CHAPTER 4: PARAMETER ESTIMATION

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<tr><th colspan="2">POINT ESTIMATE</th></tr>
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<tr><th colspan="2">CONFIDENCE INTERVAL FOR POPULATION MEAN $\mathbb{E}[X] = \mu$</th></tr>
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<tr><th>σ^2: KNOWN</th><th>σ^2: UNKNOWN</th></tr>
<tr><td>$Z-$statistic: $Z = \dfrac{\overline{X} - \mu}{\sigma}\sqrt{n} \sim N(0,1)$
• Two-sided confidence interval for μ with a confidence level $1-\alpha$:
$\big(\overline{X} - z_{\frac{\alpha}{2}}\dfrac{\sigma}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}}\dfrac{\sigma}{\sqrt{n}}\big)$
where $\Phi(z_{\frac{\alpha}{2}}) = \mathbb{P}(Z < z_{\frac{\alpha}{2}}) = 1 - \dfrac{\alpha}{2}$.
Error of confidence interval estimation:
$\varepsilon = z_{\frac{\alpha}{2}}\dfrac{\sigma}{\sqrt{n}}$.
• One-sided confidence interval for μ with a confidence level $1-\alpha$:
<u>Case 1:</u> Lower interval $\big(-\infty, \overline{X} + z_{\alpha}\dfrac{\sigma}{\sqrt{n}}\big)$
<u>Case 2:</u> Upper interval $\big(\overline{X} - z_{\alpha}\dfrac{\sigma}{\sqrt{n}}, +\infty\big)$
where $\Phi(z_{\alpha}) = \mathbb{P}(Z < z_{\alpha}) = 1 - \alpha$.</td><td>$T-$statistic: $T = \dfrac{\overline{X} - \mu}{S}\sqrt{n} \sim t(n-1)$
• Two-sided confidence interval for μ with a confidence level $1-\alpha$:
$\big(\overline{X} - t_{\frac{\alpha}{2},n-1}\dfrac{S}{\sqrt{n}}, \overline{X} + t_{\frac{\alpha}{2},n-1}\dfrac{S}{\sqrt{n}}\big)$
where $\mathbb{P}(T > t_{\frac{\alpha}{2},n-1}) = \dfrac{\alpha}{2}$.
Error of confidence interval estimation:
$\varepsilon = t_{\frac{\alpha}{2},n-1}\dfrac{S}{\sqrt{n}}$.
• One-sided confidence interval for μ with a confidence level $1-\alpha$:
<u>Case 1:</u> Lower interval $\big(-\infty, \overline{X} + t_{\alpha,n-1}\dfrac{S}{\sqrt{n}}\big)$
<u>Case 2:</u> Upper interval $\big(\overline{X} - t_{\alpha,n-1}\dfrac{S}{\sqrt{n}}, +\infty\big)$
where $\mathbb{P}(T > t_{\alpha,n-1}) = \alpha$.</td></tr>
<tr><th colspan="2">POPULATION X is non-normal with $\mathbb{E}[X] = \mu$ and Var$(X) = \sigma^2$
Sample size n large enough: $n \geq 30$</th></tr>
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CONFIDENCE INTERVAL FOR POPULATION PROPORTION p
Let p be the population proportion. Let $\widehat{p}$ be the sample proportion. For n large enough (check $n\widehat{p} \geq 5$ and $n(1-\widehat{p}) \geq 5$), we have $Z = \dfrac{\widehat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \approx N(0,1)$ • Two-sided confidence interval for p with a confidence level $1-\alpha$: $\left(\widehat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}, \widehat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}\right)$. Error of confidence interval estimation: $\varepsilon = z_{\frac{\alpha}{2}}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}$ • One-sided confidence interval for p: **Case 1:** Lower interval $\left(0, \widehat{p} + z_{\alpha}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}\right)$ **Case 2:** Upper interval $\left(\widehat{p} - z_{\alpha}\sqrt{\frac{\widehat{p}(1-\widehat{p})}{n}}, 1\right)$

Problem 4.1. Let $(X_1, X_2, \ldots, X_n)$ be a random sample of size n taken from the population X.
(1) $X \sim Poi(\lambda)$.

(a) Find a method of moments estimator $\widehat{\lambda}$ of the parameter λ. Is the MoM estimator $\widehat{\lambda}$ unbiased ?

(b) Find the maximum likelihood estimator $\widetilde{\lambda}$ of λ. Is the MLE $\widetilde{\lambda}$ unbiased ?

(2) $X \sim Exp\left(\dfrac{1}{\lambda}\right)$.

(a) Find a method of moments estimator $\widehat{\lambda}$ of the parameter λ. Is the MoM estimator $\widehat{\lambda}$ unbiased ?

(b) Find the maximum likelihood estimator $\widetilde{\lambda}$ of λ. Is the MLE $\widetilde{\lambda}$ unbiased ?

(3) $X \sim N(\mu, \sigma^2)$.

