1 Symbolic Logic, Sets, Maps, and Complex Numbers

1.1 Symbolic Logic

Exercise 1.1. *Determine the truth table of the following propositions*

a) $[A \wedge (B \vee C)] \to C$

b) $[\overline{A} \wedge (B \vee C)] \wedge B$.

Exercise 1.2. *Let* p, q *are propositions. Are* $(p \to q) \to q$ *and* $p \wedge q$ *logically equivalent? Explain why.*

Exercise 1.3. *Assume that* $(A \wedge B) \to (B \wedge C)$ *and* $(A \vee B) \to (B \vee C)$ *are tautology propositions. Prove that* $A \to B$ *is a tautology proposition.*

Exercise 1.4. *Which of the following propositions are tautology, contradiction*

a) $(p \vee q) \to (p \wedge q)$,

b) $(p \wedge q) \vee (p \to q)$,

c) $p \to (q \to p)$,

d) $(q \to (q \to p))$,

e) $(p \to q) \to q$,

f) $[(p \to q) \wedge (q \to r)] \to (p \to r)$.

Exercise 1.5. *Prove that*

a) $A \leftrightarrow B$ *and* $(A \wedge B) \vee (\overline{A} \wedge \overline{B})$ *are logically equivalent.*

b) $(A \to B) \to C$ *and* $A \to (B \to C)$ *are not logically equivalent.*

Exercise 1.6. *Find the negation* p *if*

a) $p = \forall \epsilon > 0, \exists \delta > 0 : \forall x, |x - x_0| < \delta, |f(x) - f(x_0)| < \epsilon$.

b) $p = \forall M > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n| > M$. [1]

c) $p = \forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - L| < \epsilon$. [2]

1.2 Sets

Exercise 1.7. *Let*

$$A = \{x \in \mathbb{R} | x^2 - 4x + 3 \leq 0\}, B = \{x \in \mathbb{R} | |x - 1| \leq 1\},$$

and

$$C = \{x \in \mathbb{R} | x^2 - 5x + 6 \leq 0\}.$$

Compute $(A \cup B) \cap C$ *and* $(A \cap B) \cup C$.

Exercise 1.8. *Let* A, B, C, D *be arbitrary sets. Prove that*

[1] this is the condition such that $\lim\limits_{n \to +\infty} x_n = \infty$

[2] this is the condition such that $\lim\limits_{n \to +\infty} x_n = L$

a) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

b) $A \cup (B \setminus A) = A \cup B$.

c) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

d) $A \setminus (A \setminus B) = A \cap B$.

e) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

f) $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D)$.

g) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

h) $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

i) *Is it true that* $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$. *If not, give a counterexample.*

j) *If* $(A \cap C) \subset (A \cap B)$ *and* $(A \cup C) \subset (A \cup B)$, *then* $C \subset B$.

1.3 Maps

Exercise 1.9. *Let* $f : X \to Y$ *be a map and* $A, B \subset X; C, D \subset Y$. *Prove that*

a) $f(A \cup B) = f(A) \cup f(B)$,

b) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$,

c) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$,

d) $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$,

e) $A \subset f^{-1}(f(A))$,

f) $C \supset f(f^{-1}(C))$.

g) $f(A \cap B) \subset f(A) \cap f(B)$. *Give an example to show that the converse is not true.*

Exercise 1.10. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (2x, 2y)$ *and* $A = \{(x, y) \in \mathbb{R}^2 | (x-4)^2 + y^2 = 4\}$. *Find* $f(A), f^{-1}(A)$.

Exercise 1.11. *Which of the following maps are injective, surjective, bijective?*

a) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3 - 2x$,

b) $f : (-\infty, 0] \to [4, +\infty), f(x) = x^2 + 4$,

c) $f : (1, +\infty) \to (-1, +\infty), f(x) = x^2 - 2x$,

d) $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{3\}, f(x) = \frac{3x+1}{x-1}$,

e) $f : [4, 9] \to [21, 96], f(x) = x^2 + 2x - 3$,

f) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x - 2|x|$,

g) $f : (-1, 1) \to \mathbb{R}, f(x) = \ln \frac{1+x}{1-x}$,

h) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \frac{1}{x}$,

Exercise 1.12. *Let* $f(x) = -x^2 - 2x + 3$.

