Introduction to Communications Engineering

IT4593E

Đỗ Công Thuần, Ph.D.

Dept. of CE, SoICT, HUST

Email: thuandc@soict.hust.edu.vn

ONE LOVE. ONE FUTURE.

Thông tin chung

- Tên học phần: **Nhập môn kỹ thuật truyền thông**
- Mã học phần: **IT4593E**
- Khối lượng: **2 TC (2-1-0-4)**
- Lý thuyết và bài tập: **10 buổi lý thuyết, 5 buổi bài tập**
- Đánh giá học phần:

  30% QT (kiểm tra + bài tập/project + chuyên cần-quiz)

  70% CK (trắc nghiệm + tự luận)
- Tài liệu tham khảo:
  - Lecture slides
  - Lecture notes
  - Textbooks, ví dụ ***Communication Systems Engineering***, 2nd Edition, by John G. Proakis Masoud Salehi
  - Internet

Lec 05:

Receiver Performance – Probability of Error

Communication over a Channel

Binary data sequence $\underline{u}_T$

Transmitted waveform $s(t)$

AWGN Channel

Received waveform $r(t) = s(t) + n(t)$

$$\boxed{\underline{u}_T \longrightarrow s(t) \longrightarrow r(t) = s(t) + n(t)}$$

Problem at the Receiver Side

$$\underline{\mathrm{u}}_{\mathrm{T}} \longrightarrow \mathrm{s}(t) \longrightarrow \mathrm{r}(t) = \mathrm{s}(t) + \mathrm{n}(t)$$

Problem $\boxed{\text{receive } \boldsymbol{r}(t) \rightarrow \text{recover } \underline{u}_T}$

Construct an orthonormal basis B from M (the space spanned by the $s_i(t)$)

Project the received waveform $r(t)$ onto B to form vector $\underline{r}$

Minimum distance criterion

$$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s}_R = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)}$$

Can be represented by the Voronoi Region criterion

C4 $$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{if} \quad \underline{\rho} \in V(\underline{s}) \ \text{choose } \underline{s}_R = \underline{s}}$$

Error probability

To determine the quality of a digital wireless link: we need to calculate the probability of detection error: **there are 2 types**

SYMBOL ERROR RATE = SER = $P_s(e)$ =

$$P_S(e) = P(\underline{s}_R[n] \neq \underline{s}_T[n])$$

BIT ERROR RATE = BER = $P_b(e)$ =

$$P(u_R[i] \neq u_T[i])$$

Some Concepts

R_b
Bit rate

$T_b = 1/ R_b$
Time to transmit 1 bit

$T = kT_b$
Time to transmit one symbol, assuming 1 symbol corresponds to k bits

$R = 1/T$
Symbol rate

E_b

Energy to transmit 1 bit

E_S

Energy to transmit 1 symbol

$S = E_b R_b = E_S R$

Signal power

N_0

Noise power spectral density (PSD)

B

Signal bandwidth

$N = N_0 B$

Noise power

S/N

Signal to Noise ratio

E_b/N_0

S/N ratio related to 1 bit of information, or in other words, the ratio of energy per bit to noise power spectral density

Relationship:

$$\frac{S}{N} = \frac{E_b}{N_0}\frac{R_b}{B} = \frac{E_b}{N_0}\eta$$

Where, $\eta = \frac{R_b}{B}$ (**spectral efficiency**)

SOICT

System performance is described as a function of E_b/N_0

This ratio is proportional to the received signal power

$$S = \frac{S}{N} N = \frac{E_b}{N_0} \frac{R_b}{B} N_0 B = \frac{E_b}{N_0} R_b N_0$$

SER computation

Concept: $P_S(e) = P(\underline{s}_R \neq \underline{s}_T)$

We can express:

$$P_S(e) = \sum_{i=1}^{m} P_S(e \mid \underline{s}_T = \underline{s}_i) P(\underline{s}_T = \underline{s}_i) = \frac{1}{m} \sum_{i=1}^{m} P_S(e \mid \underline{s}_T = \underline{s}_i)$$

