ĐẠI HỌC BÁCH KHOA HÀ NỘI
VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

Discrete Mathematics

Course preparation group:
Nguyễn Khánh Phương
Đỗ Phan Thuận
Phạm Quang Dũng
Huỳnh Thanh Bình
Trần Vĩnh Đức
Bùi Quốc Trung
Đinh Viết Sang
Ban Hà Bằng

Contents of Part 1

Chapter 0: Sets, Relations

Chapter 1: Counting problem

Chapter 2: Existence problem

Chapter 3: Enumeration problem

Chapter 4: Combinatorial optimization problem

PART 1

COMBINATORIAL THEORY

(Lý thuyết tổ hợp)

PART 2

GRAPH THEORY

(Lý thuyết đồ thị)

Content

1. **Introduction to existence problems**
2. Basic proof methods
3. Dirichlet principle (pigeonhole principle)

1. Introduction to existence problems

- In the "Counting problem" chapter, we focused on counting the combinatorial configurations. In those problems, the existence of the configurations is obvious, and the main object is to count the number of elements that satisfy the given properties.
- However, in many combinatorial problems, it is very difficult to point out the existence of a configuration that satisfies given properties:
  - For example, when a player needs to calculate his moves to answer whether there is a possibility of winning or not?
  - A person needs to search for the key to decipher a secret code that he does not know if this is really the opponent's encrypted message, or just the secret code issued by the opponent to ensure the safety of real telegrams ...
- In combinatorics, besides the counting problem, there is another very important problem is considering the existence of combinatorial configurations satisfying given properties - the problem of existence.

Content

1. Introduction to existence problems
2. **Basic proof methods**
3. Dirichlet principle (pigeonhole principle)

The 36 officers problem

- This problem is proposed by Euler, it is described as following:

"You're in command of an army that consists of six regiments, each containing six officers of six different ranks. Can you arrange the officers in a 6x6 square so that each row and also each column of the square holds officers of all 6 ranks and all 6 regiments?

2. Basic proof methods

- **2.1. Direct Proof**

2.2. Proof by Contradiction

2.3. Proof by Contrapositive

2.4. Proof by Mathematical Induction

2.1. Direct proofs

We begin with an example demonstrating the transitivity of divisibility.

Theorem. If a divides b and b divides c then a divides c.

Prove. By using the definition of the divisibility, there exist integers k_1 and k_2 such that

$a = b\, k_1$ and $b = c\, k_2$.

Then

$a = b\, k_2 = c\, k_1\, k_2$.

Let $k = k_1 k_2$. We have k as an integer, and $a = ck$, so by the definition of divisibility, a divides c.

2. Basic proof methods

2.1. Direct Proof
2.2. Proof by Contradiction
2.3. Proof by Contrapositive
2.4. Proof by Mathematical Induction

2.1. Direct proofs

If P, Then Q

- In most theorems, exercises or tests, you need to prove the form "If P, Then Q".
- In this example: "if a divides b and b divides c, then a divides c"
  - "P" is "If a divides b and b divides c" and "Q" is "a divides c".
- This is the standard state of many theorems.
- The direct proof can be conceived as a series of inferences beginning with "P" and ending with "Q":

$$P \Rightarrow \ldots \Rightarrow Q$$

2.2. Proof by Contradiction

Requirement: Proof the statement P

Proof by contradiction:

- Assume it is false (Assume ¬P)
- Prove that ¬P cannot occur
  - (it means a contradiction exists: not satisfying the properties given in the problem or come to the absurd such as 1 = 0)

Requirement: Proof "If P, Then Q",

Proof by contradiction:

- Assume it is false (Assume that "P and Not Q" are true).
- It thus means a contradiction exists.

2. Basic proof methods

2.1. Direct Proof
2.2. Proof by Contradiction
2.3. Proof by Contrapositive
2.4. Proof by Mathematical Induction

2. Basic proof methods

2.1. Direct Proof
2.2. Proof by Contradiction
2.3. Proof by Contrapositive
2.4. Proof by Mathematical Induction

2.3. Proof by Contrapositive (Chứng minh bằng phản đề)

Proof by contrapositive uses the logically equivalence of two statement "If P then Q" ($P \Rightarrow Q$) and "If not Q then not P ($\neg Q \Rightarrow \neg P$):

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

Example:

The statement "If this is my car, then its color is red"

is equivalent to

"If its color is not red, then it is not my car".

- Thus, to prove "If P, then Q" by using contrapositive proof, we prove "If not Q then not P").

2.4. Proof by Mathematical Induction

- This is a very useful proof technique when we have to prove that the proposition $P(n)$ is true for all natural numbers $n \geq n_0$.
- Similar to the "domino effect" principle.

Outline of proof by Induction:

- Basic step: Prove the first statement $P(n_0)$ is true
- Inductive step: Given any integer $n \geq n_0$, prove that $P(n) \rightarrow P(n+1)$ (Assuming $P(n)$ is true and showing it forces $P(n+1)$ is true)
- Conclusion: $P(n)$ is true $\forall n \geq n_0$

(The assumption that $P(n)$ is true is called the inductive hypothesis)

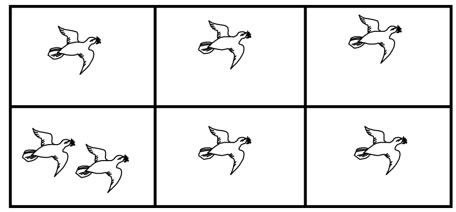
Content

1. Introduction to existence problems
2. Basic proof methods
3. **Dirichlet principle (pigeonhole principle)**

3.1. Dirichlet principle

If putting more than n objects into n boxes then at least one box has at least 2 objects (≥ 2).

- 7 objects
- 6 boxes



Proof. (Contradiction).

3. Dirichlet principle

3.1. Principle statement

3.2. Application examples

3.1. Dirichlet principle

The above principle has been successfully applied by the German mathematician Dirichlet to solving many existence problems in combinatorics.

It is also presented in the language of pigeons:

"*If one put more than n pigeons into n pigeonholes, then at least one hole has more than one pigeon* (**≥ 2)**."

So the principle is also known as " **Pigeonhole principle** ".

Example

If putting more than n objects into n boxes then at least one box has at least 2 objects (≥ 2).

Example 1. Among 13 people, there are always 2 people born in the same month as there are only 12 months.

Example 2. In the exam, the test score is assessed by an integer between 0 and 100. Then at least how many students must take the test so that it is certainly to exist 2 students get same result ?

Solution. There are 101 different results

➔ Using Dirichlet principle, the number of students is 102

VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

Example

Dirichlet principle: "If putting n objects into k boxes, one could always find at least one box containing $\geq \lceil n/k \rceil$ objects".

Example 3. In a group of 100 people, what is the minimum number of people that were born in the same month ?.

Solution: Putting people born in the same month into one group. There are 12 months. Therefore, according to the Dirichlet principle, there exists at least one group consisting $\geq \lceil 100/12 \rceil = 9$ people

VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

Generalized Pigeonhole Principle

Dirichlet principle: If putting more than n objects into n boxes then at least one box has ≥ 2 objects.

When the number of objects putting into k boxes is much larger than the k, it is obviously that the claim in the principle about the existence of a box containing at least two objects is too small. In such a case, we use the following generalized Dirichlet principle:

"If putting n objects into k boxes, one could always find at least one box containing $\geq \lceil n/k \rceil$ objects".

Here the symbol $\lceil \alpha \rceil$ is the least integer greater than or equal to α.

e.g.: $\lceil 3.14 \rceil = 4$

VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG