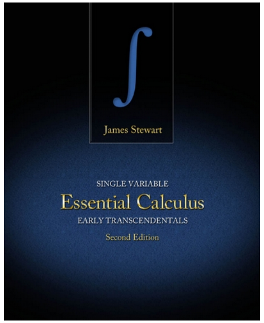
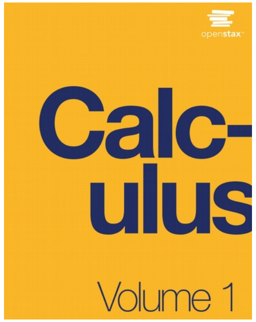
Chapter 7: Applications of Integration





7.1 Areas between Curves

7.2 Volumes

7.3 Volumes by Cylindrical Shells

7.4 Arc Length

7.5 Area of a Surface of Revolution

7.6 Applications to Physics and Engineering

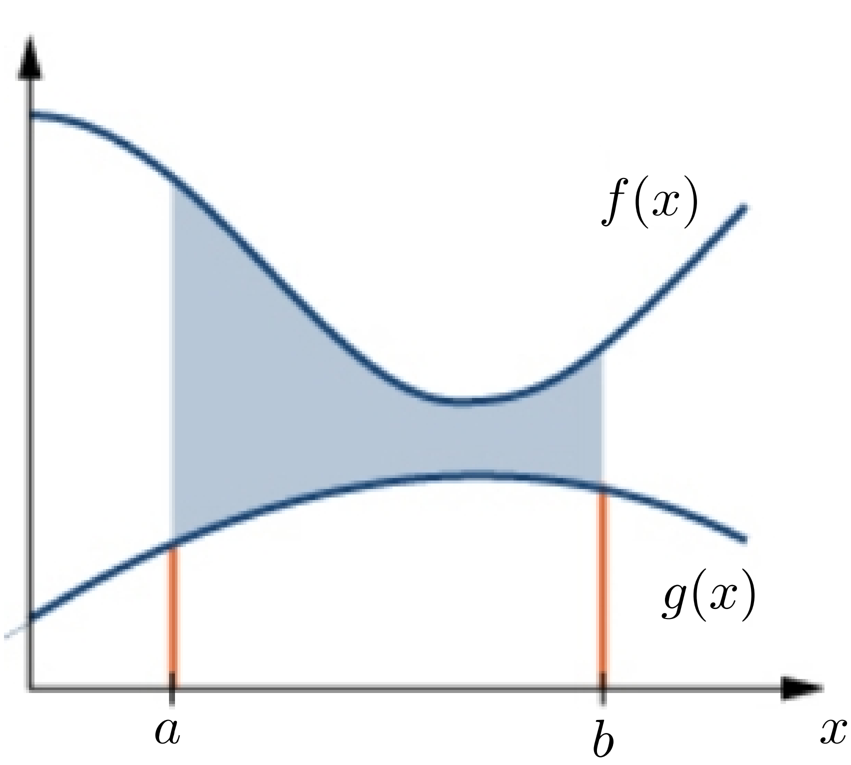
7.7 Differential Equations

The pictures are taken from the books:

[1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280
2) G. Strang and E. J. Herman, Calculus 1, https://openstax.org/details/books/calculus-volume-1]

7.1 Areas between Curves

• Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions.



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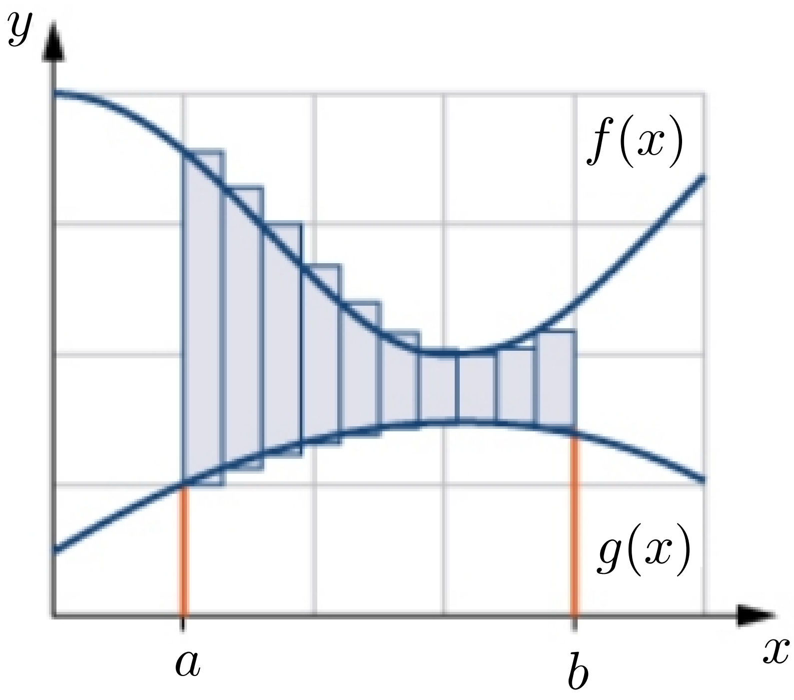
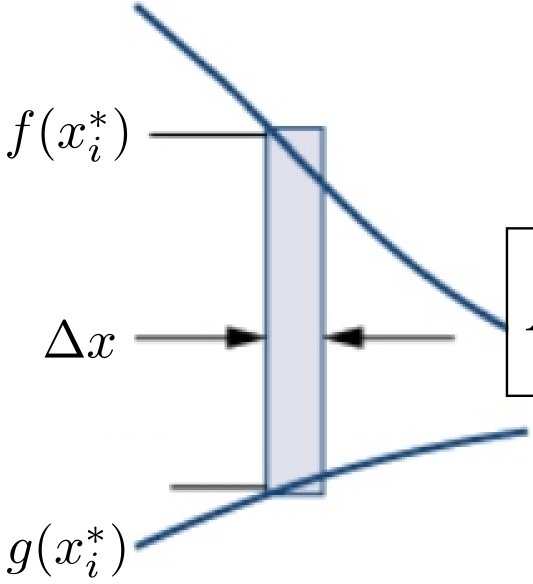




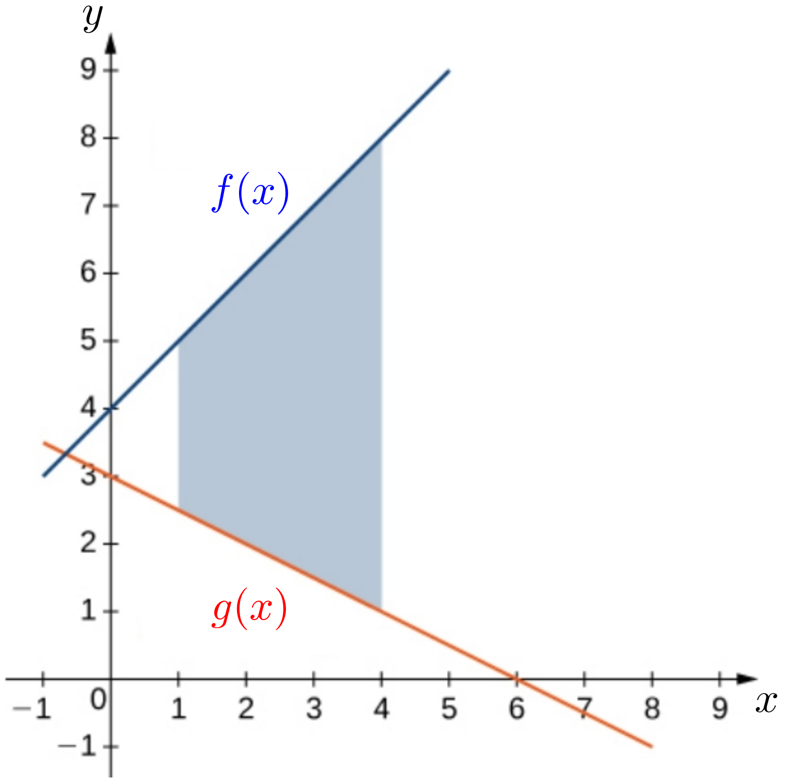
$$A \approx \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right] \Delta x$$

$$\Big\downarrow \; n \to \infty$$

$$\boxed{A = \int_a^b \left[f(x) - g(x)\right] \mathrm{d}x}$$

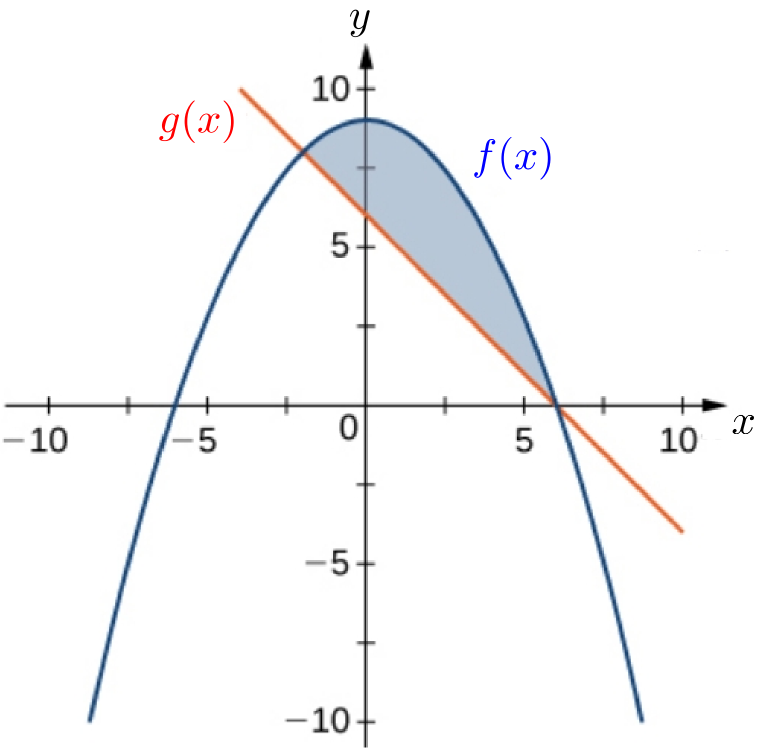
7.1 Examples

1. If R is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \dfrac{x}{2}$ over the interval $[1, 4]$, find the area of region R.