Therefore, we need to calculate:

$$P_S(e \mid \underline{s}_T = \underline{s}_i) = P(\underline{s}_R \neq \underline{s}_T \mid \underline{s}_T = \underline{s}_i)$$

SER computation

First expression:

$$P_S(e\,|\,\underline{s}_T=\underline{s}_i)=P(\underline{s}_R\neq\underline{s}_T\,|\,\underline{s}_T=\underline{s}_i)=1-P(\underline{s}_R=\underline{s}_T\,|\,\underline{s}_T=\underline{s}_i)=$$

$$=1-P(\underline{\rho}\in V(\underline{s}_i)\,|\,\underline{s}_T=\underline{s}_i)$$

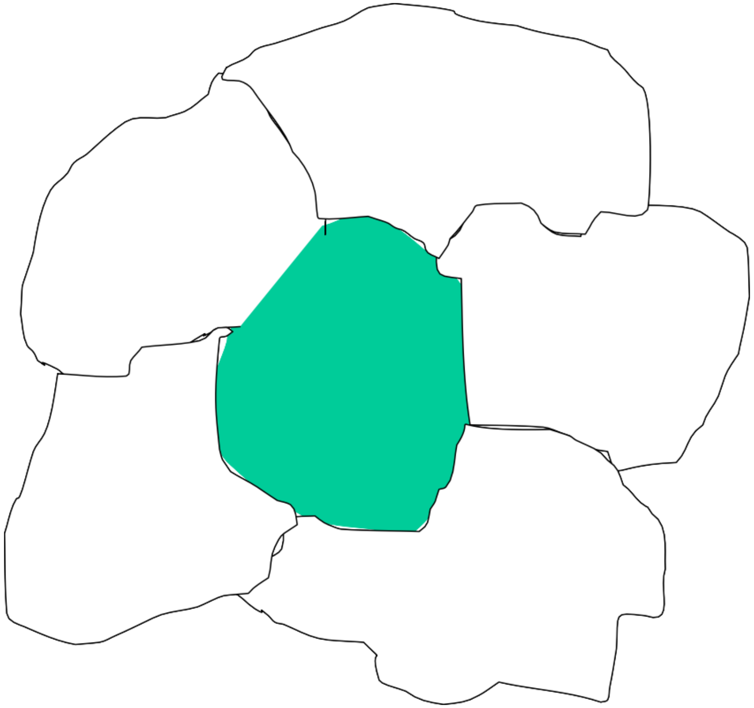
Second expression:

$$P_S(e\,|\,\underline{s}_T=\underline{s}_i)=P(\underline{s}_R\neq\underline{s}_T\,|\,\underline{s}_T=\underline{s}_i)=P(\underline{\rho}\notin V(\underline{s}_i)\,|\,\underline{s}_T=\underline{s}_i)=$$

$$=\sum_{j\neq i}P(\underline{s}_R=\underline{s}_j\,|\,\underline{s}_T=\underline{s}_i)=\sum_{j\neq i}P(\underline{\rho}\in V(\underline{s}_j)\,|\,\underline{s}_T=\underline{s}_i)$$

