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Discrete Mathematics

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1.1. The sum rule

1.2. The product rule

1.1. The sum rule

Generalized sum rule: If we have tasks $T_1, T_2, \ldots, T_k$ that can be done in $m_1, m_2, \ldots, m_k$ ways, respectively, and any two of these tasks can not be done at the same time, then there are $m_1 + m_2 + \ldots + m_k$ ways to do one of these tasks.

➔The sum rule can also be phrased in terms of *set theory:* The size of the union on k finite pair wise disjoint sets is the sum of their sizes:

- Let $A_1, A_2, \ldots, A_k$ be disjoint sets. Then the number of ways to choose any element from one of these sets is

$$|A_1 \cup A_2 \cup \ldots \cup A_k| = |A_1| + |A_2| + \ldots + |A_k|.$$

1.1. The sum rule

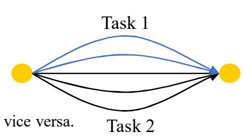
Let us consider two tasks:

m_1 is the number of ways to do **task 1**

m_2 is the number of ways to do **task 2**

Tasks are <u>independent</u> of each other, i.e.,

Performing **task 1** does not accomplish **task 2** and vice versa.



<u>*Sum rule*</u>: the number of ways that "**either** task 1 **or** task 2 can be done, but **not both**", is $m_1 + m_2$.

Generalizes to multiple tasks ...

Things	1	2	3	…	k
ways	m_1	m_2	m_3	…	m_k

select one of them: $m_1 + m_2 + m_3 + \ldots + m_k$ **ways**

1. Basic counting principles

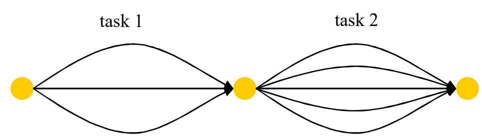
1.1. The sum rule

1.2. The product rule

The product rule

- Consider two tasks:
  - m_1 is the number of ways to do task 1
  - m_2 is the number of ways to do task 2
  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing task 1does not accomplish task 2 and vice versa.

<u>Product rule</u>: the number of ways that "both tasks 1 and 2 can be done" in m_1m_2.



- Generalize to multiple tasks ...

1.2. The product rule

Generalized product rule: If we have a procedure consisting of sequential tasks $T_1, T_2, \ldots, T_k$ that can be done in $m_1, m_2, \ldots, m_k$ ways, respectively, then there are $m_1 * m_2 * \ldots * m_k$ ways to carry out the procedure.

➔The product rule can also be phrased in terms of *set theory:* Let $A_1, A_2, \ldots, A_k$ be finite sets. Then the number of ways to choose one element from each set in the order of $A_1, A_2, \ldots, A_k$ is

$$|A_1 \text{ x } A_2 \text{ x} \ldots \text{ x } A_k| = |A_1| * |A_2| * \ldots * |A_k|.$$

1.2. The product rule

- If each element a_i of k- tuple $(a_1, a_2, \ldots, a_k)$ has m_i number of ways to select ($i = 1, 2, \ldots, k$), then the number of tuples could be generated is the product of these $m_1m_2 \ldots m_k$

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2.1. Permutation

2.2. Combination

2. Elementary combinatorial configuration

2.1. Permutation

- 2.1.1. Permutation
- 2.1.2. Circulation permutation
- 2.1.3. Permutation of multisets

2.2. Combination

2.1.1. Permutation

A *permutation* of a set A of objects is an ordered arrangement of the elements of A where each element appears only once

Example: If $A = \{a, b, c\}$, then the permutations of A are

1. *abc*
2. *acb*
3. *bac*
4. *bca*
5. *cab*
6. *cba*

The number of permutations of any set with n elements is

$$P(n) = n! = n(n-1) \cdots 2 \cdot 1$$

(*Note that by definition* $0! = 1$)

2. Elementary combinatorial configuration

2.1. Permutation

- 2.1.1. Permutation
- **2.1.2. Circulation permutation**
- 2.1.3. Permutation of multisets

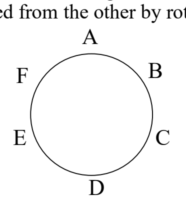
2.2. Combination

2.1.2. Circulation permutation

Circulation permutation $n!/n=(n-1)!$

Example 3: 6 people A, B, C, D, E, F are seated around a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotations?

