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HUST

MI1120Q - Calculus 2 Exercise

Chapter 1

VECTORS AND THE GEOMETRY OF SPACE

Reference: James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

1.1 Three-dimensional coordinate systems

1. Find the lengths of the sides of the triangle PQR. Is it a right triangle? Is it an isosceles triangle?

a) $P(3;-2;-3)$, $Q(7;0;1)$, $R(1;2;1)$.

b) $P(2;-1;0)$, $Q(4;1;1)$, $R(4;-5;4)$.

2. Find an equation of the sphere with center $(1;-4;3)$ and radius 5. Describe its intersection with each of the coordinate planes.

3. Find an equation of the sphere that passes through the origin and whose center is $(1;2;3)$.

4. Find an equation of a sphere if one of its diameters has end points $(2;1;4)$ and $(4;3;10)$.

5. Find an equation of the largest sphere with center $(5,4,9)$ that is contained in the first octant.

6. Write inequalities to describe the following regions

a) The region consisting of all points between (but not on) the spheres of radius r and R centered at the origin, where $r < R$.

b) The solid upper hemisphere of the sphere of radius 2 centered at the origin.

7. Consider the points P such that the distance from P to $A(-1;5;3)$ is twice the distance from P to $B(6;2;-2)$. Show that the set of all such points is a sphere, and find its center and radius.

8. Find an equation of the set of all points equidistant from the points $A(-1;5;3)$ and $B(6;2;-2)$. Describe the set.

1.2 Vectors

9. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2;4)$.

10. Find the unit vectors that are parallel to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6;1)$.

11. Find the unit vectors that are perpendicular to the tangent line to the curve $y = 2\sin x$ at the point $(\pi/6;1)$.

12. Let C be the point on the line segment AB that is twice as far from B as it is from A. If $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$, show that $\mathbf{c} = \mathbf{\frac{2}{3}a} + \mathbf{\frac{1}{3}b}$.

1.3 The dot product

13. Determine whether the given vectors are orthogonal, parallel, or neither

a) $a = (-5;3;7), \quad b = (6;-8;2)$

b) $a = (4;6), \quad b = (-3;2)$

c) $a = -i + 2j + 5k, \quad b = 3i + 4j - k$

d) $u = (a,b,c), \quad v = (-b;a;0)$

14. For what values of b are the vectors $(-6;b;2)$ and $(b;b^2;b)$ orthogonal?

15. Find two unit vectors that make an angle of 60^o with $v = (3;4)$.

16. If a vector has direction angles $\alpha = \pi/4$ and $\beta = \pi/3$, find the third direction angle γ.

17. Find the angle between a diagonal of a cube and one of its edges.

18. Find the angle between a diagonal of a cube and a diagonal of one of its faces.

1.4 The cross product

19. Find the area of the parallelogram with vertices $A(-2;1)$, $B(0;4)$, $C(4;2)$, and $D(2;-1)$.

20. Find the area of the parallelogram with vertices $K(1;2;3)$, $L(1;3;6)$, $M(3;8;6)$ and $N(3;7;3)$.

21. Find the volume of the parallelepiped determined by the vectors a, b, and c.

a) $a = (6;3;-1), \quad b = (0;1;2), \quad c = (4;-2;5).$

b) $a = i + j - k, \quad b = i - j + k, \quad c = -i + j + k.$

22. Let $v = 5j$ and let u be a vector with length 3 that starts at the origin and rotates in the xy-plane. Find the maximum and minimum values of the length of the vector $u \times v$. In what direction does $u \times v$ point?

1.5 Equations of lines and planes

23. Determine whether each statement is true or false.

a) Two lines parallel to a third line are parallel.

b) Two lines perpendicular to a third line are parallel.

c) Two planes parallel to a third plane are parallel.

d) Two planes perpendicular to a third plane are parallel.

e) Two lines parallel to a plane are parallel.

f) Two lines perpendicular to a plane are parallel.

g) Two planes parallel to a line are parallel.

h) Two planes perpendicular to a line are parallel.

i) Two planes either intersect or are parallel.

j) Two lines either intersect or are parallel.

k) A plane and a line either intersect or are parallel.

24. Find a vector equation and parametric equations for the line.

a) The line through the point $(6;-5;2)$ and parallel to the vector $(1;3;-2/3)$.

b) The line through the point $(0;14;-10)$ and parallel to the line $x = -1+2t; y = 6-3t; z = 3+9t$.

c) The line through the point $(1,0,6)$ and perpendicular to the plane $x + 3y + z = 5$.

25. Find parametric equations and symmetric equations for the line of intersection of the plane $x+y+z=1$ and $x+z=0$.

26. Find a vector equation for the line segment from $(2;-1;4)$ to $(4;6;1)$.

27. Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

a) $L_1 : x = -6t, y = 1+9t, z = -3t;\quad L_2 : x = 1+2s, y = 4-3s, z = s$.

b) $L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3};\quad L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$.

28. Find an equation of the plane.

a) The plane through the point $(6;3;2)$ and perpendicular to the vector $(-2;1;5)$

b) The plane through the point $(-2;8;10)$ and perpendicular to the line $x = 1+t, y = 2t, z = 4-3t$.

c) The plane that contains the line $x = 3+2t, y = t, z = 8-t$ and is parallel to the plane $2x+4y+8z = 17$.

29. Find the cosine of the angle between the planes $x+y+z=0$ and $x+2y+3z=1$.

30. Find parametric equations for the line through the point $(0;1;2)$ that is perpendicular to the line $x = 1+t, y = 1-t, z = 2t$, and intersects this line.

31. Find the distance between the skew lines with parametric equations $x = 1+t, y = 1+6t, z = 2t$ and $x = 1+2s, y = 5+15s, z = -2+6s$.

1.6 Quadric surfaces

32. Find an equation for the surface obtained by rotating the parabola $y = x^2$ about the y-axis.

33. Find an equation for the surface consisting of all points that are equidistant from the point $(-1;0;0)$ and the plane $x = 1$. Identify the surface.

Chapter 2

VECTOR FUNCTIONS

Reference: James Stewart. *Calculus*, sixth edition. Thomson, USA 2008.

2.1 Vector functions

34. Find the domain of the vector function.

a) $r(t) = (\sqrt{4-t^2}, e^{-3t}, \ln(t+1))$

b) $r(t) = \frac{t-2}{t+2}i + \sin t j + \ln(9-t^2)k$

35. Find the limit

a) $\lim\limits_{t\to 0}(\frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{t+1})$

b) $\lim\limits_{t\to\infty}(\arctan t, e^{-2t}, \frac{\ln t}{t+1})$

36. Find a vector function that represents the curve of intersection of the two surfaces.

a) The cylinder $x^2+y^2=4$ and the surface $z=xy$.

b) The paraboloid $z=4x^2+y^2$ and the parabolic cylinder $y=x^2$.

