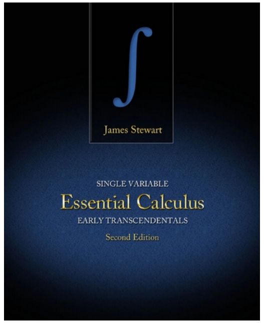
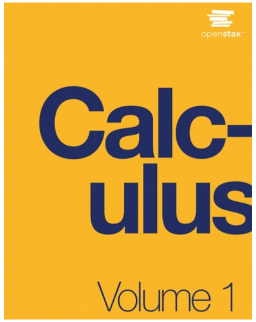
Chapter 2: Derivatives





2.1 Derivatives and Rates of Change

2.2 The Derivative as a Function

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2.5 The Chain Rule

2.6 Implicit Differentiation

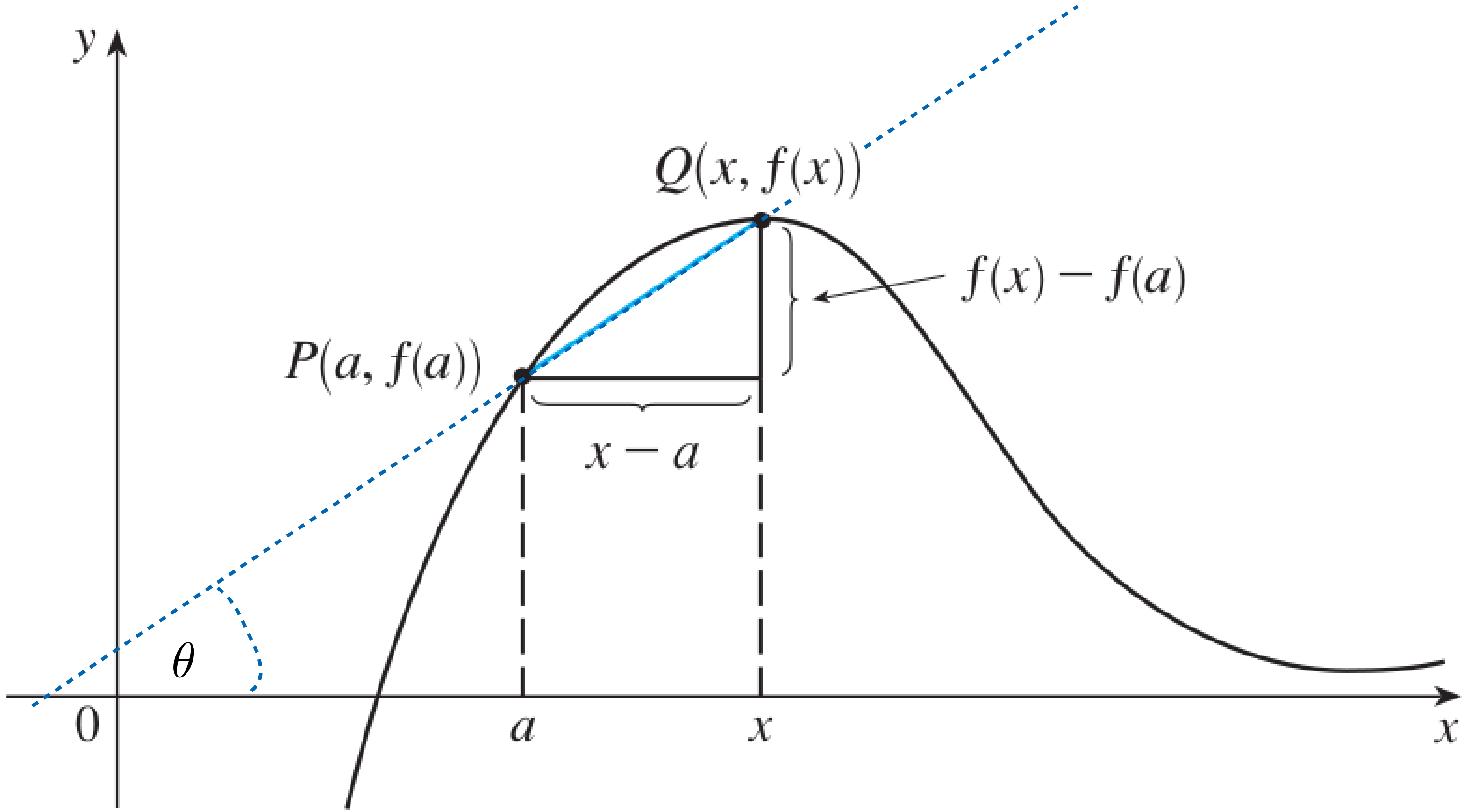
2.7 Related Rates

2.8 Linear Approximations and Differentials

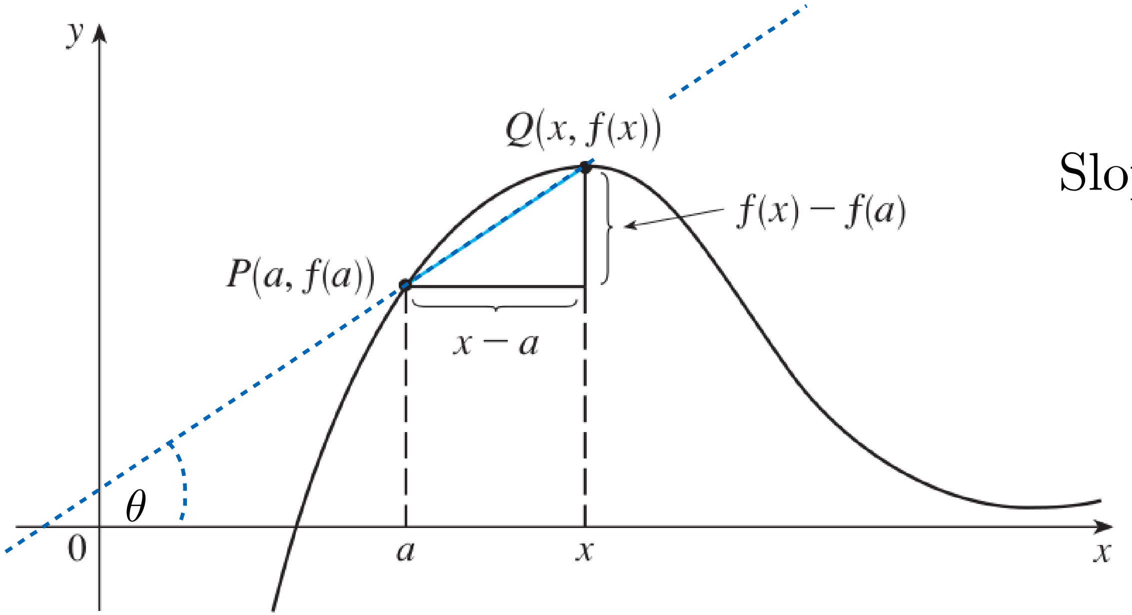
The pictures are taken from the books:

