient statistic for solving the problem
- Sufficient to solve the problem (determine the transmitted signal) in space S
- Other spatial dimensions contain no useful information, only noise

Decision problem: Vector representation

$\textcircled{P2}$

Problem

given $r_S(t) = s(t) + n_S(t)$ → recover $s(t)$

All the 3 signals belong to S

↓

Vector representation

$$\boxed{r_S(t) = s(t) + n_S(t)} \Rightarrow \boxed{\underline{r} = \underline{s}_T + \underline{n}}$$

$$\underline{r} = (r_1, \ldots, r_j, \ldots, r_d) \qquad r_j = \int_0^T r(t) b_j(t) dt$$

$$\underline{s_T} = (s_1, \ldots, s_j, \ldots, s_d) \qquad s_j = \int_0^T s(t) b_j(t) dt$$

$$\underline{n} = (n_1, \ldots, n_j, \ldots, n_d) \qquad n_j = \int_0^T n(t) b_j(t) dt$$

Received vector

The received vector $\underline{r}$ (in space S) has the representation:

$$\underline{r} = \underline{s}_T + \underline{n}$$

where $\underline{s}_T = (s_1, \dots, s_j, \dots, s_d) \in M$ is the transmitted signal

and $\underline{n} = (n_1, \dots, n_j, \dots, n_d)$ is the noise vector in space S

For each component of the vector we have: $r_j = s_j + n_j$

$$r_j = s_j + n_j$$

Therefore, the component r_j is Gaussian random variables with:

- Mean: $E[r_j]=s_j$
- Variance: $\sigma^2[r_j]=N_0/2$
- Statistically independent

$$\left(E[r_i r_j] = s_i s_j = E[r_i]E[r_j]\right)$$

$(P2)$ Problem:
given $r_S(t) = s(t) + n_S(t) \rightarrow$ recover $s(t)$

$\Downarrow$

$(P3)$ Problem:
given $\underline{r} = \underline{s}_T + \underline{n} \rightarrow$ recover $\underline{s}_T$

Important note:

given $r(t)$, the vector $\underline{r}$ is easily computed (since the orthonormal basis signals are known

Decision criterion

$\textcircled{P3}$ $\boxed{\text{Problem: given } \underline{r}=\underline{s}_T+\underline{n} \rightarrow \text{ recover } \underline{s}_T}$

At the receiver side, given $\underline{r}$

We want to choose the received signal $\boxed{\underline{s}_R \in M}$

Goal: make a correct decision: $\underline{s}_R = \underline{s}_T$

However, this is not always possible, due to the presence of noise.

given $\underline{r}$, we want to create decision criteria to determine $\underline{s}_R$

$\Downarrow$

Minimize the probability of symbol (signal) detection error

$$\boxed{P_S(e) = P(\underline{s}_R \neq \underline{s}_T)}$$

Decision criterion

$P3$ Problem:

given $\underline{r}=\underline{s}_T+\underline{n} \rightarrow$ recover $\underline{s}_T$

Suppose we receive $\underline{r}=\underline{\rho} \in R^d$

$\rightarrow$ choose $\underline{s}_R \in M$ such that $P_S(e)$ is minimized

Decision Criterion:

$C1$

$$\underline{s}_R = \arg\min_{\underline{s}_i \in M} \left[P(\underline{s}_R \neq \underline{s}_T \mid \underline{r} = \underline{\rho}) \right]$$

SOICT

Detection

The problem of deciding which possibility, among a set of possibilities, is true.

- A random variable X with m possible values occurring with a **priori probability**: $P(X=x)$
- We observe a variable Y connected to X by the probabilities:

  $P(Y=y|X=x)$, called **likelihoods**

When an experiment is performed, we obtain 2 samples:

$x \in X$ and $y \in Y$.

The decision maker will observe the value of y not x.

Given y, the observer makes a decision $d(y)=x'$

This decision is correct if $x'=x$

The accepted decision criterion to make decision $d(y)$:

Maximize correct decisions $P(x'= x)$

=

Minimize wrong decisions $P(x'\neq x)$

SOICT

MAP criterion

This is equivalent to the criterion:
a MAXIMUM A POSTERIORI (MAP)

$$\boxed{d(y) = \arg\max_{x} \left[P(X = x \mid Y = y) \right]}$$

MAP criterion

Proof:

$$P(X' \neq X) = \sum_x \sum_y P(X' \neq X, X = x, Y = y) =$$

$$= \sum_x \sum_y P(X' \neq X \mid X = x, Y = y) P(X = x, Y = y) =$$

$$= \sum_x \sum_y P(X'(y) \neq x \mid X = x, Y = y) P(X = x \mid Y = y) P(Y = y) =$$

$$= \sum_y \left[\sum_x (1 - \delta_{X'(y),x}) P(X = x \mid Y = y) \right] P(Y = y)$$

$$X'(y) = \arg\min_z \sum_x (1 - \delta_{z,x}) P(X = x \mid Y = y) = \arg\max_x P(X = x \mid Y = y)$$

