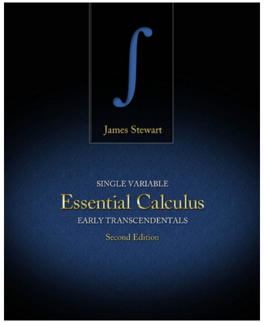
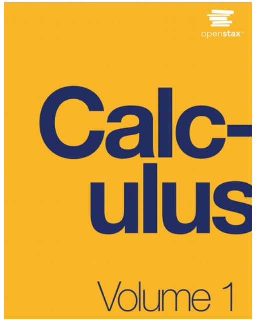
Chapter 5: Integrals





5.1 Areas and Distances

5.2 The definite Integral

5.3 Evaluating Definite Integrals

5.4 The Fundamental Theorem of Calculus

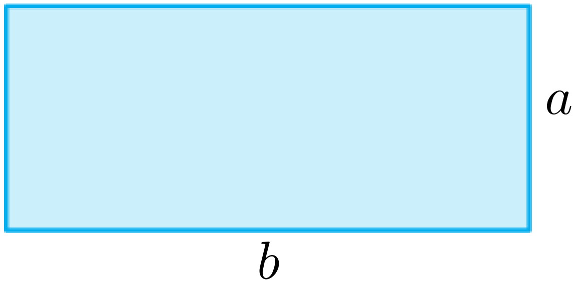
5.5 The Substitution Rule

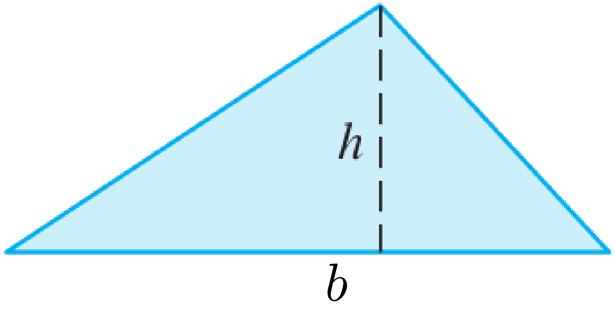
The pictures are taken from the books:

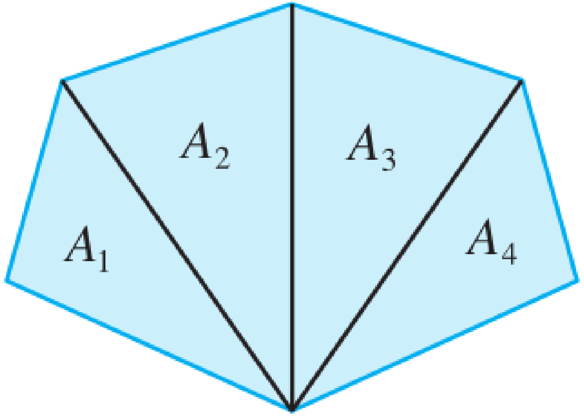
[1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280
2) G. Strang and E. J. Herman, Calculus 1, https://openstax.org/details/books/calculus-volume-1]

5.1 Areas and Distances

- Find the area of the geometric objects below.







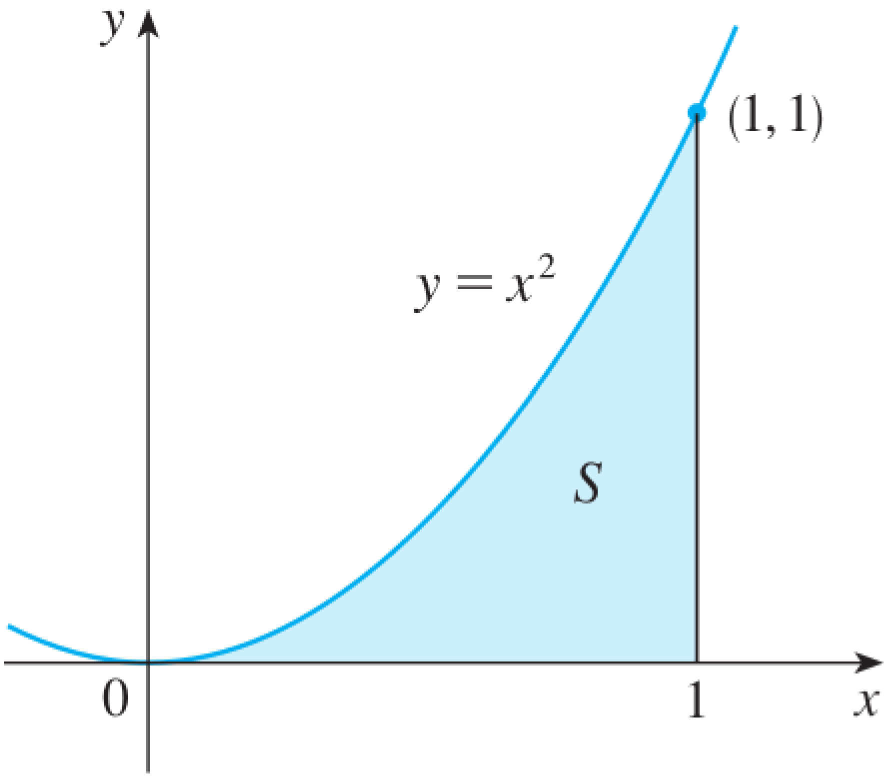
$$A = ab$$

$$A = \frac{1}{2}ab$$

$$A = A_1 + A_2 + A_3 + A_4 + A_5$$

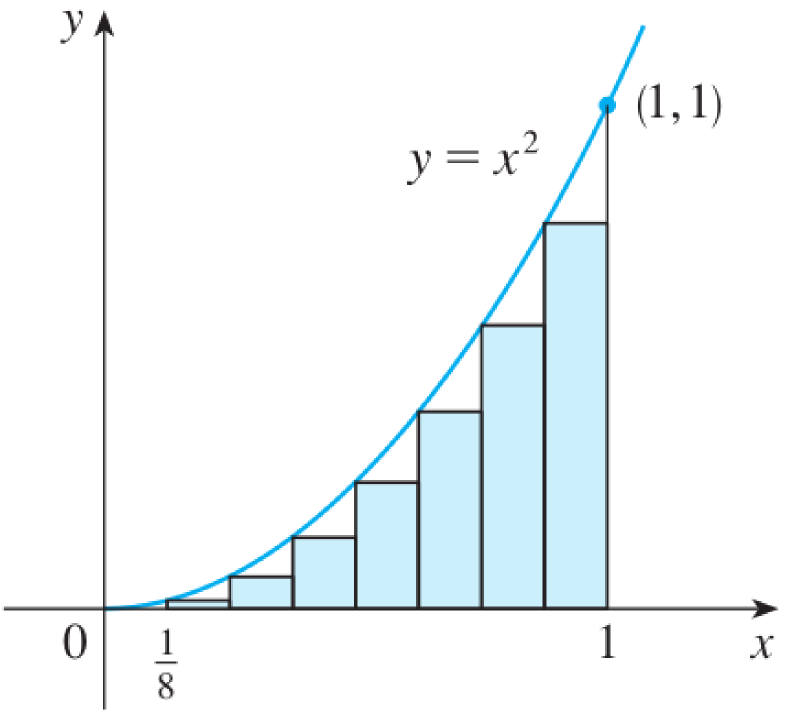
5.1 Areas and Distances

- Find the area of the region S that lies under the curve $y = x^2$ from $a = 0$ to $b = 1$.