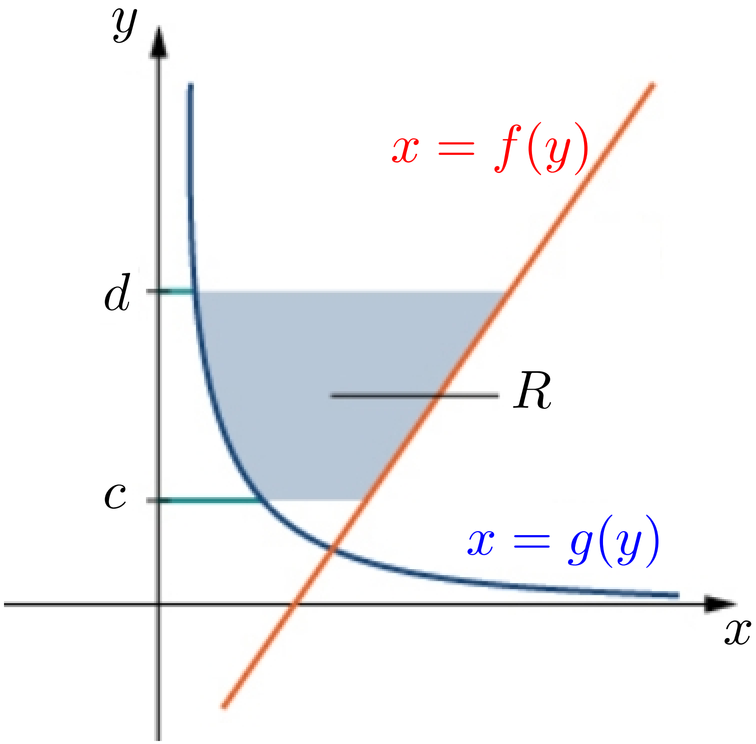
7.1 Examples

2. If R is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.



7.1 Areas between Curves

• Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$, then its area is



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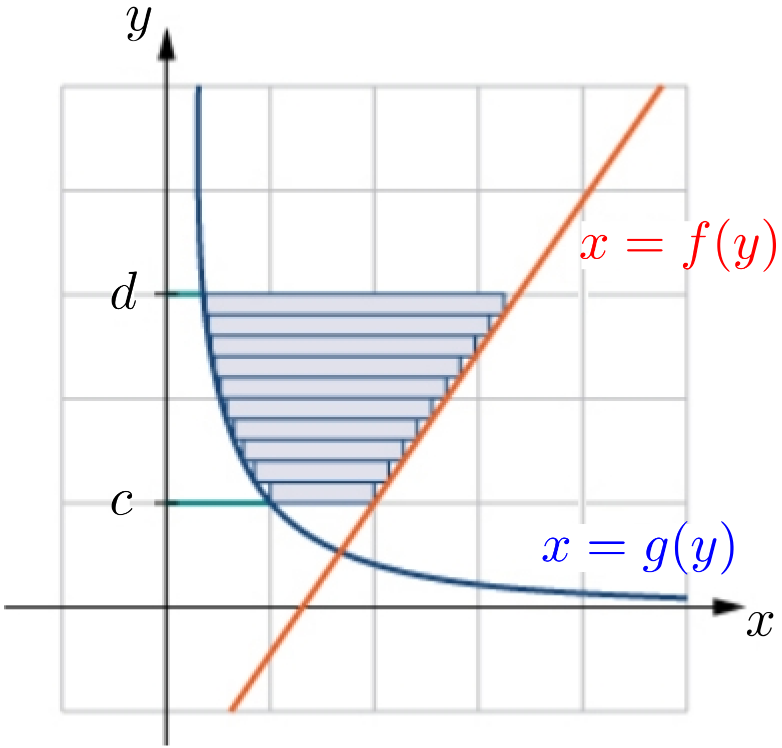
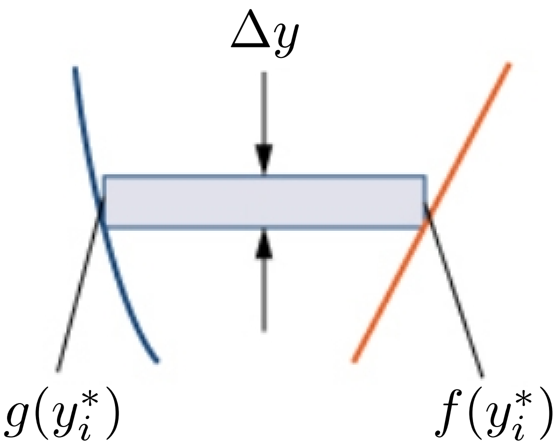
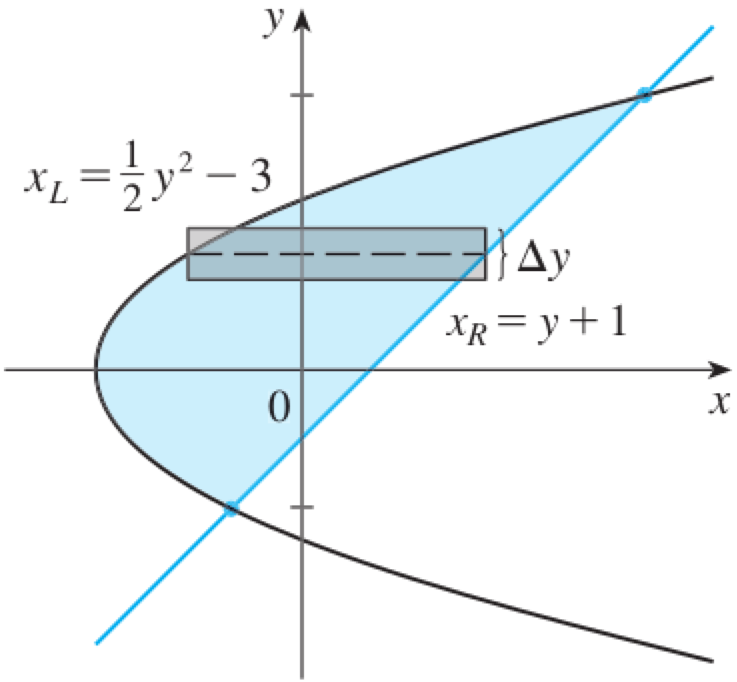




$$\boxed{\begin{aligned} A &= \int_c^d \left[f(y) - g(y)\right] \mathrm{d}y \\ &= \int_c^d \left[x_R - x_L\right] \mathrm{d}y \end{aligned}}$$

7.1 Examples

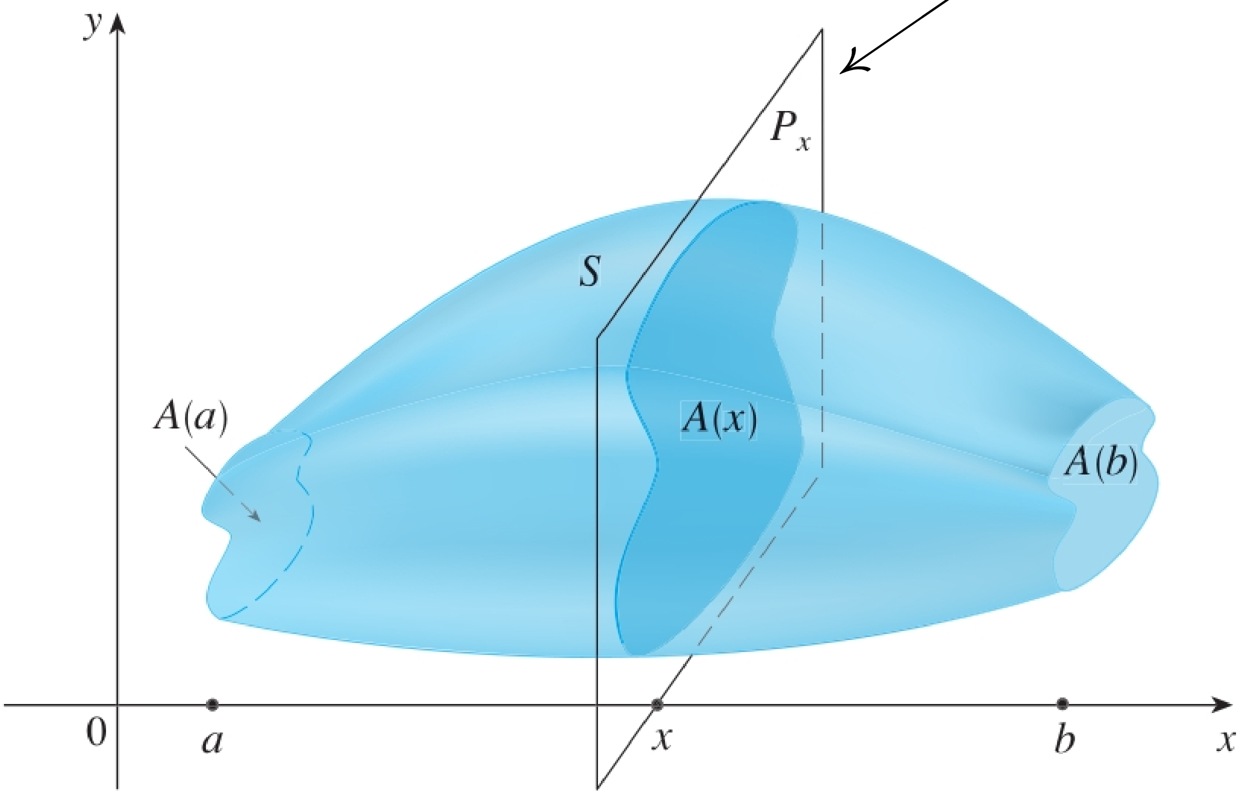
- Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



7.2 Volumes

- Using Calculus to define the **Volume** of a solid S.

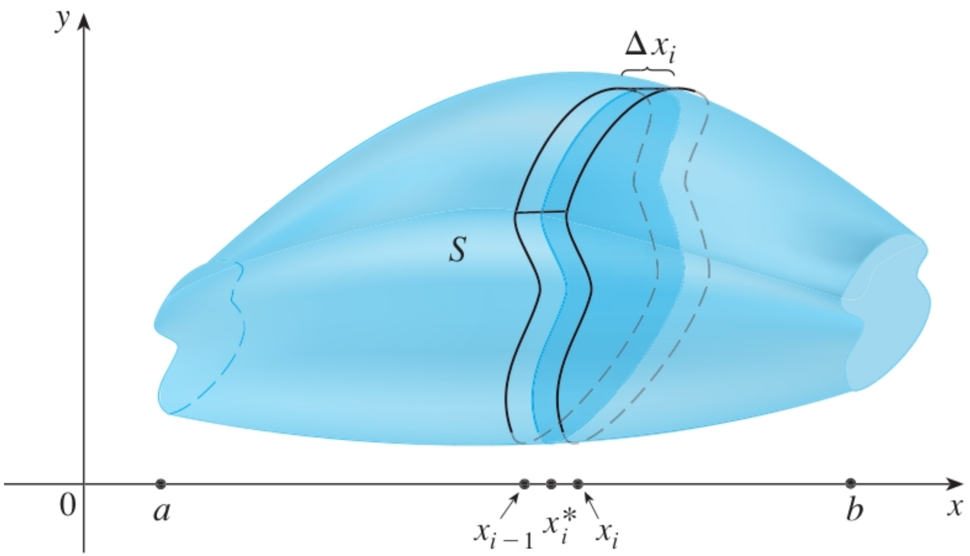
Cross-section

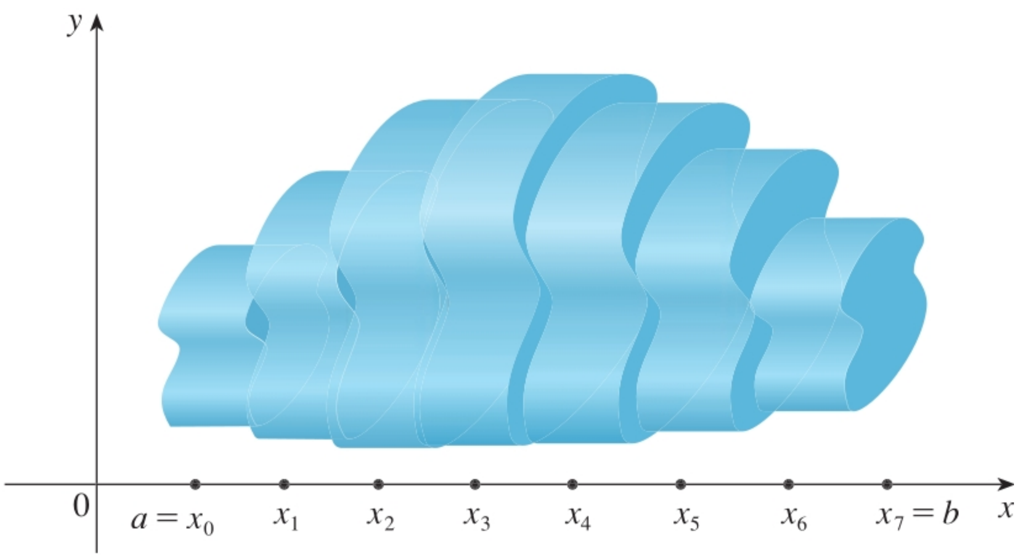


7.2 Volumes

- Using Calculus to define the **Volume** of a solid S.

$$V(S_i) \approx A(x_i^*)\Delta x_i \quad \Rightarrow \quad V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x_i \quad \Rightarrow \quad V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i$$

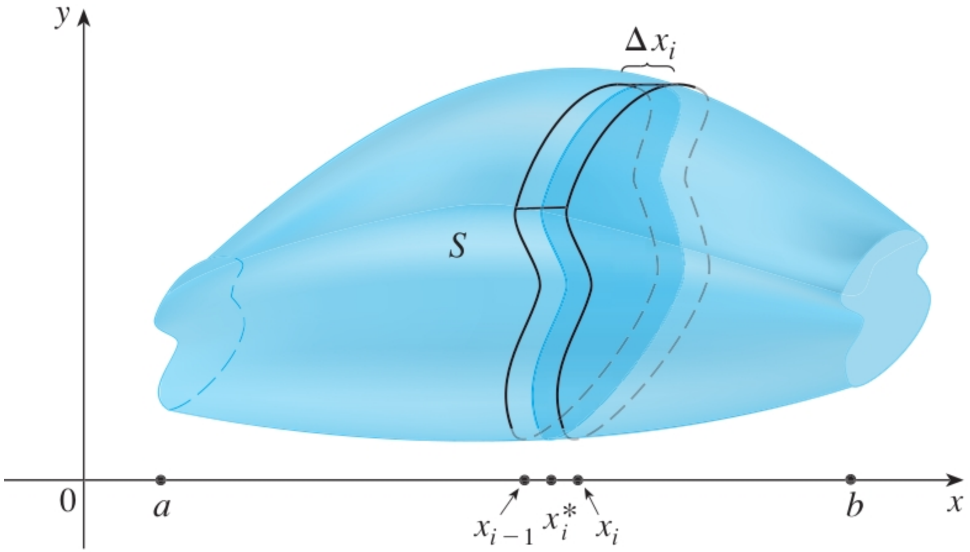


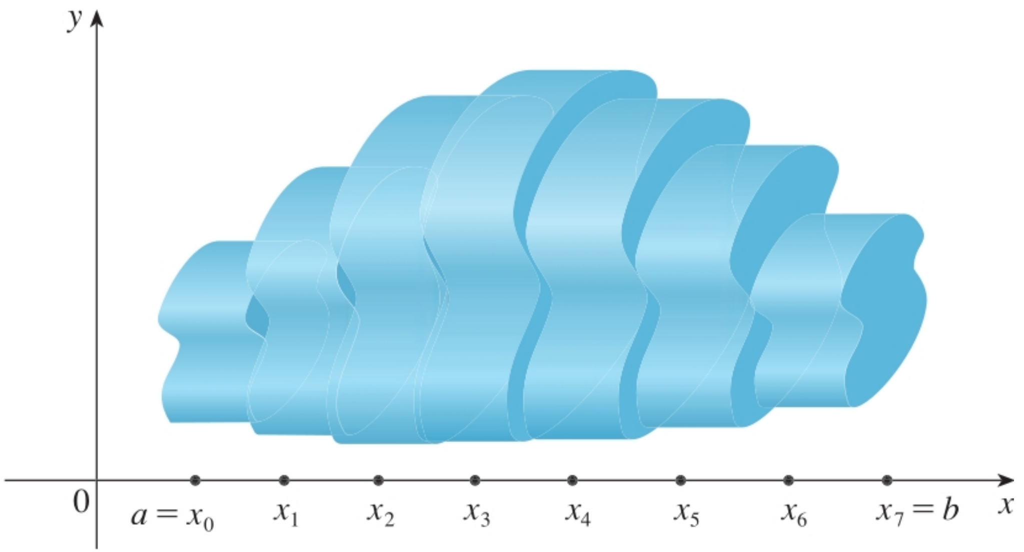


7.2 Volumes

- Using Calculus to define the **Volume** of a solid S.

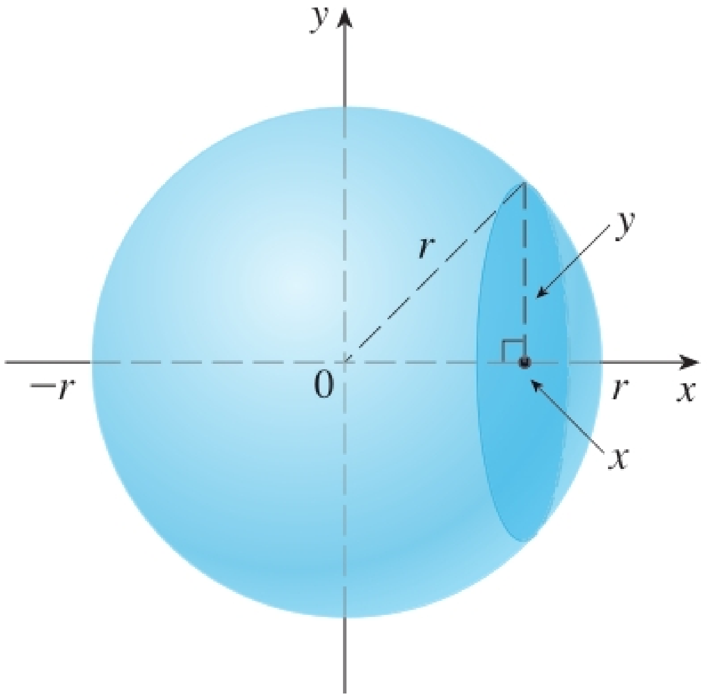
$$\boxed{V = \int_a^b A(x)\mathrm{d}x}$$





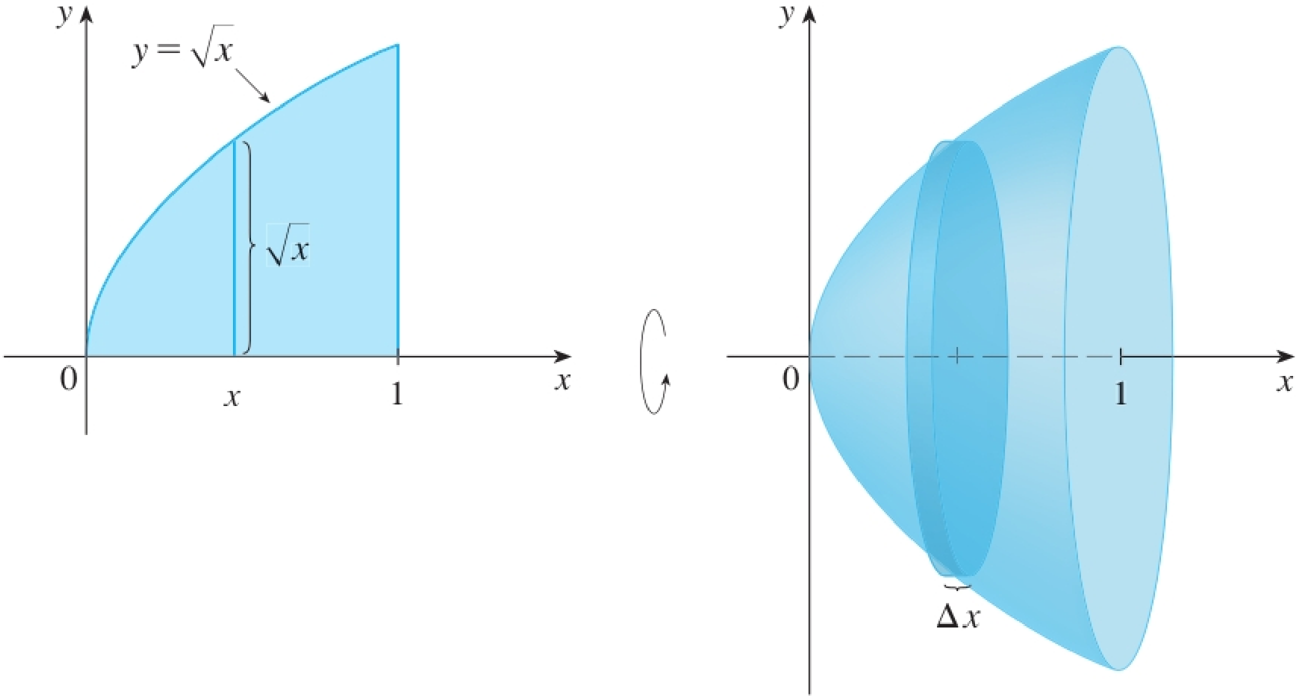
7.2 Examples

1. Show that the volume of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$.



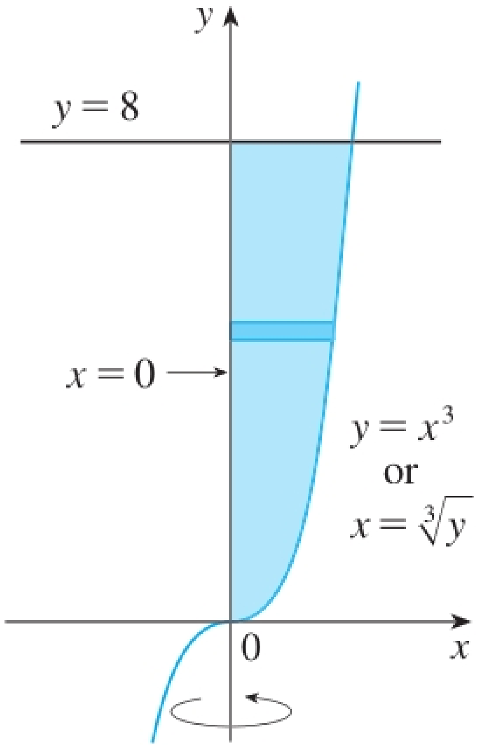
7.2 Examples

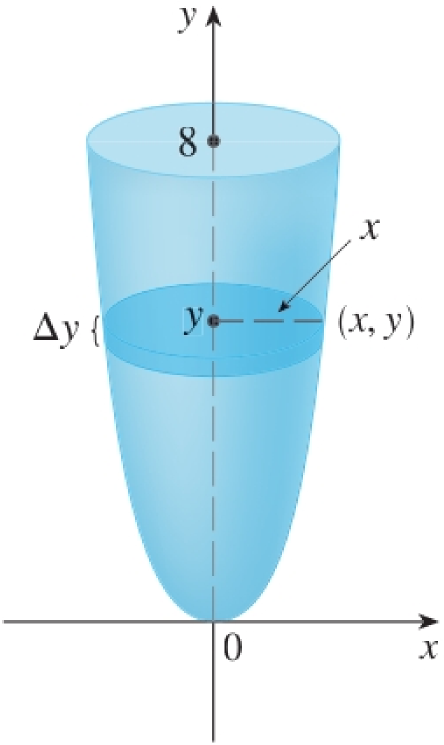
2. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.