a) *Find* a *such that* $f : \mathbb{R} \to (-\infty, a]$ *is surjective.*

b) *Find* b *such that* $f : [b, +\infty) \to (-\infty, 3]$ *is injective.*

Exercise 1.13. *Let* X, Y, Z *be sets and* $f : X \to Y, g : Y \to Z$ *be maps. Prove that*

a) *If* f *and* g *are injective, then* $g \circ f$ *is injective.*

b) *If* f *and* g *are surjective, then* $g \circ f$ *is surjective.*

c) *If* f *and* g *are bijective, then* $g \circ f$ *is bijective.*

d) *If* f *is surjective and* $g \circ f$ *is injective, then* g *is injective.*

e) *Give an example to show that $g \circ f$ is injective, but g is not.*

f) *If g is is injective and $g \circ f$ is surjective, then f is surjective.*

g) *Give an example to show that $g \circ f$ is surjective but f is not.*

1.4 Algebraic Structures

Exercise 1.14. *Consider the commutativity, associativity of the following binary operator $*$ on $\mathbb{R}$ and $\circ$ on $\mathbb{R}^2$ and find the identity element, the inverse element.*

a) $x * y := xy + 1$,

b) $x * y := \frac{1}{2}xy$,

c) $(x_1, x_2) \circ (y_1, y_2) := \left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}\right)$.

Exercise 1.15. *Let X, Y be sets, $* : Y \times Y \to Y$ is a commutative, associative binary operator with identity element e and $f : X \to Y$ be an bijection. Consider the binary operator on X as follow: $x_1 \circ x_2 = f^{-1}(f(x_1) * f(x_2))$. Prove that $\circ$ is a commutative, associative binary operator with identity element.*

Exercise 1.16. *Which of the following sets is a group?*

a) $(m\mathbb{Z}, +)$, *where* $m\mathbb{Z} = \{n \in \mathbb{Z} | n$ *is divisible by* $m\}$.

b) $(2^{\mathbb{Z}}, \times)$, *where* $2^{\mathbb{Z}} = \{2^n, n \in \mathbb{Z}\}$.

c) $(P_n(X), +)$, *where* $P_n(X)$ *is the all real polynomials of degree not exceeding* n.

Exercise 1.17. *Let X be arbitrary set and consider the binary operator $x * y = x, \forall x, y \in X$. Prove that $(X, *)$ is a semigroup.*

Exercise 1.18. *Lett X be a semigroup with the multiplication.*

a) *Prove that if $ab = ba \forall a, b \in X$, then $(ab)^n = a^n b^n, n > 1$.*

b) *Let $a, b \in X$ such that $(ab)^2 = a^2b^2$. Can we conclude that $ab = ba$?*

Exercise 1.19. *Prove that*

a) $(\mathbb{Q}, +, \times)$ *is a field.*

b) *The* $(\mathbb{Z}, +, \times)$ *is a ring but not a field.*

Exercise 1.20. *Which of the following sets is a ring? a field?*

a) $X = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$,

b) $Y = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$

where the addition and multiplication are the common addition and multiplication

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2},$$
$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

1.5 Complex Numbers

Exercise 1.21. *Find the canonical forms of the following complex numbers.*

a) $(1+i\sqrt{3})^9$,

b) $\frac{(1+i)^{21}}{(1-i)^{13}}$,

c) $(2+i\sqrt{12})^5(\sqrt{3}-i)^{11}$,

d) $\frac{2+3i}{5+4i}$.

Exercise 1.22. *Solve the following equations in the field of complex numbers.*

a) $z^2+z+1=0$,

b) $z^2+2iz-5=0$,

c) $z^4-3iz^2+4=0$,

d) $z^6-7z^3-8=0$,

e) $\frac{(z+i)^4}{(z-i)^4}=1$,

f) $z^8(\sqrt{3}+i)=1-i$,

g) $\overline{z^7}=\frac{1}{z^3}$,

h) $z^4=z+\overline{z}$.

2 Matrices

2.1 Matrix Operations

Exercise 2.1. *Let* $A=\begin{bmatrix}1 & -3 & 2\\ 2 & 1 & -1\\ 0 & 3 & -2\end{bmatrix}, B=\begin{bmatrix}2 & 1 & 1\\ -2 & 3 & 0\\ 1 & 2 & 4\end{bmatrix}, C=\begin{bmatrix}-1 & 2 & 1\\ 3 & 4 & 1\\ 2 & 0 & 2\end{bmatrix}$.
Compute $A+BC, A^TB-C, A(BC), (A+3B)(B-C)$.