(a) Find a method of moments estimator $(\widehat{\mu}, \widehat{\sigma}^2)$ of the parameter (μ, σ^2). Is the MoM estimator $(\widehat{\mu}, \widehat{\sigma}^2)$ unbiased ?

(b) Find the maximum likelihood estimator $(\widetilde{\mu}, \widetilde{\sigma}^2)$ of (μ, σ^2). Is the MLE $(\widetilde{\mu}, \widetilde{\sigma}^2)$ unbiased ?

Problem 4.2. The following measurements were recorded for the drying time, in hours, of a certain brand of latex paint:

$$3.4, 2.5, 4.8, 2.9, 3.6, 2.8, 3.3, 5.6, 3.7, 2.8, 4.4, 4.0, 5.2, 3.0, 4.8$$

Assume that the measurements represent a random sample taken from a normal population $N(\mu, \sigma^2)$.
(1) The standard deviation σ is supposed to be 0.9.

(a) Find a 95% confidence interval estimate of μ.

(b) Will the width of a confidence interval be narrower or wider for 99% confidence than 95% confidence ?

(2) The standard deviation σ is supposed to be unknown.

(a) Find a 99% confidence interval estimate of μ.

(b) Will the width of a confidence interval be narrower or wider for 95% confidence than 99% confidence ?

Problem 4.3. (a) The number of cars sold annually by used car salespeople is normally distributed with a standard deviation of 15. A random sample of 15 salespeople was taken, and the number of cars each sold is listed here:

$$79, 43, 58, 66, 101, 63, 79, 33, 58, 71, 60, 101, 74, 55, 88$$

Find the 95% confidence interval estimate of the population mean μ.

Problem 4.4. The heights of students at a college follow a normal distribution $N(\mu, \sigma^2)$. The following measurements were recorded for a sample of 20 students:

$$160.3, 162.1, 165.8, 161.1, 170.9, 165.0, 168.3, 166.9, 158.8, 166.8,$$
$$159.2, 167.7, 169.8, 169.4, 168.4, 160.7, 171.6, 167.3, 158.3, 163.4$$

(a) Find a 95% confidence interval estimate of μ.
(b) Find a 99% confidence interval estimate of μ.

Problem 4.5. The contents (X) of similar containers of sulfuric acid have a mean of μ and a standard deviation of 0.3 (liters). The content of a random sample of 40 similar containers was recorded and the data are shown here

X	[9.4,9.6)	[9.6,9.8)	[9.8,10.0)	[10.0,10.2)	[10.2,10.4)	[10.4,10.6)
Frequency	6	6	9	14	4	1

Find a 95% confidence interval estimate for μ.

Problem 4.6. A random sample of delivery times (X) for 35 deliveries to an address across town by a courier service was recorded. These data (in hours) are shown here

X	[3.5,4.0)	[4.0,4.5)	[4.5,5.0)	[5.0,5.5)	[5.5,6.0)	[6.0,6.5)
Frequency	4	5	9	8	6	3

Find a 98% confidence interval estimate for the average delivery time.

Problem 4.7. In a random sample of 500 families owning television sets in the city of Hamilton, Canada, it is found that 340 families subscribe to HBO.
(a) Find a 95% confidence interval for the actual proportion p of families with television sets in this city that subscribe to HBO.
(b) How large a sample is required if we want to be 95% confident that our estimate of p is

within 0.02 of the true value?
(c) What is the error of 90% confidence interval estimate of p?

Problem 4.8. (a) A random sample of 200 voters in a town is selected, and 114 are found to support an annexation suit. Find the 96% confidence interval for the fraction of the voting population favoring the suit.
(b) What can we assert with 96% confidence about the possible size of our error if we estimate the fraction of voters favoring the annexation suit to be 0.57?

Problem 4.9. The following data present the diameter of forged piston rings used in an automobile engine

X (mm)	23.94-23.96	23.96-23.98	23.98-24	24-24.02	24.02-24.04	24.04-24.06
Frequency	1	19	28	32	17	3

(1) Suppose that the diameters follow a normal distribution $N(\mu, \sigma^2)$. Find a 90% confidence interval on μ.
(2) A forged piston is called oversized if its diameter is bigger than 24.02 mm.

(a) Find a 95% confidence interval on the population proportion of oversized pistons.

(b) Find a 95% confidence interval on the maximum population proportion of oversized pistons.

(c) Find a 95% confidence interval on the minimum population proportion of oversized pistons.

Problem 4.10. The following data present the diameter of holes drilled by a machine

X (mm)	93.94-93.96	93.96-93.98	93.98-94	94-94.02	94.02-94.04	94.04-94.06
Frequency	4	68	120	136	60	12

(1) Suppose that the diameters follow a normal distribution $N(\mu, \sigma^2)$. Find a 95% confidence interval on μ.
(2) A hole is called undersized if its diameter is smaller than 93.98 mm.

(a) Find a 95% confidence interval on the population proportion of undersized holes.

(b) Find a 95% confidence interval on the maximum population proportion of undersized holes.

(c) Find a 95% confidence interval on the minimum population proportion of undersized holes.