Lời giải bài tập chương 6

(Trường vô hướng)

Câu 1:

Ta có : $\overrightarrow{l} = (1; -2; 2)$

$$\Longrightarrow \cos\alpha = \frac{l_x}{\left|\overrightarrow{l}\right|} = \frac{1}{3}; \quad \cos\beta = \frac{l_y}{\left|\overrightarrow{l}\right|} = \frac{-2}{3}; \quad \cos\gamma = \frac{l_z}{\left|\overrightarrow{l}\right|} = \frac{2}{3}$$

$$\Longrightarrow \frac{\partial u}{\partial \overrightarrow{l}} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma$$

$$= \frac{2x}{3} - \frac{4y}{3} + \frac{4z}{3}$$

$$\Longrightarrow \frac{\partial u}{\partial \overrightarrow{l}}(1; 0; -1) = \frac{-2}{3}$$

Câu 2:

Ta có : $\overrightarrow{l} = (2; 1)$

$$\Longrightarrow \cos\alpha = \frac{l_x}{\left|\overrightarrow{l}\right|} = \frac{2}{\sqrt{5}}; \quad \cos\beta = \frac{l_y}{\left|\overrightarrow{l}\right|} = \frac{1}{\sqrt{5}}$$

Đặt $F(x; y; z) = z^3 + 2xy - y$. Khi đó:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-2y}{3z^2}; \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{1-2x}{3z^2}$$

$$\Longrightarrow \frac{\partial z}{\partial \overrightarrow{l}} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\cos\beta = \frac{-4y}{3\sqrt{5}z^2} + \frac{1-2x}{3\sqrt{5}z^2}$$

$$\Longrightarrow \frac{\partial z}{\partial \overrightarrow{l}}(-1; -1) = \frac{-4(-1)}{3\sqrt{5}\cdot 1^2} + \frac{1-2(-1)}{3\sqrt{5}\cdot 1^2} = \frac{7\sqrt{5}}{15}$$

Câu 3:

Ta có:

$$\overrightarrow{\text{grad}}u = \frac{\partial u}{\partial x}\overrightarrow{i} + \frac{\partial u}{\partial y}\overrightarrow{j} + \frac{\partial u}{\partial z}\overrightarrow{k} = \frac{x^3\overrightarrow{i} + y^3\overrightarrow{j} + z^3\overrightarrow{k}}{(x^4+y^4+z^4)^{\frac{3}{4}}}$$

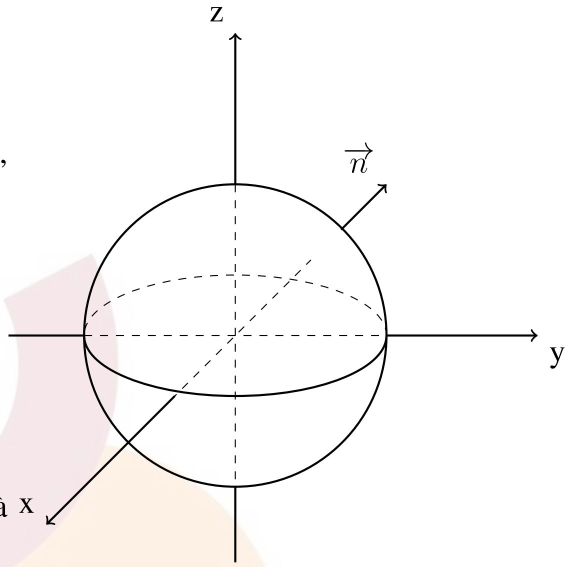
$$\Longrightarrow \overrightarrow{\text{grad}}u(1; 1; 1) = \frac{\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}}{3^{\frac{3}{4}}} = (3^{\frac{-3}{4}}; 3^{\frac{-3}{4}}; 3^{\frac{-3}{4}})$$

(Trường vector)

Câu 1:

Thông lượng của $\overrightarrow{F} = x^4\overrightarrow{i} + y^4\overrightarrow{j} + z^4\overrightarrow{k}$ là:

$$\phi = \iint\limits_S x^4dydz + y^4dzdx + z^4dxdy$$



Trong đó S là mặt cầu $x^2 + y^2 + z^2 = 4$, mặt S kín, hướng ra ngoài

Áp dụng công thức Ostrogradsky ta có:

$$\phi = \iiint\limits_V 4(x^3 + y^3 + z^3)dxdydz$$

Trong đó miền V là $x^2 + y^2 + z^2 \leq 4$

Ta thấy x^3, y^3, z^3 là hàm lẻ đối với biến x, y, z mà miền V là miền đối xứng.

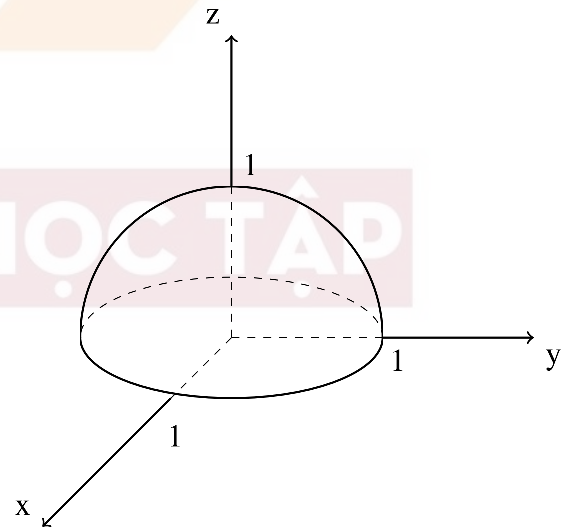
$\Rightarrow \phi = 0$

Vậy $\phi = 0$

Câu 2:

Thông lượng của trường $\overrightarrow{F}$ là:

$$\phi = \iint\limits_S xz^2dydz + x^2ydzdx + y^2(z+1)dxdy$$



Dựng mặt S' $\begin{cases} Oxy, & \text{hướng xuống dưới} \\ x^2 + y^2 \leqslant 1 \end{cases}$

Ta có: $\phi = \iint\limits_{S+S'} - \iint\limits_{S'} = I_1 - I_2$

Tính I_1

Áp dụng công thức Ostrogradski:

$$I_1 = \iiint\limits_V (x^2 + y^2 + z^2)dxdydz \quad \text{trong đó } V \begin{cases} x^2 + y^2 + z^2 \leqslant 1 \\ z \geqslant 0 \end{cases}$$

$$\text{Đặt} \begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases} \Rightarrow \begin{cases} 0 \leqslant \varphi \leqslant 2\pi \\ 0 \leqslant \theta \leqslant \dfrac{\pi}{2} \\ 0 \leqslant r \leqslant 1 \end{cases}, |J| = r^2 sin\theta$$

Do đó $I_1 = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{2}} d\theta \int\limits_0^1 r^4 \sin\theta dr = \dfrac{2\pi}{5}$

Tính I_2

Mặt S' $\begin{cases} z = 0 \\ x^2 + y^2 \leqslant 1 \end{cases} \Rightarrow I_2 = \iint\limits_{S'} y^2 dxdy$

Đặt $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \begin{cases} 0 \leqslant \varphi \leqslant 2\pi \\ 0 \leqslant r \leqslant 1 \end{cases}, |J| = r$

$I_2 = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 r^3 \sin^2\varphi dr = \dfrac{\pi}{4}$

Vậy $\phi = I_1 - I_2 = \dfrac{3\pi}{20}$

Câu 3:

Điểm không xoáy của trường vector $\overrightarrow{F}$ là điểm M thỏa mãn

$$\overrightarrow{\text{rot}}\overrightarrow{F}(M) = 0$$

Ta có:

$$\begin{aligned}
\overrightarrow{\text{rot}}\overrightarrow{F} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\overrightarrow{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\overrightarrow{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{k} \\
&= (z^2 + 2xy - xy)\overrightarrow{i} + (y^2 - y^2)\overrightarrow{j} + (yz - x^2 - 2yz)\overrightarrow{k} \\
&= (z^2 + xy)\overrightarrow{i} - (x^2 + yz)\overrightarrow{k} = \overrightarrow{0}
\end{aligned}$$

$$\Leftrightarrow \begin{cases} z^2 + xy = 0 \\ x^2 + yz = 0 \end{cases} \Leftrightarrow \begin{cases} z^3 + xyz = 0 \\ x^3 + xyz = 0 \end{cases} \Leftrightarrow \begin{cases} x = z \\ z^2 + zy = 0 \end{cases} \Leftrightarrow \begin{cases} x = z \\ \left[\begin{array}{l} z = 0 \\ z = -y \end{array}\right. \end{cases}$$

Vậy những điểm không xoáy sẽ có dạng $(0; k; 0)$ hoặc $(k; -k; k)$ với $k \in \mathbb{R}$

Câu 4:

(a)

NOTE: $\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$

$$\overrightarrow{F} = \frac{-m}{r^3}\overrightarrow{r} = -m\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\overrightarrow{i} + \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\overrightarrow{j} + \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\overrightarrow{k}\right)$$

$$= \overrightarrow{\text{grad}}\left(-\frac{m}{\sqrt{x^2+y^2+z^2}}\right) = \overrightarrow{\text{grad}}\left(-\frac{m}{r}\right)$$

Do đó $\overrightarrow{F}$ là trường thế với hàm thế vị $u = -\dfrac{m}{r}$

(b)

Ta có : $\overrightarrow{\text{grad}}\, u = \dfrac{\partial u}{\partial x}\overrightarrow{i} + \dfrac{\partial u}{\partial y}\overrightarrow{j} + \dfrac{\partial u}{\partial z}\overrightarrow{k}$

Khi đó:

$$\overrightarrow{\text{rot}}(\overrightarrow{\text{grad}} u) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z \partial y}\right)\overrightarrow{i} + \left(\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial z \partial x}\right)\overrightarrow{j} + \left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x}\right)\overrightarrow{k}$$

Do u có đạo hàm riêng cấp 2 liên tục nên

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y};\quad \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x};\quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Do đó $\overrightarrow{\text{rot}}(\overrightarrow{\text{grad}} u) = 0$ hay $\overrightarrow{\text{grad}} u$ là trường thế.

