HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
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EXERCISES OF PROBABILITY AND STATISTICS

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Chapter 2

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS.

2.1 DISCRETE RANDOM VARIABLES.

Problem 2.1.

A civil engineer is studying a left-turn lane that is long enough to hold 7 cars. Let X be the number of cars in the lane at the end of a randomly chosen red light. The engineer believes that the probability that $X = x$ is proportional to $(x+1)(8-x)$ for $x = 0, 1, ..., 7$.

(a) Find the probability mass function of X.
(b) Find the probability that X will be at least 5.

Problem 2.2.

A midterm test has 4 multiple choice questions with four choices with one correct answer each. If you just randomly guess on each of the 4 questions, what is the probability that you get exactly 2 questions correct? Assume that you answer all and you will get (+5) points for 1 question correct, (-2) points for 1 question wrong. Let X is number of points that you get. Find the probability mass function of X and the expected value of X.

Problem 2.3.

The random variable N has the following pmf $f_N(n) = \begin{cases} c\left(\frac{1}{2}\right)^2, & n = 0,1,2 \\ 0, & \text{otherwise} \end{cases}$

(a) What is the value of the constant c?
(b) What is $P[N \leq 1]$?
(c) What is the cdf of N?

Problem 2.4.

The random variable V has the following pmf $f_V(n) = \begin{cases} cv^2, & v = 1,2,3,4 \\ 0, & \text{otherwise} \end{cases}$

(a) Find the value of the constant c.
(b) Find $P[V \in u^2 \mid u = 1, 2, 3, ...]$.
(c) Find the probability that V is an even number.
(d) Find $P[V > 2]$.

Problem 2.5.

Suppose when a baseball player gets a hit, a single is twice as likely as a double which is twice as likely as a triple which is twice as likely as a home run. Also, the player's batting average, i.e., the probability the player gets a hit, is 0.300. Let B denote the number of bases touched safely during an at-bat. For example, B = 0 when the player makes an out, B = 1 on a single, and so on. What is the pmf of B?

Problem 2.6.

There are two boxes, the first box consists of 7 red balls and 3 white balls, the second box consists of 5 red balls and 2 white balls. Draw randomly 2 balls fron the first box to second one, then continue to draw randomly 2 balls from the second one. Let X be the number of white balls out of these 2 balls. Find the probability distribution of X.

Problem 2.7.

When a conventional paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. To be confident that a message is received at least once, a system transmits the message n times.

(a) Assuming all transmissions are independent, what is the pmf of K, the number of times the pager receives the same message?
(b) Assume $p = 0.8$. What is the minimum value of n that produces a probability of 0.95 of receiving the message at least once?

Problem 2.8.

When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the pager receives the message, it transmits an acknowledgment signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again.

(a) What is the pmf of N, the number of times the system sends the same message?
(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$. What is the minimum value of p necessary to achieve the goal?

Problem 2.9.

The random variable X has the following cdf: $F_X(x) = \begin{cases} 0, & x \le -1, \\ 0.2, & -1 < x \le 0, \\ 0.7, & 0 < x \le 1, \\ 1, & x > 1. \end{cases}$

(a) Draw a graph of the cdf.
(b) Write $f_X(x)$, the pmf of X. Be sure to write the value of $f_X(x)$ for all x from $-\infty$ to $+\infty$.
(c) Write the probability distribution table of X.

Problem 2.10.

The random variable X has the following cdf: $F_X(x) = \begin{cases} 0, & x \le -3, \\ c, & -3 < x \le 5, \\ 0.8, & 5 < x \le 7, \\ 1, & x > 7. \end{cases}$

(a) Find the constant c given that $P(X = -3) = P(X = 5)$.
(b) Draw a graph of the cdf.
(c) Write $f_X(x)$, the pmf of X.
(d) Write the probability distribution table of X.

Problem 2.11.
In **Problem 2.5**, find and sketch the cdf of B, the number of bases touched safely during an at-bat.

Problem 2.12.
A shipment of 7 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. Let X be the number of defective sets purchased by the hotel.

(a) Find the probability distribution of X.
(b) Find the cumulative distribution function of the random variable X.
(c) Find find $P(X = 1)$ and $P(0 < X \le 2)$.

Problem 2.13.
Voice calls cost 20 cents each and data calls cost 30 cents each. X is the cost of one telephone call. The probability that a call is a voice call is $P[V] = 0.6$. The probability of a data call is $P[D] = 0.4$.

(a) Find the pmf of X.
(b) What is $E[X]$, the expected value of X?