37. Suppose u and v are vector functions that possess limits as $t\to a$ and let c be a constant. Prove the following properties of limits.

a) $\lim\limits_{t\to a}[u(t)+v(t)] = \lim\limits_{t\to a} u(t) + \lim\limits_{t\to a} v(t)$

b) $\lim\limits_{t\to a} cu(t) = c\lim\limits_{t\to a} u(t)$

c) $\lim\limits_{t\to a}[u(t).v(t)] = \lim\limits_{t\to a} u(t). \lim\limits_{t\to a} v(t)$

d) $\lim_{t \to a}[u(t) \times v(t)] = \lim_{t \to a} u(t) \times \lim_{t \to a} v(t)$

38. Find the derivative of the vector function.

a) $r(t) = (t \sin t, t^3, t \cos 2t)$.

b) $r(t) = \arcsin t i + \sqrt{1 - t^2} j + k$

c) $r(t) = e^{t^2} i - \sin^2 t j + \ln(1 + 3t)$

39. Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point. Illustrate by graphing both the curve and the tangent line on a common screen.

a) $x = t, y = e^{-t}, z = 2t - t^2; (0; 1; 0)$

b) $x = 2 \cos t, y = 2 \sin t, z = 4 \cos 2t; (\sqrt{3}, 1, 2)$

c) $x = t \cos t, y = t, z = t \sin t; (-\pi, \pi, 0)$

40. Find the point of intersection of the tangent lines to the curve $r(t) = (\sin \pi t, 2 \sin \pi t, \cos \pi t)$ at the points where $t = 0$ and $t = 0.5$

41. Evaluate the integral

a) $\int_0^{\pi/2} (3 \sin^2 t \cos t\, i + 3 \sin t \cos^2 t\, j + 2 \sin t \cos t\, k) dt$

b) $\int_1^2 (t^2\, i + t\sqrt{t - 1}\, j + t \sin \pi t\, k) dt$

c) $\int (e^t\, i + 2t\, j + \ln t\, k) dt$

d) $\int (\cos \pi t\, i + \sin \pi t\, j + t^2\, k) dt$

42. If a curve has the property that the position vector $r(t)$ is always perpendicular to the tangent vector $r'(t)$, show that the curve lies on a sphere with center the origin.

2.2 Arc length and curvature

43. Find the length of the curve.

a) $r(t) = (2 \sin t, 5t, 2 \cos t), \quad -10 \leq t \leq 10$

b) $r(t) = (2t, t^2, \frac{1}{3}t^3), \quad 0 \leq t \leq 1$

c) $r(t) = \cos t\, i + \sin t\, j + \ln \cos t\, k, \quad 0 \le t \le \pi/4$

44. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6; 18; 36)$.

45. Suppose you start at the point $(0; 0; 3)$ and move 5 units along the curve $x = 3\sin t, y = 4t, z = 3\cos t$ in the positive direction. Where are you now?

46. Reparametrize the curve

$$r(t) = \left(\frac{2}{t^2+1} - 1\right) i + \frac{2t}{t^2+1} j$$

with respect to arc length measured from the point $(1; 0)$ in the direction of increasing . Express the reparametrization in its simplest form. What can you conclude about the curve?

47. Find the curvature

a) $r(t) = t^2\, i + t\, k$

b) $r(t) = t\, i + t\, j + (1 + t^2)\, k$

c) $r(t) = 3t\, i + 4\sin t\, j + 4\cos t\, k$

d) $x = e^t \cos t, y = e^t \sin t$

e) $x = t^3 + 1, y = t^2 + 1$

48. Find the curvature of $r(r) = (e^t \cos t, e^t \sin t, t)$ at the point (1, 0, 0).

49. Find the curvature of $r(r) = (t, t^2, t^3)$ at the point (1, 1, 1).

50. Find the curvature

a) $y = 2x - x^2$, b) $y = \cos x$, c) $y = 4x^{5/2}$.

51. At what point does the curve have maximum curvature? What happens to the curvature as $x \to \infty$?

a) $y = \ln x$, b) $y = e^x$.

52. Find an equation of a parabola that has curvature 4 at the origin.

Chapter 3

Multiple Integrals

3.1 Double Integrals

3.1.1 Double Integrals in Cartesian coordinate

53. *Evaluate*

a) $\iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} x\sin(x+y)dxdy,$

b) $\iint\limits_{[0,2]\times[1,2]} (x-3y^2)dxdy,$

c) $\iint\limits_{[1,2]\times[0,\pi]} y\sin(xy)dxdy,$

d) $\iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin(x-y)dxdy,$

e) $\iint\limits_{[0,2]\times[1,2]} (y+xy^{-2})dxdy,$

f) $\iint\limits_{[0,1]\times[-3,2]} \frac{xy^2}{x^2+1}dxdy,$

g) $\iint\limits_{[0,1]\times[0,1]} \frac{1+x^2}{1+y^2}dxdy,$

h) $\iint\limits_{[0,\frac{\pi}{6}]\times[0,\frac{\pi}{3}]} x\sin(x+y)dxdy,$

i) $\iint\limits_{[0,1]\times[0,1]} \frac{x}{1+xy}dxdy,$

j) $\iint\limits_{[0,2]\times[0,3]} ye^{-xy}dxdy,$

k) $\iint\limits_{[1,3]\times[1,2]} \frac{1}{1+x+y}dxdy.$

54. *Evaluate*

a) $\iint\limits_{D} x^2\,(y-x)\,dxdy$ *where* D *is the region bounded by* $y=x^2$ *and* $x=y^2$.

b) $\iint\limits_{D} |x+y|dxdy, D:=\{(x,y)\in\mathbb{R}^2\,||x\leq 1|\,,|y|\leq 1\}$

c) $\iint\limits_{D} \sqrt{|y-x^2|}dxdy, D:=\{(x,y)\in\mathbb{R}^2\,||x|\leq 1, 0\leq y\leq 1\}$

d) $\iint\limits_{[0,1]\times[0,1]} \frac{ydxdy}{(1+x^2+y^2)^{\frac{3}{2}}}$

e) $\iint\limits_D \frac{x^2}{y^2}dxdy$, *where* D *is bounded by the lines* $x=2, y=x$ *and the hyperbola* $xy=1$.

f) $\iint\limits_D \frac{y}{1+x^5}dxdy$, *where* $D=\{(x,y)|0\le x\le 1, 0\le y\le x^2\}$,

g) $\iint\limits_D y^2e^{xy}dxdy$, *where* $D=\{(x,y)|0\le y\le 4, 0\le x\le y\}$,

h) $\iint\limits_D x\sqrt{y^2-x^2}dxdy$,*where* $D=\{(x,y)|0\le y\le 1, 0\le x\le y\}$,

i) $\iint\limits_D (x+y)dxdy$, *where* D *is bounded by* $y=\sqrt{x}$ *and* $y=x^2$,

j) $\iint\limits_D y^3dxdy$, *where* D *is the triangle region with vertices* $(0,2),(1,1)$ *and* $(3,2)$,

k) $\iint\limits_D xy^2dxdy$, *where* D *is enclosed by* $x=0$ *and* $x=\sqrt{1-y^2}$.

Change the order of integration

55. *Change the order of integration*

a) $\int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy.$

b) $\int\limits_{0}^{1} dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx.$

c) $\int\limits_{0}^{2} dx \int\limits_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)\,dx.$

d) $\int\limits_{0}^{4} dx \int\limits_{0}^{\sqrt{x}} f(x,y)dy,$

e) $\int\limits_{0}^{1} dx \int\limits_{4x}^{4} f(x,y)dy,$

f) $\int\limits_{0}^{3} dy \int\limits_{-\sqrt{9-y^2}}^{9-y^2} f(x,y)dx,$

g) $\int\limits_{0}^{3} dy \int\limits_{0}^{\sqrt{9-y}} f(x,y)dx,$

h) $\int\limits_{0}^{2} dx \int\limits_{0}^{\ln x} f(x,y)dy,$

i) $\int\limits_{0}^{1} dx \int\limits_{\arctan x}^{\frac{\pi}{4}} f(x,y)dy,$

j) $\int\limits_{0}^{\sqrt{2}} dy \int\limits_{0}^{y} f(x,y)\,dx + \int\limits_{\sqrt{2}}^{2} dy \int\limits_{0}^{\sqrt{4-y^2}} f(x,y)\,dx.$

56. *Evaluate the integral by reversing the order of integration*

a) $\int\limits_{0}^{1} dy \int\limits_{3y}^{3} e^{x^2}dx,$

b) $\int\limits_{0}^{\sqrt{\pi}} dy \int\limits_{y}^{\sqrt{\pi}} \cos(x^2)dx,$

c) $\int\limits_{0}^{4} dx \int\limits_{\sqrt{x}}^{2} \frac{1}{y^3+1}dy,$

d) $\int\limits_{0}^{1} dx \int\limits_{x}^{1} e^{\frac{x}{y}}dy,$

e) $\int\limits_{0}^{1} dy \int\limits_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}dx,$

f) $\int\limits_{0}^{8} dy \int\limits_{\sqrt[3]{y}}^{2} e^{x^4}dx.$

Change of variables

57. *Evaluate* $I = \iint\limits_D (4x^2 - 2y^2)\,dxdy$, *where* $D: \begin{cases} 1 \le xy \le 4 \\ x \le y \le 4x. \end{cases}$

58. *Evaluate*

$$I = \iint\limits_D \frac{x^2 \sin xy}{y} dxdy,$$

where D *is bounded by parabolas*

$$x^2 = ay, x^2 = by, y^2 = px, y^2 = qx, \ (0 < a < b, 0 < p < q).$$

59. *Evaluate* $I = \iint\limits_D xydxdy$, *where* D *is bounded by the curves*

$$y = ax^3, y = bx^3, y^2 = px, y^2 = qx, \ (0 < b < a, 0 < p < q).$$

Hint: Change of variables $u = \frac{x^3}{y}, v = \frac{y^2}{x}$.

60. *Prove that*

$$\int\limits_0^1 dx \int\limits_0^{1-x} e^{\frac{y}{x+y}} dy = \frac{e-1}{2}.$$

Hint: Change of variables $u = x + y, v = y$.

61. *Find the area of the domain bounded by* $xy = 4, xy = 8, xy^3 = 5, xy^3 = 15$.

Hint: Change of variables $u = xy, v = xy^3$, $(S = 2\ln 3)$.

62. *Find the area of the domain bounded by* $y^2 = x, y^2 = 8x, x^2 = y, x^2 = 8y$.

Hint: Change of variables $u = \frac{y^2}{x}, v = \frac{x^2}{y}$, $(S = \frac{279\pi}{2})$.

63. *Find the area of the domain bounded by* $y = x^3, y = 4x^3, x = y^3, x = 4y^3$.

Hint: Change of variables $y = x^3, y = 4x^3, x = y^3, x = 4y^3$.

64. *Prove that*

$$\iint\limits_{x+y\le 1, x\ge 0, y\ge 0} \cos\left(\frac{x-y}{x+y}\right) dxdy = \frac{\sin 1}{2}.$$

Hint: Change of variables $u = x - y, v = x + y$.

65. *Evaluate*

$$I = \iint\limits_D \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}\right) dxdy,$$

where D *is bounded by the axes and the parabola* $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$.

Double Integrals in polar coordinate

66. *Express the double integral* $I = \iint\limits_D f(x,y)\,dxdy$ *in terms of polar coordinates, where* D *is given by* $x^2+y^2 \geq 4x, x^2+y^2 \leq 8x, y \geq x, y \leq \sqrt{3}x$.

67. *Evaluate* $\iint\limits_D xy^2 dxdy$ *where* D *is bounded by* $\begin{cases} x^2+(y-1)^2 = 1 \\ x^2+y^2-4y = 0. \end{cases}$

68. *Evaluate*

a) $\iint\limits_D |x+y|dxdy,$ *b)* $\iint\limits_D |x-y|dxdy,$

where $D : x^2+y^2 \leq 1$.

69. *Evaluate* $\iint\limits_D \frac{dxdy}{(x^2+y^2)^2}$, *where* $D : \begin{cases} 4y \leq x^2+y^2 \leq 8y \\ x \leq y \leq x\sqrt{3}. \end{cases}$

70. *Evaluate* $\iint\limits_D \frac{xy}{x^2+y^2}dxdy$, *where* $D : \begin{cases} x^2+y^2 \leq 12, x^2+y^2 \geq 2x \\ x^2+y^2 \geq 2\sqrt{3}y, x \geq 0, y \geq 0. \end{cases}$

71. *Evaluate* $\iint\limits_D (x+y)dxdy$, *where* D *is the region that lies to the left of the y-axis, between the circles* $x^2+y^2 = 1$ *and* $x^2+y^2 = 4$.

72. *Evaluate* $\iint\limits_D \cos(x^2+y^2)dxdy$, *where* D *is the region that lies above the* x*-axis within the circle* $x^2+y^2 = 9$.

Evaluate $\iint\limits_D \sqrt{4-x^2-y^2}dxdy$, *where* $D = \{(x,y)|x^2+y^2 \leq 4, x \geq 0\}$.

73. *Evaluate* $\iint\limits_D ye^x dxdy$, *where* D *is the region in the first quadrant enclosed by the circle* $x^2+y^2 = 25$.

74. *Evaluate* $\iint\limits_D \arctan\frac{y}{x}dxdy$, *where* $D = \{(x,y)|1 \leq x^2+y^2 \leq 4, 0 \leq y \leq x\}$.

75. *Evaluate* $\iint\limits_D xdxdy$, *where* D *is the region in the first quadrant that lies between the circles* $x^2+y^2 = 4$ *and* $x^2+y^2 = 2x$.

3.1.2 Applications of Double Integrals

76. *Compute the area of the domain* D *bounded by*

a) $\begin{cases} y = 2^x, y = 2^{-x}, \\ y = 4. \end{cases}$

b) $\begin{cases} y^2 = x, y^2 = 2x \\ x^2 = y, x^2 = 2y. \end{cases}$

c) $\begin{cases} y = 0, y^2 = 4ax \\ x + y = 3a, \quad (a > 0). \end{cases}$

d) $\begin{cases} x^2 + y^2 = 2x, x^2 + y^2 = 4x \\ x = y, y = 0. \end{cases}$

e) $r = 1, r = \frac{2}{\sqrt{3}} \cos\varphi.$

f) $\left(x^2 + y^2\right)^2 = 2a^2xy \ \ (a > 0).$

g) $x^3 + y^3 = axy \ \ (a > 0)$ *(Descartes leaf)*

h) $r = a\,(1 + \cos\varphi) \ \ (a > 0)$ *(Cardioids)*

77. *Compute the volume of the object given by*

a) $\begin{cases} 3x + y \geq 1, y \geq 0 \\ 3x + 2y \leq 2, \\ 0 \leq z \leq 1 - x - y. \end{cases}$

b) $\begin{cases} 0 \leq z \leq 1 - x^2 - y^2, \\ x \leq y \leq x\sqrt{3}. \end{cases}$

78. *Compute the volume of the object bounded by the surfaces*

a) $\begin{cases} z = 4 - x^2 - y^2 \\ 2z = 2 + x^2 + y^2 \end{cases}$

b) $\begin{cases} z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}, z = 0 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2x}{a} \end{cases}$

c) $\begin{cases} az = x^2 + y^2 \\ z = \sqrt{x^2 + y^2}. \end{cases}$

79. *Find the area of the part of the paraboloid $x = y^2 + z^2$ that satisfies $x \leq 1$.*

3.1.3 Triple Integrals

Triple Integrals in Cartesian coordinate

80. *Evaluate*

a) $\iiint\limits_V (x^2 + y^2)\, dxdydz$, *where V is bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $x^2 + y^2 - z^2 = 0$.*

b) $\iiint\limits_E y dxdydz$, *where E is bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.*

c) $\iiint\limits_E x^2e^y dxdydz$, *where* E *is bounded by the parabolic cylinder* $z = 1 - y^2$ *and the planes* $z = 0, x = 1$ *and* $x = -1$.

d) $\iiint\limits_E xydxdydz$, *where* E *is bounded by the parabolic cylinder* $y = x^2$ *and* $x = y^2$ *and the planes* $z = 0$ *and* $z = x + y$.

e) $\iiint\limits_E xyzdxdydz$, *where* E *is the solid tetrahedron with vertices* $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ *and* $(0, 0, 1)$.

f) $\iiint\limits_E xdxdydz$, *where* E *is the bounded by the paraboloid* $x = 4y^2 + 4z^2$ *and the plane* $x = 4$.

g) $\iiint\limits_E zdxdydz$, *where* E *is the bounded by the cylinder* $y^2 + z^2 = 9$ *and the planes* $x = 0, y = 3x$ *and* $z = 0$ *in the first octant.*

Change of variables

81. *Evaluate*

a) $\iiint\limits_V (x + y + z)dxdydz$, *where* V *is bounded by* $\begin{cases} x + y + z = \pm 3 \\ x + 2y - z = \pm 1 \\ x + 4y + z = \pm 2 \end{cases}$.

b) $\iiint\limits_V (3x^2 + 2y + z)dxdydz$, *where* $V : |x - y| \le 1, |y - z| \le 1, |z + x| \le 1$.

c) $\iiint\limits_V dxdydz$, *where* $V : |x - y| + |x + 3y| + |x + y + z| \le 1$.

Triple Integrals in Cylindrical Coordinates

82. *Evaluate* $\iiint\limits_V (x^2 + y^2)\, dxdydz$, *where* $V : \begin{cases} x^2 + y^2 \le 1 \\ 1 \le z \le 2 \end{cases}$

83. *Evaluate* $\iiint\limits_V z\sqrt{x^2 + y^2}dxdydz$, *where:*

a) V *is bounded by:* $x^2 + y^2 = 2x$ *and* $z = 0, z = a$ $(a > 0)$.

b) V *is a half of the sphere* $x^2 + y^2 + z^2 \le a^2, z \ge 0$ $(a > 0)$

84. *Evaluate* $I = \iiint\limits_V \sqrt{x^2 + y^2}dxdydz$ *where* V *is bounded by:* $\begin{cases} x^2 + y^2 = z^2 \\ z = 1. \end{cases}$

85. *Evaluate* $\iiint\limits_V \frac{dxdydz}{\sqrt{x^2+y^2+(z-2)^2}}$, *where* $V : \begin{cases} x^2 + y^2 \le 1 \\ |z| \le 1. \end{cases}$

Triple Integrals in Spherical Coordinates

86. *Evaluate* $\iiint\limits_V (x^2+y^2+z^2)\,dxdydz$, *where* $V: \begin{cases} 1 \le x^2+y^2+z^2 \le 4 \\ x^2+y^2 \le z^2. \end{cases}$

87. *Evaluate* $\iiint\limits_V \sqrt{x^2+y^2+z^2}dxdydz$, *where* $V:\ x^2+y^2+z^2 \le z.$

88. *Evaluate* $\iiint\limits_V z\sqrt{x^2+y^2}dxdydz$, *where* V *is a half of the ellipsoid* $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} \le 1, z \ge 0, (a,b>0)$.

