Chapter 1. Probability

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- An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment**.
- The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S.

Random experiment and Sample space

Example 2.1

Consider the experiment of:

- Flipping a coin: the sample space $S = \{H, T\}$

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An experiment consists of flipping a coin and then flipping it a second time if a head occurs, if a tail occurs on the first flip, then a die is tossed once.

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- The sample space $S = \mathbb{R}^+ = \{x|x > 0\}$.
- If it is known that all connectors will be between 10 and 11 millimeters thick, the sample space could be $S = \{x|10 < x < 11\}$.

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Consider an experiment in which you select a molded plastic part, such as a connector, and measure its thickness.

- The sample space $S = \mathbb{R}^+ = \{x | x > 0\}$.
- If it is known that all connectors will be between 10 and 11 millimeters thick, the sample space could be $S = \{x | 10 < x < 11\}$.
- If the objective of the analysis is to consider only whether a particular part is low, medium, or high for thickness, then the sample space is $S = \{\text{low, medium, high}\}$.

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A sample space is **continuous** if it contains an interval (either finite or infinite) of real numbers.

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In Example 2.3, the choice $S = \{x | x > 0\}$ is an example of a continuous sample space, whereas $S = \{\text{low, medium, high}\}$ is a discrete sample space.

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In Example 2.3, the choice $S = \{x|x > 0\}$ is an example of a continuous sample space, whereas $S = \{\text{low, medium, high}\}$ is a discrete sample space.

Remark

The best choice of a sample space depends on the objectives of the study.

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Example 2.5

- Randomly select a camera and record the recycle time of a flash. Then $S = R^{+} = \{x|x > 0\}$, the positive real numbers

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- Suppose it is known that all recycle times are between 1.5 and 5 seconds. Then $S = \{x|1.5 < x < 5\}$ is continuous.
- It is known that the recycle time has only three values (low, medium or high). Then $S = \{\text{low, medium, high}\}$ is discrete.
- Does the camera conform to minimum recycle time specifications? Then $S = \{\text{yes, no}\}$ is discrete.

Events

Definition 2.3

- An **event** is a subset of the sample space of a random experiment, and is denoted by $A, B, C, \ldots$.
- Each outcome is called a simple event.
- The entire sample space S is called the certain event. The null event, denoted by the symbol $\emptyset$, is an empty subset of S, and has no outcomes.

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- The simple events are $A_i = \{i\}, i = 1, 2, ..., 6$

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- Let A be the event that the component fails before the end of the fifth year: then $A = \{t | 0 \leq t < 5\}$
- Let B be the event that the component will not fail before the end of the sixth year: then $B = \{t | t \geq 6\}$

Events relation: complement

Definition 2.4

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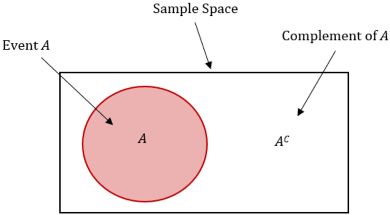
Venn diagram of complement



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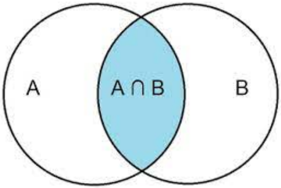

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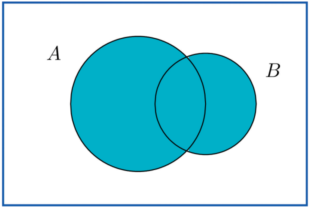

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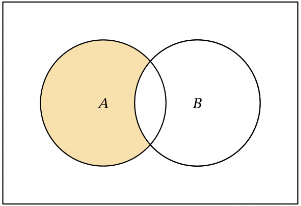

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Consider the experiment of tossing a 6-sided die. The sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{3, 6\}$ (outcome is divisible by 3) and $B = \{2, 4, 6\}$ (outcome is even).

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- The complement of A is: $\bar{A} = \{1, 2, 4, 5\}$

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- The intersection of A and B is: $AB = \{6\}$
- The union of A and B is: $A + B = \{2, 3, 4, 6\}$
- The difference of A and B is: $A - B = \{3\}$. The difference of B and A is: $B - A = \{2, 4\}$.

Events relation

Example 2.9

Consider the experiment of observing the life in years of a certain electronic component. The sample space $S = \{t|t \geq 0\}$, where t is the life of an electronic component. Let $A = \{t|0 \leq t < 5\}$, $B = \{t|3 < t < 8\}$ and $C = \{t|t > 4\}$.

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- $AB = \{t | 3 < t < 5\}$, $AC = \{t | 4 < t < 5\}$; $BC = \{t | 4 < t < 8\}$ and $ABC = \{t | 4 < t < 5\}$
- $A + B = \{t | 0 \leq t < 8\}$; $A + C = S$; $B + C = \{t | t > 3\}$ and $A + B + C = S$.

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- $A + B = \{t | 0 \leq t < 8\}$; $A + C = S$; $B + C = \{t | t > 3\}$ and $A + B + C = S$.
- $A - B = \{t | 0 \leq t \leq 3\}$ and $B - A = \{t | 5 \leq t < 8\}$;

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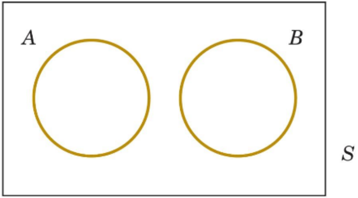
- Symbolically, $A \cap B = \emptyset$



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The set of n events $\{A_1, A_2, ..., A_n\}$ are mutually exclusive and exhaustive if and only if:

- Any 2 events A_i, A_j, for all $i \neq j$, are mutually exclusive ($A_i \cap A_j = \emptyset$).
- Union of all events A_i are the sample space: $A_1 \cup A_2 \cup ... \cup A_n = S$.

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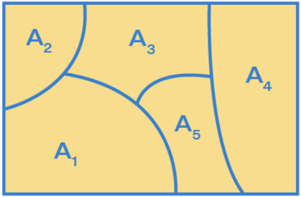

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- B and C are mutually exclusive and exhaustive.
- $\{A_1, A_2, ..., A_6\}$ are mutually exclusive and exhaustive.

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Consider the experiment of observing the life in years of a certain electronic component. Then $S = \{t | t \geq 0\}$ and let $A = \{t | 0 \leq t < 5\}$, $B = \{t | 5 \leq t < 8\}$ and $C = \{t | t \geq 8\}$.

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- A and C are mutually exclusive but not exhaustive.
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  $(A \cup B) \cup C = A \cup (B \cup C)$
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- DeMorgan's law:
  $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

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  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- Associative law (like in algebra):
  $(A \cup B) \cup C = A \cup (B \cup C)$
  $(A \cap B) \cap C = A \cap (B \cap C)$
- DeMorgan's law:
  $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$
- Complement law: $(A^c)^c = A$.

Laws of Events

- Commutative law (event order is unimportant):
  $A \cap B = B \cap A; A \cup B = B \cup A$
- Distributive law (like in algebra):
  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
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  $(A \cup B) \cup C = A \cup (B \cup C)$
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- DeMorgan's law:
  $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$
- Complement law: $(A^c)^c = A$.
- Others: $A \cap \emptyset = \emptyset; A \cup \emptyset = A; A \cap A^c = \emptyset; A \cup A^c = S; S^c = \emptyset; \emptyset^c = S$

Counting Techniques

- There are three special rules, or counting techniques, used to determine the number of outcomes in events.

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- There are three special rules, or counting techniques, used to determine the number of outcomes in events.
- They are:
  1. Multiplication rule
  2. Permutation rule
  3. Combination rule
- Each has its special purpose that must be applied properly – the right tool for the right job.

Counting – Multiplication Rule

Multiplication rule:

- Let an operation consist of k steps and there are:
  - n_1 ways of completing step 1,
  - n_2 ways of completing step $2, \ldots$ and
  - n_k ways of completing step k.

Counting – Multiplication Rule

Multiplication rule:

- Let an operation consist of k steps and there are:
  - n_1 ways of completing step 1,
  - n_2 ways of completing step $2, \ldots$ and
  - n_k ways of completing step k.
- Then, the total number of ways to perform k steps is $n_1.n_2 \ldots n_k$

Counting – Multiplication Rule

Example 2.12 - Web Site Design

- In the design for a website, we can choose to use among:
  - 4 colors,
  - 3 fonts, and
  - 3 positions for an image.

  How many designs are possible?

Counting – Multiplication Rule

Example 2.12 - Web Site Design

- In the design for a website, we can choose to use among:
  - 4 colors,
  - 3 fonts, and
  - 3 positions for an image.

  How many designs are possible?
- Answer: via the multiplication rule, there are $4.3.3 = 36$ possible designs.

Counting – Permutation Rule

Permutation Rule:

- A permutation is a unique sequence of distinct items.

Counting – Permutation Rule

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- A permutation is a unique sequence of distinct items.
- If $S = a, b, c$, then there are 6 permutations:
    - Namely: abc, acb, bac, bca, cab, cba (order matters)

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  $P_n = n! = n(n-1)(n-2)\ldots 2.1$

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- Number of permutations for a set of n items is:
  $P_n = n! = n(n-1)(n-2)\dots 2.1$
- $7! = 7.6.5.4.3.2.1 = 5040$
- By definition: $0! = 1$

Counting–Subset Permutations

Subset Permutations

- For a sequence of r items from a set of n items:

$$A_n^r = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Example 2.13: Printed Circuit Board