7.2 Examples

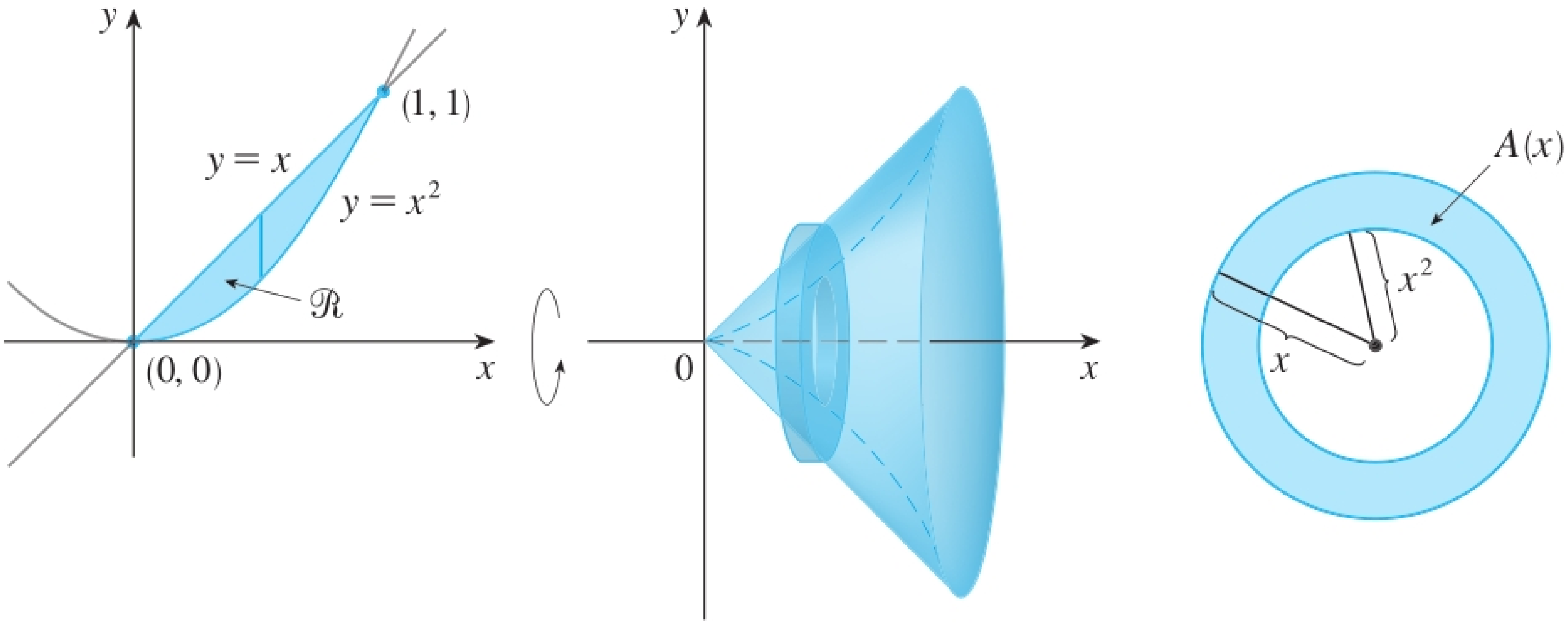
3. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.





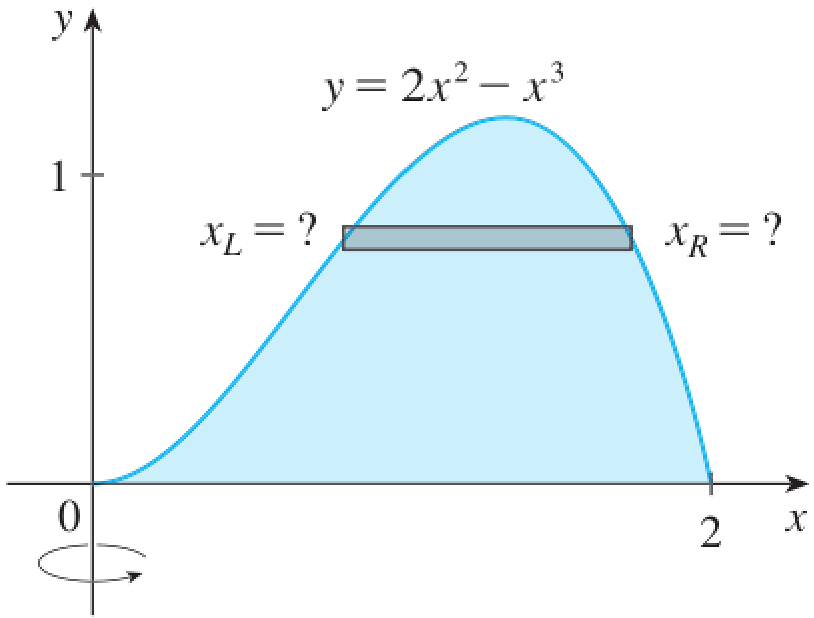
7.2 Examples

4. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



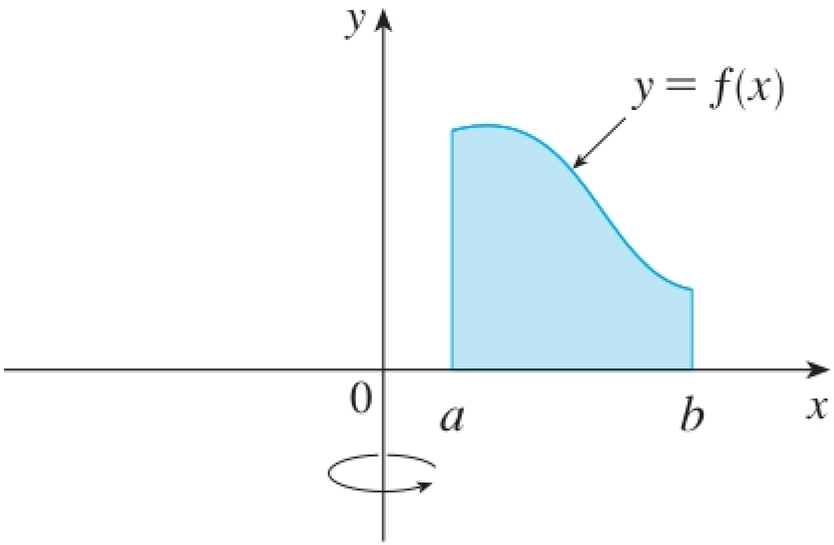
7.3 Volumes by Cylindrical Shells

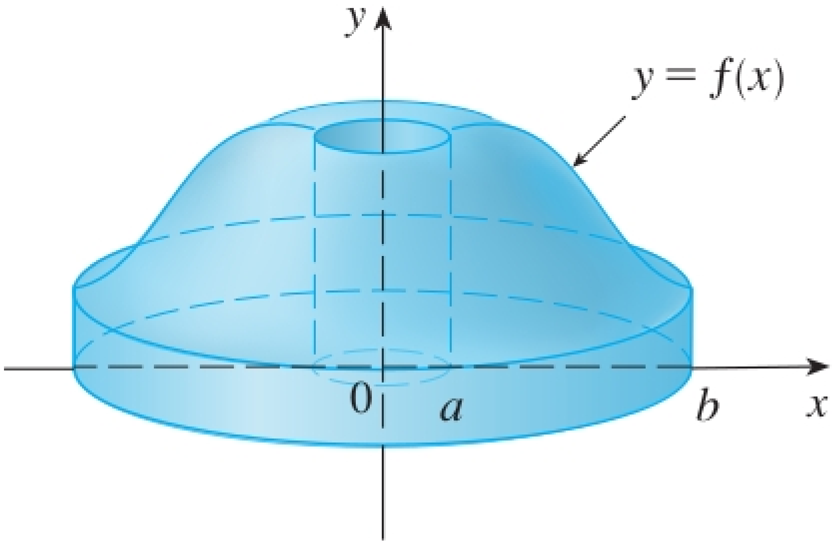
• Some volume problems are very difficult to handle by the methods of the preceding section.



7.3 Volumes by Cylindrical Shells

- Better method: **Method of cylindrical shells**





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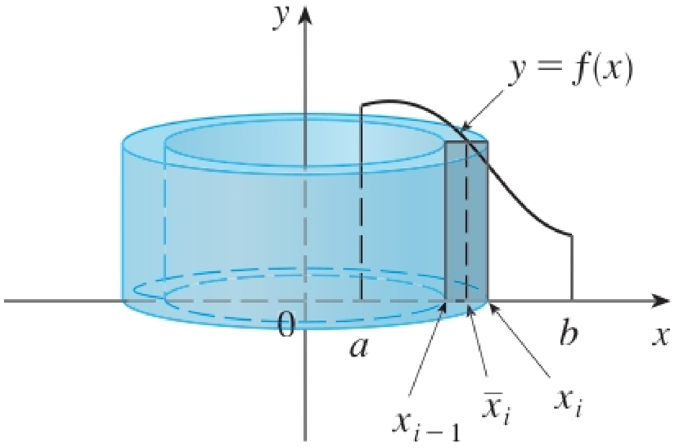
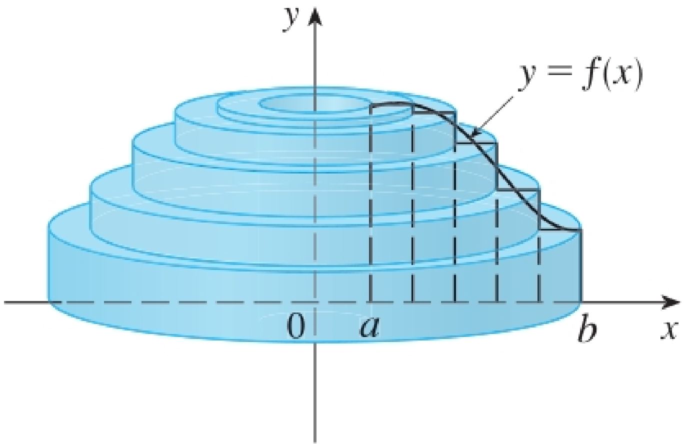
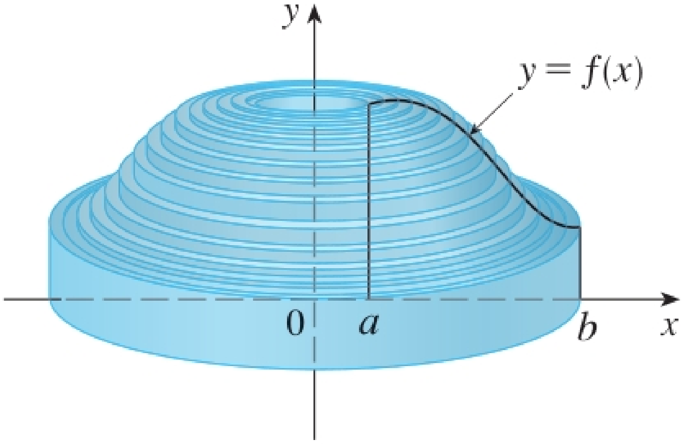
- Better method: **Method of cylindrical shells**



