Linear Algebra

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Chapter 3: Vector Space

1. Vector Space

2. Linear subspace
   - Basis and coordinates of linear spaces
   - Dimension of linear spaces

Vector Space

Definition

A vector space over a field $\mathbb{R}$ *is a set* V *together with two operations*

a) *The vector addition or simply addition*

$$\begin{aligned} + : V \times V &\to V \\ (\alpha, \beta) &\mapsto \alpha + \beta \end{aligned}$$

b) *The scalar multiplication*

$$\begin{aligned} \times : \mathbb{R} \times V &\to V \\ (a, \alpha) &\mapsto a\alpha \end{aligned}$$

that satisfy the eight axioms listed below.

Vector Space

Eight axioms

Axiom	Meaning
Associativity of addition	$(u+v)+w = u+(v+w)$
Commutativity of addition	$u+v = v+u$
Identity element of addition	$\exists 0 \in V : 0+v = v+0 = v$
Inverse elements of addition	$\forall v \in V, \exists v' \in V : v'+v = v+v' = 0$
Compatibility	$a(bv) = (ab)v$
Identity element of scalar multiplication	$1v = v$
Distributivity	$(a+b)v = av+bv$
Distributivity	$a(u+v) = au+av$

where $a, b \in \mathbb{R}, u, v, w \in V$.

Vector space

Example

Is V with operations a vector space?

a) $V = \{(x, y, z) \,|x, y, z \in \mathbb{R}\}$

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$$

$$k(x, y, z) = (|k|\, x, |k|\, y, |k|\, z)$$

b) $V = \{x = (x_1, x_2) \,|x_1 > 0, x_2 > 0\} \subset \mathbb{R}^2$

$$(x_1, x_2) + (y_1, y_2) = (x_1 y_1, x_2 y_2)$$

$$k(x_1, x_2) = (x_1^k, x_2^k)$$

where k is any real number.

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Linear subspace

Definition

Let V be a vector space over $\mathbb{R}$, and let $W \neq \emptyset$ be a subset of V. Then W is a subspace if

$$\begin{cases} u + v \in W, & \forall u, v \in W \\ av \in W, & \forall a \in \mathbb{R}, v \in W \end{cases}$$

Properties of subspaces

A way to characterize subspaces is that they are closed under linear combinations. That is, a nonempty set W is a subspace if and only if every linear combination of (finitely many) elements of W also belongs to W.

Span of vectors

Span of vectors

Let V be a linear space and $S = \{v_1, v_2, \ldots, v_n\} \subset V$.

i) A linear combination of vectors $v_1, v_2, \ldots, v_n$ is any vector of the form $c_1v_1 + c_2v_2 + \ldots c_nv_n$, where $c_1, \ldots, c_n \in \mathbb{R}$.

ii) The set of all possible linear combinations is called the span:

$$\text{span}(S) = \{c_1v_1 + c_2v_2 + \ldots c_nv_n | c_1, \ldots, c_n \in \mathbb{R}\}.$$

Theorem

$W = \text{span}(V)$ *is a subspace of* V.

Generators of vector space

Let V be a linear space and $S = \{v_1, v_2, \ldots, v_n\} \subset V$. If $\text{span}(S) = V$, then we say that S is a generator of V or V is generated by S.

Direct sum of linear subspaces

Example

Let V_1, V_2 be linear subspaces of V and
$V_1 + V_2 := \{x_1 + x_2 \,|x_1 \in V_1, x_2 \in V_2\}$. Prove that:

a) $V_1 \cap V_2$ is a linear subspace of V.

b) $V_1 + V_2$ is a linear subspace of V.

Definition

Let V_1, V_2 be linear subspaces of V. We say that V is a direct sum of V_1 and V_2 and write $V = V_1 \oplus V_2$ if $V_1 + V_2 = V, V_1 \cap V_2 = \{0\}$.

Example

Prove that $V = V_1 \oplus V_2$ if and only if each $v \in V$ has a unique representation $v = v_1 + v_2$, $(v_1 \in V_1, v_2 \in V_2)$.

Linear independence

Linear independence

A set of vectors $(v_1, \ldots, v_n)$ is said to be

a) *linearly independent* if the equation

$$a_1 v_1 + a_2 v_2 + \ldots a_n v_n = 0$$

can only be satisfied by $a_1 = a_2 = \ldots = a_n = 0$.

b) *linearly dependent* if otherwise. It means that there exist scalars $a_1, a_2, \ldots, a_k$, not all zero, such that $a_1 v_1 + a_2 v_2 + \ldots a_n v_n = 0$.

Notice that if not all of the scalars are zero, then at least one is non-zero, say a_1, in which case this equation can be written in the form

$$v_1 = \frac{-a_2}{a_1} v_2 + \cdots + \frac{-a_n}{a_1} v_n$$

Thus, v_1 is shown to be a linear combination of the remaining vectors.

