Hanoi University of Science and Technology
School of Information and Communications Technology

Discrete Mathematics

Nguyễn Khánh Phương

Department of Computer Science
School of Information and Communication Technology
E-mail: phuongnk@soict.hust.edu.vn

Discrete Mathematics

Discrete Mathematics deals with

- "Separated" or discrete sets of objects (rather than continuous sets)
- Processes with a sequence of individual steps (rather than continuously changing processes)

Kind of problems solved by discrete mathematics

- How many ways are there to choose a computer password?
- What is the probability of winning a lottery?
- Is there a link between two users in a social network?
- What is the shortest path between two cities using a transportation system?
- How can a list of integers sorted in increasing order? How many steps are required to do such a sorting?

Importance of Discrete Mathematics

- Information is stored and manipulated by computers in a discrete fashion
- Applications in many different areas
- Discrete mathematics is a gateway to more advanced courses
- Develops mathematical reasoning skills
- Emphasizes the new role of mathematics

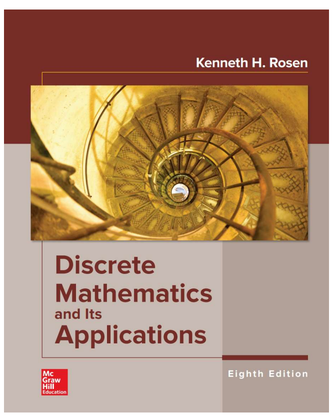
The new role of Mathematics

- Make the computer to solve the problem for you
- Modeling (vs. calculations)
- Using logic
  - to choose the right model
  - to write a correct computer program
  - to justify answers
- Efficiency
  - make the computer to solve the problem fast
  - choose the more efficient model

Goals of this course

- Study of standard facts of discrete mathematics
- Development of mathematical reasoning skills (emphasis on modeling, logic, efficiency)
- Discussion of applications

Text book

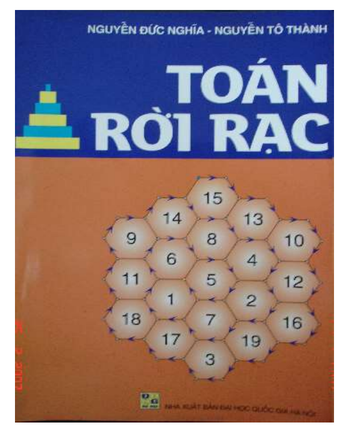


Rosen K.H. *Discrete Mathematics and its Applications (8th Editions)*. McGraw - Hill Book Company, 2019.

Use lecture notes as study guide.

Text book

Nguyễn Đức Nghĩa,
Nguyễn Tô Thành
TOÁN RỜI RẠC
(in lần thứ ba)
Nhà xuất bản Đại học Quốc gia Hà nội, 2003, 290 trang



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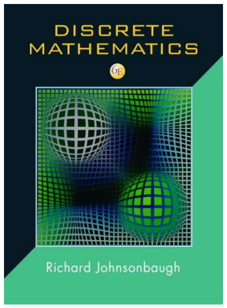
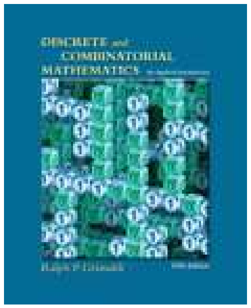
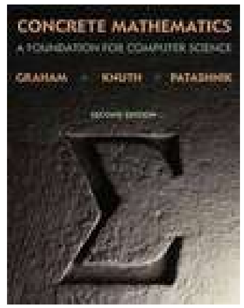
2. **Grimaldi R.P.** *Discrete and Combinatorial Mathematics (an Applied Introduction),* Addison-Wesley, 5th edition, 2004.
3. **R. Graham, O. Patashnik, and D.E. Knuth.** *Concrete Mathematics*, Second Edition. Addison-Wesley, 1994.







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5. Nguyễn Xuân Quỳnh. *Cơ sở Toán rời rạc và ứng dụng*. NXB KHKT, Hà nội, 1996.

6. Đỗ Đức Giáo. *Toán rời rạc*. NXB KHKT, Hà nội, 2001.

PART 1

COMBINATORIAL THEORY

(Lý thuyết tổ hợp)

PART 2

GRAPH THEORY

(Lý thuyết đồ thị)

Contents of Part 1: Combinatorial Theory

Chapter 1. Counting problem

- This is the problem aiming to answer the question: "How many ways are there that satisfy given conditions?" The counting method is usually based on some basic principles and some results to count simple configurations .
- Counting problems are effectively applied to evaluation tasks such as calculating the probability of an event, calculating the complexity of an algorithm (how long the algorithm will take to run), ...

Street art

Given N paintings in a row over a distance of M centimeters. Each painting i ($1 \le i \le N$) will be drawn on a length of t_i cm, so $t_1+t_2+..+t_n = M$. The K city's most famous artists have been selected to do this work, each artist will be assigned to draw at least one painting. To facilitate the artist's work, if someone is assigned to draw more than one painting, the paintings must be adjacent to each other on the street art

Contents of Part 1: Combinatorial Theory

Chapter 1. Counting problem

- This is the problem aiming to answer the question: “How many ways are there that satisfy given conditions?” The counting method is usually based on some basic principles and some results to count simple configurations .
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Chapter 2. Existence problem

In the counting problem, configuration existence is obvious; in the existence problem, we need to answer the question: "Is there a combinatorial configuration that satisfies given properties ?”

Chapter 3. Enumeration problem

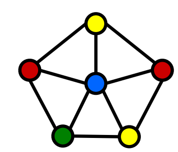
This problem is interested in giving all the configurations that satisfy given conditions.

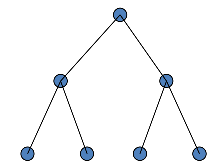
Chapter 4. Combinatorial optimization problem

- Unlike the enumeration problem, this problem only concerns the "best" configuration in a certain sense.
- In the optimization problems, each configuration is assigned a numerical value (which is the use value or the cost to construction the configuration), and the problem is that among the configurations that satisfy the given conditions, find the configuration with the maximum or minimum value assigned to it

Contents of Part 2: Graph Theory

- Graphs
- Degree sequence, Eulerian graphs, isomorphism
- Trees
- Matching
- Coloring



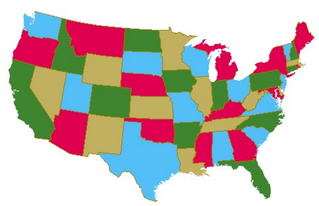


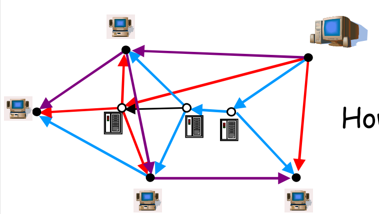
Computer networks, data structures

Contents of Part 2: Graph Theory

How to color a map?

How to schedule exams/projects?





How to send data efficiently?

Contents of Part 1

Chapter 0: Sets, Relations

Chapter 1: Counting problem

Chapter 2: Existence problem

Chapter 3: Enumeration problem

Chapter 4: Combinatorial optimization problem

Contents of Part 1

Chapter 0: Sets, Relations

Chapter 1: Counting problem

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Chapter 4: Combinatorial optimization problem

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Chapter 0

SETS, RELATIONS

Contents

1. Definitions

2. Set operations

3. The algebra of sets

4. Computer representation of sets

5. Relations

6. Functions

1.Definitions

- We have already implicitly dealt with sets
  – Integers (Z), rationals (Q), naturals (N), reals (R), etc.
- We will develop more fully
  – The definitions of sets
  – The properties of sets
  – The operations on sets

1. Definitions

1.1 Set and element

1.2. Specification of set

1.1 Set and element

- **Definition**:
  - A set is an unordered collection of (unique) objects
  - The objects in a set are called elements or members of a set.

  A set is said to contain its elements.
- Notation, for a set A:
  - $x \in A$: x is an element of A
  - $x \notin A$: x is not an element of A

Example:

V={a, e, i, o, u} (vowels in English)

C = all students subscribed to IT3020E in Winter 2020

Note:

- We often denote sets with capitals
- Brackets are used to define the set. {.}

1.1 Set and element

- **Definition**: A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where
  - m_1 occurs a_1 times
  - m_2 occurs a_2 times
  - …
  - m_r occurs a_r times

1. Definitions

1.1 Set and element

1.2. Specification of set

1.2 Specification of set

Our first concern will be how to describe a set; that is, how do we most conveniently describe a set and the elements that are in it? Sets can be defined in various ways.

At first we consider two ways:

1. Set extension
2. Set intension

1.2 Specification of set

- A set is defined in **extension** when you enumerate all the elements:

$$O=\{0,2,4,6,8\}$$

- The **set-builder** notation

$$A = \{x \mid \text{conditions}(x)\}.$$

this could be read as "all x such that the conditions hold true".

Example: $O=\{ x \mid (x\in Z) \wedge (x=2k) \text{ for some } k\in Z\}$

reads: O is the set that contains all x such that x is an integer and x is even

- A set is defined in **intension** when you give its set-builder notation

$$O=\{ x \mid (x\in Z) \wedge (0 \le x \le 8) \wedge (x=2k) \text{ for some } k \in Z \}$$

1.2 Specification of set

Well-known sets in math:

- $N = \{0,1,2,3,\ldots\}$
- $Z = \{\ldots,-2,-1,0,1,2,\ldots\}$
- $Z^+ = \{1,2,3,\ldots\}$
- $Q = \{p/q \mid p \text{ in } Z, q \text{ in } Z, q \text{ is not } 0\}$
- $R = \{x \mid x \text{ is a real number}\}$.