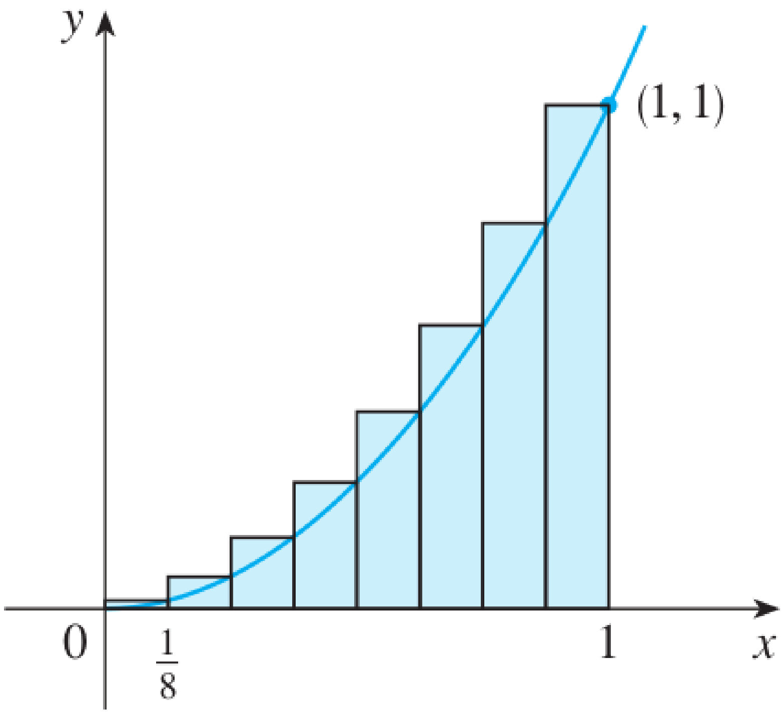


5.1 Areas and Distances

- Approximate S with $n = 8$ **rectangles** of **equal width**:



