EXERCISES - CALCULUS 3

Chapter 1

Series

1.1 Number series

Exercise 1.1. Test for convergence and find the sum (if exists):

a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

b) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

c) $\sum_{n=1}^{\infty} (\sin n + 1 - \sin n)$

d) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

e) $\sum_{n=1}^{\infty} \left(\frac{9}{10^n} - \frac{2}{5^n}\right)$

f) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}.3^n}{10^{n+2}}$

g) $\sum_{n=1}^{\infty} \arctan \frac{1}{n^2+n+1}$

Exercise 1.2. Test for convergence:

1. Divergence test

a) $\sum_{n=1}^{\infty} \frac{(-1)^n.n}{n+1}$

b) $\sum_{n=1}^{\infty} \frac{2n+3}{6n-1}$

c) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$

d) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n$

e) $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right)$

2. Comparison tests

a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+n+1}}{n^2\sqrt{n}+2}$

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}-\sqrt{n}}{2n+1}$

c) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

d) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{e}-1}{n}$

e) $\sum_{n=2}^{\infty} \arctan(2^{-n})$

f) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

g) $\sum_{n=1}^{\infty} \frac{2}{\ln(2n+1)}$

h) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sin\frac{1}{\sqrt{n}}\right)$

i) $\sum_{n=1}^{\infty} \frac{4+\cos n}{n^2(1+e^{-n})}$

3. Ratio test

a) $\sum\limits_{n=1}^{\infty} \frac{2019^n}{n!}$ c) $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n+1)!}$ e) $\sum\limits_{n=1}^{\infty} \frac{n^n}{4^n.n!}$

b) $\sum\limits_{n=2}^{\infty} \frac{1}{3^n} \frac{(2n+1)!}{n^2-1}$ d) $\sum\limits_{n=1}^{\infty} \frac{n!}{3^{n^2}}$ f) $\sum\limits_{n=2}^{\infty} \frac{e^n n!}{n^n}$

4. Root test

a) $\sum\limits_{n=1}^{\infty} \left(\frac{3n+1}{3n+2}\right)^{n^2}$ c) $\sum\limits_{n=2}^{\infty} \left(\frac{n}{n+2}\right)^{n^2-1}$ e) $\sum\limits_{n=2}^{\infty} \left(\cos\frac{1}{n}\right)^{n^3}$

b) $\sum\limits_{n=2}^{\infty} \frac{1}{4^n}\left(1-\frac{1}{n}\right)^{n^2}$ d) $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}\left(\frac{n-2}{n}\right)^{n^2+1}$

5. Integral test

a) $\sum\limits_{n=2}^{\infty} \frac{\ln n}{n^2}$ c) $\sum\limits_{n=4}^{\infty} \frac{1}{n\ln n\ln(\ln n)}$ e) $\sum\limits_{n=2}^{\infty} \frac{1}{\ln(n!)}$

b) $\sum\limits_{n=2}^{\infty} \frac{\ln n}{n}$ d) $\sum\limits_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

6. Series with sign-changing terms

a) $\sum\limits_{n=1}^{\infty} \frac{\cos n}{\sqrt{n^3+1}}$ d) $\sum\limits_{n=1}^{\infty} \left(\frac{3-2n}{2n+5}\right)^{n^2}$ g) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{3^n+\cos n}$

b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+1}$ e) $\sum\limits_{n=1}^{\infty} \frac{n\cos(n\pi)}{2n^2+1}$ h) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n+\cos n}{n\ln^2 n}$

c) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n.n^3}{2^n-1}$ f) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}.n^3}{(n^2+1)^{\frac{4}{3}}}$ i) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n+1}\sin\frac{1}{\sqrt{n}}$

Exercise 1.3. Test for absolute and conditional convergence:

a) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$ c) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ e) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$

b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{n+100}$ d) $\sum\limits_{n=1}^{\infty} (-1)^n\left(\frac{2n+100}{3n+1}\right)^n$