$$V_i = (2\pi\overline{x}_i)[f(\overline{x}_i)]\Delta x$$



$$V \approx \sum_{i=1}^{n}(2\pi\overline{x}_i)[f(\overline{x}_i)]\Delta x$$

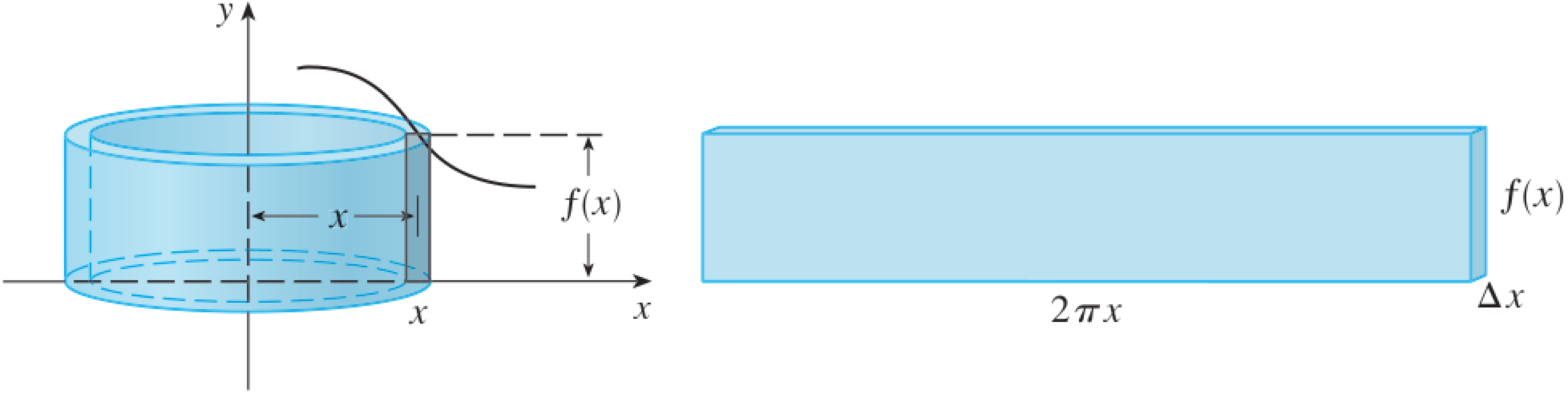


$$\Rightarrow \qquad V = \lim_{\substack{n\to\infty \\ \max \Delta x \to 0}} \sum_{i=1}^{n}(2\pi\overline{x}_i)[f(\overline{x}_i)]\Delta x = \boxed{\int_a^b (2\pi x)f(x)\mathrm{d}x}$$

7.3 Volumes by Cylindrical Shells

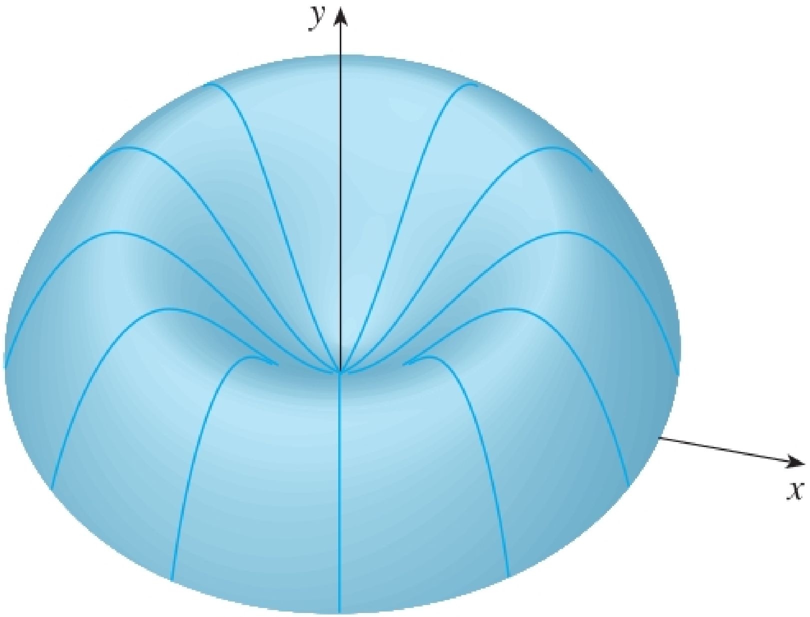
- The best way to remember this formula:

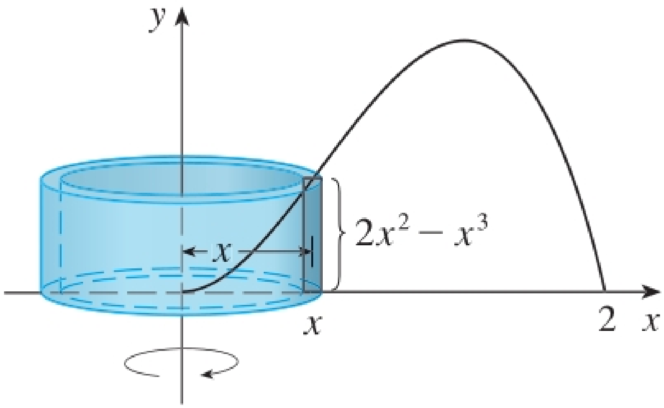
$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \quad \underbrace{[f(x)]}_{\text{height}}\, dx$$



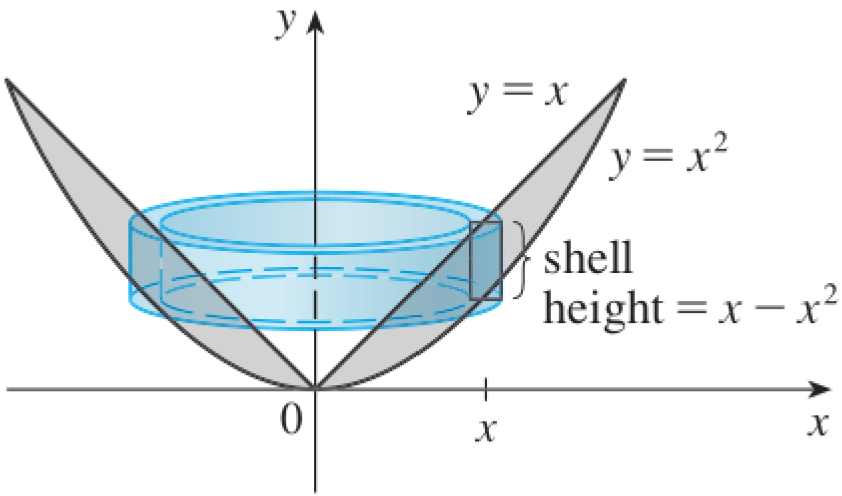
7.3 Examples

1. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

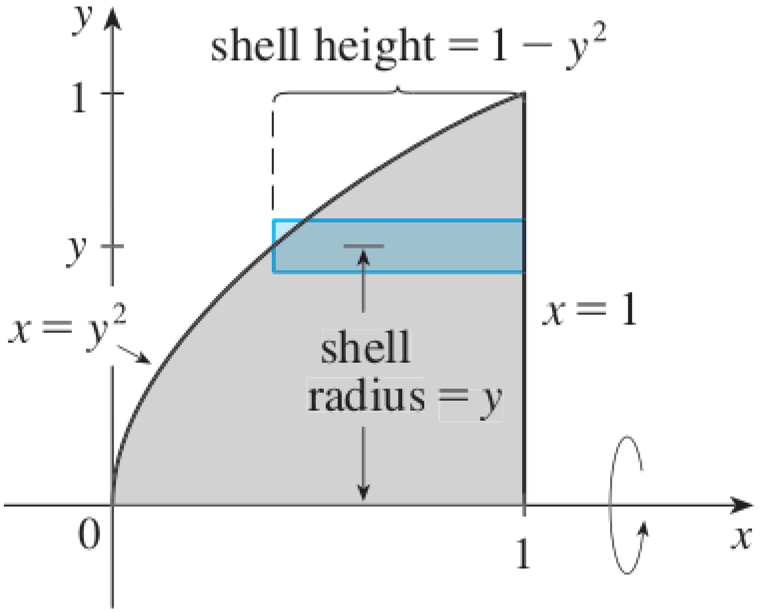




2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

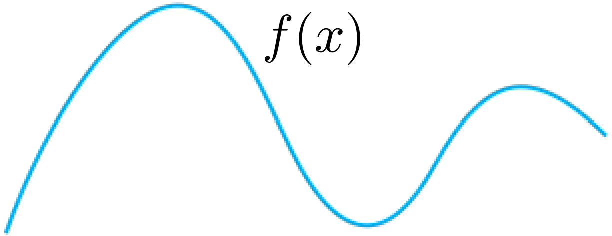


3. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



7.4 Arc Length

- What is the length of this curve?



