Chapter 4. Estimation

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Population and sample

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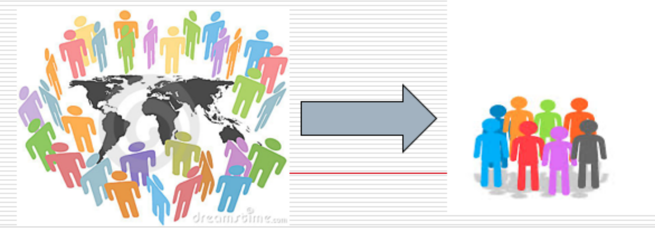
- A population is the set of all individuals of interest. The number of individuals N is called the population size.
- In practice, we usually study on a characteristic X of individuals in a population. Let x_i be the value of X of the i^{th} individual, the set $\{x_1, x_2, \ldots, x_N\}$ or X is also called the population.
- A subset of n individuals taken from a population X is called a sample of size n. A sample of size n is a vetor of n obsevations $(x_1, x_2, .., x_n)$.



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- A sample $(x_1, x_2, .., x_n)$ is called a representation of random sample $(X_1, X_2, \ldots, X_n)$.
- The joint probability distribution (pdf or pmf) of the random sample $(X_1, X_2, \ldots, X_n)$ is

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n)$$

  and is called the likelihood function.

Example

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- A random vector $(X_1, X_2, \ldots, X_{50})$, where X_i are i.i.d random variable having the same normal distribution $f(x; \mu, \sigma^2)$, is called a random sample of size 50 drawn from the population X.

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- A random vector $(X_1, X_2, \ldots, X_{50})$, where X_i are i.i.d random variable having the same normal distribution $f(x; \mu, \sigma^2)$, is called a random sample of size 50 drawn from the population X.
- Observed the electricity bills of 50 households from this region and onbtained the following sample $(x_1, x_2, \ldots, x_{50}) = (255, 367, \ldots, 423)$, this sample is a representation of the random sample $(X_1, X_2, \ldots, X_{50})$.

Statistic

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  is a statistic.
- A statistic is a random variable and the distribution of a statistic is called a sampling distribution.

Some important statistic

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- The adjusted random sample standard deviation $S = \sqrt{S^2}$

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$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

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- **Theorem 4.1**: For all distribution of X, we have $E(\bar{X}) = \mu$ and $V(\bar{X}) = \frac{\sigma^2}{n}$.
- **Theorem 4.2**: If X is normal: $X \sim N(\mu, \sigma^2)$ then $\bar{X}$ is also normal: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$. So the Z-statistic is standard normal: $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$.

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- **Theorem 4.3**:The central limit theorem:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{n \to +\infty} N(0, 1)$$

When n is large enough, $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \approx N(0,1)$ or $\bar{X} \approx N(\mu, \frac{\sigma^2}{n})$, for all distribution of X.

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- **Theorem 4.4**: If X is normal: $X \sim N(\mu, \sigma^2)$ then the T-statistic follows the Student's distribution with $n-1$ degrees of freedom:

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- **Theorem 4.5** (The central limit theorem + Slutsky's theorem): For all distribution of X:

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- **Definition 4.4**:
  - A point estimator of θ is a statistic $\hat{\theta} = h(X_1, X_2, \ldots, X_n)$.
  - A confidence interval estimation of θ with a confidence level $1-\alpha$ is a random interval $[\hat{\theta}_1; \hat{\theta}_2] = [h_1(X_1, X_2, \ldots, X_n); h_2(X_1, X_2, \ldots, X_n)]$ such that:

$$P(\hat{\theta}_1 \leq \theta \leq \hat{\theta}_2) = 1 - \alpha$$

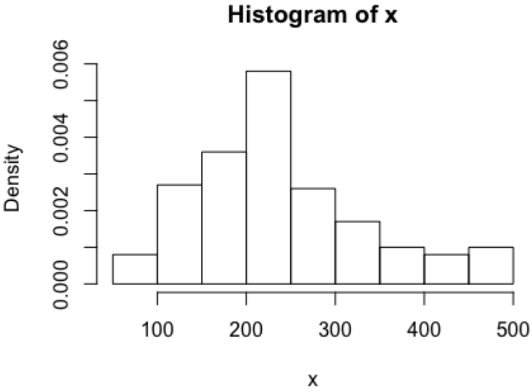
Example

Let X be the electricity bills (thousands dong) of households in a region of Vietnam (in June 2020). Observed the electricity bills of 200 households from this region and onbtained the following data:

196.65	468.75	320.50	300.50	213.05	140.60	290.00	216.95	360.50	317.95	195.55
220.50	255.60	289.00	194.55	374.25	382.05	185.55	219.10	215.60	220.00	186.75
97.80	340.50	88.50	209.50	234.04	333.00	291.10	108.50	245.00	184.00	153.50
219.50	214.15	155.20	140.40	108.50	410.00	125.50	220.30	160.00	300.50	310.20
244.40	194.50	210.20	360.00	456.50	237.40	235.00	203.25	109.20	240.15	260.50
275.50	101.55	455.50	246.25	291.55	262.00	378.65	194.50	248.00	262.92	85.75
248.00	204.75	310.70	213.10	320.50	125.60	110.25	77.35	119.50	313.50	222.00
388.10	110.50	160.00	210.00	310.30	380.10	281.00	105.35	280.15	188.80	272.50
103.40	213.50	280.50	119.50	166.10	180.50	212.00	154.75	100.50	452.60	436.35
225.00	124.30	170.00	127.35	107.90	140.00	195.00	315.10	241.05	168.00	120.50
223.95	237.05	285.45	100.50	228.55	248.70	175.80	466.05	219.00	216.00	425.50
390.00	176.85	240.50	226.00	108.70	160.00	470.50	225.00	440.00	265.00	162.80
260.50	175.80	73.05	460.50	263.60	59.50	198.00	416.50	315.50	155.00	190.00
158.50	225.00	266.70	153.60	238.00	297.60	201.75	240.50	270.90	196.65	299.20
70.50	125.60	100.40	240.00	240.00	224.05	194.00	247.00	325.40	102.20	166.10
361.00	430.00	240.00	250.50	470.00	157.75	98.40	236.50	230.85	317.65	200.70
165.00	350.50	319.15	275.88	203.05	234.50	220.75	180.50	436.50	403.00	460.50
220.00	103.50	222.15	170.50	224.15	460.00	260.40	200.50	311.40	260.00	251.55
100.60	212.20									