{,...} is used to indicate the the rest of the sequence once it's clear how to proceed

Example: {1,2,3,4,...}

1.2 Specification of set

- There is a set with no elements. It is called the ***empty set*** (or *null set*) and denoted {} or $\varnothing$.
- A set that has one element is called a ***singleton set***.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}
- Note the subtlety in $\varnothing \neq \{\varnothing\}$ **??why**
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and this set contains a set

1.2 Specification of set

- If there are exactly n distinct elements in a set S, with n is a nonnegative integer, we say that:
  - S is a finite set, and
  - The cardinality of S is n. Notation: $|S| = n$.
- **Definition.** A set is a ***finite set*** if it has a finite number of elements. A set that is not finite is an *infinite set*.
- Let A be a finite set. The number of different elements in A is called its *cardinality* and is denoted by $|A|$. Other notations commonly used for the cardinality of A are $N(A)$, $\#A$.

  If A be a infinite set, then we write $|A| = \infty$.

Example:

- $|\varnothing| = 0$ since $\varnothing$ contains no elements.
- $|\{\pi, 2, \text{Newton}\}| = 3$.
- If $N_n = \{0, 1, \ldots, n\}$ then $|N_n| = n + 1$.
- $|\{n: n \text{ is a prime number}\}| = \infty$.
- The sets N, Z, Q, R are all infinite

1.2 Specification of set

- Sets can be elements of other sets

Example:

- $S_1 = \{\varnothing,\{a\},\{b\},\{a,b\},c\}$
- $S_2=\{\{1\},\{2,4,8\},\{3\},\{6\},4,5,6\}$

Example: What is the cardinality of the set ?

- $X = \{\{a, b\}\}$
- $A = \{1, 2, \{a, b\}\}$

1.2 Specification of set

- Let $B = \{x \mid (x\leq 100) \wedge (x \text{ is prime})\}$

  the cardinality of B is $|B|=$?

25 because there are 25 primes less than or equal to 100.

Contents

1. Definitions
2. Set operations
3. The algebra of sets
4. Computer representation of sets
5. Relations
6. Functions

2. Set operations

2.1 Set comparison

2.2 Venn diagram

2.3 Set operations

2.4 Partition and cover

2.1. Set comparison

- **Definition**: Two sets, A and B, are equal if they contain the same elements. We write A=B.

Example:

- {2,3,5,7}={3,2,7,5}, because a set is unordered
- Also, {2,3,5,7}={2,2,3,5,3,7} because a set contains unique elements
- However, {2,3,5,7} $\neq$ {2,3}

2.1. Set comparison

if $P(x)$ and $Q(x)$ are propositional functions which are true for the same objects x, then the sets they define are equal, i.e.

$$\{x : P(x)\} = \{x : Q(x)\}.$$

Example: there are 2 sets

A = $\{x: (x-4)^2 = 25\}$

B = $\{x: (x+1)(x-9) = 0\}$

Question: A = B ?

Yes: A = B, since the two propositional functions $P(x)$: $(x-4)^2 = 25$ and $Q(x)$: $(x+1)(x-9) = 0$ are true for the same values of x, namely -1 and 5.

2.1. Set comparison

- **Definition**: A is said to be a **subset** of B, if and only if every element of A is also an element of B

that is: $\forall x\ (x \in A \Rightarrow x \in B)$

Denote: $A \subseteq B$ or $B \supseteq A$,

Example: $S = \{ 1, 2, 3, \ldots, 11, 12 \}$ and $T = \{ 1, 2, 3, 6 \}$ then $T \subseteq S$.

- **Theorem**: For any set S
  - $\varnothing \subseteq S$ and
  - $S \subseteq S$

2.1. Set comparison

- **Definition**: If $A \subseteq B$ and $A \neq B$ then set A is called a **proper subset** of set B.

(that is there is an element $x \in B$ such that $x \notin A$)

Denote: $A \subset B$

Example 1: $A = \{ 1, 2, 3 \}$, $B = \{ 2, 3, 1 \}$, $C = \{ 3 \}$. Then:

$$B = A,\ C \subset A,\ C \subset B.$$

Example 2:

$\{1, 4, 9, 16, ...\} \subseteq \{1, 2, 3, ...\} \subseteq \{0, 1, 2, ...\}$. ?

$\{1, 4, 9, 16, ...\} \subset \{1, 2, 3, ...\} \subset \{0, 1, 2, ...\}$.

Here, we could have used the proper subset symbol $\subset$ to link these three sets instead.

2.1. Set comparison

- You may be asked to show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$
  - To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B.
  - Any proof method can be used.
- To prove that A is a proper subset of B ($A \subset B$), you must prove
  - A is a subset of B **and**
  - $\exists x\ (x \in B) \wedge (x \notin A)$

2.1. Set comparison

- Finally to show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  - $A \subseteq B$ and
  - $B \subseteq A$
- Logically speaking, you must show the following quantified statements:

$$(\forall x\ (x \in A \Rightarrow x \in B)) \wedge (\forall x\ (x \in B \Rightarrow x \in A))$$

we will see an example later..

2.1. Set comparison

Examples:

- $N = \{0, 1, 2, 3, ...\}$ the set of *natural numbers*.
- $Z = \{..., -2, -1, 0, 1, 2, ...\}$ the set of *integers*.
- Z^+ : the set of positive integers
- $Q = \{p/q : p, q \in Z \text{ and } q \neq 0\}$ the set of fractions or *rational numbers*.
- Q^+ : the set of positive rational numbers
- R = the set of *real numbers*;
- R^+ : the set of positive real numbers
- $C = \{x + iy : x, y \in R \text{ and } i^2 = -1\}$ the set of *complex numbers*.

Clearly the following subset relations hold amongst these sets:

$$N \subseteq Z \subseteq Q \subseteq R \subseteq C.$$

Question: ??? $N = Z^+$

Note that N is *not* equal to Z^+ since 0 belongs to the first but not the second

2.1. Set comparison

We shall sometimes use E and O to denote the sets of even and odd integers respectively:

- $E = \{2n : n \in Z\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$
- $O = \{2n+1 : n \in Z\} = \{\dots, -3, -1, 1, 3, 5, \dots\}$

Universal set **(U)** : contains as subsets all sets relevant to the current task or study. Anything outside the universal set is simply not considered. The universal set is not something fixed for all time -we can change it to suit different contexts.

2. Set operations

2.1 Set comparison

2.2 Venn diagram

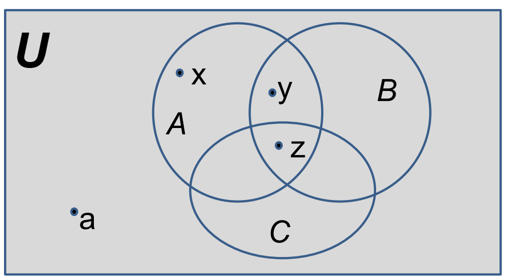
2.3 Set operations

2.4 Partition and cover

2.2. Venn diagram

John Venn
1834-1923

- A set can be represented graphically using a Venn Diagram

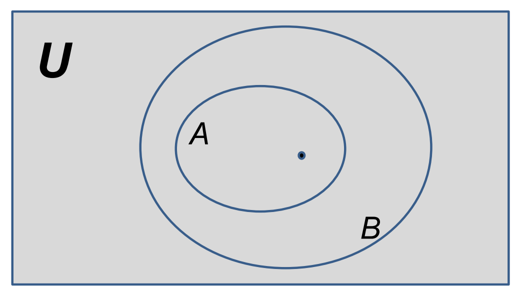


- The universal set U is represented by the interior of a rectangle
- The sets by region inside the rectangle and elements which belong to a given set are placed inside the region representing it.
- If an element belongs to more than one set in the diagram, the two regions representing the sets concerned must overlap and the element is placed in the overlapping region.

In this way the picture represents the relationships between the sets concerned.

2.2. Venn diagram

Example:



if $A \subseteq B$: the region representing A may be enclosed inside the region representing B to ensure that every element in the region representing A is also inside that representing B

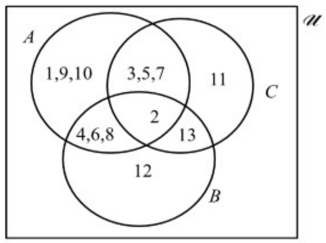
2.2. Venn diagram

Example: Draw the Venn diagram that represents 3 sets:

$A = \{1, 2, \ldots, 10\}$,

$B = \{2, 4, 6, 8, 12, 13\}$,

$C = \{2, 3, 5, 7, 11, 13\}$



2. Set operations

2.1 Set comparison

2.2 Venn diagram

2.3 Set operations

2.4 Partition and cover

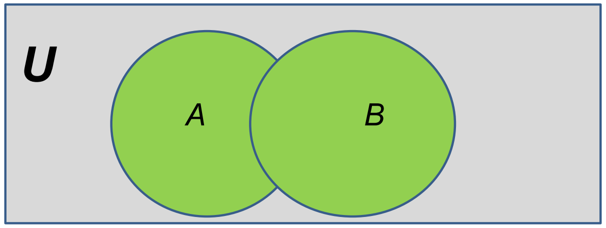
2.3. Set operations

- Arithmetic operators (+,-, × ,÷) can be used on pairs of numbers to give us new numbers
- Similarly, set operators exist and act on two sets to give us new sets:
  1. Union
  2. Generalized union
  3. Intersection
  4. Generalized intersection
  5. Set difference
  6. Set complement

Set Operators: Union

- **Definition**: The union of two sets A and B is the set that contains all elements in A, B, or both. We write:

$$A \cup B = \{ x \mid (x \in A) \vee (x \in B) \}$$



Set Operators: Generalized Union

- **Definition**: The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection

Let $A_1, A_2, \ldots, A_n$ be sets. Their union is:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$
$$= \{x \mid x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots \text{ or } x \in A_n\}$$
$$= \{x \mid x \text{ belongs to at least one set } A_i, i = 1, \ldots, n\}$$

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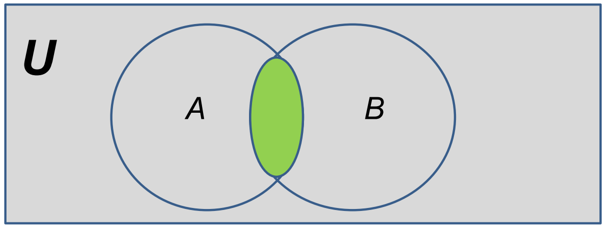
Set Operators: Intersection

- **Definition**: The intersection of two sets A and B is the set that contains all elements that are element of both A and B.