Problem 2.14.

Given the random variable X in Problem 2.9, let $V = g(X) = |X|$.

(a) Find the pmf $f_V(v)$ of V.
(b) Find the cdf $F_V(v)$ of V.
(c) Find $E[V]$.

Problem 2.15.

In a certain lottery game, the chance of getting a winning ticket is exactly one in a thousand. Suppose a person buys one ticket each day (except on the leap year day February 29) over a period of fifty years. What is the expected number $E[T]$ of winning tickets in fifty years? If each winning ticket is worth \$1000, what is the expected amount $E[R]$ collected on these winning tickets? Lastly, if each ticket costs \$2, what is your expected net profit $E[Q]$?

Problem 2.16.

In an experiment to monitor two calls, the pmf of N, the number of voice calls, is

$$f_N(n) = \begin{cases} 0.2, & n = 0, \\ 0.7, & n = 1, \\ 0.1, & n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $E[N]$, the expected number of voice calls.
(b) Find $E[N^2]$, the second moment of N.
(c) Find $Var[N]$, the variance of N.
(d) Find σ_N, the standard deviation of N.

Problem 2.17.

Show that the variance of $Y = aX + b$ is $Var[Y] = a^2 Var[X]$

Problem 2.18.

Given a random variable X with mean μ_X and variance σ_X^2 , find the mean and variance of the standardized random variable $Y = \dfrac{X - \mu_X}{\sigma_X}$.

2.2 CONTINUOUS RANDOM VARIABLES.

Problem 2.19.
The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0, & x \leq -1, \\ \dfrac{x+1}{2}, & -1 < x \leq 1, \\ 1, & x > 1 \end{cases}$$

(a) What is $P[X \leq 1/2]$?
(b) What is $P[-1/2 \leq X < 3/4]$?
(c) What is $P[|X| > 1/2]$?
(d) What is the value of a such that $P[X < a] = 0.8$?

Problem 2.20.
The cumulative distribution function of the continuous random variable V is

$$F_V(v) = \begin{cases} 0, & v \leq -5, \\ c(v+5)^2, & -5 < x \leq 7, \\ 1, & x > 7 \end{cases}$$

(a) What is c?
(b) What is $P[V \geq 4]$?
(c) What is $P[-3 \leq v < 0]$?
(d) What is the value of a such that $P[V \geq a] = 2/3$?

Problem 2.21.
The random variable X has the following probability density function $f_X(x) = \begin{cases} cx, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$. Use the pdf to find

(a) the constant c.
(b) $P[0 \leq X \leq 1]$.
(c) $P\left[-\frac{1}{2} \leq X \leq \frac{1}{2}\right]$.
(d) the cdf $F_X(x)$.

Problem 2.22.
The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0, & x \leq -1, \\ \dfrac{x+1}{2}, & -1 < x \leq 1, \\ 1, & x > 1 \end{cases}$$

Find the pdf $f_X(x)$ of X.

Problem 2.23.

Continuous random variable X has the following pdf $f_X(x) = \begin{cases} 1/4, & -1 \leq x \leq 3, \\ 0, & \text{otherwise.} \end{cases}$.

Define the random variable Y by $Y = h(X) = X^2$.

(a) Find $E[X]$ and $V[X]$.
(b) Find $h(E[X])$ and $E[h(X)]$.
(c) Find $V[Y]$.

Problem 2.24.

Random variable X has the following cdf $F_X(x) = \begin{cases} 0, & x \leq 0, \\ \dfrac{x}{2}, & 0 < x \leq 2, \\ 1, & x > 2 \end{cases}$.

(a) What is $E[X]$?
(b) What is $V[X]$?

Problem 2.25.
A continuous random variable X has the following pdf $f_X(x) = Ce^{-|x|}$ for all $x \in R$..

(a) Find the normalizing constant C.
(b) Find the cdf of X.
(c) Find the cdf and pdf of $Y = X^2$.
(d) Find $E(X); V(X)$.
(e) Find the probability that out of 5 independent trials of observing the value of X, there are exactly 3 times that X takes value in the interval $[2; 3]$.

Problem 2.26.

The cumulative distribution function of the continuous random variable X is

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{2} - k\cos x, & 0 < x \leq \pi, \\ 1, & x > \pi. \end{cases}$$

(a) What is k?
(b) What is $\mathrm{P}\left[0 < x < \frac{\pi}{2}\right]$?
(c) What is $E[X]$?

Problem 2.27.