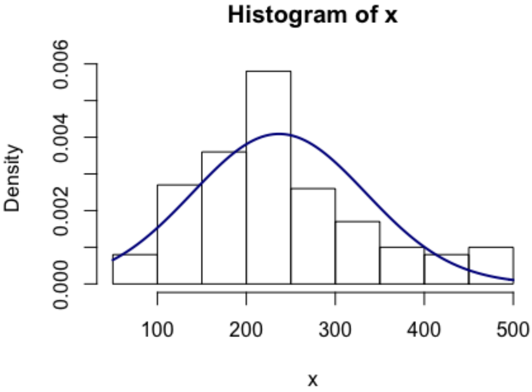
Example

The histogram for these data is the following:



Example

The distribution of data can be approximated by a normal distribution:



Example

- Modelling: We can suppose that the electricity bills of households in this region follows a normal distribution with parameter $\theta = (\mu, \sigma^2)$ and the probability density function:

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)$$

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For the given sample, the sample mean

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \ldots + x_n) = 236.78$$

is also called a point estimate of μ.

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- Maximum likelihood estimator: To find θ that maximizes the likelihood function $L(\theta)$ or $\log L(\theta)$:

$$\hat{\theta} = \operatorname{argmax} L(\theta) = \operatorname{argmax} \log L(\theta)$$

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- Step 4: Solve the equation

$$\frac{\partial \log L(\theta)}{\partial \theta} = 0$$

and let $\hat{\theta}$ be the solution, then prove that

$$\frac{\partial^2 \log L(\theta)}{\partial \theta^2}\Big|_{\theta=\hat{\theta}} < 0$$

.

Maximum likelihood estimation (MLE)

Example 4.1: Let X be the lifetime of a type of batteries produced by a factory and suppose that X follows an exponential distribution with a parameter $\lambda > 0$. Find the maximum likelihood estimator of λ.

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- Step 3: The log- likelihood function:

$$\log L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} X_i$$

Maximum likelihood estimation (MLE)

- Step 4: Solve the equation

$$\frac{\partial \log L(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0$$

we obtain the solution

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}}.$$

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- Since

$$\frac{\partial^2 \log L(\lambda)}{\partial \lambda^2} = -\frac{n}{\lambda^2} < 0, \text{ for all } \lambda > 0,$$

then the maximum likelihood estimator of λ is

$$\hat{\lambda} = \frac{1}{\bar{X}}.$$

Maximum likelihood estimation (MLE) of normal distribution

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- The likelihood function is

$$L(\theta) = \Pi_{i=1}^n f(X_i;\theta) = \Pi_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(X_i-\mu)^2}{2\sigma^2}\Big)$$

Maximum likelihood estimation (MLE) of normal distribution

- Consider a random sample $(X_1, X_2, \ldots, X_n)$ drawn from a normal population X with a mean of μ and a variance of σ^2. Find the MLE of $\theta = (\mu, \sigma^2)$.
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$$f(x;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)$$

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$$\log L(\theta) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$$

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- Obtain the MLE of μ and σ^2 as follows:

$$\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

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$$L(p) = \Pi_{i=1}^n f(X_i;p) = \Pi_{i=1}^n p^{X_i}(1-p)^{1-X_i} = p^{\sum_{i=1}^n X_i}(1-p)^{n-\sum_{i=1}^n X_i}$$

Maximum likelihood estimation (MLE) of noraml distribution

- The log-likelihood function is

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$$\hat{p} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Method of moments

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- Method of moments: Let X be a population with a probability distribution $f(x;\theta)$, where θ is an unknown parameter in R^r. The estimator of θ by the method of moment is the solution of the following system of equations:

$$E[X^k] = \frac{1}{n}\sum_{i=1}^{n} X_i^k, k = 1, .., r.$$

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$$E[X] = \frac{1}{n}\sum_{i=1}^{n} X_i \Leftrightarrow \frac{1}{\lambda} = \bar{X} \Leftrightarrow \lambda = \frac{1}{\bar{X}}.$$

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- The estimator of λ by the method of moments is $\hat{\lambda}_{MM} = \dfrac{1}{\bar{X}} = \hat{\lambda}_{MLE}$.

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- We adjusted $\hat{S}^2$ to obtain an unbiased estimator of σ^2 as follows:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Confidence interval estimation

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- Procedure of finding a confidence interval estimation:
  - Find a point estimator $\hat{\theta}$ of θ.
  - Using the sampling distribution of $\hat{\theta}$ or the central limit theorem:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \approx N(0;1)$$

to find an interval $[\hat{\theta_1}, \hat{\theta_2}]$ such that $P[\hat{\theta_1} < \theta < \hat{\theta_2}] = 1-\alpha$ (where μ and σ are functions of θ).