We write:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$



Two sets A and B are *disjoint* if $A \cap B = \varnothing$.

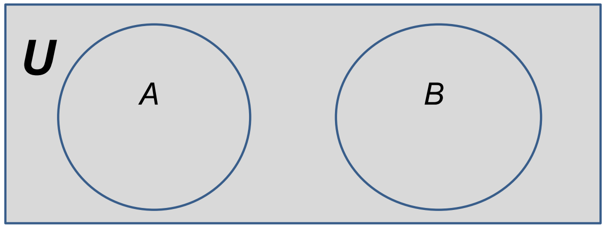
Set Operators: Generalized Intersection

- **Definition**: The intersection of a collection of sets is the set that contains those elements that are members of <u>every</u> set in the collection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$
$$= \{x \mid x \in A_i \text{ for all } i = 1, 2, \ldots, n\}$$

Disjoint Sets

- **Definition**: Two sets are said to be disjoint if their intersection is the empty set: $A \cap B = \varnothing$



Set Operators: Generalized Intersection

- $\bigcap_{i\in I} A_i = \{x | x \in A_i \text{ for all } i \in I\}$ is used for the intersection of the family of sets A_i indexed by the set I.
- A collection of sets $\{A_i \mid i \in I\}$ is *disjoint* if

$$\bigcap_{i\in I} A_i = \emptyset$$

A collection of sets is *pairwise disjoint* (or *mutually disjoint*) if every pair of sets in the collection are disjoint.

Example:

$A = A_1 \cup A_2 \cup A_3 \cup A_4$

$A_1 = \{1, 3, 5, 7\}$

$A_2 = \{2, 4, 6\}$

$A_3 = \{-1, -3, -5, -7\}$

$A_4 = \{-2, -4, -6\}$

$A_5 = \{2\}$

$B = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$

? A is not disjoint

? A is disjoint

? A is pairwise disjoint

? B is not disjoint

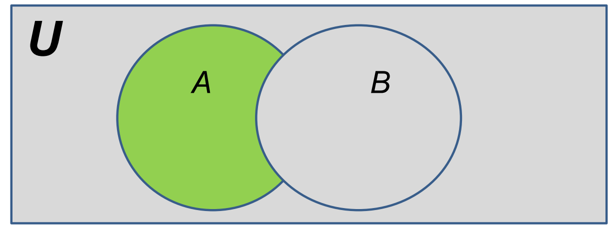
? B is disjoint

? B is pairwise disjoint

Set Operators: Set Difference

- **Definition**: The difference of two sets A and B is the set containing those elements that are in A but not in B.

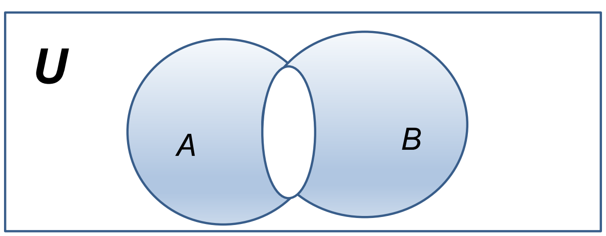
Denote: A\B or A−B



Set Operators: Set Difference

- The *symmetric difference* of two sets A and B, denoted $A \oplus B$, is defined as follows

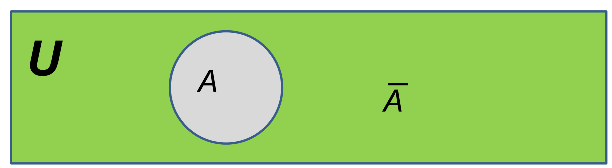
$$A \oplus B = (A \cup B) \setminus (A \cap B)$$



Set Operators: Set Complement

- **Definition**: The complement of a set A, denoted $\bar{A}$, or A^c or $\neg$ A consists of all elements not in A. That is the difference of the universal set and U: U\A

$$\bar{A} = A^C = \{x \mid x \notin A\}$$



$A - B = \{x \mid x \in A \text{ and } x \notin B\} = A \cap B^c$.

Set Complement

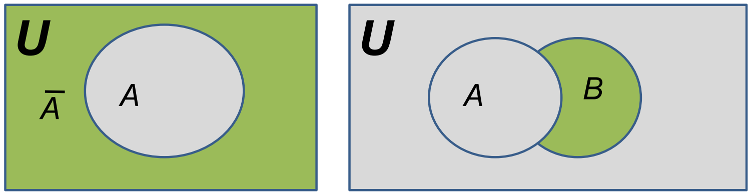
Examples:

Let $A=\{1, 2, 3\}$, $B = \{3, 4, 5\}$. Then

- $A \cup B =$
- $A \cap B =$
- $A \setminus B =$
- $A \oplus B =$

Set Complement: Absolute & Relative

- Given the Universe U, and A,B ⊂ U.
- The (absolute) complement of A is $\bar{A} = U \backslash A$
- The (relative) complement of A in B is B\A



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2. Set operations

2.1 Set comparison

2.2 Venn diagram

2.3 Set operations

2.4 Partition and cover

2.4. Partition and cover

Let $\mathcal{E} = \{E_i\}_{i \in I}$ be a collection of subsets of the set M, $E_i \subseteq M$.

Collection $\mathcal{E}$ will be called a cover of M if each element of M must be an element of at least one of the sets of $\mathcal{E}$:

$$M \subset \bigcup_{i \in I} E_i \Leftrightarrow \forall x \in M \ \exists i \in I \ x \in E_i .$$

The disjoint cover $\mathcal{E}$ of M is called a partition of M, i.e.

$$\mathcal{E} \text{ is a partition of } M \Leftrightarrow M = \cup_{i \in I} E_i,\ E_i \cap E_j = \emptyset,\ i \neq j.$$

Example: M = {1, 2, 3, 4}

$\mathcal{E}_1 = \{\{1, 2\}, \{3, 4\}\}$ is a partition of M

$\mathcal{E}_2 = \{\{1, 2, 3\}, \{3, 4\}\}$ is not a partition of M

2.4. Partition and cover

Example: M= {11, 12, 13, 14}

$\mathcal{E}_1$ = {{11, 12}, {11, 13}, {12, 14}}

$\mathcal{E}_2$ = {{11, 12}, {13, 14}}

$\mathcal{E}_3$ = {{11, 12}, {13}}

$\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ is cover / partition of M ?

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  3.2 Properties of set operations
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3.1. Power Set

- **Definition**: The power set of a set A, denoted P(A), is the set of all subsets of A.

Examples

Let A = {∅ } →P (A) = {∅}
Let A = {a} →P (A) = {∅, {a}}
Let A = {a, b} → P(A)= {∅, {a}, {b}, {a, b}}
Let A = {a, b, c} → P(A) = ?

{
∅,
{a}, {b}, {c},
{a, b}, {a, c}, {b, c},
{a, b, c}
}

- Note: the empty set ∅ and the set itself are always elements of the power set.

3.1. Power Set

- **Theorem**: Let A be a set such that $|A|=n$, then

$$|P(A)| = 2^n$$

Proof: Let A be the set $\{a_1, a_2, ..., a_n\}$.

- We can form a subset of A by considering each element a_i in turn and either including it or not in the subset.
- For each element there are two choices (either include it or don't) and the choice for each element is independent of the choices for the other elements, so there are 2^n choices altogether.
- Each of these 2^n choices gives a different subset and every subset of A can be obtained in this way.

Let A = $\{a, b, c\}$ → P(A) = ?

0 1 0 1 0 1

{
$\varnothing$,
$\{a\}$, $\{b\}$, $\{c\}$,
$\{a, b\}$, $\{a, c\}$, $\{b, c\}$,
$\{a, b, c\}$
}

3.1. Power Set

Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Determine whether each of the following is true or false and give a brief justification.

1. $B \in P(A)$
2. $B \in A$
3. $A \in P(A)$
4. $A \subseteq P(A)$
5. $B \subseteq P(A)$
6. $\{\{a\}, B\} \subseteq P(A)$
7. $\varnothing \in P(A)$
8. $\varnothing \subseteq P(A)$.

3.1. Power Set

Theorem.

For all sets A and B:

1. $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.
2. $P(A) \cap P(B) = P(A \cap B)$.
3. $P(A) \cup P(B) \subseteq P(A \cup B)$.

3.1. Power Set

Theorem.

For all sets A and B:

1. $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.

We prove the two statements:

$A \subseteq B \Rightarrow P(A) \subseteq P(B)$ and

$P(A) \subseteq P(B) \Rightarrow A \subseteq B$.

