VINUNIVERSITY

VIN πONEERS

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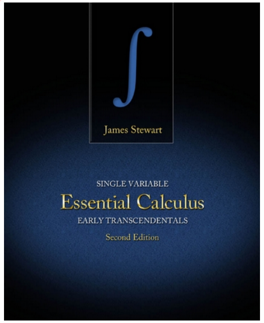
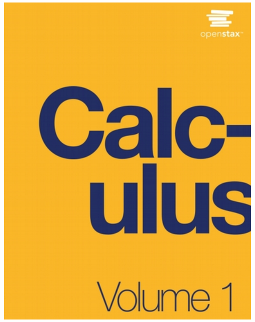
Chapter 1: Functions and Limits





The pictures are taken from the books:

[1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280
2) G. Strang and E. J. Herman, Calculus 1, https://openstax.org/details/books/calculus-volume-1]

Chapter 1.1

Functions and Their Representations

Chapter 1.1: Functions

- Consider two variables x and y. How does x affects y?

Domain

Range (Co-Domain)

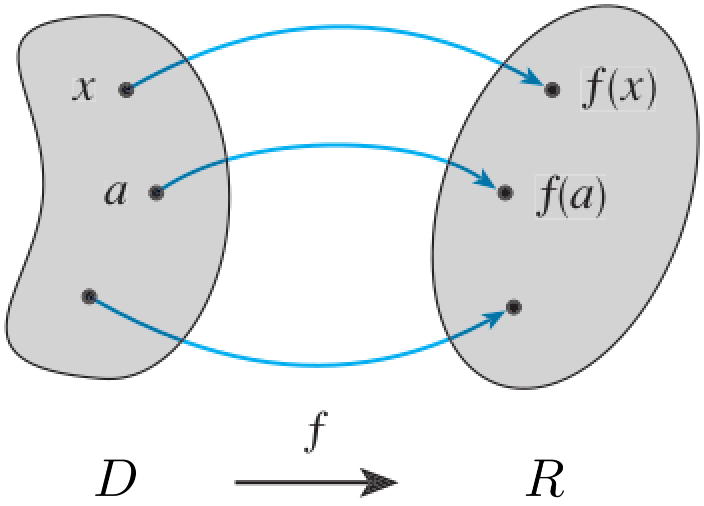
$x \in D$

$y \in R$

$$x \xrightarrow{f} y = f(x)$$

Independent Variable

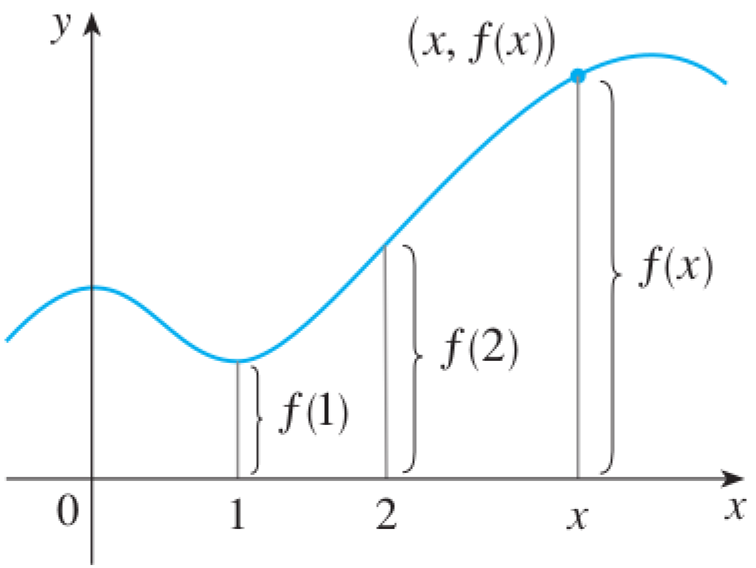
Dependent Variable

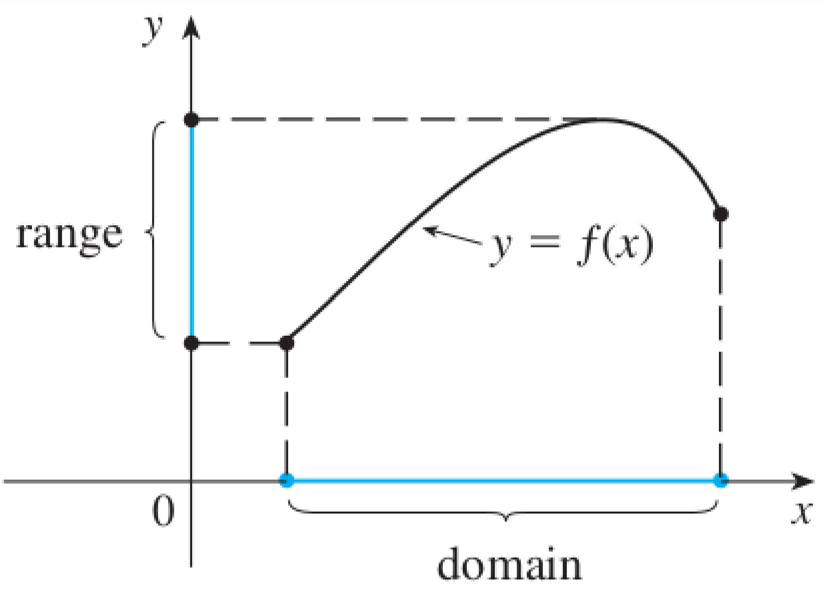


Chapter 1.1: Functions

Definition:

A **function** $f : D \to R$ is a rule that assigns to each element x in a set D **exactly** one element, called $f(x)$, in a set R.





