UNIT 9

INVERSE Z-TRANSFORM

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Contents

- Inverse Z-transform
- The method of finding discrete-time signals in the time domain through their representation in the Z-domain.

❑ Learning Objectives

Upon completing this lesson, students will have a grasp of the following concepts:

1. The inverse Z-transform using the method of partial fraction expansion to obtain simple rational expressions.
2. The inverse Z-transform using the method of power series expansion.

Linearity

If

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

Then

$$x(n) = ax_1(n) + bx_2(n) \xrightarrow{z} aX_1(z) + bX_2(z)$$

- Observation: The linearity property allows for the representation of $X(z)$ into a linear combination of components whose corresponding discrete-time signals in the time domain are already known.

1. Partial fraction expansion into simple rational fractions

- Inverse Z-transform:: $X(z)$, region of convergence → x(n) ?

$$X(z) = \frac{P(z)}{Q(z)} = S(z) + \frac{P_0(z)}{Q(z)}$$

- $P(z), Q(z)$ are polynomials of degree M and N, respective.
- $S(z)$ in a polynomial of degree $M - N$ $(M < N \rightarrow S(z) = 0)$. The degree of $P_0(z)$ is smaller than the degree of $Q(z)$

$$\frac{P_0(z)}{Q(z)} = \sum_{i=1}^{N} \frac{A_i}{z - z_i}$$

- z_i : single zero point of $Q(z)$: $A_i = (z - z_i)\frac{P_0(z)}{Q_0(z)}\Big|_{z=z_i}$

Partial fraction expansion into simple rational fractions

$$X(z) = \frac{P(z)}{Q(z)} = S(z) + \frac{P_0(z)}{Q(z)}$$

- If the zero point z_n of $Q(z)$ has the degree of q

$$\frac{P_0(z)}{Q(z)} = \sum_{\substack{i=1 \\ i \neq n}}^{N} \frac{A_i}{z - z_i} + \sum_{j=1}^{q} \frac{B_j}{(z - z_n)^j}$$

$$B_j = \frac{1}{(q-j)!} \frac{d^{q-j}}{dz^{q-j}} \left[(z - z_n)^q \frac{P_0(z)}{Q(z)} \right] \Bigg|_{z=z_n}$$

Example

- Given $X(z)$ with $|z| > 2$. Find $x(n)$?

$$X(z) = \frac{1}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^2}{z^2 - 3z + 2}$$

- The denominator has two roots: $z = 1$ và $z = 2$

$$X(z) = a\frac{z}{z-1} + b\frac{z}{z-2}$$

$$\Rightarrow \qquad a = -1, \qquad b = 2$$

$$\Rightarrow \qquad x(n) = 2.2^n u(n) - u(n) = (2^{n+1} - 1)\,u(n)$$

2. Expansion by power series

- $X(z)$ has the form of a ratio of two polynomials in z^{-1}. Perform polynomial devision to obtain each samples of $x(n)$
- Example:

$$X(z) = \frac{z^{-1}}{1 - 1{,}414z^{-1} + z^{-2}}$$

$$
\begin{array}{l|l}
z^{-1} & 1-1{,}414z^{-1}+z^{-2} \\
\hline
\underline{z^{-1} - 1{,}414z^{-2} + z^{-3}} & z^{-1} + 1{,}414z^{-2} + z^{-3} - z^{-5} - 1{,}414\, z^{-6} \ldots \\
1{,}414z^{-2} - z^{-3} & \\
\underline{1{,}414z^{-2} - 2z^{-3} + 1{,}414z^{-4}} & \\
z^{-3} - 1{,}414z^{-4} & \\
\underline{z^{-3} - 1{,}414z^{-4} + z^{-5}} & \\
-z^{-5} & \\
\underline{-z^{-5} + 1{,}414z^{-6} - z^{-7}} & \\
-1{,}414z^{-6} + z^{-7} &
\end{array}
$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(0) = 0, x(1) = 1, x(2) = 1{,}414,$$

$$x(3) = 1, x(4) = 0, x(5) = -1 \ldots.$$

$$n < 0, x(n) = 0$$

Several common Z-transform (1/2)

Signal	Z-transform	ROC
$\delta(n)$	1	Toàn mf z
$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
-$u(-n-1)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta(n-m)$	z^{-m}	Toàn mf z trừ 0 nếu $m > 0$, trừ ∞ nếu $m < 0$
$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$ 7

Several common Z-transform (2/2)

Signal	Z-transform	ROC
$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\Omega n)\, u(n)$	$\frac{1-(\cos\Omega)z^{-1}}{1-2(\cos\Omega)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\Omega n) u(n)$	$\frac{1-(\sin\Omega)z^{-1}}{1-2(\sin\Omega)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$

4. Summary

- Inverse Z-transform is applied to find discrete signals in the time domain from their representation in the complex domain.
- Representation in the Z-domain is often decomposed into basic components through expansion as proper fractions or power series. Utilizing the linearity property of the Z-transform, the inverse Z-transform can be computed easily from these basic components..

5. Assignment

- Excercise 1
  - ❑ Compute the inverse Z-transform:

a. $X(z) = \frac{Z^2+4Z}{Z^2-3Z+2}, |z| > 2$

b. $X(z) = \frac{Z+5}{Z^2-3Z+2}, |z| > 2$

Homework

- Excercise 2
  - Compute the inverse Z-transform of the following signals:

a. $X(z) = \frac{1}{z-a}, |z| > a$

b. $X(z) = \frac{1}{(z-a)^2}, |z| > a$

c. $X(z) = \frac{1}{(z-a)^M}, |z| > a$

Homework

- Excercise 3
  - Calculate convolution of the following signals:

$$x_1(n) = 2^n u(n) \qquad x_2(n) = 3^n u(n)$$

The next unit 10

IMPLEMENTATION OF THE SYSTEM IN Z DOMAIN

References:

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wishing you all the best in your studies!