ĐỀ THI THỬ GIỮA KỲ GIẢI TÍCH II 20192

Nhóm ngành 1, nhóm ngành 3
Thời gian làm bài: 60 phút

ĐÁP ÁN

Câu 1. $F(x,y,z) = x^2 - 4y^2 + 2z^2 - 6$
Ta có $F'_x(M) = 4; F'_y(M) = -16, F'_z(M) = 12$
Phương trình tiếp diện tại M là $4(x-2) - 16(y-2) + 12(z-3) = 0$
Phương trình pháp tuyến tại M là $\frac{x-2}{4} = \frac{y-2}{-16} = \frac{z-3}{12}$

Câu 2. $\begin{cases} x = 2(t - \sin t) \\ y = 2(1 - \cos t) \end{cases}$

Ta có $x' = 2(1-\cos t), y' = 2\sin t, x" = 2\sin t, y" = 2\cos t$

$$\begin{vmatrix} x' & y' \\ x" & y" \end{vmatrix} = 4(\cos t - \cos^2 t) - 4\sin^2 t = 4(\cos t - 1)$$

$$x'^2 + y'^2 = 4(1-\cos t)^2 + 4\sin^2 t = 8(1-\cos t)$$

$$C(M) = \left| \frac{\begin{vmatrix} x' & y' \\ x" & y" \end{vmatrix}}{(x'^2+y'^2)^{\frac{3}{2}}} \right| = \frac{4(1-\cos t)}{16\sqrt{2}(1-\cos t)^{\frac{3}{2}}} = \frac{1}{4\sqrt{2}\sqrt{1-\cos t}} = \frac{1}{4\sqrt{2}}$$

Câu 3. $(a+x)(y-c)^2 = x^2(x-a) (a \neq 0)$
Đặt $F(x,y,c) = (a+x)(y-c)^2 - x^2(x-a)$

Xét $\begin{cases} F(x,y,c) = 0 \\ F'_c(x,y,c) = 0 \end{cases} \Leftrightarrow \begin{cases} (a+x)(y-c)^2 - x^2(x-a) = 0 \\ -2(a+x)(y-c) = 0 \end{cases}$

$\Leftrightarrow \left[\begin{array}{l} y = c, x = 0 \\ y = c, x = a \end{array} \right.$

$\begin{cases} F'_x(x,y,c) = 0 \\ F'_y(x,y,c) = 0 \end{cases} \Leftrightarrow \begin{cases} (y-c)^2 - x^2 - 2x(x-a) = 0 \\ 2(a+x)(y-c) = 0 \end{cases}$

Ta có $F_x'^2(0,c,c) + F_y'^2(0,c,c) = 0, F_x'^2(a,c,c) + F_y'^2(a,c,c) = -a^2 \neq 0$
$\Rightarrow$ (a,c) là tập hợp điểm chính quy, và hình bao của họ đường cong là $x = a(a \neq 0)$

Câu 4.

$$I = \int_{-\pi}^{\pi} dx \int_0^{\cos x} f(x,y)dy$$
$$= -\int_{-\pi}^{-\frac{\pi}{2}} dx \int_{\cos x}^{0} f(x,y)dy + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \int_0^{\cos x} f(x,y)dy - \int_{\frac{\pi}{2}}^{\pi} dx \int_{\cos x}^{0} f(x,y)dy$$
$$= -\int_{-1}^{0} dy \int_{-\pi}^{-\arccos y} f(x,y)dx + \int_0^1 dy \int_{-\arccos y}^{\arccos y} f(x,y)dx$$
$$- \int_{-1}^{0} dy \int_{\arccos y}^{\pi} f(x,y)dx$$

Câu 5.

1)
$$I = \iint\limits_D (x+2y)dxdy, D \text{ giới hạn bởi } y = 2x^2, y = 1 + x^2$$

$$I = \int_{-1}^{1} dx \int_{2x^2}^{1+x^2} (x+2y)dy$$
$$= \int_{-1}^{1} (xy + y^2)\Big|_{2x^2}^{1+x^2} dx$$
$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx$$
$$= \left(-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x\right)\Big|_{-1}^{1} = \frac{32}{15}$$

2)

$$I = \iint\limits_D \frac{xy}{x^2+y^2}dxdy, D: \begin{cases} 2x \le x^2+y^2 \le 12 \\ x^2+y^2 \ge 2\sqrt{3}y \\ x, y \ge 0 \end{cases}$$