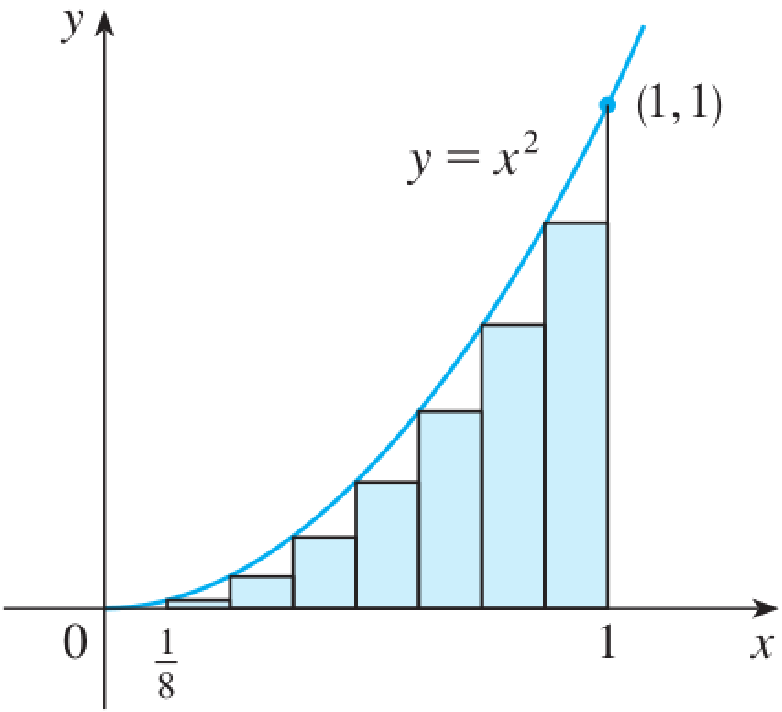
Left endpoints, area L_8



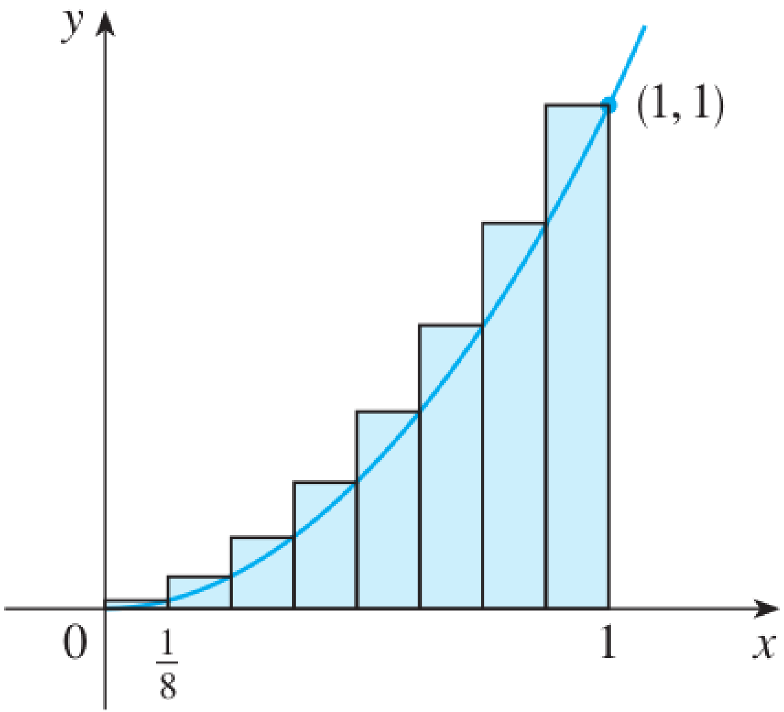
Right endpoints, area R_8

5.1 Areas and Distances

$$L_8 = 0.2734375 < A < 0.3984375 = R_8$$



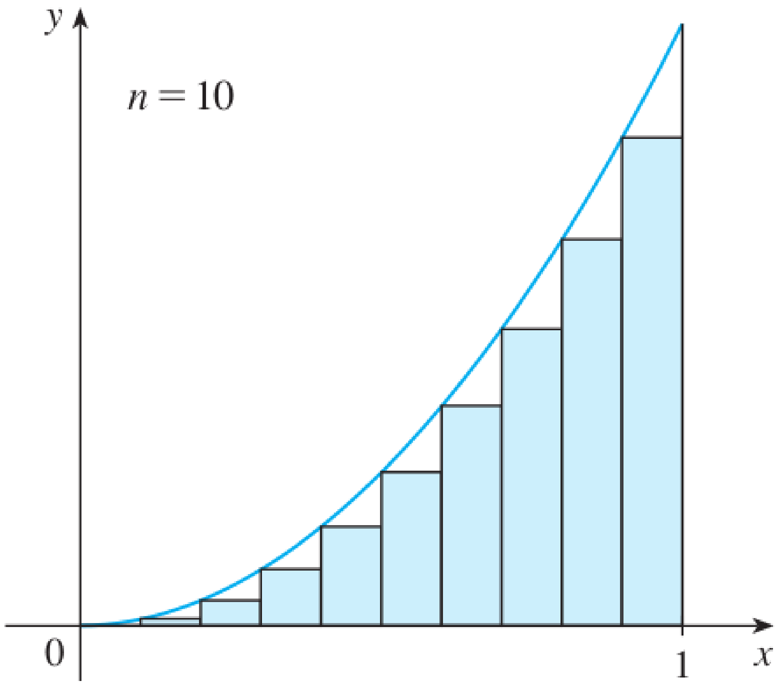
Left endpoints, area L_8



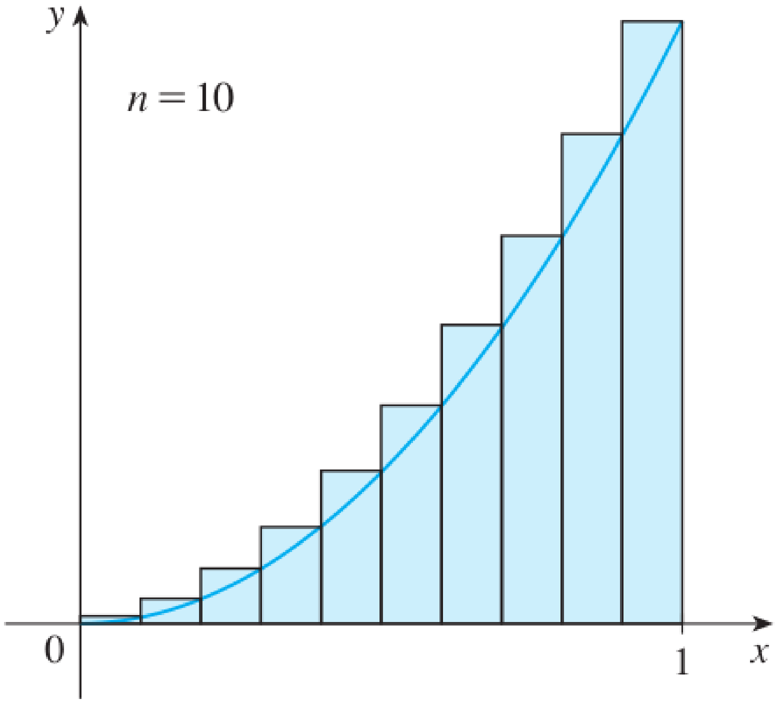
Right endpoints, area R_8

5.1 Areas and Distances

$$L_{10} = 0.285 < A < 0.385 = R_{10}$$

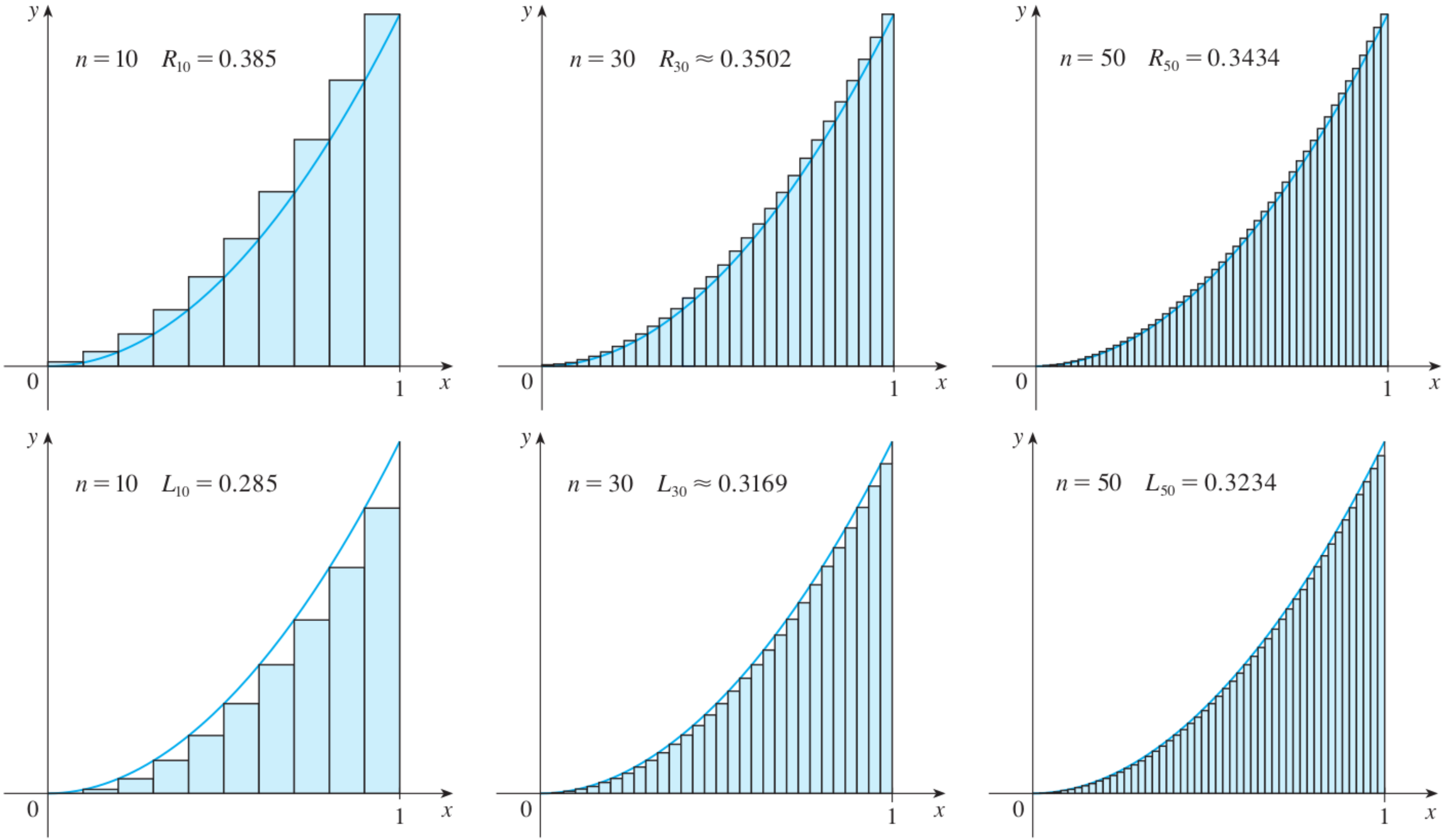


Left endpoints, area L_{10}