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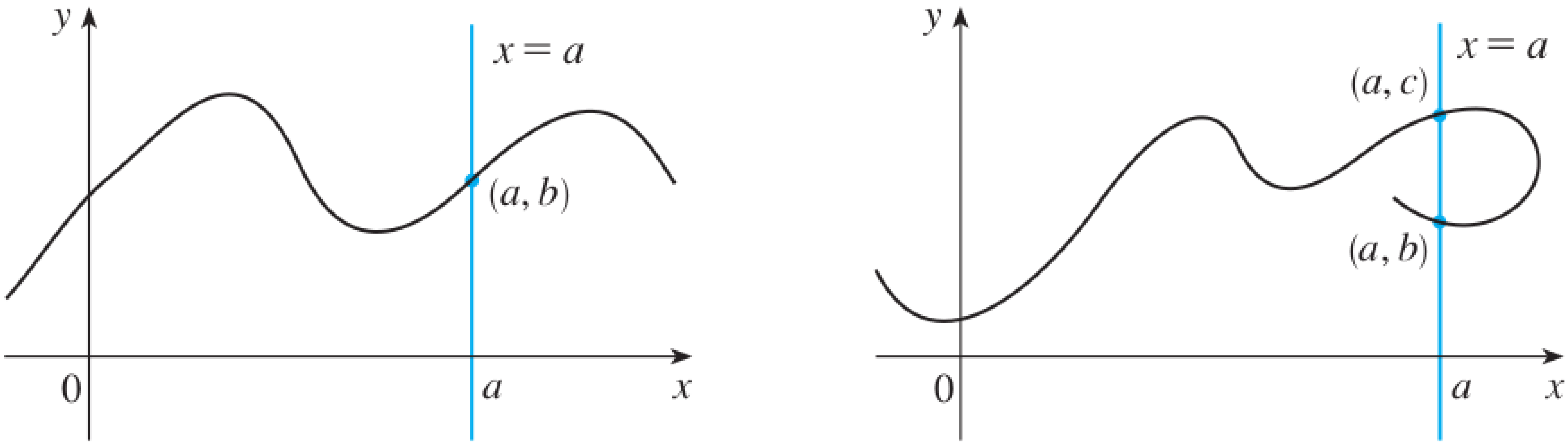


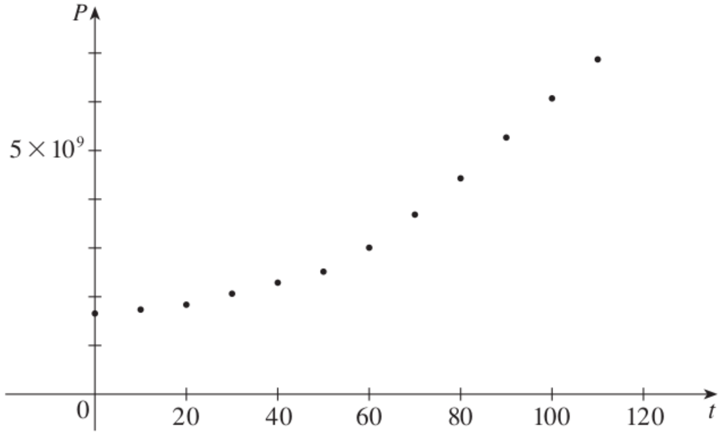
The vertical test line

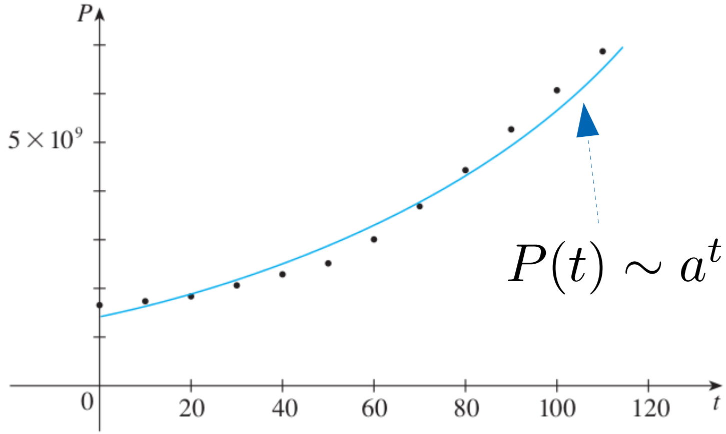
Chapter 1.1: Representing Functions

There are four possible ways to represent a function:

- verbally (description in words)
- visually (graph)
- numerically (table of values)
- algebraically (formula)

t	Population (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870



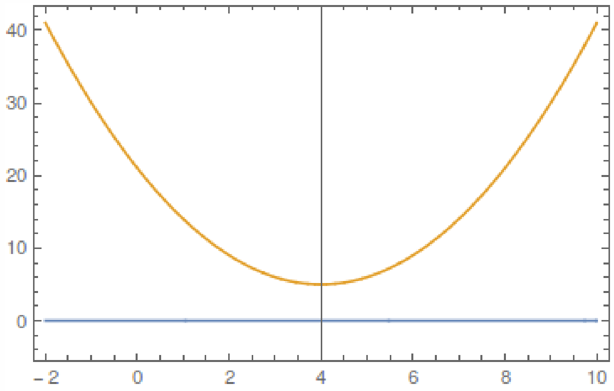


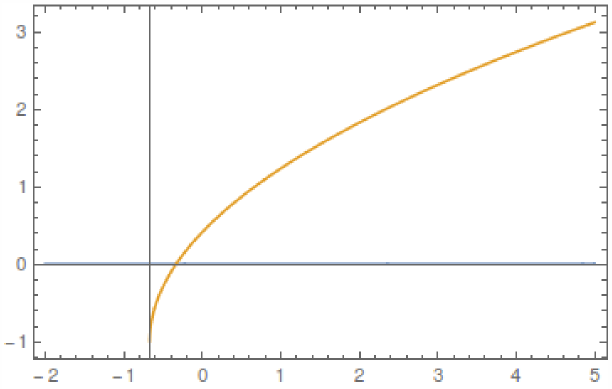
$P(t) \sim a^t$

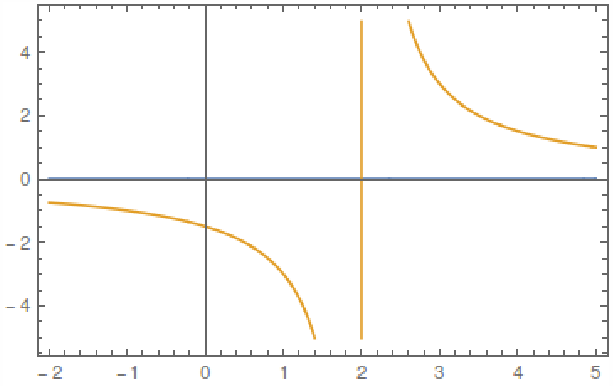
1.1 Examples

For each of the following functions, determine the i. **domain** and ii. **range**.