Basis of linear space

Basis

A basis B of a vector space V is a linearly independent subset of V that spans V,i.e.,

i) the linear independence property,i.e.,

$$a_1 v_1 + a_2 v_2 + \dots a_n v_n = 0 \Leftrightarrow a_1 = a_2 = \dots = a_n = 0.$$

ii) the spanning property,

$$\forall v \in V, v = a_1 v_1 + a_2 v_2 + \dots a_n v_n.$$

Coordinates

The numbers a_i above are called the coordinates of the vector v with respect to the basis B, and by the first property they are uniquely determined.

Dimension of linear spaces

Definition

A vector space that has a finite basis is called finite-dimensional.

Theorem

Every vector space has a basis. All bases of a vector space have the same cardinality (number of elements), called the dimension of the vector space.

Definition

a) *The number of vectors in each basis of V is called the dimension of V, denoted by $\dim V$. If $V = \{0\}$, then $\dim V = 0$ by convention.*

b) *If V has no finite basis, then it is called infinite dimensional.*

Dimension of linear spaces

Example

Let $v_1 = (2, 0, 1, 3, -1)$, $v_2 = (1, 1, 0, -1, 1)$, $v_3 = (0, -2, 1, 5, -3)$, $v_4 = (1, -3, 2, 9, -5)$.

a) Find the dimension and a basis of $\text{span}(v_1, v_2, v_3, v_4)$.

b) Let $V_1 = \text{span}(v_1, v_2)$, $V_2 = \text{span}(v_3, v_4)$. Find the dimension and a basis of $V_1 + V_2$, $V_1 \cap V_2$.

Theorem

Let V_1, V_2 be finite dimensional spaces. Then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Homogeneous systems

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases} \tag{1}$$

or in matrix form $Ax = 0$.

Properties

i) The zero solution (or trivial solution): $x_1 = x_2 = \cdots = x_n = 0$.

ii) If $\det(A) \neq 0$ then it is also the only solution.

iii) If $\det(A) = 0$ then there is a solution set with an infinite number of solutions. Moreover:

   a) If u, v are solutions, then $u + v$ is also a solution.

   b) If v is a solution, then kv is also a solution for every $k \in \mathbb{R}$.

The solution set to a homogeneous system is a linear subspace of $\mathbb{R}^n$.

Homogeneous systems

Homogeneous $Ax = 0$ vs non-homogeneous systems $Ax = b$

If p is any specific solution to the linear system $Ax = b$, then the entire solution set can be described as

$$\{p + v : v \text{ is any solution to } Ax = 0\}.$$

Theorem

Let A be a $m \times n$ matrix. The dimension of the set of solutions of the homogeneous system $Ax = 0$ is

$$n - \operatorname{rank} A.$$

Rank of a set of vectors

Definition

Let $S = \{u_1, u_2, \ldots, u_p\} \subset V$. The maximum number of linearly independent vectors in S is called the rank of S and denoted by $r(S)$.

Rank of a set of vectors

Let $S = \{u_1, u_2, \ldots, u_p\} \subset V$.

i) $r = r(S) = \dim \operatorname{span}(S)$ and any r linearly independent vectors in S are a basis of $\operatorname{span}(S)$.

ii) If B is any basis of V, then $r(S) = r(A)$, where A is the coordinate matrix of S.

Change of basis

Problem

Let V be a n-dimensional linear space with two bases

$$\mathcal{B} = (e_1, e_2, \ldots, e_n) \text{ and } \mathcal{B}' = (e'_1, e'_2, \ldots, e'_n)$$

Denote by $[v]_{\mathcal{B}} = [v_1, v_2, \ldots, v_n]^T$ the coordinate vector of $v \in V$ (as column) in $\mathcal{B}$. Find the relation between $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{B}'}$

The transformation matrix

The matrix P that satisfies

$$[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'} \text{ for each } v \in V$$

is called the transformation matrix from $\mathcal{B}$ to $\mathcal{B}'$.

Change of basis

Theorem

For each pair of bases $\mathcal{B}$ and $\mathcal{B}'$ of V, the transformation matrix from $\mathcal{B}$ to $\mathcal{B}'$ is uniquely determined by

$$P = \left[[e_1']_{\mathcal{B}}[e_2']_{\mathcal{B}} \dots [e_n']_{\mathcal{B}}\right].$$

Theorem

If P is the transformation matrix from $\mathcal{B}$ to $\mathcal{B}'$, then

(a) P is invertible ($\det P \neq 0$),

(b) P^{-1} is the transformation matrix from $\mathcal{B}'$ to $\mathcal{B}$

Example

Let $P_3[x]$ and the standard basis $E = \{1, x, x^2, x^3\}$ and basis $B = \{1, 1+x, (1+x)^2, (1+x)^3\}$. Find the transformation matrix from E to B and B to E. Find the coordinates of $v = 2 + 2x - x^2 + 3x^3$ w.r.t B.