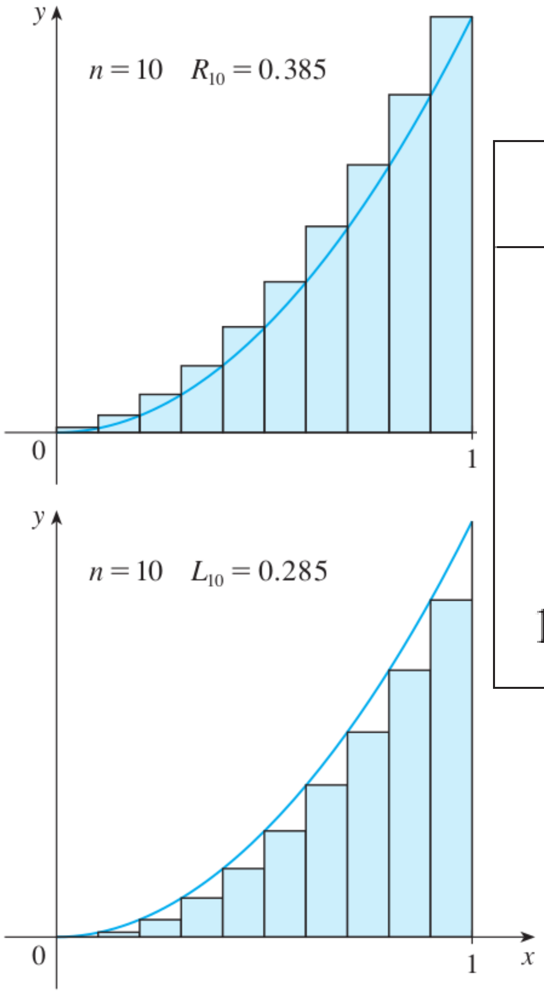


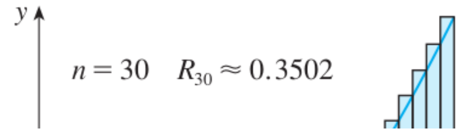
Right endpoints, area R_{10}

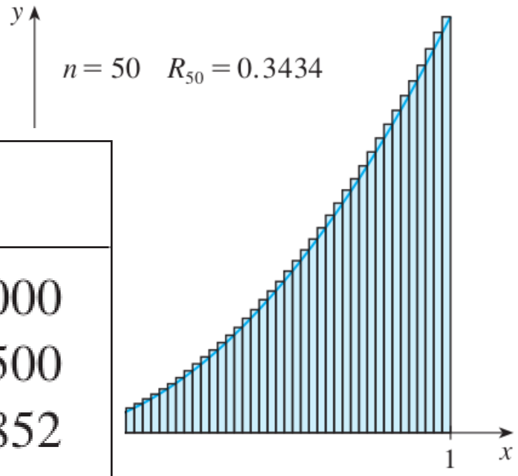
5.1 Areas and Distances



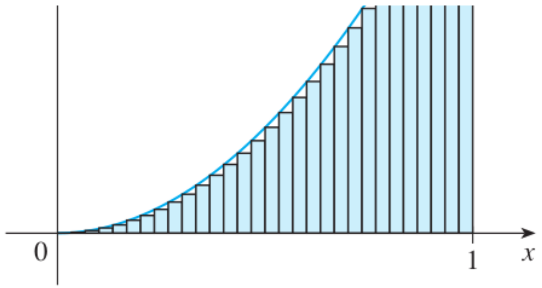
5.1 Areas and Distances

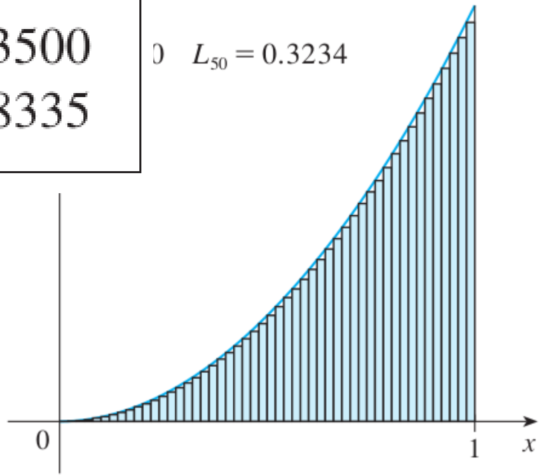






n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335





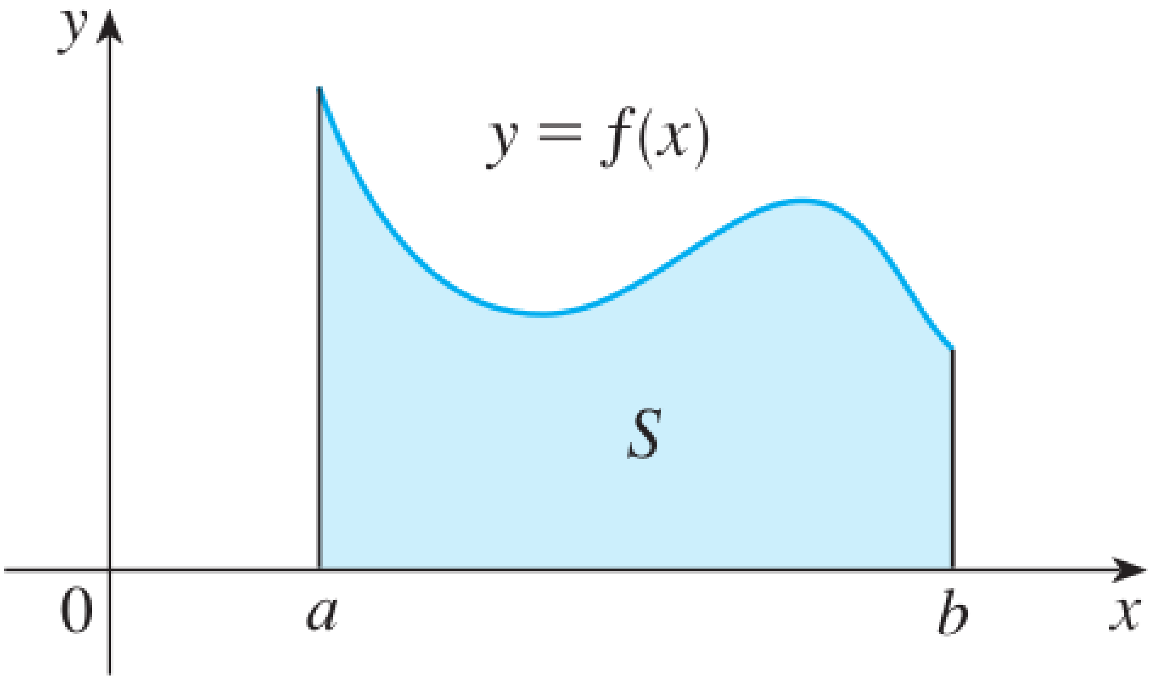
5.1 Areas and Distances

- We identify $$\boxed{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n}$$

