HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
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EXERCISES OF PROBABILITY AND STATISTICS

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Contents

Chapter 1

PROBABILITY.

1.1 EXPERIMENTS

Problem 2.1.

A fax transmission can take place at any of three speeds depending on the condition of the phone connection between the two fax machines. The speeds are high (h) at 14400b/s, medium (m) at 9600b/s, and low (l) at 4800b/s. In response to requests for information, a company sends either short faxes of two (t) pages, or long faxes of four (f) pages. Consider the exp. of monitoring a fax trans and observing the trans speed and length. An observation is a two-letter word, e.g., a high-speed, two-page fax is ht.

(a) What is the sample space of the experiment?
(b) Let A_1 be the event "medium-speed fax." What are the outcomes in A_1?
(c) Let A_2 be the event "short fax." What are the outcomes in A_2?
(d) Let A_3 be the event "high-speed fax or low-speed fax." What are the outcomes in A_3?
(e) Are A_1, A_2, and A_3 mutually exclusive?
(f) Are A_1, A_2, and A_3 exhaustive?

Problem 2.2.

An integrated circuit factory has 3 machines X, Y, and Z. Test 1 integrated circuit produced by each machine. Either a circuit is acceptable (a) or fails (f). An observation is a sequence of 3 test results corr. to the circuits from machines X, Y, and Z, respectively. E.g., aaf is the observation that the circuits from X and Y pass and the circuit from Z fails.

(a) What are the elements of the sample space of this experiment?
(b) What are the elements of the sets Z_F = {circuit from Z fails}, X_A = {circuit from X is acceptable}.
(c) Are Z_F and X_A mutually exclusive?
(d) Are Z_F and X_A collectively exhaustive?
(e) What are the elements of the sets C = {more than one circuit acceptable}, D = {at least two circuits fail}.
(f) Are C and D mutually exclusive and exhaustive?

Problem 2.3.
Find out the birthday (month and day but not year) of a randomly chosen person. What is the sample space of the experiment. How many outcomes are in the event that the person is born in July?

Problem 2.4.
Let the sample space of the experiment consist of the measured resistances of two resistors. Give four examples of mutually exclusive and exhaustive set of events.

Problem 2.5.
An experiment involves tossing a pair of dice, one green and one red, and recording the numbers that come up. Let x equal the outcome on the green die and y the outcome on the red die.

(a) Describe the sample space S by listing the elements (x, y).
(b) List the elements corresponding to the event A that the sum is greater than 8.
(c) List the elements corresponding to the event B that a 2 occurs on either die.
(d) List the elements corresponding to the event C that a number greater than 4 comes up on the green die.
(e) List the elements corresponding to the events $A \cap B; B \cap C; C \cap A$.
(f) Construct a Venn diagram to illustrate the intersections and unions of the events A, B, and C.

Problem 2.6.
A coin is flipped tiwce. Let A be the event that the first flip is head, let B be the event that the second flip is head and let C be the event that the two flips are the same.

(a) Describe the sample space S.
(b) List the elements corresponding to the events $A \cap B; B \cap C; C \cap A$ and $A \cap B \cap C$.

1.2 Counting Methods.

Problem 2.7.
Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. How many different code words are there? How many code words have exactly three 0's?

Problem 2.8.
Consider a language containing four letters: A, B, C, D. How many three-letter words can you form in this language? How many four-letter words can you form if each letter appears only once in each word?

Problem 2.9.
On an American League baseball team with 15 field players and 10 pitchers, the manager must select for the starting lineup, 8 field players, 1 pitcher, and 1 designated hitter. A starting lineup specifies the players for these positions and the positions in a batting order for the 8 field players and designated hitter. If the designated hitter must be chosen among all the field players, how many possible starting lineups are there?

Problem 2.10.
A basketball team has three pure centers, four pure forwards, four pure guards, and one swingman who can play either guard or forward. A pure position player can play only the designated position. If the coach must start a lineup with one center, two forwards, and two guards, how many possible lineups can the coach choose?

Problem 2.11.
A train consists of four carriages. There are six passengers standing on the station platform.

(a) How many ways are there to arrange six passengers in four carriages?
(b) How many ways are there to arrange six passengers in four carriages such that the first carriage consists of 3 passengers, the second carriage consists of 2 passengers and the third carriage consists of 1 passenger?
(c) How many ways are there to arrange six passengers in four carriages such that one carriage consists of 3 passengers, one carriage consists of 2 passengers and one carriage consists of 1 passenger?
(d) How many ways are there to arrange six passengers in four carriages such that each carriage consists of at least one passenger?

Problem 2.12.
Four married couples have bought 8 seats in the same row for a concert. In how many different ways can they be seated

(a) with no restrictions?
(b) if each couple is to sit together?

Problem 2.13.
How many distinct permutations can be made from the letters of the word INFINITY?

Problem 2.14.
(a) In how many ways can 6 people be lined up to get on a bus?
(b) If 3 specific persons, among 6, insist on following each other, how many ways are possible?
(c) If 2 specific persons, among 6, refuse to follow each other, how many ways are possible?

1.3 Probability of events and Probability Rules.

Problem 2.15.
In a certain city, three newspapers A, B, and C are published. Suppose that 60 percent of the families in the city subscribe to newspaper A, 40 percent of the families subscribe to newspaper B, and 30 percent of the families subscribe to newspaper C. Suppose also that 20 percent of the families subscribe to both A and B, 10 percent subscribe to both A and C, 20 percent subscribe to both B and C, and 5 percent subscribe to all three newspaper A, B, and C. What percentage of the families in the city subscribe to at least one of the three newspapers?

Problem 2.16.
From a group of 3 freshmen, 4 sophomores, 4 juniors and 3 seniors a committee of size 4 is randomly selected. Find the probability that the committee will consist of

(a) 1 from each class;
(b) 2 sophomores and 2 juniors;
(c) Only sophomores and juniors.

Problem 2.17.
A box contains 24 light bulbs of which four are defective. If one person selects 10 bulbs from the box in a random manner, and a second person then takes the remaining 14 bulbs, what is the probability that all 4 defective bulbs will be obtained by the same person?

Problem 2.18.
Suppose that three runners from team A and three runners from team B participate in a race. If all six runners have equal ability and there are no ties, what is the probability that three runners from team A will finish first, second, and third, and three runners from team B will finish fourth, fifth, and sixth?

Problem 2.19.
Suppose that a school band contains 10 students from the freshman class, 20 students from the sophomore class, 30 students from the junior class, and 40 students from the senior class. If 15 students are selected at random from the band, what is the probability that at least one students from each of the four classes?

Problem 2.20.
Suppose that 10 cards, of which 5 are red and 5 are green, are placed at random in 10 envelopes, of which 5 are red and 5 are green. Determine the probability that exactly x envelopes will contain a card with a matching color $(x = 0, 1, 2, ..., 10)$.

Problem 2.21.
Consider two events A and B with $P(A) = 0.4$ and $P(B) = 0.7$. Determine the maximum and minimum possible values of $P(A \cap B)$ and the conditions under which each of these values is attained.

Problem 2.22.
Suppose that four guests check their hats when they arrive at a restaurant, and that these hats are returned to them in a random order when they leave. Determine the probability that no guest will receive the proper hat.

Problem 2.23.
Suppose that four guests check their hats when they arrive at a restaurant, and that these hats are returned to them in a random order when they leave. Determine the probability that at least 2 guests will receive the proper hat.

Problem 2.24.
Suppose that A, B and C are three independent events such that $P(A) = \frac{1}{4}, P(B) = \frac{1}{3}$ and $P(C) = \frac{1}{2}$.

(a) What is the probability that none of these three events will occur?
(b) Determine the probability that exactly one of these three events will occur.

Problem 2.25.
Three players A, B and C take turns tossing a fair coin. Suppose that A tosses the coin first, B tosses the second and C tosses third and cycle is repeated indefinitely until someone wins by being the first player to obtain a head. Determine the probability that each of the three players will win.

Problem 2.26.
Computer programs are classified by the length of the source code and by the execution time. Programs with ≥ 150 lines in the source code are big (B). Programs with ≤ 150 lines are little (L). Fast programs (F) run in less than 0.1s. Slow programs (W) require at least 0.1s. Monitor a program executed by a computer. Observe the length of the source code and the run time. The probability model for this experiment contains the following information: $P[LF] = 0.5, P[BF] = 0.2, P[BW] = 0.2$. What is the sample space of the experiment? Calculate the following probabilities:

(a) $P[W]$
(b) $P[B]$;
(c) $P[W \cup B]$.

Problem 2.27.
You have a six-sided die that you roll once and observe the number of dots facing upwards. What is the sample space? What is the probability of each sample outcome?What is the probability of E, the event that the roll is even?

Problem 2.28.
A student's score on a 10-point quiz is equally likely to be any integer between 0 and 10. What is the probability of an A, which requires the student to get a score of 9 or more? What is the probability the student gets an F by getting less than 4?

Problem 2.29.
Mobile telephones perform hand-offs as they move from cell to cell. During a call, a telephone either performs zero hand-offs (H0), one hand-off (H1), or more than one hand-off (H2). In addition, each call is either long (L), if it lasts more than three minutes, or brief (B). The following table describes the probabilities of the possible types of calls.

	H_0	H_1	H_2
L	0.1	0.1	0.2
B	0.4	0.1	0.1

(a) What is the probability $P[H_0]$ that a phone makes no hand-offs?
(b) What is the probability a call is brief?
(c) What is the probability a call is long or there are at least two hand-offs?

Problem 2.30.
Proving the following facts: (a) $P[A \cup B] \geq P[A]$; (b) $P[A \cup B] \geq P[B]$; (c) $P[A \cap B] \leq P[A]$; (d) $P[A \cap B] \leq P[B]$.

Problem 2.31.
Proving by induction the union bound: For any collection of events $A_1, ..., A_n$,

$$P[A_1 \cup A_2 \cup ... \cup A_n] \leq \sum_{i=1}^{n} P[A_i]$$

Problem 2.32.
A manufacturer of a flu vaccine is concerned about the quality of its flu serum. Batches of serum are processed by three different departments having rejection rates of 0.15, 0.12, and 0.09, respectively. The inspections by the three depart- ments are sequential and independent.

(a) What is the probability that a batch of serum survives the first departmental inspection but is rejected by the second department?
(b) Given that a batch of serum has been rejected, what is the probability that it is rejected by the second department?

Problem 2.33.
An aerospace company has submitted bids on two separate federal government defense contracts. The company president believes that there is a 40% probability of winning the first contract. If they win the first contract, the probability of winning the second is 70%. However, if they lose the first contract, the president thinks that the probability of winning the second contract decreases to 50%.

(a) What is the probability that they win both contracts?
(b) What is the probability that they lose both contracts?
(c) What is the probability that they win only one contract?

Problem 2.34.
An investor believes that on a day when the Dow Jones Industrial Average (DJIA) increases, the probability that the NASDAQ also increases is 77%. If the investor believes that there is a 60% probability that the DJIA will increase tomorrow, what is the probability that the NASDAQ will increase as well?

Problem 2.35.
A union's executive conducted a survey of its members to determine what the membership felt were the important issues to be resolved during upcoming negotiations with management. The results indicate that 74% of members felt that job security was an important issue, whereas 65% identified pension benefits as an important issue. Of those who felt that pension benefits were important, 60% also felt that job security was an important issue. One member is selected at random.

(a) What is the probability that he or she felt that both job security and pension benefits were important?
(b) What is the probability that the member felt that at least one of these two issues was important?

Problem 2.36.
A potential investor investigates the relationship between the performance of mutual funds and the university where the fund manager earned his/her MBA. Let A be the event that "a mutual fund outperformed the market" and let B be the event that "the fund manager graduated from top 20 MBA program". The joint probability table based on the data collected from 200 mutual funds for the investigation are given below:

	A	$\bar{A}$
B	0.11	0.29
$\bar{B}$	0.06	0.54

Select at random a mutual fund.

(a) What is the probability that the mutual fund outperformed the market?
(b) What is the probability that the mutual fund outperformed the market or the fund manager graduated from top 20 MBA program?
(c) What is the probability that the mutual fund outperformed the market given that the fund manager has graduated from top 20 MBA program?
(d) What is the probability that the mutual fund outperformed the market or the fund manager did not graduate from top 20 MBA program?

1.4 INDEPENDENCE

Problem 2.37.
Is it possible for A and B to be independent events yet satisfy $A = B$?

Problem 2.38.
Use a Venn diagram in which the event areas are proportional to their probabilities to illustrate two events A and B that are independent.

Problem 2.39.
In an experiment, A, B, C, and D are events with probabilities $P[A] = \frac{1}{4}$, $P[B] = \frac{1}{8}$, $P[C] = \frac{5}{8}$, and $P[D] = \frac{3}{8}$. Furthermore, A and B are disjoint, while C and D are independent.
(a) Find $P[A \cap B], P[A \cup B], P[A \cap B^c]$, and $P[A \cup B^c]$.
(b) Are A and B independent?
(c) Find $P[C \cap D], P[C \cap D^c]$, and $P[C^c \cap D^c]$.
(d) Are C^c and D^c independent?

Problem 2.40.
In an experiment, A, B, C, and D are events with probabilities $P[A \cup B] = \frac{5}{8}, P[A] = \frac{3}{8}, P[C \cap D] = \frac{1}{3}, P[C] = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find $P[A \cap B], P[B], P[A \cap B^c]$, and $P[A \cup B^c]$.
(b) Are A and B independent?
(c) Find $P[D], P[C \cap D^c], P[C^c \cap D^c], P[C|D]$.
(d) Find $P[C \cup D]$ and $P[C \cup D^c]$.
(e) Are C and D^c independent?

Problem 2.41.
Consider the data given in the problem 2.36

(e) Are the performance of mutual funds and the university where the fund manager earned his/her MBA independent?

1.5 Law of Total Probability and Bayes' Rule.

Problem 2.42.
Given the model of hand-offs and call lengths in Problem 2.29,
(a) What is the probability that a brief call will have no hand-offs?
(b) What is the probability that a call with one hand-off will be long?
(c) What is the probability that a long call will have one or more hand-offs?

Problem 2.43.

You have a six-sided die that you roll once. Let R_i denote the event that the roll is i. Let G_j denote the event that the roll is greater than j. Let E denote the event that the roll of the die is even-numbered.
(a) What is $P[R_3|G_1]$, the conditional probability that 3 is rolled given that the roll is greater than 1?
(b) What is the conditional probability that 6 is rolled given that the roll is greater than 3?
(c) What is $P[G_3|E]$, the conditional probability that the roll is greater than 3 given that the roll is even?
(d) Given that the roll is greater than 3, what is the conditional probability that the roll is even?

Problem 2.44.

You have a shuffled deck of three cards: 2, 3, and 4. You draw one card. Let C_i denote the event that card i is picked. Let E denote the event that card chosen is a even-numbered card.
(a) What is $P[C_2|E]$, the probability that the 2 is picked given that an even-numbered card is chosen?
(b) What is the conditional probability that an evennumbered card is picked given that the 2 is picked?

Problem 2.45.

Two different suppliers, A and B, provide a manufacturer with the same part. All suppliers of this part are kept in a large bin. In the past, 5 percent of the parts supplied by A and 9 percent of the parts supplied by B have been defective. A supplies four times as many parts as B. Suppose you reach into the bin and select a part and find it is non-defective. What is the probability that it was supplied by A?

Problem 2.46.

Suppose that 30 percent of the bottles produced in a certain plant are defective. If a bottle is defective, the probability is 0.9 that an inspector will notice it and remove it from the filling line. If a bottle is not defective, the probability is 0.2 that the inspector will think that it is defective and remove it from the filling line.
(a) If a bottle is removed from the filling line, what is the probability that it is defective?
(b) If a customer buys a bottle that has not been removed from the filling line, what is the probability that it is defective?