[1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280
2) G. Strang and E. J. Herman, Calculus 1, https://openstax.org/details/books/calculus-volume-1]

2.1 Derivatives and Rates of Change



2.1 Derivatives and Rates of Change



$$PQ: \quad y = f(x) = mx + b$$

Slope $\dashrightarrow$ $$m = \tan(\theta) = \frac{f(x) - f(a)}{x - a}$$

Existence of

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

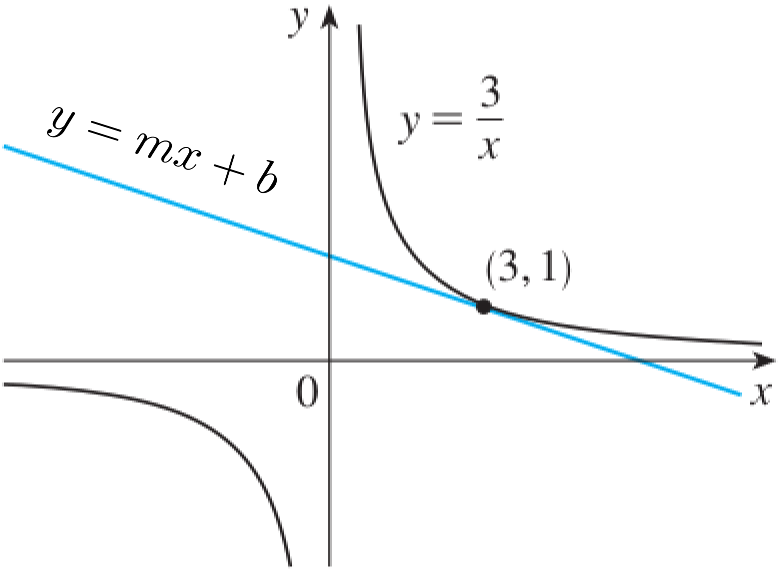
$$\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

Example: $f(x) = \dfrac{3}{x} \quad \Rightarrow \quad m = -\dfrac{1}{3}$

http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_at_a_point.html

2.1 Example

- Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3,1)$.



$$m = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{3}{3+h}\right) - 1}{h} = \lim_{h\to 0} -\frac{1}{3+h} = -\frac{1}{3}$$

$$1 = 3m + b \quad \Rightarrow \quad b = 2 \quad \Rightarrow \quad \boxed{y = -\frac{1}{3}x + 2}$$

2.1 Motivation

The r.h.s. of the slope depends only on the function f. Then, we can introduce it as property only of f.

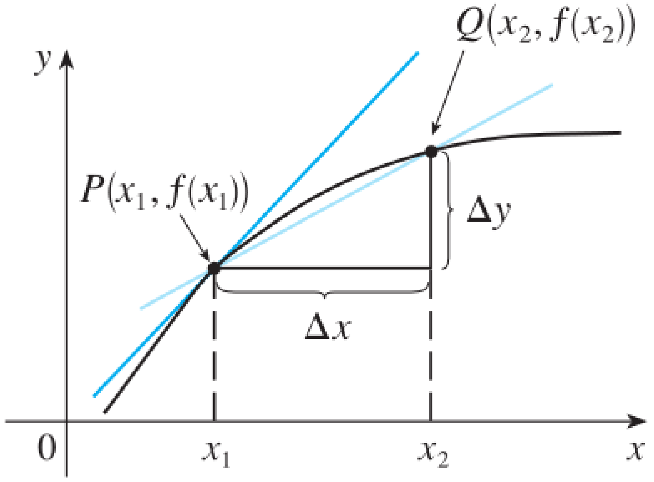
Definition

The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

2.1 Rate of Change (RoC)



- If x changes from x_1 to x_2, $\Delta x = x_2 - x_1$ and the corresponding change in y is $\Delta y = f(x_2) - f(x_1)$, then the quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

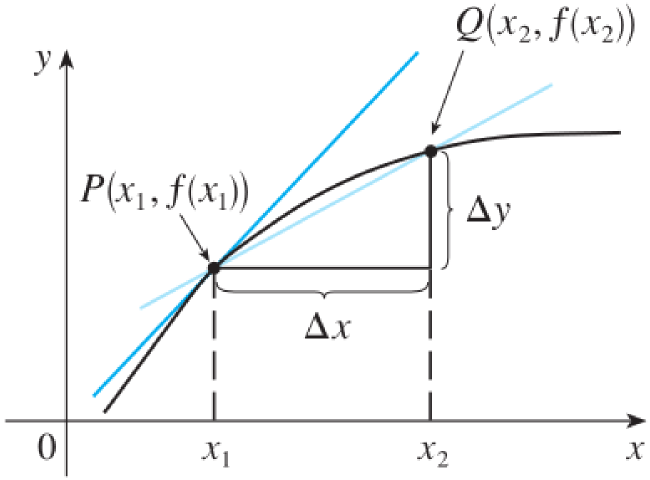
is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$.