Exercise 2.2. *Let* $A=\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}, B=\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}$.

a) Compute $F=A^2-3A$,

b) Find the matrix X *satisfies* $(A^2+5I)X=B^T(3A-A^2)$.

Exercise 2.3. *Find the matrix* X *such that:*

a) $\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}3 & 0\\ 2 & 1\end{bmatrix}+2X=\begin{bmatrix}1 & -2\\ 5 & 7\end{bmatrix}$.

b) $\frac{1}{2}X-\begin{bmatrix}1 & -3 & 2\\ 3 & -4 & 1\\ 2 & -5 & 3\end{bmatrix}\begin{bmatrix}2 & 5 & 6\\ 1 & 2 & 5\\ 1 & 3 & 2\end{bmatrix}=\begin{bmatrix}0 & -6 & 6\\ -2 & 9 & 2\\ -4 & -8 & 6\end{bmatrix}$.

Exercise 2.4. *Find all* 2×2 *matrices such that*

a) $A^2=I$.

b) $A^2=0$.

Exercise 2.5. *Compute* A^n, *where*

a) $A = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}$

b) $A = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$

Exercise 2.6. *Show that the linear transformation $y = Ax$ with matrix*

$$A = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}, \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is a counterclockwise rotation in the Cartesian x_1x_2-coordinate system in the plane about the origin, where a is angle of rotation.

2.2 Determinants

Exercise 2.7. *Compute the following determinants*

a) $A = \begin{vmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{vmatrix}$

c) $C = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}$

b) $B = \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$

d) $D = \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix}.$

Exercise 2.8. *Prove that if A is a skew-symmetric (or antisymmetric or antimetric) matrix of order n, where n is odd, then* $\det(A) = 0$.

Exercise 2.9. *Let A be a square matrix of order* 2017*. Prove that*

$$\det(A - A^T)^{2017} = 2017(\det A - \det A^T).$$

Exercise 2.10. *Let A, B be square matrices of order* 2017 *satisfy $AB + B^T A^T = 0$. Prove that* $\det A = 0$ *or* $\det B = 0$.

Exercise 2.11. *Let A, B be real square matrices of the same order. Prove that*

$$\det(A^2 + B^2) \geq 0.$$

Exercise 2.12. *Let $A = [a_{ij}]_{n\times n}$ be a complex matrix such that $a_{ij} = -\overline{a_{ji}}$. Prove that* $\det(A)$ *is a real number.*

Exercise 2.13. *Let A be an $n \times n$ square matrix satisfies $A^2 + 2017I = 0$. Prove that* $\det A > 0$.

Exercise 2.14. *Prove that if A is a real square matrix satisfies $A^3 = A + I$, then* $\det A > 0$.

2.3 Rank of matrices

Exercise 2.15. *Find the rank of the following matrices*

a) $A = \begin{bmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{bmatrix}$.

b) $B = \begin{bmatrix} 4 & 3 & -5 & 2 & 3 \\ 8 & 6 & -7 & 4 & 2 \\ 4 & 3 & -8 & 2 & 7 \\ 4 & 3 & 1 & 2 & -5 \\ 8 & 6 & -1 & 4 & -6 \end{bmatrix}$.

2.4 Inverse of a Matrix

Exercise 2.16. *Find the inverses of the matrices*

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 9 & 8 & 7 \end{bmatrix}$,

c) $C = \begin{bmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -a & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Exercise 2.17. *Let A, B be square matrices of the same order satisfy $AB = A+B$. Prove that $AB = BA$.*

2.5 Systems of Linear Equations

Exercise 2.18. *Solve the following systems of linear equations*

a) $\begin{cases} x_1 - 2x_2 + x_3 &= 4 \\ 2x_1 + x_2 - x_3 &= 0 \\ -x_1 + x_2 + x_3 &= -1 \end{cases}$

b) $\begin{cases} 3x_1 - 5x_2 - 7x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 2 \\ -2x_1 + x_2 + 5x_3 &= 2. \end{cases}$

c) $\begin{cases} ax_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + ax_2 + x_3 + x_4 = a \\ x_1 + x_2 + ax_3 + x_4 = a^2 \end{cases}$