$a) \quad f(x) = (x-4)^2 + 5, \qquad b) \quad f(x) = \sqrt{3x+2} - 1, \qquad c) \quad f(x) = \dfrac{3}{x-2}$

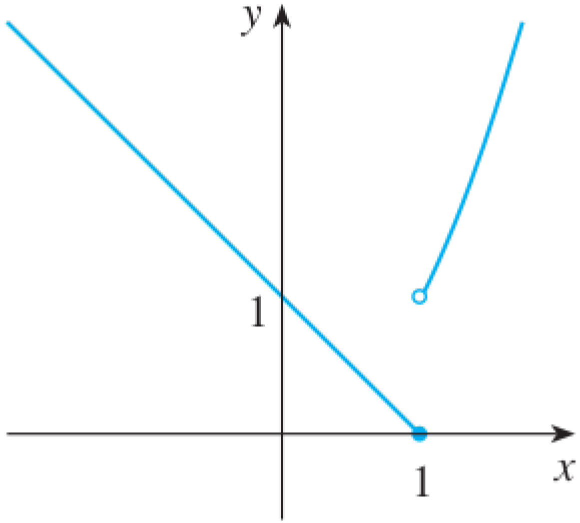




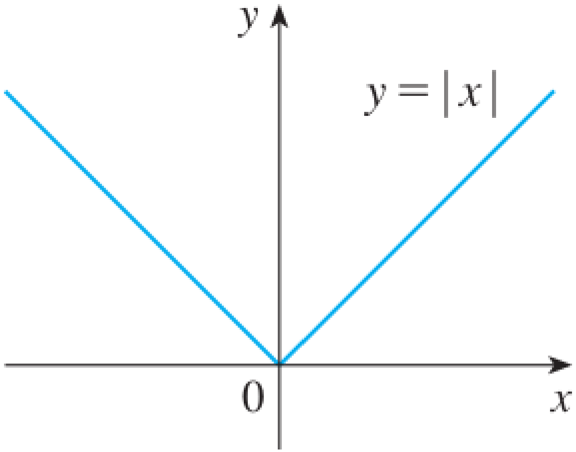


Chapter 1.1: Piecewise defined Functions

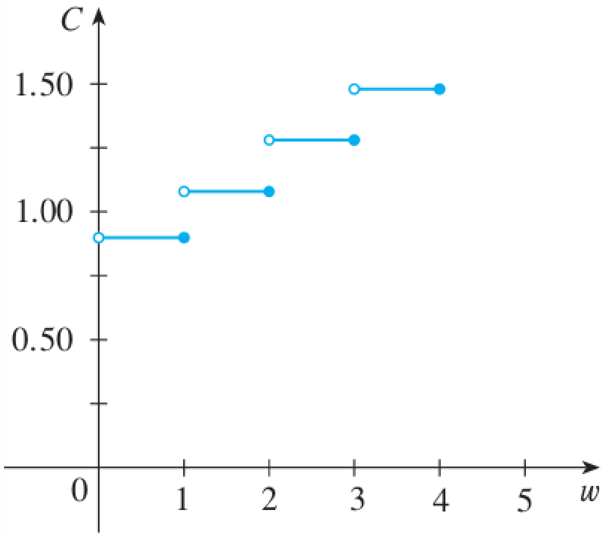
$$f(x) = \begin{cases} 1 - x\,, & x \leq 1 \\ x^2\,, & x > 1 \end{cases}$$



$$f(x) = |x| = \begin{cases} x\,, & x \geq 0 \\ -x\,, & x < 0 \end{cases}$$



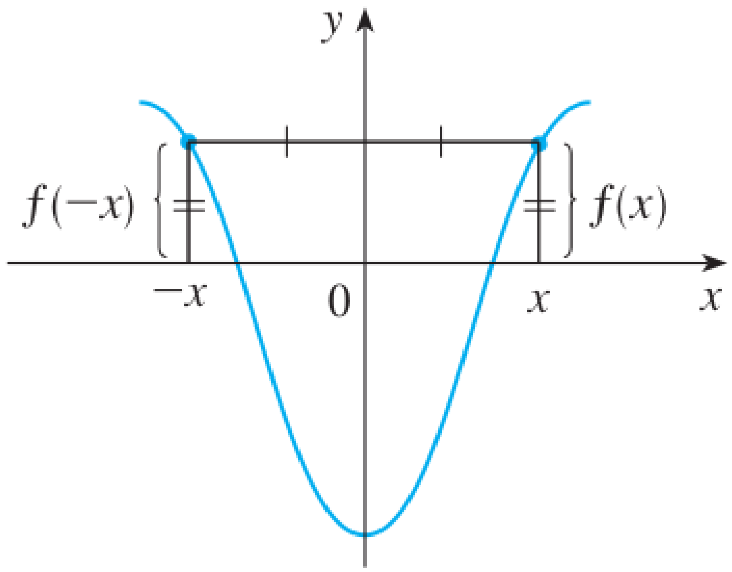
$$C(w) = \begin{cases} 0.88\,, & 0 < w \leq 1 \\ 1.08\,, & 1 < w \leq 2 \\ 1.28\,, & 2 < w \leq 3 \\ 1.48\,, & 3 < w \leq 4 \end{cases}$$