- Then, the quotient

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

is called the **instantaneous rate of change of y with respect to x**.

2.1 Rate of Change (RoC)



- We indetify the derivative $f'(a)$ with the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

2.2 The derivative as a function

http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_as_a_function.html

- Generalization of $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ for any number x:

$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

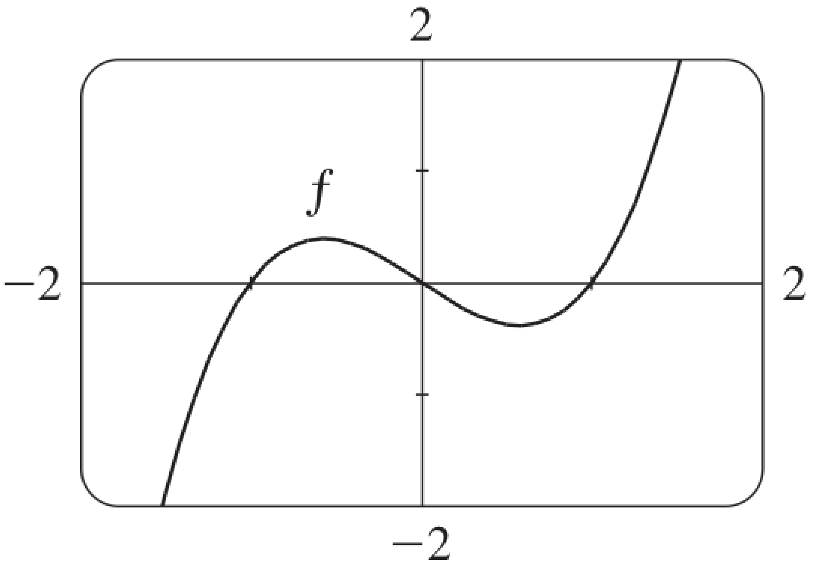
Other notations

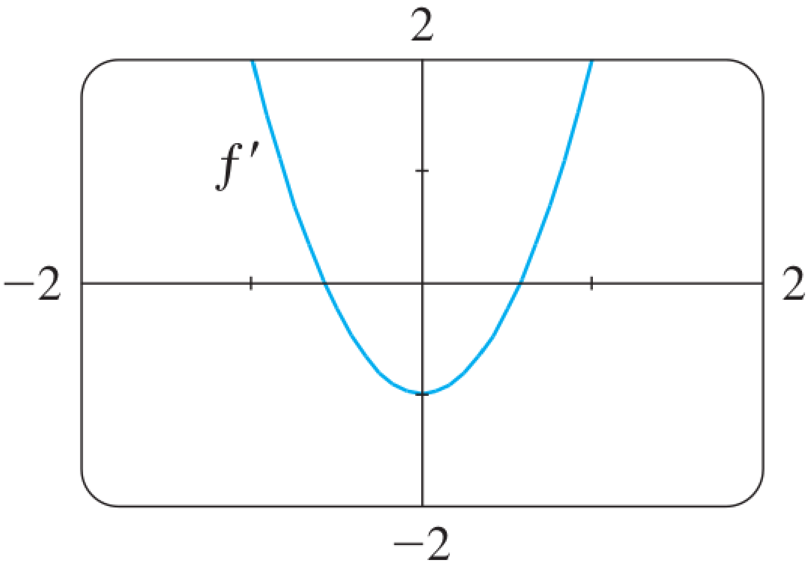
$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f(x)}{\mathrm{d}x} \qquad\qquad f'(a) = \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=a}$$

2.2 Example

- $f(x) = x^3 - x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^2 - 1\right) = 3x^2 - 1$$



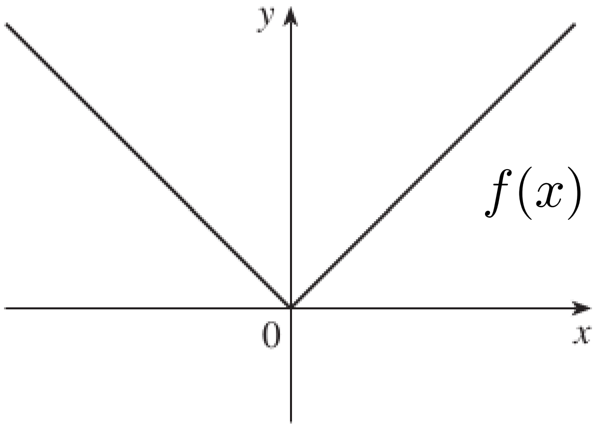


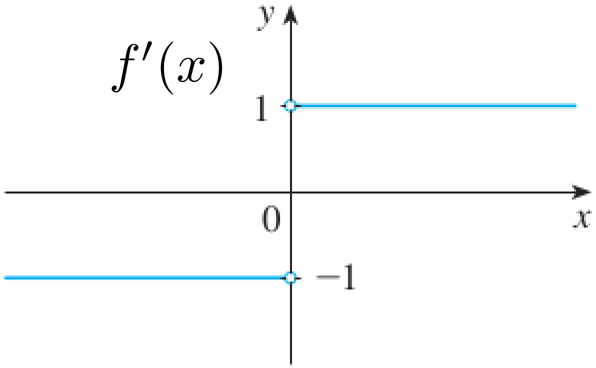
2.2 Differentiable function

Definition A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a,b) [or (a,∞),or $(-\infty,a)$ or $(-\infty,\infty)$] if it is differentiable at every number in the interval.

Example: Where is the function $f(x)=|x|$ differentiable?