ABCDEF,
BCDEFA,
CDEFAB,
DEFABC,
EFABCD,
FABCDE



are the same arrangements circularly

➔ There are 6! ways to seat 6 people around the table

For each seating, there are 6 "rotations" of the seating

Thus, the final answer is 6!/6 = 5! = 120

2. Elementary combinatorial configuration

2.1. Permutation

2.1.1. Permutation

2.1.2. Circulation permutation

2.1.3. Permutation of multisets

2.2. Combination

2.1.4. Permutations of multisets

A *multiset* M is a collection whose members need not be distinct.

Example: The collection

$M = (a, a, a, b, b, c, d, d, d, 1, 2, 2, 2, 3, 3, 3, 3)$

is a multiset; and sometimes it is convenient to write

$M = (3a, 2b, c, 3d, 1, 3\bullet 2, 4\bullet 3)$.

A multiset M over a set S can be viewed as a function $v : S \to \mathbb{N}$ from S to the set $\mathbb{N}$ of nonnegative integers; each element $x \in S$ is repeated $t(x)$ times in M; we write $M = (S; t)$.

2.1.4. Permutation of multisets

Let M be a multiset and $|M| = n$.

Proposition 1. *Let M be a multiset of r different types where each type has infinitely elements. Then the number of k-permutations of M equals r^k*

Example 1. What is the number of binary numerals with at most 4 digits?

Solution: The question is to find the number of 4-permutations of the multiset $(\infty 0, \infty 1)$. Thus the answer is $2^4 = 16$.

2.1.4. Permutation of multisets

Proposition 2. *Let M be a multiset of r different types with repetition numbers $n_1, n_2, \ldots, n_r$ respectively. Let $n = n_1 + n_2 + \ldots + n_r$. Then the number of permutations of M equals*

$$\frac{n!}{n_1!\, n_2! \ldots n_r!}$$

Proof.

2. Elementary combinatorial configuration

2.1. Permutation

2.2. Combination

2.2. Combination

2.1.1. Definitions

2.2.2. Binomial coefficients

2.2.3. Combinations of Multisets

2.2.4. Multinomial coefficients

2.2.1. Definitions

A k-combination of a set of n elements is a subset of size k of n elements.

(Note: A permutation is a sequence while a combination is a set)

Example:

The 2-permutation (sequence) of SOHN is:

SO, SH, SN, OH, ON, OS, HN, HS, HO, NS, NO, NH

The 2-combination (set) of SOHN is:

{S,O},{S,H},{S,N},{O,H},{O,N},{H,N}

2.2.1. Definitions

The number of all k-combinations of a set of n elements denoted

$$\binom{n}{k},\ C_n^k \text{ or } C(n,k)$$

and read "*n choose k*".

$$\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$$

Proof ???

This number is also called a *binomial coefficient* because such numbers occur as coefficients in the expansions of powers of binomial expressions such as $(a+b)^n$

Binomial Coefficients

$(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$

$$= \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

Binomial Theorem: Let x and y be variables, and let n be any nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

2.2. Combination

2.1.1. Definitions

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2.2.3. Combinations of Multisets

2.2.4. Multinomial coefficients

2.2. Combination

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2.2.4. Multinomial coefficients

2.2.3. Combinations of multisets

Let M be a multiset $\{\infty a_1, \infty a_2, \ldots, \infty a_n\}$ (M has n distinct objects):

A *k-combination* of M is an unordered collection of k objects selected from n types of objects of M.

A *k-combination* of M is also called an k-combination with repetition allowed.

The number of k-combination of M is C(n+ k-1, k) = C(n+ k-1, n-1)

(the number of selections, ***with repetitions***, of k objects from n distinct objects)

Need to divide k candies for n kids $B_1, B_2, \ldots, B_n$. How many different ways to divide?

Let t_j be the number of candies for kid B_j, j=1,…,n. At this point, the above problem leads to the problem:

Let k and n be non-negative integers. How many non-negative integers in the following equation have?

$$t_1 + t_2 + t_3 + \cdots + t_n = k$$

$$t_1, t_2, \cdots, t_n \in Z^+$$

2.2. Combination

2.2.4. Multinomial coefficients

The number of ordered arrangements of n objects, in which there are k_1 objects of type 1, k_2 objects of type 2, ..., and k_m objects of type m and where $k_1+k_2+ \ldots +k_m = n$, is

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\ldots k_m!}$$

2.2.4. Multinomial coefficients

The Multinomial Theorem:

$$(x_1 + x_2 + \ldots x_m)^n = \sum \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

where the summation is over all sequences of non-negative integers $(k_1, k_2, \ldots, k_m)$ such that $k_1 + k_2 + \ldots + k_m = n$.

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3. Inclusion-exclusion principle

3.1. Inclusion-exclusion principle

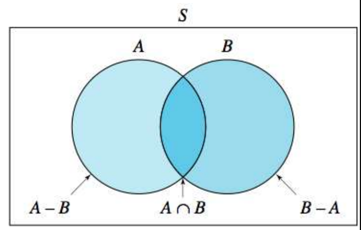
3.2. Derangement

3.1. Inclusion-exclusion principle

The inclusion–exclusion principle is an equation relating the sizes of two sets and their union. It states that if A and B are two (finite) sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

The meaning of the statement is that the number of elements in the union of the two sets is the sum of the elements in each set, respectively, minus the number of elements that are in both.



Similarly, for three sets A, B and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

3.1. Inclusion-exclusion principle

More generally, for finite sets $A_1, A_2, \ldots, A_m$

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = N_1 - N_2 + \cdots + (-1)^{m+1} N_m$$

where:

$$N_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le m} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|, \quad k = 1, 2, \ldots, m$$

Note: N_k is the sum of the cardinalities of all intersections of k from m given sets. For example:

$$N_1 = |A_1| + \ldots + |A_m|$$

$$N_m = |A_1 \cap A_2 \cap \ldots \cap A_m|$$

3. Inclusion-exclusion principle

3.1. Inclusion-exclusion principle

3.2. Derangement

3.2. Derangement

A ***derangement*** of $\{1, 2, \ldots, n\}$ is a permutation on the set such that none of elements i is placed at position ith in the permutation.

(In other words, derangement is a permutation that has no fixed points)

Example: $n = 5$

Permutation (2,3,5,4,1) is not a derangement

Permutation (2,3,5,1,4) is a derangement

Denote D_n the number of derangements of $\{1, 2, \ldots, n\}$ ➔ D_n = ??

3.2. Derangement

Denote D_n the number of derangements of $\{1, 2, \ldots, n\}$ ➔ D_n = ??

Let S = set of all permutations of $\{1, 2, \ldots, n\}$ ➔ $|S| = n!$

Let A_i = subset of permutations of $\{1, 2, \ldots, n\}$ such that the ith element = i in the permutation. $|A_1 \cup A_2 \cup \cdots \cup A_n|$???

➔ $|A_1 \cup A_2 \cup \cdots \cup A_n|$ counts the number of permutations in which at least one object i of the n objects appears in the ith position (its original position).

➔ D_n: number of permutations such that none of the n objects appears in their original positions. Therefore:

$$D_n = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

Remember the inclusion-exclusion principle:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \boxed{\sum_{i=1}^{n} |A_i|} - \boxed{\sum_{i<j} |A_i \cap A_j|} + \boxed{\sum_{i<j<k} |A_i \cap A_j \cap A_k|} - \boxed{\sum_{i<j<k<\ell} |A_i \cap A_j \cap A_k \cap A_\ell|} + \ldots + (-1)^{n+1} \boxed{|A_1 \cap A_2 \cap \cdots \cap A_n|}$$

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4. Recurrence relations

4.1. Recurrence relations

4.2. Solve recurrence relations

4.1. Recurrence relations

A recurrence **without specifying any initial values (initial conditions)**.

➔ can have (and usually has) **multiple solutions**.

Example: $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4$. It has the following sequences a_n as solution:

a_n=5

a_n=3n

a_n=n+1

If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Example: $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4$

where a_0=0; a_1 = 3

➔The sequence a_n=5 is not the solution

➔The sequence a_n=3n is the unique solution

4.1. Recurrence relations

Definition:

A recurrence relation for the sequence $\{a_n\}$ is the equation that expresses a_n in terms of one or more of the previous terms in the sequence namely, a_0, a_1, …, a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a **solution** to a recurrence relation if its terms satisfy the recurrence relation.

