Calculus 2

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Bibliography

- James Stewart, Calculus Early Transcendentals, Brooks Cole Cengage Learning, 2012.
  1. Vectors and the geometry of space: Chapter 12,
  2. Vector functions: Chapter 13,
  3. Multiple Integrals: Chapter 15,
  4. Line Integrals: Chapter 16,
  5. Surface Integrals: Chapter 16,
- http://bit.ly/bai-giang

1. Vectors and vector functions
   - Vectors
   - Equations of Lines and Planes
   - Cylinders and quadratic surfaces

2. Vector Functions
   - Vector functions and space curves
   - Curvature
   - Motion in space: Velocity and acceleration

Vector and Geometry of Space

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Vectors

The term vector is used to indicate a quantity that has both magnitude and direction (velocity, force,. . .).

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Definition

1. *A n-dimensional vector is an ordered* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *of real numbers. The numbers* $a_1, a_2, \ldots a_n$ *are called the components of* $\mathbf{a}$.
2. *2D,* $\mathbf{i} = (1, 0), \mathbf{j} = (0, 1)$: **standard basic vectors**.
3. *3D,* $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$: **standard basic vectors**.

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The length of vectors

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The length of vectors

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \Rightarrow |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Operations on vectors

$$\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \Rightarrow \mathbf{a} + \mathbf{b} = ?, \mathbf{a} - \mathbf{b} = ? c\mathbf{a} = ?$$

Vectors

Properties of vectors

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
3. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
4. $(cd)\mathbf{a} = c(d\mathbf{a})$
5. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
6. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
7. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
8. $1\ \mathbf{a} = \mathbf{a}$

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Dot product

If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

1. $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \Rightarrow \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$
2. $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \Rightarrow \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$

The dot product

Properties of the dot product

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
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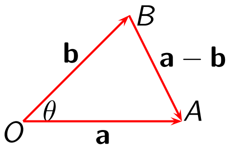
Theorem

If $\theta, 0 \le \theta \le \pi$, is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$.

Corollary:

1. $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$,
2. $\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$.



Vectors

Direction Angles and Direction Cosines

1. The **direction angles** of $\mathbf{a} \neq \mathbf{0}$ are the angles $\alpha, \beta, \gamma \in [0, \pi]$ that the vector **a** makes with the positive *x*- , *y*- , and *z*- axes.
2. The cosines of these direction angles, $\cos\alpha, \cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector **a**.

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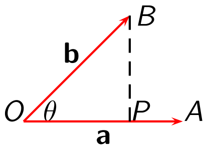
Projection

- The **scalar projection** of **b** onto **a** is $|\overrightarrow{OP}| = |\mathbf{b}|\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$.
- The **vector projection** of **b** onto **a** is $\overrightarrow{OP} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$.



Cross Product

$$\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \Rightarrow \mathbf{a} \times \mathbf{b} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right).$$

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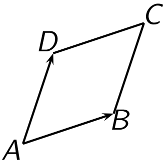
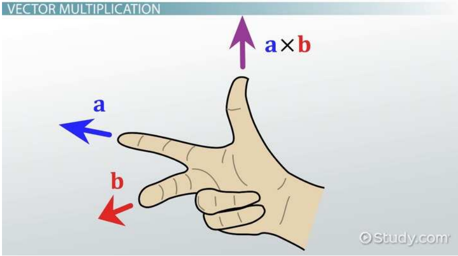
Properties

1. $\mathbf{a} \times \mathbf{b}$ is orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$.
2. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, where $\theta, 0 \le \theta \le \pi$, is the angle between $\mathbf{a}$, $\mathbf{b}$.
3. The direction of $\mathbf{a} \times \mathbf{b}$ is given by the right-hand rule.

Corollary.

1. $\mathbf{a} // \mathbf{b} \Leftrightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$,
2. $|\mathbf{a} \times \mathbf{b}|$ = the area of the parallelogram.





The cross product

Properties

If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Scalar triple product

Definition (Scalar triple product)

The scalar triple product of three vectors $\mathbf{a}, \mathbf{b}$, *and* $\mathbf{c}$, *denoted by* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, *is a number that is defined by* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Theorem

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \qquad (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{b}, \mathbf{c}, \mathbf{a}) = (\mathbf{c}, \mathbf{a}, \mathbf{b}) = -(\mathbf{b}, \mathbf{a}, \mathbf{c}).$$

Properties

1. The volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is $V = |(\mathbf{a}, \mathbf{b}, \mathbf{c})|$.
2. The vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are coplanar if and only if $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$.

