Chapter 2: Series of functions

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Definition

Given a **sequence of functions** $\{u_n(x)\}_{n\geq 1}$ defined on a set X. Series of functions is the sum

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots$$

The $n-$th partial sum is

$$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

Domain of convergence

Definition

$\sum_{n=1}^{\infty} u_n(x)$ converges at x_0 if $\sum_{n=1}^{\infty} u_n(x_0)$ **converges**.

$\sum_{n=1}^{\infty} u_n(x)$ diverges at x_0 if $\sum_{n=1}^{\infty} u_n(x_0)$ **diverges**.

The set of all x_0 at which the series of functions $\sum_{n=1}^{\infty} u_n(x)$ converges is called the domain of convergence of the series.

For x in the domain of convergence: $\sum_{n=1}^{\infty} u_n(x) = S(x)$, $S(x)$ is called the sum of the series.

$$S(x) = \lim_{n\to\infty} S_n(x).$$

Example

Find the domain of convergence

$$\text{a) } \sum_{n=1}^{\infty} x^n \qquad \text{b) } \sum_{n=1}^{\infty} n^x \qquad \text{c) } \sum_{n=1}^{\infty} \frac{1}{1+x^n} \qquad \text{d) } \sum_{n=1}^{\infty} \frac{x^n}{n}$$

- Find $D = \lim_{n\to\infty} \left| \frac{u_{n+1}(x)}{u_n(x)} \right|$ or $C = \lim_{n\to\infty} \sqrt[n]{|u_n(x)|}$.
- Find x such that $D < 1$ or $C < 1$, the series converges.
- Test for convergence at **endpoints**.
  At these points $D = 1$ (or $C = 1$), we have to use other criteria.

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Uniform convergence

$$\sum_{n=1}^{\infty} u_n(x) = S(x) \Leftrightarrow \lim_{n\to\infty} S_n(x) = S(x) \Leftrightarrow$$

$$\forall\, \varepsilon > 0, \exists\, N_0(\varepsilon, x) \in \mathbb{N} : \forall\, n \geq N_0 : |S_n(x) - S(x)| < \varepsilon.$$

Definition

The series of functions $\sum\limits_{n=1}^{\infty} u_n(x)$ converges uniformly to $S(x)$ **on the set** X if

$$\forall\, \varepsilon > 0, \exists\, N_0(\varepsilon) \in \mathbb{N} \,\Big|\, \forall\, n \geq N_0 : |S_n(x) - S(x)| < \varepsilon, \forall x \in X.$$

Illustration

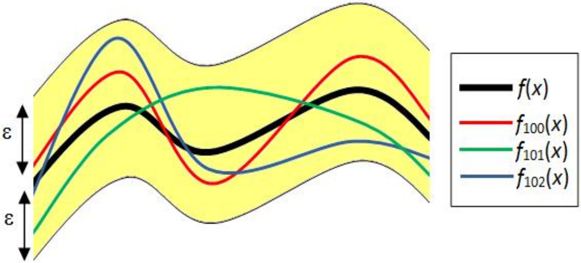

Figure: Source: simomaths.wordpress.com

Example

Test for uniform convergence

1. $S_n(x) = \dfrac{x^n}{n}$, $-1 \leq x \leq 1$.
2. $S_n(x) = x^n$, $0 \leq x \leq 1$.

Weierstrass test

Proposition

If

- $|u_n(x)| \leq a_n, \forall n \in \mathbb{N}, \forall x \in X$, $a_n \in \mathbb{R}$,
- *the number series* $\sum\limits_{n=1}^{\infty} a_n$ *converges,*

then the series $\sum\limits_{n=1}^{\infty} u_n(x)$ *converges uniformly on the set* X.

Example

Test for uniform convergence.

1. $\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2 + x^2}$ on $\mathbb{R}$.
2. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^{2n}}{n\sqrt{n}}$ on $(-1, 1)$.

Properties of uniformly convergent series of functions

Theorem (Continuity)

If $\sum_{n=1}^{\infty} u_n(x)$ **converges uniformly** *to* $S(x)$ *on the set* X *and* $u_n(x)$ *are continuous functions on* X*, then* $S(x)$ *is continuous on* X.

$X = [a, b]$, continuity implies integrability.

Theorem (Integrability)

If $\sum_{n=1}^{\infty} u_n(x)$ *converges uniformly to* $S(x)$ *on* $[a, b]$, $u_n(x)$ *are continuous functions on* $[a, b]$. *Then* $S(x)$ *is integrable on* $[a, b]$. *Moreover,*

$$\int_a^b S(x)dx = \int_a^b \Big(\sum_{n=1}^{\infty} u_n(x)\Big) dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx.$$

Theorem (Differentiability)

If $\sum_{n=1}^{\infty} u_n(x)$ *converges to* $S(x)$ *on* (a,b), $u_n(x)$ *are continuously differentiable on* (a,b), $\sum_{n=1}^{\infty} u_n'(x)$ *converges uniformly on* (a,b) *then* $S(x)$ *is differentiable on* (a,b). *Moreover,*

$$S'(x) = \Big(\sum_{n=1}^{\infty} u_n(x)\Big)' = \sum_{n=1}^{\infty} u_n'(x).$$

Example

Prove that the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$ is differentiable on $\mathbb{R}$.

Example

Find the sum $\sum_{n=1}^{\infty} \dfrac{x^n}{n}$, $x \in [-1, 1)$.