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Problem 1: Consider a random sample $(X_1, X_2, \ldots, X_n)$ taken from a population X with a mean of $\mu = E(X)$ and a variance of $\sigma^2 = V(X)$. Find a $1 - \alpha$ confidence interval estimation of μ.

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- Then a $1-\alpha$ confidence interval (CI) estimation of μ is:

$$\left[\bar{X} - Z_{\alpha/2}\frac{S}{\sqrt{n}}; \bar{X} + Z_{\alpha/2}\frac{S}{\sqrt{n}}\right] = \bar{X} \mp \epsilon,$$

where $\epsilon = Z_{\alpha/2}\frac{S}{\sqrt{n}}$ is called the error of CI.

Confidence interval estimation of μ

Example 4.3: Let X be the amount of telephone bills (USD) of customers in a city. Suppose that X follows a normal distribution $N(\mu, \sigma^2)$. Observed a sample of 20 customers, we obtained the following data:

$$31.3, 28.8, 30.8, 29.6, 32.5, 30.1, 28.6, 32.2, 30.8, 32.6,$$

$$31.8, 28.5, 29.9, 27.2, 36.0, 30.6, 29.2, 30.9, 31.0, 30.8$$

- Find the point estimate of μ and σ^2 by method of moments.
- Find the point estimate of μ and σ^2 by MLE method.
- Find a 90% confidence interval estimate of μ.
- Suppose that the standard deviation σ is known to equal to 1.5. Find a 90% confidence interval estimate of μ.

Confidence interval estimation of μ

Solution of Example: The point estimator of μ and σ^2 by method of moments:

- The parameter $\theta = (\mu, \sigma^2)$ is in R^2, so the dimension $r = 2$.
- We have $E(X) = \mu$ and $E(X^2) = \mu^2 + \sigma^2$.

Confidence interval estimation of μ

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- The parameter $\theta = (\mu, \sigma^2)$ is in R^2, so the dimension $r = 2$.
- We have $E(X) = \mu$ and $E(X^2) = \mu^2 + \sigma^2$.
- We solve the following system of equations:

$$E[X] = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } E(X^2) = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Confidence interval estimation of μ

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- The point estimator of μ and σ^2 by method of moments are:

$$\hat{\mu}_{MM} = \bar{X} \text{ and } \hat{\sigma}^2_{MM} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 = \hat{S}^2.$$

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- For the given sample, the point estimate of μ and σ^2 by method of moments are:

$$\hat{\mu}_{MM} = \bar{x} = \frac{1}{20}(31.3 + ... + 30.8) = 30.66$$

and

$$\hat{\sigma}^2_{MM} = \hat{s}^2 = \frac{1}{20}(31.3^2 + ... + 30.8^2 - 20 * 30.66^2) = 3.4234$$

Confidence interval estimation of μ

Solution of Example:

- The point estimator of μ and σ^2 by the maximum likelihood estimation method are:

$$\hat{\mu}_{MLE} = \bar{X} \text{ and } \hat{\sigma}^2_{MLE} = \hat{S}^2.$$

Confidence interval estimation of μ

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- The point estimator of μ and σ^2 by the maximum likelihood estimation method are:

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For the given sample, the point estimate of μ and σ^2 by the maximum likelihood estimation method are:

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and

$$\hat{\sigma}^2_{MLE} = \hat{s}^2 = \frac{1}{20}(31.3^2 + ... + 30.8^2 - 20 * 30.66^2) = 3.4234$$

Confidence interval estimation of μ

Solution of Example: Find a 90% confidence interval estimate of μ.

- Since $X \sim N(\mu; \sigma^2)$ where σ^2 is unknown then we use the following statistic

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1},$$

so a $1 - \alpha$ confidence interval (CI) estimation of μ is:

$$\left[\bar{X} - t_{n-1;\alpha/2}\frac{S}{\sqrt{n}}; \bar{X} + t_{n-1;\alpha/2}\frac{S}{\sqrt{n}}\right]$$

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so a $1 - \alpha$ confidence interval (CI) estimation of μ is:
$$\left[\bar{X} - t_{n-1;\alpha/2}\frac{S}{\sqrt{n}}; \bar{X} + t_{n-1;\alpha/2}\frac{S}{\sqrt{n}}\right]$$
- For the given sample, we have
$n = 20; \bar{x} = 30.66; s^2 = \frac{n}{n-1}\hat{s}^2 = 3.604; s = \sqrt{3.604} = 1.9;$
$1 - \alpha = 90\%$ then $t_{n-1;\alpha/2} = t_{19;0.05} = 1.73$

Confidence interval estimation of μ

Solution of Example: Find a 90% confidence interval estimate of μ.

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$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1},$$
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$$\left[\bar{X} - t_{n-1;\alpha/2}\frac{S}{\sqrt{n}}; \bar{X} + t_{n-1;\alpha/2}\frac{S}{\sqrt{n}}\right]$$
- For the given sample, we have
$n = 20; \bar{x} = 30.66; s^2 = \frac{n}{n-1}\hat{s}^2 = 3.604; s = \sqrt{3.604} = 1.9;$
$1 - \alpha = 90\%$ then $t_{n-1;\alpha/2} = t_{19;0.05} = 1.73$
- So the CI of μ is
$$30.66 \mp 1.73\frac{1.9}{\sqrt{20}} = 30.66 \mp 0.735 = [29.925; 31.395]$$

Confidence interval estimation of μ

Solution of Example: Find a 90% confidence interval estimate of μ when $\sigma = 1.5$.

- Since $X \sim N(\mu; \sigma^2)$ where σ is known to equal to 1.5 then we use the following statistic
$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0; 1),$$
so a $1 - \alpha$ confidence interval (CI) estimation of μ is:
$$\left[\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}; \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

Confidence interval estimation of μ

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$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0; 1),$$
so a $1 - \alpha$ confidence interval (CI) estimation of μ is:
$$\left[\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}; \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$
- For the given sample, we have $n = 120; \bar{x} = 30.66$; $1 - \alpha = 90\%$ then $Z_{\alpha/2} = Z_{0.05} = 1.645$

Confidence interval estimation of μ

Solution of Example: Find a 90% confidence interval estimate of μ when $\sigma = 1.5$.

- Since $X \sim N(\mu; \sigma^2)$ where σ is known to equal to 1.5 then we use the following statistic

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0; 1),$$

  so a $1 - \alpha$ confidence interval (CI) estimation of μ is:

$$\left[\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}; \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

- For the given sample, we have $n = 120; \bar{x} = 30.66$; $1 - \alpha = 90\%$ then $Z_{\alpha/2} = Z_{0.05} = 1.645$
- So the CI of μ is

$$30.66 \mp 1.645\frac{1.5}{\sqrt{20}} = 30.66 \mp 0.55 = [30.11; 31.21]$$

Confidence interval estimation of σ^2

Problem 2: Let $(X_1, X_2, .., X_n)$ be a random sample taken from a normal population $X \sim N(\mu, \sigma^2)$. Find a $1 - \alpha$ confidence interval estimate of σ^2.

Confidence interval estimation of σ^2

Problem 2: Let $(X_1, X_2, .., X_n)$ be a random sample taken from a normal population $X \sim N(\mu, \sigma^2)$. Find a $1-\alpha$ confidence interval estimate of σ^2.

- The point estimator of σ^2 is S^2.

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- The point estimator of σ^2 is S^2.
- The sampling distribution of S^2 is the following

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1},$$

  where χ^2_{n-1} is the Chi-squared distribution with $n-1$ degrees of freedom.

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- Let $\chi^2_{n-1;1-\alpha/2}$ and $\chi^2_{n-1;\alpha/2}$ be the critical value of Chi-squared distribution χ^2_{n-1} at level $\alpha/2$ and $1-\alpha/2$.

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  where χ^2_{n-1} is the Chi-squared distribution with $n-1$ degrees of freedom.
- Let $\chi^2_{n-1;1-\alpha/2}$ and $\chi^2_{n-1;\alpha/2}$ be the critical value of Chi-squared distribution χ^2_{n-1} at level $\alpha/2$ and $1-\alpha/2$.
- We have

$$P\left(\chi^2_{n-1;1-\alpha/2} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n-1;\alpha/2}\right) = 1-\alpha$$

Confidence interval estimation of σ^2

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Then

$$P\Big(\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}\Big) = 1-\alpha$$

Confidence interval estimation of σ^2

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  Then

$$P\Big(\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}\Big) = 1-\alpha$$

- A $1-\alpha$ confidence interval estimate of σ^2 is

$$\Big[\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}}; \frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}\Big]$$

Confidence interval estimation of σ^2

Example 4.4: Let X be the amount of telephone bills (USD) of customers in a city. Suppose that X follows a normal distribution $N(\mu, \sigma^2)$. Observed a sample of 20 customers, we obtained the following data:

$$31.3, 28.8, 30.8, 29.6, 32.5, 30.1, 28.6, 32.2, 30.8, 32.6,$$

$$31.8, 28.5, 29.9, 27.2, 36.0, 30.6, 29.2, 30.9, 31.0, 30.8$$

Find a 90% confidence interval estimate of σ^2.

Confidence interval estimation of σ^2

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$$31.8, 28.5, 29.9, 27.2, 36.0, 30.6, 29.2, 30.9, 31.0, 30.8$$

Find a 90% confidence interval estimate of σ^2.

- A $1-\alpha$ confidence interval estimate of σ^2 is

$$\Big[\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}}; \frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}\Big],$$

  where $n = 20; s^2 = 3.604; 1-\alpha = 0.9$ then
  $\chi^2_{n-1;1-\alpha/2} = \chi^2_{19,0.95} = 10.12; \chi^2_{n-1;\alpha/2} = \chi^2_{19,0.05} = 30.14.$

Confidence interval estimation of σ^2

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Find a 90% confidence interval estimate of σ^2.

- A $1-\alpha$ confidence interval estimate of σ^2 is

$$\left[\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}}; \frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}\right],$$

  where $n = 20; s^2 = 3.604; 1-\alpha = 0.9$ then
  $\chi^2_{n-1;1-\alpha/2} = \chi^2_{19,0.95} = 10.12; \chi^2_{n-1;\alpha/2} = \chi^2_{19,0.05} = 30.14.$
- Then a 90% confidence interval estimate of σ^2 is

$$\left[\frac{19 * 3.604}{30.14}; \frac{19 * 3.604}{10.12}\right] = [2.27; 6.77].$$

Confidence interval estimation of population proportion p

Problem 3: Let p be a population proportion, for example, p is the proportion of defective items in a production line. Find a $1-\alpha$ confidence interval estimate of p.

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- Consider a random sample of size n from the population.

Confidence interval estimation of population proportion p

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Confidence interval estimation of population proportion p

Problem 3: Let p be a population proportion, for example, p is the proportion of defective items in a production line. Find a $1-\alpha$ confidence interval estimate of p.