d) $\begin{cases} (2-a)x_1 + x_2 + x_3 = 0 \\ x_1 + (2-a)x_2 + x_3 = 0 \\ x_1 + x_2 + (2-a)x_3 = 0 \end{cases}$

e) $\begin{cases} 3x_1 - 5x_2 + 2x_3 + 4x_4 &= 2 \\ 7x_1 - 4x_2 + x_3 + 3x_4 &= 5 \\ 5x_1 + 7x_2 - 4x_3 - 6x_4 &= 3 \end{cases}$

f) $\begin{cases} 3x_1 - x_2 + 3x_3 &= 1 \\ -4x_1 + 2x_2 + x_3 &= 3 \\ -2x_1 + x_2 + 4x_3 &= 4 \\ 10x_1 - 5x_2 - 6x_3 &= -10. \end{cases}$

Exercise 2.19. *Let A be a $m \times n$ matrix. Prove that the dimension of the set of solutions of the homogeneous system $Ax = 0$ is*

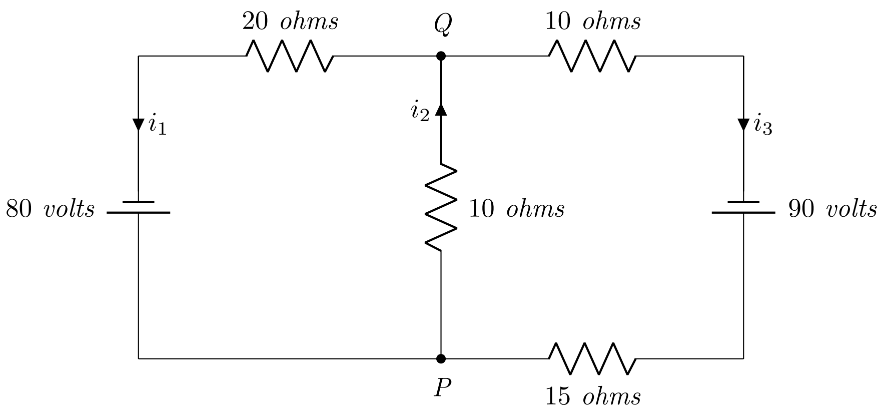
$$n - \operatorname{rank} A.$$

Exercise 2.20. *Find the dimension and a basis of the set of solutions of the homogeneous system*

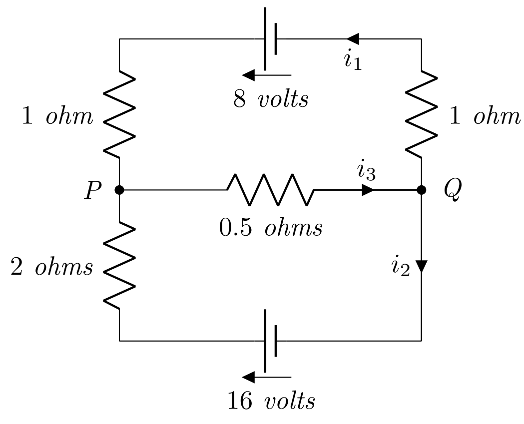
$$\begin{cases} x_1 - x_2 + 2x_3 + 2x_4 - x_5 = 0 \\ x_1 - 2x_2 + 3x_3 - x_4 + 5x_5 = 0 \\ 2x_1 + x_2 + x_3 + x_4 + 3x_5 = 0 \\ 3x_1 - x_2 - 2x_3 - x_4 + x_5 = 0 \end{cases}$$

Exercise 2.21. *Using Kirchhoff's laws and Gauss elimination method, find the currents in the following networks.*

a)



b)



3 Vector Space

3.1 Basic concepts

Exercise 3.1. *Determine whether V is a vector space?*

a) $V = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$, *the operations are defined as*

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$$

$$k(x, y, z) = (|k|\, x, |k|\, y, |k|\, z), \quad k \in \mathbb{R}.$$

b) $V = \{x = (x_1, x_2) \,|x_1 > 0, x_2 > 0\} \subset \mathbb{R}^2$, *the operations are defined as*

$$(x_1, x_2) + (y_1, y_2) = (x_1y_1, x_2y_2)$$

$$k(x_1, x_2) = (x_1^k, x_2^k), \quad k \in \mathbb{R}.$$

3.2 Subspaces

Exercise 3.2. *Let* V_1, V_2 *be linear subspaces of* V *and* $V_1 + V_2 := \{x_1 + x_2 \,|x_1 \in V_1, x_2 \in V_2\}$. *Prove that:*

a) $V_1 \cap V_2$ *is a linear subspace of* V.

b) $V_1 + V_2$ *is a linear subspace of* V.

