**Problem 1:**

Solve the differential equation

$$y' = \sqrt{\frac{5}{y} - 5}\,.$$

**Solution:**

$$y' = \sqrt{\frac{5}{y} - 5} = \sqrt{5}\sqrt{\frac{1-y}{y}} \quad \Rightarrow \quad \sqrt{\frac{y}{1-y}}\mathrm{d}y = \sqrt{5}\mathrm{d}x(1m)$$

We substitute $y = \sin^2(\phi)(1m) \Rightarrow \mathrm{d}y = 2\sin(\phi)\cos(\phi)\mathrm{d}\phi$ so that

$$2\sin(\phi)\cos(\phi)\sqrt{\frac{\sin^2(\phi)}{1-\sin^2(\phi)}}\mathrm{d}\phi = \sqrt{5}\mathrm{d}x \quad \Rightarrow \quad 2\sin^2(\phi)\mathrm{d}\phi = \sqrt{5}\mathrm{d}x \quad \Rightarrow$$
$$I := \int \sin^2(\phi)\mathrm{d}\phi = \frac{\sqrt{5}}{2}\int \mathrm{d}x \quad (1m)$$

Solving the intergal $I$,

$$\begin{aligned} I &:= \int \sin^2(\phi)\mathrm{d}\phi = -\int \sin(\phi)[\cos(\phi)]'\mathrm{d}\phi \\ &= -\sin(\phi)\cos(\phi) + \int \cos^2(\phi)\mathrm{d}\phi \\ &= -\sin(\phi)\cos(\phi) + \int [1 - \sin^2(\phi)]\mathrm{d}\phi \\ &= -\sin(\phi)\cos(\phi) + \phi - I \\ \Rightarrow \quad I &= \frac{1}{2}\big[\phi - \frac{1}{2}\sin(2\phi)\big](1m) \end{aligned}$$

we have

$$\phi - \frac{1}{2}\sin(2\phi) = \sqrt{5}x + C_2$$

Using the initial value $y(0) = 1$ we determine the integration constant as $C_2 = \sin^{-1}(1) - \frac{1}{2}\sin\left(2\sin^{-1}(1)\right) = \frac{\pi}{2}$. We finally obtain

$$\phi - \frac{1}{2}\sin(2\phi) = \sqrt{5}x + \frac{\pi}{2}\,, \quad (1m)$$
$$y = \sin^2(\phi) \quad (1m)\,.$$

**Problem 2:**

Suppose the function $f(x) = \sin(2x + \frac{\pi}{2})$.

a) simplify $f(x)$ using trigonometrical identities,

b) Calculate explicitely its Maclaurin series.

c) Prove that the former series converges and give the convergence interval.

**Solution:**

a) Using the relation $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ for $a = 2x$ and $b = \pi/2$, we have $f(x) = \sin(2x + \frac{\pi}{2}) = \cos(2x)(1m)$.

b) We have (2m)

$$\left.\begin{aligned} f(x) &= \cos(2x) \\ f'(x) &= -2\sin(2x) \\ f''(x) &= -4\cos(2x) \\ f'''(x) &= 8\sin(2x) \\ f^{(4)}(x) &= 16\cos(2x) \\ f^{(5)}(x) &= -32\sin(2x) \\ f^{(6)}(x) &= -64\cos(2x) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} f(0) &= \cos(0) = 1 \\ f'(0) &= -2\sin(0) = 0 \\ f''(0) &= -4\cos(0) = -4 \\ f'''(0) &= 8\sin(2x) = 0 \\ f^{(4)}(0) &= 16\cos(2x) = 16 = 4^2 \\ f^{(5)}(0) &= -32\sin(2x) = 0 \\ f^{(6)}(0) &= -64\cos(2x) = -64 = -4^3 \end{aligned}\right\}$$

Thus (1m)

$$\text{MCLSeries} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$

c) Apply the ratio test (1m)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{2(n+1)} x^{2(n+1)}}{(2n+2)!} \frac{(2n)!}{2^{2n} x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{4x^2}{(2n+1)(2n+2)} \right| \to 0 < 1$$

Thus, it converges for $x \in \mathbb{R}$ (1m).

**Problem 3**

Check if the series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+2}\right)$ converges.

**Solution:**

We have

$$\begin{aligned}
(1m) \lim_{n\to\infty} \sum_{i=1}^{n} \ln\left(\frac{i}{i+2}\right) &= (1m) \lim_{n\to\infty} \left[\sum_{i=1}^{n} \ln(i) - \sum_{i=1}^{n} \ln(i+2)\right] \\
&= (1m) \lim_{n\to\infty} \left[\sum_{i=1}^{n} \ln(i) - \sum_{j=3}^{n+2} \ln(j)\right] \\
&= \ln(1) + \ln(2) + \lim_{n\to\infty} \left[\sum_{i=3}^{n} \ln(i) - \sum_{j=3}^{n} \ln(j) - \ln(n+1) - \ln(n+2)\right] \\
&= (1m) \ln(2) - \lim_{n\to\infty} \ln[(n+1)(n+2)] \\
&\leq (1m) \ln(2) - \lim_{n\to\infty} \ln(n^2) \to -\infty
\end{aligned}$$

It diverges.

**Problem 4:**

The curve $y = \sqrt{R^2 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = R^2$, where $R > 1$ is the radius of the circle. Find the area of the surface obtained by rotating this arc about the $x$-axis.