89. *Evaluate* $\iiint\limits_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dxdydz$, *where* $V: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1, (a,b,c>0)$.

90. *Evaluate* $\iiint\limits_V \sqrt{z-x^2-y^2-z^2}dxdydz$, *where* $V:\ x^2+y^2+z^2 \le z.$

91. *Evaluate* $\iiint\limits_V (4z-x^2-y^2-z^2)dxdydz$, *where* V *is the sphere* $x^2+y^2+z^2 \le 4z.$

92. *Evaluate* $\iiint\limits_V xzdxdydz$, *where* V *is the domain* $x^2+y^2+z^2-2x-2y-2z \le -2.$

93. *Evaluate*

$$I = \iiint\limits_V \frac{dxdydz}{(1+x+y+z)^3},$$

where V *is bounded by* $x=0, y=0, z=0$ *and* $x+y+z=1.$

94. *Evaluate*

$$\iiint\limits_V zdxdydz,$$

where V *is a half of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2} \le 1, (z \ge 0).$$

95. *Evaluate*

a) $I_1 = \iiint\limits_B \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$, *where* B *is the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1.$

b) $I_2 = \iiint\limits_C zdxdydz$, *where* C *is the domain bounded by the cone* $z^2 = \frac{h^2}{R^2}(x^2+y^2)$ *and the plane* $z=h.$

c) $I_3 = \iiint\limits_D z^2dxdydz$, *where* D *is bounded by the sphere* $x^2+y^2+z^2 \le R^2$ *and the sphere* $x^2+y^2+z^2 \le 2Rz.$

d) $I_4 = \iiint\limits_V (x+y+z)^2dxdydz$, *where* V *is bounded by the paraboloid* $x^2+y^2 \le 2az$ *and the sphere* $x^2+y^2+z^2 \le 3a^2.$

96. *Find the volume of the object bounded by the planes* $Oxy, x = 0, x = a, y = 0, y = b$*, and the paraboloid elliptic*

$$z = \frac{x^2}{2p} + \frac{y^2}{2y}, \ (p > 0, q > 0).$$

97. *Evaluate*

$$I = \iiint\limits_V \sqrt{x^2 + y^2 + z^2} dxdydz,$$

where V *is the domain bounded by* $x^2 + y^2 + z^2 = z$.

98. *Evaluate*

$$I = \iiint\limits_V zdxdydz,$$

where V *is the domain bounded by the surfaces* $z = x^2 + y^2$ *and* $x^2 + y^2 + z^2 = 6$.

99. *Evaluate*

$$I = \iiint\limits_V \frac{xyz}{x^2 + y^2} dxdydz,$$

where V *is the domain bounded by the surface* $(x^2 + y^2 + z^2)^2 = a^2xy$ *and the plane* $z = 0$.

Chapter 4

Integrals depending on a parameter

4.1 Definite Integrals depending on a parameter

100. *Compute*

a) $\lim\limits_{y\to 0} \int\limits_{y}^{1+y} \frac{dx}{1+x^2+y^2}$. *b)* $\lim\limits_{y\to 0} \int\limits_{0}^{2} x^2 \cos xy dx$.

101. *Evaluate*

a) $I(y) = \int\limits_{0}^{1} \arctan \frac{x}{y} dx$. *b)* $J(y) = \int\limits_{0}^{1} \ln(x^2+y^2)\, dx$. *c)* $K = \int\limits_{0}^{1} \frac{x^b - x^a}{\ln x}$, $(0 < a < b)$.

4.2 Improper Integrals depending on a parameter

102. *Show that the integral*

a) $I(y) = \int\limits_{1}^{\infty} \sin(yx) dx$ *is convergent if* $y = 0$ *and is divergent if* $y \neq 0$.

b) $I(y) = \int\limits_{0}^{\infty} \frac{\cos \alpha x}{x^2+1}$ *is uniformly convergent on* $\mathbb{R}$.

c) $I(y) = \int\limits_{0}^{1} x^{-y} dx = \int\limits_{1}^{\infty} t^{y-2} dt$ *is convergent if* $y < 1$ *and divergent if* $y \geq 1$.

d) $I(y) = \int\limits_{0}^{+\infty} e^{-yx} \frac{\sin x}{x}$ *is uniformly convergent on* $[0, +\infty)$.

e) $I(y) = \int\limits_{0}^{\infty} \frac{\cos \alpha x}{x^2+1}$ *is uniformly convergent on* $\mathbb{R}$.

103. *a) Evaluate* $I(y) = \int\limits_0^{+\infty} ye^{-yx}dx \quad (y > 0)$.

b) Prove that $I(y)$ *converges to* 1 *uniformly on* $[y_0, +\infty)$ *for all* $y_0 > 0$.

c) Explain why $I(y)$ *is not uniformly convergent on* $(0, +\infty)$.

104. *Prove that*

a) $\int\limits_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

b) $\int\limits_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$.

c) $\int\limits_0^{\infty} \sin(x^2)dx = \int\limits_0^{\infty} \cos(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

d) $\int\limits_0^{+\infty} e^{-yx}\frac{\sin x}{x} = \frac{\pi}{2} - \arctan y$.

e) $\int\limits_0^{\infty} \frac{\sin yx}{x(1+x^2)}dx = \frac{\pi}{2}(1 - e^{-y}), \quad y \geq 0$.

f) $\int\limits_0^{\infty} \frac{1-\cos yx}{x^2} = \frac{\pi}{2}|y|$.

g) $\int\limits_0^{\infty} \frac{x \sin yx}{a^2+x^2}dx = \frac{\pi}{2}e^{-ay}, \quad a, y \geq 0$.

h) $\int\limits_0^{\infty} e^{-yx^2}dx = \frac{\sqrt{\pi}}{2\sqrt{y}}, \quad y > 0$.

i) $\int\limits_0^{+\infty} \left(e^{-\frac{a}{x^2}} - e^{-\frac{b}{x^2}}\right) dx = \sqrt{\pi b} - \sqrt{\pi a}, (a, b > 0)$.

j) $\int\limits_0^{+\infty} \frac{\arctan\frac{x}{a} - \arctan\frac{x}{b}}{x}dx = \frac{\pi}{2}\ln\frac{b}{a}, \quad (a, b > 0)$.