- A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many designs are possible?
- Answer: Order is important, so use the permutation formula with $n = 8, r = 4$

$$A_8^4 = \frac{8!}{(8-4)!} = \frac{8.7.6.5.4!}{4!} = 8.7.6.5 = 1680$$

Counting - Similar Item Permutations

Similar Item Permutations

- Used for counting the sequences when some items are identical.
- The number of permutations of: $n = n_1 + n_2 + \ldots + n_r$ items of which $n_1, n_2, \ldots, n_r$ are identical is calculated as:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Counting - Similar Item Permutations

Example 2.14: Hospital Schedule

- In a hospital, an operating room needs to schedule three knee surgeries and two hip surgeries in a day. The knee surgery is denoted as k and the hip as h.
- How many sequences are there?
  Since there are 2 identical hip surgeries and 3 identical knee surgeries, then
  $$\frac{5!}{3!2!} = \frac{5.4.3!}{2.1.3!} = 10$$
- What is the set of sequences?
  $\{kkkhh, kkhkh, kkhhk, khkkh, khkhk, khhkk, hkkkh, hkkhk, hkhkk, hhkkk\}$

Counting – Combination Rule

Combination Rule

- A combination is a selection of r items from a set of n where order does not matter.
- If $S = \{a, b, c\}, n = 3$, then
  – If $r = 3$, there is 1 combination, namely: abc
  – If $r = 2$, there are 3 combinations, namely ab, ac, and bc
- $\#$ of permutations $\geq$ $\#$ of combinations
- Since order does not matter with combinations, we are dividing the $\#$ of permutations by $r!$, where $r!$ is the $\#$ of arrangements of r elements.:

$$C_n^r = \frac{n!}{r!(n-r)!}$$

Counting – Combination Rule

Example 2.15: Sampling w/o Replacement-1

- A bin of 50 parts contains 3 defectives and 47 non-defective parts. A sample of 6 parts is selected from the 50 without replacement. How many samples of size 6 contain 2 defective parts?
- First, how many ways are there for selecting 2 parts from the 3 defective parts?
  Answer:
  $$C_3^2 = \frac{3!}{2!1!} = 3 \text{ different ways}$$

Counting – Combination Rule

Example 2.15: Sampling w/o Replacement-1

- Now, how many ways are there for selecting 4 parts from the 47 non-defective parts?
  Answer:
  $$C_{47}^{4} = \frac{47!}{4!43!} = \frac{47.46.45.44.43!}{4.3.2.1.43!} = 178365 \text{ different ways}$$
- Now, how many ways are there to obtain:
  – 2 from 3 defectives, and
  – 4 from 47 non-defectives?
  Answer:
  $$C_3^2 C_{47}^4 = 3.178365 = 535095 \text{ different ways}$$

Introduction to Probability

It is often necessary to "guess" about the outcome of an event in order to make a decision:

- Politicians study polls to guess their likelihood of winning an election.
- Doctors choose the treatments needed for various diseases based on their assessment of likely results.
- You may have visited a casino where people play games chosen because of the belief that the likelihood of winning is good.
- You may have chosen your course of study based on the probable availability of jobs.

Probability deals with the chance of an event occurring (Probability is a numerical measure of the likelihood that an event will occur).

Approaches to Determine Probability

Approaches to Determine Probability:

The way that we calculate the probability of an event depends on the situation we are analyzing.

- Classical Method Approach to Probability
- Empirical Method Approach to Probability

Classical Method Approach to Probability

Classical Method Approach to Probability:

- The classical method approach requires that an experiment results n equally like outcomes (equally likely means that each outcome of the experiment occurs with equal probability of $\frac{1}{n}$).
- When all outcomes in the sample space are equally likely, calculate the probability of an event A as follow:

$$P(A) = \frac{\text{number of outcomes in event A}}{\text{total number of outcomes in the sample space S}} = \frac{\#A}{n}$$

- Note that the outcomes in the sample space S must be mutually exclusive and exhaustive.

Classical Method Approach to Probability

Example 2.16

Consider the experiment of tossing a fair 6-sided die printed with little dots numbering $1, 2, 3, 4, 5$, and 6.

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- The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
- The 6 outcomes are equally likely.
- Let A be the event that the outcome is divisible by 3: $A = \{3, 6\}$. The probability of A is

$$P(A) = \frac{\#A}{n} = \frac{2}{6} = \frac{1}{3}$$

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- Let B be the event that the outcome is even: $B = \{2, 4, 6\}$. The probability of B is

$$P(B) = \frac{\#B}{n} = \frac{3}{6} = \frac{1}{2}$$

Classical Method Approach to Probability

Example 2.17

A fair coin is tossed twice. What is the probability of getting "exactly one head"? What is the probability that at least one head occurs?

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- The 4 outcomes are equally likely.
- Let A be the event of "getting exactly one head": $A = \{HT, TH\}$. The probability of A is $P(A) = \frac{\#A}{n} = \frac{2}{4} = 0.5$

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- The sample space is $S = \{HH, HT, TH, TT\}$.
- The 4 outcomes are equally likely.
- Let A be the event of "getting exactly one head": $A = \{HT, TH\}$. The probability of A is $P(A) = \frac{\#A}{n} = \frac{2}{4} = 0.5$
- Let B be the event that at least one head occurs: $B = \{HH, HT, TH\}$. The probability of B is $P(B) = \frac{\#B}{n} = \frac{3}{4} = 0.75$

Classical Method Approach to Probability

When the outcomes are not equally likely and we can assign a probility to each outcome. Then probability of an event A is: $P(A) = \sum_{\omega \in A} P(\omega)$.

Example 2.18

Consider the experiment of tossing a fair 6-sided die that 2 faces printed with 2 dots and the other 4 faces printed with 3, 4, 5, and 6 dots respectively.

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Consider the experiment of tossing a fair 6-sided die that 2 faces printed with 2 dots and the other 4 faces printed with 3, 4, 5, and 6 dots respectively.

- The sample space $S = \{2, 3, 4, 5, 6\}$.
- The 5 outcomes are not equally likely: $P(\{2\}) = \frac{2}{6}$ and $P(\{i\}) = \frac{1}{6}$ for $i = 3, .., 6$

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- The sample space $S = \{2, 3, 4, 5, 6\}$.
- The 5 outcomes are not equally likely: $P(\{2\}) = \frac{2}{6}$ and $P(\{i\}) = \frac{1}{6}$ for $i = 3, .., 6$
- Let A be the event of "getting an even number": $A = \{2, 4, 6\}$. The probability of A is

$$P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{2}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

Classical Method Approach to Probability

Example 2.19

An experiment consists of flipping a fair coin and then flipping it a second time if a head occurs, if a tail occurs on the first flip, then a fair die is tossed once.

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- Let A be the event that the second toss is tail or an even number: $A = \{HT, T2, T4, T6\}$.

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- The outcomes are not equally likely: $P(HH) = P(HT) = \frac{1}{4}$ and $P(Ti) = \frac{1}{12}$ for $i = 1, .., 6$
- Let A be the event that the second toss is tail or an even number: $A = \{HT, T2, T4, T6\}$.
- The probability of A is

$$P(A) = P(HT) + P(T2) + P(T4) + P(T6) = \frac{1}{4} + \frac{1+1+1}{12} = \frac{1}{2}$$

Classical Method Approach to Probability

Practice 2.1

Choose randomly two ball from a box that consists of 4 red balls and 6 white balls with the same size. Find the probability of the following events.

- Two balls chosen are red.
- Exactly one ball is red.
- At least one ball is red.

Empirical Method Approach to Probability

Empirical Method Approach to Probability:

- The empirical or relative frequency approach to probability uses results from identical previous experiments that have been performed many times.
- Probabilities are based on historical or previously recorded data by determining the proportion of times an event occurs within the data.

$$P(A) \approx f_n(A) = \frac{\text{number of times A occurs}}{\text{total number of trials}}$$

- To get an accurate probability using this approach, it is important that the experiment is repeated a very large number of times.
- The law of large numbers: as the number of repetitions of an experiment increases, the relative frequency obtained in the experiment tends to the theoretical probability.

Empirical Method Approach to Probability

Example 2.20

An online retailer wants to know the probability that a transaction will be less than \$30. Suppose that in 2000 transactions, 650 are less than \$30.

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A computer shop tracks the number of desktop computer systems it sold over a month (30 days) and there are 12 days it sold 3 desktops per day.

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Example 2.21

A computer shop tracks the number of desktop computer systems it sold over a month (30 days) and there are 12 days it sold 3 desktops per day.

- We can approximate that

$$P(\text{the shop will sell 3 desktops on a given day}) = \frac{12}{30} = 0.4$$

Probability of Complementary Events

The Complementary Rule:

- In any experiment, an event A or its complement $\bar{A}$ must occur. This means that $P(A) + P(\bar{A}) = 1$.
- Probability of the complementary event: $P(\bar{A}) = 1 - P(A)$.

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An online retailer knows that 30% of customers spend more than $100 per transaction. What is the probability that a customer spends at most $100 per transaction?

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- Let A be the event of spending more than $100 per transaction.

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- Let A be the event of spending more than \$100 per transaction.
- The event of spending at most \$100 per transaction is the complement of A.
- Then $P(\bar{A}) = 1 - P(A) = 1 - 0.3 = 0.7$.

Probability of the Union of Events – Addition Rule

Addition Rule:

- The probability of the union of event A and B is $P(A+B) = P(A) + P(B) - P(AB)$
- If A and B are mutually exclusive then $P(A+B) = P(A) + P(B)$
- $P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC)$ $-P(CA) + P(ABC)$

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- $P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC)$ $-P(CA) + P(ABC)$
- In genenral, $P(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) +$ $\sum_{i<j<k} P(A_iA_jA_k) - ... + (-1)^{n-1}P(A_1A_2...A_n)$

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- Since $A\bar{B} = A - B = A - AB$ then $P(A\bar{B}) = P(A) - P(AB)$.

Probability of the Union of Events – Addition Rule

Example 2.23:

At a local language school, 40% of the students are learning Spanish, 20% of the students are learning German, and 8% of the students are learning both Spanish and German. Select at random a student at this school.

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- Let A be the event that he/she is learning Spanish and let B be the event that he/she is learning German.
- We have $P(A) = 0.4; P(B) = 0.2$ and $P(AB) = 0.08$

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- We have $P(A) = 0.4; P(B) = 0.2$ and $P(AB) = 0.08$
- The probability that he/she is learning Spanish or German is $P(A + B) = P(A) + P(B) - P(AB) = 0.4 + 0.2 - 0.08 = 0.52$

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- The probability that he/she is learning Spanish or German is $P(A + B) = P(A) + P(B) - P(AB) = 0.4 + 0.2 - 0.08 = 0.52$
- The probability that he/she is learning Spanish but is not learning German is $P(A\bar{B}) = P(A) - P(AB) = 0.4 - 0.08 = 0.32$

Probability of the Union of Events – Addition Rule

Example 2.24:

Suppose that 4 guests check their hats when they arrive at a restaurant, and that these hats are returned to them in a random order when they leave. Determine the probability that no guest will receive the proper hat.

Probability of the Union of Events – Addition Rule

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Suppose that 4 guests check their hats when they arrive at a restaurant, and that these hats are returned to them in a random order when they leave. Determine the probability that no guest will receive the proper hat.

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Example 2.24:

Suppose that 4 guests check their hats when they arrive at a restaurant, and that these hats are returned to them in a random order when they leave. Determine the probability that no guest will receive the proper hat.

- Let A be the event that no guest will receive the proper hat and let A_i be the event that the i^{th} guest receives his proper hat.
- Then $\bar{A}$ is the event that at least one guest will receive the proper hat and $A = A_1 + A_2 + A_3 + A_4$.

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Suppose that 4 guests check their hats when they arrive at a restaurant, and that these hats are returned to them in a random order when they leave. Determine the probability that no guest will receive the proper hat.

- Let A be the event that no guest will receive the proper hat and let A_i be the event that the i^{th} guest receives his proper hat.
- Then $\bar{A}$ is the event that at least one guest will receive the proper hat and $A = A_1 + A_2 + A_3 + A_4$.
- So $P(\bar{A}) = \sum_{i=1}^{4} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k)$ $-P(A_1A_2A_3A_4)$.

Probability of the Union of Events – Addition Rule

Example 2.24:

Suppose that 4 guests check their hats when they arrive at a restaurant, and that these hats are returned to them in a random order when they leave. Determine the probability that no guest will receive the proper hat.

- Let A be the event that no guest will receive the proper hat and let A_i be the event that the i^{th} guest receives his proper hat.
- Then $\bar{A}$ is the event that at least one guest will receive the proper hat and $A = A_1 + A_2 + A_3 + A_4$.
- So $P(\bar{A}) = \sum_{i=1}^{4} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k)$ $-P(A_1A_2A_3A_4)$.
- We have $P(A_i) = \frac{3!}{4!} = \frac{1}{4}; P(A_iA_j) = \frac{2!}{4!} = \frac{1}{12}; P(A_iA_jA_k) = \frac{1!}{4!} = \frac{1}{24}; P(A_1A_2A_3A_4) = \frac{1}{4!} = \frac{1}{24}$.

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- We have $P(A_i) = \frac{3!}{4!} = \frac{1}{4}; P(A_iA_j) = \frac{2!}{4!} = \frac{1}{12}; P(A_iA_jA_k) = \frac{1!}{4!} = \frac{1}{24}; P(A_1A_2A_3A_4) = \frac{1}{4!} = \frac{1}{24}$.
- Then $P(\bar{A}) = 4.\frac{1}{4} - 6.\frac{1}{12} + 4.\frac{1}{24} - \frac{1}{24} = \frac{5}{8}$ and $P(A) = 1 - P(\bar{A}) = \frac{3}{8}$.

Conditional Probability

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- For example, we would certainly like to know the probability that a person has lung disease given that he smokes. This is called a conditional probability.
- Definition 2.8: A **conditional probability** is the probability of an event A **given** that another event B has already occurred, and denoted by $P(A|B)$.

Conditional Probability

NOTE:

The conditional probability $P(A|B)$ is NOT the same as $P(AB)$.

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Conditional Probability Rule:

The conditional probability of event A given event B is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Conditional Probability

Example 2.25:

At a local language school, 40% of the students are learning Spanish, 20% of the students are learning German, and 8% of the students are learning both Spanish and German. Select at random a student at this school and suppose that he is learning Spanish. What is the probability that he is also learning German.