The cumulative distribution function of the continuous random variable X is

$$F(x) = \begin{cases} 0, & x \leq -a, \\ A + B \arcsin \frac{x}{a}, & -a < x < a, \\ 1, & x \geq a. \end{cases}$$

(a) What are A and B?
(b) What is the pdf $f_X(x)$?

Problem 2.28.

The cumulative distribution function of the continuous random variable X is

$$F(x) = a + b \arctan x, \ (-\infty < x < +\infty)$$

(a) What are a and b?
(b) What is pdf $f_X(x)$?
(c) What is $P[-1 < X < 1]$?

Problem 2.29.

The cumulative distribution function of the continuous random variable X is

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{2}$$

What is the value of x_1 such that $P(X > x_1) = \frac{1}{4}$?

Problem 2.30.
The continuous random variable X has probability density function

$$f(x) = \begin{cases} k\sin 3x, & x \in \left(0; \dfrac{\pi}{3}\right), \\ 0, & x \notin \left(0; \dfrac{\pi}{3}\right). \end{cases}$$

Use the pdf to find

(a) the constant k.
(b) cdf $F_X(x)$.
(c) $P\left[\dfrac{\pi}{6} \leq X < \dfrac{\pi}{3}\right]$.

Problem 2.31.
The continuous random variable X has a pdf $f(x) = \dfrac{c}{e^x + e^{-x}}$. What is $E[X]$?

2.3 Some Important Probability Distributions.

Problem 2.32.
In a package of M&Ms, Y, the number of yellow M&Ms, is uniformly distributed between 5 and 15.

(a) What is the pmf of Y?
(b) What is $P[Y < 10]$?
(c) What is $P[Y > 12]$?
(d) What is $P[8 \leq Y \leq 12]$?

Problem 2.33.
The number of bits B in a fax transmission is a geometric $(p = 2.5 \times 10^{-5})$ random variable. What is the probability $P[B > 500,000]$ that a fax has over 500,000 bits?

Problem 2.34.
X is a continuous uniform random variable with expected value $\mu_X = 7$ and variance $Var[X] = 3$. What is the pdf of X?

Problem 2.35.
In a package of M&Ms, Y, the number of yellow M&Ms, is uniformly distributed between 5 and 15.

(a) What is the PMF of Y?
(b) What is $P[Y < 10]$?
(c) What is $P[Y > 12]$?
(d) What is $P[8 \leq Y \leq 12]$?

Problem 2.36.
Let X have the binomial pmf $f_X(x) = C_4^x \left(\frac{1}{2}\right)^4$.

(a) Find the standard deviation of the random variable X.
(b) What is $P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value?

Problem 2.37.
Give examples of practical applications of probability theory that can be modeled by the following pmf. In each case, state an experiment, the sample space, the range of the random variable, the pmf of the random variable, and the expected value: (a) Bernoulli;(b) Binomial; (c) Poisson. Make up your own examples.

Problem 2.38.
X is a continuous uniform random variable on $(-5, 5)$.

(a) What is the pdf $f_X(x)$?
(b) What is the cdf $F_X(x)$?
(c) What is $E[X]$?
(d) What is $E[X^5]$?
(e) What is $E[e^X]$?

Problem 2.39.
When a conventional paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. To be confident that a message is received at least once, a system transmits the message n times.

(a) Assuming all transmissions are independent, what is the pmf of K, the number of times the pager receives the same message?
(b) Assume $p = 0.8$. What is the minimum value of n that produces a probability of 0.95 of receiving the message at least once?

Problem 2.40.

When a two-way paging system transmits a message, the probability that the message will be received by the pager it is sent to is p. When the pager receives the message, it transmits an acknowledgment signal (ACK) to the paging system. If the paging system does not receive the ACK, it sends the message again.

(a) What is the pmf of N, the number of times the system sends the same message?
(b) The paging company wants to limit the number of times it has to send the same message. It has a goal of $P[N \leq 3] \geq 0.95$. What is the minimum value of p necessary to achieve the goal?

Problem 2.41.

The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is the Gaussian $(85, 10)$ random variable. What is $P[T > 100]$, $P[T < 60]$, and $P[70 \leq T \leq 100]$?

Problem 2.42.

What is the pdf of Z, the standard normal random variable?

Problem 2.43.

X is a Gaussian random variable with $E[X] = 0$ and $P[|X| \leq 10] = 0.1$. What is the standard deviation σ_X?

Problem 2.44.

Y is an exponential random variable with variance $V[Y] = 25$.

