Calculus 2 Midterm mock exam Solution

Q1. (1pt)

Let $F(x,y,z) = x^2 + 3y^2 - z^2 - 3 = 0$

We have that : $F'_x(A) = 2x = 2, F'_y(A) = 6y = 6, F'_z(A) = -2z = -2$

Tangent plane: $2(x-1) + 6(y-1) - 2(z-1) = 0$

Normal plane: $\dfrac{x-1}{2} = \dfrac{y-1}{6} = \dfrac{z-1}{-2}$

Q2. (1pt)

The parametric form of the curve (P): $\vec{r}(x) = \left(x, -2x^2 - 4x, 0\right)$

$$\Rightarrow \begin{cases} \vec{r}'(x) = \left(1, -4x - 4, 0\right) \\ \vec{r}''(x) = \left(1, -4, 0\right) \end{cases} \Rightarrow \vec{r}'(x) \times \vec{r}''(x) = (0, 0, -4)$$

The curvature of the curve P at the point M is: $\mathscr{K} = \dfrac{\left|\vec{r}'(x) \times \vec{r}''(x)\right|}{\left|\vec{r}'(x)\right|^3}$

$\Leftrightarrow \mathscr{K} = \dfrac{4}{\sqrt{1 + 16(x+1)^2}}$ reaches the maximum value at $x = -1$

$\Rightarrow$ At the point $M = (-1, 2)$, the curvature of the curve (P) reaches the maximum value

Q3. (1pt)

We have that: $I = \iint\limits_D 4xydxdy = \iint\limits_D ((x+y)^2 - (x-y)^2)dxdy$

Let: $\begin{cases} u = x + y \\ v = x - y \end{cases}$

$\Rightarrow$ D $\leftrightarrow$ D': $\begin{cases} 0 \leq u \leq 1 \\ 1 \leq v \leq 2 \end{cases}$

and $|J| = \dfrac{1}{2}$

We have that:

$$I = \frac{1}{2}\iint\limits_{D'}(u^2 - v^2)dudv$$
$$= \frac{1}{2}\int\limits_0^1 du\int\limits_1^2 (u^2 - v^2)dv$$
$$= \frac{1}{2}\int\limits_0^1 \left(u^2v - \frac{v^3}{3}\right)\Big|_1^2 du$$
$$= \frac{1}{2}\int\limits_0^1 (u^2 - \frac{7}{3})du$$
$$= -1$$

Q4. (1pt)

Let: $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$ $\Rightarrow |J| = r^2\cos\theta$, and $\begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq \dfrac{\pi}{3} \\ 0 \leq \varphi \leq 2\pi \end{cases}$

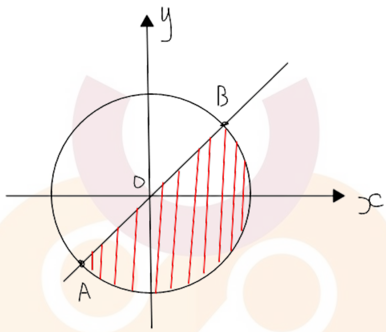
We have that:

$$I = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\frac{\pi}{3}} d\theta \int\limits_0^2 \frac{(r\sin\theta\cos\varphi)^2}{r}.r^2\sin\theta dr$$

$$= \int_0^{2\pi} (cos\varphi)^2 d\varphi \int_0^{\frac{\pi}{3}} (\sin\theta)^3 d\theta \int_0^2 r^3 dr$$

$$= \frac{5\pi}{6}$$

Q5. (1pt)



Choose $C_1 : y = x$ from B to A

$$\Rightarrow C_1 : \begin{cases} x = t \\ y = t \end{cases}, \ t : \frac{1}{\sqrt{2}} \to -\frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \int_{C_1} y^3 dx + 2x^3 dy = \int_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}} (t^3 + 2t^3) dt = 3.\frac{t^4}{4}\Big|_{\frac{1}{\sqrt{2}}}^{-\frac{1}{\sqrt{2}}} = 0$$

Let $I = \int_C y^3 dx + 2x^3 dy = (\int_C + \int_{C_1}) - \int_{C_1}$

Since the curve $C + C_1$ is closed, positive oriented

$$\Rightarrow I = \int_C + \int_{C_1} = \iint_D (6x^2 - 3y^2) dxdy \ \text{ with D} : \begin{cases} x^2 + y^2 \le 1 \\ y \le x \end{cases} \quad \text{Let} \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Leftrightarrow$$

$$\begin{cases} |J| = r \\ 0 \le r \le 1 \\ -\dfrac{3\pi}{4} \le \varphi \le \dfrac{\pi}{4} \end{cases}$$

$$\Rightarrow I = 3\int_{\frac{-3\pi}{4}}^{\frac{\pi}{4}} (2cos^2\varphi - sin^2\varphi)d\varphi \int_0^1 r^3 dr = \frac{3\pi}{8}$$