- Firstly, suppose $A \subseteq B$. Let $X \in P(A)$. This means $X \subseteq A$. Since $A \subseteq B$, it follows that $X \subseteq B$, which means that $X \in P(B)$. Since $X \in P(A)$ implies $X \in P(B)$, we conclude that $P(A) \subseteq P(B)$, which completes the proof of the first statement.
- To prove the converse statement, suppose $P(A) \subseteq P(B)$. Since $A \in P(A)$, it follows that $A \in P(B)$. This means that $A \subseteq B$, which completes the proof.

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3.2 Properties of set operations

Let A, B and C be any sets. The following laws hold

Equality	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws (Đồng nhất)
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws (Trội)
$A \cup A = A$ $A \cap A = A$	Idempotent laws (Lũy đẳng)
$A \cup \bar{A} = U$ $A \cap \bar{A} = \varnothing$ $\bar{\varnothing} = U$ $\bar{U} = \varnothing$	Complementation laws (Bù)
$\overline{(\bar{A})} = A$	Involution laws (Bù kép)

3.2 Properties of set operations

Equality	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws (Giao hoán)
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws (Kết hợp)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws (Phân phối)
$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$	De Morgan's laws (Luật De Morgan)
$A \cap (A \cup B) = A$ $A \cup (A \cap B) = A.$	Absorption laws

Proving Set Equivalences

To prove set equivalence

$$A = B,$$

We could use common techniques:

1. Proof $A \subseteq B$ and $B \subseteq A$.
2. Using definitions and equivalence of logical propositions that define the set.
3. Use the truth table.

Proving Set Equivalences

To prove set equivalence

$A = B$,

We could use common techniques:

1. **Proof $A \subseteq B$ and $B \subseteq A$.**
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Proving Set Equivalences

- Recall that to prove such identity A = B, we must show that:
  1. The left-hand side is a subset of the right-hand side
  2. The right-hand side is a subset of the left-hand side
  3. Then conclude that the two sides are thus equal

Example: Let

- A={x |x is even}
- B={x |x is a multiple of 3}
- C={x |x is a multiple of 6}

- Show that A∩B=C

Show that A∩B=C

Example: Let

- A={x |x is even}
- B={x |x is a multiple of 3}
- C={x |x is a multiple of 6}

- Show that A∩B=C

- The left-hand side is a subset of the right-hand side

A∩B ⊆ C: $\forall x \in A\cap B$

$\Rightarrow x$ is a multiple of 2 and x is a multiple of 3

$\Rightarrow$ we can write $x = 2*3*k$ for some integer k

$\Rightarrow x = 6k$ for some integer $k \Rightarrow x$ is a multiple of 6

$\Rightarrow x \in C$

- The right-hand side is a subset of the left-hand side

C ⊆A∩B: $\forall x \in C$

$\Rightarrow x$ is a multiple of 6 $\Rightarrow x = 6k$ for some integer k

$\Rightarrow x = 2(3k)=3(2k) \Rightarrow x$ is a multiple of 2 and of 3

$\Rightarrow x \in A\cap B$

Proving Set Equivalences

Example 2: Proof that: $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

- **Part 1:** Proof $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
  - Assume $x\in A\cap(B\cup C)$, need to proof $x\in(A\cap B)\cup(A\cap C)$.
  - As $x\in A$, and either $x\in B$ or $x\in C$.
    - Case1: $x\in B$. Then $x\in A\cap B$, therefore $x\in(A\cap B)\cup(A\cap C)$.
    - Case2: $x\in C$. Then $x\in A\cap C$, therefore $x\in(A\cap B)\cup(A\cap C)$.
  - Thus, $x\in(A\cap B)\cup(A\cap C)$.
  - So $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
- **Part 2:** Proof $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$.

Proving Set Equivalences

To prove set equivalence

$A = B$,

We could use common techniques:

1. Proof $A \subseteq B$ and $B \subseteq A$.
2. **Using definitions and equivalence of logical propositions that define the set.**
3. Use the truth table.

NGUYỄN KHÁNH PHƯƠNG
CS-SOICT-HUST

Proving Set Equivalences

Example 3: Proof: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	According to complementation definition
$= \{x \mid \neg(x \in (A \cap B))\}$	According to definition of $\notin$
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	According to definition of intersection
$= \{x \mid x \notin A \vee x \notin B)\}$	According to De Morgan law
$= \{x \mid x \in \overline{A} \vee x \in \overline{B})\}$	According to complementation definition
$= \{x \mid x \in \overline{A} \cup \overline{B})\}$	According to union definition

Proving Set Equivalences

To prove set equivalence

$A = B$,

We could use common techniques:

1. Proof $A \subseteq B$ and $B \subseteq A$.
2. Using definitions and equivalence of logical propositions that define the set.
3. **Use the truth table.**

Truth table

- Building tables:
  - The columns correspond to set expressions.
  - The rows correspond to all possible combinations of membership in the set.
- Fill in the table: Use "1" to indicate a member, "0" to indicate non-member.
- Equality is proven if two columns corresponding to two expressions on both sides are **identical**.

Example 4: Proof: $(A \cup B) - B = A - B$.

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Proving Set Equivalences

Example 5: Using the truth table, proof that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

A B C	B∪C	A∩(B∪C)	A∩B	A∩C	(A∩B)∪(A∩C)
1 1 1	1				
1 1 0	1				
1 0 1	1				
1 0 0	0				
0 1 1	1				
0 1 0	1				
0 0 1	1				
0 0 0	0				

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4.1. Characteristic vector

Suppose that we have U = $\{u_1, u_2, ..., u_n\}$, where n is not too large. Then each subset $M \subset$ U can be represented by a vector $b = (b_1, b_2, ..., b_n)$ where

$b_i = 1 \leftrightarrow u_i \in M, i = 1, 2, ..., n.$

$= 0$ otherwise

- Vector b constructed by this rule is called characteristic vector of the set M.
- It is clear that each subset $M \subset$ U corresponds to unique characteristic vector b, and on the other hand, each binary n-vector b corresponds to unique subset of U

Example:

0 1 1 0 1 0 1 0 0 0 1

Suppose that U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.

Consider the subsets $S, Q \subseteq$ U.

- S = {2, 3, 5, 7, 11} ↔ 01101010001
- Q = {1, 2, 4, 11} ↔ 11010000001

4.1. Characteristic vector

Note that all the set operation ∪ (Union), ∩ (Intersection), ¬(complement) can be done by correspondently logic operation OR, AND, NOT

Example: Suppose that U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.

Consider the subsets $S, Q \subseteq$ U.

- S = {2, 3, 5, 7, 11} ↔ 01101010001
- Q = {1, 2, 4, 11} ↔ 11010000001
- $S \cup Q \leftrightarrow 01101010001 \vee 11010000001$

  → $S \cup Q \leftrightarrow 11111010001$
- $S \cap Q \leftrightarrow 01101010001 \wedge 11010000001$

  → $S \cap Q \leftrightarrow 01001000001$
- $S = 01101010001$

  → $\neg S \leftrightarrow 10010101110$

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4.2. Subset enumeration

In many practical situation, we have to examine all the subsets of a given set U = $\{u_1, u_2, ..., u_n\}$.

For example: Enumerate all subsets of U = {1, 2, 3}

1. ∅
2. {1}
3. {2}
4. {3}
5. {1, 2}
6. {1, 3}
7. {2, 3}
8. {1, 2, 3}

4.2. Subset enumeration

In many practical situation, we have to examine all the subsets of a given set U = $\{u_1, u_2, ..., u_n\}$. How to do it?

- Answer: Each subset of U ~ a characteristic vector

→Enumeration of all the subsets of U ~ enumeration of all binary n-vector.

1 2

Since each binary n-vector can be considered as the binary representation of a nonnegative integer $\alpha(b) = b_1b_2...b_n$, $0 \le \alpha(b) \le 2^n$-1

→ enumeration of all binary n-vector 2

~ enumeration of binary representation for all nonnegative integer from 0 to 2^n-1.

3

1.4.2. Subset enumeration

For example: Enumerate all subsets of U = {1, 2, 3}

0) 000 ∅
1) 001 {3}
2) 010 {2}
3) 011 {2,3}
4) 100 {1}
5) 101 {1,3}
6) 110 {1,2}
7) 111 {1,2,3}

Enumeration of all the subsets of U ❶
~ enumeration of all binary n-vector ❷
~ enumeration of binary representation for all nonnegative integer from 0 to 2^n-1. ❸

4.2. Subset enumeration

Subset Enumeration Algorithm:

- **Step** $k = 0, 1, ..., 2^n-1$: Output binary representation of the number k.
- Clearly, if we have the binary representation of the number k ($b_1b_2...b_n$) then the binary representation of the number $k+1$ can be obtained by binary addition $b_1b_2...b_n$ to 1.