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- Let A be the event that he/she is learning Spanish and let B be the event that he/she is learning German.
- We have $P(A) = 0.4; P(B) = 0.2$ and $P(AB) = 0.08$

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- The probability that he is learning German given that he is learning Spanish is

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.08}{0.4} = 0.2$$

Multiplication Rule

Multiplication Rule:

From the conditional probability rule:

$$P(A|B) = \frac{P(AB)}{P(B)} \text{ and } P(B|A) = \frac{P(AB)}{P(A)}$$

we have

$$P(AB) = P(A)P(B|A) = P(B)P(A|B)$$

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In general, we have

$$P(A_1A_2...A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)...P(A_n|A_1...A_{n-1})$$

Multiplication Rule

Example 2.26:

An aerospace company has submitted bids on two separate federal government defense contracts. The company president believes that there is a 40% probability of winning the first contract. If they win the first contract, the probability of winning the second is 70%. What is the probability that they win both contracts?

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- We have $P(A) = 0.4; P(B|A) = 0.7$
- The probability that that they win both contracts is

$$P(AB) = P(A)P(B|A) = 0.4 * 0.7 = 0.28$$

Independence

Independent events:

- **Definition 2.9**: Two events A and B are said to be independent if the probability of one event is not affected by the occurrence of the other event:

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- The events $A_1, A_2, ..., A_n$ are called totally independent if for any events $A_{i_1}, A_{i_2}, ..., A_{i_k}, k \geq 2$, we have

$$P(A_{i_1}A_{i_2}...A_{i_k}) = P(A_{i_1})P(A_{i_2})...P(A_{i_k})$$

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- Let A be the event that he/she is learning Spanish and let B be the event that he/she is learning German.
- We have $P(A) = 0.4; P(B) = 0.2$ and $P(AB) = 0.08$
- The probability that he is learning German given that he is learning Spanish is $P(B|A) = \frac{P(AB)}{P(A)} = \frac{0.08}{0.4} = 0.2 = P(B)$
  So A and B are independent.

Independent events

Example 2.28:

A fair coin is tossed tiwce. Let A be the event that the fist toss is head, B is the event that the second toss is head and C be the event that all two tosses are the same. Are the events A, B, C independent in pairs or totally independent?

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- We have $AB = BC = CA = ABC = \{HH\}$, then $P(AB) = P(BC) = P(CA) = P(ABC) = 0.25$
- Hence, $P(AB) = P(A)P(B); P(BC) = P(B)P(C)$ and $P(AC) = P(A)P(C)$, then A, B, C are independent in pairs.

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- We have $AB = BC = CA = ABC = \{HH\}$, then $P(AB) = P(BC) = P(CA) = P(ABC) = 0.25$
- Hence, $P(AB) = P(A)P(B); P(BC) = P(B)P(C)$ and $P(AC) = P(A)P(C)$, then A, B, C are independent in pairs.
- Since $P(ABC) = 0.25 \neq P(A)P(B)P(C) = 0.125$, then A, B, C are not totally independent.

Practice

Practice 2.2:

Suppose that out of all customers entered a bookstore, there are 30% of them who asked the salesperson, 20% of them who bought books and 15% of them who did both of these 2 things. Select at random a customer entered the store.

- What is the probability that he/she did not neither ask the salesperson nor buy books?
- What is the probability that he/she did not buy books given that he/she has asked the salesperson?

Practice

Practice 2.3: The personnel department of an insurance company has compiled data on promotion, classified by gender. The joint probability table based on the data collected from 200 managers for the investigation are given below:

Manager	Promoted	Not promoted
Male	0.17	0.68
Female	0.03	0.12

Select at random a manager.

- What is the probability that the manager selected is promoted?
- What is the probability that the manager selected is not promoted or a female?
- What is the probability that the manager selected is a male given that he is promoted ?
- Are promotion and gender dependent on each another?

The Probability Tree

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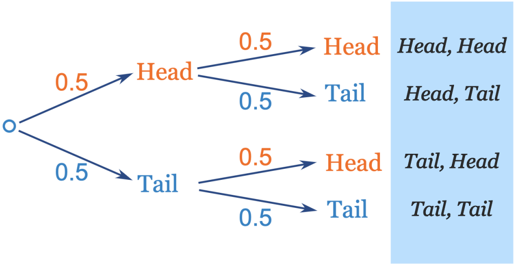
We can extend the tree diagram to two tosses of a coin:



The Probability Tree

Example 2.29:

It is known that 83% of regularly scheduled flights depart on time. Of those flights depart on time, 90% of them arrive on time and of those flights do not depart on time, 40% them arrive on time.

- Find the probability that a plane arrives on time.
- Find the probability that a plane departed on time, given that it has arrived on time.

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Aanlysis of Example 2.29:

- When we observe a flight, there are outcomes: the flight departs on time or not.

