# Chapter 1: Applications of differential calculus in geometry

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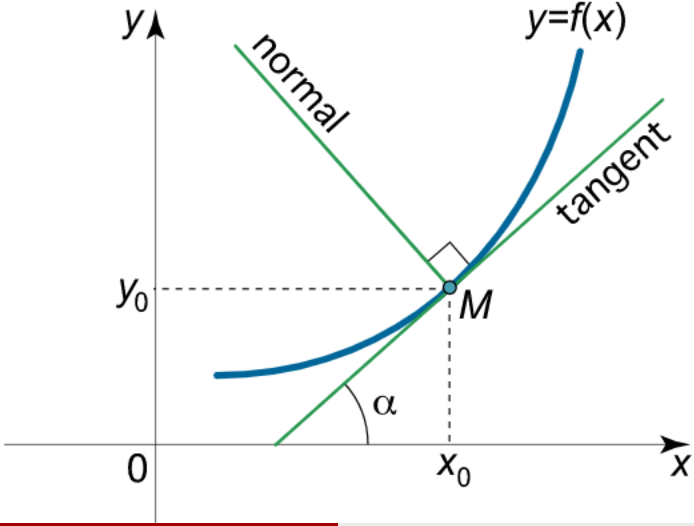
# Normal line and tangent line



# 1.1.1. Normal line and tangent line

Problem:

In the coordinate plane $Oxy$, given a curve $L$ and a point $M \in L$. Find the equations of the tangent line and the normal line of $L$ at $M$.

**Remark:** The curve $L$ is defined by $y = f(x)$. Let $M(x_0, y_0) \in L$.

• The equation of the tangent line to $L$ at $M$ is

$$\boxed{y - y_0 - f'(x_0)(x - x_0) = 0.}$$

• The equation of the normal line to $L$ at $M$ is

$$\boxed{f'(x_0)(y - y_0) + x - x0 = 0.}$$

# The curve given by $F(x, y) = 0$

### Non-singular point (điểm chính quy)

The curve $L$ is given by $F(x, y) = 0$. The point $M(x_0, y_0) \in L$ is called a non-singular point

$$\boxed{(F'_x(M))^2 + (F'_y(M))^2 \neq 0.}$$

A point that is not non-singular is called singular (điểm kỳ dị).

- The equation of the tangent line to $L$ at $M(x_0, y_0)$ is

$$\boxed{F'_x(x_0, y_0)(x - x_0) + F'_y(x_0, y_0)(y - y_0) = 0.}$$

- The equation of the normal line to $L$ at $M$ is

$$\boxed{\frac{x - x_0}{F'_x(x_0, y_0)} = \frac{y - y_0}{F'_y(x_0, y_0)}.}$$

# Parametrized curve $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

## Non-singular point

A point $M(x(t_0), y(t_0)) \in L$ is called non-singular (điểm chính quy) if at least one of $x'(t_0), y'(t_0)$ is not 0.

- The equation of the tangent line at $M(x(t_0), y(t_0))$ is

$$\boxed{\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)}.}$$

- The equation of the normal line at $M(x(t_0), y(t_0))$ is

$$\boxed{x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) = 0.}$$

## Example (GK20192)

Find the equations of the tangent line and normal line to $x^3 + y^3 = 9xy$ at $(2, 4)$.

Solution:

- $x^3 + y^3 = 9xy \Leftrightarrow F(x, y) := x^3 + y^3 - 9xy$.
- $F'_x(x, y) = 3x^2 - 9y$ và $F'_y(x, y) = 3y^2 - 9x$.
- At $(2, 4)$:
  $F'_x(2, 4) = 3 \cdot 2^2 - 9 \cdot 4 = -24$, $F'_y(2, 4) = 3 \cdot 4^2 - 9 \cdot 4 = 30$.
- The equation of the tangent line at (2,4):

$$-24(x-2) + 30(y-4) = 0 \text{ or } 4x - 5y + 12 = 0.$$

- The equation of the normal line at (2,4):

$$\frac{x-2}{-24} = \frac{y-4}{30} \text{ or } 5x + 4y - 26 = 0.$$

### Example (GK20181)

Find the equations of the tangent line and the normal line of $x = (t^2 - 1)e^{2t}$, $y = (t^2 + 1)e^{3t}$ at $t = 0$.