Exercise 1.4. Test for convergence

a) $\sum\limits_{n=1}^{\infty} \frac{n+1}{(n^2+2)\ln(n+3)}$ c) $\sum\limits_{n=1}^{\infty} \frac{n^5}{3^n+2^n}$ e) $\sum\limits_{n=2}^{\infty} \left(e^{\frac{(-1)^n}{\sqrt{n}}}-1\right)$

b) $\sum\limits_{n=1}^{\infty} \frac{2-n^2.3^{-n^2}}{n^2}$ d) $\sum\limits_{n=1}^{\infty} \left(\cos\frac{1}{n+1}-\cos\frac{1}{n}\right)$ f) $\sum\limits_{n=2}^{\infty} \frac{(-1)^n(n-1)}{n^2+1}$

1.2 Function series

Exercise 1.5. Determine the domain of convergence of the following function series:

a) $\sum\limits_{n=1}^{\infty} \dfrac{1}{1+n^{-x}}$ d) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{x^{2n}+1}$ g) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n^3+1}}{(x^2+1)n^x}$

b) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n^x}$ e) $\sum\limits_{n=1}^{\infty} \dfrac{n^x+(-1)^n}{n}$ h) $\sum\limits_{n=1}^{\infty} n.e^{-nx}$

c) $\sum\limits_{n=1}^{\infty} \dfrac{1}{x^n+1}$ f) $\sum\limits_{n=1}^{\infty} \left(x+\dfrac{1}{n}\right)^n$ i) $\sum\limits_{n=1}^{\infty} \dfrac{(n!)^2}{(2n)!}x^n$

Exercise 1.6. Determine the domain of convergence of the following power series:

a) $\sum\limits_{n=1}^{\infty} \dfrac{x^{2n}}{n}$ d) $\sum\limits_{n=1}^{\infty} \dfrac{e^{nx}}{n^2+n+1}$ g) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n.n}{\sqrt{n^3+1}}(1-3x)^n$

b) $\sum\limits_{n=1}^{\infty} \dfrac{(x+2)^n}{n\sqrt{n}}$ e) $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{1+n^3}(2x+1)^n$ h) $\sum\limits_{n=1}^{\infty} \left(\dfrac{1-2n}{2n+3}\right)^n x^{2n+1}$

c) $\sum\limits_{n=1}^{\infty} \dfrac{n}{2n+1}(x-2)^n$ f) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{2^n+3^n}$

Exercise 1.7. Test for uniform convergence on the given set of the following series:

a) $\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{2x^2+n^2}$, on $\mathbb{R}$ e) $\sum\limits_{n=1}^{\infty} \dfrac{x}{1+n^4x^2}$, on $[0,\infty)$

b) $\sum\limits_{n=1}^{\infty} \dfrac{e^{-nx}+1}{n^2}$, on $[0,\infty)$ f) $\sum\limits_{n=1}^{\infty} \dfrac{x}{(1+(n-1)x)(1+nx)}$, on $(0,1]$

c) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{(4x^2+9)^n}$, $x\in\mathbb{R}$ g) $\sum\limits_{n=1}^{\infty} (1-x)x^n$, on $[0,1]$

d) $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n}\left(\dfrac{2x+1}{x+2}\right)^n$, $x\in[-1;1]$ h) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{x^2+n+2}$, on $\mathbb{R}$.

Exercise 1.8. 1. Let $F(x)=\sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n^3}$. Prove that

(a) $F(x)$ is continuous $\forall x$ (b) $\lim\limits_{x\to 0} F(x)=0$ (c) $F'(x)=\sum\limits_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$

2. Prove that $\displaystyle\int_0^{\pi} \left(\frac{\cos 2x}{1.3}+\frac{\cos 4x}{3.5}+\frac{\cos 6x}{5.7}+\dots\right)dx = 0.$

