Chương 2: Matrices - Determinants - Systems of linear equations

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2.1.1. Definitions

Let K be the field of real numbers or the field of complex numbers.

- A *matrix* (over K) of size $m \times n$ is a rectangular array of numbers (in K), which has m *row* và n *column*:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

  where $a_{ij} \in K$ (for $i = 1, \dots, m$, $j = 1, \dots, n$).
  The numbers a_{ij} are called the entries of A.
- If $m = n$ then A is called a square matrix. The entries $a_{11}, a_{22}, \dots, a_{nn}$ are call diagonal entries. Diagonal entries form the *main diagonal* of A.
- Matrices are usually written in square brackets as above, or parentheses, and may be abberivated by writing only single generic term, such as $A = [a_{ij}]_{m\times n}$ or $A = (a_{ij})_{m\times n}$.
- The set of matrices of size $m \times n$ with entries in K is denoted by $\mathcal{M}_{m\times n}(K)$, or $\mathcal{M}_{m,n}(K)$, or $M(m \times n, K)$. In the case $m = n$, we also use the notation $\mathcal{M}_n(K)$ to denote the set of square matrices of order n (with entries in K).

Some special matrices

- A matrix of size $1 \times n$ is called a *row matrix*.
- A matrix of size$m \times 1$ is called a *column matrix*.
- A matrix $A = [a_{ij}]_{m \times n}$ where all entries $a_{ij} = 0$ $(\forall i, j)$, is called the *zero matrix*, usually denoted by $\mathcal{O}$, or $\mathcal{O}_{m \times n}$.

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

- A square matrix $A = [a_{ij}]_{n\times n}$ is called an *upper triangular matrix* if $a_{ij} = 0$, for all $i > j$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

- A square matrix $A = [a_{ij}]_{n\times n}$ is called a *lower triangular matrix* if $a_{ij} = 0$, for all $i < j$.

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

- A square matrix $A = [a_{ij}]_{n\times n}$ is called a *diagonal matrix* if $a_{ij} = 0$, for all $i \neq j$.

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

- A square $A = [a_{ij}]_{n\times n}$ is called the *identity matrix (of order nn)* if it is a diagonal matrix and $a_{ii} = 1$, for all i. The identity matrix of order n is usually denoted by I_n (or I), or E_n (or E).

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Equality of matrices

Defintion

Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ are equal, written $A = B$, if

- they have the same size: $m = p$ và $n = q$;
- $a_{ij} = b_{ij}$ for all $i = 1, \dots, m$, $j = 1, \dots, n$.

2.1.2. Operations with matrices

Matrix addition

Consider two matrices of the same size $m \times n$: $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$. Their sum $A + B$ is the $m \times n$ matrix given by:

$$A + B = [a_{ij} + b_{ij}]_{m\times n}.$$

The sume of two matrices of different size is undefined.

To add two matrices of same size, we add their corresponding entries.

Example:

$$\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -5 \\ 2 & 2 & 1 \end{bmatrix}.$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

For a matrix $A = [a_{ij}]_{m\times n}$, the *nagative* of A, written as $-A$, is defined by $-A = [-a_{ij}]_{m\times n}$. We also define $A - B = A + (-B)$.

Properties

On the set of $m \times n$ matrices (over K), we have

- $(A + B) + C = A + (B + C)$,
- $A + \mathcal{O} = \mathcal{O} + A = A$,
- $A + (-A) = (-A) + A = \mathcal{O}$,
- $A + B = B + A$.

In other words, the set $\mathcal{M}_{m\times n}(K)$ together with matrix addition is an abelian group.

Scalar multiplication

Definition

The product of a number (scalar) k and an $m \times n$ matrix $A = [a_{ij}]_{m \times n}$ is the matrix kA of size $m \times n$ given by

$$kA = [ka_{ij}]_{m \times n}.$$

To multiply a matrix A by a scalar k, we multiplying each entry in A by k.

Example:

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Remark: We have $(-1)A = -A$.

Properties

Properties

Let A, B be in $\mathcal{M}_{m\times n}(K)$ and $c, d \in K$. We have

- $(cd)A = c(dA)$,
- $1A = A$,
- $c(A+B) = cA + cB$,
- $(c+d)A = cA + dA$.