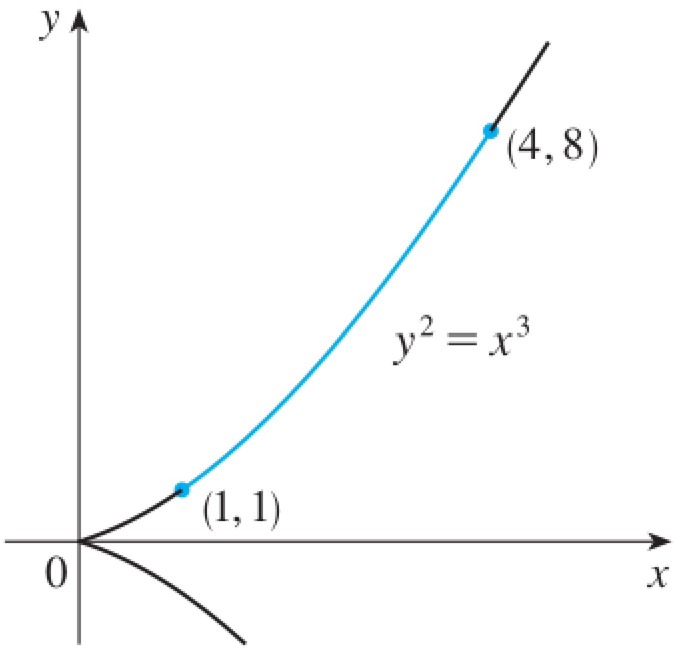
The Arc Length Formula: If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$,$a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \mathrm{d}x = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x$$

(Proof on whiteboard)

7.4 Example

- Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$.



7.4 Arc Length

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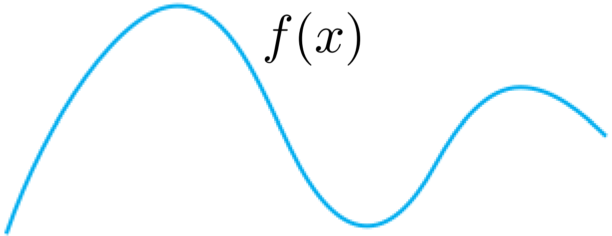


The Arc Length Formula: If a curve has the equation $x = g(y)$, $c \leq y \leq d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\,\mathrm{d}y = \int_c^d \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$$

7.4 Examples

1. Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

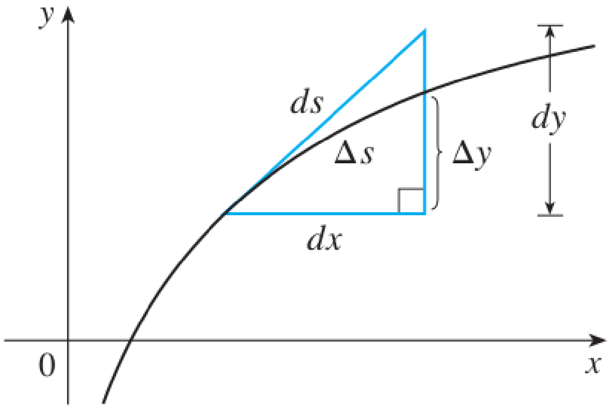
2. **(a)** Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$. **(b)** Use Simpson's Rule with $n = 10$ to estimate the arc length.

7.4 Arc Length

- The Arc Length Function: $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\mathrm{d}t$

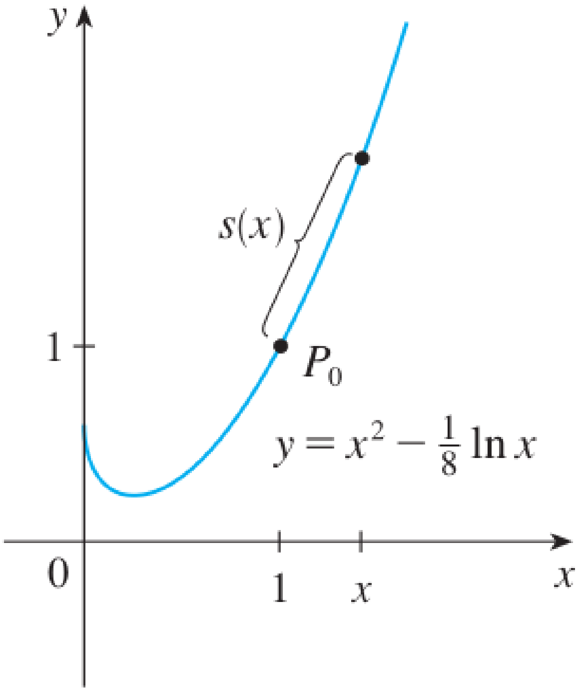
with

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \qquad \Leftrightarrow \qquad (\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$$



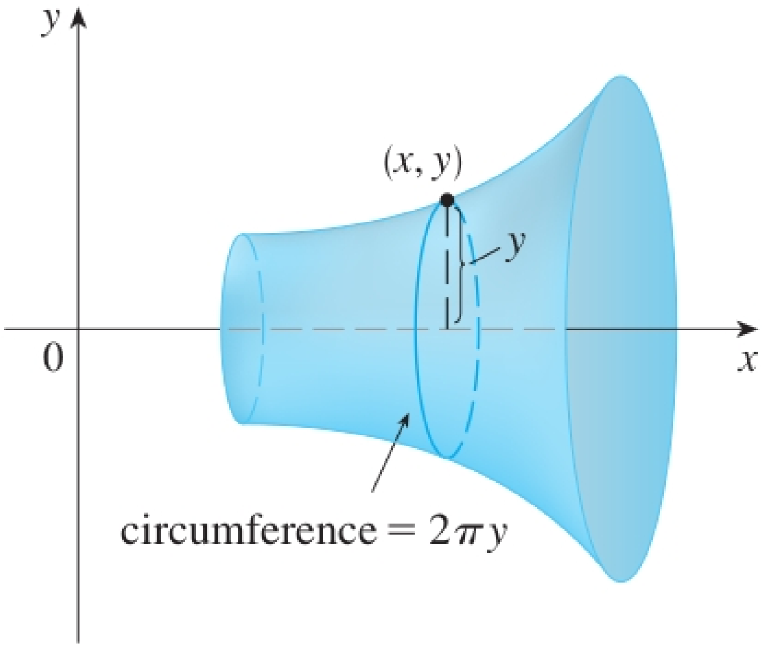
7.4 Example

- Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln(x)$ taking $P_0(1,1)$ as the starting point.

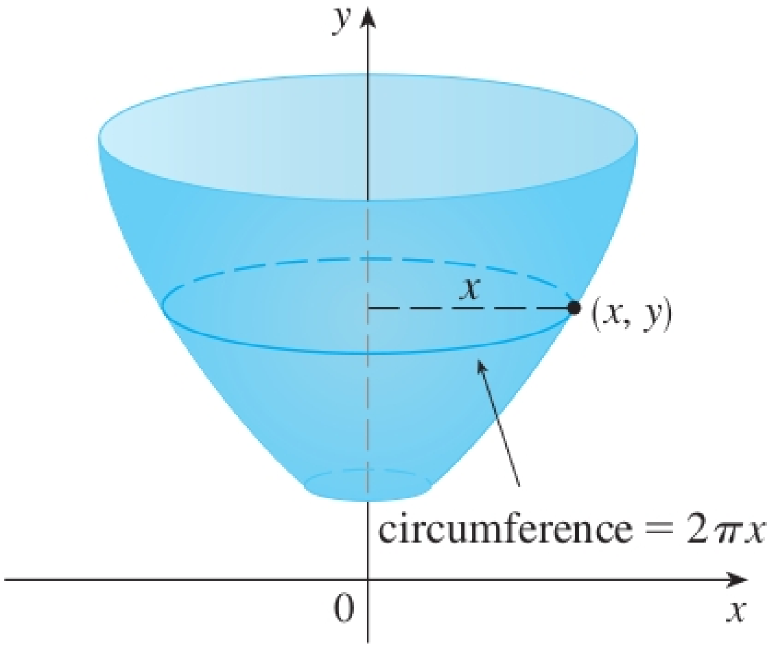


7.5 Area of a Surface of Revolution

• A surface of revolution is formed when a curve $y = f(x)$ is rotated about a line.