Solution:

- At $t = 0$, we have a point $M(-1, 1)$.
- $x' = 2te^{2t} + 2(t^2 - 1)e^{2t}$, $y' = 2te^{3t} + 3(t^2 + 1)e^{3t}$.
- At $t = 0$: $x' = -2$, $y' = 3$.
- The equation of the tangent line:

$$\frac{x+1}{-2} = \frac{y-1}{3} \text{ or } 3x + 2y + 1 = 0.$$

- The equation of the normal line:

$$-2(x+1) + 3(y-1) = 0 \text{ or } 2x - 3y + 5 = 0$$

# 1.1.2. Curvature

- The curvature measures how fast a curve is changing direction at a given (regular) point.
- The curvature of a curve at any point is always non-negative.
- The curvature of the curve $L$ at $M$, is denoted by $C(M)$.

**Example**

The curvature of any line is 0.

**Example**

The curvature of a circle with radius $R$ at any point is $1/R$.

# Curvature: Formulas

- The curve $L$ is given by $y = f(x)$. Let $M(x_0, y_0)$ in $L$. The curvature at $M$ is given by

$$\boxed{C(M) = \frac{|y''|}{(1 + y'^2)^{3/2}}.}$$

- Let $L$ be a curve given by $x = x(t)$, $y = y(t)$. Let $M = (x(t_0), y(t_0))$ be a point in $L$. Then

$$\boxed{C(M) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \frac{|x'(t_0)y''(t_0) - y'(t_0)x''(t_0)|}{(x'(t_0)^2 + y'(t_0)^2)^{3/2}}.}$$

- Let $L$ be a curve defined by $r = f(\varphi)$ (in cylindrical coordinates). Let $M$ be a point in $L$. Then

$$\boxed{C(M) = \frac{|r^2 + 2r'^2 - rr''|}{(r^2 + r'^2)^{3/2}}.}$$

### Example (GK20201)

Find the curvature of $y = x^3 + x$ at $M(1.2)$.

Solution:

- $y' = 3x^2 + 1,\ y'' = 6x.$
- At $M(1,2)$: $y'(1) = 3 \cdot 1^2 + 1 = 4,\ y''(1) = 6.$
- The curvature at $M(1,2)$ is

$$C(M) = \frac{|y''(1)|}{(1 + y'(1)^2)^{3/2}} = \frac{|6|}{(1 + 4^2)^{3/2}} = \frac{6}{17\sqrt{17}}.$$

### Example (GK20192)

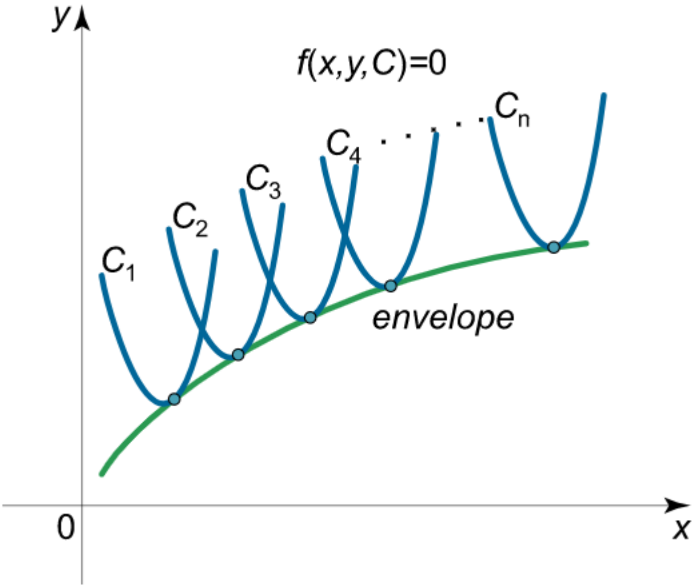
Find the curvature of $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ when $t = -\pi/2$.

Solution:

- $x'(t) = 2(1 - \cos t)$, $x''(t) = 2\sin t$, $y'(t) = 2\sin t$, $y''(t) = 2\cos t$.
- $t = -\pi/2$: $x'(-\pi/2) = 2(1 - \cos(-\pi/2)) = 2$, $x''(t) = 2\sin(-\pi/2) = -2$, $y'(-\pi/2) = 2\sin(-\pi/2) = -2$, $y''(-\pi/2) = 2\cos(-\pi/2) = 0$.
- Then
$$C(M) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}} = \frac{|2 \cdot 0 - (-2)(-2)|}{(2^2 + (-2)^2)^{3/2}} = \frac{4}{8\sqrt{8}} = \frac{1}{4\sqrt{2}}.$$