Exercise 1.9. Find the sum of the following function series or number series:

a) $\sum\limits_{n=1}^{\infty} nx^n$, $x\in(-1;1)$ d) $\sum\limits_{n=1}^{\infty} \dfrac{x^{n+1}}{n(n+1)}$, $x\in(-1;1)$

b) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n+1}, x\in(-1,1)$ e) $\sum\limits_{n=1}^{\infty} \dfrac{x^{4n-3}}{4n-3}$, $x\in(-1;1)$

c) $\sum\limits_{n=1}^{\infty} (n^2+n)x^{n+1}, x\in(-1,1)$ f) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n.\pi^{2n+1}}{(2n+1)!}$

g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1).2^n}$

h) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)3^n}$

i) $\sum_{n=1}^{\infty} \frac{3n+1}{8^n}$

j) $\sum_{n=1}^{\infty} \frac{1}{(2n)!!}$

Exercise 1.10. Find the Maclaurin series of the following functions:

a) $y = \sin^2 x \cos^2 x$

b) $y = \sin x \sin 3x$

c) $y = e^{2x} + 3x \cos x$

d) $y = \frac{2x+1}{x^2 - 3x + 2}$

e) $y = \frac{2x-1}{x^2 + 2x - 3}$

f) $y = \frac{1}{x^2 + x + 1}$

g) $y = \frac{1}{\sqrt{4 - x^2}}$

h) $y = \ln(1 + 2x)$

i) $y = x \ln(x + 2)$

j) $y = \ln(1 + x - 2x^2)$

k) $y = \arcsin x$

Exercise 1.11. Find the Taylor series of y at the given point:

a) $y = \frac{1}{2x+3}$, $x_0 = 4$

b) $y = \sin \frac{\pi x}{3}$, $x_0 = 1$

c) $y = \sqrt{x}$, $x_0 = 4$

Exercise 1.12. Graph each of the following periodic functions and find compute corresponding Fourier series

a) $y = x$, $x \in (-\pi, \pi)$, $T = 2\pi$

b) $y = |x|$, $x \in (-\pi, \pi)$, $T = 2\pi$

c) $y = \begin{cases} 4, & 0 < x < 2, \\ -4, & 2 < x < 4 \end{cases}$, $T = 4$

d) $y = \begin{cases} 2x, & 0 \le x < 3, \\ 0, & -3 < x < 0 \end{cases}$, $T = 6$

e) $y = 2x, 0 < x < 10$, $T = 10$

f) $y = \begin{cases} 2 - x, & 0 < x < 4, \\ x - 6, & 4 < x < 8 \end{cases}$, $T = 8$

In each part, find the points of discontinuity of the function. To what value does the series converge at those points?

Exercise 1.13. Expand the function into a Fourier series

a) $f(x) = x, x \in [0, \pi]$, $f(x)$ is an odd and periodic function of $T = 2\pi$.

b) $f(x) = 2 - x, x \in (0, 2)$, $f(x)$ is an even and periodic of $T = 4$.

c) $f(x) = x + 1$, $x \in [0, \pi)$.

d) $f(x) = x - 1$, $x \in (0, \pi)$ into a Fourier sine series.

e) $f(x) = x(\pi - x)$, $x \in [0, \pi]$ into a Fourier cosine series. Then prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Chapter 2

Ordinary differential equations

2.1 First order ODEs

Exercise 2.1. 1) Separable equations

a) $2y(x^2+4)dy = (y^2+1)dx$

b) $y' + e^{y+x} = 0$

c) $1 + x + xy'y = 0$

d) $y' = \cos^2 x \cos^2(2y)$

e) $y' = x^2 y$, $y(1) = 1$

f) $xdx + ye^{-x}dy = 0, y(0) = 1.$

g) $y^2\sqrt{1-x^2}dy = \arcsin x dx, y(0) = 0$

h) $y' = \dfrac{2x}{y + x^2 y}, y(0) = -2.$

2) Homogeneous equations:

a) $y' = \dfrac{y}{x} + \dfrac{x}{y} + 1$

b) $xy' = x\sin\dfrac{y}{x} + y$

c) $2y' + \left(\dfrac{y}{x}\right)^2 = -1$

d) $(x+2y)dx - xdy = 0$

e) $xy' = y + e^{\frac{y}{x}}, y(1) = 0$

f) $xy' = y + 2x^3 \sin^2\dfrac{y}{x}, y(1) = \frac{\pi}{2}$

g) $y' = \dfrac{y^2}{x^2} - \dfrac{y}{x} + 1, y(1) = 2$

h) $(2x - y + 4)dx + (x + 2y - 3)dy = 0.$

3) Linear equations:

a) $xy' - 4y = 4x^8$

b) $(x^2+1)y' + 2xy = e^x$

c) $xy' - y = x^2\cos x, y(\pi) = \pi$

d) $y' + y\sin x = \sin x$, $y(0) = 0$

e) $y' - \dfrac{3}{x}y = 2x^2, y(1) = 2$

f) $(2xy+3)dy - y^2dx = 0.$

4) Bernoulli equations:

a) $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$

b) $xy' + y = -x^3y^2, y(1) = 1$

c) $y' + xy = \dfrac{xe^{-2x^2}}{y}$

d) $xy' = \dfrac{x^3}{y^2} - 2y, y(1) = 2.$

5) Exact equations:

a) $(x^2 + y)dx = (2y - x)dy$

b) $e^y dx = (2y - xe^y)dy$

c) $(3x^2y^2 + 2y + 1)dx + 2(x + x^3y)dy = 0$

d) $\dfrac{y}{x}dx + (e^y + 1 + \ln x)\, dy = 0$

e) $(e^x \sin y + y^2)dx + (e^x \cos y + 2xy)dy = 0$

Exercise 2.2. In each of the following parts, find an integrating factor and solve the given equation

1. $\left(\dfrac{y}{x} - 1\right) dx + \left(\dfrac{y}{x} + 1\right) dy = 0$
2. $(3x^2y + 2xy + y^3)\, dx + (x^2 + y^2)\, dy = 0$
3. $ydx + (2xy - e^{-2y})\, dy = 0$

Exercise 2.3. Solve the ODE $y' = y^2 - \dfrac{2}{x^2}$ by change of function $y = \dfrac{z}{x}$.

Exercise 2.4. Solve the ODE $xy' - (2x + 1)y + y^2 + x^2 = 0$ by change of function $y = z + x$.

Exercise 2.5. Solve the following ODEs:

a) $y' = (x + y)^2$

b) $y' = 1 + x + y + xy$

c) $(2xy^2 - 3y^3)dx = (3xy^2 - y)dy$

d) $xy' = y + x^3 \sin x, y(\pi) = 0$

e) $(x^2y^2 - x)dy = ydx$

f) $3xy^2y' - y^3 = x, y(1) = 3$

g) $(8xy^2 - y)dx + xdy = 0, y(1) = 1$

h) $x = (y')^2 - y' + 2$

2.2 Second order ODEs

Exercise 2.6. Solve the following ODEs

a) $xy'' + 2y' = 12x^2$

b) $\begin{cases} (1 - x^2)y'' - xy' = 2, \\ y(0) = 0, y'(0) = 0 \end{cases}$

c) $2yy'' = (y')^2 + 1$

d) $\begin{cases} (1 + x)y'' + x(y')^2 = y', \\ y(0) = 1, y'(0) = 2 \end{cases}$

Exercise 2.7. Solve the following equations

1. $(x - 1)^2y'' + 4(x - 1)y' + 2y = 0$, given a particular solution $y_1 = \dfrac{1}{1 - x}$.
2. $xy'' + 2y' + xy = 0$, given a particular solution $y_1 = \dfrac{\sin x}{x}$.
3. $y'' - \dfrac{2xy'}{x^2 + 1} + \dfrac{2y}{x^2 + 1} = 0$ given a particular solution $y_1 = x$.
4. $x^2y'' + xy' + \left(x^2 - \dfrac{1}{4}\right) y = 0, x > 0$, given a particular solution $y_1 = \dfrac{\cos x}{\sqrt{x}}$.

Exercise 2.8. Solve the ODEs with constant coefficients:

a) $y'' - 4y' + 3y = (15x + 37)e^{-2x}$

b) $y'' - y = 4(x + 1)e^x$

c) $y'' - 2y' + y = (12x + 4)e^x$

d) $y'' - y' - 2y = xe^x \cos x$

e) $y'' + 2y' + 10y = e^{-x} \cos 3x$

f) $y'' + y = 2 \cos x \cos 2x$

g) $y'' + 2y' + 2y = 8 \cos x - \sin x$

h) $y'' + y' - 2y = x + \sin 2x$

i) $y'' + 3y' - 4y = 3 \sin^2 x$

j) $y'' + 4y = e^{3x} + x \sin 2x$

Exercise 2.9. Solve the ODEs using the method of variation of parameters:

a) $y'' - 2y' + y = \dfrac{e^x}{x}$

b) $y'' - 3y' + 2y = \dfrac{1}{1 + e^{-x}}$

Exercise 2.10. Solve the ODE $(2x - x^2)y'' + 2(x - 1)y' - 2y = -2$, given two particular solutions $y_1 = 1, y_2 = x$.

Exercise 2.11. Solve the following Euler equations

1. $x^2y'' - 3xy' + 4y = x^3, y(1) = 1, y'(1) = 2$
2. $y'' - \dfrac{y'}{x+1} + \dfrac{y}{(x+1)^2} = \dfrac{2}{x+1}, \ x > -1.$

2.3 Systems of first order ODEs

Exercise 2.12. Solve the following systems of ODEs

a) $\begin{cases} \dfrac{dy}{dx} = 5y + 4z \\ \dfrac{dz}{dx} = 4y + 5z \end{cases}$

b) $\begin{cases} \dfrac{dy}{dx} = y + 5z \\ \dfrac{dz}{dx} = -y - 3z \end{cases}$

c) $\begin{cases} \dfrac{dx}{dt} = \dfrac{y}{x - y} \\ \dfrac{dy}{dt} = \dfrac{x}{x - y} \end{cases}$

d) $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + \dfrac{1}{\cos t} \end{cases}$

2.4 Power series method

Exercise 2.13. Solve the following series by the power series method

a) $(x^2 + 1)y'' - 4xy' + 6y = 0$

b) $y'' + xy' + y = 0, y(0) = 0, y'(0) = 1.$

Chapter 3

Laplace transform

3.1 Laplace and inverse Laplace transforms

Exercise 3.1. Using the definition, find the Laplace transforms of the following functions:

a) $f(t) = t$ b) $f(t) = e^{2t+3}$ c) $f(t) = \sin(2t)$.

Exercise 3.2. Find the Laplace transforms of the following functions:

a) $f(t) = \sqrt{t} + 3t - 2t^2\sqrt{t}$ c) $f(t) = (e^t + e^{-2t})^2$ e) $f(t) = 2\sin\left(t + \dfrac{\pi}{3}\right)$

b) $f(t) = (t+2)^2 - 2e^{3t}$ d) $f(t) = 2\sin 3t.\cos 5t$ f) $f(t) = e^{-2t} - 3u(t-2)$

Exercise 3.3. Find the inverse Laplace transforms of the following functions:

a) $F(s) = \dfrac{3}{s^4} - \dfrac{2}{s^{\frac{5}{2}}} + \dfrac{4}{s}$ c) $F(s) = \dfrac{5-3s}{s^2+9}$ e) $F(s) = \dfrac{e^{-2s}+5}{s}$

b) $F(s) = \dfrac{3}{s-4} + \dfrac{10}{s+2}$ d) $F(s) = \dfrac{10s-3}{s^2+25}$ f) $F(s) = \dfrac{e^{-\pi s}}{s} - \dfrac{2s+3}{s^2+4}$

3.2 Transformation of initial value problems

Exercise 3.4. Solve the following IVPs:

a) $\begin{cases} x^{(3)} - x'' - x' + x = e^{2t} \\ x(0) = x'(0) = x''(0) = 0 \end{cases}$ c) $\begin{cases} x^{(4)} - 16x = 240\cos t \\ x(0) = x'(0) = x''(0) = x^{(3)} = 0 \end{cases}$

b) $\begin{cases} x^{(3)} - 6x'' + 11x' - 6x = 0 \\ x(0) = x'(0) = 0, x''(0) = 2 \end{cases}$ d) $\begin{cases} x^{(4)} + 8x'' + 16x = 0 \\ x(0) = x'(0) = x''(0) = 0, x^{(3)}(0) = 1 \end{cases}$