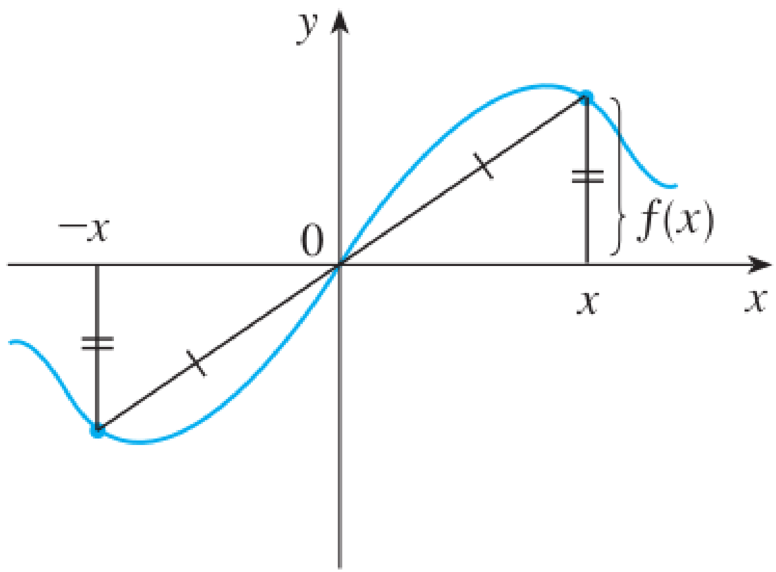


Chapter 1.1: Symmetry

i. Even functions: $f(-x) = f(x)$,

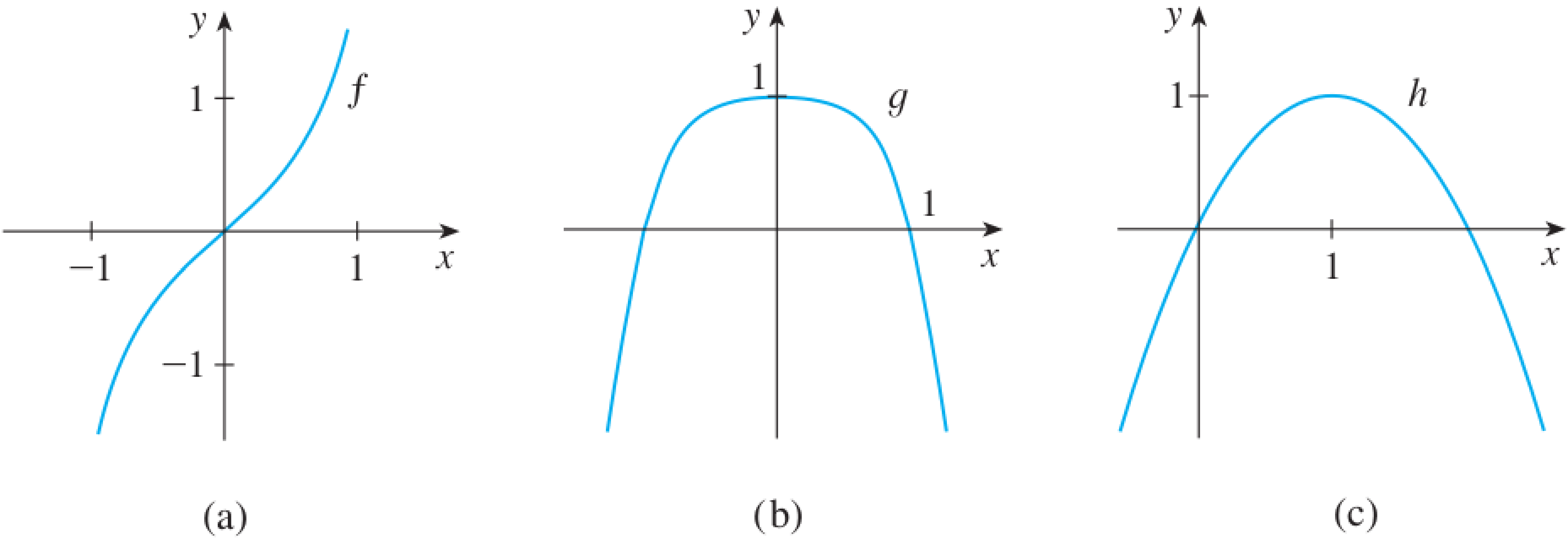
ii. Odd functions: $f(-x) = -f(x)$





Chapter 1.1: Symmetry

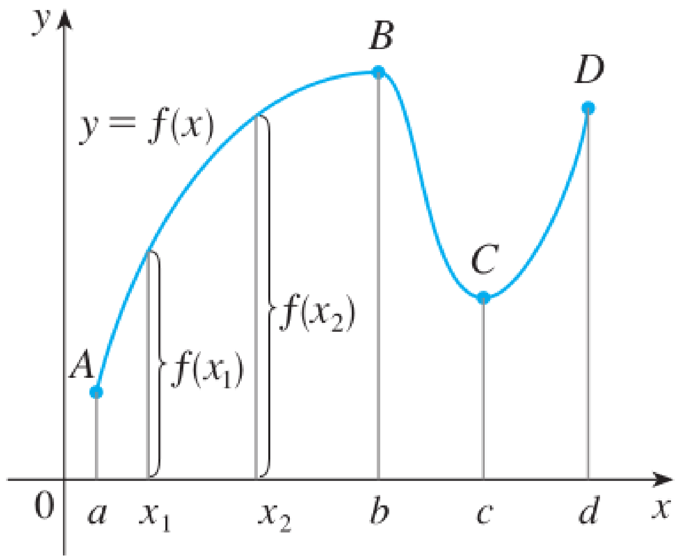
Examples: (a) $f(x) = x^5 + x$, (b) $g(x) = 1 - x^4$, (c) $h(x) = 2x - x^2$

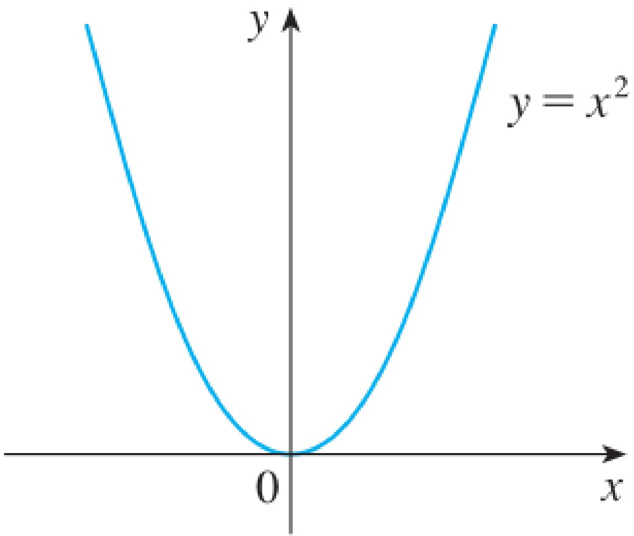


Chapter 1.1: Increasing and Decreasing Functions

Definition:

- A function $f(x)$ is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
- A function $f(x)$ is called **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.





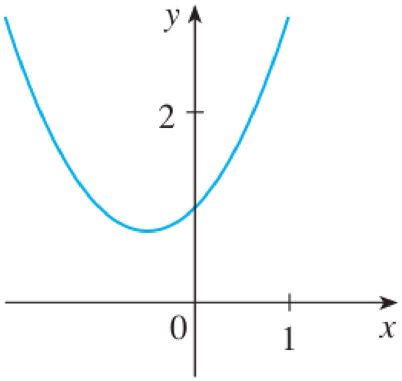
Chapter 1.2

Basic Classes of Functions

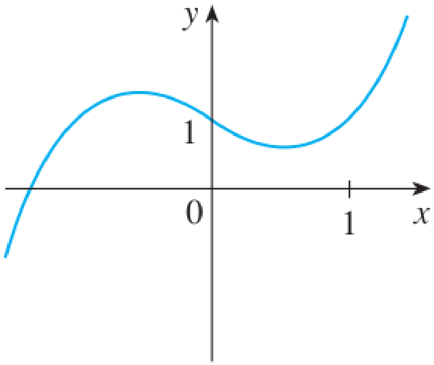
Chapter 1.2: Basic Functions

- Linear Function: $f(x) = ax + b,$ $\quad$ slope$=a = \tan(\theta) = \dfrac{y_2 - y_1}{x_2 - x_1}$

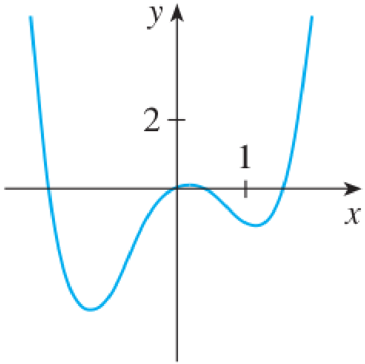
- Polynomial: $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$