Example: Calculate for $y = x^2$: $$A = \lim_{n\to\infty} R_n$$

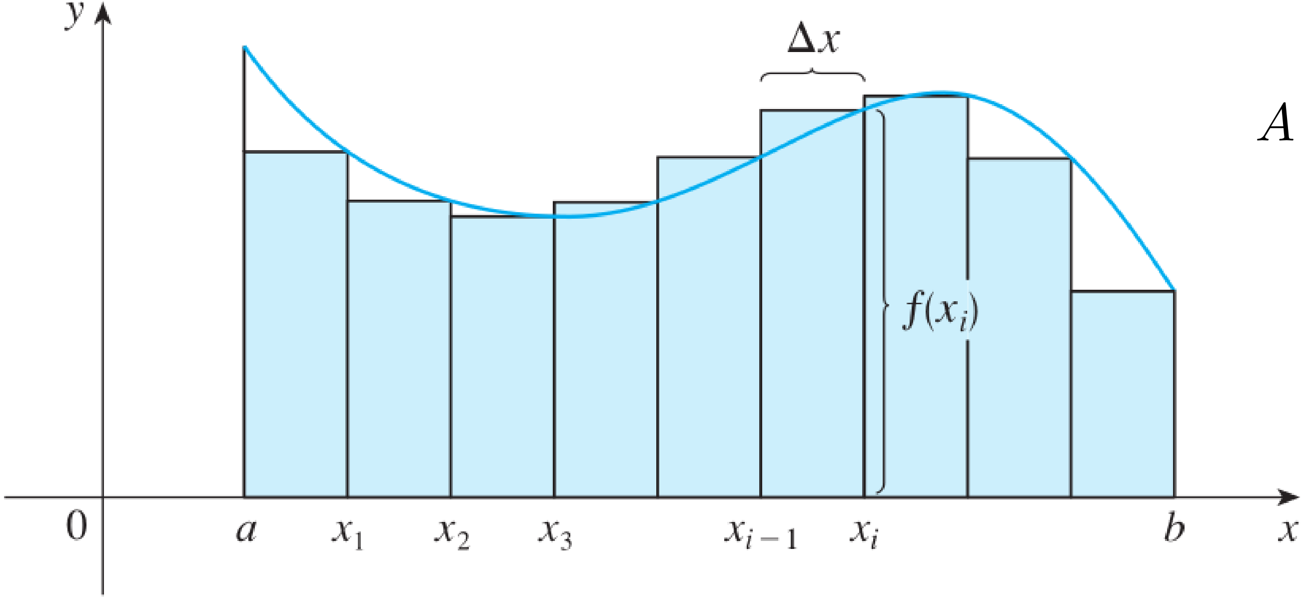
5.1 Areas and Distances

- Find the area of the region S that lies under the curve $y = f(x)$ from a to b.



5.1 Areas and Distances

- Find the area of the region S that lies under the curve $y = f(x)$ from a to b.



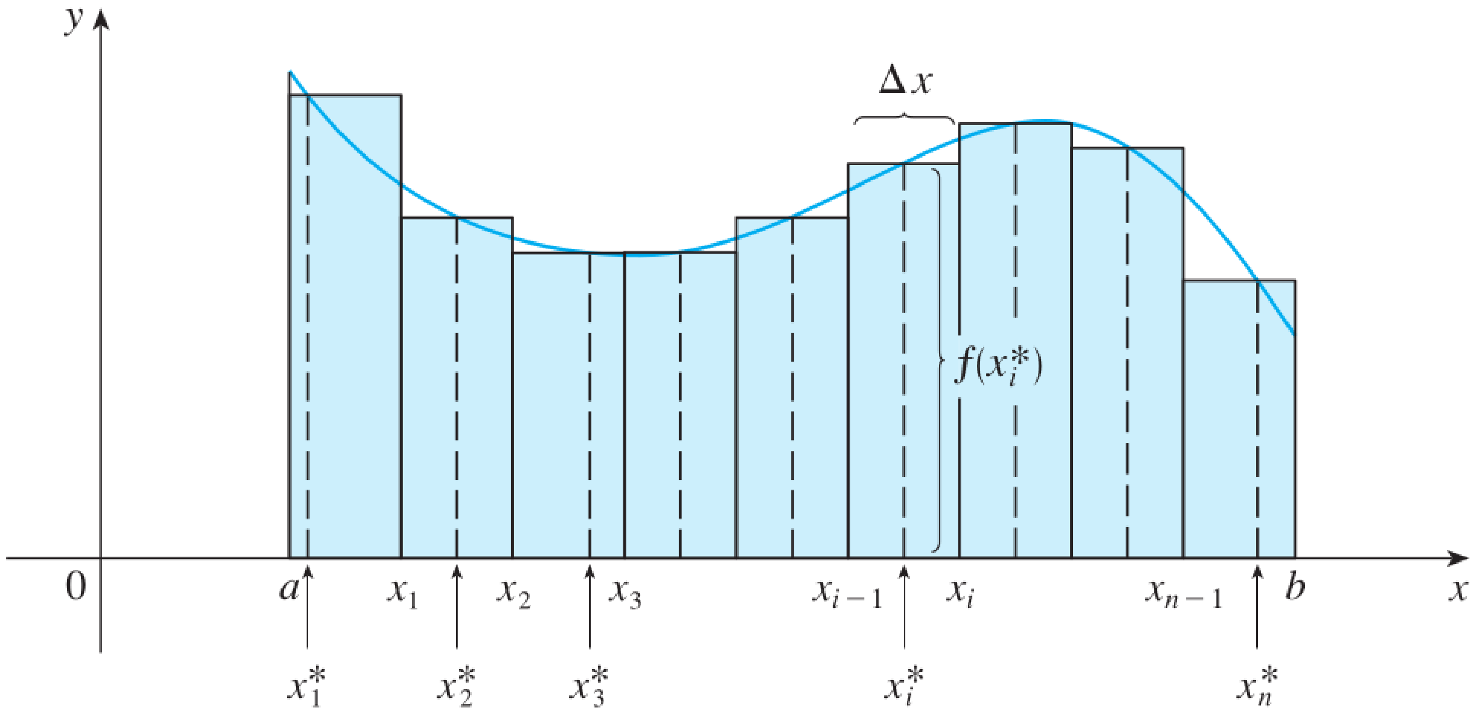
$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n}$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