Câu 5:

$\overrightarrow{F} = y\overrightarrow{i} + 2z\overrightarrow{j} + 3x\overrightarrow{k}$ và $C : \begin{cases} x^2 + y^2 + z^2 = 9 \\ x + 2y + z = 0 \end{cases}$

Hoàn lưu của $\overrightarrow{F}$ dọc theo C là:

$$I = \int\limits_C ydx + 2zdy + 3xdz \quad \Longrightarrow \begin{cases} P = y \\ Q = 2z \\ R = 3x \end{cases}$$

Áp dụng công thức Stoke:

$$I = \iint\limits_S (-1)dxdy + (-3)dzdx + (-2)dydz$$

S là hình tròn tâm O bán kính 3 nằm trên mặt phẳng $x + 2y + z = 0$ bao bởi C, chiều dương hướng theo tia Oz.

Vecto pháp tuyến hướng theo chiều dương của mặt S là: $\overrightarrow{n} = (\cos\alpha; \cos\beta; \cos\gamma) = (\frac{1}{\sqrt{6}}; \frac{2}{\sqrt{6}}; \frac{1}{\sqrt{6}})$.

Sử dụng mối liên hệ giữa tích phân mặt loại 1 và loại 2:

$$\begin{aligned} I &= \iint\limits_S (-1)\cos\gamma + (-3)\cos\beta + (-2)\cos\alpha \, dS \\ &= \iint\limits_S \frac{-9}{\sqrt{6}} dS \\ &= \frac{-27\sqrt{6}}{2}\pi \end{aligned}$$

Vậy $I = \dfrac{-27\sqrt{6}}{2}\pi$

Câu 6:

(a)

$$\overrightarrow{\text{grad}} u = \frac{\partial u}{\partial x}\overrightarrow{i} + \frac{\partial u}{\partial y}\overrightarrow{j} + \frac{\partial u}{\partial z}\overrightarrow{k}$$

$$\Longrightarrow \ \text{div}(\overrightarrow{\text{grad}}\, u) = \text{div}(\frac{\partial u}{\partial x}\overrightarrow{i} + \frac{\partial u}{\partial y}\overrightarrow{j} + \frac{\partial u}{\partial z}\overrightarrow{k})$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \Delta u$$

(b)

$$u\overrightarrow{F} = uF_x\overrightarrow{i} + uF_y\overrightarrow{j} + uF_z\overrightarrow{k}$$

$$\overrightarrow{\text{rot}}(u\overrightarrow{F}) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ uF_x & uF_y & uF_z \end{vmatrix}$$

$$= \left(\frac{\partial(uF_z)}{\partial y} - \frac{\partial(uF_y)}{\partial z}\right)\overrightarrow{i} + \left(\frac{\partial(uF_x)}{\partial z} - \frac{\partial(uF_z)}{\partial x}\right)\overrightarrow{j} + \left(\frac{\partial(uF_y)}{\partial x} - \frac{\partial(uF_x)}{\partial y}\right)\overrightarrow{k}$$

$$= \left(F_z\frac{\partial u}{\partial y} + u\frac{\partial F_z}{\partial y} - F_y\frac{\partial u}{\partial z} - u\frac{\partial F_y}{\partial z}\right)\overrightarrow{i} + \left(F_x\frac{\partial u}{\partial z} + u\frac{\partial F_x}{\partial z} - F_z\frac{\partial u}{\partial x} - u\frac{\partial F_z}{\partial x}\right)\overrightarrow{j}$$
$$+ \left(F_y\frac{\partial u}{\partial x} + u\frac{\partial F_y}{\partial x} - F_x\frac{\partial u}{\partial y} - u\frac{\partial F_x}{\partial y}\right)\overrightarrow{k}$$

$$= u\left(\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\overrightarrow{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\overrightarrow{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\overrightarrow{k} + \right)$$
$$+ \left(\left(F_z\frac{\partial y}{\partial y} - F_y\frac{\partial u}{\partial z}\right)\overrightarrow{i} + \left(F_x\frac{\partial u}{\partial z} - F_z\frac{\partial u}{\partial x}\right)\overrightarrow{j} + \left(F_y\frac{\partial u}{\partial x} - F_x\frac{\partial u}{\partial y}\right)\overrightarrow{k}\right)$$

$$= u\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial x} \\ F_x & F_y & F_z \end{vmatrix} + (F_x; F_y; F_z) \wedge (\frac{\partial u}{\partial x}; \frac{\partial u}{\partial y}; \frac{\partial u}{\partial z}) = \text{u.rot}\overrightarrow{F} + \overrightarrow{\text{grad}} u \wedge \overrightarrow{F}$$