Exercise 3.3. *Let* V_1, V_2 *be subspaces of* V. *Assume that*

i) $\{v_1, v_2, \cdots, v_m\}$ *be a generator of* V_1, *and*

ii) $\{u_1, u_2, \cdots, u_n\}$ *be a generator of* V_2.

Prove that $\{v_1, \cdots, v_m, u_1, u_2, \cdots, u_n\}$ *is a generator of* $V_1 + V_2$.

Exercise 3.4. *Prove that* $V = V_1 \oplus V_2$ [3] *if and only if each* $v \in V$ *has a unique representation*

$$v = v_1 + v_2, \, (v_1 \in V_1, v_2 \in V_2).$$

Exercise 3.5. *Express* $v = (1, 2, 5) \in \mathbb{R}^3$ *as a linear combination of the vectors* u_1, u_2, u_3 *where* $u_1 = (1, -3, 2), u_2 = (2, -4, -1), u_3 = (1, -5, 7)$.

Exercise 3.6. *Express the polynomial* $v = t^2 + 4t - 3$ *over* $\mathbb{R}$ *as a linear combination of the polynomials* $p_1 = t^2 - 2t + 5, p_2 = 2t^2 - 3t, p_3 = t + 3$.

3.3 Linear Dependence and Independence

Exercise 3.7. *Determine whether the following vectors are linearly dependent or linearly independent.*

a) $v_1 = (1, 2, 3), v_2 = (3, 6, 7)$.

b) $v_1 = (4, -2, 6)\,, v_2 = (-6, 3, -9)$.

c) $v_1 = (2, 3, -1), v_2 = (3, -1, 5), v_3 = (-1, 3, -4)$.

d) $u = t^3 + 4t^2 - 2t + 3, v = t^3 + 6t^2 - t + 4, w = 3t^3 + 8t^2 - 8t + 7$.

[3] We say that V is a direct sum of V_1 and V_2 and write $V = V_1 \oplus V_2$ if $V_1 + V_2 = V, V_1 \cap V_2 = \{0\}$.

3.4 Bases and dimension

Exercise 3.8. *Let* $v_1 = (2, 0, 1, 3, -1), v_2 = (1, 1, 0, -1, 1), v_3 = (0, -2, 1, 5, -3), v_4 = (1, -3, 2, 9, -5)$.

a) *Find the dimension and a basis of* $\operatorname{span}(v_1, v_2, v_3, v_4)$.

b) *Let* $V_1 = \operatorname{span}(v_1, v_2), V_2 = \operatorname{span}(v_3, v_4)$. *Find the dimension and a basis of* $V_1 + V_2, V_1 \cap V_2$.

Exercise 3.9. *Let* $u_1 = (1, 3, -2, 1), u_2 = (-2, 3, 1, 1), u_3 = (2, 1, 0, 1), u = (1, -1, -3, m)$. *Find* m *such that* $u \in \operatorname{span}(u_1, u_2, u_3)$.

Exercise 3.10. *Let* V_1, V_2 *be finite dimensional spaces. Then*

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Exercise 3.11. *Let*

$$v_1 = 1 + x^2 + x^3, v_2 = x - x^2 + 2x^3, v_3 = 2 + x + 3x^3, v_4 = -1 + x - x^2 + 2x^3$$

be vectors on $P_3[x]$.

a) *Find the rank of* $\{v_1, v_2, v_3, v_4\}$.

b) *Find the dimension and a basis of* $\operatorname{span}(v_1, v_2, v_3, v_4)$.

Exercise 3.12. *Let* $v_1 = 1, v_2 = 1 + x, v_3 = x + x^2, v_4 = x^2 + x^3$ *be vectors on* $P_3[x]$.

a) *Prove that* $\mathbb{B} = \{v_1, v_2, v_3, v_4\}$ *is a basis of* $P_3[x]$.

b) *Find the coordinates of* $v = 2 + 3x - x^2 + 2x^3$ *with respect to this basis.*

c) *Find the coordinates of* $v = a_0 + a_1x + a_2x^2 + a_3x^3$ *with respect to this basis.*

Exercise 3.13. *Let* $E = \left\{1, x, x^2, x^3\right\}$ *be the standard basis of* $P_3[x]$ *and* $B = \{1, 1 + x, (1 + x)^2, (1 + x)^3\}$.

a) *Prove that* B *is a basis of* $P_3[x]$.

b) *Find the transformation matrix from* E *to* B *and* B *to* E.

c) *Find the coordinates of* $v = 2 + 2x - x^2 + 3x^3$ *with respect to the basis* B.