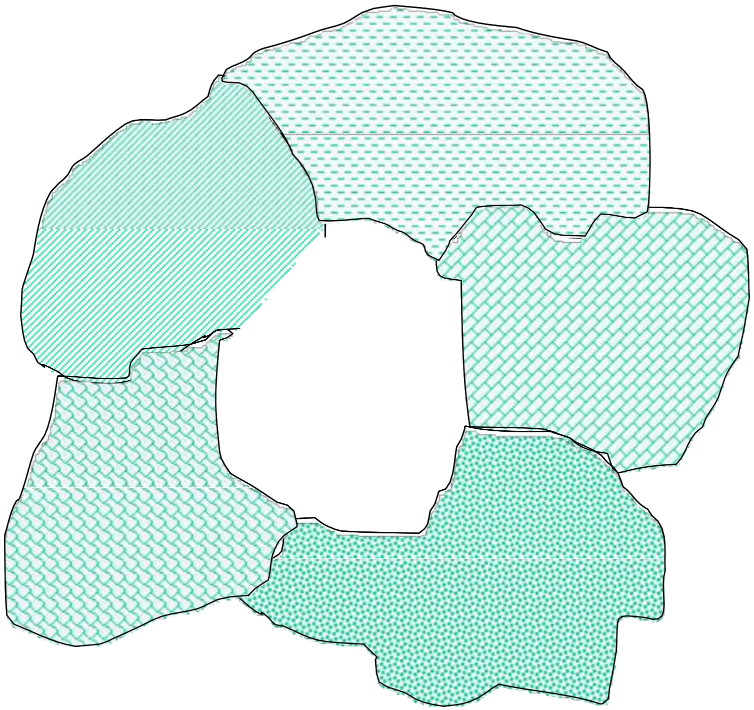
First expression:

$$\boxed{P_S(e \mid \underline{s}_T = \underline{s}_i) = 1 - P(\underline{\rho} \in V(\underline{s}_i) \mid \underline{s}_T = \underline{s}_i)}$$



Second expression:

$$P_S(e \mid \underline{s}_T = \underline{s}_i) = P(\underline{\rho} \notin V(\underline{s}_i) \mid \underline{s}_T = \underline{s}_i) = \sum_{j \neq i} P(\underline{\rho} \in V(\underline{s}_j) \mid \underline{s}_T = \underline{s}_i)$$



SOICT

BER computation

When the received signal is correct ($\underline{s}_R = \underline{s}_T$), then the binary sequence (the data of interest) will be correct ($\underline{v}_R = \underline{v}_T$).

When the received signal is wrong ($\underline{s}_R \neq \underline{s}_T$), then the received binary sequence will certainly also be wrong ($\underline{v}_R \neq \underline{v}_T$), ut the number of erroneous bits will depend on the **Hamming labeling** and is represented by:

$$\frac{d_H(\underline{v}_R, \underline{v}_T)}{k}$$

Where d_H is the Hamming distance between $\underline{v}_R$ and $\underline{v}_T$ (number of differing bits between these two vectors/bit clusters)

BER computation

We have $$P_b(e) = \frac{1}{m}\sum_{i=1}^{m} P_b(e \mid \underline{s_T} = \underline{s_i})$$

Where $$P_b(e \mid \underline{s_T} = \underline{s_i}) = \sum_{j \neq i} P_b(e, \underline{s_R} = \underline{s_j} \mid \underline{s_T} = \underline{s_i}) =$$

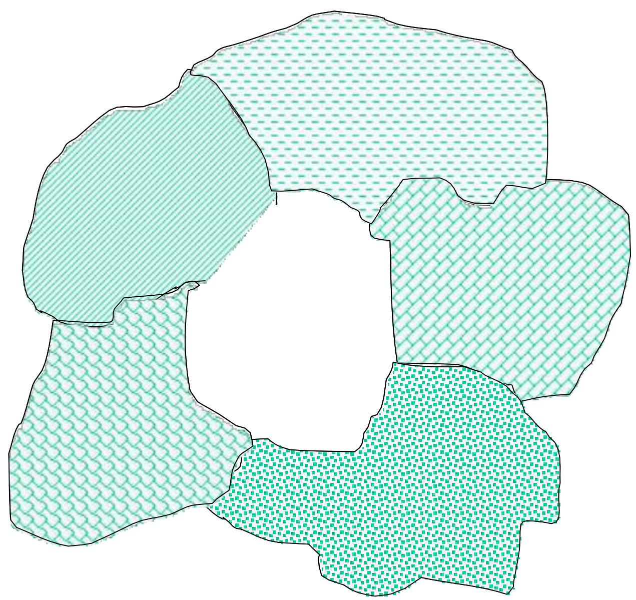
$$= \sum_{j \neq i} \frac{d_H(\underline{v_j}, \underline{v_i})}{k} P(\underline{s_R} = \underline{s_j} \mid \underline{s_T} = \underline{s_i}) =$$