k) $\lim\limits_{y\to 0^+}\left(\int\limits_0^{+\infty} ye^{-yx}dx\right) \neq \int\limits_0^{+\infty}\left(\lim\limits_{y\to 0^+} ye^{-yx}\right)dx$ *and explain why?*

105. *Evaluate* $(a, b, \alpha, \beta > 0)$*:*

a) $\int\limits_0^{+\infty} \frac{e^{-\alpha x} - e^{-\beta x}}{x}dx$.

b) $\int\limits_0^{+\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2}dx$.

c) $\int\limits_0^{+\infty} \frac{dx}{(x^2+y)^{n+1}}$.

d) $\int\limits_0^{+\infty} e^{-ax}\frac{\sin bx - \sin cx}{x}$.

e) $\int\limits_0^{+\infty} e^{-ax}\frac{\cos bx - \cos cx}{x}, \quad (a > 0)$.

f) $\int\limits_0^{+\infty} e^{-ax}\cos yx$.

g) $\int\limits_0^{+\infty} e^{-x^2}\cos(yx)\,dx$.

h) $\int\limits_{-\infty}^{+\infty} \frac{\arctan(x+y)}{1+x^2}dx$.

i) $\int\limits_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x}dx$, *where* $a, b > 0$.
$\int\limits_0^{+\infty} \frac{e^{-ax^3} - e^{-bx^3}}{x}dx$, *where* $a, b > 0$.

j) $\int\limits_0^{\infty} \frac{e^{-ax^2} - \cos bx}{x^2}dx, (a > 0)$

k) $\int\limits_0^{\pi} \ln(1 + y\cos x)dx$,

l) $\int\limits_0^{\infty} e^{-x^2}\sin ax dx$,

m) $\int\limits_0^{\infty} \frac{\sin xy}{x}dx, y \geq 0$, *n)* $\int\limits_0^\infty e^{-ax^2}\cos bx dx \ \ (a > 0)$,

o) $\int\limits_0^\infty x^{2n}e^{-x^2}\cos bx dx, n \in \mathbb{N}$.

p) $\int\limits_0^\infty \frac{\sin ax \cos bx}{x} dx$,

q) $\int\limits_0^\infty \frac{\sin ax \sin bx}{x} dx$.

4.3 Euler Integral

106. *Evaluate*

a) $\int\limits_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x dx$.

b) $\int\limits_0^a x^{2n}\sqrt{a^2 - x^2} dx \ \ (a > 0)$.

c) $\int\limits_0^{+\infty} x^{10}e^{-x^2} dx$.

d) $\int\limits_0^{+\infty} \frac{\sqrt{x}}{(1+x^2)^2} dx$.

e) $\int\limits_0^{+\infty} \frac{1}{1+x^3} dx$.

f) $\int\limits_0^{+\infty} \frac{x^{n+1}}{(1+x^n)} dx, \ \ (2 < n \in \mathbb{N})$.

g) $\int\limits_0^1 \frac{1}{\sqrt[n]{1-x^n}} dx, \ \ n \in \mathbb{N}^*$.

h) $\int\limits_0^{+\infty} \frac{x^4}{(1+x^3)^2} dx$,

Chapter 5

Line integrals

87. Evaluate the line integral, where C is the given curve

a) $\int\limits_C x \sin y ds$, C is the line segment from $(0, 3)$ to $(4, 6)$.

b) $\int\limits_C (x^2y^3 - \sqrt{x})dy$, C is the arc of the curve $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$.

c) $\int\limits_C xe^y dx$, C is the arc of the curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

d) $\int\limits_C \sin x dx + \cos y dy$, C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and the line segment from $(-1, 0)$ to $(-2, 3)$.

e) $\int\limits_C xyzds$, $C : x = 2\sin t, y = t, z = -2\cos t$, $0 \le t \le \pi$.

f) $\int\limits_C xyz^2 ds$, C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.

g) $\int\limits_C x^2y\sqrt{z}dz$, $C : x = t^3, y = t, z = t^2$, $0 \le t \le 1$.

h) $\int\limits_C zdx + xdy + ydz$, $C : x = t^2, y = t^3, z = t^2$, $0 \le t \le 1$.

k) $\int\limits_C (x + yz)dx + 2xdy + xyzdz$, C consists of line segments from $(1, 0, 1)$ to $(2, 3, 1)$ and from $(2, 3, 1)$ to $(2, 5, 2)$.

l) $\int\limits_C x^2dx + y^2dy + z^2dz$, C consists of line segments from $(0, 0, 0)$ to $(1, 2, -1)$ and from $(1, 2, -1)$ to $(3, 2, 0)$.

88. Evaluate the following line integrals

a) $\int\limits_C (x - y)ds$, where C is the circle $x^2 + y^2 = 2x$.

b) $\int\limits_C (x^2+y^2+z^2)ds$, where C is the helix $x = a\cos t$, $y = a\sin t$, $z = bt$, $(0 \le t \le 2\pi)$.

89. Evaluate the line integral $\int_C F \cdot dr$, where $F(x, y, z) = xi - zj + yk$ and C is given by $r(t) = 2ti + 3tj - t^2k$, $-1 \le t \le 1$.

90. Find the work done by the force field $F(x, y, z) = (y + z, x + z, x + y)$ on a particle that moves along the line segment from $(1; 0; 0)$ to $(3; 4; 2)$.

91. Evaluate the line integral by two methods: (a) directly and using Green's Theorem

a) $\oint_C (x-y)dx + (x+y)dy$, C is the circle with center the origin and radius 2.

b) $\oint_C xydx + x^2dy$, C is the rectangle with vertices $(0; 0)$, $(3; 0)$, $(3; 1)$, and $(0; 1)$.

c) $\oint_C ydx + xdy$, C consists of the line segments from $(0; 1)$ to $(0; 0)$ and from $(0; 0)$ to $(1; 0)$ and the parabola $y = 1 - x^2$ from $(1; 0)$ to $(0; 1)$.

92. Use Green's Theorem to evaluate the line integral along given positively oriented curve

a) $\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y)dy$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

b) $\int_C xe^{-2x}dx + (x^4 + 2x^2y^2)dy$, C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

c) $\int_C (e^x + x^2y)dx + (e^y - xy^2)dy$, C is the circle $x^2 + y^2 = 25$.

d) $\int_C (2x - x^3y^5)dx + x^3y^8dy$, C is the ellipse $4x^2 + y^2 = 4$.

