Iterated Integrals

Suppose that *f* is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

We use the notation $\int_c^d f(x, y)\, dy$ to mean that *x* is held fixed and $f(x, y)$ is integrated with respect to *y* from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.)

Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of *x*, so it defines a function of *x*:

$$A(x) = \int_c^d f(x, y)\, dy$$

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If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

1
$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

2
$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

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Similarly, the iterated integral

$$\boxed{3} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$.

Notice that in both Equations 2 and 3 we work *from the inside out*.

Example 1

Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Solution:

(a) Regarding *x* as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2}$$

$$= x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right)$$

$$= \tfrac{3}{2} x^2$$

Example 1 – *Solution*

cont'd

Thus the function *A* in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example.

We now integrate this function of *x* from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2 \, dx$$

$$= \frac{x^3}{2} \Big]_0^3$$

$$= \frac{27}{2}$$

Example 1 – *Solution*

(b) Here we first integrate with respect to *x*:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy$$

$$= 9 \frac{y^2}{2} \bigg]_1^2 = \frac{27}{2}$$

Iterated Integrals

Notice that in Example 1 we obtained the same answer whether we integrated with respect to y or x first.

In general, it turns out (see Theorem 4) that the two iterated integrals in Equations 2 and 3 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

Iterated Integrals

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

$\boxed{4}$ **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Iterated Integrals

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form.

To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$.

Then Fubini's Theorem gives

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

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In the inner integral, *y* is a constant, so *h*(*y*) is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)\,h(y)\,dx \right] dy = \int_c^d \left[h(y) \left(\int_a^b g(x)\,dx \right) \right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant.

Therefore, in this case, the double integral of *f* can be written as the product of two single integrals:

$$\boxed{5} \qquad \iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$