For example: Enumerate all subsets of U = {1, 2, 3}

0) 000 ∅
1) 001 {3}
2) 010 {2}
3) 011 {2,3}
4) 100 {1}
5) 101 {1,3}
6) 110 {1,2}
7) 111 {1,2,3}

$$\begin{array}{r} 101 \\ +\quad 1 \\ \hline 110 \end{array}$$

4.2. Subset enumeration

Subset Enumeration Algorithm

- **Step** $k = 0, 1, ..., 2^n-1$: Output binary representation of the number k.
- Clearly, if we have the binary representation of the number k ($b_1b_2...b_n$) then the binary representation of the number $k+1$ can be obtained by binary addition $b_1b_2...b_n$ to 1.

```
Algorithm: BINARY_INCREMENT(b1,b2,...,bn)
i = n;
while (n>=1) and (bi==1)
      bi = 0;
      i = i - 1 ;
endwhile
      bi = 1;
```

$$\begin{array}{r} 100101111111111 \\ +\qquad\qquad\qquad 1 \\ \hline 100110000000000 \end{array}$$

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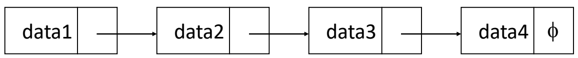
4.3. List of elements

- When the set U contain a large number of elements, but considered subset U have a small cardinality, binary representation is not reasonable. In this case we can represent the subset by list of all its elements.
- This list is usually implemented as the **linked list structure**. Each element of list is a record that consists of two fields, one of which contains the information of the element and the other one is a pointer to the next element:

```
class ListNode {
    Object data ; //element information
    ListNode next; //pointer to the next element
}
```



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5. Relations

5.1. Ordered pair
5.2. Cartesian product
5.3. Binary relation
5.4. Relation representation
5.5. Operations on relations
5.6. Properties of relations

5.1. Ordered pair

- An *ordered pair* is a set of a pair of objects with an order associated with them.
- In general (x, y) is different from (y, x).
- **Definition (equality of ordered pairs):** Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Example: if the ordered pair (a, b) is equal to $(1, 2)$, then a=1, and b=2.
$(1, 2)$ is not equal to the ordered pair $(2, 1)$.

5.2. Cartesian product

René Descartes (1596-1650)

- Let $A_1, A_2, \ldots, A_n$ be any sets, where $n \in \mathbb{Z}^+$ and $n \geq 3$.

Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is defined as follows:

$$A_1 \times A_2 \times \ldots \times A_n \equiv_{\text{def}} \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, 1 \leq i \leq n\}.$$

When $A_1 = A_2 = \ldots = A_n = A$, it is usually to denote $A \times A \times \ldots \times A$ by A^n

An element of $A_1 \times A_2 \times \ldots \times A_n$ is called an ordered n-tuple.
When n=3, we have a *triple*.

Example: $A = \{1, 2\}$, $B = \{a, b\}$ and $C = \{\alpha, \beta\}$ then
$A \times B \times C = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}$.

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5.2. Cartesian product

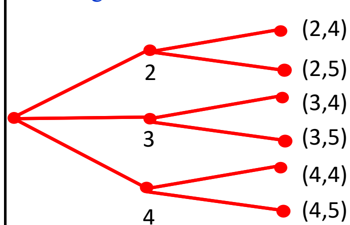
Theorem : If $X_1, X_2, \ldots, X_n$ are finite sets then

$$|X_1 \times X_2 \times \ldots \times X_n| = |X_1| \times |X_2| \times \ldots \times |X_n|$$

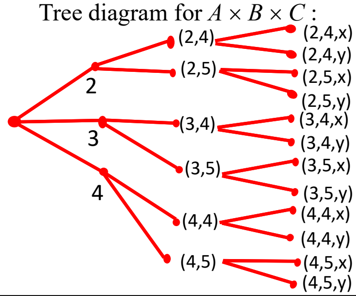
Enumeration: To enumerate all elements of Cartesian product of the sets we can use *tree diagram*.

Example: $A = \{2, 3, 4\}$, $B = \{4, 5\}$, and $C = \{x, y\}$.

Tree diagram for $A \times B$:



Tree diagram for $A \times B \times C$:



5.2. Cartesian product

How the Cartesian product operation behaves with respect to the other set theory operations such as intersection and union?

Theorem : For any three sets A, B, C:

$$A \times (B \cap C) = (A \times B) \cap (A \times C);$$
$$A \times (B \cup C) = (A \times B) \cup (A \times C);$$
$$(A \cap B) \times C = (A \times C) \cap (B \times C);$$
$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

5. Relations

5.1. Ordered pair

5.2. Cartesian product

5.3. Binary relation

5.4. Relation representation

5.5. Operations on relations

5.6. Properties of relations

5.3. Binary Relation

Let A and B be sets:

- **Definition (binary relation):** A binary relation from a set A to a set B is a set of ordered pairs (a, b) where a is an element of A and b is an element of B.
  - A binary relation from A to B is a subset R $\subseteq A \times B$]
  - A relation on a set A is a relation from A to A, i.e., a subset R $\subseteq A \times A$
- Notation: When an ordered pair (a, b) is in a relation R, we write $a\ R\ b$, or $(a, b) \in R$. It means that element a is related to element b in relation R. We will write $a\ \overline{R}\ b$ when a element a is not related to element b in relation R.

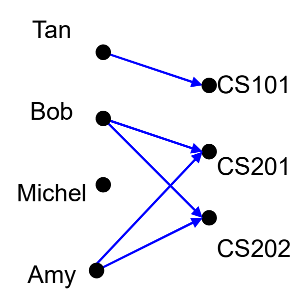
5.3. Binary Relation

Example:

- Let A be the students in a the CS major
  - A = {Tan, Bob, Michel, Amy}
- Let B be the courses the department offers
  - B = {CS101, CS201, CS202}
- We specify relation $R = A \times B$ as the set that lists all students $a \in A$ enrolled in class $b \in B$
  - R = { (Tan, CS101), (Bob, CS201), (Bob, CS202), (Amy, CS201), (Amy, CS202) }

Representing relations

We can represent relations graphically:



We can represent relations in a table:

	CS101	CS201	CS202
Tan	X		
Bob		X	X
Michel			
Amy		X	X

Relations *on* a Set

- A relation *on* a set A is a relation from A to A.

Examples of relations on Z^+: $R_<$, $R_\geq$, $R_>$:

- $R_< = \{(x, y) | x < y\}$ ($R_<$ is relation "strictly less than").
- $R_\geq = \{(x, y) | x \geq y\}$ ($R_\geq$ is relation "greater or equal").
- $R_> = \{(x, y) | x > y\}$ ($R_>$ is relation "strictly greater than").

Relations *on* a Set

Consider the following relations on Z:

- $R_1 = \{(a, b) | a \leq b\}$
- $R_2 = \{(a, b) | a > b\}$
- $R_3 = \{(a, b) | a = b \text{ or } a = -b\}$
- $R_4 = \{(a, b) | a = b\}$
- $R_5 = \{(a, b) | a = b+1\}$
- $R_6 = \{(a, b) | a + b \leq 3\}$

	(1,1)	(1,2)	(2,1)	(1,-1)	(2,2)
R1	√	√			√
R2			√	√	
R3	√			√	√
R4	√				√
R5			√		
R6	√	√	√	√	

For each the following ordered pairs

(1,1), (1,2), (2,1), (1,-1), and (2,2)

show which relation it belongs to.

5.3. Binary relation

A binary relation is a set of ordered pairs (x, y)

The **domain** is the set of all x values in the relation

domain = {-1,0,2,4,9}

These are the x values written in a set from smallest to largest

{(2,3), (-1,5), (4,-2), (9,9), (0,-6)} ← This is a relation

These are the y values written in a set from smallest to largest

range = {-6,-2,3,5,9}

The **range** is the set of all y values in the relation

5. Relations

5.1. Ordered pair

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5.4. Relation representation

1. Set of ordered pairs
2. Mapping
3. Table
4. Grid graph
5. Binary matrix
6. Directed graph   For relations on a set

5.4. Relation representation: set of ordered pairs

A relation is a set of ordered pairs (x, y).

Relation = { (3,5),(-2,4),(-3,4),(0,-4) }

The domain is the set of x values.

The range is the set of y values.

EXAMPLE

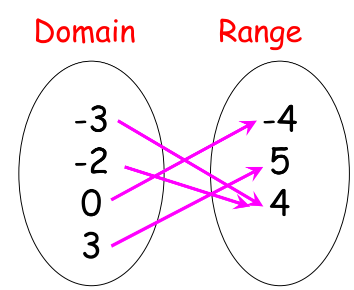
Relation = { (3,5),(-2,4),(-3,4),(0,-4) }

Domain: { -3 , -2 , 0 , 3 }

Range: { -4 , 5 , 4 }

Relation = { (3,5),(-2,4),(-3,4),(0,-4) }

2. Mapping:



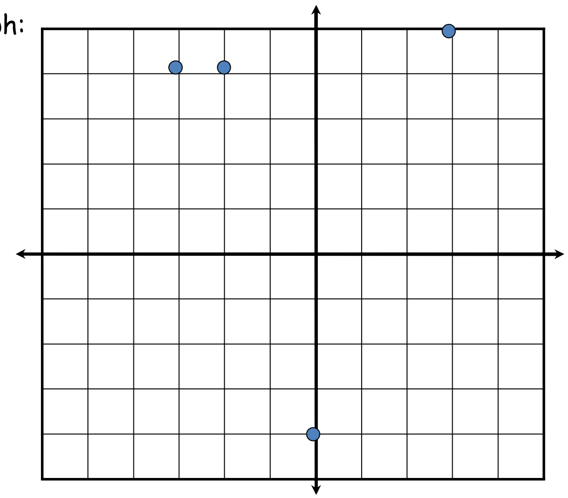
Relation = { (3,5),(-2,4),(-3,4),(0,-4) }

3. Table:

x (domain)	y (range)
-3	4
-2	4
0	-4
3	5

Relation = { (3,5),(-2,4),(-3,4),(0,-4) }

4. Grid graph:



Relation = { (3,5),(-2,4),(-3,4),(0,-4) }

5. Binary matrix

Domain a: { -3 , -2 , 0 , 3 }

Range b: { -4 , 5 , 4 }

$M_R = [m_{ij}]_{n\times m}$ of zeros and ones with n (=4) rows and m (=3)columns

$$m_{ij} = \begin{cases} 1, & \text{if } a_i \; R \; b_j \\ 0, & \text{if } a_i \; \overline{R} \; b_j \end{cases}$$

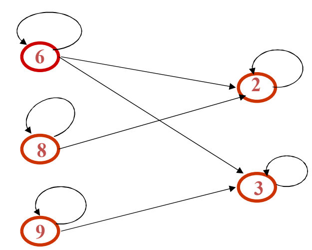
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

5.4. Relation representation

6. Directed graph

- Let $A = \{2, 3, 6, 8, 9\}$. Consider relation R: a is related to b iff *a is divisible by b*. We have:

$R = \{(2,2), (3,3), (6,2), (6,3), (6,6), (8,2), (8,8), (9,3), (9,9)\}$.