4 Linear Transformation

4.1 Kernel, Image

Exercise 4.1. *Let* $T : V \to W$ *be a linear map. Prove that*

a) $\operatorname{Ker}(T)$ *is a subspace of* V.

b) $\operatorname{Im}(T)$ *is a subspace of* W.

c) f *is injective if and only if* $\operatorname{Ker} f = \{0\}$.

d) f *is surjective if and only if* $\operatorname{Im} f = W$.

e) $\dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T) = \dim V$ *(the rank-nullity theorem).*

Exercise 4.2. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear map. Prove that the following are equivalent*

a) f is injective. b) f is surjective. c) f is bijective.

4.2 Matrices and Linear Mappings

Exercise 4.3. *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a function defined by $f(x_1, x_2, x_3) = (3x_1 + x_2 - x_3, 2x_1 + x_3)$.*

a) *Prove that f is a linear transformation.*

b) *Find the matrix of f with respect to the standard bases.*

c) *Find a basis of* $\operatorname{Ker} f$.

Exercise 4.4. *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be be a function defined by*

$$f(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 - x_2 + x_3, -x_1 + x_2 + x_3).$$

Find the matrix of f with respect to the basis $B = \{v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1)\}$.

Exercise 4.5. *Let the function $f :\ P_2[x] \to P_4[x]$ be a map defined as: $f(p) = p + x^2 p, \forall p \in P_2$.*

a) *Prove that f is a linear map.*

b) *Find the matrix of f with respect to the bases $E_1 = \{1, x, x^2\}$ of $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ of $P_4[x]$.*

c) *Find the matrix of f with respect to the bases $E_1' = \{1 + x, 2x, 1 + x^2\}$ of $P_2[x]$ and $E_2 = \{1, x, x^2, x^3, x^4\}$ of $P_4[x]$.*

Exercise 4.6. *Let $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 6 & -2 & 4 \end{bmatrix}$ be the matrix of the linear transformation $f :\ P_2[x] \to P_2[x]$ with respect to the basis $B = \{v_1, v_2, v_3\}$, where*

$$v_1 = 3x + 3x^2, v_2 = -1 + 3x + 2x^2, v_3 = 3 + 7x + 2x^2.$$

a) *Find $f(v_1), f(v_2), f(v_3)$.* b) *Find $f(1 + x^2)$.*

Exercise 4.7. *Let A be an $m\times n$ matrix and B be an $n\times p$ matrix. Prove that* $\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

Exercise 4.8. *Let A, B be $m \times n$ matrices. Prove that* $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

4.3 Eigenvalues and Eigenvectors

Exercise 4.9. *Find eigenvalues and a basis for each eigenspace of the following matrices:*

a) $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ b) $B = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ c) $C = \begin{bmatrix} 2 & -1 & 0 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

d) $D = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix}$ e) $E = \begin{bmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{bmatrix}$ f) $F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Exercise 4.10. *Let $f : P_2[x] \to P_2[x]$ be a linear transformation defined by*

$$f(a_0 + a_1x + a_2x^2) = (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2$$

Find eigenvalues and eigenvectors of f.

4.4 Diagonalizations

Exercise 4.11. *Diagonalization the following matrices*

a) $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

e) $E = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$.

f) $F = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

g) $G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Exercise 4.12. *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a function defined as*

$$f(x_1, x_2, x_3) = (2x_1 - x_2 - x_3, x_1 - x_2, -x_1 + x_2 + 2x_3).$$

Diagonalization the transformation f.