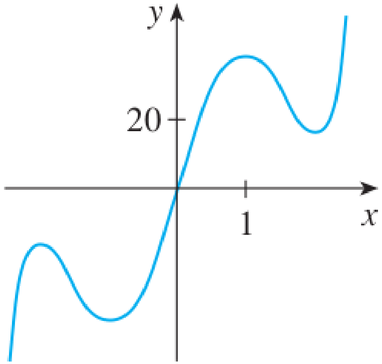
$y = x^2 + x + 1$



$y = x^3 - x + 1$



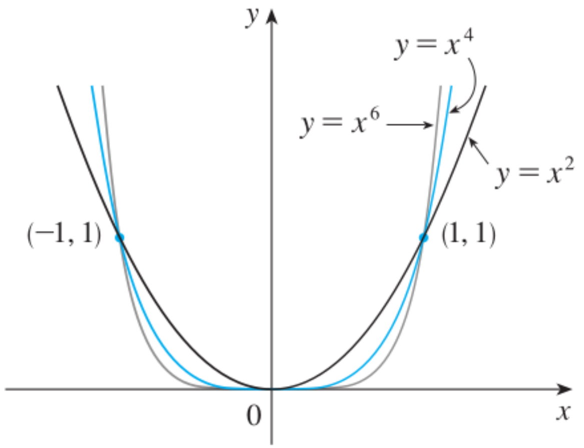
$y = x^4 - 3x^2 + x$

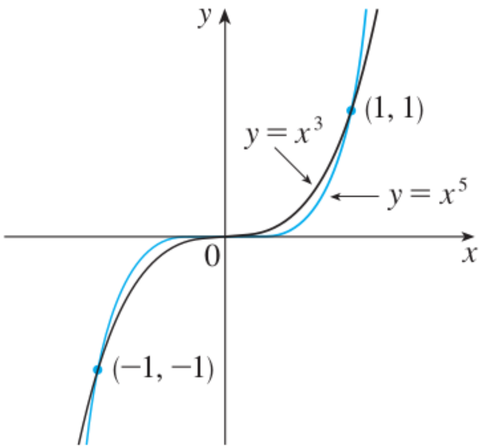


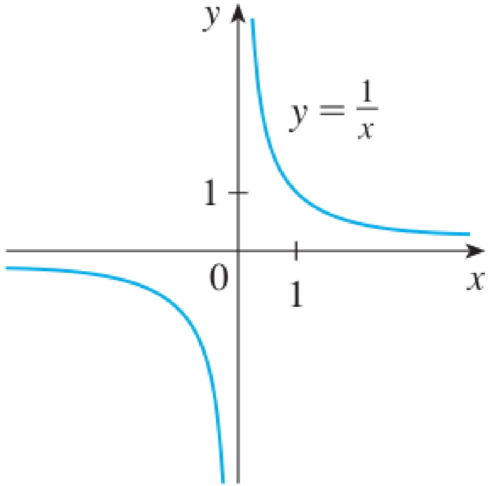
$y = 3x^5 - 25x^3 + 60x$

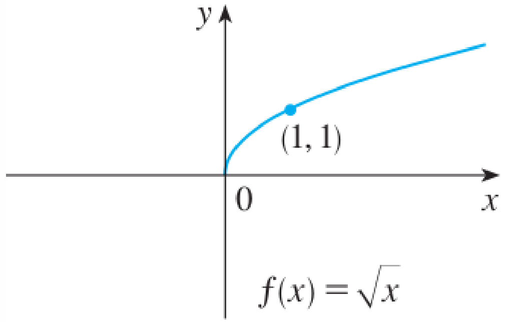
Chapter 1.2: Basic Functions

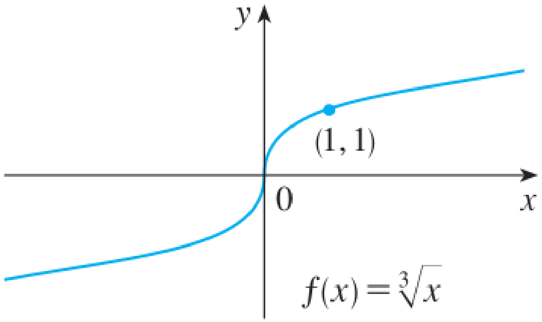
- Power Functions: $f(x) = x^a,\ x^{1/a},\ x^{-1}, \qquad a = 1, 2, 3, 4, \ldots$





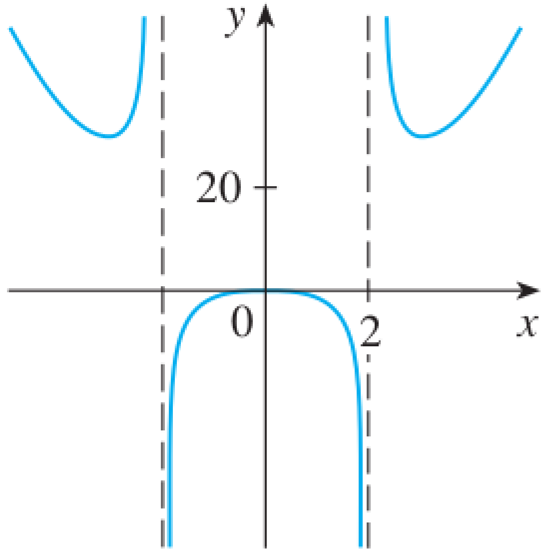






Chapter 1.2: Basic Functions

- Rational Functions: $f(x) = \frac{P(x)}{Q(x)}$, where P, Q polynomials

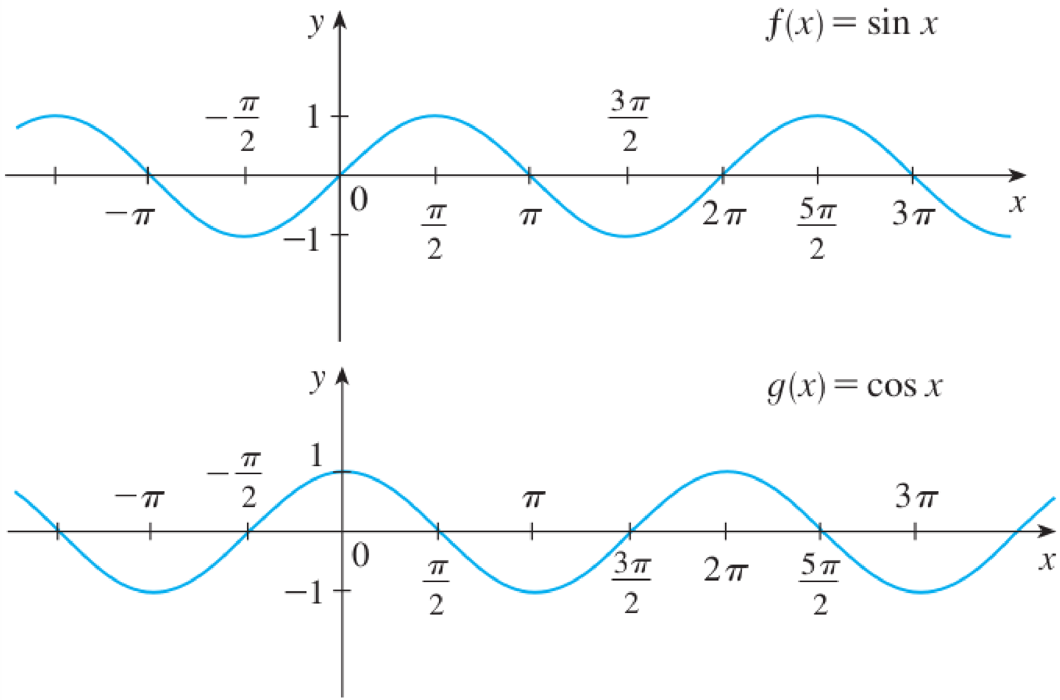


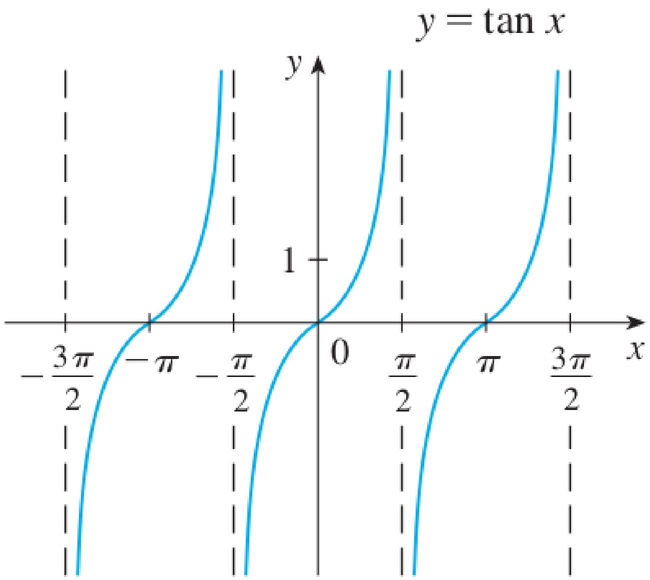
$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

$$\{x \mid x \neq \pm 2\}$$

Chapter 1.2: Basic Functions

- Trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$





- Trigonometric functions: $\sin(x),\ \cos(x),\ \tan(x)$

$$\sin^2(a) + \cos^2(a) = 1$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$$

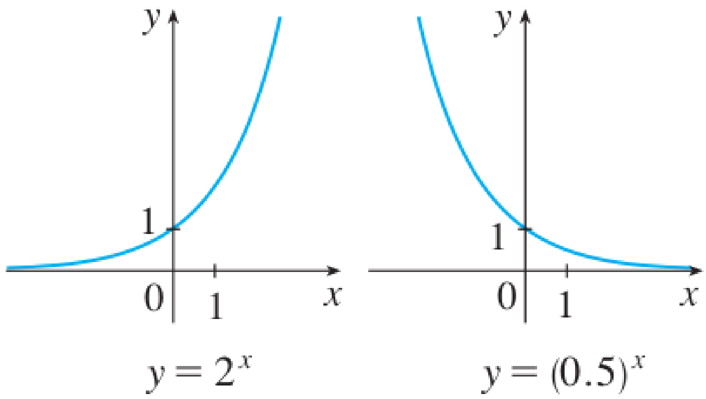
$$\xrightarrow{b=a} \quad \sin(2a) = 2\sin(a)\cos(a)$$

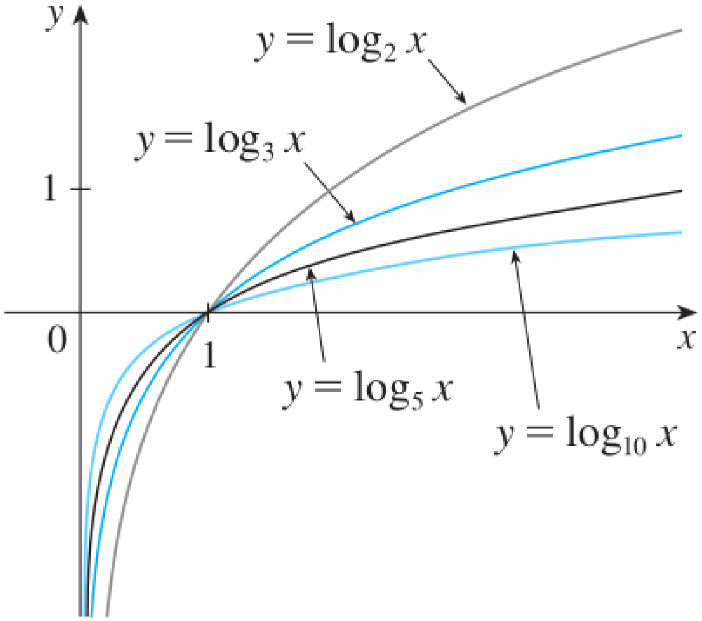
$$\xrightarrow{b=\pi} \quad \sin(a+\pi) = -\sin(a)$$