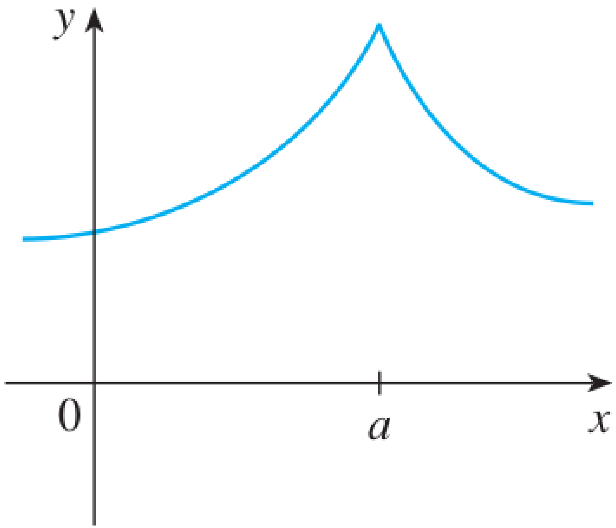
$$f'(x \gtrless 0)=\lim_{h\to 0}\frac{|x+h|-|x|}{h}=\pm 1 \qquad f'(0)=\lim_{h\to 0^\pm}=\frac{|x+h|-|x|}{h}=\pm 1$$



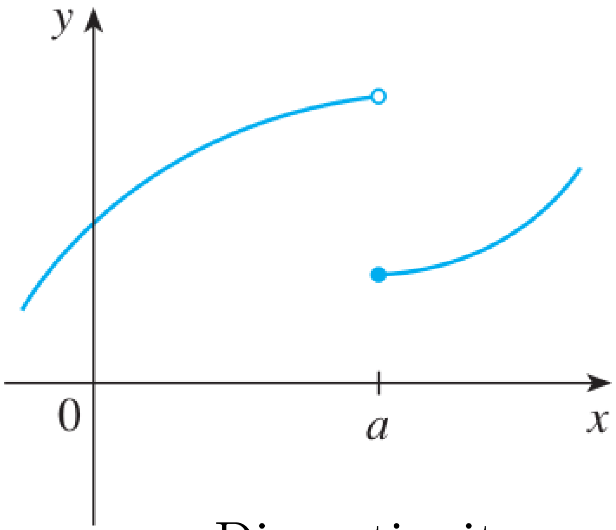


2.2 Differentiable function

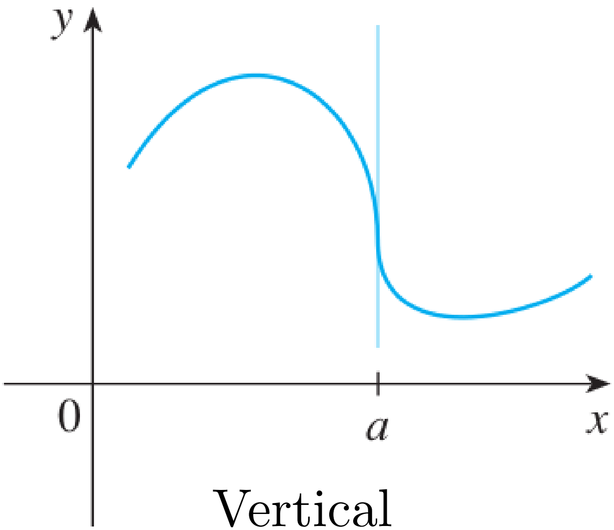
- When is a function non differentiable?



Corner



Discontinuity



Vertical tangent

Theorem If f is differentiable at a, then f is continuous at a.

2.2 Higher Order Derivatives

$$(\cdots)' = \frac{\mathrm{d}}{\mathrm{d}x}(\cdots) \quad \longrightarrow \quad ((\cdots)')' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}(\cdots)\right) \quad \Leftrightarrow \quad (\cdots)'' = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\cdots)$$

$$\big(f'(x)\big)' = \Bigg(\underbrace{\lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}}_{g(x)}\Bigg)' = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\lim_{h\to 0} \frac{f(x+2h) - f(x+h)}{h} - \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

2.2 Higher Order Derivatives

- Derivative of n^{th} order

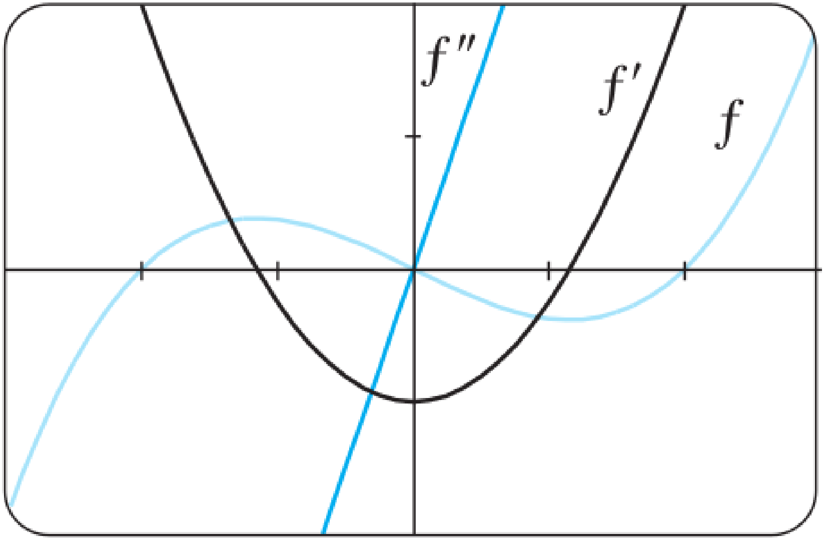
$$\underbrace{\left(\cdots\left(\left([f(x)]'\right)'\right)'\cdots\right)'}_{n\ \text{times}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\cdots\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\cdots\right) \quad \Leftrightarrow \quad f^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x)$$

2.2 Example

$$f(x) = x^3 - x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 3x^2 - 1$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = 6x$$



2.2 Example

$$f(x) = x^3 - x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 3x^2 - 1$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = 6x$$

$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h} = 6$$

$$f^{(4)}(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h} = 0$$

2.2 Example

- Physical quantities?

2.2 Example

- Physical quantities?

