LESSON 16

DISCRETE SYSTEM IN FREQUENCY DOMAIN

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❑ CONTENT

1. The frequency response of the system.
2. Determine the frequency response from the constant coefficient linear difference equation.

❑ Lesson Objectives

After completing this lesson, you will be able to understand the following topics:

- Concept, meaning and parameters of frequency response of discrete systems.
- Method for determining frequency response from constant coefficient linear difference equation.

1. Frequency response of a discrete-invariant linear system

- The excitation signal is a complex exponential signal:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$x(n) = Ae^{j\omega_0 n} \Rightarrow y(n) = \sum_{k=-\infty}^{\infty} h(k)\left[Ae^{j\omega_0 (n-k)}\right]$$

$$= A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega_0 k}\right]e^{j\omega_0 n}$$

$$\Rightarrow y(n) = AH(\omega_0)e^{j\omega_0 n}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

Expressions of Frequency Response

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k - j\sum_{k=-\infty}^{\infty} h(k)\sin\omega k$$

$$= H_R(\omega) + jH_I(\omega) = |H(\omega)|e^{j\theta(\omega)}$$

Magnitude Response

$$|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}$$

Phase Response

$$\theta(\omega) = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)}$$

$$x(n) = Ae^{j\omega_0 n} \quad \Rightarrow \quad y(n) = A.H(\omega_0).e^{j\omega_0 n} = A.|H(\omega)|.e^{j(\omega_0 n + \theta(\omega))}$$

For example, determine the response $y(n)$

$$x(n) = Ae^{-J\pi n/2}$$

$$h(n) = \left(\frac{1}{2}\right)^{n} u(n) \quad \Longrightarrow \quad H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$\omega = \frac{n}{2}: \qquad H\left(\frac{\pi}{2}\right) = \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}}e^{-j26.6^{o}}$$

$$\Rightarrow \qquad y(n) = A\left(\frac{2}{\sqrt{5}}e^{-j26.6^{o}}\right)e^{-j\pi n/2}$$

$$\Rightarrow \qquad y(n) = \frac{2}{\sqrt{5}}Ae^{j\left(\frac{\pi n}{2} - 26.6^{o}\right)}$$

Response of the real system to trigonometric functions

$$h(n) \text{ real} \Rightarrow \begin{matrix} |H(\omega)| = |H(-\omega)| \\ \theta(\omega) = -\theta(-\omega) \end{matrix}$$

$$\Rightarrow \begin{matrix} x_1(n) = Ae^{j\omega n} \rightarrow y_1(n) = A.H(\omega).e^{j\omega n} = A.|H(\omega)|.e^{j(\omega n + \theta(\omega))} \\ x_2(n) = Ae^{-j\omega n} \rightarrow y_2(n) = A.H(-\omega).e^{-j\omega n} = A.|H(\omega)|.e^{j(\omega n - \theta(\omega))} \end{matrix}$$

$$x(n) = \frac{1}{2}[x_1(n) + x_2(n)] = A\cos\omega n \quad \rightarrow \quad y(n) = A|H(\omega)|\cos[\omega n + \theta(\omega)]$$

$$x(n) = \frac{1}{j2}[x_1(n) - x_2(n)] = A\sin\omega n \quad \rightarrow \quad y(n) = A|H(\omega)|\sin[\omega n + \theta(\omega)]$$

- The response of the system to the sum of trigonometric functions

$$x(n) = \sum_{i=1}^{L} A_i\cos(\omega_i n + \varphi_i) \Rightarrow y(n) = \sum_{i=1}^{L} A_i|H(\omega_i)|\cos[\omega_i n + \varphi_i + \theta(\omega_i)]$$

2. Determination of frequency response from constant coefficient linear difference equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

- Take the Z transform on both sides of the differential equation:

$$\sum_{n=-\infty}^{\infty}\left[\sum_{k=0}^{N} a_k y(n-k)\right] z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=0}^{M} b_k x(n-k)\right] z^{-n}$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \Rightarrow \quad H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

- Represent $H(z)$ through zeros z_r and poles p_k

$$H(z) = H_0 \frac{\prod_{r=1}^{M}(z - z_r)}{\prod_{k=1}^{N}(z - p_k)} \quad \Rightarrow \quad H(e^{j\omega}) = H_0 \frac{\prod_{r=1}^{M}(e^{j\omega} - z_r)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

4. Summary

- The frequency response of a system represents the frequency domain relationship between the input signal and the output signal of the system, which characterizes the change (in terms of amplitude and phase) of the frequency components of the system. signal as it passes through the system.
- From the differential equation representing the system, the frequency response can be determined through the relationship with the transfer function.

5. Exercise 1

- An LTI system is described by the following differential equation:

$$y(n) = ay(n-1) + bx(n), \qquad 0 < a < 1$$

a. Determine the amplitude response and phase response of the system.

b. Select parameter b to $|H(\omega)| = 1$. Then plot the amplitude and phase spectrum with $a = 0.9$

c. Determine the output of the system with the input signal:

$$x(n) = 5 + 12\sin\left(\frac{\pi}{2}n\right) - 20\cos\left(\pi n + \frac{\pi}{4}\right)$$

Exercise 2

- Determine the frequency response of the following causal systems:

a. $y(n) = 2.x(n) + 3.x(n-1)$

b. $y(n) - 0.5y(n-1) = x(n)$

Exercise 3

- Determine $y(n)$, know

$$x(n) = 2 + 2.\cos\left(\frac{\pi n}{2}\right) + 2.\cos(\pi n)$$

a. $h(n) = rect_2(n)$

b. $h(n) = \{1, -1\}$

c. Compare the results of sentences a and b

Next lesson. Lesson 17

DIGITAL FILTERS CONCEPT

Tài liệu tham khảo:

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wish you all good study!