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

Chapter 1.2: Basic Functions

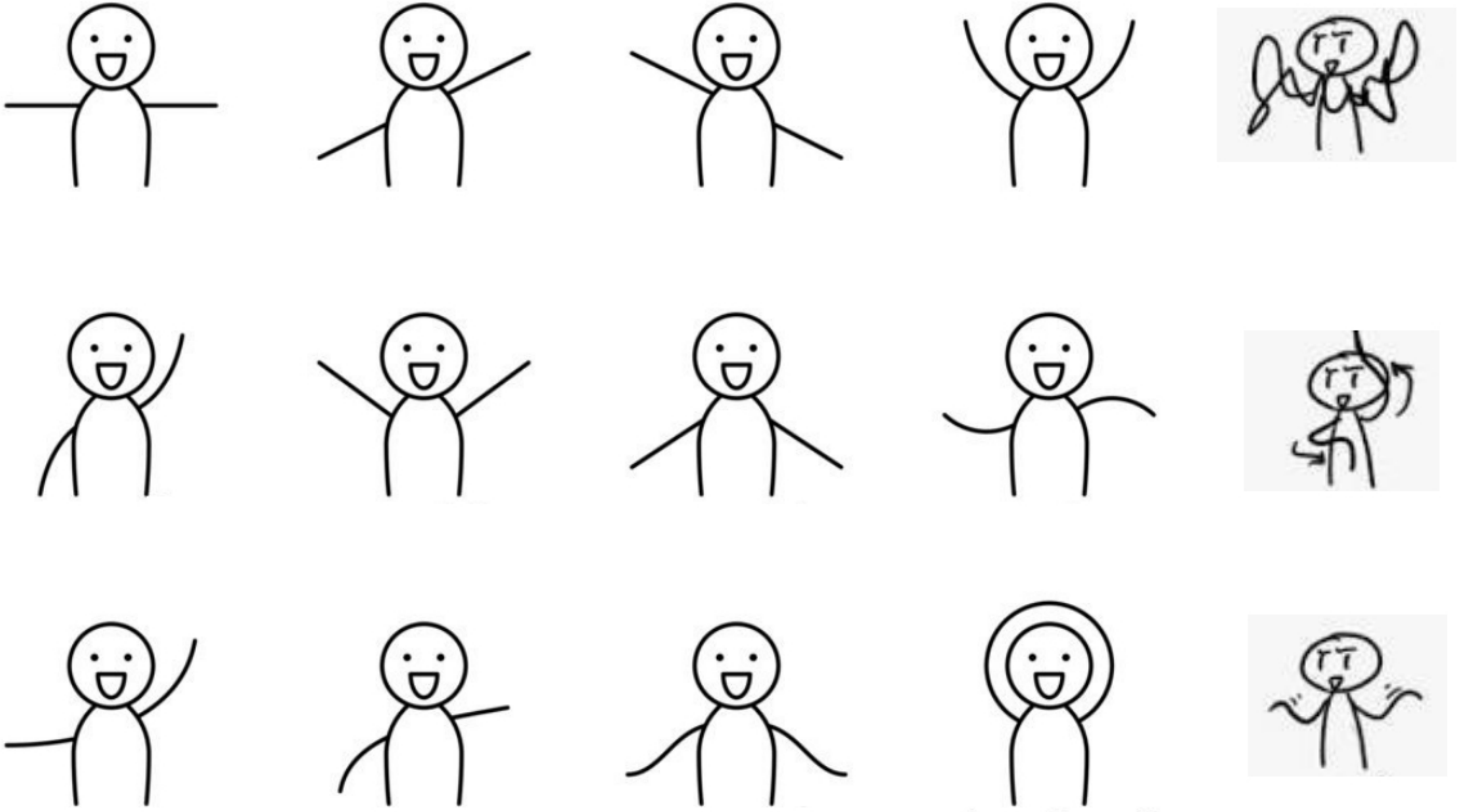
- Exponential and Logarithmic functions: $f(x) = a^x \qquad g(x) = \log_a(x)$





Chapter 1.2: Basic Functions

Examples: Dancing with Engineers



Chapter 1.2: Conmination of Functions

$$
\begin{array}{llcll}
\text{i.} & (f+g)(x) & = & f(x)+g(x) & \text{Addition} \\
\text{ii.} & (f-g)(x) & = & f(x)-g(x) & \text{Subtraction} \\
\text{iii.} & (f\cdot g)(x) & = & f(x)\cdot g(x) & \text{Multiplication} \\
\text{iv.} & \left(\dfrac{f}{g}\right)(x) & = & \dfrac{f(x)}{g(x)} & \text{Division} \\
\text{v.} & (f\circ g)(x) & = & f\big(g(x)\big) & \text{Composition}
\end{array}
$$

Chapter 1.2: Combination of Functions

Example:

$f(x)=\sqrt{x}, \quad D=[0,\infty), \quad R=[0,\infty)$

$g(x)=\sqrt{2-x}, \quad D=(-\infty,2], \quad R=[0,\infty)$

$$
\begin{aligned}
f(x)+g(x) &= \sqrt{x}+\sqrt{2-x}\,, & D&=[0,2]\,, & R&=[\sqrt{2},2] \\
f(x)-g(x) &= \sqrt{x}-\sqrt{2-x}\,, & D&=[0,2]\,, & R&=[-\sqrt{2},\sqrt{2}] \\
f(x)\cdot g(x) &= \sqrt{x}\cdot\sqrt{2-x}\,, & D&=[0,2]\,, & R&=[0,1] \\
\frac{f(x)}{g(x)} &= \frac{\sqrt{x}}{\sqrt{2-x}}\,, & D&=[0,2)\,, & R&=[0,\infty) \\
f\big(g(x)\big) &= (2-x)^{1/4}\,, & D&=(-\infty,2]\,, & R&=[0,\infty) \\
g\big(f(x)\big) &= \sqrt{2-\sqrt{x}}\,, & D&=[0,4]\,, & R&=[0,\sqrt{2}]
\end{aligned}
$$

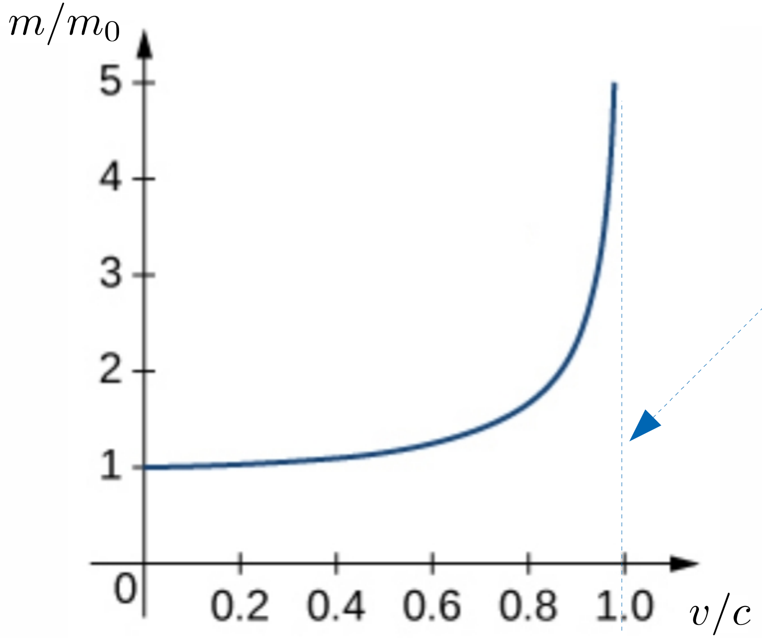
Chapter 1.3

The Limit of a Function

Chapter 1.3: The Limit of a Function

http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_intuitive_one_side.html