Velocity

Position

$$a(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) \qquad \overset{\text{math.not.}}{\Leftrightarrow} \qquad a(t) = v'(t) = x''(t)$$

Acceleration

$$\overset{\text{phys.not.}}{\Leftrightarrow} \qquad a(t) = \dot{v}(t) = \ddot{x}(t)$$

2.3-2.4 Differentiation Formulas

http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_elementary_functions.html

1. $$\frac{\mathrm{d}}{\mathrm{d}x}c = 0$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x}x = 1$$

3. $$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

4. $$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$$

5. $$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$$

6. $$\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

7. $$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \pm \frac{\mathrm{d}}{\mathrm{d}x}g(x)$$

8. $$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \cdot g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) \cdot g(x) + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}g(x)$$

9. $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}f(x) \cdot g(x) - f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}g(x)}{g^2(x)}$$

10. $$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]^n = n[f(x)]^{n-1}\frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

2.5 The Chain Rule

• If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f\big(g(x)\big)$, then F is differentiable and F' is given by the product

$$F'(x) = f'\big(g(x)\big)g'(x)$$

In Leibniz notation, if $g(x) = u$ and $f(u) = y$ are both differentiable functions, then

$$F'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_intuitive_chain_rule.html

2.5 Examples

1. Find F' if $F(x) = \sqrt{x^2+1}$

2. Differentiate a) $y = \sin(x^2)$, b) $y = \sin^2(x)$

3. Differentiate $y = (x^3 - 1)^{10}$

4. Differentiate $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$

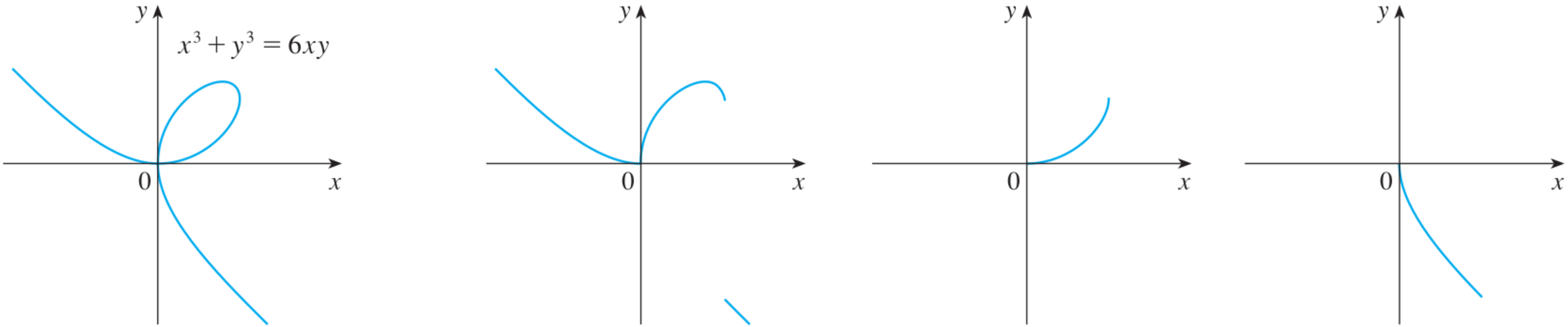
5. Differentiate $f(t) = \left(\dfrac{t-2}{2t+1}\right)^9$

6. Differentiate $f(x) = \sin\Big(\cos\big(\tan(x)\big)\Big)$

2.6 Implicit Differentiation

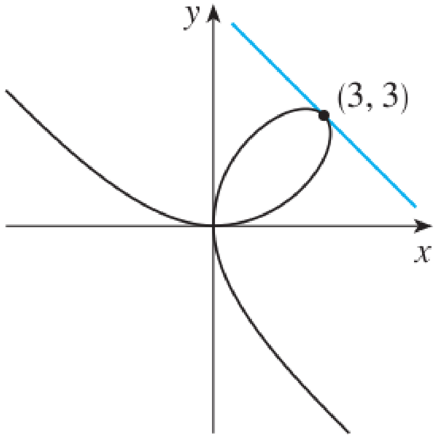
The variables x and y are related by $f(x, y) = 0$ rather than $y = f(x)$

Example: **Folium of Descartes** $\quad x^3 + y^3 = 6xy$

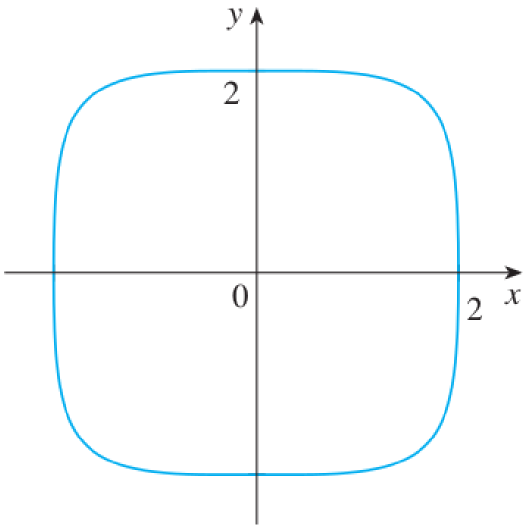


2.6 Examples

1a. Find y' if $x^3 + y^3 = 6xy$.
1b. Find the tangent to the folium of Descartes at the point $(3, 3)$.
1c. At what point (x_0, y_0) in the first quadrant is the tangent line horizontal?



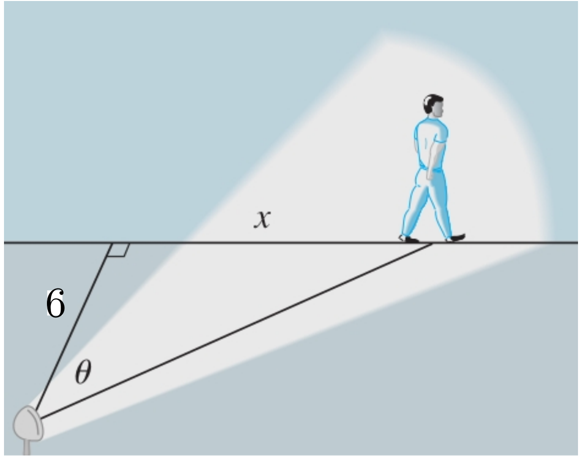
2. Find y'' if $x^4 + y^4 = 16$



2.7 Related Rates

1. Air is being pumped into a spherical balloon so that its volume increases at a rate of $100cm^3/s$. How fast is the radius of the balloon increasing when the diameter is $50cm$?