Đặt $\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \quad (r \ge 0) \Rightarrow |J| = r; x, y \ge 0 \Rightarrow 0 \le \phi \le \frac{\pi}{2}$

$$\begin{cases} 2x \le x^2+y^2 \le 12 \\ 2\sqrt{3}y \le x^2+y^2 \end{cases} \Rightarrow \begin{cases} 2r\cos\phi \le r^2 \le 12 \\ 2\sqrt{3}r\sin\phi \le r^2 \end{cases} \Rightarrow \begin{cases} 2\cos\phi \le r \le 2\sqrt{3} \\ 2\sqrt{3}\sin\phi \le r \le 2\sqrt{3} \end{cases}$$

Ta có $2\cos\phi \ge 2\sqrt{3}\sin\phi$ trên $\left[0, \frac{\pi}{6}\right]$, $2\cos\phi \le 2\sqrt{3}\sin\phi$ trên $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$
Vậy ta có $D = D_1 \cup D_2$ với
$D_1 : \begin{cases} 0 \le \phi \le \frac{\pi}{6} \\ 2\cos\phi \le r \le 2\sqrt{3} \end{cases}, D_2 : \begin{cases} \frac{\pi}{6} \le \phi \le \frac{\pi}{2} \\ 2\sqrt{3}\sin\phi \le r \le 2\sqrt{3} \end{cases}$

$$\Rightarrow I = \iint\limits_{D_1} r\sin\phi\cos\phi drd\phi + \iint\limits_{D_2} r\sin\phi\cos\phi drd\phi$$
$$= \int_0^{\frac{\pi}{6}} d\phi \int_{2\cos\phi}^{2\sqrt{3}} r\sin\phi\cos\phi dr + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\phi \int_{2\sqrt{3}\sin\phi}^{2\sqrt{3}} r\sin\phi\cos\phi dr$$
$$= \int_0^{\frac{\pi}{6}} \left(\frac{r^2}{2}\sin\phi\cos\phi\right)\Bigg|_{2\cos\phi}^{2\sqrt{3}} d\phi + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{r^2}{2}\sin\phi\cos\phi\right)\Bigg|_{2\sqrt{3}\sin\phi}^{2\sqrt{3}} d\phi$$
$$= \int_0^{\frac{\pi}{6}} \left(6\sin\phi\cos\phi - 2\sin\phi\cos^3\phi\right) d\phi + 6\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin\phi\cos\phi - \sin^3\phi\cos\phi) d\phi$$
$$= 6\int_0^{\frac{\pi}{2}} \sin\phi\cos\phi d\phi + 2\int_0^{\frac{\pi}{6}} \cos^3\phi d(\cos\phi) - 6\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^3\phi d(\sin\phi)$$
$$= 3 + \left(\frac{-7}{32}\right) - \frac{45}{32} = \frac{11}{8}$$

Câu 6. 1) Đặt $\begin{cases} x = r\cos\phi\sin\theta \\ y = 2r\sin\phi\sin\theta \\ z = 3r\cos\phi \end{cases} \quad (0 \le \phi \le 2\pi, 0 \le \theta \le \pi, 0 \le r \le 1)$
$J(r,\phi,\theta) = 1.2.3.(-r^2\sin\theta) = -6r^2\sin\theta$
Vậy
$$I = 6\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^1 r^2.r^2\sin\theta dr$$
$$= 6.2\pi \int_0^{\pi} \sin\theta d\theta \int_0^1 r^4 dr$$
$$= 6.2\pi.2.\frac{1}{5} = \frac{24\pi}{5}$$

2) Đặt $x = r\cos\phi, y = r\sin\phi (0 \le \phi \le 2\pi, r \ge 0)$ $V' : \begin{cases} 0 \le r \le 2 \\ 0 \le \phi \le 2\pi \\ \frac{r^2}{2} \le z \le 2 \end{cases}$

$$\begin{aligned} I &= \iiint_{V}' r^2.r\,dr\,d\phi\,dz \\ &= \int_0^{2\pi} d\phi \int_0^2 r^3 dr \int_{\frac{r^2}{2}}^2 dz \\ &= 2\pi \int_0^2 r^3 \left(2 - \frac{r^2}{2}\right) \\ &= 2\pi \left(\frac{2r^4}{4} - \frac{r^6}{12}\right)\Bigg|_0^2 = \frac{16}{3}\pi \end{aligned}$$