- Consider a random sample of size n from the population.
- A point estimator of p is $\hat{p}$, the sample proportion (example: the proportion of defective items in a sample of n items).
- By the following limit theorem

$$Z = \frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \approx N(0,1)$$

  we obtain the following $1-\alpha$ confidence interval estimate of p;

$$\hat{p} \mp \epsilon = \hat{p} \mp Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

  where $\epsilon = Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ is the error of the CI.

Confidence interval estimation of population proportion p

Example 4.5: Let p be the proportion of defective items in a production line. Examined a random sample of 120 items from the line and there were 6 defective items. Find a 90% confidence interval estimate of p.

Confidence interval estimation of population proportion p

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- A 90% confidence interval estimate of p is

$$\hat{p} \mp \epsilon = \hat{p} \mp Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where $n = 20; \hat{p} = 6/120 = 0.05; 1 - \alpha = 0.9$ then
$Z_{\alpha/2} = Z_{0.025} = 1.96.$

Confidence interval estimation of population proportion p

Example 4.5: Let p be the proportion of defective items in a production line. Examined a random sample of 120 items from the line and there were 6 defective items. Find a 90% confidence interval estimate of p.

- A 90% confidence interval estimate of p is

$$\hat{p} \mp \epsilon = \hat{p} \mp Z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

  where $n = 20; \hat{p} = 6/120 = 0.05; 1-\alpha = 0.9$ then $Z_{\alpha/2} = Z_{0.025} = 1.96.$

- So the CI of p is

$$0.05 \mp 1.96\sqrt{\frac{0.05 * 0.95}{120}} = 5\% \mp 3.9\% = [1.1\%; 8.9\%].$$

Confidence interval estimation

General problem: Observe a population X with the pdf (ou pmf) $f(x;\theta)$, where θ is unknown parameter to estimate. Find a $1-\alpha$ confidence interval estimation of θ.

- Consider a random sample $(X_1, X_2, ..., X_n)$ taken from the population X.

Confidence interval estimation

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- Find a point estimator $\hat{\theta}$ of θ.

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- Consider a random sample $(X_1, X_2, ..., X_n)$ taken from the population X.
- Find a point estimator $\hat{\theta}$ of θ.
- Use the sampling distribution of $\hat{\theta}$ or a limit theorem, for example:

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-g_1(\theta)}{g_2(\theta)/\sqrt{n}} \approx N(0;1)$$

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$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - g_1(\theta)}{g_2(\theta)/\sqrt{n}} \approx N(0;1)$$

- From the equation

$$\mathbb{P}\Big(-Z_{\alpha/2} \leq \frac{\bar{X} - g_1(\theta)}{g_2(\theta)/\sqrt{n}} \leq Z_{\alpha/2}\Big) = 1-\alpha$$

  we find an interval $[\hat{\theta}_1; \hat{\theta}_2]$ such that $P(\hat{\theta}_1 \leq \theta \leq \hat{\theta}_2) = 1-\alpha$.

Confidence interval estimation

Example 4.6: Let X be the lifetime (in years) of a mechanical part. Suppose that X follows an exponential distribution with a rate parameter of λ.

- Construct a $1 - \alpha$ confidence interval estimation of λ.
- Given the following sample:

X	[0,1]	(1, 2]	(2, 3]	(3, 4]	(4, 5]	(5, 6]	(6, 7]
N^o of parts	20	12	8	3	3	2	2

Find a 90% confidence interval estimate of λ for this sample.

Confidence interval estimation

Solution:

- Since $X \sim \mathcal{E}(\lambda)$ then the pdf of X is $f(x;\lambda) = \lambda e^{-\lambda x}$, for $x > 0$ and $\mu = E(X) = 1/\lambda$; $\sigma^2 = V(X) = 1/\lambda^2$ then $\sigma = 1/\lambda$.

Confidence interval estimation

Solution:

- Since $X \sim \mathcal{E}(\lambda)$ then the pdf of X is $f(x;\lambda) = \lambda e^{-\lambda x}$, for $x > 0$ and $\mu = E(X) = 1/\lambda$; $\sigma^2 = V(X) = 1/\lambda^2$ then $\sigma = 1/\lambda$.
- By the central limit theorem:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - 1/\lambda}{(1/\lambda)/\sqrt{n}} = (\bar{X}\lambda - 1)\sqrt{n} \approx N(0;1)$$

Confidence interval estimation

Solution:

- Since $X \sim \mathcal{E}(\lambda)$ then the pdf of X is $f(x;\lambda) = \lambda e^{-\lambda x}$, for $x > 0$ and $\mu = E(X) = 1/\lambda$; $\sigma^2 = V(X) = 1/\lambda^2$ then $\sigma = 1/\lambda$.
- By the central limit theorem:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - 1/\lambda}{(1/\lambda)/\sqrt{n}} = (\bar{X}\lambda - 1)\sqrt{n} \approx N(0;1)$$

- From the equation

$$\mathbb{P}\Big(- Z_{\alpha/2} \leq (\bar{X}\lambda - 1)\sqrt{n} \leq Z_{\alpha/2}\Big) = 1 - \alpha$$

Confidence interval estimation

Solution:

- Since $X \sim \mathcal{E}(\lambda)$ then the pdf of X is $f(x;\lambda) = \lambda e^{-\lambda x}$, for $x > 0$ and $\mu = E(X) = 1/\lambda$; $\sigma^2 = V(X) = 1/\lambda^2$ then $\sigma = 1/\lambda$.
- By the central limit theorem:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - 1/\lambda}{(1/\lambda)/\sqrt{n}} = (\bar{X}\lambda - 1)\sqrt{n} \approx N(0;1)$$

- From the equation

$$\mathbb{P}\Big(- Z_{\alpha/2} \leq (\bar{X}\lambda - 1)\sqrt{n} \leq Z_{\alpha/2}\Big) = 1 - \alpha$$

or

$$\mathbb{P}\Big(\frac{1 - \frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}} \leq \lambda \leq \frac{1 + \frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}}\Big) = 1 - \alpha$$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1-\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}}; \frac{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}}\right]$$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1-\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}};\frac{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}}\right]$$

- For the given sample, we have $n=50; 1-\alpha=0.9$ then
  $Z_{\alpha/2}=Z_{0.05}=1.645$;
  $\bar{x}=(20*0.5+12*1.5+...+2*6.5)/50=1.92.$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1-\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}};\frac{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}}\right]$$

- For the given sample, we have $n=50; 1-\alpha=0.9$ then
$Z_{\alpha/2}=Z_{0.05}=1.645;$
$\bar{x}=(20*0.5+12*1.5+...+2*6.5)/50=1.92.$
- So the 90% CI of λ is

$$\frac{1\mp\frac{1.645}{\sqrt{50}}}{1.92}=[0.4;0.64]$$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1-\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}}; \frac{1+\frac{Z_{\alpha/2}}{\sqrt{n}}}{\bar{X}}\right]$$

- For the given sample, we have $n = 50; 1-\alpha = 0.9$ then
  $Z_{\alpha/2} = Z_{0.05} = 1.645$;
  $\bar{x} = (20 * 0.5 + 12 * 1.5 + ... + 2 * 6.5)/50 = 1.92.$
- So the 90% CI of λ is

$$\frac{1 \mp \frac{1.645}{\sqrt{50}}}{1.92} = [0.4; 0.64]$$

- We are 90% confident that the parameter λ is between 0.4 and 0.64.

Confidence interval estimation

Solution: 2nd method

- Using the following limit theorem:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - 1/\lambda}{S/\sqrt{n}} \approx N(0;1)$$

Confidence interval estimation

Solution: 2nd method

- Using the following limit theorem:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - 1/\lambda}{S/\sqrt{n}} \approx N(0;1)$$

- From the equation

$$\mathbb{P}\Big(- Z_{\alpha/2} \leq \frac{\bar{X} - 1/\lambda}{S/\sqrt{n}} \leq Z_{\alpha/2}\Big) = 1 - \alpha$$

Confidence interval estimation

Solution: 2^{nd} method

- Using the following limit theorem:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - 1/\lambda}{S/\sqrt{n}} \approx N(0;1)$$

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$$\mathbb{P}\Big(- Z_{\alpha/2} \leq \frac{\bar{X} - 1/\lambda}{S/\sqrt{n}} \leq Z_{\alpha/2}\Big) = 1 - \alpha$$

or

$$\mathbb{P}\Big(\frac{1}{\bar{X} + Z_{\alpha/2}\frac{S}{\sqrt{n}}} \leq \lambda \leq \frac{1}{\bar{X} - Z_{\alpha/2}\frac{S}{\sqrt{n}}}\Big) = 1 - \alpha$$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1}{\bar{X}+Z_{\alpha/2}\frac{S}{\sqrt{n}}};\frac{1}{\bar{X}-Z_{\alpha/2}\frac{S}{\sqrt{n}}}\right]$$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1}{\bar{X}+Z_{\alpha/2}\frac{S}{\sqrt{n}}};\frac{1}{\bar{X}-Z_{\alpha/2}\frac{S}{\sqrt{n}}}\right]$$

- For the given sample, we have $n = 50$; $Z_{\alpha/2} = Z_{0.05} = 1.645$; $\bar{x} = 1.92$; $s^2 = (20 * 0.5^2 + ... + 2 * 6.5^2 - 50 * 1.92^2)/49 = 2.861$; $s = \sqrt{2.861} = 1.69$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1}{\bar{X}+Z_{\alpha/2}\frac{S}{\sqrt{n}}};\frac{1}{\bar{X}-Z_{\alpha/2}\frac{S}{\sqrt{n}}}\right]$$

- For the given sample, we have $n=50$; $Z_{\alpha/2}=Z_{0.05}=1.645$; $\bar{x}=1.92$; $s^2=(20*0.5^2+...+2*6.5^2-50*1.92^2)/49=2.861$; $s=\sqrt{2.861}=1.69$
- So the 90% CI of λ is

$$\frac{1}{1.92\mp 1.645*\frac{1.69}{\sqrt{50}}}=[0.43;0.65]$$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\left[\frac{1}{\bar{X}+Z_{\alpha/2}\frac{S}{\sqrt{n}}};\frac{1}{\bar{X}-Z_{\alpha/2}\frac{S}{\sqrt{n}}}\right]$$

- For the given sample, we have $n=50$; $Z_{\alpha/2}=Z_{0.05}=1.645$; $\bar{x}=1.92$; $s^2=(20*0.5^2+...+2*6.5^2-50*1.92^2)/49=2.861$; $s=\sqrt{2.861}=1.69$
- So the 90% CI of λ is

$$\frac{1}{1.92\mp 1.645*\frac{1.69}{\sqrt{50}}}=[0.43;0.65]$$

- We are 90% confident that the parameter λ is between 0.43 and 0.65.

Confidence interval estimation

Example 4.7: Let X be the number of accidents per week in a small city. Suppose that X follows a Poisson distribution with a mean parameter of λ.