Q6. (1pt)

The mass of the curve is determined from: $m = \int\limits_L \rho(x,y)ds = \int\limits_L \frac{1}{y}ds$

We have that: $y'_x = \frac{1}{2}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)$

$$\Rightarrow ds = \sqrt{1 + y'^2_x}dx = \sqrt{1 + \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2}dx = \frac{1}{2}\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)dx = \frac{1}{2}ydx$$

$$\Rightarrow m = \int\limits_0^2 \frac{1}{y}\sqrt{1 + y'^2_x}dx = \int\limits_0^2 \frac{1}{y}.\frac{1}{2}ydx = 1$$

Q7. (1pt)

Divide S into S_1, S_2, S_3 such that:

$$\begin{cases} S_1 : x^2 + y^2 = 4; x - 3 \le z \le x + 2 \\ S_2 : z = x + 2; x^2 + y^2 \le 4 \\ S_3 : z = x - 3; x^2 + y^2 \le 4 \end{cases}$$

$$\Rightarrow I = \iint\limits_S = \iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3}$$

We have that:

- $I_1 = \iint\limits_{S_1} (x-z)dS$. Put $\begin{cases} x = 2\cos\varphi \\ y = 2\sin\varphi \\ z = z \end{cases} \Rightarrow \begin{cases} 0 \le \varphi \le 2\pi \\ x-3 \le z \le x+2 \end{cases}$

  and $\vec{r}(z,\varphi) = (2\cos\varphi, 2\sin\varphi, z) \Rightarrow \begin{cases} \vec{r}_z = (0,0,1) \\ \vec{r}_\varphi = (-2\sin\varphi, 2\cos\varphi, 0) \end{cases}$

  $\Rightarrow |\vec{r}_z \times \vec{r}_\varphi| = \sqrt{4\left(\cos\varphi^2 + \sin\varphi^2\right)} = 2$

  $\Rightarrow$

$$
\begin{aligned}
I_1 &= \iint\limits_{D_z\varphi} (2\cos\varphi - z).zdzd\varphi \\
&= 2\int_0^{2\pi} d\varphi \int_{2\cos\varphi-3}^{2\cos\varphi+2} (2cos\varphi - z)dz \\
&= 2\int_0^{2\pi} d\varphi \left(2\cos\varphi.z - \frac{z^2}{2}\right)\Big|_{\cos\varphi-3}^{2\cos\varphi+2} \\
&= 2\int_0^{2\pi} \left(10\cos\varphi + \frac{(2\cos\varphi-3)^2-(2\cos\varphi+2)^2}{2}\right)d\varphi \\
&= 2\int \frac{5}{2}d\varphi = 10\pi
\end{aligned}
$$

- $I_2 = \iint\limits_{S_2} (z-x)ds$

  We have that: $z = x+2 \Rightarrow \begin{cases} z'_x = 1 \\ z'_y = 0 \end{cases} \Rightarrow \sqrt{1+(z'_x)^2+(z'_y)^2} = \sqrt{2}$

$$\Rightarrow I_2 = \iint\limits_D -2\sqrt{2}dS = -2\sqrt{2}.4\pi = -8\sqrt{2}\pi,\ D: x^2+y^2 \leq 4$$

- $I_3 = \iint\limits_{S_3} (z-x)dS$. We have that $z = x-3 \Rightarrow \sqrt{1+(z'_x)^2+(z'_y)^2} = \sqrt{2} \Rightarrow I_3 =$
  $\iint\limits_D 3\sqrt{2}dS = 3\sqrt{2}.4\pi = 12\sqrt{2}\pi$, with D: $x^2+y^2 \leq 4$

So $I = I_1 + I_2 + I_3 = (10+4\sqrt{2})\pi$

Q8. (1pt)

We have $\mathbf{F} = (x^2-y, x+2y, x+y+z) = (P,Q,R)$
The flux of $\mathbf{F}$ through the surface (S) is:

$$\phi = \iint_S \left(Pdydz + Qdxdz + Rdxdy\right)$$

The surface (S) is closed with outward (positive) orientation. Let V be the simple solid region bounded by this surface. Applying Ostrogradsky's Theorem, we have:

$$\phi = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) = \iiint_V (2x+3)dxdydz$$

with $V: |x-y| + |x+2y| + |x+y+z| \leq 1$

Let $\begin{cases} u = x-y \\ v = x+2y \\ w = x+y+z \end{cases} \Rightarrow J^{-1} = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 3 \Rightarrow \begin{cases} |J| = \dfrac{1}{3} \\ x = \dfrac{2u+v}{3} \end{cases}$

$$\Rightarrow \phi = \iiint_{V_{uvw}} \frac{1}{3}\left(\frac{4u}{3}+\frac{2v}{3}+3\right)dudvdw \qquad V_{uvw}: |u|+|v|+|w| \leq 1$$

$$\Rightarrow \phi = \iiint_{V_{uvw}} \left(\frac{4u}{9}\right)dudvdw + \iiint_{V_{uvw}} \left(\frac{2v}{9}\right)dudvdw + \iiint_{V_{uvw}} dudvdw = I_1 + I_2 + I_3$$