$$= \sum_{j \neq i} \frac{d_H(\underline{v_j}, \underline{v_i})}{k} P(\underline{\rho} \in V(\underline{s_j}) \mid \underline{s_T} = \underline{s_i})$$

$$\left[\text{where } \underline{v_i} = e^{-1}\left(\underline{s_i}\right) \text{ and } \underline{v_j} = e^{-1}\left(\underline{s_j}\right)\right]$$

$$P_b(e) = \frac{1}{m}\sum_{i=1}^{m} P_b(e \mid \underline{s}_T = \underline{s}_i)$$

$$P_b(e \mid \underline{s}_T = \underline{s}_i) = \sum_{j \neq i} \frac{d_H(\underline{v}_j\ , \underline{v}_i)}{k}\ P(\underline{\rho} \in V(\underline{s}_j) \mid \underline{s}_T = \underline{s}_i)$$



Introduction: The erfc function

Consider a Gaussian random variable n with

- Mean: μ
- Variance: σ^2
- PDF:

$$f_n(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

We have

$$P(n > x) = \int_x^{+\infty} f_n(x)\, dx = \frac{1}{2} erfc\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$$

The erfc function

With the definition $\mathrm{erfc}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_x^{+\infty} e^{-t^2} dt$

We have

$$\mathrm{P}(n > x) = \int_x^{+\infty} f_n(x) dx = \int_x^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})\, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{(x-\mu)}{\sqrt{2}\sigma}}^{+\infty} e^{-t^2} dt = \frac{1}{2} \mathrm{erfc}(\frac{x-\mu}{\sqrt{2}\sigma})$$

In the case of mean = 0 and variance = $N_0/2$, we have:

$$P(n > x) = \frac{1}{2} \mathrm{erfc}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) = \frac{1}{2} \mathrm{erfc}(\frac{x}{\sqrt{N_0}})$$

SER/BER computation for binary antipodal signals

Consider a 1-dimensional signal space (d=1) with 2 signals (m=2), symmetric about the origin:

$$M = \{\underline{s_1} = (+A), \underline{s_2} = (-A)\}$$

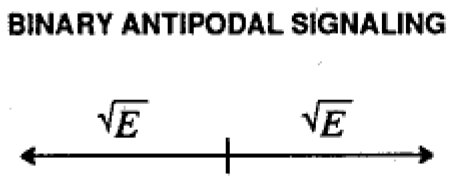
The Voronoi region for each signal is defined as follows:

$$V(\underline{s_1}) = \{\underline{\rho} = (\rho_1), \rho_1 \geq 0\}$$

$$V(\underline{s_2}) = \{\underline{\rho} = (\rho_1), \rho_1 \leq 0\}$$

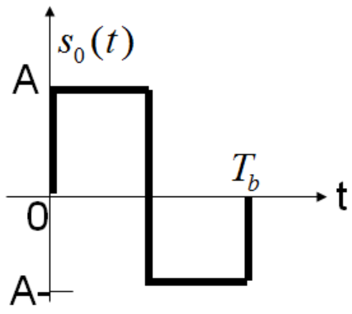
SOICT

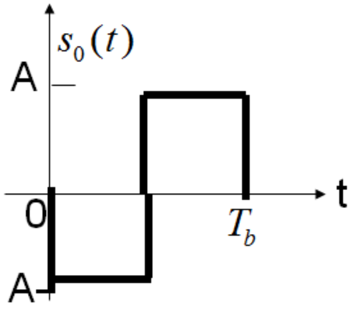
Antipodal signals it's that signal 180 degree opposite to each other. One signal have value on -1 and other on 1 (as exapthis is the mathematical from):



This signal shown in left, telecommunication used it in digital systems like B-PSK (Binary Phase Shift Keying); the idea from this signal in telecom receiver to know if we get 0 or we get 1. If the value are closer to +E the receiver understand this as 1; if the value are closer to -E the receiver understand it as 0.