5.4. Relation representation

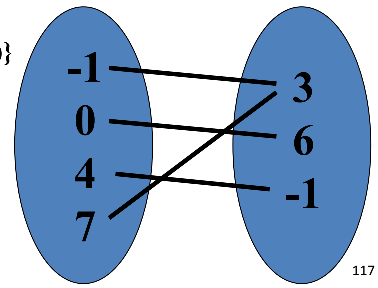
Exercise: Given the following table, show the relation as set of ordered pairs, domain, range and mapping

X	-1	0	4	7
Y	3	6	-1	3

Relation = **{(-1,3), (0,6), (4,-1), (7,3)}**

Domain = **{-1, 0, 4, 7}**

Range = **{-1, 3, 6}**



5. Relations

5.1. Ordered pair
5.2. Cartesian product
5.3. Binary relation
5.4. Relation representation
5.5. Operations on relations
5.6. Properties of relations

5.5. Operations on relations

- A relation is a set. It is a set of ordered pairs if it is a binary relation. Thus all the set operations apply to relations such as union, intersection and complementing.

Example:

- The union of the "less than" and "equality" relations on the set of integers is the "less than or equal to" relation on the set of integers.
- The intersection of the "less than" and "less than or equal to" relations on the set of integers is the "less than" relation on the same set.
- The complement of the "less than" relation on the set of integers is the "greater than or equal to" relation on the same set.

1. **Complementary Relations**
2. **Inverse Relations**
3. **Identity relation**
4. ***n*-ary Relations**
5. **Composite Relation**

5.5. Operations on relations

1. Complementary Relations

Let $R \subseteq A \times B$ be a binary relation. *Complementary relation* $\overline{R}$ of R is defined as the set

$$\overline{R} \equiv_{\text{def}} \{(a,b) \mid (a,b) \notin R\} = (A \times B) - R .$$

Example:

$$\overline{R}_< = \{(a,b) \mid (a,b) \notin R_<\} = \{(a,b) \mid a \ge b\} = R_\ge .$$

2. Inverse Relations

Each binary relation $R \subseteq A \times B$ has inverse relation R^{-1}, which defined by

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}.$$

Example1:

$$(R_<)^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = R_>.$$

Example 2: Let B be the set of jobs, and A be the set of the workers. Consider R as relation from A to B that defined as follows

$aRb \Leftrightarrow a$ perform b.

Then $b\,R^{-1}\,a \Leftrightarrow b$ is performed by a.

Note: We have $(R^{-1})^{-1} = R$.

5.5. Operations on relations

3. Identity Relation

Identity relation I_A on the set A is defined as

$$I_A = \{(a, a) \mid a \in A\}.$$

***4. n*-ary Relations**

An *n-ary* relation on sets $A_1, ..., A_n$ is a set of ordered n-tuples $(a_1, ..., a_n)$ where a_i is an element of A_i for all i, $1 \le i \le n$. Thus an n-ary relation R on sets $A_1, ..., A_n$ is a subset of Cartesian product $A_1 \times A_2 \times ... \times A_n$:

$$R \subseteq A_1 \times A_2 \times ... \times A_n.$$

Note that the sets $A_1, ..., A_n$ are not to be different.

Example: Application of n-ary Relations

Example: Teaching Assignments

Professor	*Department*	*Course Number*
Cruz	Chemistry	335
Cruz	Chemistry	412
Farber	Psychology	501
Farber	Psychology	617
Grammar	Physics	544
Grammar	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

- A relational database models a database as a relation.
- The relation's domains are called its attributes.
- How many attributes in the Teaching Assignment table?

The relation's primary key is an attribute whose value uniquely determines an element in the relation.

In general, a primary key may consist of > 1 attribute.

What single attribute could serve as the primary key in the Teaching Assignment table?

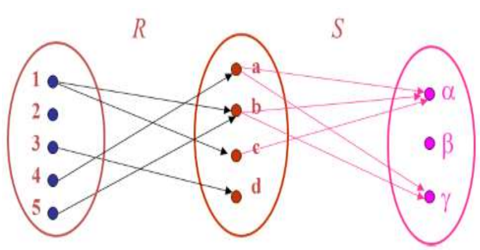
5.5. Operations on relations

5. Composite Relations

Suppose $R \subseteq A \times B$ and $S \subseteq B \times C$. Composite (or product) relation $R \circ S$ of two relations R and S is the following

$$R \circ S = \{(a,c) \mid aRb \wedge bSc\}$$

Example: Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\}$, $C = \{\alpha, \beta, \gamma\}$. Consider the relations R, S which are displayed in the following diagram



$R \circ S = ?$

We have: $R \circ S = \{(1, \alpha), (1, \gamma), (4, \alpha), (4, \gamma), (5, \alpha), (5, \gamma)\} \subseteq A \times C$

Example

- Let S be a set of students:

  { Bill, Jill, Will }.
- Let C be a set of courses:

  { 16, 24, 32, 40, 48, 56 }
- Let

  R = { (s, c) | student s has taken course c }.
  - Many students may have taken the same course.
  - A student may have taken many courses.

Matrix representation of relation R

Relation R: Student *s* has taken course *c*

STUDENT × COURSE

	16	24	32	40	48	56
Bill	1	1	0	0	0	0
Jill	1	1	1	0	0	1
Will	1	0	0	1	0	0

(row i, column j) = 1 ⟷ student i has taken course j.

Matrix representation of relation S

Relation S: Course c has been taught by teacher t

COURSE × TEACHER

	Mike	Diana	Pete
16	1	1	0
24	1	1	1
32	0	1	0
40	0	0	0
48	1	0	1
56	1	0	0

$S \circ R = M_R \times M_S$ "Boolean" matrix product

$S \circ R = \{ (s, t) \mid \exists c (sRc \wedge cSt) \}$.

Describe in English what $S \circ R$ represents ?

$$\begin{array}{c} \text{STUDENT} \end{array} \overset{\text{COURSE}}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}} \begin{array}{c} \text{COURSE} \end{array} \overset{\text{TEACHER}}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} = \begin{array}{c} \text{STUDENT} \end{array} \overset{\text{TEACHER}}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

5. Relations

5.1. Ordered pair
5.2. Cartesian product
5.3. Binary relation
5.4. Relation representation
5.5. Operations on relations
5.6. Properties of relations

5.6. Properties of relations

Six properties of relations we will study:

1. Reflexive
2. Irreflexive
3. Symmetric
4. Asymmetric
5. Anti-symmetric
6. Transitive

5.6. Properties of relations

Let R be a relation on set A. We say that R is:

1. *reflexive* if and only if $a\ R\ a$ for every $a \in A$;
2. *irreflexive* if and only if its complementary relation is reflexive.
3. *symmetric* if and only if $a\ R\ b$ implies $b\ R\ a$ for every $a, b \in A$;
4. *asymmetric* if and only if $(a, b) \in R \Rightarrow (b, a) \notin R$ (Asymmetry is the opposite of symmetry)
5. *anti-symmetric* if and only if $a\ R\ b$ and $b\ R\ a$ implies $a = b$ for every $a, b \in A$ (Antisymmetry is *not* the opposite of symmetry)
6. *transitive* if and only if $a\ R\ b$ and $b\ R\ c$ implies $a\ R\ c$ for every $a, b, c \in A$.

Examples of reflexive relations: $=$, $\leq$, $\geq$

Examples of irreflexive relations (relations that are not reflexive): $<$, $>$

Examples of symmetric relations: $=$

Examples of asymmetric relations: $<$, $>$

Examples of anti-symmetric relations: $=$, $\leq$, $\geq$

Examples of transitive relations:

- If $a < b$ and $b < c$, then $a < c$ ➔ Thus, $<$ is transitive
- If $a = b$ and $b = c$, then $a = c$ ➔ Thus, $=$ is transitive

Notes on *symmetric relations

Let R be a relation on set A. We say that R is:

1. *symmetric* if and only if $a\ R\ b$ implies $b\ R\ a$ for every $a, b \in A$;
2. *asymmetric* if and only if $(a, b) \in R \Rightarrow (b, a) \notin R$ (Asymmetry is the opposite of symmetry)
3. *anti-symmetric* if and only if $a\ R\ b$ and $b\ R\ a$ implies $a = b$ for every $a, b \in A$ (Antisymmetry is *not* the opposite of symmetry)

Example: A relation can be neither symmetric or asymmetric

R = { (a,b) | a=|b| }

- This is not symmetric

  -4 is not related to itself
- This is not asymmetric

  4 is related to itself
- Note that it is antisymmetric

Properties of relations summary

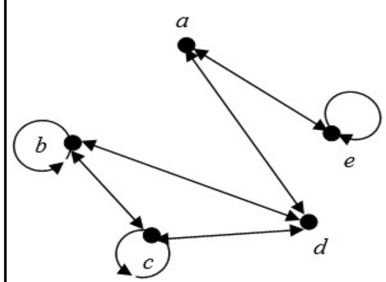
	=	<	>	≤	≥
Reflexive					
Irreflexive					
Symmetric					
Asymmetric					
Antisymmetric					
Transitive					

Properties of relations summary

	=	<	>	≤	≥
Reflexive	X			X	X
Irreflexive		X	X		
Symmetric	X				
Asymmetric		X	X		
Antisymmetric	X			X	X
Transitive	X	X	X	X	X

5.6. Properties of relations

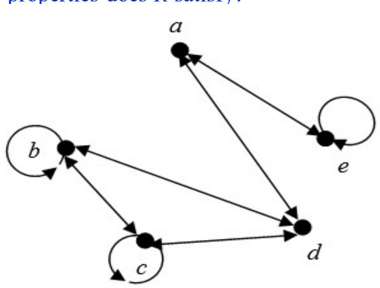
Example: Consider the directed graph of a relation R on the set $A = \{a, b, c, d, e\}$. Which of the properties does R satisfy?