- Find the point estimator of λ by the method of moment and by the MLE method.
- Construct a $1-\alpha$ confidence interval estimation of λ.
- Given the following sample:

X	0	1	2	3	4
N^o of weeks	7	15	10	12	6

Find a 90% confidence interval estimate of λ for this sample.

Confidence interval estimation

Solution: The point estimator of λ by the method of moment:

- The parameter $\lambda \in R_{+}^{*}$ then the dimension of parameter space is $r = 1$.

Confidence interval estimation

Solution: The point estimator of λ by the method of moment:

- The parameter $\lambda \in R_{+}^{*}$ then the dimension of parameter space is $r = 1$.
- The first moment of X is $E(X) = \lambda$.

Confidence interval estimation

Solution: The point estimator of λ by the method of moment:

- The parameter $\lambda \in R_+^*$ then the dimension of parameter space is $r = 1$.
- The first moment of X is $E(X) = \lambda$.
- The first sample moment of X is $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$.

Confidence interval estimation

Solution: The point estimator of λ by the method of moment:

- The parameter $\lambda \in R_+^*$ then the dimension of parameter space is $r = 1$.
- The first moment of X is $E(X) = \lambda$.
- The first sample moment of X is $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$.
- We solve the following equation:

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ or } \lambda = \bar{X}$$

So the point estimator of λ by the method of moment is $\hat{\lambda}_{MM} = \bar{X}$.

Confidence interval estimation

Solution: The point estimator of λ by the MLE method.:

- The pmf of X is

$$f(x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}, x = 0, 1, 2, ...$$

Confidence interval estimation

Solution: The point estimator of λ by the MLE method.:

- The pmf of X is

$$f(x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}, x = 0, 1, 2, ...$$

- The likelihood function of λ is

$$L(\lambda) = \prod_{i=1}^{n} f(X_i;\lambda) = \prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{X_i}}{X_i!} = e^{-n\lambda}\frac{\lambda^{\sum_{i=1}^{n} X_i}}{\prod_{i=1}^{n} X_i!}$$

Confidence interval estimation

Solution: The point estimator of λ by the MLE method.:

- The pmf of X is

$$f(x;\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}, x = 0, 1, 2, \dots$$

- The likelihood function of λ is

$$L(\lambda) = \prod_{i=1}^{n} f(X_i;\lambda) = \prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{X_i}}{X_i!} = e^{-n\lambda}\frac{\lambda^{\sum_{i=1}^{n} X_i}}{\prod_{i=1}^{n} X_i!}$$

- The log-likelihood function of λ is

$$\log L(\lambda) = -n\lambda + \sum_{i=1}^{n} X_i \log(\lambda) - \log(\prod_{i=1}^{n} X_i!)$$

Confidence interval estimation

Solution: The point estimator of λ by the MLE method:

- Solve the following equation:

$$\frac{\partial \log L(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} X_i}{\lambda} = 0$$

we have $\lambda = \bar{X}$.

Confidence interval estimation

Solution: The point estimator of λ by the MLE method:

- Solve the following equation:

$$\frac{\partial \log L(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} X_i}{\lambda} = 0$$

  we have $\lambda = \bar{X}$.

- Since

$$\frac{\partial^2 \log L(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^{n} X_i}{\lambda^2} < 0$$

  then the likelihood function attains maximum at $\lambda = \bar{X}$. So $\hat{\lambda}_{MLE} = \bar{X}$.

Confidence interval estimation

Solution: Construct a $1-\alpha$ confidence interval estimation of λ:

- Since $X \sim P(\lambda)$ then $\mu = E(X) = \lambda; \sigma^2 = V(X) = \lambda; \sigma = \sqrt{\lambda}$.

Confidence interval estimation

Solution: Construct a $1-\alpha$ confidence interval estimation of λ:

- Since $X \sim P(\lambda)$ then $\mu = E(X) = \lambda; \sigma^2 = V(X) = \lambda; \sigma = \sqrt{\lambda}$.
- By the central limit theorem:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - \lambda}{\sqrt{\lambda}}\sqrt{n} \approx N(0;1)$$

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- From the equation

$$\mathbb{P}\left(\frac{|\bar{X}-\lambda|}{\sqrt{\lambda}}\sqrt{n} \leq Z_{\alpha/2}\right) = 1-\alpha$$

Confidence interval estimation

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- Since $X \sim P(\lambda)$ then $\mu = E(X) = \lambda; \sigma^2 = V(X) = \lambda; \sigma = \sqrt{\lambda}$.
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$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - \lambda}{\sqrt{\lambda}}\sqrt{n} \approx N(0;1)$$

- From the equation

$$\mathbb{P}\Big(\frac{|\bar{X} - \lambda|}{\sqrt{\lambda}}\sqrt{n} \leq Z_{\alpha/2}\Big) = 1-\alpha$$

or

$$\mathbb{P}\Big(n(\bar{X} - \lambda)^2 \leq \lambda Z^2_{\alpha/2}\Big) = 1-\alpha$$

Confidence interval estimation

Solution: Construct a $1-\alpha$ confidence interval estimation of λ:

- or

$$\mathbb{P}\Big(n\lambda^2 - (2n\bar{X} + Z^2_{\alpha/2})\lambda + n\bar{X}^2\Big) = 1-\alpha$$

Confidence interval estimation

Solution: Construct a $1-\alpha$ confidence interval estimation of λ:

- or
$$\mathbb{P}\Big(n\lambda^2 - (2n\bar{X} + Z^2_{\alpha/2})\lambda + n\bar{X}^2\Big) = 1-\alpha$$

- or
$$\mathbb{P}\left(\lambda \in \frac{2n\bar{X} + Z^2_{\alpha/2} \mp Z_{\alpha/2}\sqrt{4n\bar{X} + Z^2_{\alpha/2}}}{2n}\right) = 1-\alpha$$

Confidence interval estimation

Solution: Construct a $1-\alpha$ confidence interval estimation of λ:

- or

$$\mathbb{P}\Big(n\lambda^2 - (2n\bar{X} + Z^2_{\alpha/2})\lambda + n\bar{X}^2\Big) = 1-\alpha$$

- or

$$\mathbb{P}\left(\lambda \in \frac{2n\bar{X} + Z^2_{\alpha/2} \mp Z_{\alpha/2}\sqrt{4n\bar{X} + Z^2_{\alpha/2}}}{2n}\right) = 1-\alpha$$

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\frac{2n\bar{X} + Z^2_{\alpha/2} \mp Z_{\alpha/2}\sqrt{4n\bar{X} + Z^2_{\alpha/2}}}{2n}$$

Confidence interval estimation

Solution:

- For the given sample, we have $n = 50; 1 - \alpha = 0.9$ then $Z_{\alpha/2} = Z_{0.05} = 1.645$;
  $\bar{x} = (7 * 0 + 15 * 1 + 10 * 2 + 21 * 3 + 6 * 4)/50 = 1.9$.

Confidence interval estimation

Solution:

- For the given sample, we have $n = 50; 1 - \alpha = 0.9$ then
  $Z_{\alpha/2} = Z_{0.05} = 1.645$;
  $\bar{x} = (7 * 0 + 15 * 1 + 10 * 2 + 21 * 3 + 6 * 4)/50 = 1.9$.
- So the 90% CI of λ is

$$\frac{2 * 50 * 1.9 + 1.645^2 \mp 1.645\sqrt{4 * 50 * 1.9 + 1.645^2}}{2 * 50} = [1.61; 2.25]$$

Confidence interval estimation

Solution:

- For the given sample, we have $n = 50; 1 - \alpha = 0.9$ then
  $Z_{\alpha/2} = Z_{0.05} = 1.645$;
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- So the 90% CI of λ is

$$\frac{2 * 50 * 1.9 + 1.645^2 \mp 1.645\sqrt{4 * 50 * 1.9 + 1.645^2}}{2 * 50} = [1.61; 2.25]$$

- We are 90% confident that the parameter λ is between 1.61 and 2.25.

Confidence interval estimation

Solution: 2^{nd} method.

- By the following limit theorem:

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - \lambda}{S}\sqrt{n} \approx N(0;1)$$

Confidence interval estimation

Solution: 2^{nd} method.

- By the following limit theorem:

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - \lambda}{S}\sqrt{n} \approx N(0;1)$$

- From the equation

$$\mathbb{P}\left(\frac{|\bar{X} - \lambda|}{S}\sqrt{n} \leq Z_{\alpha/2}\right) = 1 - \alpha$$

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- By the following limit theorem:

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - \lambda}{S}\sqrt{n} \approx N(0;1)$$

- From the equation

$$\mathbb{P}\left(\frac{|\bar{X} - \lambda|}{S}\sqrt{n} \leq Z_{\alpha/2}\right) = 1 - \alpha$$

or

$$\mathbb{P}\left(\bar{X} - Z_{\alpha/2}\frac{S}{\sqrt{n}} \leq \lambda \leq \bar{X} + Z_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1 - \alpha$$

Confidence interval estimation

Solution:

- Then a $1 - \alpha$ confidence interval estimation of λ is

$$\bar{X} \mp Z_{\alpha/2} \frac{S}{\sqrt{n}}$$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

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- For the given sample, we have $n = 50$; $Z_{\alpha/2} = Z_{0.05} = 1.645$; $\bar{x} = 1.9; s^2 = (7*0^2+15*1^2+10*2^2+21*3^2+6*4^2-50*1.9^2) = 1.602; s = \sqrt{1.602} = 1.27.$

Confidence interval estimation

Solution:

- Then a $1-\alpha$ confidence interval estimation of λ is

$$\bar{X} \mp Z_{\alpha/2}\frac{S}{\sqrt{n}}$$

- For the given sample, we have $n = 50$; $Z_{\alpha/2} = Z_{0.05} = 1.645$; $\bar{x} = 1.9$; $s^2 = (7*0^2 + 15*1^2 + 10*2^2 + 21*3^2 + 6*4^2 - 50*1.9^2) = 1.602$; $s = \sqrt{1.602} = 1.27$.
- So the 90% CI of λ is

$$1.9 \mp 1.645\frac{1.27}{\sqrt{50}} = [1.6; 2.2]$$

Confidence interval estimation

Solution:

- Then a $1 - \alpha$ confidence interval estimation of λ is

$$\bar{X} \mp Z_{\alpha/2} \frac{S}{\sqrt{n}}$$

- For the given sample, we have $n = 50$; $Z_{\alpha/2} = Z_{0.05} = 1.645$; $\bar{x} = 1.9; s^2 = (7 * 0^2 + 15 * 1^2 + 10 * 2^2 + 21 * 3^2 + 6 * 4^2 - 50 * 1.9^2) = 1.602; s = \sqrt{1.602} = 1.27$.
- So the 90% CI of λ is

$$1.9 \mp 1.645 \frac{1.27}{\sqrt{50}} = [1.6; 2.2]$$

- We are 90% confident that the parameter λ is between 1.6 and 2.2.