# Some exercises

- (GK20213) Find the curvature of $(x-2)^2+(y-1)^2=5$ at $M(4,2)$.
- (GK20212) Find the curvature of the curve defined by $z=x^2+y^2$, $z=2x$ at $A(1,1,2)$.
- (CK20212) Let $(E)$ be the curve defined by $\dfrac{x^2}{16}+\dfrac{y^2}{36}=1$. Find the curvature of $(E)$ at $A(4,0)$.
- (GK20192) Find the curvature of $y=e^{2x}$ at $A(0;1)$.
- (GK20182) Find the curvature of $x=t^2$, $y=t\ln t$, $t>0$ when $t=e$.


y
f(x,y,C)=0
$C_1$
$C_2$
$C_3$
$C_4$
$C_n$
envelope
0
x

# 1.1.3. Envelope of a family of curves

Given a family of curves $F(x, y, c) = 0$, where $c$ is a parameter. The envelope of this family of curves is a curve $E$ such that

- every curve in the family touches tangentially to $E$;
- and at each point of $E$, $E$ touches tangentially to one of the curves of the family.

### Example

Consider the circle $\mathcal{C} : (x - c)^2 + y^2 = R^2$, where $c$ is a parameter. The envelope of this family has two lines $y = \pm R$.

# Rules to find the envelope

**Theorem**

Let $F(x, y, c) = 0$ be a family of curve, where $c$ is a parameter. If any curve in the family has no singular points then the parametric equations of the envelope are defined by the system of equations

$$\begin{cases} F(x, y, c) = 0 \\ F'_c(x, y, c) = 0. \end{cases}$$

Remark: If any curve has a singular point, we must exclude the singular points.

## Example (GK20201)

Find the envelope of $(\Gamma_c)$: $2x\cos c + y\sin c = 1$.

Solution:

- $2x\cos c + y\sin c = 1 \Leftrightarrow F(x,y,c) := 2x\cos c + y\sin c - 1 = 0$.
- $F'_x = 2\cos c$, $F'_y = \sin c$. The systems $F'_x = F'_y = 0$ has no solutions. The family $(\Gamma_c)$ has no singular points.
-
$$\begin{cases} F(x,y,c) = 0 \\ F'_c(x,y,c) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x\cos c + y\sin c = 1 \\ -2x\sin c + y\cos c = 0 \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{1}{2}\cos c \\ y = \sin c \end{cases}$$

- The envelope is the ellipse
$$4x^2 + y^2 = 1.$$

# Some exercises

- (GK20213) Find the envelope of $\dfrac{x}{c^4} + \dfrac{y}{(1-c)^4} = 1$, where $c$ is a parameter.
- (GK20212) Find the envelope of $y = 2cx^2 + c^2 + 1$, where $c \leq 0$ is a parameter.
- (GK20192) Find the envelope of $x^2 + y^2 - 4yc + 2c^2 = 0$, where $c \neq 0$ is a parameter.
- (GK20192) Find the envelope of $y = 4cx^3 + c^4$, where $c \neq 0$ is a parameter.
- (GK20182) Find the envelope of $(x+c)^2 + (y-c)^2 = 2$, where $c \neq 0$ is a parameter.
- (GK20181) Find the envelope of $x = 2cy^2 + 3c^2$, where $c \neq 0$ is a parameter.

# 1.2.1. Vector functions

Let $I$ be an interval in $\mathbb{R}$.

- A map $\vec{r}\colon I \to \mathbb{R}^n, t \mapsto \vec{r}(t)$ is called a vector function of $t$ defined in $I$.
- Let $n = 3$ and write $\vec{r}(t) = (x(t), y(t), z(t)) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$. The set of all points $M(x(t), y(t), z(t))$ with $t$ in $I$ is called the graph of the function $r$. We also say that a space curve $L$ has the equation $x = x(t), y = y(t), z = z(t)$.
- Limit: The function $\vec{r}(t)$ has limit $\vec{a}$ as $t$ approaches to $t_0$ if $\lim\limits_{t\to t_0} ||\vec{r}(t) - \vec{a}|| = 0$, denoted by $\lim\limits_{t\to t_0} \vec{r}(t) = \vec{a}$.
- Continuous: The function $\vec{r}(t)$ defined in $I$ is continuous at $t_0 \in I$ if $\lim\limits_{t\to t_0} \vec{r}(t) = \vec{r}(t_0)$. (This is the same as $x(t), y(t), z(t)$ are continuous at $t_0$.)