**Solution:**

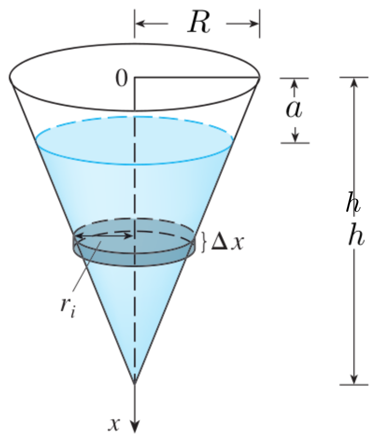
The domain is $(-R, 1]$ (1m). Then,

$$
\begin{aligned}
S = \int_{-1}^{1} 2\pi y \mathrm{d}s &= (1m) \int_{-1}^{1} 2\pi y(x) \sqrt{1 + [y'(x)]^2} \mathrm{d}x = \\
&= (1m) \int_{-1}^{1} 2\pi \sqrt{R^2 - x^2} \sqrt{1 + \left[\frac{-2x}{2\sqrt{R^2 - x^2}}\right]^2} \mathrm{d}x \\
&= \int_{-1}^{1} 2\pi \sqrt{R^2 - x^2} \sqrt{1 + \frac{x^2}{R^2 - x^2}} \mathrm{d}x \\
&= \int_{-1}^{1} 2\pi \sqrt{R^2 - x^2} \sqrt{\frac{R^2}{R^2 - x^2}} \mathrm{d}x \\
&= 4\pi R (1m)
\end{aligned}
$$

**Problem 5:**

A tank has the shape of an inverted circular cone with height $h$ and base radius $R$. It is filled with water to a height of $a$. a) Find the work required to empty the tank by pumping all of the water to the top of the tank. b) Find the value of $a$ for which the work is maximized and give $W_{\max}$.

$R$




**Solution:**

a) We first calculate the force required to raise the layer. This must at least equal to gravity(1m)

$$F_i \approx m_i g = (1m)\varrho V_i g = (1m)\varrho g A_i \Delta x_i = (1m)\varrho g \pi r_i^2 \Delta x_i$$

Triangle similarity: $\dfrac{r_i}{h - x_i} = \dfrac{R}{h}$, (1m) so that

$$F_i = \varrho g \pi \frac{R^2}{h^2}(h - x_i)^2 \Delta x_i$$

Then, we calculate the work done to raise the layer to the top. This is approximately equal to

$$W_i \approx F_i x_i = \varrho g \pi \frac{R^2}{h^2}(h - x_i)^2 x_i \Delta x_i (1m)$$

To find the total work done in emptying the entire tank, we add the contributions of each of the $n$ layers and then take the limit as $n \to \infty$:

$$\begin{aligned}
W &= \lim_{n\to\infty} \sum_{i=1}^{n} F_i x_i = \varrho g \pi \frac{R^2}{h^2} \lim_{n\to\infty} \sum_{i=1}^{n} (h - x_i)^2 x_i \Delta x_i \\
&= (1m) \varrho g \pi \frac{R^2}{h^2} \int_a^h (h-x)^2 \, x \mathrm{d}x \\
&= \varrho g \pi \frac{R^2}{h^2} \int_a^h \left(h^2 x - 2hx^2 + x^3\right) \mathrm{d}x \\
&= \varrho g \pi \frac{R^2}{h^2} \left( \frac{h^2}{2} x^2 - \frac{2}{3} h x^3 + \frac{1}{4} x^4 \right) \Big|_a^h \\
&= \varrho g \pi \frac{R^2}{h^2} \left[ \frac{1}{12} h^4 - \left( \frac{h^2}{2} a^2 - \frac{2}{3} h a^3 + \frac{1}{4} a^4 \right) \right] \\
&= \varrho g \pi R^2 \left[ \frac{1}{12} h^2 - \frac{1}{2} a^2 + \frac{2}{3h} a^3 - \frac{1}{4h^2} a^4 \right] (1m)
\end{aligned}$$

b) We have

$$\begin{aligned}
W(a) &= \varrho g \pi R^2 \left[ \frac{1}{12} h^2 - \frac{1}{2} a^2 + \frac{2}{3h} a^3 - \frac{1}{4h^2} a^4 \right] \\
W'(a) &= \varrho g \pi R^2 \left[ -1 + \frac{2}{h} a - \frac{1}{h^2} a^2 \right] a \ (1m)
\end{aligned}$$

Setting $W'(a) = 0$ (1m)we obtain $a = 0$ and/or $a = h$. From the second derivative test we observe that for $a = 0$ (1m)the work is maximized, being equal to $W_{\max} = \frac{1}{12} \varrho g \pi R^2 h^2$.

**Problem 6**

Suppose the function $f(x) = \int_0^x g(z)\mathrm{d}z$ with $x \geq 0$ and $g$ is a continuous and differentiable function. Given that $g'(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, express $f$ and $f'$ as the sum of a power series and determine their domain.

**Solution:**

From the Fundamental Theorem of Calculus, we have

$$(1m)\ f'(x) = g(x)\,, \quad (1m) \Rightarrow \quad f''(x) = g'(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Integrating, we have

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} + C_1\ (1m)$$

Also $f'(0) = 0 \ \Rightarrow \ C_1 = 0(1m)$. Integrating one more time, we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2} + C_2(1m)$$

with $C_2 = 0$(1m), since $f(0) = 0$. To calculate the radius of the series we consider the Ratio test of convergence for $g'(x)$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \left| x^2 \right| < 1$$

from which we read $-1 < x < 1$ (1m). However, since $x \geq 0$ we finally have $x \in [0, 1)$ (1m).