NOTE: There is another form for antipodal signals like this:





We have:

$$P_S(e) = \frac{1}{m}\sum_{i=1}^{m} P_S(e \mid \underline{s_T} = \underline{s_i}) = \frac{1}{2}\Big[P_S(e \mid \underline{s_T} = \underline{s_1}) + P_S(e \mid \underline{s_T} = \underline{s_2})\Big]$$

Therefore, we need to calculate:

$$P_S(e \mid \underline{s_T} = \underline{s_1})$$

And:

$$P_S(e \mid \underline{s_T} = \underline{s_2})$$

For $\underline{s}_T = \underline{s}_1$

$$P_S(e \mid \underline{s}_T = \underline{s}_1) = P(\underline{\rho} \in V(\underline{s}_2) \mid \underline{s}_T = \underline{s}_1) = P(\rho_1 < 0 \mid \underline{s}_T = \underline{s}_1)$$

We have:

$$\boxed{\underline{r} = \underline{s}_T + \underline{n} \qquad \underline{r} = \underline{\rho} \qquad \underline{s}_T = \underline{s}_1}$$

Where $\underline{\rho} = (\rho_1)$ $\qquad \underline{s}_1 = (s_{11}) = (+A)$ $\qquad \underline{n} = (n_1)$

Therefore:

$$\rho_1 = A + n_1$$

$$P_S(e \mid \underline{s}_T = \underline{s}_1) = P(\rho_1 < 0 \mid \underline{s}_T = \underline{s}_1) = P(A + n_1 < 0) = P(n_1 < -A)$$

n_1 is a Gaussian random variable, with mean = 0 and variance = $N_0/2$

$$P_S(e \mid \underline{s}_T = \underline{s}_1) = P(n_1 < -A) = P(n_1 > A) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}} \right)$$

For $\underline{s}_T = \underline{s}_2$

$$P_S(e \mid \underline{s_T} = \underline{s_2}) = P(\underline{\rho} \in V(\underline{s_1}) \mid \underline{s_T} = \underline{s_2}) = P(\rho_1 > 0 \mid \underline{s_T} = \underline{s_2})$$

We have:

$$\underline{r} = \underline{s_T} + \underline{n} \qquad \underline{r} = \underline{\rho} \qquad \underline{s_T} = \underline{s_2}$$

Therefore:

$$\underline{\rho} = (\rho_1) \qquad \underline{s_2} = (s_{21}) = (-A) \qquad \underline{n} = (n_1)$$

$$\rho_1 = -A + n_1$$

SOICT

$$\mathrm{P_S(e \mid \underline{s_T} = \underline{s_2}) = P(-A + n_1 > 0)} = P(n_1 > A)$$

$$P_S(e \mid \underline{s_T} = \underline{s_2}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

We have $$\mathrm{P_S(e \mid \underline{s_T} = \underline{s_1}) = P_S(e \mid \underline{s_T} = \underline{s_2})}$$

Therefore:

$$P_S(e) = \frac{1}{2}\left[P_S(e \mid \underline{s_T} = \underline{s_1}) + P_S(e \mid \underline{s_T} = \underline{s_2})\right] = P_S(e \mid \underline{s_T} = \underline{s_1})$$

Therefore:

$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

Note:

$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{d}{2\sqrt{N_0}}\right)$$

We have:

$$P_S(e) = P_S(e \mid \underline{s}_T = \underline{s}_1) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

Write as a function of E_b/N_0:

$$E(\underline{s}_1) = E(\underline{s}_2) = A^2$$

$$E_S = \frac{E(\underline{s}_1) + E(\underline{s}_2)}{2} = A^2$$

$$E_b = \frac{E_S}{k} = E_S = A^2$$

We have:

$$P_S(e) = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

For this signal space, we can establish a binary labeling scheme:

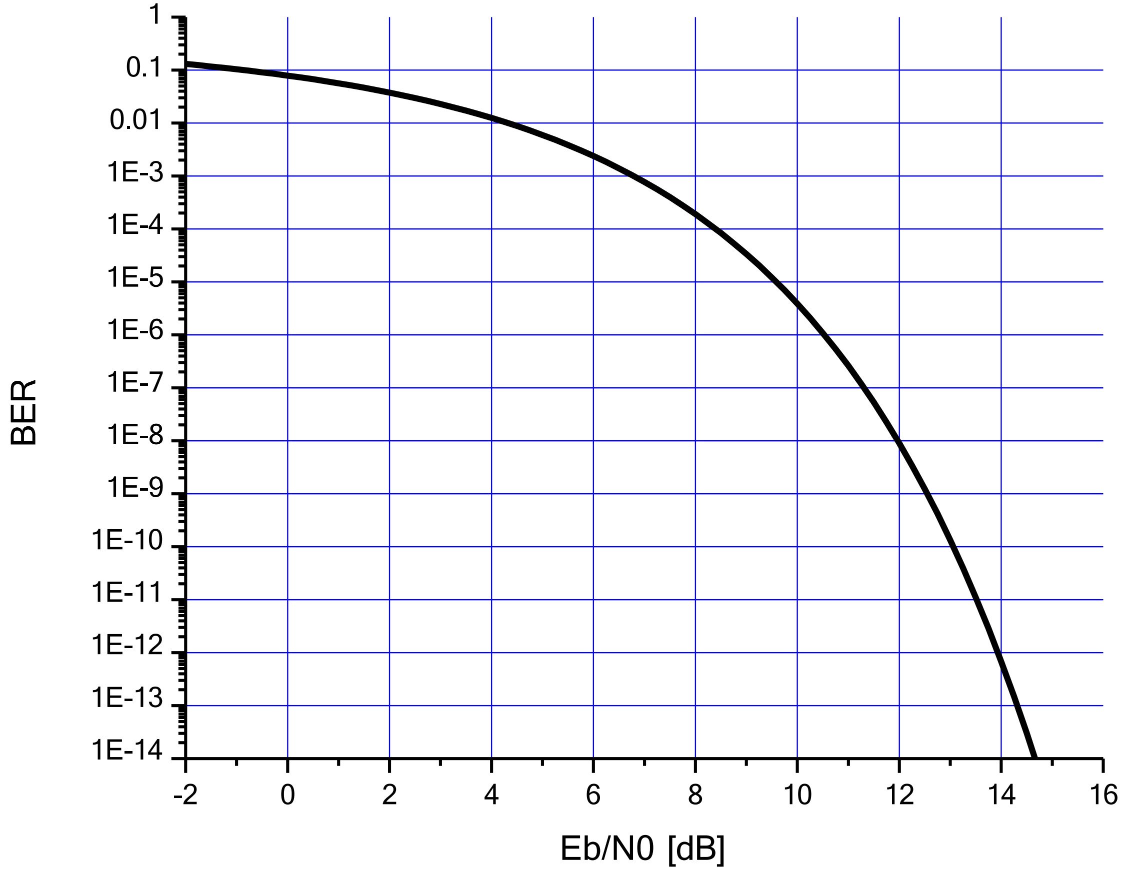
$$\mathrm{e} : \mathrm{H}_1 \Leftrightarrow \mathrm{M}$$

$$\underline{\mathrm{v}_1} = (0) \Leftrightarrow \underline{\mathrm{s}_1}$$

$$\underline{\mathrm{v}_2} = (1) \Leftrightarrow \underline{\mathrm{s}_2}$$

And in this scheme, if the signal is wrong then the binary data is certainly also wrong, therefore:

$$P_b(e) = P_S(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$



BER
1
0.1
0.01
1E-3
1E-4
1E-5
1E-6
1E-7
1E-8
1E-9
1E-10
1E-11
1E-12
1E-13
1E-14
-2
0
2
4
6
8
10
12
14
16
Eb/N0 [dB]

Different signal spaces that share the same vector space have the same BER value!!

In the last example, BER does not depend on the waveform of the orthonormal vectors:

$$b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$b_1(t) = \sqrt{\frac{2}{T}} P_T(t) \cos(2\pi f_0 t)$$