Example: Consider the recurrence relation

$a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$

The sequence a_n=3n is a solution of this recurrence relation?

For $n \geq 2$ we see that: $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$

Therefore, the sequence a_n=3n is a solution of the recurrence relation

The sequence a_n=5 is also a solution of the this recurrence relation?

For $n \geq 2$ we see that: $2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$

Therefore, the sequence a_n=5 is also a solution of the recurrence relation

Modeling with Recurrence Relations

Example 1: Someone deposits \$10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

- Let P_n denote the amount in the account after n years.

How can we determine P_n on the basis of P_{n-1}?

- We can derive the following **recurrence relation**:

$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$.

- The initial condition is $P_0 = 10{,}000$.

$P_{30} = (1.05)^{30} \cdot 10{,}000 = 43{,}219.42$

Then we have:

- $P_1 = 1.05P_0$
- $P_2 = 1.05P_1 = (1.05)^2P_0$
- $P_3 = 1.05P_2 = (1.05)^3P_0$
- …
- $P_n = 1.05P_{n-1} = (1.05)^nP_0$

➔ We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

4. Recurrence relations

4.1. Recurrence relations

4.2. Solving recurrence relations

4.2. Solving Recurrence Relations

Solving the recurrence relation for a sequence $a_0, a_1, a_2, a_3, \ldots a_n, \ldots$ is to give the explicit formula to compute the value for the general term a_n, i.e., to find an expression for an that does not involve any other a_i

Example: Given the recurrence relation:

$a_n = 2a_{n-1} - a_{n-2}$ where $n = 2, 3, 4, \ldots$

$a_0=0$; $a_1 = 3$

➔The explicit formula for the above recurrence relation is $a_n=3n$

➔$a_n=3n$ is the solution to the above recurrence relation

- Does not exist the method to solve all types of recurrence relation.
- Consider method to solve the recurrence relation with following types:
  - A linear homogeneous recurrence relation of degree k with constant coefficients
  - A linear nonhomogeneous recurrence relation of degree k with constant coefficients

4.2. Solving Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real number constants, and $c_k \neq 0$.

• A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation if it also satisfying k initial conditions:

$$a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots, a_{k-1} = C_{k-1}$$

where $C_0, C_1, \ldots, C_{k-1}$ are constants.

4.2. Solving Recurrence Relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real number constants, and $c_k \neq 0$.

Explain:

- Linear: the right-hand side is the sum of the terms before the term a_n in the sequence where the coefficients ($c_1, c_2, \ldots, c_k$) are constant (not a function dependent on n)
- Homogeneous: the right-hand side has no additional terms other than the terms a_i of the sequence
- Degree k: the right hand side has the (n-k)th term of the sequence

A linear homogeneous recurrence relation of degree k with constant coefficients

- We try to find solution of the form $a_n = r^n$, where r is a constant.
- The sequence $\{a_n = r^n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$$

if and only if r satisfying:

$r^n = c_1 r^{n-1} + \dots + c_k r^{n-k}$, or (subtract the right-hand side from the left and × by r^{k-n})

$$\boxed{r^k - c_1 r^{k-1} - \dots - c_k = 0}$$

this is called the **characteristic equation** of the recurrence relation, and its solution is called **characteristic roots** of the recurrence relation.

- We will use characteristic roots to obtain explicit formula for the sequence.

A linear homogeneous recurrence relation of degree k with constant coefficients

- **Theorem 1.** Let c_1 and c_2 be real numbers.

Suppose that $r^2 - c_1 r - c_2 = 0$ *has 2 distinct roots* r_1 *and* r_2. *Then the sequence* $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n \qquad (1)$$

$n = 0, 1, \dots,$ *where* α_1 *and* α_2 *are constants.*

Example 1

Fibonacci sequence is given by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,$$
$$F_0 = 0,\ F_1 = 1.$$

Find the explicit formula for F_n.

Solution: Solve the characteristic equation:

$$r^2 - r - 1 = 0,$$

its characteristic roots are:

$$r_1 = \frac{1+\sqrt{5}}{2};\quad r_2 = \frac{1-\sqrt{5}}{2}$$

Leonardo Fibonacci 1170-1250

Example 1

- The explicit formula:

$$F_n = \alpha_1.(r_1)^n + \alpha_2.(r_2)^n$$

where α_1, α_2 are constants and could be determined by using initial conditions F_0, F_1.

$F_0 = \alpha_1 + \alpha_2 = 0$

$F_1 = \alpha_1 r_1 + \alpha_2 r_2 = 1$

Solving these two equations, we have: $\alpha_1 = \frac{1}{\sqrt{5}};\quad \alpha_2 = -\frac{1}{\sqrt{5}}$

Therefore

Muavre formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right),\quad n \geq 0.$$

The case: one characteristic root with multiplicities 2

But what happens if the characteristic equation has only one root?

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

if and only if

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

$n = 0, 1, \ldots,$ *where* α_1 , α_2 *are constants.*

Example 2

Solving the following recurrence relation

$$a_n = 6 a_{n-1} - 9 a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$.

Solution:

Characteristic equation:

$r^2 - 6r + 9 = 0$ has one root $r = 3$. The solution is:

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

To determine α_1, α_2, using the initial conditions, we have:

$$a_0 = 1 = \alpha_1 ,$$

$$a_1 = 6 = \alpha_1 * 3 + \alpha_2 *1* 3$$

Solving these equations, we have $\alpha_1 = 1$ and $\alpha_2 = 1$.

Therefore, the solution of the recurrence relation is:

$$\mathbf{a_n = 3^n + n\,3^n}$$

General case

Theorem 3. Let $c_1, c_2, \ldots, c_k$ be real numbers. Assume the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. *Then a sequence* $\{a_n\}$ *is a solution of the recurrence relation:*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

where $n = 0, 1, 2, \ldots,$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are constants*

Example 3

Solving the recurrence relation:

$$a_n = 6 a_{n-1} - 11 a_{n-2} + 6 a_{n-3}$$

where initial conditions

$$a_0 = 2, \ a_1 = 5, \ a_2 = 15.$$

Solution: Characteristic equation

$$r^3 - 6r^2 + 11r - 6 = 0$$

has 3 distinct roots $r_1 = 1, \ r_2 = 2, r_3 = 3$.

Therefore, the solution is

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n \textit{ for some constants } \alpha_1, \alpha_2, \alpha_3$$

Example 3

Using the initial conditions, we have following equations to determine the value for constants α_1, α_2, α_3:

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 5 = \alpha_1 + \alpha_2.2 + \alpha_3.3$$

$$a_2 = 15 = \alpha_1 + \alpha_2.4 + \alpha_3.9.$$

Solving above equations, we have

$$\alpha_1 = 1,\ \alpha_2 = -1 \text{ and } \alpha_3 = 2.$$

Therefore, the solution of the recurrence relation is

$$a_n = 1 - 2^n + 2.\ 3^n$$

General case

Given linear homogeneous recurrence relation of degree k with constant coefficients

$$a_n = \sum_{i=1}^{k} c_i a_{n-i}$$

Its characteristic equation is:

$$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$

Theorem 4: If the characteristic equation has t distinct roots $r_1,\ldots,r_t$ with multiplicities $m_1,\ldots,m_t$ ($m_1+\ldots+m_t = k$). Then:

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m_1-1}n^{m_1-1})\, r_1^n +$$
$$(\alpha_{2,0} + \alpha_{2,1}n + \ldots + \alpha_{2,m_2-1}n^{m_2-1})\, r_2^n + \ldots$$
$$(\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,m_t-1}n^{m_t-1})\, r_t^n$$

$$a_n = \sum_{i=1}^{t}\left(\sum_{j=0}^{m_i-1}\alpha_{i,j}n^j\right) r_i^n$$

where $n \geq 0$, and α_{ij} are constants.