93. Show that the line integral is independent of path and evaluate the integral

a) $\int_C (1 - ye^{-x})dx + e^{-x}dy$, C is any path from $(0, 1)$ to $(1, 2)$.

b) $\int_C 2y^{3/2}dx + 3x\sqrt{y}dy$, C is any path from $(1, 1)$ to $(2, 4)$.

Curl and Divergence

94. Determine whether or not F is a conservative vector field. If it is, find a function f such that $F = \nabla f$.

a) $F(x,y) = (2x - 3y)i + (-3x + 4y - 8)j$

b) $F(x,y) = e^x \cos yi + e^x \sin yj$

c) $F(x,y) = (xy \cos xy + \sin xy)i + (x^2 \cos xy)j$

d) $F(x,y) = (\ln y + 2xy^3)i + (3x^2y^2 + x/y)j$

e) $F(x,y) = (ye^x + \sin y)i + (e^x + x \cos y)j$

95. Find a function f such that $F = \nabla f$ and then evaluate $\int_C F \cdot dr$ along the given curve C.

a) $F(x,y) = xy^2 i + x^2 y j$, $C:\ r(t) = (t + \sin \frac{1}{2}\pi t, t + \cos \frac{1}{2}\pi t)$, $0 \le t \le 1$.

b) $F(x,y) = \dfrac{y^2}{1+x^2} i + 2y \arctan x j$, $C:\ r(t) = t^2 i + 2tj$, $0 \le t \le 1$.

c) $F(x,y) = (2xz + y^2)i + 2xyj + (x^2 + 3z^2)k$, $C:\ x = t^2, y = t+1, z = 2t-1$, $0 \le t \le 1$.

d) $F(x,y) = e^y i + xe^y j + (z+1)e^z k$, $C:\ x = t, y = t^2, z = t^3$, $0 \le t \le 1$.

Chapter 6

Surface Integrals

96. Evaluate the surface integral

a) $\iint\limits_S xy dS$, S is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

b) $\iint\limits_S yz dS$, S is the part of the plane $x + y + z = 1$ that lies in the first octant.

c) $\iint\limits_S yz dS$, S is the surface with parametric equations $x = u^2$, $y = u \sin v$, $z = u \cos v$, $0 \leq u \leq 1, 0 \leq v \leq \pi/2$.

d) $\iint\limits_S z dS$, S is the surface $x = y + 2z^2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

e) $\iint\limits_S y^2 dS$, S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the $xy-$plane.

97. Evaluate the surface integral $\iint\limits_S F \cdot dS$ for the given vector field F and the oriented surface S. In other words, find the flux of F across S. For closed surfaces, use the positive (outward) orientation.

a) $F(x, y, z) = xze^y i - xze^y j + zk$, S is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation.

b) $F(x, y, z) = xi + yj + z^4 k$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.

c) $F(x, y, z) = xzi + xj + yk$, S is the hemisphere $x^2 + y^2 + z^2 = 25, y \geq 0$, oriented in the direction of the positive $y-$axis.

d) $F(x, y, z) = xyi + 4x^2j + yzk$, S is the surface $z = xe^y$, $0 \leq x \leq 1, 0 \leq y \leq 1$, with upward orientation.

e) $F(x, y, z) = x^2i + y^2j + z^2k$, S is the boundary of the solid half-cylinder $0 \leq z \leq \sqrt{1 - y^2}$, $0 \leq x \leq 2$.

98. a) Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if it has constant density.
b) Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density function is $\rho(x, y, z) = 10 - z$.

Stokes Theorem

99. Use Stokes Theorem to evaluate $\iint\limits_S \operatorname{curl} F \cdot dS$

a) $F(x, y, z) = 2y \cos zi + e^x \sin zj + xe^y k$, S is the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, oriented upward.

b) $F(x, y, z) = x^2z^2i + y^2z^2j + xyzk$, S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward.

The Divergence Theorem

100. Use the Divergence Theorem to calculate the surface integral $\iint\limits_S F \cdot dS$; that is, calculate the flux of F across S

a) $F(x, y, z) = x^3yi - x^2y^2j - x^2yzk$, S is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$ and $z = 2$.

b) $F(x, y, z) = (\cos z + xy^2)i + xe^{-z}j + (\sin y + x^2z)k$, S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

c) $F(x, y, z) = 4x^3zi + 4y^3zj + 3z^4k$, S is the sphere with radius R and center the origin.

Chapter 5

Line Integrals

5.1 Line Integrals of scalar Fields

107. *Evaluate*

a) $\int\limits_C (x-y)\,ds$, *where C is the circle* $x^2+y^2=2x$.

b) $\int\limits_C y^2 ds$, *where C is the curve* $\begin{cases} x = a\,(t-\sin t) \\ y = a\,(1-\cos t) \end{cases}, 0 \le t \le 2\pi, a > 0.$

c) $\int\limits_C \sqrt{x^2+y^2}ds$, *where C is the curve* $\begin{cases} x = (\cos t + t\sin t) \\ y = (\sin t - t\cos t) \end{cases}, 0 \le t \le 2\pi.$

d) $\int\limits_C (x+y)ds$, *where C is the circle* $x^2+y^2=2y$.

e) $\int\limits_L xyds$, *where L is the part of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, x\ge 0, y \ge 0$.

f) $I = \int\limits_L |y|ds$, *where L is the Cardioid curve* $r = a(1+\cos\varphi)$ $(a>0)$.

g) $I = \int\limits_L |y|ds$, *where L is the Lemniscate curve* $(x^2+y^2)^2 = a^2(x^2-y^2)$.

5.2 Line Integrals of vector Fields

108. *Evaluate* $\int\limits_{ABCA} 2\,(x^2+y^2)\,dx + x\,(4y+3)\,dy$, *where $ABCA$ is the quadrangular curve,* $A(0,0), B(1,1), C(0,2)$.