Extra property: If A is a matrix of size $m \times n$ and $\mathcal{O}$ is the zero matrix of size $m \times n$, then

$$cA = \mathcal{O} \Leftrightarrow \left[\begin{array}{l} c = 0 \\ A = \mathcal{O} \end{array}\right. .$$

Matrix multplication

Definition

Let $A = [a_{ij}]_{m \times n}$ be a matrix of size $m \times n$ and $B = [b_{ij}]_{n \times p}$ a matrix of size $n \times p$. The product AB is the matrix $C = [c_{ij}]_{m \times p}$ of size $m \times p$ given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \quad (\forall i = 1, \ldots, m, j = 1, \ldots, p).$$

Remarks

- The product AB is defined only in the case that the number of columns of A is equal to the number of rows of B.
- To obtain the entry c_{ij} of the product AB, we multiply the entries in the ith row of A by the corresponding entries in the jth column of B and then add the results.

$$\begin{array}{ccc}
 & b_{1j} & \\
 & b_{2j} & \\
a_{i1}\ a_{i2}\ \ldots\ a_{in} & \vdots & \Longrightarrow\ c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\
\text{the } i\text{th row of } A & & \\
 & b_{nj} & \\
 & \text{the } j\text{th column of } B &
\end{array}$$

- The product AB could be defined meanwhile the product BA is not defined. Even in the case that both AB and BA are defined, in general we still have $AB \neq BA$.

Remarks

- In general, $AB = \mathcal{O}$ does not imply that $A = \mathcal{O}$ or $B = \mathcal{O}$.
- In general, $AC = BC$ (or $CA = CB$) and $C \neq \mathcal{O}$ do not imply that $A = B$.

Example:

- $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$: $A \neq \mathcal{O}$, $B \neq \mathcal{O}$ but $AB = \mathcal{O}$.
- $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$:

$$AC = BC = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \quad \text{but } A \neq B.$$

$=$

Example

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$. Compute AB.

- $C = AB$ of size 2×2, $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$.
- $c_{11} = 1 \cdot 1 + (-1) \cdot 2 + 2 \cdot 3 = 5.$
- $c_{12} = 1 \cdot 2 + (-1) \cdot (-1) + 2 \cdot 1 = 5.$
- $c_{21} = 0 \cdot 1 + 1 \cdot 2 + (-2) \cdot 3 = -4.$
- $c_{22} = 0 \cdot 2 + 1 \cdot (-1) + (-2) \cdot 1 = -3.$
- $AB = C = \begin{bmatrix} 5 & 5 \\ -4 & -3 \end{bmatrix}$.

Properties

Let A, B, C be matrices (with sizes such the given operations are defined) and $c \in K$. Then we have following properties.

- $(AB)C = A(BC)$
- $A(B+C) = AB + AC$, $(B+C)A = BA + CA$
- $(cA)B = A(cB) = c(AB)$
- If A is of size $m \times n$ then $AI_n = A$ and $I_mA = A$.

Remark: The set $\mathcal{M}_n(K)$ of square matrices of order n together with matrix addition and multiplication is a ring (with unit).

Powers of a matrix

Let A be a square matrix of order n.

- For $k \geq 1$ being a positive interger, we define

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ times}}.$$

- Properties: $A^{k+l} = A^k A^l$, $A^{kl} = (A^k)^l$, với mọi k, l nguyên dương.
- If $f(x) = a_k x^k + \cdots + a_1 x + a_0$ is a polynomial of degree k, we define

$$f(A) = a_k A^k + \cdots + a_1 A + a_0 I_n.$$

Example (GK20161)

Let $A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$ and $P(x) = x^2 + 2x + 1$. Compute $P(A)$.

Solution 1:

- $A^2 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & -4 \\ -2 & 1 & -1 \end{bmatrix}$.
- $P(A) = A^2 + 2A + I_3 = \begin{bmatrix} 1 & 2 & -2 \\ 2 & -1 & -4 \\ -2 & 1 & -1 \end{bmatrix} + 2\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 2 & -2 & 0 \\ 0 & -1 & -2 \end{bmatrix}$.