Example 4

Solve the following recurrence relation:

$$c_n = -4c_{n-1} + 3c_{n-2} + 18c_{n-3}, \quad n \geq 3,$$

$$c_0 = 1;\ c_1 = 2;\ c_2 = 13.$$

Solution: Characteristic equation

$$r^3 + 4r^2 - 3r - 18 = (r-2)(r+3)^2 = 0$$

Hence, the solution of the recurrence relation:

$$c_n = \alpha_{10}\, 2^n + (\alpha_{20} + \alpha_{21}\, n)(-3)^n$$

where α_{10}, α_{20}, α_{21} are constants

Example 4

These constants are determined by using initial conditions:

$0 = c_0 = \alpha_{10}\, 2^0 + (\alpha_{20} + \alpha_{21}.0)\ (-3)^0 = \alpha_{10} + \alpha_{20}$

$2 = c_1 = \alpha_{10}\, 2^1 + (\alpha_{20} + \alpha_{21}.1)\ (-3)^1 = 2\alpha_{10} - 3\alpha_{20} - 3\alpha_{21}$

$13 = c_2 = \alpha_{10}\, 2^2 + (\alpha_{20} + \alpha_{21}.2)\ (-3)^2 = 4\alpha_{10} + 9\alpha_{20} + 18\alpha_{21}$

Solving three equations above, we have: $\alpha_{10} = 1$; $\alpha_{20} = -1$; $\alpha_{21} = 1$

The solution of recurrence relation is:

$$c_n = 2^n + (-1+n)(-3)^n$$

Linear nonhomogeneous recurrence relation with constant coefficients

- Linear *nonhomogeneous* recurrence relation with constant coefficients) has the term $F(n)$ dependent on n (and independent on any value of a_i) :

$$\underbrace{a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}}_{\text{Linear homogeneous recurrence relation}} + \underbrace{F(n)}_{\text{Nonhomogeneous term}}$$

2.5. Recurrence relations and generating functions

2.5.1. Recurrence relations

2.5.2. Generating functions

2.5.2. Generating functions

Generating functions are a tool to solve a wide variety of counting problems and recurrence relations.

Example 1:

How many ways to give 12 oranges for three children: A, B and C such that: A gets at least four, and B and C gets at least two, but C gets no more than five

***a*, *b* , *c* are the number of oranges that A, B and C gets, respectively**

Find the number of integer solutions to

$$a + b + c = 12 \text{ where } a \geq 4,\ b \geq 2,\ 2 \leq c \leq 5$$

2.5.2. Generating functions

Example 1:

Find the number of integer solutions to

$$a + b + c = 12 \text{ where } a \geq 4,\ b \geq 2,\ 2 \leq c \leq 5$$

n=12

Let

a: $a(x) = x^4 + x^5 + x^6 + x^7 + x^8$ (since $b + c \geq 4$ ➔ $a \leq 8$)

b : $b(x) = x^2 + x^3 + x^4 + x^5 + x^6$ (since $a + c \geq 6$ ➔ $b \leq 6$)

c : $c(x) = x^2 + x^3 + x^4 + x^5$ (since $2 \leq c \leq 5$)

The coefficient of x^{12} in $g(x) = a(x) \times b(x) \times c(x)$

$$=(x^4+ x^5+ x^6+ x^7+ x^8)(x^2+ x^3+x^4+ x^5+ x^6)(x^2+ x^3+x^4+ x^5)$$

=???

is the solution to the problem

Why???

$g(x)$ is called a *generating function*.

Assume x^a, x^b, x^c are terms derived from $(x^4+ x^5+ x^6+ x^7+ x^8)$, $(x^2+ x^3+x^4+ x^5+ x^6)$, $(x^2+ x^3+x^4+ x^5)$ respectively when developing the right-hand side ➔ $4 \leq a \leq 8,\ 2 \leq b \leq 6,\ 2 \leq c \leq 5$

When developing the right-hand side, these three terms give us the term x^n where $n = a + b + c$:

$$g(x) = (x^4+ x^5+ x^6+ x^7+ x^8)(x^2+ x^3+x^4+ x^5+ x^6)(x^2+ x^3+x^4+ x^5)$$

$$= \ldots. + Kx^n + \ldots..$$

➔The coefficient of x^n in $f(x)$ is the number of positive integer solutions to

$$a + b + c = n \text{ where } 4 \leq a \leq 8,\ 2 \leq b \leq 6,\ 2 \leq c \leq 5$$

2.5.2. Generating functions

Example 1: **Find the number of integer solutions to** ***a + b + c* = 12 where *a* ≥ 4, *b* ≥ 2, 2≤ *c* ≤ 5**