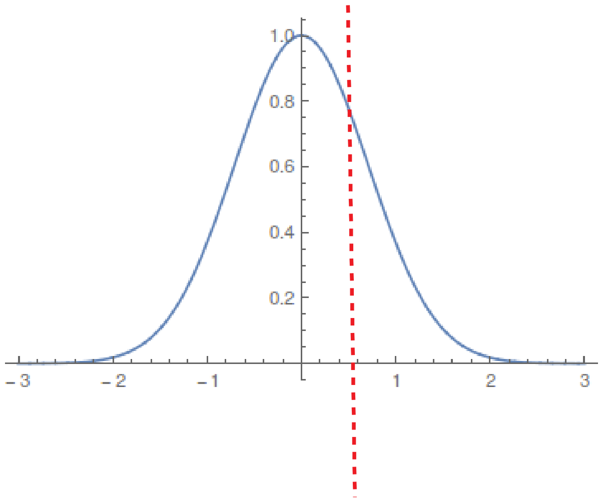
2. A man walks along a straight path at a speed of $1m/s$. A search-light is located on the ground $6m$ from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is $8m$ from the point on the path closest to the searchlight?



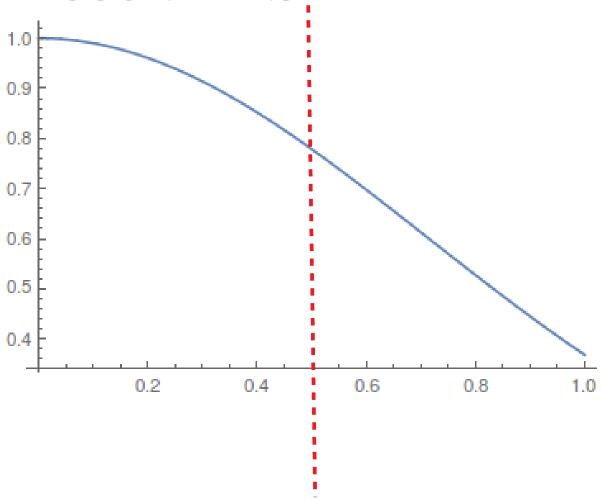
2.8 Linear Approximations and Differentials

$$f(x) = e^{-x^2}, \qquad a = 0.5 \quad \text{(red line)}$$

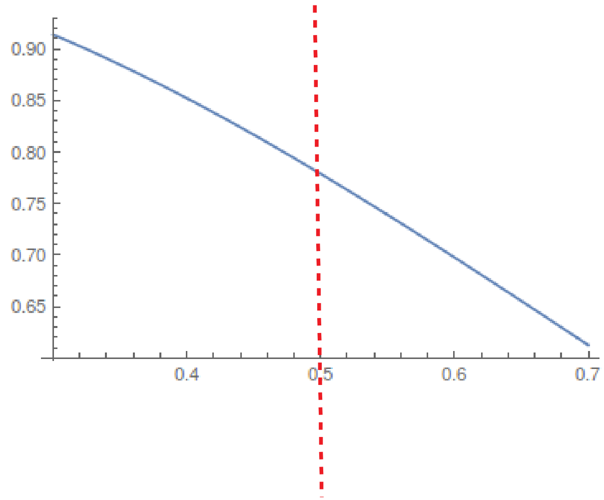
In[27]:= `Plot[f[x], {x, -3, 3}]`



In[29]:= `Plot[f[x], {x, 0, 1}]`



In[30]:= `Plot[f[x], {x, 0.3, 0.7}]`

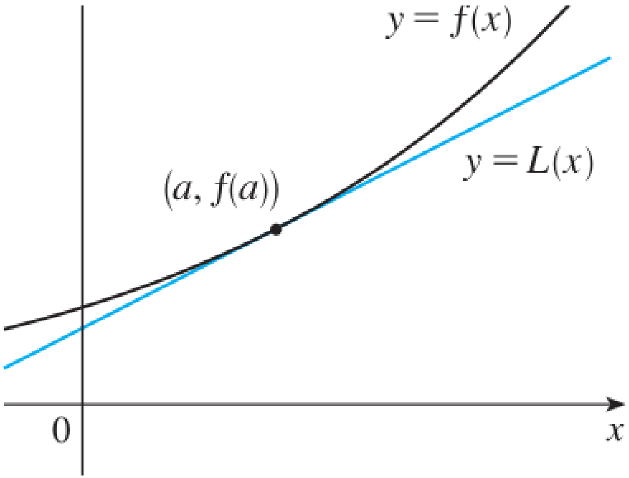


2.8 Linear Approximations and Differentials

- We use the tangent line at $(a, f(a))$ as an **approximation** to the curve $y = f(x)$ when x **is near** a. Thus,

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{L(x)}$$

Linearization function of f at a

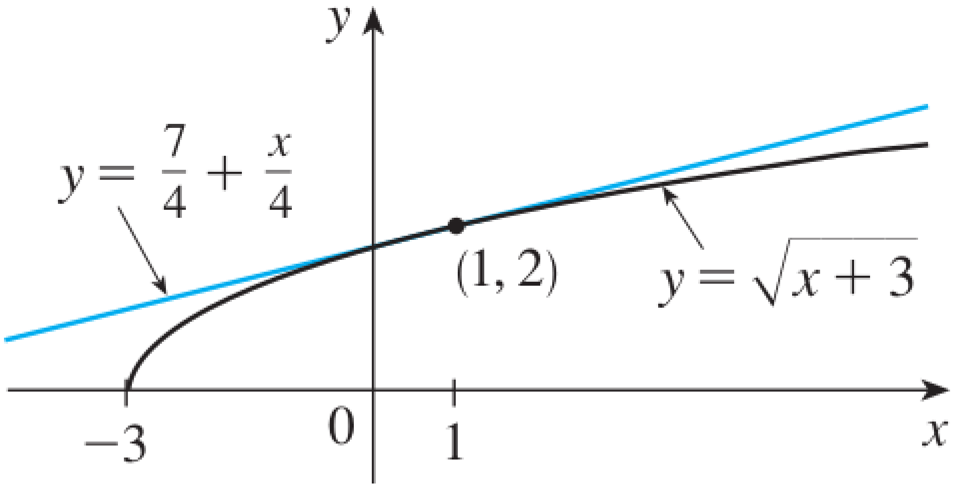


2.8 Example

- Linearize $f(x)=\sqrt{x+3}$ at $x_0=1$.