Exercise 3.5. Solve the following IVPs

a) $\begin{cases} y' = 2y + z \\ z' = y + 2z \\ y(0) = 1, z(0) = 3 \end{cases}$ b) $\begin{cases} z' + 2y = e^x \\ y' - 2z = 1 + x \\ y(0) = 1, z(0) = 2(HW - submit) \end{cases}$

3.3 Translation. Rational functions

Exercise 3.6. Find the Laplace transforms of the following functions:

a) $f(t) = t^4 e^{\pi t}$ b) $f(t) = e^{-2t}\sin 3t$ c) $f(t) = e^t \sin\left(t + \frac{\pi}{3}\right)$

Exercise 3.7. Find the inverse Laplace transforms of the following functions:

a) $F(s) = \dfrac{2+s}{s^2-3s+2}$

b) $F(s) = \dfrac{3s-2}{s(s^2+4)}$

c) $F(s) = \dfrac{2s+1}{s^2(s^2+1)}$

d) $F(s) = \dfrac{1}{s(s+1)(s+2)}$

e) $F(s) = \dfrac{2s-1}{s^2-4}$

f) $F(s) = \dfrac{5-2s}{s^2+7s+10}$

g) $F(s) = \dfrac{1}{s^3-5s^2}$

h) $F(s) = \dfrac{1}{s^4-16}$

i) $F(s) = \dfrac{s^2-2s}{s^4+5s^2+4}$

j) $F(s) = \dfrac{3s+1}{s^2+4s+4}$

k) $F(s) = \dfrac{3s+5}{s^2-6s+25}$

l) $F(s) = \dfrac{s^2+3}{(s^2+2s+2)^2}$

3.4 Derivatives, integrals and products of Laplace transforms

Exercise 3.8. Find the Laplace transforms of the following functions:

a) $f(t) = t\cos^2 t$

b) $f(t) = (t - e^{2t})^2$

c) $f(t) = te^{2t}\sin 3t$

d) $f(t) = (2t - \sin 3t)^2$

e) $f(t) = \dfrac{e^{2t}-1}{t}$

f) $f(t) = \dfrac{1-\cos 2t}{t}$

Exercise 3.9. Find the inverse Laplace transforms of the following functions:

a) $F(s) = \arctan\dfrac{1}{s}$ b) $F(s) = \ln\dfrac{s-2}{s+2}$ c) $F(s) = \ln\dfrac{s^2+1}{(s+2)(s-3)}$

Exercise 3.10. Solve the following IVPs:

a) $\begin{cases} tx'' + (t-2)x' + x = 0 \\ x(0) = 0 \end{cases}$

b) $\begin{cases} tx'' - (4t+1)x' + 2(2t+1)x = 0 \\ x(0) = 0 \end{cases}$

c) $\begin{cases} tx'' + (4t-2)x' + (13t-4)x = 0 \\ x(0) = 0 \end{cases}$

d) $\begin{cases} ty'' - ty' + y = 2 \\ y(0) = 2, y'(0) = -4 \end{cases}$

Exercise 3.11. Solve the following IVPs:

a) $\begin{cases} x'' - 3x' + 2x = u(t-2) \\ x(0) = 0, x'(0) = 1 \end{cases}$

b) $\begin{cases} x'' + 4x = \sin t - u(t-2\pi)\sin(t-2\pi) \\ x(0) = 0, x'(0) = 0 \end{cases}$

c) $\begin{cases} y'' + 2y' + 2y = e^{-(t-1)}u(t-1), \\ y(0) = y'(0) = 0. \end{cases}$

d) $\begin{cases} x'' + 4x' + 4x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$ where $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

e) $\begin{cases} x'' + x = f(t) \\ x(0) = 0, x'(0) = 1 \end{cases}$ where $f(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$

f) $\begin{cases} x'' + x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$ where $f(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi. \end{cases}$