Let

a: $a(x) = x^4 + x^5 + x^6 + x^7 + x^8$ (since $b + c \geq 4$ ➔ $a \leq 8$)

b : $b(x) = x^2 + x^3 + x^4 + x^5 + x^6$ (since $a + c \geq 6$ ➔ $b \leq 6$)

c : $c(x) = x^2 + x^3 + x^4 + x^5$ (since $2 \leq c \leq 5$)

The coefficient of x^{12} in $g(x) = a(x) \times b(x) \times c(x)$

$=(x^4+ x^5+ x^6+ x^7+ x^8) (x^2+ x^3+x^4+ x^5+ x^6)(x^2+ x^3+x^4+ x^5)$

which is 14, is the solution

$g(x)$ is called a *generating function*.

1. $x^4*x^3*x^5$
2. $x^4*x^4*x^4$
3. $x^4*x^5*x^3$
4. $x^4*x^6*x^2$
5. $x^5*x^2*x^5$
6. $x^5*x^3*x^4$
7. $x^5*x^4*x^3$
8. $x^5*x^5*x^2$
9. $x^6*x^2*x^4$
10. $x^6*x^3*x^3$
11. $x^6*x^4*x^2$
12. $x^7*x^2*x^3$
13. $x^7*x^3*x^2$
14. $x^8*x^2*x^2$

2.5.2. Generating functions

Definition: Let a_0, a_1, a_2, ... be a sequence of real numbers. The function

$$g(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

A finite sequence

$a_0, a_1, a_2, \ldots, a_n$

can be regarded as the infinite sequence

$a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots$ [set all terms higher than n to 0]

and its generating function

$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

is a polynomial.

2.5.2. Generating functions

Example 2: For the binomial coefficients we already know that:

$$(x+y)^n = \sum_{k=0}^{n} C(n,k) x^k y^{n-k} \Rightarrow (x+1)^n = \sum_{k=0}^{n} C(n,k) x^k = g(x)$$

$(x+1)^n$ is the generating function for the sequence

C(n, 0), C(n, 1),…,C(n, k),…, C(n, n), 0,0,0…

2.5.2. Generating functions

Definition: Let a_0, a_1, a_2, ... be a sequence of real numbers. The function

$$g(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

Some useful generating function:

Ex1: $(1 - x^{n+1}) = (1 - x)(1 + x + x^2 + x^3 + \ldots + x^n)$.

So $(1-x^{n+1})/(1-x) = 1 + x + x^2 + x^3 + \ldots + x^n$,

➔ $(1-x^{n+1})/(1-x)$ is the generating function for the sequence:

1, 1, 1, …, 1, 0, 0, 0, … . (where the first n+1 terms are 1)

Ex2: From Ex1, If $n \to \infty$ and $|x| < 1$, then $1 = (1 - x)(1 + x + x^2 + x^3 + \ldots)$.

So, $1/(1-x) = 1 + x + x^2 + x^3 + \ldots$

➔ $1/(1-x)$ where $|x| < 1$ is the generating function for the sequence:

1, 1, 1, …, 1, … .

2.5.2. Generating functions

Generating functions are a tool to solve a wide variety of counting problems and **recurrence relations**.

Definition: Let a_0, a_1, a_2, ... be a sequence of real numbers. The function

$$g(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{i=0}^{\infty} a_i x^i$$

is called the *generating function* for the given sequence.

2.5.2. Generating functions

Example: Solve the recurrence relation a_n-$3a_{n-1} = n$, $n \geq 1$, $a_0=1$

$$g(x) - 1 - 3xg(x) = \frac{x}{(1-x)^2}$$

$$g(x) = \frac{1}{1-3x} + \frac{x}{(1-3x)(1-x)^2}$$

$$g(x) = \frac{1}{1-3x} + \frac{A}{(1-3x)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

$$\Rightarrow A = \frac{3}{4}; B = -\frac{1}{4}; C = -\frac{1}{2};$$

$$g(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{i=0}^{\infty} a_i x^i$$

$$g(x) = \frac{7/4}{(1-3x)} + \frac{-1/4}{(1-x)} + \frac{-1/2}{(1-x)^2}$$

We find a_n by determining the coefficient of x^n in $g(x)$

➔ determining the coefficient of x^n in each of the three summands