5.1 Areas and Distances

- A more general expression for the area of the region S is: $A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$

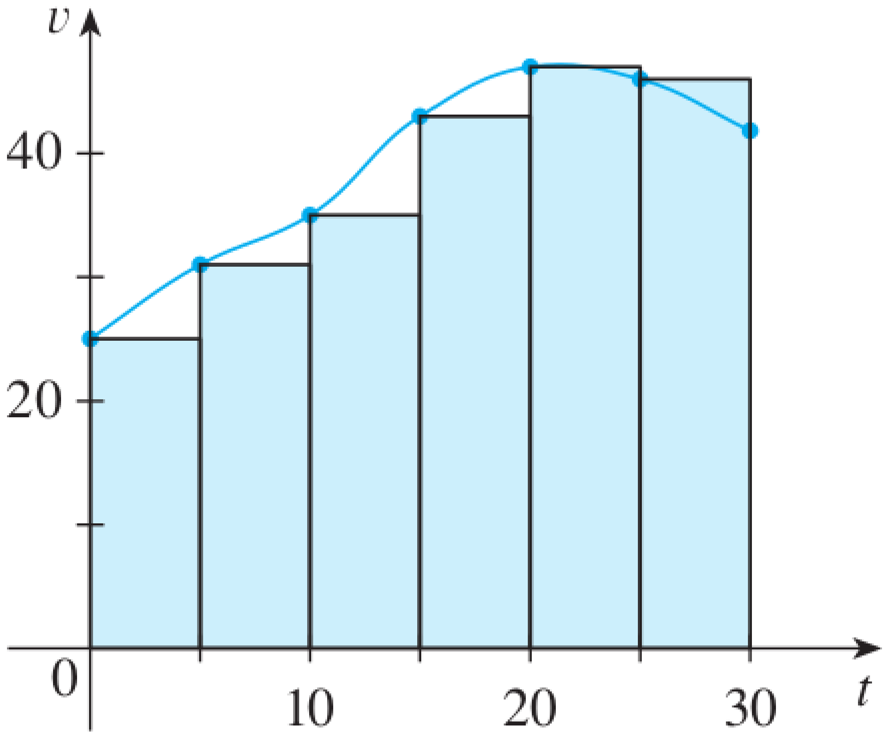


5.1 Example

• Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$. **(a)** Using right endpoints, find an expression for A as a limit. Do not evaluate the limit. **(b)** Estimate the area by taking the sample points to be midpoints and using four sub intervals and then ten subintervals.

5.1 The Distance Problem

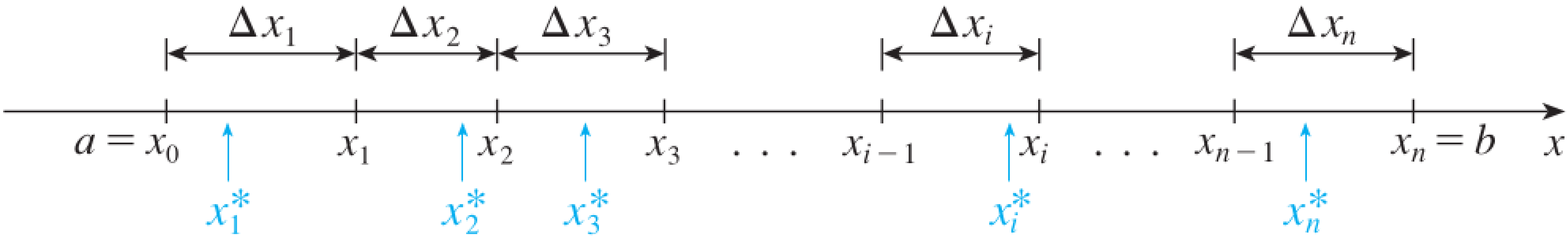
Area of a rectangle$= vt = \frac{\text{m}}{\text{sec}}\text{sec} = \text{m} = $ Distance



$$\text{Distance} = \lim_{n\to\infty}\sum_{i=1}^{n} v(t_i^*)\Delta t_i$$

5.2 The definite Integral

• **Generalization**: We consider limits similar in which f need not be positive or continuous and the subintervals don't necessarily have the same length.



• A **partition** P of $[a, b]$: $[x_0, x_1], [x_1, x_2,], \ldots, [x_{n-1}, x_n]$

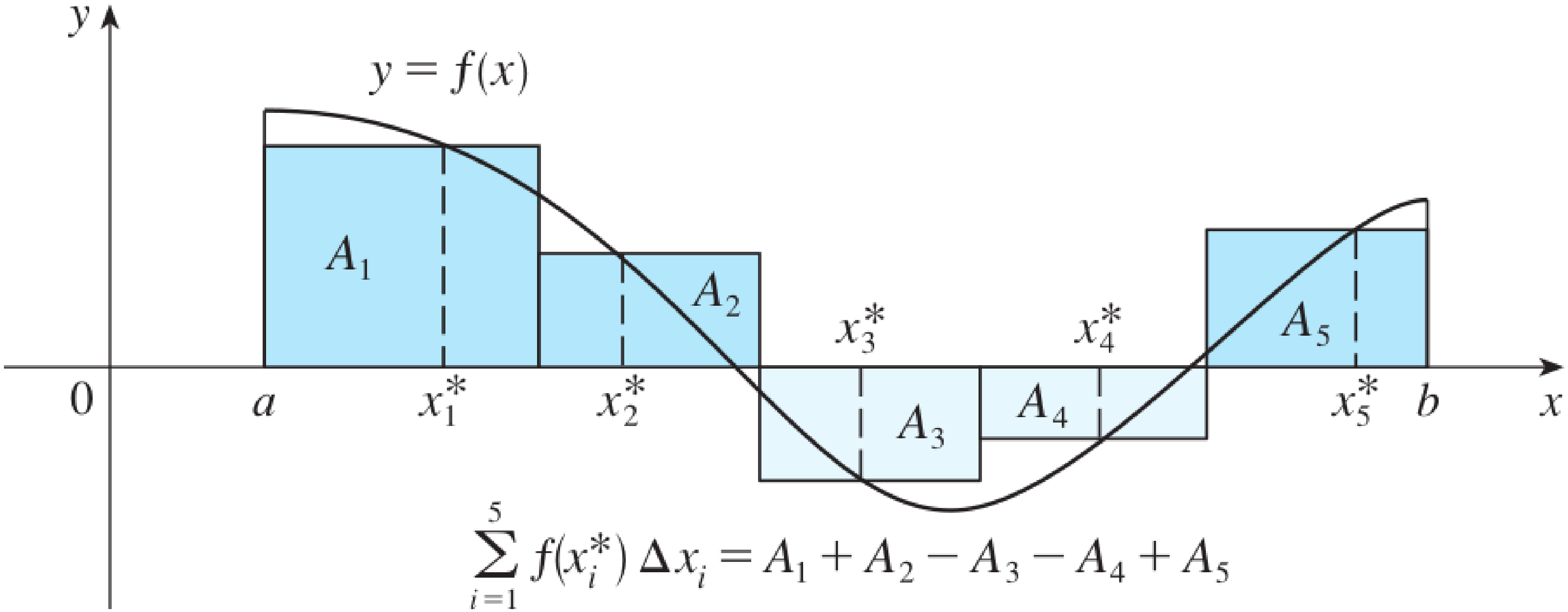
• Choose **sample points (taggs)**: $x_1^*, x_2^*, \ldots, x_n^*$ with $x_i^* \in [x_{i-1}, x_i]$