# Derivative

- The limit (if exists)

$$\boxed{\lim_{h\to 0}\frac{\Delta\vec{r}}{h}=\lim_{h\to 0}\frac{\vec{r}(t_0+h)-\vec{r}(t_0)}{h}}$$

  is called the derivative o $\vec{r}(t)$ at $t_0$, denoted by $\vec{r}'(t_0)$ or $\dfrac{d\vec{r}(t_0)}{dt}$.
- When $\vec{r}(t)$ has the derivative at $t_0$, we say $\vec{r}(t)$ is differentiable t $t_0$.
- **Remark:** If $x(t), y(t), z(t)$ are differentiable at $t_0$, then $\vec{r}(t)$ is differentiable at $t_0$ and

$$\boxed{\vec{r}'(t_0)=x'(t_0)\vec{i}+y'(t_0)\vec{j}+z'(t_0)\vec{k}.}$$

# 1.2.2. Space curves: Tangent lines

- Let $L$ be a space curve with parametrized equations $x = x(t)$, $y = y(t)$, $z = z(t)$. The corresponding vector function is $\vec{r}(t) = (x(t), y(t), z(t))$.
- Let $M(x(t_0), y(t_0), z(t_0)) \in L$ be a non-singular point (at least one of $x'(t_0), y'(t_0), z'(t_0)$ is non-zero).
- Then $\vec{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$ is called the tangent vector of $L$ at $M$. „
- The equation of the tangent line at $M$:

$$\boxed{\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}.}$$

# Curves: Normal planes

- Let $L$ be a space curve with parametrized equations $x = x(t)$, $y = y(t)$, $z = z(t)$. The corresponding vector function is $\vec{r}(t) = (x(t), y(t), z(t))$. Let $M(x(t_0), y(t_0), z(t_0)) \in L$ be a non-singular point.
- The plane passing through $M$ and is perpendicular (vuông góc) to the tangent line of $L$ at $M$ is called the *normal plane* (pháp diện) of the curve $L$ at $M$.
- The normal plane of $L$ at $M$ consists of all points $P$ such that the vector $\overrightarrow{MP}$ is perpendicular to the vector $\vec{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$. The equation of the normal plane of $L$ at $M$ is

$$\boxed{x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) + z'(t_0)(z - z(t_0)) = 0.}$$

# Curvature

The curvature measures how fast a curve is changing direction at a given point.
Let $L$ be a space curve defined by $x = x(t)$, $y = y(t)$, $z = z(t)$. Let $M(x(t_0), y(t_0), z(t_0))$ in $L$. The curvature of $L$ at $M$ is

$$\boxed{C(M) = \frac{\sqrt{\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^2 + \begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}^2 + \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}^2}}{(x'^2 + y'^2 + z'^2)^{3/2}}}$$

**Remark**: Let $\vec{r}(t) = (x(t), y(t), z(t))$. Then

$$\boxed{C(M) = \frac{||\vec{r}' \wedge \vec{r}''||}{||\vec{r}'||^3}.}$$

## Example (CK20182)

Find the equations of the tangent line and the normal plane of the curve defined by $x = t\cos 2t$, $y = t\sin 2t$, $z = 3t$ at $t = \pi/2$.

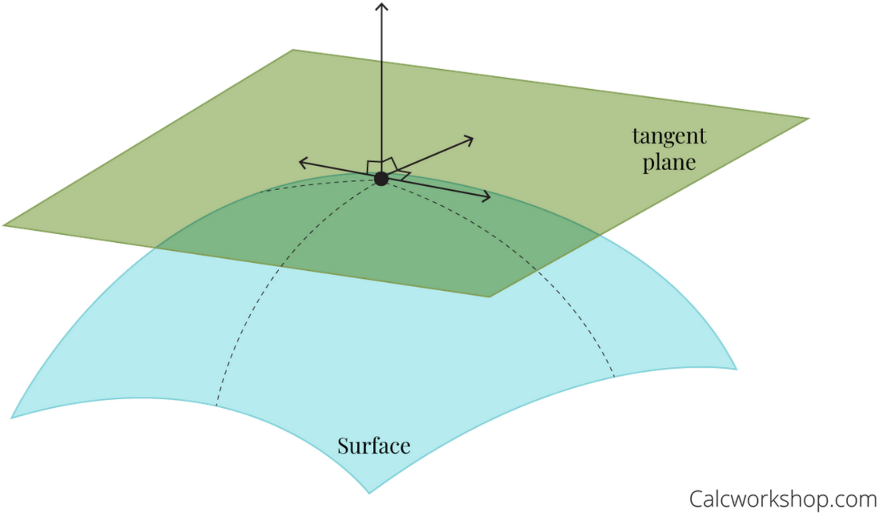
Giải:

- At $t = \pi/2$, we have a point $M(-\pi/2, 0, 3\pi/2)$.
- $x'(t) = \cos 2t - 2t\sin 2t$, $y'(t) = \sin 2t + 2t\cos 2t$, $z'(t) = 3$.
- At $t = \pi/2$: $x'(\pi/2) = -1$, $y'(\pi/2) = -\pi$, $z'(\pi/2) = 3$.
- The equation of the tangent line is

$$\frac{x + \pi/2}{-1} = \frac{y}{-\pi} = \frac{z - 3\pi/2}{3}.$$

- The equation of the normal plane is

$$(-1)(x + \pi/2) - \pi y + 3(z - 3\pi/2) = 0 \text{ hay } -x - \pi y + 3z - 5\pi = 0.$$




Tangent Plane To A Surface

# 1.2.3. Tangent planes and normal line

- Given the surface $S$ and a point $M \in S$. The line $MT$ is called a tangent line of $S$ at $M$ if it is a tangent line at $M$ to a curve lying in $S$.
- Let $S$ be the surface defined by the equation $f(x, y, z) = 0$. A point $M \in S$ is called non-singular (điểm chính quy) if at least one of $f'_x(M), f'_y(M), f'_z(M)$ is not zero.

**Theorem**

The set of all tangent lines of $S$ at a non-singular point $M$ forms a plane.

- The plane containing all tangent lines of $S$ at a non-singular point M is called the *tangent plane* (*tiếp diện*) of $S$ at $M$.
- The line through $M$ and is perpendicular to the tangent plane is called the *normal line* (*pháp tuyến*) of $S$ at $M$.

# Formula

Given the surface $S$ defined by the equation $f(x, y, z) = 0$. Let $M(x_0, y_0, z_0) \in S$ be a non-singular point.

- The equation of the tangent plane of $S$ at $M$ is

$$\boxed{f'_x(M)(x - x_0) + f'_y(M)(y - y_0) + f'_z(M)(z - z_0) = 0.}$$

- The equation of the normal line of $S$ at $M$ is

$$\boxed{\frac{x - x_0}{f'_x(M)} = \frac{y - y_0}{f'_y(M)} = \frac{z - z_0}{f'_z(M)}.}$$

## Example (GK20201)

Find the equations of the tangent plane and the normal line of $z = \ln(2x + y)$ at $M(-1, 3, 0)$.

Solution:

- Let $F(x, y, z) = \ln(2x + y) - z$.
- $F'_x = 2/(2x + y)$, $F'_y = 1/(2x + y)$, $F'_z = -1$.
- At $M(-1, 3, 0)$: $F'_x(M) = 2$, $F'_y(M) = 1$, $F'_z(M) = -1$.
- The equation of the tangent plane

$$2(x + 1) + (y - 3) - z = 0 \text{ or } 2x + y - z - 1 = 0.$$

- The equation of the normal line

$$\frac{x+1}{2} = \frac{y-3}{1} = \frac{z}{-1}.$$

# Some exercises

- (GK20213) Find the equations of the tangent plane and the normal line of$z = 2x^2 + y^2$ at $M(1, 1, 3)$.
- (GK20212) Find the equations of the tangent plane and the normal line of $\arctan(x + y^2) + z = 0$ at $M(-1, 1, 0)$.
- (CK20193) Find the equations of the tangent plane and the normal line of $x^2 - 2y^3 + 3z^2 = 11$ at $A(1; 1; 2)$.
- (GK20192) Find the equations of the tangent plane and the normal line of $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ at $t = \pi/2$.
- (GK20182) Find the equations of the tangent plane and the normal line of $x = \sin t$, $y = \cos t$, $z = e^{2t}$ at $M(0; 1; 1)$.
- (GK20182) Find the equations of the tangent plane and the normal line of $x^2 + y^2 - e^z - 2xyz = 0$ at $M(1; 0; 0)$.
- (GK20172) Find the equations of the tangent plane and the normal line of $\ln(2x + y^2) + 3z^3 = 3$ at $M(0; -1; 1)$.
- (CK20171) Find the equations of the tangent plane and the normal line of $z = \ln(4 - x^2 - 2y^2)$ at $A(-1; 1; 0)$.

# Curves defined by intersection of surfaces

- (CK20181) Find the tangent vector at $M(1;-1;1)$ of the curve defined by

$$\begin{cases} x+y+2z-2=0 \\ x^2+2y^2-2z^2-1=0 \end{cases}$$

- (CK20142) Find the equations of the tangent line and the normal plane at $A(1;-2;5)$ of the curve defined by $z=x^2+y^2$, $z=2x+3$.