(a) What is the pdf of Y?
(b) What is $E[Y^2]$?
(c) What is $P[Y > 5]$?

Problem 2.45.

X is an exponential random variable where the pdf is $f_X(x) = \begin{cases} 5e^{-5x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$.

(a) What is $E[X]$?
(b) What is $P[0.4 < X < 1]$?

Problem 2.46.
X is a Gaussian random variable with $E[X] = 0$ and $\sigma_X = 0.4$.

(a) What is $P[X > 3]$?
(b) What is the value of c such that $P[3 - c < X < 3 + c] = 0.9$?

Problem 2.47.
The peak temperature T, in degrees Fahrenheit, on a July day in Antarctica is a Gaussian random variable with a variance of 225. With probability $\frac{1}{2}$, the temperature T exceeds 10 degrees. What is $P[T > 32]$, the probability the temperature is above freezing? What is $P[T < 0]$? What is $P[T > 60]$?

Problem 2.48.
The voltage X across a 1Ω resistor is a uniform random variable with parameters 0 and 1. The instantaneous power is $Y = X^2$. Find the cdf $F_Y(y)$ and the pdf $f_Y(y)$ of Y.

Problem 2.49.
X is uniform random variable with parameters 0 and 1. Find a function $g(x)$ such that the pdf of $Y = g(X)$ is $f_Y(y) = \begin{cases} 3y^2, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$

Problem 2.50.
Four microchips are to be placed in a computer. Two of the four chips are randomly selected for inspection before assembly of the computer. Let X denote the number of defective chips found among the two chips inspected. Find the probability mass and distribution function of X if

(a) Two of the microchips were defective.
(b) One of the microchips was defective.
(c) None of the microchips was defective.

Problem 2.51.
A four engine plane can fly if at least two engines work.

(a) If the engines operate independently and each malfunctions with probability q, what is the probability that the plane will fly safely?
(b) A two engine plane can fly if at least one engine works and if an engine malfunctions with probability q, what is the probability that plane will fly safely?
(c) Which plane is the safest?

Problem 2.52.

A rat maze consists of a straight corridor, at the end of which is a branch; at the branching point the rat must either turn right or left. Assume 10 rats are placed in the maze, one at a time.

(a) If each is choosing one of the two branches at random, what is the distribution of the number that turn right?
(b) What is the probability at least 9 will turn the same way?

Problem 2.53.

A student who is trying to write a paper for a course has a choice of two topics, A and B. If topic A is chosen, the student will order 2 books through inter-library loan, while if topic B is chosen, the student will order 4 books. The student feels that a good paper necessitates receiving and using at least half the books ordered for either topic chosen.

(a) If the probability that a book ordered through inter-library loan actually arrives on time is 0.9 and books arrive independently of one another, which 2 topics should the student choose to maximize the probability of writing a good paper?
(b) What if, the arrival probability is only 0.5 instead of 0.9?

Problem 2.54.

The number of phone calls at a post office in any time interval is a Poisson random variable. A particular post office has on average 2 calls per minute.

(a) What is the probability that there are 5 calls in an interval of 2 minutes?
(b) What is the probability that there are no calls in an interval of 30 seconds?
(c) What is the probability that there are no less than one call in an interval of 10 seconds?

Problem 2.55.

An airline sells 200 tickets for a certain flight on an airplane that has only 198 seats because, on the average, 1 percent of purchasers of airline tickets do not appear for the departure of their flight. Determine the probability that everyone who appears for the departure of this flight will have a seat.

Problem 2.56.
Let X be an exponential random variable with parameter and define $Y = [X]$, the largest integer in X, (ie. $[x] = 0$ for $0 \leq x < 1$, $[x] = 1$ for $1 \leq x < 2$ etc.)

(a) Find the probability function for Y.
(b) Find $E(Y)$.
(c) Find the distribution function of Y.
(d) Let Y represent the number of periods that a machine is in use before failure. What is the probability that the machine is still working at the end of 10th period given that it does not fail before 6th period?

Problem 2.57.
Starting at 5:00 am, every half hour there is a flight from San Francisco airport to Los Angeles International Airport. Suppose that none of these planes sold out and that they always have room for passengers. A person who wants to fly LA arrives at the airport at a random time between 8:45 - 9:45 am. Find the probability that she waits at most 10 minutes and at least 15 minutes.

Problem 2.58.
The probability of having a new born boy is 0.51. Take the survey of 1000 new born babies in a hospital. Calculate the probability that the number of boys is smaller than the number of girls.