We have: $I_1 = 0$ since V_{uvw} is symmetric over Ovw and $f = \dfrac{4u}{9}$ is an odd function with respect to u. Similarly for $I_2 = 0$

$$\Rightarrow \phi = I_3 = V(V_{uvw}) = \frac{4}{3}$$

Q9. (1pt)

Let $\begin{cases} P = xz^2 \\ Q = z^2 y R = y^2(z+2) \end{cases} \quad \Rightarrow \phi = \iint\limits_S P\,dydz + Q\,dzdx + R\,dxdy$

We add to surface S another surface $\bar{S}$: $\begin{cases} z = 0 \\ x^2 + y^2 \leq 1 \end{cases}$, oriented upward to make a closed surface $S \cup \bar{S}$

$$\Rightarrow \phi = \iint\limits_{S\cup\bar{S}} P\,dydz + Q\,dzdx + R\,dxdy - \iint\limits_{\bar{S}} P\,dydz + Q\,dzdx + R\,dxdy$$

+) $I_1 = \iint\limits_{S\cup\bar{S}} P\,dydz + Q\,dzdx + R\,dxdy$

Applying Divergence's Theorem we obtain:

$$I_1 = \iiint\limits_V (P'_x + Q'_y + R'_z)\,dxdydz = \iiint\limits_V (z^2 + x^2 + y^2)\,dxdydz$$

We have $V:\begin{cases}x^2+y^2+z^2\le 1\\ z\le 0\end{cases}$, let $\begin{cases}x=r\sin\theta\cos\varphi\\ y=r\sin\theta\sin\varphi\\ z=r\cos\theta\end{cases}\Rightarrow |J|=r^2\sin\theta$

Hence, $V\to V':\begin{cases}0\le\varphi\le 2\pi\\ \dfrac{\pi}{2}\le\theta\le\pi\\ 0\le r\le 1\end{cases}$

$$\Rightarrow I_1=\int_0^{2\pi}d\varphi\int_{\frac{\pi}{2}}^{\pi}d\theta\int_0^1 r^4\sin\theta\,dr=2\pi\int_{\frac{\pi}{2}}^{\pi}\sin\theta\,d\theta\int_0^1 r^4\,dr=\frac{2\pi}{5}$$

+) $I_2=\iint\limits_{\bar{S}}P\,dydz+Q\,dzdx+R\,dxdy$ với $\bar{S}:\begin{cases}z=0\\ x^2+y^2\le 1\end{cases}$, oriented upward

$$\Rightarrow I_2=\iint\limits_{\bar{S}}y^2(z+2)\,dxdy=2\iint\limits_{\bar{S}}y^2\,dxdy$$

Let $\begin{cases}x=r\cos\varphi\\ y=r\sin\varphi\end{cases}\Rightarrow |J|=r$ and $\bar{S}\to D:\begin{cases}0\le\varphi\le 2\pi\\ 0\le r\le 1\end{cases}$

$$\Rightarrow I_2=\int_0^{2\pi}d\varphi\int_0^1 2r^3\sin^2\varphi\,dr=\frac{\pi}{2}$$

$$\Rightarrow I=I_1-I_2=\frac{2\pi}{5}-\frac{\pi}{2}=-\frac{\pi}{10}$$

Q10. (1pt)

Let: $\begin{cases}P=x^2+y^2+z^2+yz\\ Q=x^2+y^2+z^2+xz\\ R=x^2+y^2+z^2+xy\end{cases}$

S: The part of the sphere $x^2+y^2+z^2=4$ whose boundary is the curve C, upward oriented. Applying Stoke's theorem:

$$\begin{aligned} I &= \iint_S (R'_y - Q'_z)dydz + (P'_z - R'_x)dxdz + (Q'_y - P'_x)dxdy \\ &= 2\iint_S (y-z)dydz + (z-x)dxdz + (x-y)dxdy \end{aligned}$$

Ta có: $z = \sqrt{4-x^2-y^2}$

Do $(\vec{n}, O_z) < \dfrac{\pi}{2} \Rightarrow \vec{n} = \left(\dfrac{x}{\sqrt{4-x^2-y^2}}, \dfrac{y}{\sqrt{4-x^2-y^2}}, 1\right) \Rightarrow |\vec{n}| = \dfrac{2}{\sqrt{4-x^2-y^2}} \Rightarrow$

$$\begin{cases} cos\alpha = \dfrac{x}{2} \\ cos\beta = \dfrac{y}{2} \\ cos\gamma = \dfrac{z}{2} \end{cases}$$

Applying the relation between surface integral type I and type II, we have:

$$I = \iint_S \left[x(y-z) + y(z-x) + z(x-y)\right] dS = 0$$