Equations of Lines and Planes

Equations of Lines

A line L is determined by a point $P_0(x_0, y_0, z_0)$ on it and its direction vector $\mathbf{v} = (a, b, c)$.

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1. The vector equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$,
2. The parametric equation $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$,
3. The symmetric equation $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.

Equations of Planes

A plane P is determined by a point $P_0(x_0, y_0, z_0)$ in it and its normal vector $\mathbf{n} = (a, b, c)$.

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Equations of Planes

A plane P is determined by a point $P_0(x_0, y_0, z_0)$ in it and its normal vector $\mathbf{n} = (a, b, c)$.

1. The vector equations $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$,
2. The scalar equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Cylinders and quadratic surfaces

A quadratic surface is the graph of a second-degree equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

By translation and rotation it can be brought into one of two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0, \qquad Ax^2 + By^2 + Iz = 0.$$

Example

Ellipsoid $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1}$

Example

Elliptic paraboloid $\boxed{\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$ Hyperbolic paraboloid $\boxed{\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}$

Cylinders and quadratic surfaces

Example

Hyperboloid of one sheet $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$

Example

Hyperboloid of two sheets $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1}$

Example

Cone $\boxed{\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}}$

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Vector functions

Definition

A function $\mathbb{R} \to \mathbb{R}^n, t \mapsto \mathbf{r}(t) \in \mathbb{R}^n$ *is called a vector function,i.e.,* $\mathbf{r}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$.

- *If* $n = 2$, *then* $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.
- *If* $n = 3$, *then* $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

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Limit, continuity and derivative

1. $\lim\limits_{t \to t_0} \mathbf{r}(t) = \mathbf{a} \Leftrightarrow \lim\limits_{t \to t_0} |\mathbf{r}(t) - \mathbf{a}| = 0$

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4. **Continuity:** $\mathbf{r}(t)$ is continuous at t_0 if $\lim\limits_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$.

Derivatives of vector functions

Definition

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

Then $\mathbf{r}(t)$ *is differentiable at* t_0.

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Definition

If the vector function $\mathbf{r}(t)$ *is differentiable, i.e.* $\mathbf{r}'(t)$ *exists, then*

1. *The vector* $\mathbf{r}'(t)$ *is called the tangent vector to the curve* C.
2. *The unit tangent vector is defined by* $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}(t)|}$.

Derivatives of vector functions

Differentiation Rules

Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$,
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\frac{d}{dt}[\mathbf{u}(t)]$,
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$,
4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$,
5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$,
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$.

Integrals of vector functions

Integrals of vector functions

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b x(t)dt\right)\mathbf{i} + \left(\int_a^b y(t)dt\right)\mathbf{j} + \left(\int_a^b z(t)dt\right)\mathbf{k}.$$

Arc length of space curves

Let C be the curve given by $\mathbf{r} = \mathbf{r}(t)$, where $a \leq t \leq b$. Then the length of C is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt = \int_a^b |\mathbf{r}'(t)|dt.$$

The arc length function

The arc length function of the curve C is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, $a \leq t \leq b$,i.e.,

$$s(t) = \int_a^t |\mathbf{r}'(\tau)|d\tau \quad (\textbf{Note: } s'(t) = |\mathbf{r}'(t)|).$$

Curvature

Let $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ be the unit tangent vector. The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Thus we can define

Definition

The curvature of a curve is defined by

$$K = \left| \frac{d\mathbf{T}}{ds} \right|,$$

where $\mathbf{T}$ *is the unit tangent vector and* s *is the arc length function.*

Theorem

$$K = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

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$$K = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Curvature

Curvature of plane curves

1. $y = f(x) \Rightarrow K = \frac{|y''|}{(1+y'^2)^{3/2}}.$

2. $$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \Rightarrow K = \left| \frac{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}{(x'^2+y'^2)^{3/2}} \right|$$

Curvature of space curves

3. $$\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t) \end{cases} \Rightarrow K = \frac{\sqrt{\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}^2 + \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}^2 + \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^2}}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}.$$

The normal and binormal vectors

Let $\mathbf{r} = \mathbf{r}(t)$ be a smooth space curve.

1. The unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$,
2. $\mathbf{T}'(t) \perp \mathbf{T}(t) \Rightarrow$ the unit normal vector $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$,
3. $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the binormal vector. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector.

Motion in space: Velocity and acceleration

Suppose a particle moves on a smooth space curve C, defined by a vector function $\mathbf{r}(t)$.

Velocity and acceleration

1. The velocity vector $\mathbf{v}(t)$ is defined from

$$\mathbf{v}(t) = \mathbf{r}'(t).$$

2. The speed is $|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}$.
3. The acceleration vector $\mathbf{a}(t)$ is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.