109. *Evaluate* $\int\limits_{ABCDA} \frac{dx+dy}{|x|+|y|}$, *where $ABCDA$ is the triangular curve,* $A(1,0), B(0,1), C(-1,0), D(0,-$

Green's Theorem

110. *Evaluate the integral* $\int\limits_C (xy + x + y)\,dx + (xy + x - y)\,dy$, *where* C *is the positively oriented circle* $x^2 + y^2 = R^2$ *by*

i) computing it directly and

ii) Green's Theorem, then compare the results,

111. *Evaluate the following integrals, where* C *is a half the circle* $x^2 + y^2 = 2x$, *traced from* $O(0,0)$ *to* $A(2,0)$.

a) $\int\limits_C (xy + x + y)\,dx + (xy + x - y)\,dy$

b) $\int\limits_C x^2\left(y + \frac{x}{4}\right)dy - y^2\left(x + \frac{y}{4}\right)dx.$

c) $\int\limits_C (xy + e^x \sin x + x + y)\,dx - (xy - e^{-y} + x - \sin y)\,dy.$

112. *Evaluate* $\oint\limits_{OABO} e^x\left[(1 - \cos y)\,dx - (y - \sin y)\,dy\right]$, *where* $OABO$ *is the triangle,* $O(0,0), A(1,1), B$

Applications of Line Integrals

113. *Find the area of the domain bounded by an arch of the cycloid* $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1 - \cos\theta) \end{cases}$ *and* Ox $(a > 0)$.

Independence of Path

114. *Evaluate* $\int\limits_{(-2,1)}^{(3,0)} (x^4 + 4xy^3)\,dx + (6x^2y^2 - 5y^4)\,dy.$

115. *Evaluate* $\int\limits_{(1,\pi)}^{(2,2\pi)} \left(1 - \frac{y^2}{x^2}\cos\frac{y}{x}\right)dx + \left(\sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x}\right)dy.$

Chapter 6

Surface Integrals

6.1 Surface Integrals of scalar Fields

116. *Evaluate* $\iint\limits_S \left(z + 2x + \frac{4y}{3}\right) dS$, *where* $S = \left\{(x, y, z) \,|\, \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1, x, y, z \geq 0\right\}$.

117. *Evaluate* $\iint\limits_S (x^2 + y^2)\, dS$, *where* $S = \{(x, y, z) \,| z = x^2 + y^2, 0 \leq z \leq 1\}$.

118. *Evaluate* $\iint\limits_S x^2y^2z dS$, *where* S *is the part of the cone* $z = \sqrt{x^2 + y^2}$ *lies below the plane* $z = 1$.

119. *Evaluate* $\iint\limits_S \dfrac{dS}{(2 + x + y + z)^2}$, *where* S *is the boundary of the triangular pyramid* $x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0$.

6.2 Surface Integrals of vector Fields

120. *Evaluate* $\iint\limits_S z\,(x^2 + y^2)\, dxdy$, *where* S *is a half of the sphere* $x^2 + y^2 + z^2 = 1, z \geq 0$, *with the outward normal vector.*

121. *Evaluate* $\iint\limits_S ydxdz + z^2dxdy$, *where* S *is the surface* $x^2 + \frac{y^2}{4} + z^2 = 1, x \geq 0, y \geq 0, z \geq 0$, *and is oriented downward.*

122. *Evaluate* $\iint\limits_S x^2y^2zdxdy$, *where* S *is the surface* $x^2 + y^2 + z^2 = R^2, z \leq 0$ *and is oriented upward.*

The Divergence Theorem

123. *Evaluate the following integrals, where* S *is the surface* $x^2+y^2+z^2 = a^2$ *with outward orientation.*

a. $\iint\limits_S xdydz + ydzdx + zdxdy$ b. $\iint\limits_S x^3dydz + y^3dzdx + z^3dxdy.$

124. *Evaluate* $\iint\limits_S y^2zdxdy + xzdydz + x^2ydxdz$*, where* S *is the boundary of the domain* $x \geq 0, y \geq 0, x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2$ *which is outward oriented.*

125. *Evaluate* $\iint\limits_S xdydz + ydzdx + zdxdy$*, where* S *the boundary of the domain* $(z-1)^2 \leq x^2 + y^2, a \leq z \leq 1, a > 0$ *which is outward oriented.*

Stokes' Theorem

126. *Use Stokes' Theorem to evaluate* $\int\limits_C F \cdot dr = \int\limits_C Pdx + Qdy + Rdz$*. In each case* C *is oriented counterclockwise as viewed from above.*

1. $F(x, y, z) = (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}$*,* C *is the triangle with vertices* $(1, 0, 0), (0, 1, 0)$ *and* $(0, 0, 1)$.

2. $F(x, y, z) = \mathbf{i} + (x + yz)\mathbf{j} + (xy - \sqrt{z})\mathbf{k}$*,* C *is the boundary of the part of the plane* $3x + 2y + z = 1$ *in the first octant.*

3. $F(x, y, z) = yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}$*,* C *is the circle* $x^2 + y^2 = 16, z = 5$.

4. $F(x, y, z) = xy\mathbf{i} + 2z\mathbf{j} + 3y\mathbf{k}$*,* C *is the curve of intersection of the plane* $x + z = 5$ *and the cylinder* $x^2 + y^2 = 9$.

6.3 Vector Calculus

6.3.1 Scalar Fields

127. *Find the directional derivative of the function* $f(x, y, z) = x^2y^3z^4$ *at the point* $M(1, 1, 1)$ *in the direction of the vector* $\vec{l} = (1, 1, 1)$.

128. *Find* ∇u*, where* $u = r^2 + \frac{1}{r} + \ln r$ *and* $r = \sqrt{x^2 + y^2 + z^2}$.

129. *In what direction from* $O(0, 0, 0)$ *does* $f = x \sin z - y \cos z$ *have the maximum rate of change.*

6.3.2 Vector Fields

130. *Let* $F = xz^2\overrightarrow{i} + yx^2\overrightarrow{j} + zy^2\overrightarrow{k}$. *Find the flux of* F *across the surface* $S : x^2+y^2+z^2 = 1$ *with the outward direction.*

131. *Let* $F = x(y + z)\overrightarrow{i} + y(z + x)\overrightarrow{j} + z(x + y)\overrightarrow{k}$ *and* L *is the intersection between the quatity* $x^2 + y^2 + y = 0$ *and a half of the sphere* $x^2 + y^2 + z^2 = 2, z \geq 0$. *Prove that the circulation of* F *across* L *is equal to* 0.

132. *Prove that* F *is a conservative vector field on* Ω *if and only if* $\operatorname{curl} F(M) = 0\ \forall M \in \Omega$.

133. *Which of the following fields are conservative and find their potential functions.*

a. $F = 5(x^2 - 4xy)\overrightarrow{i} + (3x^2 - 2y)\overrightarrow{j} + \overrightarrow{k}$.

b. $G = yz\overrightarrow{i} + xz\overrightarrow{j} + xy\overrightarrow{k}$.

c. $H = (x + y)\overrightarrow{i} + (x + z)\overrightarrow{j} + (z + y)\overrightarrow{k}$.