Solution 2: $P(A) = A^2 + 2A + I_3 = (A + I_3)^2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 2 & -2 & 0 \\ 0 & -1 & -2 \end{bmatrix}$.

Example (GK20161*)

Let $A = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$. Compute A^3 and A^{27}.

- $A^2 = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = \begin{bmatrix} -2 & -2i \\ 2i & -2 \end{bmatrix} = 2i\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} = (2i)A.$
- $A^3 = A^2 \cdot A = (2i)A \cdot A = (2i)A^2 = (2i)\cdot(2i)A = -4A = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix}.$
- By induction on k: $A^k = (2i)^{k-1}A$, for every natural number $k \geq 1$.
- $A^{27} = (2i)^{26}A = -2^{26}A.$

The transpose of a matrix

Transpose of a matrix

Let $A = [a_{ij}]_{m\times n}$ be an $m \times n$ matrix. The transpose of A, denote by $A^T = [b_{ij}]$, is the $n \times m$ matrix given by

$$b_{ij} = a_{ji}, \quad \forall i = 1, \dots n, j = 1, \dots, m.$$

Thus, the columns of A^T are the rows of A, the rows of A^T are the columns of A.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Symmetric and skew-symmetric matrices

Definition

- A matrix A is said to be symmetric if $A^T = A$.
- A matrix A is said to be skew-symmetric if $A^T = -A$.

Thus

- Matrix $A = [a_{ij}]$ is symmetric if and only if A is a square matrix and $a_{ij} = a_{ji}$, $\forall i, j$.
- Matrix $A = [a_{ij}]$ is skew-symmetric if and only if A is a square matrix and $a_{ij} = -a_{ji}$, $\forall i, j$. (In particular if A is skew-symmetric then $a_{ii} = 0$, $\forall i$.)

Example: Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 4 \end{bmatrix}$ is symmetric; and $\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$ is skew-symmetric.

Properties

Let A, B be matrices (of suitable sizes) and c a scalar. We have:

1. $(A^T)^T = A$,
2. $(A+B)^T = A^T + B^T$,
3. $(cA)^T = cA^T$,
4. $(AB)^T = B^T A^T$,
5. AA^T and $A^T A$ are symmetric, for every matrix A of arbitary size $m \times n$.

2.2.1. Definition

Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. We shall define recursively the determinant of A, denoted by $\det(A)$ or $|A|$.

The determinant of a square matrix of order 1 or 2

- If $A = [a_{11}]$ is a square matrix of order 1, then $\det(A) = a_{11}$.
- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Determinant of square matrices of order $n \geq 3$

Suppose we have defined the determinant of any square matrix of order $n - 1$.

- Consider a square matrix $A = [a_{ij}]_{n \times n}$ of order n.
- For each i, j, let M_{ij} be the matrix obtained from A by deleting the ith row and the jth column. The matrix M_{ij} is a square matrix of order $n - 1$.
- Set $C_{ij} = (-1)^{i+j} \det(M_{ij})$, and C_{ij} is called the *cofactor* of a_{ij}. ($\det(M_{ij})$ is called the minor of a_{ij}.)

Định nghĩa

The determinant of $A = [a_{ij}]_{n \times n}$ is

$$\det(A) = |A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Example

Evaluate the determinant of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$.

- $M_{11} = \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow C_{11} = +\det(M_{11}) = \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = -4.$
- $M_{12} = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix} \Rightarrow C_{12} = -\det(M_{12}) = -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -(-2) = 2.$
- $M_{13} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \Rightarrow C_{13} = +\det(M_{13}) = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 5.$
- $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 1 \cdot (-4) + 2 \cdot 2 + (-1) \cdot 5 = -5.$

In short, we have

$$\begin{aligned} |A| &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13}) \\ &= 1 \cdot \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} \\ &= 1 \cdot (-4) + 2 \cdot 2 + (-1) \cdot 5 = -5. \end{aligned}$$

The Laplace expansion

Theorem (The Laplace expansion of a determinant)

Let $A = [a_{ij}]$ be a square matrix of order n. For any i and j, we have :

- The Laplace expansion along the ith row:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in} .$$

- The Laplace expansion along the jth column:

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj} .$$

Remark: We usually use the Laplace expansion along a row or column which has many zeroes.