1. *reflexive* if and only if $a\ R\ a$ for every $a \in A$;
2. *irreflexive* if and only if its complementary relation is reflexive.
3. *symmetric* if and only if $a\ R\ b$ implies $b\ R\ a$ for every $a, b \in A$;
4. *asymmetric* if and only if $(a, b) \in R \Rightarrow (b, a) \notin R$ (Asymmetry is the opposite of symmetry)
5. *anti-symmetric* if and only if $a\ R\ b$ and $b\ R\ a$ implies $a = b$ for every $a, b \in A$ (Antisymmetry is *not* the opposite of symmetry)
6. *transitive* if and only if $a\ R\ b$ and $b\ R\ c$ implies $a\ R\ c$ for every $a, b, c \in A$.

5.6. Properties of relations

Example: Consider the directed graph of a relation R on the set $A = \{a, b, c, d, e\}$. Which of the properties does R satisfy?



- R is not reflexive, since there is no arrow from d to itself, for example.
- R is symmetric, but not anti-symmetric, since every arrow connecting distinct points is bidirectional.
- R is not transitive since, for instance, there are arrows from a to d, and from d to b, but not from a to b.

Equivalence relation

Let R be a binary relation on the set S.

- (1) R is reflexive if sRs $\forall s \in S$
- (2) R is symmetric if s_1Rs_2 ➔ s_2Rs_1 $\forall s_1, s_2 \in S$
- (3) R is transitive if s_1Rs_2 and s_2Rs_3 ➔ s_1Rs_3 $\forall s_1, s_2, s_3 \in S$
- (4) R is equivalence relation if it is reflexive, symmetric, and transitive

➔ A binary relation is an equivalence relation on a non-empty set S if and only if the relation is reflexive(R), symmetric(S) and transitive(T).

Example 1: S = {All people}. Define xRy if x has the same parents as y
➔ R is equivalence relation on S.
Example 2: S =ℝ. Define xRy if x <y
As $x \not< x$ ➔ R is not reflexive ➔ R is not equivalence relation on S.

Equivalence relation

Let R be a binary relation on the set S.

- (1) R is reflexive if sRs $\forall s \in S$
- (2) R is symmetric if s_1Rs_2 ➔ s_2Rs_1 $\forall s_1,s_2 \in S$
- (3) R is transitive if s_1Rs_2 and s_2Rs_3 ➔ s_1Rs_3 $\forall s_1,s_2,s_3 \in S$
- (4) R is equivalence relation if it is reflexive, symmetric, and transitive

➔ A binary relation is an equivalence relation on a non-empty set S if and only if the relation is reflexive(R), symmetric(S) and transitive(T).

➔ A binary relation is a **partial order** if and only if the relation is reflexive(R), antisymmetric(A) and transitive(T).

Contents

1. Definitions
2. Set operations
3. The algebra of sets
4. Computer representation of sets
5. Relations
6. Functions

6. Functions

6.1. Definitions
6.2. Properties of function
6.3. Injective, surjective and bijective function
6.4. Function representation

6. Functions

- **Definition**: A function f from a set A to a set B, denote it by $f: A \to B$, is a relation from A to B that satisfies:
  - for each element a in A, there is an element b in B such that (a, b) is in the relation, and
  - if (a, b) and (a, c) are in the relation, then $b = c$. ➔ **1 to 1**

A function is also called a *mapping* or a *transformation*.

The set A in the above definition is called the *domain* of the function and B its *codomain*.

Thus, f is a function if it *covers* the domain (maps every element of the domain) and it is *single valued.*

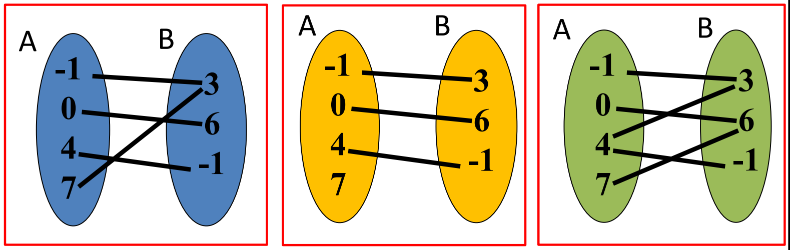
The relation given by f between a and b represented by the ordered pair (a, b) is denoted as $f(a) = b$, and b is called the image of a under f.

- **Proposition**: If $|A| = m$, $|B| = n$, then the number of possible functions from A to B is n^m

6. Functions

Thus, $f: A \rightarrow B$ is a function if it *covers* the domain (maps every element of the domain) and it is *single valued.*

- Single valued: each element in the domain is used only once
- Not allowed: 1 – many and 1 to empty



Which relation is function?

6. Functions

- The image of the set S under function $f : A \rightarrow B$, denoted by $f(S)$ is:

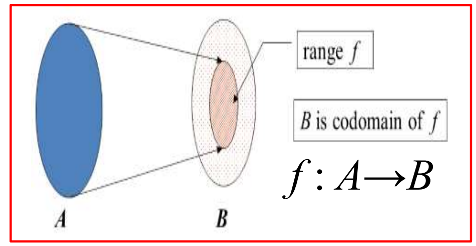
$$f(S) = \{ f(a) \mid a \in S \}$$

- The image of the domain under function $f : A \rightarrow B$, denoted by range f is:

$$\text{range } f = f(A)$$

(is also called the range of f)

In general case: range $f = f(A) \subseteq B$.



6. Functions

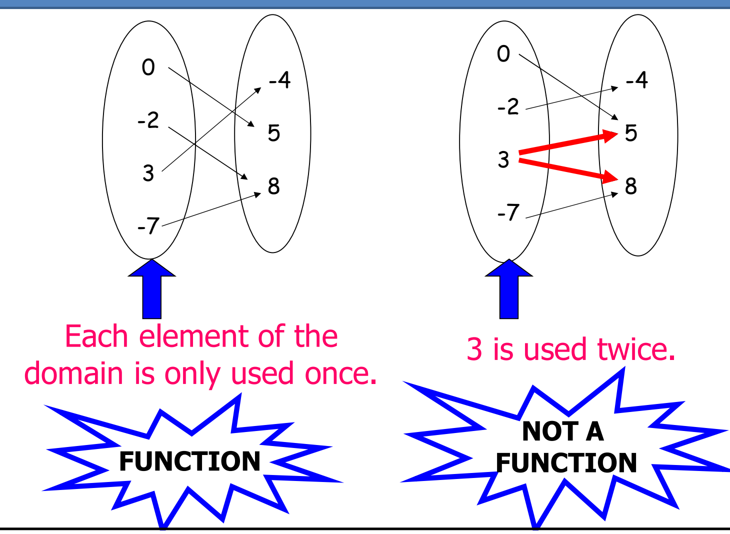
Example 1: Let $A = \{a, b\}$, $B = \{1, 2, 3\}$. Which following relations from A to B are functions from A to B ?

- $P = \{(a,1), (b,1)\}$
- $Q = \{(a,2), (b,3)\}$
- $S = \{(a,1)\}$
- $T = \{(a,2), (b,1), (b,3)\}$

Relation is function if:

- Single valued: each element in the domain is used only once
- Not allowed: 1 – many and 1 to empty

Example 2: Are the following relations also functions?



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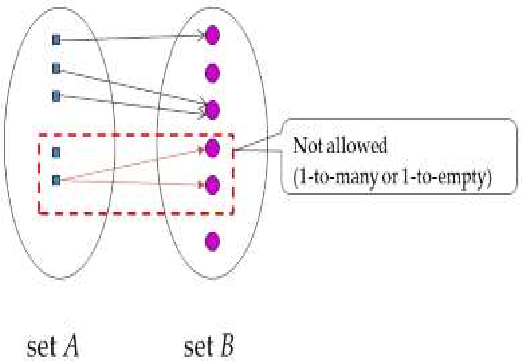

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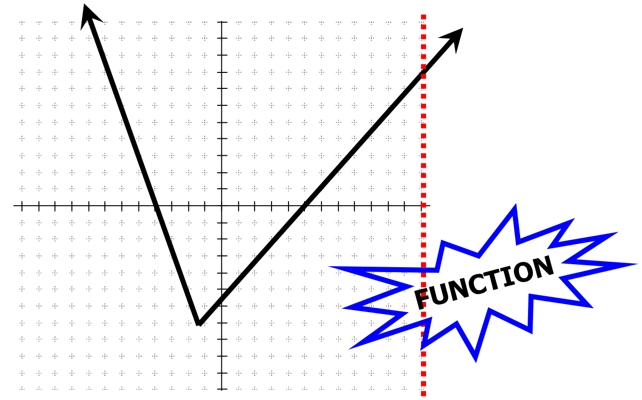
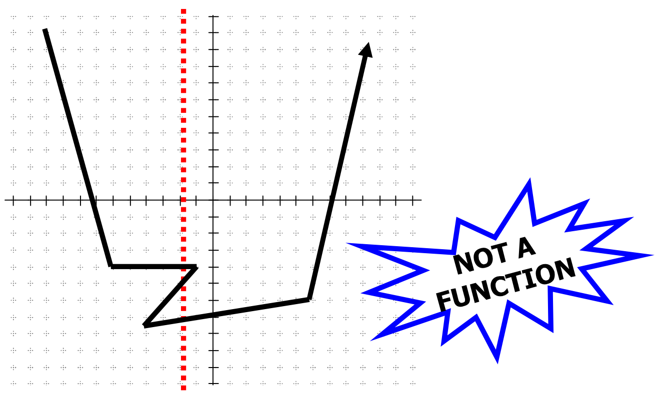


Will all vertical lines cross the graph at **only** one point?