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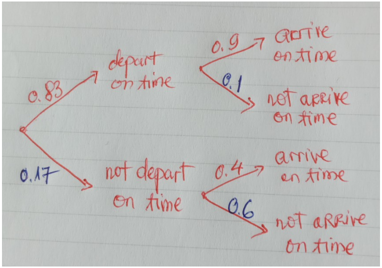
- When we observe a flight, there are outcomes: the flight departs on time or not.
- Given that the flight has departed on time or not, it will arrive on time or not.
- We have the following probability tree:



The Probability Tree

Problem:

Solution:

- Let A be the event that "the flight departs on time" and let B be the event that "the flight arrives on time".

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- Let A be the event that "the flight departs on time" and let B be the event that "the flight arrives on time".
- Then the system $\{A, \bar{A}\}$ is mutually exclusive and exhaustive.

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- Let A be the event that "the flight departs on time" and let B be the event that "the flight arrives on time".
- Then the system $\{A, \bar{A}\}$ is mutually exclusive and exhaustive.
- We have $B = B \cap S = B \cap (A + \bar{A}) = AB + \bar{A}B$, then $P(B) = P(AB) + P(\bar{A}B) = P(A)P(B|A) + P(\bar{A})P(B|\bar{A})$

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- Let A be the event that "the flight departs on time" and let B be the event that "the flight arrives on time".
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- So the probability that a plane arrives on time is $P(B) = 0.83 * 0.9 + 0.17 * 0.4 = 0.815$
- 81.5% of flights will arrive on time.

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- Then the system $\{A, \bar{A}\}$ is mutually exclusive and exhaustive.
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- So the probability that a plane arrives on time is $P(B) = 0.83 * 0.9 + 0.17 * 0.4 = 0.815$
- 81.5% of flights will arrive on time.
- The probability that a plane departed on time, given that it has arrived on time is

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B|A)}{P(B)} = \frac{0.83 * 0.9}{0.815} = 0.917$$

Total Probability Rule and Bayes' Rule

Total Probability Rule and Bayes' Rule

- The formula: $P(B) = P(AB) + P(\bar{A}B) = P(A)P(B|A) + P(\bar{A})P(B|\bar{A})$ is called the Total Probability Rule .
- The formula: $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$ is called the Bayes' Rule.

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- The formula: $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$ is called the Bayes' Rule.

In general:

Let $\{A_1, ..., A_n\}$ be a system of mutually exclusive and exhautive events.

- Total Probability Rule:
  $P(B) = P(A_1)P(B|A_1) + ... + P(A_n)P(B|A_n) = \sum_{i=1}^n P(A_i)P(B|A_i)$
- Bayes' Rule: $P(A_j|B) = \frac{P(A_j)P(B|A_j)}{P(B)} = \frac{P(A_j)P(B|A_j)}{\sum_{i=1}^n P(A_i)P(B|A_i)}$

Total Probability Rule and Bayes' Rule

Example 2.30:

In a certain assembly plant, three machines, B_1, B_2, and B_3, make $30\%, 45\%$, and 25%, respectively, of the products. It is known from past experience that $2\%, 3\%$, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected.

- What is the probability that it is defective?
- If a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

Total Probability Rule and Bayes' Rule

- Let B_i be the event that "the product chosen is made by machine B_i; with $i = 1, 2, 3$" and let A be the event that "the product chosen is defective".

Total Probability Rule and Bayes' Rule

- Let B_i be the event that "the product chosen is made by machine B_i; with $i = 1, 2, 3$" and let A be the event that "the product chosen is defective".
- We have: $\{B_1, B_2, B_3\}$ is a system of mutually exclusive and exhautive events and: $P(B_1) = 0.3; P(B_2) = 0.45; P(B_3) = 0.25$ and $P(A|B_1) = 0.02; P(A|B_2) = 0.03; P(A|B_3) = 0.02$

Total Probability Rule and Bayes' Rule

- Let B_i be the event that "the product chosen is made by machine B_i; with $i = 1, 2, 3$" and let A be the event that "the product chosen is defective".
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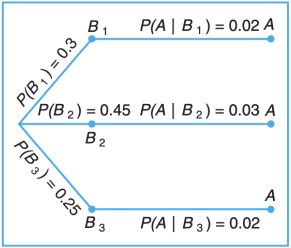
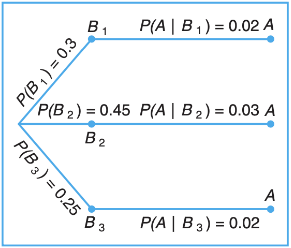
- We have the following probability tree:



Total Probability Rule and Bayes' Rule

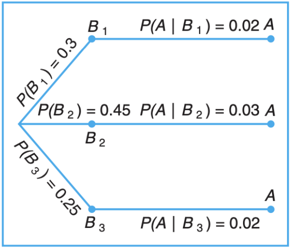
- By the Total Probability Rule, the probability that "the product chosen is defective" is
  $P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$
  $= 0.3 * 0.02 + 0.45 * 0.03 + 0.25 * 0.02 = 0.0245$



Total Probability Rule and Bayes' Rule

- By the Bayes' Rule, the probability that the product chosen was made by machine B_3, given that it was found to be defective is:

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(A)} = \frac{0.25 * 0.02}{0.0245} = \frac{10}{49}$$



Total Probability Rule and Bayes' Rule

Practice 2.4:

A paint-store chain produces and sells latex and semigloss paint. Based on long-range sales, the probability that a customer will purchase latex paint is 0.75. Of those that purchase latex paint, 60% also purchase rollers. But only 30% of semigloss paint buyers purchase rollers. A randomly selected buyer purchases a roller and a can of paint. What is the probability that the paint is latex?