$$f'(x)=\frac{1}{2\sqrt{x+3}}$$

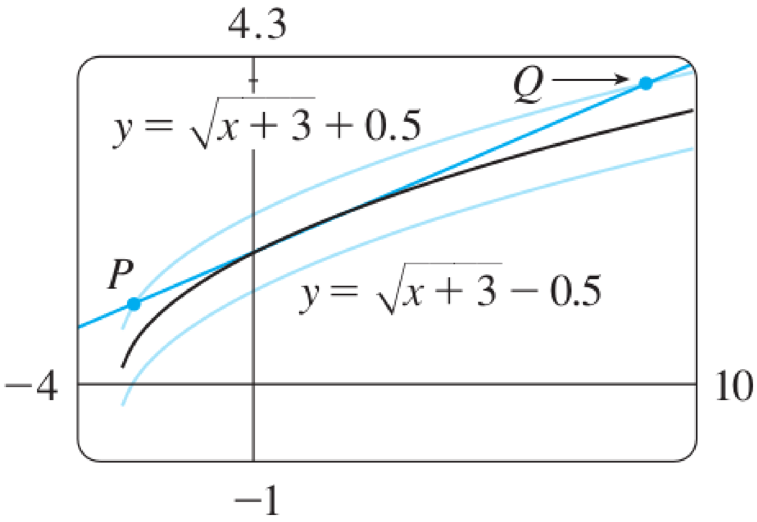
$$f(x)\approx f(x_0)+f'(x_0)(x-x_0)=\frac{7}{4}+\frac{x}{4} \quad \Rightarrow \quad \boxed{\sqrt{x+3}\approx\frac{x+7}{4}}$$



2.8 Example

- For what values of x is the linear approximation $\sqrt{x+3} \approx (x+7)/4$ accurate to within 0.5?

Accuracy to within 0.5 means that the functions should differ by less than 0.5.

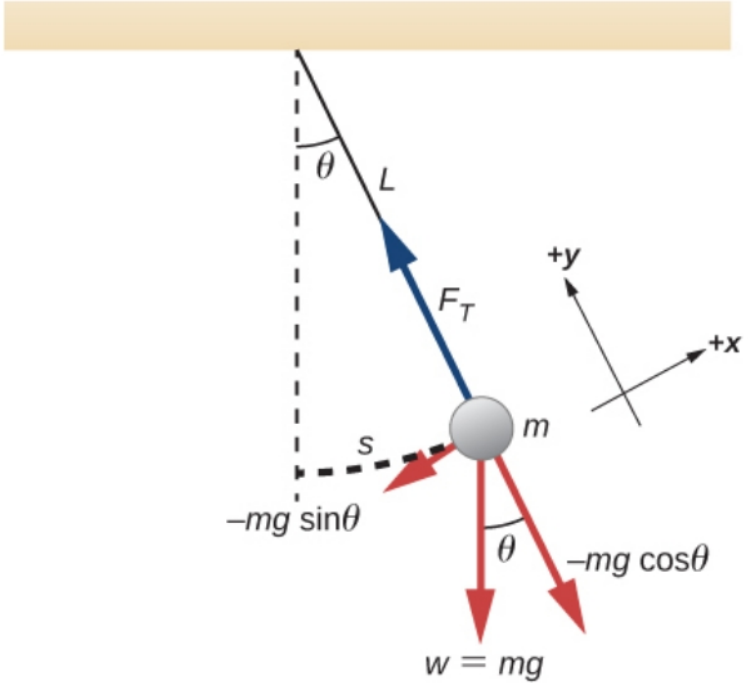


$$\left|\sqrt{x+3} - \frac{x+7}{4}\right| < 0.5 \quad \Rightarrow \quad -2.6 < x < 8.6$$

2.8 Application in Physics

- Pendulum: $a_{\mathrm{T}} = -\frac{g}{\ell}\sin(\theta) \quad \Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\sin(\theta)$

  Approximation at $\theta_0 = 0$: $\quad \sin(\theta) \approx \theta \quad \Rightarrow \quad \ddot{\theta} = -\frac{g}{\ell}\theta$