Exercise 4.13. *Find a basis of $\mathbb{R}^3$ such that the matrix of $f : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to this basis is a diagonal matrix, where*

$$f(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + 2x_2 + x_3, x_1 + x_2 + 2x_3).$$

Exercise 4.14. *Prove that if A is an n-by-n matrix with real or complex entries and if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A (with multiplicities), then*

a) *The eigenvalues of A^{-1} (assume that A is invertible) are $\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}$ (with multiplicities),*

b) *The eigenvalues of A^2 are $\lambda_1^2, \cdots, \lambda_n^2$ (with multiplicities),*

c) *The eigenvalues of A^p are $\lambda_1^p, \cdots, \lambda_n^p$ (with multiplicities), where $1 \leq p \in \mathbb{N}$.*

Exercise 4.15. *Let* $A = \begin{pmatrix} 4 & -12 \\ -12 & 11 \end{pmatrix}$. *Compute* A^n.

Exercise 4.16. *The Fibonacci sequence is defined by:* $F_0 = 0, F_1 = 1$ *and* $F_{n+1} = F_n + F_{n-1}$ *if* $n \geq 1$. *Prove the following Cauchy-Binet formula*

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

5 Quadratic Form- Euclidean Space

5.1 Inner product spaces

Exercise 5.1. *Determine if the following are inner products on* $P_3[x]$*?*

a) $p \cdot q = p(0)q(0) + p(1)q(1) + p(2)q(2)$

b) $p \cdot q = p(0)q(0) + p(1)q(1) + p(2)q(2) + p(3)q(3)$

c) $p \cdot q = \int\limits_{-1}^{1} p(x)q(x)dx$.

In case it is, compute $p \cdot q$*, where* $p = 2 - 3x + 5x^2 - x^3, q = 4 + x - 3x^2 + 2x^3$.

Exercise 5.2. *Let* $\mathbb{B} = \{e_1, e_2, ..., e_n\}$ *be a basis of an* n*-dimensional vector space* V*. If* $u, v \in V$*, then let* $\begin{cases} [u]_{\mathbb{B}} = (a_1, a_2, \dots, a_n), \\ [v]_{\mathbb{B}} = (b_1, b_2, \dots, b_n) \end{cases}$ *be the coordinate columns of* u *and* v*. We define*

$$u \cdot v = a_1b_1 + a_2b_2 + \dots + a_nb_n.$$

a) Prove that this is an inner product on V*.*

b) Apply for $V = \mathbb{R}^3$*, where* $e_1 = (1,0,1), e_2 = (1,1,-1), e_3 = (0,1,1), u = (2,-1,-2), v = (2,0,5)$ *and compute* $u \cdot v$*.*

c) Apply for $V = P_2[x]$*, where* $\mathbb{B} = \{1, x, x^2\}, u = 2 + 3x^2, v = 6 - 3x - 3x^2$ *and compute* $u \cdot v$*.*

d) Apply for $V = P_2[x]$*, where* $\mathbb{B} = \{1 + x, 2x, x - x^2\}, u = 2 + 3x^2, v = 6 - 3x - 3x^2$ *and compute* $u \cdot v$*.*

5.2 Length (Norm) of vectors

Exercise 5.3. *Let* V *be an Euclidean space. Prove that for all* $u, v \in V$*,*

$$\begin{cases} \|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right), \\ u \perp v \Leftrightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2. \end{cases}$$

5.3 Orthogonality

Exercise 5.4. *Apply the Gram-Schmidt process to the vectors* $\{u_1, u_2, u_3, u_4\}$*, where*

$$u_1 = (1,1,1,1)\,, u_2 = (0,1,1,1)\,, u_3 = (0,0,1,1)\,, u_4 = (0,0,0,1)\,.$$

Exercise 5.5. *Let the inner product on* $P_2[x]$ *be defined as* $p \cdot q = \int\limits_{-1}^{1} p(x)q(x)dx$*, where* $p, q \in P_2[x]$*.*

a) *Apply the Gram-Schmidt process to the basis* $\mathbb{B} = \left\{1, x, x^2\right\}$ *to get an orthonormal basis* $\mathcal{A}$*.*

b) *Find the change of basis matrix for converting the basis* $\mathbb{B}$ *to the basis* $\mathcal{A}$

c) *Find the coordinate vector* $[r]_{\mathcal{A}}$ *if* $r = 2 - 3x + 3x^2$

Exercise 5.6. *Let*

$$v_1 = (1,1,0,0,0)\,, v_2 = (0,1,-1,2,1)\,, v_3 = (2,3,-1,2,1)$$

and $V = \left\{x \in \mathbb{R}^5 \middle|\, x \perp v_i, i = 1,2,3\right\}$

a) *Prove that* V *is a subspace of* $\mathbb{R}^5$*.*

b) *Find* $\dim V$*.*

5.4 Projection

Exercise 5.7. *Let* $v_1 = (6,3,-3,6)\,, v_2 = (5,1,-3,1)$*. Find the projection of* $v = (1,2,3,4)$ *onto* $U = \operatorname{span}(v_1, v_2)$*.*