Example

Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$.

- By the Laplace expansion along the 2nd row, we have

$$\begin{aligned} |A| &= -a_{21}\det(M_{21}) + a_{22}\det(M_{22}) - a_{23}\det(M_{23}) \\ &= (-2)\cdot\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + (-1)\cdot\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} - 2\cdot\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ &= (-2)\cdot 5 + (-1)\cdot 5 - 2\cdot(-5) = -5. \end{aligned}$$

- By the Laplace expansion along the 3rd column, we have

$$\begin{aligned} |A| &= a_{13}\det(M_{13}) - a_{23}\det(M_{23}) + a_{33}\det(M_{33}) \\ &= (-1)\cdot\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 2\cdot\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \\ &= (-1)\cdot 5 - 2\cdot(-5) + 2\cdot(-5) = -5. \end{aligned}$$

The determinant of a 3×3 matrix

The determinant of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} ?$$

Method 1: Expansion along a row or column.

Method 2:

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \,. \end{aligned}$$

Example

Evaluate the determinant of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$.

$$\begin{array}{rrr|rr} 1 & 2 & -1 & 1 & 2 \\ 2 & -1 & 2 & 2 & -1 \\ 3 & 1 & 2 & 3 & 1 \end{array}$$

$$\begin{aligned} |A| &= 1 \cdot (-1) \cdot 2 + 2 \cdot 2 \cdot 3 + (-1) \cdot 2 \cdot 1 - 3 \cdot (-1) \cdot (-1) - 1 \cdot 2 \cdot 1 - 2 \cdot 2 \cdot 2 \\ &= (-2) + 12 + (-2) - 3 - 2 - 8 \\ &= -5. \end{aligned}$$

Some exercises

- (GK20161) Find x such tha $\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & -3 \\ 4 & x^2 & 9 \end{vmatrix} = 0.$
- (GK20171) b) Solve for x: $\begin{vmatrix} 3-x & 2 & 2 \\ 2 & 3-x & 2 \\ 2 & 2 & 3-x \end{vmatrix} = 0.$
- (GK20191) Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. Find $\lambda \in \mathbb{R}$ such that $\det(A - \lambda E) = 0$, where E is the identity matrix of order 3.
- (GK20201) Find a condition a, b, c to ensure that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 0.$

The determinant of a triangular matrix

Consider triangular matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}.$$

We have

$$\det(A) = a_{11}a_{22}\dots a_{nn}\,, \quad \det(B) = b_{11}b_{22}\dots b_{nn}\,.$$

Specal case: This formular can be applied for diagonal matrices. In particular, $\det(I_n) = 1$.

2.2.2. Some properties of determinants

Determinant of transpose

$$\det(A^T) = \det(A)$$

.

Remark: In the following, we only state properties of determinants in terms of "rows". But these properties still holds true if we replace "rows" by "column".

Interchange two rows

If we interchange two rows of A to obtain B then $\det(B) = -\det(A)$.

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = -\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 10 \end{vmatrix} (H_2 \leftrightarrow H_1)$$

Corollary

If A has two equal rows then $\det(A) = 0$.

Multiply a row by a scalar

If B is obtained from A by multiplying a row of $A$by a scalar k, then $\det(B) = k\det(A)$.

Example:

$$\begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} (H_1 \leftarrow \tfrac{1}{2}H_1)$$

Corollary

- if one row of A is a multiple of another row then$\det(A) = 0$.
- If A has a zero row then $\det(A) = 0$.
- If A is a square matrix of order n and k is a scalar then $\det(kA) = k^n\det(A)$.

Property

If a square matrix $A = [a_{ij}]_{n\times n}$ has some ith row such that $a_{ij} = b_j + c_j$ ($\forall j = 1, \ldots, n$) then $\det(A)$ is the sum of two determinants

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1+3 & (-1)+6 & 2+4 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 7 & 8 & 10 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 4 \\ 7 & 8 & 10 \end{vmatrix}$$

Corollary

If B is obtained from $A$by adding a multiple of a row A to another row of A then $\det(B) = \det(A)$.

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 7 & 8 & 10 \end{vmatrix} (H_2 \leftarrow H_2 + 2H_1)$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 2 & 1 \end{vmatrix} (H_3 \leftarrow H_3 - 3H1)$$

The determiant of the product

$$\det(AB) = \det(A)\det(B).$$

Example (GK20201-N3)

Show there is no real square matrix A of order 2019 such that $A^{2020} + E = O$, where E is the identity matrix of order 2019.

- Suppose there is a real matrix A of order 2019 such that $A^{2020} + E = O$. Then $A^{2020} = -E$.
- Hence $\det(A^{2020}) = \det(A)^{2020} = \det(-E) = (-1)^{2019} \det(E) = -1$.
- This is impossible sinceì $\det(A) \in \mathbb{R}$ and $\det(A)^{2020} \geq 0$.

Some exercises*

- (GK20213) Let A, B be real square matrices of order 2023. Show that there is no real matrix X such that $(AX)^T B^{2022} XA + 3E = \mathcal{O}$.
- (GK20191) Let A, B be two square matrices of the same orders such that $A^{2019} = 0$ and $AB = A + B$. Show that $\det(B) = 0$. [Hint: $\det(A) = 0$ và $A(B - I) = B$, where I is the identity matrix.]
- (GK20181) Let $A \neq O$ and $n \in \mathbb{N}$, $n \geq 2$ such that $A^n = O$. Show that $\det(A - E) \neq 0$, where E is the identity matrix. [Hint: $(A - E)(A^{n-1} + \cdots + A + E) = A^n - E$.]
- (GK20181) Let A be a square matrix and $\lambda \in \mathbb{R}$ such that $\det(A - \lambda E) = 0$, where E is the identity matrix. Show that
$$\det(A^2 + 2A - (\lambda^2 + 2\lambda)E) = 0.$$
- (GK2017) Let A, B be real square matrices of order n, $n \geq 2$, such that $AB = BA$. Show that $\det(A^2 + B^2) \geq 0$. [Hint: since $AB = BA$, hence $A^2 + B^2 = (A + iB)(A - iB)$.]
- (CK20161) Let A be a real square matrix of order 2017. Show that
$$\det(A - A^T)^{2017} = 2017(\det A - \det A^T).$$

2.2.3. Evaluation of a determinant using elementary row operations

Elementary row operations

The following operations on rows of matrices are called elementary row operations.

- Interchange two rows. ($H_i \leftrightarrow H_j$.)
- Multiply a row by a nonzero constant. ($H_i \leftarrow kH_i$, $k \neq 0$.)
- Add a multiple of a row to another row. ($H_i \leftarrow H_i + kH_j$.)

Similarly, replacing "rows" by "columns" we obtain elementary column operations.

Effects of elementary row operations on a determinant

Elementary row operation	**The effect on a determinant**
Interchange two rows	Change sign
Multiply a row by a nonzero constant $k \neq 0$	Multiply by k
Add a multiple of a row to another row.	Unchange

We have a similar table, if we replace "row(s)" by "column(s)".

Evaluating of a determinant using elementary row operations

Use elementary (row, column) operations to transform a given determinant

- to a determinant of a triangular matrix,
- or to a determiant of a simpler matrix (for example, a matrix that has a column with all zeroes except at one entry, we can use the Laplace expansion along that row to reduce the determinant to a determinant of smaller order).

Example

Evaluate $\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{vmatrix}$.

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$$\begin{aligned}
\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{vmatrix} &= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 4 \\ 3 & 1 & 2 \end{vmatrix} (H_2 \leftarrow H_2 - 2H_1) \\
&= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 4 \\ 0 & -5 & 5 \end{vmatrix} (H_3 \leftarrow H_3 - 3H_1) \\
&= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 4 \\ 0 & 0 & 1 \end{vmatrix} (H_3 \leftarrow H_3 - H_2) \\
&= -5.
\end{aligned}$$

2.3.1. Rank of a matrix

2.3.2. Inverse of a matrix