Total Probability Rule and Bayes' Rule

Practice 2.5:

Researchers have developed statistical models based on financial ratios that predict whether a company will go bankrupt over the next 12 months. In a test of one such model, the model correctly predicted the bankruptcy of 85% of firms that did in fact fail, and it correctly predicted nonbankruptcy for 74% of firms that did not fail. Suppose that we expect 8% of the firms in a particular city to fail over the next year.

- What is the probability that model predicts bankruptcy for a firm that you own?
- Suppose that the model predicts bankruptcy for a firm that you own. What is the probability that your firm will fail within the next 12 months?

Bernoulli Rule

Bernoulli Trial:

A Bernoulli trial is a random experiment that

- results two posibble outcomes labeled by "success" and "failure"

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- the probability of success is a constant p and the probability of failure is a constant $q = 1 - p$.

Bernoulli Rule

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- results two posibble outcomes labeled by "success" and "failure"
- the probability of success is a constant p and the probability of failure is a constant $q = 1 - p$.

Exapmle 2.31:

Tossing a fair coin is a Bernoulli trial:

- results two posibble outcomes: Head (labeled by "success") or Tail (labeled by "failure")
- $\mathrm{P}(\text{success}) = p = 0.5$ and $\mathrm{P}(\text{failure}) = q = 1 - p = 0.5$.

Bernoulli Rule

Exapmle 2.32:

The defective rate in a production line is 5%; testing an item is a Bernoulli trial:

- results two posibble outcomes: the item is defective (labeled by "success") or it is non-defective (labeled by "failure")
- $\mathrm{P}(\text{success}) = p = 0.05$ and $\mathrm{P}(\text{failure}) = q = 1 - p = 0.95$.

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Exapmle 2.32:

The defective rate in a production line is 5%; testing an item is a Bernoulli trial:

- results two posibble outcomes: the item is defective (labeled by "success") or it is non-defective (labeled by "failure")
- P(success) $= p = 0.05$ and P(failure) $= q = 1 - p = 0.95$.

Exapmle 2.33:

"A salesperson called to a customer" is a Bernoulli trial:

- results two posibble outcomes: she closed a sale (labeled by "success") or she did not close a sale (labeled by "failure")
- P(success) $= p = 0.6$ and P(failure) $= q = 1 - p = 0.4$.

Bernoulli Rule

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- The Bernoulli rule: $P_n(k) = C_n^k p^k q^{n-k}$

Bernoulli Rule

Exapmle 2.34:

- Consider the Bernoulli trial of tossing a fair coin: $p = P(\text{Head}) = 0.5$ and $q = P(\text{Tail}) = 0.5$
- The probability that there are exactly three heads out of 4 tosses is $P_4(3) = C_4^3 0.5^3 0.5^1 = 0.25$

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Explanation:

- The event "there are exactly three heads out of 4 tosses" $=$ HHHT $+$ HHTH $+$ HTHH $+$ THHH.

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Explanation:

- The event "there are exactly three heads out of 4 tosses" = HHHT + HHTH + HTHH + THHH.
- Then $P_4(3) = P(HHHT) + P(HHTH) + P(HTHH) + P(THHH) = C_4^3 0.5^3 0.5^1 = 0.25$

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The defective rate in a production line is supposed to be 5%. Tested 10 items from the production line.

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- The probability that there are exactly 2 defective items out of 10 selected items is: $P_{10}(2) = C_{10}^{2} 0.05^{2} 0.95^{8} = 0.075$
- The probability that there are at least one defective item out of 10 selected items is: $1 - P_{10}(0) = 1 - C_{10}^{0} 0.05^{0} 0.95^{10} = 0.401$

Bernoulli Rule

Exapmle 2.36:

Suppose that the probability that a salesperson closed a sale in each call to a customer is 70%. How many calls will she have to make such that the probability that she will close at least one sale is more than 0.99?

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- The probability she will close at least one sale out of n calls is $1 - P_n(0) = 1 - C_n^0 0.7^0 0.3^n = 1 - 0.3^n > 0.99 \Leftrightarrow 0.3^n < 0.01 \Leftrightarrow n > 3.82$

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- So she will have to make at leat $n = 4$ calls.