• **Riemann Sum**: $\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$

5.2 The definite Integral

- Geometric Interpretation of the **Riemann Sum**:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$



$$\sum_{i=1}^{5} f(x_i^*)\,\Delta x_i = A_1 + A_2 - A_3 - A_4 + A_5$$

5.2 The definite Integral

Definition of a definite Integral: If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is the number

$$\int_a^b f(x)\mathrm{d}x = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.

5.2 The definite Integral

Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\mathrm{d}x$ exists.

Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \,,$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

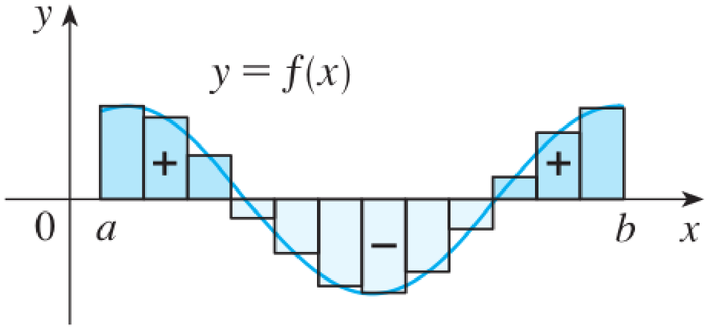
5.2 The definite Integral

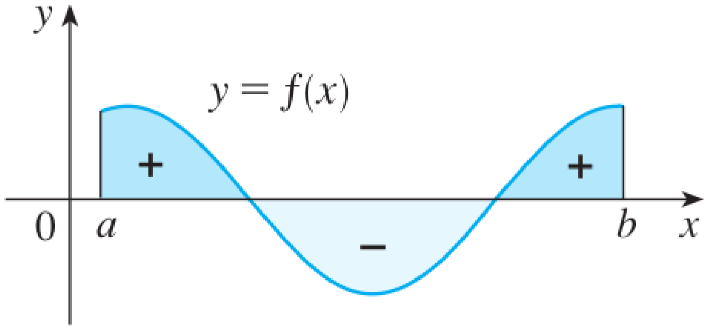
Note 1 The definite integral $\int_a^b f(x)\mathrm{d}x$ is a number, thus

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(r)\mathrm{d}r$$

Note 2 A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\mathrm{d}x = (\text{Area above the } x\text{-axis}) - (\text{Area below the } x\text{-axis})$$





5.2 Examples

1. (a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$. **(b)** Evaluate $y = \int_0^3 (x^3 - 6x)\mathrm{d}x$.

2. Evaluate the following integrals by interpreting each in terms of areas:

(a) $\displaystyle\int_0^1 \sqrt{1 - x^2}\mathrm{d}x$, **(b)** $\displaystyle\int_0^3 (x - 1)\mathrm{d}x$.

5.2 Properties of the definite Integral

1. $$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$$

2. $$\int_a^a f(x)\mathrm{d}x = 0$$

3. $$\int_a^b c\,\mathrm{d}x = c(b-a)\,, \qquad c \text{ constant}$$

4. $$\int_a^b [f(x) \pm g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \pm \int_a^b g(x)\mathrm{d}x$$

5. $$\int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$$

6. $$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x\,, \qquad c \in (a,b)$$

5.2 Properties of the definite Integral

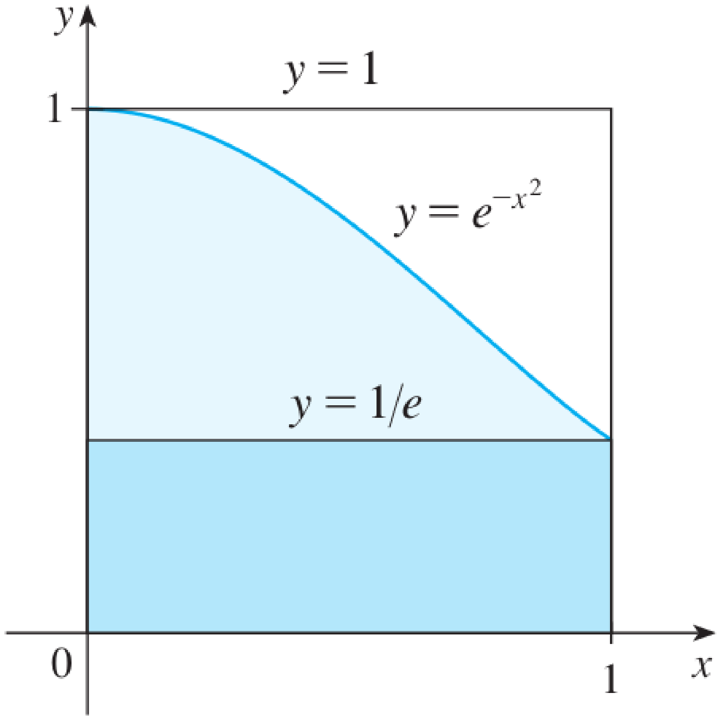
7. If $f(x) \geq 0$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq 0$

8. If $f(x) \geq g(x)$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq \int_a^b g(x)\mathrm{d}x$

9. If $m \leq f(x) \leq M$ for $x \in [a, b]$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$

5.2 Example

- Use Property 9 (p. 22) to estimate $\displaystyle\int_0^1 e^{-x^2}\,\mathrm{d}x$.



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5.3 Evaluating definite Integrals

Evaluation Theorem

If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

5.3 Indefinite Integrals

$$\boxed{\int f(x)\mathrm{d}x = F(x) \quad \Rightarrow \quad \frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x)}$$

Note Distinguish between **definite** and **indefinite** integrals. A definite integral $\int_a^b f(x)\mathrm{d}x$ is a number, whereas an indefinite integral $\int f(x)\mathrm{d}x$ is a function (or family of functions).