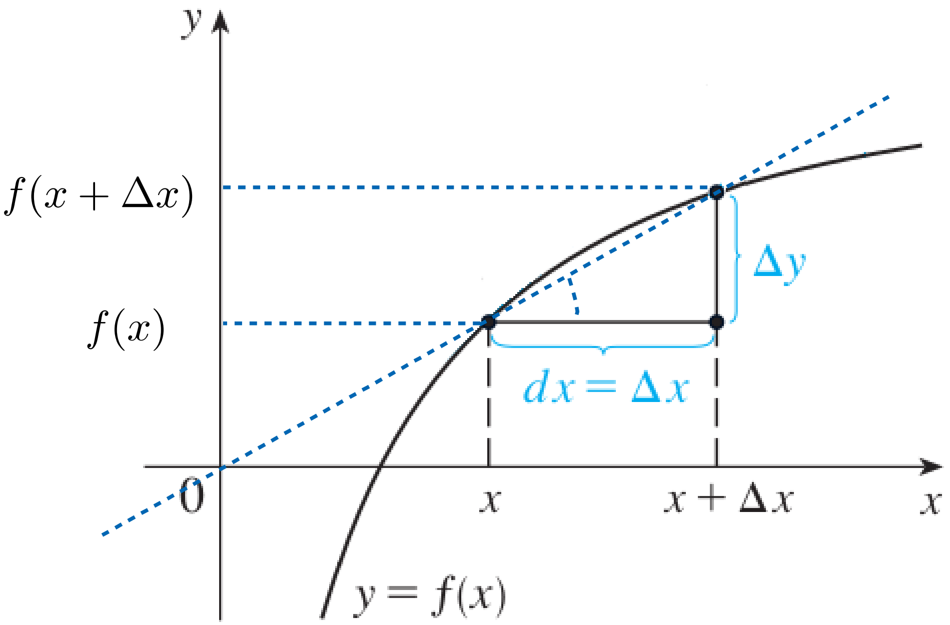


2.8 Differentials

$$\Delta y = f(x + \Delta x) - f(x) \qquad \Rightarrow \qquad \mathrm{d}y = \lim_{\Delta \to 0} \Delta y = \lim_{\Delta \to 0} [f(x + \Delta x) - f(x)]$$

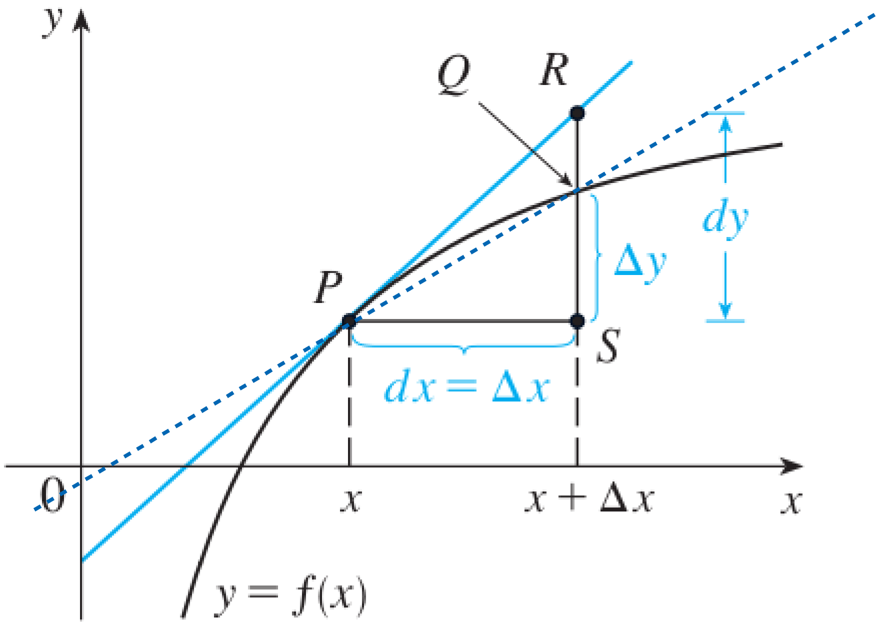
$$= \lim_{\Delta \to 0} [f(x) + f'(x)\Delta x - f(x)]$$

$$= f'(x)\mathrm{d}x$$



$$\Rightarrow \quad \boxed{\mathrm{d}y = f'(x)\mathrm{d}x} \quad \text{The differential}$$

2.8 Approximate Δy through dy



$$\lim_{\Delta \to 0} f(x + \Delta x) = \lim_{\Delta \to 0} [f(x) + f'(x)\Delta x]$$

For small Δ :

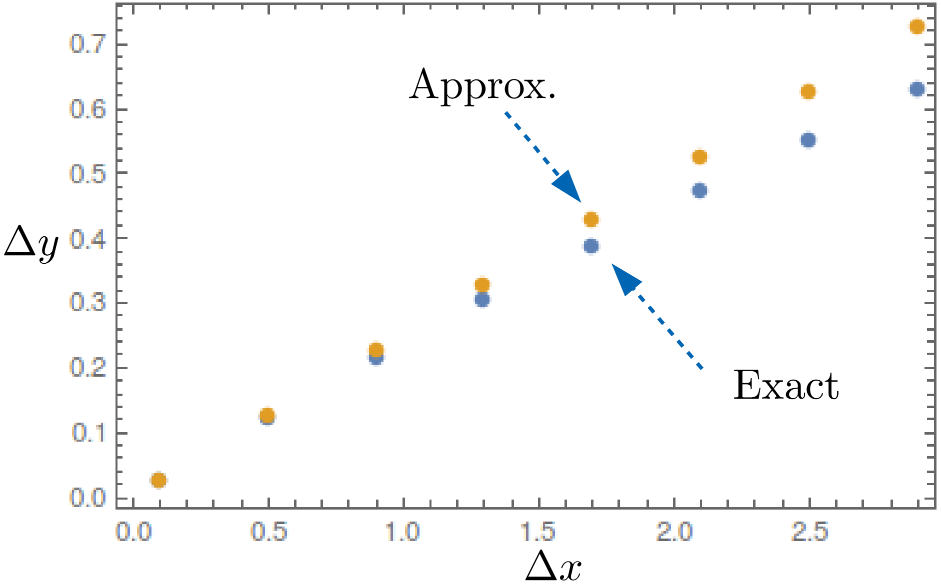
$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

$$\Rightarrow \quad \boxed{\Delta y \approx f'(x)\Delta x}$$

2.8 Example

- Consider $f(x) = \sqrt{x+3} \quad \Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{x+3}}$, starting point $x_0 = 1$

Δx	Exact Δy	Approx. Δy
0.1	0.024846	0.025
0.5	0.121320	0.125
0.9	0.213594	0.225
1.3	0.302173	0.325
1.7	0.387467	0.425
2.1	0.469818	0.525
2.5	0.549510	0.625
2.9	0.626785	0.725



2.8 Example

• The radius of a sphere was measured and found to be $21cm$ with a possible error in measurement of at most $0.05cm$. What is the maximum error in using this value of the radius to compute the volume of the sphere?