Fourier series

Nguyen Thu Huong

School of Applied Mathematics and Informatics
Hanoi University of Science and Technology

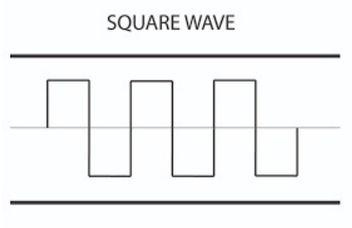
May 10, 2023

Content

Content

Recall: Taylor series

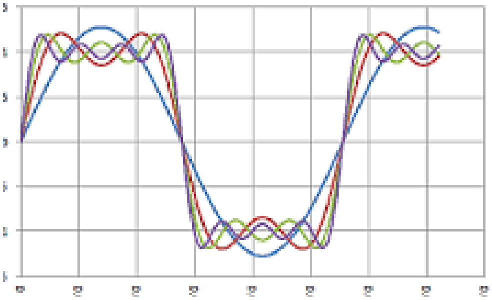
Expand an **infinitely differentiable** function $f(x)$ in a **neighborhood of** x_0 into power series of $x - x_0$.
Can we relax the smoothness of the function?
In physics, electrical engineering, one deals with periodic phenomenon.



What is Fourier series?

Fourier series is a mathematical way to express a **nontrigonometric periodic** function in terms of trigonometric functions, if $f(x)$ is periodic with period T, we want to expand

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega x + b_n \sin n\omega x \right), \; \omega = \frac{2\pi}{T}.$$



Trigonometric series

Basic case: $T = 2\pi$.

Definition

Trigonometric series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Proposition (Sufficient condition for convergence)

If $\sum_{n=1}^{\infty} |a_n|, \sum_{n=1}^{\infty} |b_n|$ *converge then the series above converges absolutely and uniformly on* $\mathbb{R}$.

Proof.

$|a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| \, \forall n \in \mathbb{N}, \, \forall x \in \mathbb{R}$.

Fourier series corresponding to a function $f(x)$

Let $f(x)$ be a periodic function with period 2π which is integrable over $[-\pi, \pi]$ and assume that we expand $f(x)$ into a trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

- Determine $a_n, b_n,\ n = 0, 1, 2 \ldots$?
- When does the series converge to $f(x)$?

Orthogonality properties for the sine and cosine functions

Lemma

For $m, n \in \mathbb{N}^*$, we have

$$\int_{-\pi}^{\pi} \sin mx dx = 0, \qquad \int_{-\pi}^{\pi} \cos mx dx = 0,$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0,$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

If $m = n \neq 0$:

$$\int_{-\pi}^{\pi} \sin^2 mx dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2mx) dx$$
$$= \frac{1}{2} \left(x - \frac{\sin 2mx}{2m} \right) \Big|_{-\pi}^{\pi} = \pi.$$

If $m \neq n$:

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x) dx$$
$$= \frac{1}{2} \left(\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right) \Big|_{-\pi}^{\pi} = 0.$$

Assume that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

we obtain (formally)

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nxdx + \int_{-\pi}^{\pi} b_n \sin nxdx\right)$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Multiplying (1) by $\sin mx$ then integrating over $[-\pi, \pi]$, we get

$$\int_{-\pi}^{\pi} f(x)\sin mxdx = \int_{-\pi}^{\pi} \frac{a_0}{2}\sin mxdx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx \sin mxdx + \int_{-\pi}^{\pi} b_n \sin nx \sin mxdx\right)$$

Hence, $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mxdx$; $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mxdx$.

Fourier series

Definition

Let $f(x)$ be a $2\pi-$periodic function and be integrable over $[-\pi, \pi]$. The Fourier series or Fourier expansion corresponding to $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2}$$

where the Fourier coefficients $\mathbf{a_n}, \mathbf{b_n}$ are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \ n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \ n \geq 1.$$

Content

Fourier expansion of $2l-$ periodic function

Assume that $f(x)$ is periodic with period $2l$.

Set $x' = \frac{\pi}{l}x$ and $f(x) = f\left(\frac{l}{\pi}x'\right) =: g(x')$.

It is obvious

$g(x' + 2\pi) = f\left(\frac{l}{\pi}(x' + 2\pi)\right) = f\left(\frac{l}{\pi}x' + 2l\right) = g(x')$,

and $g(x')$ is periodic with period 2π.

The Fourier series corresponding to $g(x')$ is

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx' + b_n \sin nx'$, the Fourier coefficients are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x') \cos nx' dx' = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x') \sin nx' dx' = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx.$$

Summary

Let f be a periodic with period $2l$. The Fourier series corresponding to $f(x)$ is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right),$$

where

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx$$

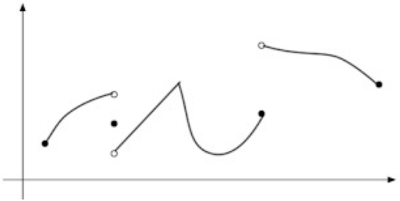
$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx,\ n\geq 1.$$

Theorem (Dirichlet conditions)

Let $f(x)$ be a periodic function with period $2l$ which is defined except possibly at a finite number of points in $(-l;l)$, and $f(x)$, $f'(x)$ are piecewise continuous in $(-l;l)$. Then the Fourier series corresponding to $f(x)$ converges to

- *$f(x)$ if x is a point of continuity,*
- *$\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity,*

where $f(x+0) = \lim\limits_{y \to x^+} f(y)$ and $f(x-0) = \lim\limits_{y \to x^-} f(y)$.



f is piecewise continuous in $[a, b]$ if there exist $a = x_0 < x_1 < \ldots < x_n = b$ such that $f(x)$ continuous in (x_{i-1}, x_i) and x_i is a point of discontinuity of the first type.

Example

Expand into Fourier series the periodic function $f(x)$ with period 2π and $f(x) = x$, $-\pi \leq x < \pi$.

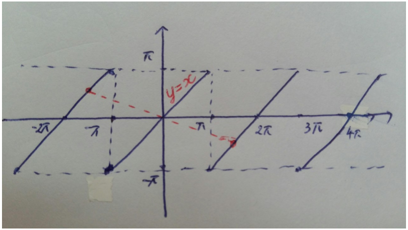
Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0;\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx dx = 0.$$

$$\begin{aligned} b_n =& \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx \\ =& \frac{2}{\pi}\int_{0}^{\pi} x \frac{d(-\cos nx)}{n} = \frac{2}{\pi}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}dx\right] \\ =& \frac{2}{\pi}\left[\frac{(-1)^{n+1}\pi}{n} + \frac{\sin nx}{n^2}\Big|_0^{\pi}\right] = \frac{2.(-1)^{n+1}}{n}. \end{aligned}$$

So $f(x) = \sum_{n=1}^{\infty} \frac{2.(-1)^{n+1}}{n} \sin nx, x \neq (2k+1)\pi, k \in \mathbb{Z}$.

The graph of $f(x)$.



At $x = \pi$, $f(\pi + 0) = -\pi$; $f(\pi - 0) = \pi$, the series converges to 0.
At $x = (2k + 1)\pi$, the series converges to 0.

Example

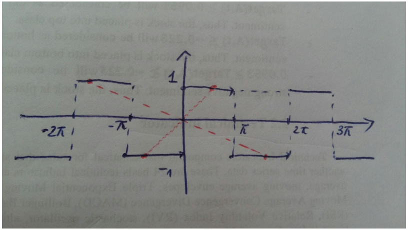
Expand into Fourier series the periodic function $f(x)$ with period 2π and
$$f(x) = \begin{cases} -1, & \text{if } -\pi \le x < 0, \\ 1, & \text{if } 0 \le x < \pi. \end{cases}$$

Fourier coefficients are
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)dx + \int_0^{\pi} dx\right] = \frac{1}{\pi}[-\pi + \pi] = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\cos nx)dx + \int_0^{\pi}\cos nx dx\right]$$
$$= \frac{1}{n\pi}\left[-\sin nx\,\Big|_{-\pi}^{0} + \sin nx\,\Big|_0^{\pi}\right] = 0.$$
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\sin nx)dx + \int_0^{\pi}\sin nx dx\right] = \frac{2(1-(-1)^n)}{n\pi}.$$

Hence, $f(x) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi}\sin nx, x \ne k\pi, k \in \mathbb{Z}.$

Graph of $f(x)$



$f(x)$ is odd so $a_n = 0, \forall n \geq 0$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\frac{-\cos(nx)}{n}\Big|_0^{\pi}$$

$$= \frac{2(1-(-1)^n)}{n\pi} = \begin{cases} 0 & \text{if } n = 2l\ , \\ \dfrac{4}{n\pi} & \text{if } n = 2l+1. \end{cases}$$

We get

$$f(x) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi}\sin nx = \sum_{l=0}^{\infty}\frac{4}{(2l+1)\pi}\sin(2l+1)x, x \neq k\pi.$$

At $x = k\pi$, the series converges to 0.