5.3 Table of definite Integrals

- $\int x^n \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + C \ \ (n \neq -1)$
- $\int \dfrac{1}{x} \mathrm{d}x = \ln(|x|) + C$
- $\int e^x \mathrm{d}x = e^x + C$
- $\int a^x \mathrm{d}x = \dfrac{a^x}{\ln(a)} + C$
- $\int \sin(x) \mathrm{d}x = \cos(x) + C$
- $\int \cos(x) \mathrm{d}x = -\sin(x) + C$
- $\int \sinh(x) \mathrm{d}x = \cosh(x) + C$
- $\int \cosh(x) \mathrm{d}x = -\sinh(x) + C$

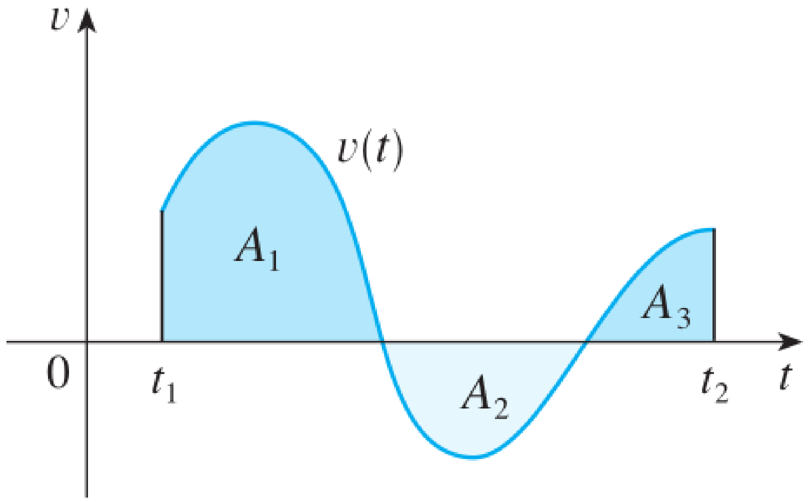
5.3 Example

- Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) \mathrm{d}x$ and interpret the result in terms of areas.

5.3 Applications

Net Change Theorem

The integral of a rate of change is the net change: $\displaystyle\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$



$$\text{Displacement} = \int_{t_1}^{t_2} v(t)\mathrm{d}t = A_1 - A_2 + A_3$$

$$\text{Disctance} = \int_{t_1}^{t_2} |v(t)|\mathrm{d}t = A_1 + A_2 + A_3$$

5.3 Example

• A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).
(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.
(b) Find the distance traveled during this time period.

5.4 The Fundamental Theorem of Calculus

- If f is continuous on $[a, b]$, then the function t defined by

$$g(x) = \int_a^x f(t)\mathrm{d}t\,, \qquad x \in [a, b]$$

is an antiderivative of f, that is,

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)\,, \qquad x \in (a, b)\,.$$

5.4 Examples

1. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\mathrm{d}t$

2. Find the derivative of the function $h(x) = \int_1^{x^4} \sec(t)\mathrm{d}t$

5.4 Average Value of a Function

- The **average value** of f on the interval $[a, b]$ is given by

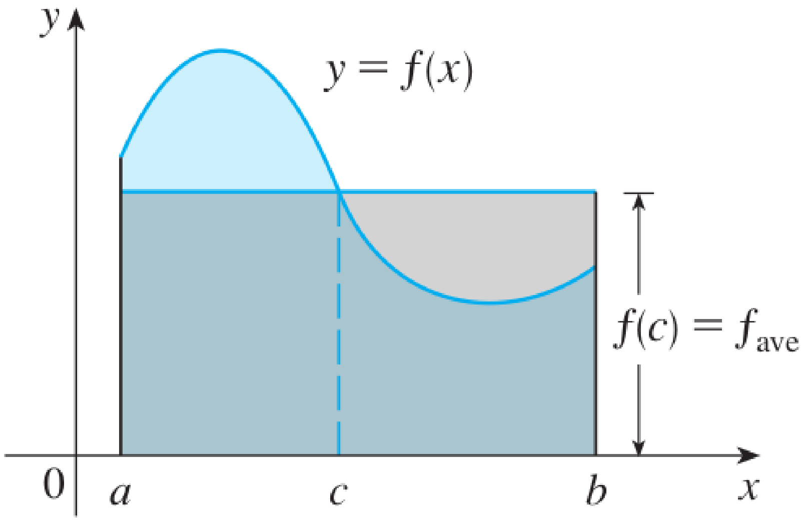
$$\langle f \rangle = \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x$$

$\left(\begin{matrix}\text{proof}\\\text{whiteboard}\end{matrix}\right)$

5.4 Average Value of a Function

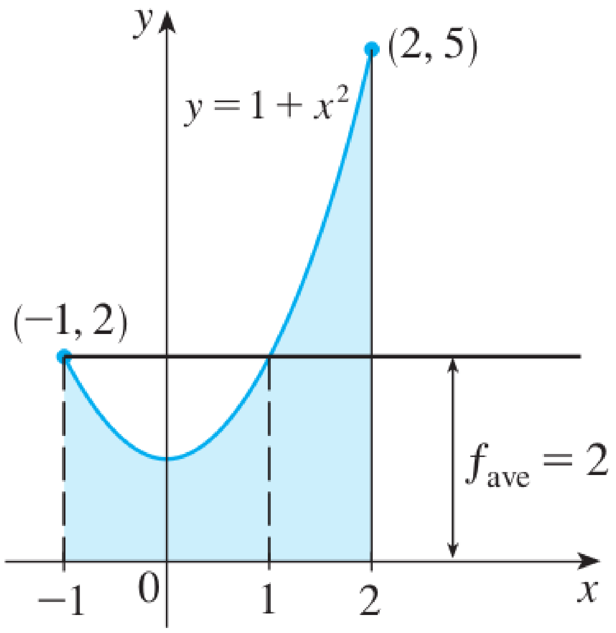
The Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$f(c) = \langle f \rangle = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \qquad \Rightarrow \qquad \int_a^b f(x)\mathrm{d}x = f(c)(b-a)$$



5.4 Example

Find the average value $\langle f \rangle$ of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$ and the value c for which $f(c) = \langle f \rangle$.



5.5 The Substitution Rule (indefinite Integrals)

- If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f\big(g(x)\big)g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u$$

Examples:

Find **(a)** $\int x^3\cos(x^4+2)\mathrm{d}x$, **(b)** $\int \tan(x)\mathrm{d}x$, **(c)** $\int \frac{x}{\sqrt{1-4x^2}}\mathrm{d}x$

5.5 The Substitution Rule (definite Integrals)

- If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f\big(g(x)\big)g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u$$

Example:

Find **(a)** $\displaystyle\int_1^2 \frac{1}{(3-5x)^2}\mathrm{d}x$, **(b)** $\displaystyle\int_1^e \frac{\ln(x)}{x}\mathrm{d}x$

5.5 Symmetry

- Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x$

(b) If f is odd $[f(-x) = -f(x)]$, then $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}x = 0$