Rotation about x-axis



Rotation about y-axis

7.5 Area of a Surface of Revolution

• In the case where f is positive and has a continuous derivative, we define the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis as

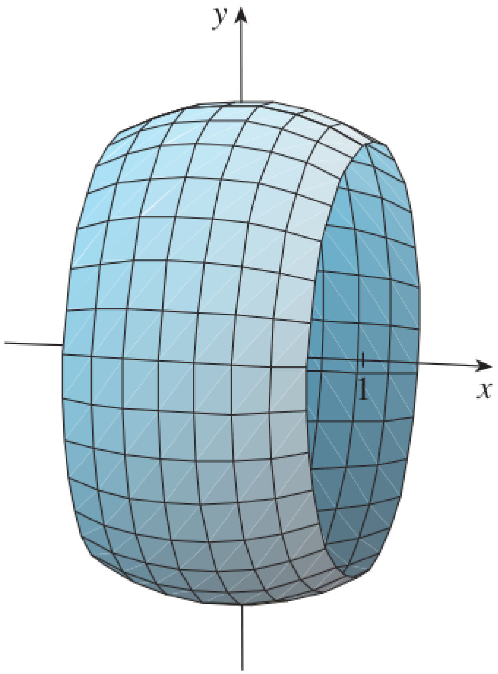
$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\mathrm{d}x = \int_a^b 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x = \int_a^b 2\pi y \mathrm{d}s$$

• If the curve is described as $x = g(y)$, $c \leq y \leq d$, then the formula for surface area becomes

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\mathrm{d}y = \int_c^d 2\pi x\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y = \int_c^d 2\pi x \mathrm{d}s$$

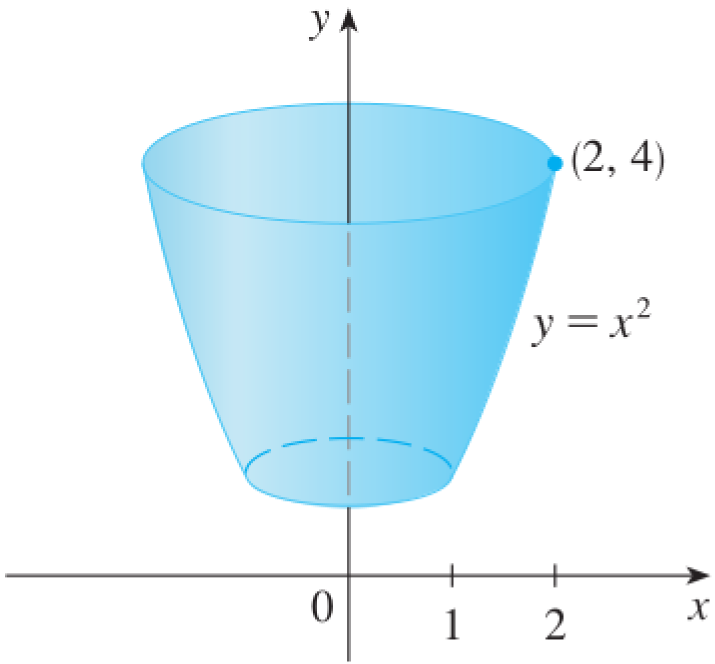
7.5 Examples

1. The curve y=$\sqrt{4-x^2}$, -1$\leq x \leq 1$, is an arc of the circle $x^2+y^2=4$. Find the area of the surface obtained by rotating this arc about the x -axis. (The surface is a portion of a sphere of radius 2.)



7.5 Examples

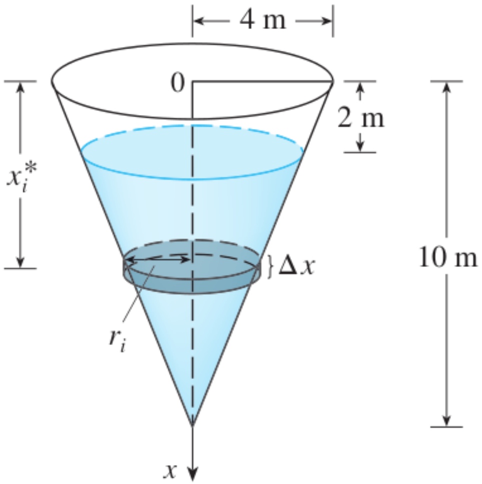
2. The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.



7.6 Applications to Physics and Engineering

1. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

2. A tank has the shape of an inverted circular cone with height 10m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)



7.7 Differential Equations

• A **differential equation** is an equation that contains an unknown function and one or more of its derivatives. Here are some examples:

$$\boxed{1} \quad y' = xy$$

$$\boxed{2} \quad y'' + 2y' + y = 0$$

$$\boxed{3} \quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - y = e^{-x}$$

7.7 Differential Equations

• A **separable differential equation** is a first-order differential equation that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)f(y) \qquad \text{or equivalently} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)}\,, \quad h(y) \neq 0$$

where $f(y) = 1/h(y)$.

• Solution: $h(y)\mathrm{d}y = g(x)\mathrm{d}x \;\Rightarrow\; \int h(y)\mathrm{d}y = \int g(x)\mathrm{d}x + C$

• Initial-value problem: Determining $C = y(x_0) = y_0$

7.7 Examples

1. (a) Solve the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2}{y^2}$. **(b)** Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

2. Solve the differential equations

$$(\mathbf{a}) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2}{2y + \cos(y)}$$

$$(\mathbf{b}) \quad \frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y$$

$$(\mathbf{c}) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

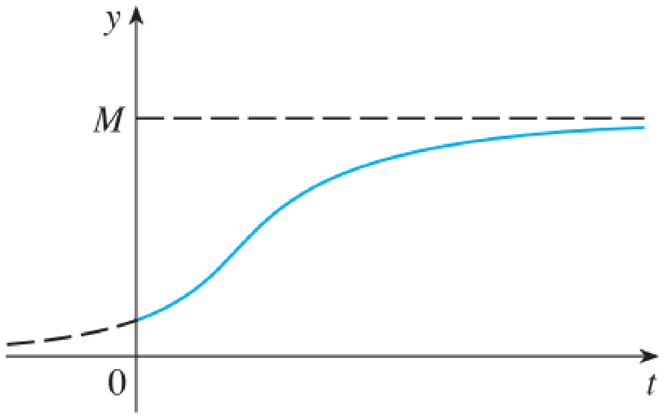
7.7 Differential Equations

• A **logistic differential equation** is a first-order differential equation the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky(M-y)\,,$$

where k is a constant and M is the **carrying capacity** (maximal population size).

• Solution: $y(t) = \dfrac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}$



7.7 Differential Equations

• Suppose we are given a first-order differential equation of the form

$$y' = F(x, y)\,,$$

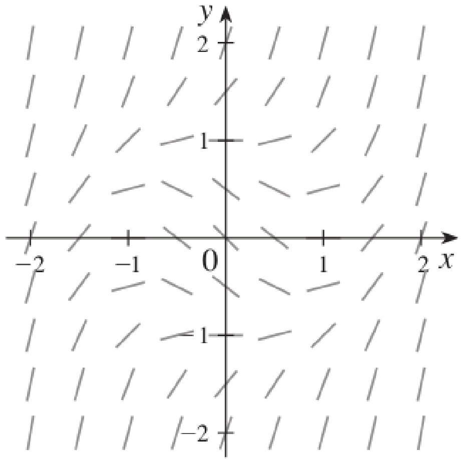
Even if it is impossible to find the solution, we can still visualize it. If we draw short line segments with slopes $F(x, y)$ at several points (x, y) , the result is called a **direction field** (or **slope field**).

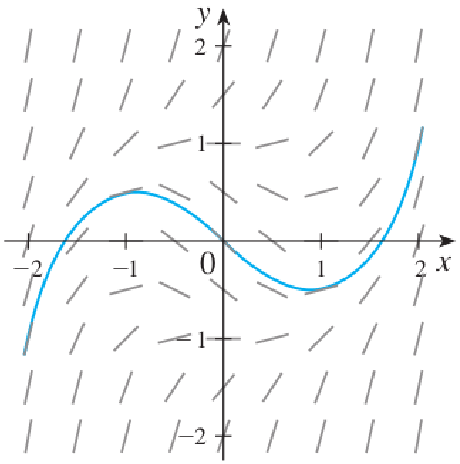
• These line segments indicate the direction in which a solution curve is heading, so the direction field helps us visualize the general shape of these curves.

7.7 Example

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$. **(b)** Use part (a) to sketch the solution curve that passes through the origin.

x	-2	-1	0	1	2	-2	-1	0	1	2	...
y	0	0	0	0	0	1	1	1	1	1	...
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	...





7.7 Example

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$. **(b)** Use part (a) to sketch the solution curve that passes through the origin.