Problem 2.47.

Suppose that traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. In particular, the timing was designed so that with probability 0.75 a driver will find the second light to have the same color as the first. Assuming the first light is equally likely to be red or green.
(a) What is the probability that the second light is green?
(b) What is the probability that you wait for at least one light?

Problem 2.48.

A factory has three machines A, B, and C. Past records show that the machine A produced 40% of the items of output, the machine B produced 35% of the items of output, and machine C produced 25% of the items. Further 2% of the items produced by machine A were defective, 1.5% produced by machine B were defective, and 1% produced by machine C were defective.
(a) If an item is drawn at random, what is the probability that it is defective?
(b) An item is acceptable if it is not defective. What is the probability that an acceptable item comes from machine A?

Problem 2.49.

A financial analyst estimates that the probability that the economy will experience a recession in the next 12 months is 25%. She also believes that if the economy encounters a recession, the probability that her mutual fund will increase in value is 20%. If there is no recession, the probability that the mutual fund will increase in value is 75%. Find the probability that the mutual fund's value will increase.

Problem 2.50.

The Rapid Test is used to determine whether someone has HIV (the virus that causes AIDS). The false-positive and false-negative rates are 2.7% and 8%, respectively. A physician has just received the Rapid Test report that his patient tested positive. Before receiving the result, the physician assigned his patient to the low-risk group (defined on the basis of several variables) with only a 0.5% probability of having HIV. What is the probability that the patient actually has HIV?

Problem 2.51.
A customer-service supervisor regularly conducts a survey of customer satisfaction. The results of the latest survey indicate that 8% of customers were not satisfied with the service they received at their last visit to the store. Of those who are not satisfied, only 22% return to the store within a year. Of those who are satisfied, 64% return within a year. A customer has just entered the store. In response to your question, he informs you that it is less than 1 year since his last visit to the store. What is the probability that he was satisfied with the service he received?

Problem 2.52.
A telemarketer sells magazine subscriptions over the telephone. The probability of a busy signal or no answer is 65%. If the telemarketer does make contact, the probability of $0, 1, 2$, or 3 magazine subscriptions is $0.5, 0.25, 0.2$, and 0.05, respectively. Find the probability that in one call she sells no magazines.

Problem 2.53.
There are 3 boxes of marbles: the first box contains 3 red marbles, 2 white marbles; the second box contains 2 red marbles, 2 white marbles; the third box has no marbles. Draw randomly 1 marble from the first box and 1 marble from the second box and put them in the third box. Then, from the third box, 1 marble is drawn at random. Given that the marble drawn from the third box is red, what is the probability that the marble drawn from the first box is red?

1.6 Bernoulli formula.

Problem 2.54.
Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is a zero with probability 0.8, independent of any other bit.

(a) What is the probability of the code word 00111?
(b) What is the probability that a code word contains exactly three ones?

Problem 2.55.
Suppose each day that you drive to work a traffic light that you encounter is either green with probability $\frac{7}{16}$, red with probability $\frac{7}{16}$, or yellow with probability $\frac{1}{8}$, independent of the status of the light on any other day. If over the course of five days, G, Y, and R denote the number of times the light is found to be green, yellow, or red, respectively, what is the probability that $P[G = 2, Y = 1, R = 2]$? Also, what is the probability $P[G = R]$?

Problem 2.56.

We wish to modify the cellular telephone coding system in example below to reduce the number of errors. In particular, if there are 2 or 3 zeroes in the received sequence of 5 bits, we will say that a deletion (event D) occurs. Otherwise, if at least 4 zeroes are received, then the receiver decides a zero was sent. Similarly, if at least 4 ones are received, then the receiver decides a one was sent. We say that an error occurs if either a one was sent and the receiver decides zero was sent or if a zero was sent and the receiver decides a one was sent. For this modified protocol, what is the prob. $P[E]$ of an error? What is the prob. $P[D]$ of a deletion?