Exercise 5.8. *Find the projection of* u *on* v*, where*

a) $u = (1,3,-2,4)$ *onto* $v = (2,-2,4,5)\,,$

b) $u = (4,1,2,3,-3)\,, v = (-1,-2,5,1,4)$*.*

5.5 Orthogonal diagonalization

Exercise 5.9. *Orthogonal diagonalization of the following symmetric matrices*

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

b) $B = \begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

5.6 Quadratic forms

Exercise 5.10. *Determine the definiteness of the following quadratic form on* $\mathbb{R}^3$*.*

a) $\omega_1(x_1,x_2,x_3) = x_1^2+5x_2^2-4x_3^2+2x_1x_2-4x_1x_3,$

b) $\omega_2(x_1,x_2,x_3) = x_1x_2 + 4x_1x_3 + x_2x_3,$

c) $2x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3,$

d) $\omega_3 = 5x^2 + 2y^2 + z^2 - 6xy + 2xz - 2yz.$

Exercise 5.11. *Find a such that the following quadratic forms are positive definite:*

a) $5x_1^2 + x_2^2 + ax_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3.$

b) $2x_1^2 + x_2^2 + 3x_3^2 + 2ax_1x_2 + 2x_1x_3.$

c) $x_1^2 + x_2^2 + 5x_3^2 + 2ax_1x_2 - 2x_1x_3 + 4x_2x_3.$

Exercise 5.12. *Lagrange reduction of quadratic forms to canonical (diagonal) form*

a) $\omega_1(x_1,x_2,x_3) = x_1^2+5x_2^2-4x_3^2+2x_1x_2-4x_1x_3,$

b) $\omega_2(x_1,x_2,x_3) = x_1x_2 + 4x_1x_3 + x_2x_3,$

c) $\omega_3 = 5x^2 + 2y^2 + z^2 - 6xy + 2xz - 2yz.$

d) $5x_1^2 + x_2^2 + ax_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3.$

e) $2x_1^2 + x_2^2 + 3x_3^2 + 2ax_1x_2 + 2x_1x_3.$

f) $x_1^2 + x_2^2 + 5x_3^2 + 2ax_1x_2 - 2x_1x_3 + 4x_2x_3.$

Exercise 5.13. *Orthogonal diagonalization of the following quadratic forms*

a) $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2$

b) $7x_1^2 - 7x_2^2 + 48x_1x_2$

c) $2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_2x_3$

d) $5x_1^2 + x_2^2 + x_3^2 - 6x_1x_2 + 2x_1x_3 - 2x_2x_3.$

5.7 Quadratic lines and surfaces

Exercise 5.14. *Classify the following quadratic curves*

a) $2x^2 - 4xy - y^2 + 8 = 0$

b) $x^2 + 2xy + y^2 + 8x + y = 0$

c) $11x^2 + 24xy + 4y^2 - 15 = 0$

d) $2x^2 + 4xy + 5y^2 = 24$

e) $x^2 + xy - y^2 = 18$

f) $x^2 - 8xy + 10y^2 = 10.$

Exercise 5.15. *Classify the following quadratic surfaces*

a) $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 = 4,$

b) $5x^2 + 2y^2 + z^2 - 6xy + 2xz - 2yz = 1,$

c) $2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 = 16,$

d) $7x^2 - 7y^2 + 24xy + 50x - 100y - 175 = 0,$

e) $7x^2 + 7y^2 + 10z^2 - 2xy - 4xz + 4yz - 12x + 12y + 60z = 24,$

f) $2xy + 2yz + 2xz - 6x - 6y - 4z = 0.$

Exercise 5.16. *Let* $Q(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3.$

a) Find $\max\limits_{x_1^2+x_2^2+x_3^2=1} Q(x_1, x_2, x_3), \min\limits_{x_1^2+x_2^2+x_3^2=1} Q(x_1, x_2, x_3).$

b) Find $\max\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3), \min\limits_{x_1^2+x_2^2+x_3^2=16} Q(x_1, x_2, x_3).$