YES. Each x is only used once.

Example 2: Are the following relations also functions?

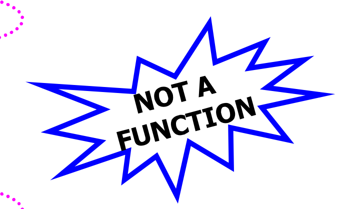


Will all vertical lines cross the graph at only one point?

NO. x =-2 is used three times.

Example 2: Are the following relations also functions?

x	y
-4	0
-2	-3
2	3
-4	1



Are all x values used only once?

NO. x = - 4 is used twice.

6. Functions

6.1. Definitions

6.2. Properties of function

6.3. Injective, surjective and bijective function

6.4. Function representation

6.2. Properties of function

$f: A \rightarrow B$ is a function from a set A to a set B, $S \subseteq A$, and $T \subseteq B$.

- **Property 1:** $f(S \cup T) = f(S) \cup f(T)$

1. **Proof for $f(S \cup T) \subseteq f(S) \cup f(T)$:**
   - Let y be an arbitrary element of $f(S \cup T)$. Then there is an element x in $S \cup T$ such that $y = f(x)$. If x is in S, then y is in $f(S)$. Hence y is in $f(S) \cup f(T)$.
   - Similarly y is in $f(S) \cup f(T)$ if x is in T.
   - Hence if $y \in f(S \cup T)$, then $y \in f(S) \cup f(T)$.
2. **Proof for $f(S) \cup f(T) \subseteq f(S \cup T)$:**
   - Let y be an arbitrary element of $f(S) \cup f(T)$. Then y is in $f(S)$ or in $f(T)$. If y is in $f(S)$, then there is an element x in S such that $y = f(x)$. Since $x \in S$ implies $x \in S \cup T$, $f(x) \in f(S \cup T)$.
   - Hence $y \in f(S \cup T)$.
   - Similarly $y \in f(S \cup T)$ if $y \in f(T)$.

Property 1 has been proven.

6.2. Properties of function

$f: A \rightarrow B$ is a function from a set A to a set B, $S \subseteq A$, and $T \subseteq B$.

- **Property 2:** $f(S \cap T) \subseteq f(S) \cap f(T)$

Proof.

- Let y be an arbitrary element of $f(S \cap T)$. Then there is an element x in $S \cap T$ such that $y = f(x)$, that is there is an element x which is in S and in T, and for which $y = f(x)$ holds. Hence $y \in f(S)$ and $y \in f(T)$, that is $y \in f(S) \cap f(T)$.
- Note here that the converse of Property 2 does not necessarily hold. For example let $S = \{1\}$, $T = \{2\}$, and $f(1) = f(2) = \{3\}$. Then $f(S \cap T) = f(\varnothing) = \varnothing$, while $f(S) \cap f(T) = \{3\}$. Hence $f(S) \cap f(T)$ cannot be a subset of $f(S \cap T)$ giving a counterexample to the converse of Property 2.

6. Functions

6.1. Definitions

6.2. Properties of function

6.3. Injective, surjective and bijective function

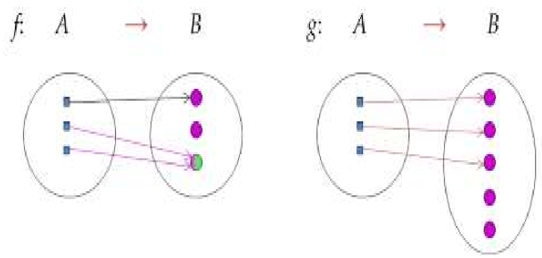
6.4. Function representation

6.3. Injective, surjective and bijective function

- A function *f* from a set *A* to a set *B* is said to be *injective* (one-to-one) if and only if:

  for all elements $a_1, a_2 \in A$

  if $f(a_1) = f(a_2)$ then $a_1 = a_2$



The function *f* is not injective

The function g is injective

$a_1, a_2 \in A, a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

6.3. Injective, surjective and bijective function

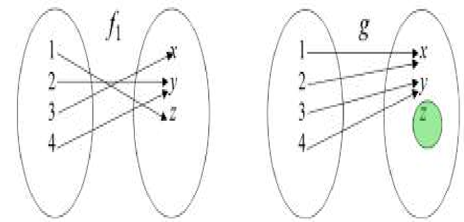
- A function *f* from a set *A* to a set *B* is said to be *surjective* (onto), if and only if: $\forall b \in B, \exists\, a \in A: b = f(a)$.

  (read: for any element $b \in B$ there is an element $a \in A$ *such that* $f(a) = b$)

that is: *f* is onto if and only if $f(A) = B$.

Example: $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Then the functions:

$f_1 = \{(1, z), (2, y), (3, x), (4, y)\}$; $g=\{(1, x), (2, x), (3, y), (4, y)\}$

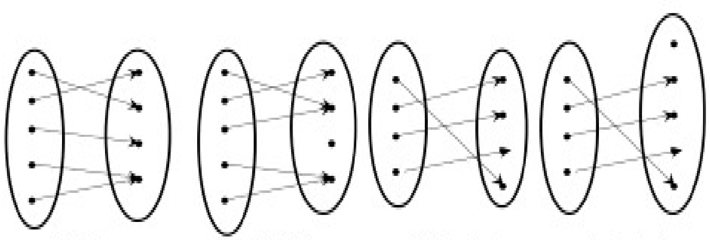


The function f_1 is onto

The function *g* is not onto because $g(A)=\{x,y\}\neq B$

6.3. Injective, surjective and bijective function

- A function is called a *bijection*, if it is injective (1-1) and surjective (onto).



Onto (but not 1-1)

Not Onto (not 1-1)

Both 1-1 and onto

1-1 but not onto

6.3. Injective, surjective and bijective function

- A function is called a *bijection*, if it is injective (1-1) and surjective (onto).

Examples:

1) Linear functions: f(x)=ax+b when a≠0

   (with domain and co-domain **R**)

2) Exponential functions: $f(x)=b^x$ (b>0, b≠1)

   (with domain **R** and co-domain $\mathbf{R}^+$)

3) Logarithmic functions: $f(x)=\log_b x$ (b>0, b≠1)

   (with domain $\mathbf{R}^+$ and co-domain **R**)

6. Functions

6.1. Definitions

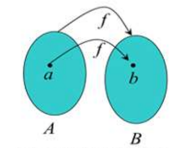
6.2. Properties of function

6.3. Injective, surjective and bijective function

6.4. Function representation

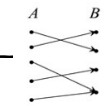
6.4. Function representation

Functions can be represented four different ways:

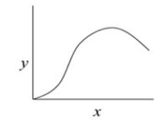


Like Venn diagrams

1. mapping
2. graph
3. table
4. matrix



Bipartite Graph



Plot

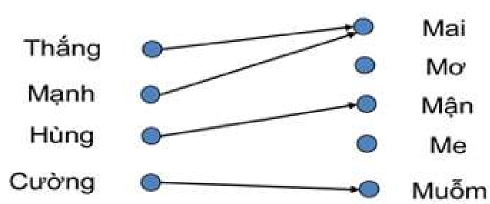
a	a_1	a_2	...	a_m
$f(a)$	$f(a_1)$	$f(a_2)$	...	$f(a_m)$

The function $f: A \to B$ can be defined by a matrix $A_f = \{a_{ij}\}$ size of $m \times n$ where

$$a_{ij} = \begin{cases} 1, \text{ if } b_j = f(a_i), & i = 1,\dots,m;\ j = 1,\dots,n. \\ 0, \text{ otherwise} \end{cases}$$

6.4. Function representation

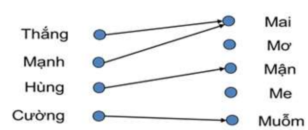
- Let A = {Thắng, Mạnh, Hùng, Cường} and B = {Mai, Mơ, Mận, Me, Muỗm}. Consider the function $f: A \to B$ defined by the following diagram:



Represent this function by table and matrix

6.4. Function representation

- Represent this function by table and matrix



$$A_f = \begin{array}{c} \begin{array}{ccccc} \text{Mai} & \text{Mơ} & \text{Mận} & \text{Me} & \text{Muỗm} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Thắng} \\ \text{Mạnh} \\ \text{Hùng} \\ \text{Cường} \end{array}$$

x	Thắng	Mạnh	Hùng	Cường
$y=f(x)$	Mai	Mai	Mận	Muỗm