Chapter 3. Vectors of Random Variables

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Introduction

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- Studying each variable separately may give incomplete information.
- For example: when we observed the characteristics of a machine, we are interested in some different variables at the same time, such as the weight, size, quality, material, . . .

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- For simplicity, we study two-dimensional random variables (X, Y), where X and Y are one-dimensional variables.
- Most results can be extended easily to n-dimensional variables.
- This chapter analyzes experiments that produce two random variables, X and Y.
- If X and Y are discrete, we have a discrete two-dimensional random variable; if they are continuous, we have a continuous two-dimensional variable.

Joint probability distrubutions

Definitions 4.1:

Let X and Y be two discrete random variables, where the range of X is $S_X = \{x_1, x_2, .., x_n\}$ and the range of Y is $S_Y = \{y_1, y_2, .., y_m\}$.

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- The range of two-dimensional random variable (X, Y) is $S_{X,Y} = \{(x_i, y_j) : i = 1, ..., n; j = 1, ..., m\}$.

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- The range of two-dimensional random variable (X, Y) is $S_{X,Y} = \{(x_i, y_j) : i = 1, ..., n; j = 1, ..., m\}$.
- The joint probability distrubution of (X, Y) is defined by the joint probability mass function (pmf):

$$f(x, y) = P(X = x, Y = y) = \begin{cases} p_{ij} \text{ if } x = x_i, y = y_j, \\ 0 \text{ otherwise} \end{cases}$$

such that $p_{ij} \geq 0 \forall i, j$ and $\sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$.

Joint probability distrubutions

Definitions 4.1:

- The joint probability distrubution of two-dimensional discrete random variable (X, Y) is also given by the following table:

	y_1	y_2	$\dots$	y_j	$\dots$	y_m
x_1	p_{11}	p_{12}	$\dots$	p_{1j}	$\dots$	p_{1m}
x_2	p_{21}	p_{22}	$\dots$	p_{2j}	$\dots$	p_{2m}
.	.	.	...	.	...	.
.	.	.	...	.	...	.
.	.	.	...	.	...	.
x_i	p_{i1}	p_{i2}	$\dots$	p_{ij}	$\dots$	p_{im}
.	.	.	...	.	...	.
.	.	.	...	.	...	.
.	.	.	...	.	...	.
x_n	p_{n1}	p_{n2}	$\dots$	p_{nj}	$\dots$	p_{nm}

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- For all $(x, y) \in R^2 : f(x, y) \geq 0$.

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Let X and Y be two continuous random variables. The joint probability distrubution of (X, Y) is defined by the joint probability density function (pdf) $f(x, y)$ that satisfies the following requirements:

- For all $(x, y) \in R^2 : f(x, y) \geq 0$.
- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dxdy = 1$.

Joint probability distrubutions

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Let X and Y be two continuous random variables. The joint probability distrubution of (X, Y) is defined by the joint probability density function (pdf) $f(x, y)$ that satisfies the following requirements:

- For all $(x, y) \in R^2 : f(x, y) \geq 0$.
- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)dxdy = 1$.
- For $D \subset R^2$: $P[(X, Y) \in D] = \int \int_D f(x, y)dxdy$.

Joint probability distrubutions

Definitions 4.3:

Let (X, Y) be a two-dimensional random variable. The joint cumulative distribution function (cdf) of (X, Y) is defined by:

$$F(x, y) = P(X < x, Y < y), \forall (x, y) \in R^2.$$

Joint probability distrubutions

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- If (X, Y) is discrete then $F(x, y) = \sum_{x_i < x} \sum_{y_j < y} P(X = x_i, Y = y_j)$.

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- If (X, Y) is discrete then $F(x, y) = \sum_{x_i < x} \sum_{y_j < y} P(X = x_i, Y = y_j)$.
- If (X, Y) is continuous then $F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) dudv$.

Marginal probability distribution

Definitions 4.4:

Let (X, Y) be a two-dimensional discrete random variable with the joint pmf

$$f(x,y) = P(X = x, Y = y) = \begin{cases} p_{ij} \text{ if } (x,y) = (x_i, y_j), \\ 0 \text{ otherwise} \end{cases}$$

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- The marginal probability distribution of X is the defined by:

X	x_1	x_2	...	x_i	...	x_n
P	$p_{1\cdot}$	$p_{2\cdot}$	...	$p_{i\cdot}$	...	$p_{n\cdot}$

where $p_{i\cdot} = \sum_{j=1}^{m} p_{ij}, i = 1, .., n$.

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P	$p_{1\cdot}$	$p_{2\cdot}$	...	$p_{i\cdot}$	...	$p_{n\cdot}$

  where $p_{i\cdot} = \sum_{j=1}^{m} p_{ij}, i = 1, .., n$.

- The marginal probability distribution of Y is the defined by:

Y	y_1	y_2	...	y_j	...	y_m
P	$p_{\cdot 1}$	$p_{\cdot 2}$	...	$p_{\cdot j}$	...	$p_{\cdot m}$

  where $p_{\cdot j} = \sum_{i=1}^{n} p_{ij}, j = 1, .., m$.

Marginal probability distrubutions

Definitions 4.4:

- The marginal probability distrubution of X and of Y:

	y_1	y_2	$\dots$	y_j	$\dots$	y_m	$P(x_i)$
x_1	p_{11}	p_{12}	$\dots$	p_{1j}	$\dots$	p_{1m}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	$\dots$	p_{2j}	$\dots$	p_{2m}	$p_{2\cdot}$
.	.	.	$\dots$	.	$\dots$	.	.
.	.	.	$\dots$	.	$\dots$	.	.
.	.	.	$\dots$	.	$\dots$	.	.
x_i	p_{i1}	p_{i2}	$\dots$	p_{ij}	$\dots$	p_{im}	$p_{i\cdot}$
.	.	.	$\dots$	.	$\dots$	.	.
.	.	.	$\dots$	.	$\dots$	.	.
.	.	.	$\dots$	.	$\dots$	.	.
x_n	p_{n1}	p_{n2}	$\dots$	p_{nj}	$\dots$	p_{nm}	$p_{n\cdot}$
$P(y_j)$	$p_{\cdot 1}$	$p_{\cdot 2}$	$\dots$	$p_{\cdot j}$	$\dots$	$p_{\cdot m}$	1

Marginal probability distribution

Definitions 4.5:

Let (X, Y) be a two-dimensional continuous random variable with the joint pdf $f(x, y)$.

- The marginal probability distribution of X is the defined by the marginal pdf:

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)dy.$$

Marginal probability distribution

Definitions 4.5:

Let (X, Y) be a two-dimensional continuous random variable with the joint pdf $f(x, y)$.

- The marginal probability distribution of X is the defined by the marginal pdf:
$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)dy.$$
- The marginal probability distribution of Y is the defined by the marginal pdf:
$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)dx.$$

Conditional probability distribution

Definitions 4.6:

Let (X, Y) be a two-dimensional discrete random variable with the joint pmf

$$f(x,y) = P(X = x, Y = y) = \begin{cases} p_{ij} \text{ if } (x,y) = (x_i, y_j), \\ 0 \text{ otherwise} \end{cases}$$

Conditional probability distribution

Definitions 4.6:

Let (X, Y) be a two-dimensional discrete random variable with the joint pmf

$$f(x,y) = P(X = x, Y = y) = \begin{cases} p_{ij} \text{ if } (x,y) = (x_i, y_j), \\ 0 \text{ otherwise} \end{cases}$$

- The conditional probability distribution of X given that $Y = y_j$ is defined by:

$X \mid Y = y_j$	x_1	x_2	...	x_i	...	x_n
P	$p_{1j}/p_{.j}$	$p_{2j}/p_{.j}$	...	$p_{ij}/p_{.j}$	...	$p_{nj}/p_{.j}$

since

$$P(X = x_i \mid Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{p_{ij}}{p_{.j}}, i = 1, .., n$$

.

Conditional probability distribution

Definitions 4.7:

Let (X, Y) be a two-dimensional discrete random variable with the joint pmf

$$f(x,y) = P(X = x, Y = y) = \begin{cases} p_{ij} \text{ if } (x,y) = (x_i, y_j), \\ 0 \text{ otherwise} \end{cases}$$

- The conditional probability distribution of Y given that $X = x_i$ is defined by:

$Y\|X = x_i$	y_1	y_2	...	y_j	...	y_m
P	$p_{i1}/p_{i\cdot}$	$p_{i2}/p_{i\cdot}$	...	$p_{ij}/p_{i\cdot}$	...	$p_{im}/p_{i\cdot}$

since

$$P(Y = y_j | X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)} = \frac{p_{ij}}{p_{i\cdot}}, j = 1, .., m$$

.

Conditional probability distribution

Definitions 4.7:

Let (X, Y) be a two-dimensional continuous random variable with the joint pdf $f(x, y)$.

- The conditional probability distribution of X givent that $Y = y$ is defined by conditional pdf:

$$f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)}.$$

Conditional probability distribution

Definitions 4.7:

Let (X, Y) be a two-dimensional continuous random variable with the joint pdf $f(x, y)$.

- The conditional probability distribution of X givent that $Y = y$ is defined by conditional pdf:

$$f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)}.$$

- The conditional probability distribution of Y givent that $X = x$ is defined by conditional pdf:

$$f_{Y|X=x}(y) = \frac{f(x, y)}{f_X(x)}.$$

Independence

Definitions 4.8:

- Let (X, Y) be a two-dimensional discrete random variable with the joint pmf $f(x, y)$. X and Y are called independent if and only if:

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$$

for all $i = 1, .., n; j = 1, ..., m$.

Independence

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- Let (X, Y) be a two-dimensional discrete random variable with the joint pmf $f(x, y)$. X and Y are called independent if and only if:

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$$

for all $i = 1, .., n; j = 1, ..., m$.

- Let (X, Y) be a two-dimensional continuous random variable with the joint pdf $f(x, y)$. X and Y are called independent if and only if:

$$f(x, y) = f_X(x)f_Y(y)$$

for all $(x, y) \in R^2$.

Examples

Example 4.1:

Let X be the number of accidents per day and Y be the number of injured or dead people per day at a small town. We observed X and Y during a period of 100 days and obtained the following data:

$X \setminus Y$	0	1	2	3
0	8	0	0	0
1	22	9	1	0
2	20	10	3	1
3	8	5	2	1
4	4	3	2	1

Examples

Example 4.1:

Let X be the number of accidents per day and Y be the number of injured or dead people per day at a small town.

- Find the joint probability distribution of (X, Y).
- Find the marginal probability distribution of X and of Y.
- Find the conditional probability distribution of X given that $Y = 2$.
- Find the conditional probability distribution of Y given that $X = 2$.
- Are X and Y independent?

Examples

Solution of Example 4.1:

The joint probability distribution of (X, Y) is the following table:

$X \setminus Y$	0	1	2	3
0	0.08	0	0	0
1	0.22	0.09	0.01	0
2	0.2	0.1	0.03	0.01
3	0.08	0.05	0.02	0.01
4	0.04	0.03	0.02	0.01

Examples

Solution of Example 4.1:

To find the marginal probability distributions of X and of Y, we add the probabilities by rows and by columns:

$X \setminus Y$	0	1	2	3	$P(x_i)$
0	0.08	0	0	0	0.08
1	0.22	0.09	0.01	0	0.32
2	0.2	0.1	0.03	0.01	0.34
3	0.08	0.05	0.02	0.01	0.16
4	0.04	0.03	0.02	0.01	0.1
$P(y_j)$	0.62	0.27	0.08	0.03	1

Examples

Solution of Example 4.1:

- The marginal probability distribution of X is the following table:

X	0	1	2	3	4
P	0.08	0.32	0.34	0.16	0.1

Examples

Solution of Example 4.1:

- The marginal probability distribution of X is the following table:

X	0	1	2	3	4
P	0.08	0.32	0.34	0.16	0.1

- The marginal probability distribution of Y is the following table:

X	0	1	2	3
P	0.62	0.27	0.08	0.03

Examples

Solution of Example 4.1:

- The conditional probability distribution of X given that $Y = 2$ is the following table:

$X\|Y = 2$	0	1	2	3	4
P	0	1/8	3/8	2/8	2/8

Examples

Solution of Example 4.1:

- The conditional probability distribution of X given that $Y = 2$ is the following table:

$X\|Y = 2$	0	1	2	3	4
P	0	1/8	3/8	2/8	2/8

- The conditional probability distribution of Y given that $X = 2$ is the following table:

$Y\|X = 2$	0	1	2	3
P	20/34	10/34	3/34	1/34

Examples

Solution of Example 4.1:

- The conditional probability distribution of X given that $Y = 2$ is the following table:

$X\|Y = 2$	0	1	2	3	4
P	0	1/8	3/8	2/8	2/8

- The conditional probability distribution of Y given that $X = 2$ is the following table:

$Y\|X = 2$	0	1	2	3
P	20/34	10/34	3/34	1/34

- Since $0.08 = P(X = 0, Y = 0) \neq P(X = 0)P(Y = 0) = 0.08 * 0.62 = 0.0496$ then X and Y are not independent.

Examples

Example 4.2:

Suppose that the two-dimensional continuous random variable (X, Y) has the following joint pdf:

$$f(x,y) = \begin{cases} Cxy \text{ if } 0 \leq x \leq 4; 1 \leq y \leq 5, \\ 0 \text{ otherwise} \end{cases}$$

- Find the normalizing constant C.
- Find the marginal probability distribution of X and of Y.
- Find the conditional probability distribution of X given that $Y = 2$.
- Find the conditional probability distribution of Y given that $X = 2$.
- Are X and Y independent?

Examples

Solution of Example 4.2:

Find the normalizing constant C.

- Since $f(x, y)$ is a joint pdf then $f(x, y) \geq 0, \forall (x, y) \in R^2 \Leftrightarrow C \geq 0$.

Examples

Solution of Example 4.2:

Find the normalizing constant C.

- Since $f(x,y)$ is a joint pdf then $f(x,y) \geq 0, \forall (x,y) \in R^2 \Leftrightarrow C \geq 0$.
- And $\int\int_{R^2} f(x,y)dxdy = \int_0^4 \int_1^5 Cxydxdy = C\int_0^4 \left(x\int_1^5 ydy\right)dx =$ $96C = 1$, so $C = 1/96$.

Examples

Solution of Example 4.2:

Find the normalizing constant C.

- Since $f(x,y)$ is a joint pdf then $f(x,y) \geq 0, \forall (x,y) \in R^2 \Leftrightarrow C \geq 0$.
- And $\int\int_{R^2} f(x,y)dxdy = \int_0^4 \int_1^5 Cxydxdy = C\int_0^4 \left(x\int_1^5 ydy\right)dx = 96C = 1$, so $C = 1/96$.
- Then

$$f(x,y) = \begin{cases} \frac{1}{96}xy \text{ if } 0 \leq x \leq 4; 1 \leq y \leq 5, \\ 0 \text{ otherwise} \end{cases}$$

Examples

Solution of Example 4.2:

Find the marginal probability distribution of X.

- The marginal pdf of X is $f_X(x) = \int_{-\infty}^{+\infty} f(x, y)dy$.

Examples

Solution of Example 4.2:

Find the marginal probability distribution of X.

- The marginal pdf of X is $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$.
- If $x \notin [0,4]$ then $f_X(x) = \int_{-\infty}^{+\infty} 0dy = 0$.

Examples

Solution of Example 4.2:

Find the marginal probability distribution of X.

- The marginal pdf of X is $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$.
- If $x \notin [0,4]$ then $f_X(x) = \int_{-\infty}^{+\infty} 0dy = 0$.
- If $x \in [0,4]$ then $f_X(x) = \int_{-\infty}^{1} 0dy + \int_{1}^{5} \frac{1}{96}xydy + \int_{5}^{+\infty} 0dy = \frac{1}{8}x$.

Examples

Solution of Example 4.2:

Find the marginal probability distribution of X.

- The marginal pdf of X is $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$.
- If $x \notin [0,4]$ then $f_X(x) = \int_{-\infty}^{+\infty} 0dy = 0$.
- If $x \in [0,4]$ then $f_X(x) = \int_{-\infty}^{1} 0dy + \int_{1}^{5} \frac{1}{96}xydy + \int_{5}^{+\infty} 0dy = \frac{1}{8}x$.
- Then

$$f_X(x) = \begin{cases} \frac{1}{8}x \text{ if } 0 \leq x \leq 4, \\ 0 \text{ otherwise} \end{cases}$$

Examples

Solution of Example 4.2:

Find the marginal probability distribution of Y.

- The marginal pdf of Y is $f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)dx$.

Examples

Solution of Example 4.2:

Find the marginal probability distribution of Y.

- The marginal pdf of Y is $f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)dx$.
- If $y \notin [1, 5]$ then $f_Y(y) = \int_{-\infty}^{+\infty} 0dx = 0$.

Examples

Solution of Example 4.2:

Find the marginal probability distribution of Y.

- The marginal pdf of Y is $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx$.
- If $y \notin [1,5]$ then $f_Y(y) = \int_{-\infty}^{+\infty} 0dx = 0$.
- If $y \in [1,5]$ then $f_Y(y) = \int_{-\infty}^{0} 0dx + \int_{0}^{4} \frac{1}{96}xydx + \int_{4}^{+\infty} 0dx = \frac{1}{12}y$.

Examples

Solution of Example 4.2:

Find the marginal probability distribution of Y.

- The marginal pdf of Y is $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx$.
- If $y \notin [1,5]$ then $f_Y(y) = \int_{-\infty}^{+\infty} 0dx = 0$.
- If $y \in [1,5]$ then $f_Y(y) = \int_{-\infty}^{0} 0dx + \int_0^4 \frac{1}{96}xydx + \int_4^{+\infty} 0dx = \frac{1}{12}y$.
- Then

$$f_Y(y) = \begin{cases} \frac{1}{12}y \text{ if } 1 \le y \le 5, \\ 0 \text{ otherwise} \end{cases}$$

Examples

Solution of Example 4.2:

Find the conditional probability distribution of X given that $Y = 2$.

- The conditional pdf of X given that $Y = 2$ is

$$f_{X|Y=2}(x) = \frac{f(x,2)}{f_Y(2)} = \begin{cases} \frac{(1/96)x.2}{2/12} = \frac{1}{8}x \text{ if } 0 \le x \le 4, \\ \frac{0}{2/12} = 0 \text{ otherwise} \end{cases}$$

Examples

Solution of Example 4.2:

Find the conditional probability distribution of Y given that $X = 2$.

- The conditional pdf of Y given that $X = 2$ is

$$f_{Y|X=2}(y) = \frac{f(2,y)}{f_X(2)} = \begin{cases} \frac{(1/96)2y}{2/8} = \frac{1}{12}y \text{ if } 1 \leq y \leq 5, \\ \frac{0}{2/8} = 0 \text{ otherwise} \end{cases}$$

Examples

Solution of Example 4.2:

Find the conditional probability distribution of Y given that $X = 2$.

- The conditional pdf of Y given that $X = 2$ is

$$f_{Y|X=2}(y) = \frac{f(2,y)}{f_X(2)} = \begin{cases} \frac{(1/96)2y}{2/8} = \frac{1}{12}y \text{ if } 1 \leq y \leq 5, \\ \frac{0}{2/8} = 0 \text{ otherwise} \end{cases}$$

Are X and Y independent?

- Since $f(x,y) = f_X(x)f_Y(y), \forall (x,y) \in R^2$ then X and Y are independent.

Functions of random vectors

Let (X, Y) be a two-dimensional random variable and $g : R^2 \to R$ be a function. The random variable Z is defined by $Z = g(X, Y)$. Then

- $E(Z) = E[g(X, Y)] = \sum_{i=1}^{n} \sum_{j=1}^{m} g(x_i, y_j) p_{ij}$ if (X, Y) is discrete.

Functions of random vectors

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- $E(Z) = E[g(X, Y)] = \sum_{i=1}^{n} \sum_{j=1}^{m} g(x_i, y_j) p_{ij}$ if (X, Y) is discrete.
- and $E(Z) = E[g(X, Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f(x, y) dxdy$ if (X, Y) is continuous.

Covariance

Definitions 4.9:

Let (X, Y) be a two-dimensional random variable. The covariance of (X, Y) is defined by:

$$cov(X, Y) = E(XY) - E(X)E(Y)$$

Covariance

Definitions 4.9:

Let (X, Y) be a two-dimensional random variable. The covariance of (X, Y) is defined by:

$$cov(X, Y) = E(XY) - E(X)E(Y)$$

where

- $E(X) = \sum_{i=1}^{n} x_i p_{i\cdot} = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i p_{ij}$,
- $E(Y) = \sum_{j=1}^{m} y_j p_{\cdot j} = \sum_{i=1}^{n} \sum_{j=1}^{m} y_j p_{ij}$,
- $E(XY) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j p_{ij}$,

if (X, Y) is discrete.

Covariance

Definitions 4.9:

Let (X, Y) be a two-dimensional random variable. The covariance of (X, Y) is defined by:

$$cov(X, Y) = E(XY) - E(X)E(Y)$$

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Let (X, Y) be a two-dimensional random variable. The covariance of (X, Y) is defined by:

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where

- $E(X) = \int_{-\infty}^{+\infty} xf_X(x)dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xf(x, y)dxdy$,
- $E(Y) = \int_{-\infty}^{+\infty} yf_Y(y)dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} yf(x, y)dxdy$,
- $E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x, y)dxdy$,

if (X, Y) is continuous.

Covariance

Properties:

- If X and Y are independent then $cov(X, Y) = 0$, but the opposite is not always true.

Covariance

Properties:

- If X and Y are independent then $cov(X, Y) = 0$, but the opposite is not always true.
- If $cov(X, Y) \neq 0$ then X and Y are called to be related and if $cov(X, Y) = 0$ then X and Y are not related.

Covariance matrix

Definitions 4.10:

Let (X, Y) be a two-dimensional random variable. The covariance matrix of (X, Y) is defined by:

$$\Gamma(X, Y) = \begin{bmatrix} V(X) & cov(X, Y) \\ cov(Y, X) & V(Y) \end{bmatrix}$$

where $cov(Y, X) = cov(X, Y)$.

Coefficient of correlation

Definitions 4.11:

Let (X, Y) be a two-dimensional random variable. The coefficient of correlation between X and Y is defined by:

$$\rho(X, Y) = \frac{cov(X, Y)}{\sqrt{V(X)V(Y)}}.$$

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Properties:

- If X and Y are independent then $\rho(X, Y) = 0$, but the opposite is not always true.
- If $\rho(X, Y) \neq 0$ then X and Y are related and if $\rho(X, Y) = 0$ then X and Y are not related.

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- If $\rho(X, Y) \neq 0$ then X and Y are related and if $\rho(X, Y) = 0$ then X and Y are not related.
- The range of $\rho(X, Y)$ is $[-1, 1]$: $-1 \leq \rho(X, Y) \leq 1$.
- If $Y = a + bX$ with $b > 0$ then $\rho(X, Y) = 1$ and if $Y = a + bX$ with $b < 0$ then $\rho(X, Y) = -1$.

Examples

Example 4.3:

Let X be the number of accidents per day and Y be the number of injured or dead people per day at a small town. We observed X and Y during a period of 100 days and obtained the following data:

$X \setminus Y$	0	1	2	3
0	8	0	0	0
1	22	9	1	0
2	20	10	3	1
3	8	5	2	1
4	4	3	2	1

- Find the covariance matrix of (X, Y).
- Find the coefficient of correlation between X and Y.
- Are X and Y related?

Examples

Solution of Example 4.3:

Find the covariance matrix of (X, Y).

- $E(XY) = 0 * 0 * 0.08 + 1 * 0 * 0.22 + ... + 4 * 3 * 0.01 = 1.25$

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Then

- The covariance of (X, Y) is $cov(X, Y) = E(XY) - E(X)E(Y) = 0.2724$

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Then

- The covariance of (X, Y) is $cov(X, Y) = E(XY) - E(X)E(Y) = 0.2724$
- The covariance matrix of (X, Y) is

$$\Gamma(X, Y) = \begin{bmatrix} V(X) & cov(X, Y) \\ cov(Y, X) & V(Y) \end{bmatrix} = \begin{bmatrix} 1.1856 & 0.2724 \\ 0.2724 & 0.5896 \end{bmatrix}$$

Examples

Solution of Example 4.3:

Find the coefficient of correlation between X and Y.

- The coefficient of correlation between X and Y is

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{0.2724}{\sqrt{1.1856 * 0.5896}} = 0.823$$

Examples

Solution of Example 4.3:

Find the coefficient of correlation between X and Y.

- The coefficient of correlation between X and Y is

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{0.2724}{\sqrt{1.1856 * 0.5896}} = 0.823$$

Are X and Y related?

- Since $\rho(X,Y) = 0.823 \neq 0$ then X and Y are related and since $\rho(X,Y) = 0.823$ is close to 1, then X and Y have a strong positive linear relationship.

Examples

Example 4.4:

Suppose that the two-dimensional continuous random variable (X, Y) has the following joint pdf:

$$f(x,y) = \begin{cases} \frac{1}{96}xy \text{ if } 0 \leq x \leq 4; 1 \leq y \leq 5, \\ 0 \text{ otherwise} \end{cases}$$

- Find the covariance matrix of (X, Y).
- Find the coefficient of correlation between X and Y.
- Are X and Y related?

Examples

Solution of Example 4.4:

We have

- $\mu_X = E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_0^4 \frac{1}{8} x^2 dx = \frac{8}{3}$

Examples

Solution of Example 4.4:

We have

- $\mu_X = E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx = \int_0^4 \frac{1}{8} x^2 dx = \frac{8}{3}$
- $V(X) = \int_{-\infty}^{+\infty} x^2 f_X(x) dx - \mu_X^2 = \int_0^4 \frac{1}{8} x^3 dx - (8/3)^2 = \frac{8}{9}$

Examples

Solution of Example 4.4:

We have

- $\mu_X = E(X) = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_0^4 \frac{1}{8}x^2 dx = \frac{8}{3}$
- $V(X) = \int_{-\infty}^{+\infty} x^2 f_X(x)dx - \mu_X^2 = \int_0^4 \frac{1}{8}x^3 dx - (8/3)^2 = \frac{8}{9}$
- $\mu_Y = E(Y) = \int_{-\infty}^{+\infty} y f_Y(y)dy = \int_1^5 \frac{1}{12}y^2 dy = \frac{31}{9}$.

Examples

Solution of Example 4.4:

We have

- $\mu_X = E(X) = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_0^4 \frac{1}{8}x^2 dx = \frac{8}{3}$
- $V(X) = \int_{-\infty}^{+\infty} x^2 f_X(x)dx - \mu_X^2 = \int_0^4 \frac{1}{8}x^3 dx - (8/3)^2 = \frac{8}{9}$
- $\mu_Y = E(Y) = \int_{-\infty}^{+\infty} y f_Y(y)dy = \int_1^5 \frac{1}{12}y^2 dy = \frac{31}{9}$.
- $V(Y) = \int_{-\infty}^{+\infty} y^2 f_Y(y)dy - \mu_Y^2 = \int_1^5 \frac{1}{12}y^3 dy - (31/9)^2 = \frac{92}{81}$.

Examples

Solution of Example 4.4:

We have

- $\mu_X = E(X) = \int_{-\infty}^{+\infty} x f_X(x)dx = \int_0^4 \frac{1}{8}x^2 dx = \frac{8}{3}$
- $V(X) = \int_{-\infty}^{+\infty} x^2 f_X(x)dx - \mu_X^2 = \int_0^4 \frac{1}{8}x^3 dx - (8/3)^2 = \frac{8}{9}$
- $\mu_Y = E(Y) = \int_{-\infty}^{+\infty} y f_Y(y)dy = \int_1^5 \frac{1}{12}y^2 dy = \frac{31}{9}$.
- $V(Y) = \int_{-\infty}^{+\infty} y^2 f_Y(y)dy - \mu_Y^2 = \int_1^5 \frac{1}{12}y^3 dy - (31/9)^2 = \frac{92}{81}$.
- $E(XY) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x,y)dxdy = \frac{1}{96} \int_0^4 \int_1^5 x^2y^2 dxdy = \frac{248}{27}$

Examples

Solution of Example 4.4:

Then

- The covariance of (X, Y) is
$cov(X, Y) = E(XY) - E(X)E(Y) = \frac{248}{27} - \frac{8}{3}\frac{31}{9} = 0.$

Examples

Solution of Example 4.4:

Then

- The covariance of (X, Y) is
  $cov(X, Y) = E(XY) - E(X)E(Y) = \frac{248}{27} - \frac{8}{3}\frac{31}{9} = 0.$
- The covariance matrix of (X, Y) is

$$\Gamma(X, Y) = \begin{bmatrix} V(X) & cov(X, Y) \\ cov(Y, X) & V(Y) \end{bmatrix} = \begin{bmatrix} \frac{8}{9} & 0 \\ 0 & \frac{92}{81} \end{bmatrix}$$

Examples

Solution of Example 4.4:

- The coefficient of correlation between X and Y is

$$\rho(X, Y) = \frac{cov(X, Y)}{\sqrt{V(X)V(Y)}} = \frac{0}{\sqrt{(8/9)(92/81)}} = 0$$

Examples

Solution of Example 4.4:

- The coefficient of correlation between X and Y is

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{0}{\sqrt{(8/9)(92/81)}} = 0$$

Are X and Y related?

- Since $\rho = 0$ then X and Y are not related.