Câu 7. Miền lấy tích phân đối xứng qua $(Oxy), (Oyz), (Ozx)$
Xét miền $V^+ = \frac{1}{8}V(x, y, z \ge 0)$
Miền V^+ có hình chiếu xuống (Oxy) là

$$\left(\frac{x^2}{1} + \frac{y^2}{4}\right)^2 = x^2 - \frac{y^2}{4}(D)$$

và có $0 \le z \le 3\sqrt{1 - x^2 - \frac{y^2}{4}}$
Biến đổi $\begin{cases} x = r\cos\phi \\ y = 2r\sin\phi \end{cases}$ $|J| = 2r, 0 \le z \le 3\sqrt{1 - r^2}$
$\Rightarrow D' : r^4 = r^2\cos 2\phi \Rightarrow r^2 = 2\cos\phi$
$\cos 2\phi \ge 0 \Rightarrow \phi \in \left[0, \frac{\pi}{4}\right]$

$$\begin{aligned}
V &= 8V^+ = 8\int_0^{\frac{\pi}{4}} d\phi \int_0^{\sqrt{\cos 2\phi}} 3\sqrt{1-r^2} 2r dr \\
&= 8\int_0^{\frac{\pi}{4}} d\phi \left(-\int_0^{\sqrt{\cos 2\phi}} 3\sqrt{1-r^2} d(1-r^2)\right) \\
&= 8\int_0^{\frac{\pi}{4}} \left(-2(1-r^2)^{\frac{3}{2}}\Big|_0^{\sqrt{\cos 2\phi}}\right) d\phi \\
&= 8\int_0^{\frac{\pi}{4}} \left(-2(1-\cos 2\phi)^{\frac{3}{2}}+2\right) d\phi = 8\int_0^{\frac{\pi}{4}} \left(-2(2\sin^2\phi)^{\frac{3}{2}}+2\right) d\phi \\
&= 8\int_0^{\frac{\pi}{4}} \left(-4\sqrt{2}\sin^3\phi+2\right) d\phi = 8\int_0^{\frac{\pi}{4}} 4\sqrt{2}(1-\cos^2\phi) d(\cos\phi) + 8\frac{\pi}{2} \\
&= 32\sqrt{2}\left(\frac{5\sqrt{2}}{12}-\frac{2}{3}\right)+4\pi = \frac{80}{3}-\frac{64\sqrt{2}}{3}+4\pi
\end{aligned}$$

Câu 8

$$I = \int_0^1 \cos\left(\ln\frac{1}{x}\right)\frac{x^b - x^a}{\ln x} dx$$

Vì $\frac{x^b - x^a}{\ln x} = \int_a^b x^y dy$

$$\Rightarrow I = \int_0^1 \left(\int_a^b \cos\left(\ln\frac{1}{x}\right) x^y dy\right) dx$$

Hàm $F(x,y) = x^y \cos\left(\ln\frac{1}{x}\right)$ liên tục trong hình chữ nhật $[0,1] * [a,b]$ (khi $x = 0$, ta đặt $f(0,y) = 0$)
Bởi vậy, có thể thực hiện phép thế tích phân

$$I = \int_a^b dy \int_0^1 x^y \cos\left(\ln\frac{1}{x}\right) dx$$

Đặt $x = e^{-t} \Rightarrow dx = -e^{-t}$

$$\Rightarrow I = \int_a^b dy \int_0^{+\infty} e^{-ty} \cos t(e^{-t})$$

$$\Rightarrow I = \int_a^b dy \int_0^{+\infty} e^{-ty} \cos t(e^{-t}) dt$$
$$= \int_a^b dy \int_0^{+\infty} e^{-t(y+1)} \cos t dt$$

Xét

$$A = \int_0^{+\infty} e^{-t(y+1)} \cos t dt = \int_0^{+\infty} e^{-t(y+1)} d(\sin t)$$
$$= \sin t . e^{-t(y+1)} \Big|_0^{+} \infty - \int_0^{+\infty} \sin t d(e^{-t(y+1)})$$
$$= 0 - \int_0^{+\infty} \sin t.(-(y+1)) e^{-t(y+1)}) dt$$
$$= (y+1) \int_0^{+\infty} e^{-t(y+1)} \sin t dt$$
$$= (y+1) \int_0^{+\infty} e^{-t(y+1)} d(-\cos t)$$
$$= (y+1) \left[-e^{-t(y+1)} \cos t \Big|_0^{+\infty} + \int_0^{+\infty} \cos t.d \left(e^{-t(y+1)} \right) \right]$$
$$= (y+1) \left[1 - (y+1) \int_0^{+\infty} e^{-t(y+1)} \cos t dt \right]$$
$$= (y+1) [1 - (y+1) A]$$

$$\Rightarrow A = \frac{y+1}{1 + (y+1)^2}$$

Do đó:

$$I = \int_a^b \frac{y+1}{1+(y+1)^2} dy = \frac{1}{2} \ln[1+(y+1)^2] \Big|_a^b = \frac{1}{2} \ln \frac{b^2+2b+2}{a^2+2a+2}$$