Example

Expand $f(x)$ to Fourier series $f(x)$, where $f(x)$ is an odd and 4−periodic function and $f(x) = 2 - x$, $x \in (0, 2)$.

$l = 2$. $f(x)$ is odd, $a_0 = a_n = 0$.

$$
\begin{aligned}
b_n &= \int_0^2 (2-x)\sin\frac{n\pi x}{2}dx = \int_0^2 (2-x)\frac{2}{n\pi}d\left(-\cos\frac{n\pi x}{2}\right) \\
&= \frac{2}{n\pi}\left[(x-2)\cos\frac{n\pi x}{2}\Big|_0^2 - \int_0^2 \cos\frac{n\pi x}{2}dx\right] \\
&= \frac{2}{n\pi}\left[2 - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2\right] = \frac{4}{n\pi}.
\end{aligned}
$$

Hence,

$$
f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi x}{2}, x \neq 2k.
$$

Fourier expansion of odd and even functions

- If $f(x)$ is an **odd function**: $a_0 = a_n = 0$,

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, n \geq 1$$

Fourier expansion $\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ consists of only sine functions.

- If $f(x)$ is an **even function**: $b_n = 0$,

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{d} x, n \geq 0.$$

Fourier expansion $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$ consists of only cosine functions.

Content

Let $f(x)$ be a function defined on the interval (a, b) that f and f' are piecewise continuous.
To expand $f(x)$ into Fourier series, we will construct a function $g(x)$ that satisfies

- $g(x) = f(x)$ for all $x \in (a, b)$(such $g(x)$ is called an extension of $f(x)$).
- $g(x)$ is periodic (with period $T \geq b - a$).

Fourier series corresponding to $g(x)$ is the Fourier series corresponding to $f(x)$ for $x \in (a, b)$.

Assume that $f(x)$ is defined on $(0, L)$.

1. Expand $f(x)$ into a Fourier cosine series:

   +) Consider $g(x) = f(|x|)$, $x \in (-L, L)$ and $g(x)$ is $2L-$periodic.

   +) $b_n = 0, a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}dx.$

   +) $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}.$

2. Expand $f(x)$ into a Fourier sine series:

   +) Consider $g(x) = \begin{cases} -f(-x), \text{ if } -L < x < 0 \\ f(x), \text{ if } 0 < x < L, \end{cases}$

   and $g(x)$ is $2L-$periodic.

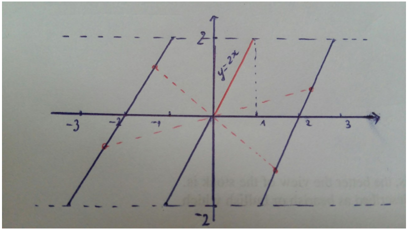
   +) $a_n = 0, b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$

   +) $f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$

Example

Expand $f(x) = 2x$, $0 \le x < 1$, into Fourier sine series.

Consider a function $g_1(x)$ whose graph is as follows.



$g_1(x) = f(x) = 2x$ for $0 \le x < 1$; $g_1(x)$ is an odd function and periodic with period $T = 2, l = 1$.

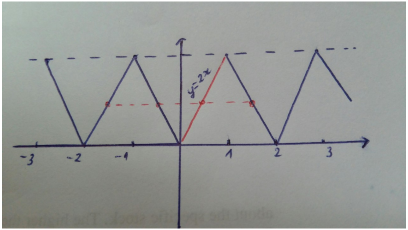
Fourier coefficients: $a_n = 0, b_n = 2\displaystyle\int_0^1 2x\sin(n\pi x)dx$, and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x), 0 \le x < 1.$$

Example

Expand $f(x) = 2x$, $0 \leq x < 1$, into Fourier cosine series.

Consider $g_2(x)$:



$g_2(x) = f(x) = 2x$ for $0 \leq x < 1$; $g_2(x)$ is an even function and periodic with $T = 2, l = 1$.

Fourier coefficients $b_n = 0, a_n = 2\int_0^1 2x\cos(n\pi x)dx$, and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x), 0 \leq x < 1.$$