SOICT

ML criterion

Bayes' Theorem $$P(X=x\,|\,Y=y)=\frac{P(Y=y\,|\,X=x)P(X=x)}{P(Y=y)}$$

$$d(y)=\arg\max_{x}\left[P(X=x\,|\,Y=y)\right]$$

$$\Rightarrow\quad d(y)=\arg\max_{x}\left[P(Y=y\,|\,X=x)P(X=x)\right]$$

Under the assumption $P(X=x)=\dfrac{1}{m}$

$$d(y)=\arg\max_{x}\left[P(Y=y\,|\,X=x)\right]$$

ML criterion

MAXIMUM LIKELIHOOD criterion

$$\boxed{d(y) = \arg\max_{x}\left[P(Y = y \mid X = x)\right]}$$

Decision problem at the receiver

RV X is the transmitted signal $\underline{s}_T \in M$

The observed RV Y is the received signal $\underline{r} = \underline{s}_T + \underline{n} \in S$

$$\underline{r} = \underline{s}_T + \underline{n}$$

The association between $\underline{r}$ and $\underline{s}_T$ $\quad f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i)$

This is a Gaussian probability density function with mean $\underline{s}_i$ and variance $N_0/2$ in each dimension.

Gaussian probability density function (PDF)

Example: r is a Gaussian random variable

- Mean: μ
- Variance: σ^2
- Probability density function (PDF):

$$f_r(\rho) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\rho - \mu)^2}{2\sigma^2}\right)$$

SOICT

Example: a pair of Gaussian random variables $r_1\ r_2$

- Mean μ
- Variance: σ^2
- Statistically independent
- PDF:

$$f_{r_1 r_2}(\rho_1\ \rho_2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\rho_1-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\rho_2-\mu)^2}{2\sigma^2}\right)$$

$$f_{r_1 r_2}(\rho_1\ \rho_2) = \frac{1}{(\sqrt{2\pi}\sigma)^2}\exp\left(-\frac{(\rho_1-\mu)^2+(\rho_2-\mu)^2}{2\sigma^2}\right)$$

Gaussian probability density function

$$f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i)$$

$\underline{r}$ = d-dimensional Gaussian RVs

- Mean: $\mu = s_{ij}$
- Variance: $\sigma^2 = N_0/2$
- Statistically independent
- PDF:

$$f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i) = \frac{1}{(\sqrt{\pi N_0})^d} \exp\left(-\frac{\sum_{j=1}^{d} (\rho_j - s_{ij})^2}{N_0}\right)$$

ML criterion

$$d(y) = \arg\max_{x} \left[P(Y = y \mid X = x) \right]$$

Becomes:

$$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s}_R = d\left(\underline{\rho}\right) = \arg\max_{\underline{s}_i \in M} \left[f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i) \right]}$$

$C2$

ML criterion

Expression $f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i)$

$$\underline{s}_R = \arg\max_{\underline{s}_i \in M} \left[\frac{1}{(\sqrt{\pi N_0})^d} \exp\left(-\frac{\sum_{j=1}^{d} (\rho_j - s_{ij})^2}{N_0} \right) \right]$$

$$\underline{s}_R = \arg\min_{\underline{s}_i \in M} \sum_{j=1}^{d} (\rho_j - s_{ij})^2$$

Minimum distance criterion

$$\underline{s_R} = \arg\min_{\underline{s}_i \in M} \sum_{j=1}^{d} (\rho_j - s_{ij})^2$$

By calculating the Euclidean distance between vectors in R^d:

$$d_E^2(\underline{\rho} - \underline{s}_i) = \sum_{j=1}^{d} (\rho_j - s_{ij})^2$$

We have:

$$\underline{s_R} = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)$$

SOICT

The ML criterion corresponds to the **minimum distance criterion**

$$C3 \quad \boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s}_R = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)}$$

SOICT

Voronoi region

$$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s}_R = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)}$$

This is the criterion associated with any vector $\underline{\rho} \in R^d$ representing the received signal $\underline{s}_R \in M$

We have the **Voronoi region (decision region)** $V(\underline{s}_i)$

= the set of all received vectors for which the choice is $\underline{s}_R = \underline{s}_i$

$$V(\underline{s}_i) = \left\{ \underline{\rho} \in R_d : \underline{s}_R = \underline{s}_i \right\}$$

Voronoi region

The set of received vectors used to make the choice: $\underline{s}_R = \underline{s}_i$

When do we have $\underline{s}_R = \underline{s}_i$ ***?***

→ When $\underline{\rho} \in R^d$ is closer to $\underline{s}$ than to all other signals in the signal space

$$\boxed{V(\underline{s}_i) = \{\underline{\rho} \in R^d : \; d_E^2(\underline{\rho}, \underline{s}_i) \le d_E^2(\underline{\rho}, \underline{s}) \quad \forall \underline{s} \in M\}}$$

Note:

If we receive $\underline{\rho} \in V(\underline{s}_i)$ $\Rightarrow$ Ta chọn $\underline{s_R} = \underline{s}_i$

minimum distance criterion

$$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s_R} = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)}$$

Can be represented by the Voronoi region criterion

C4 $$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{if} \quad \underline{\rho} \in V(\underline{s}) \ \text{choose } \underline{s_R} = \underline{s}}$$