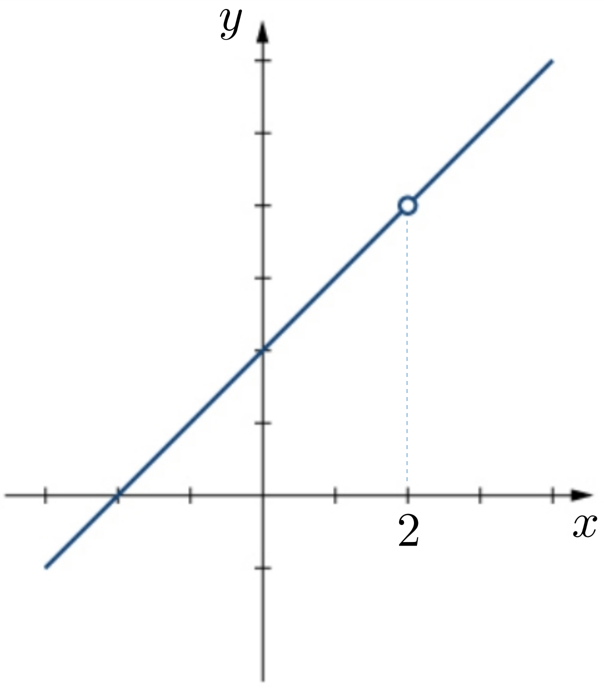
Motivation: Mass-Energy Equivalence: $E = mc^2 \quad \to \quad m = \dfrac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$

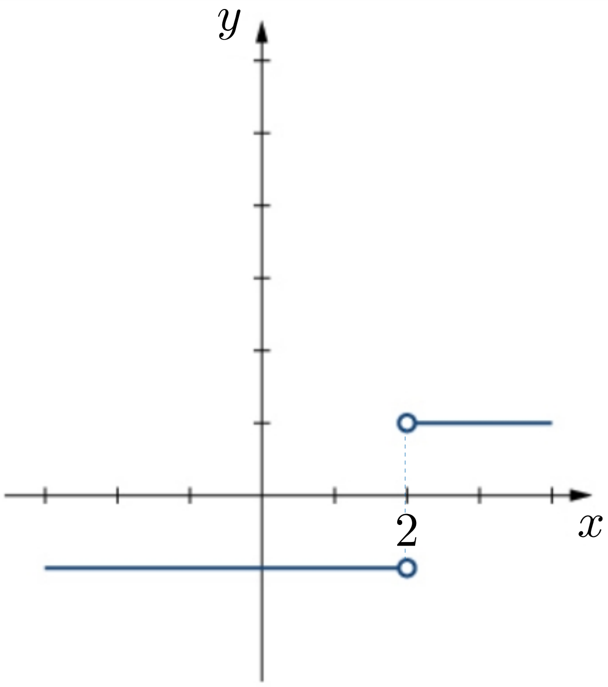


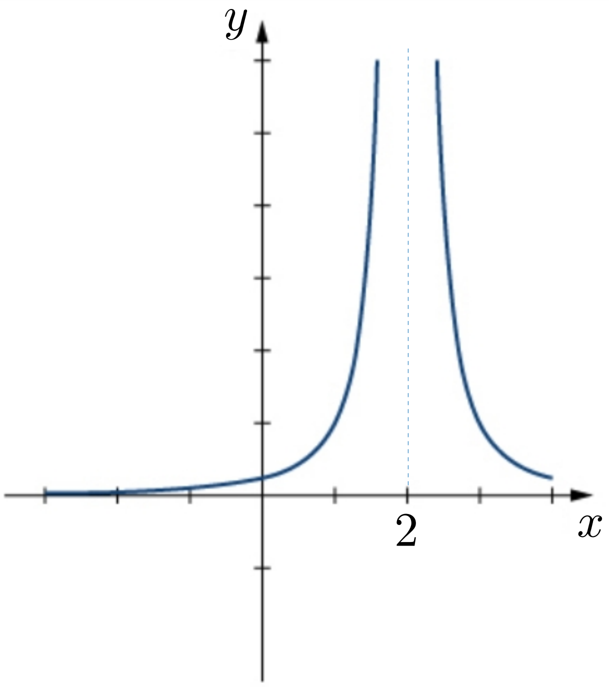
What happens as $v \to c$?

Chapter 1.3: The Limit of a Function

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \frac{|x - 2|}{x - 2}, \qquad h(x) = \frac{1}{(x-2)^2}$$





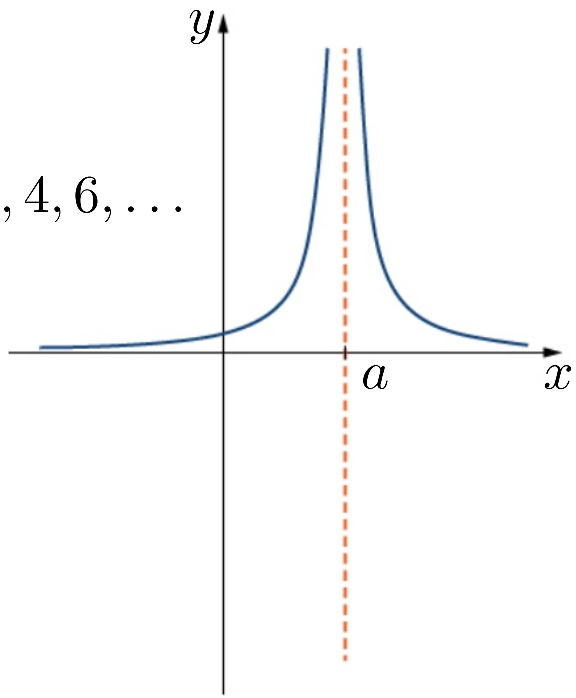


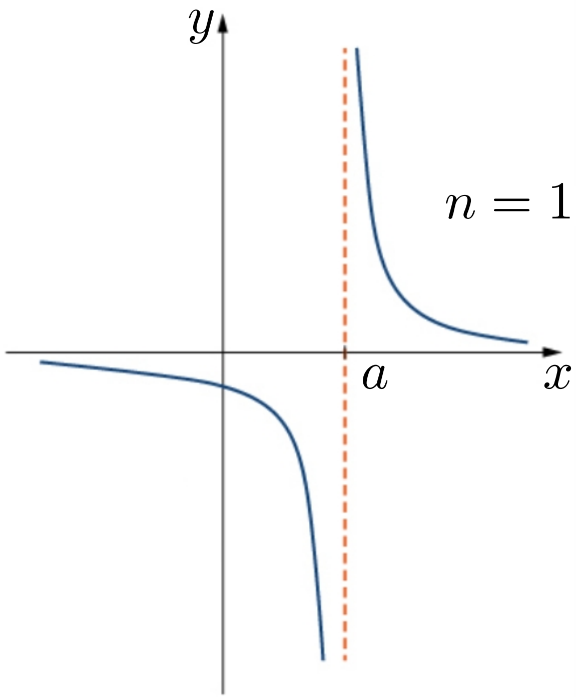
http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_intuitive_one_side.html

Chapter 1.3: The Limit of a Function

$$f(x) = \frac{1}{(x-a)^n}$$

$n = 2, 4, 6, \ldots$





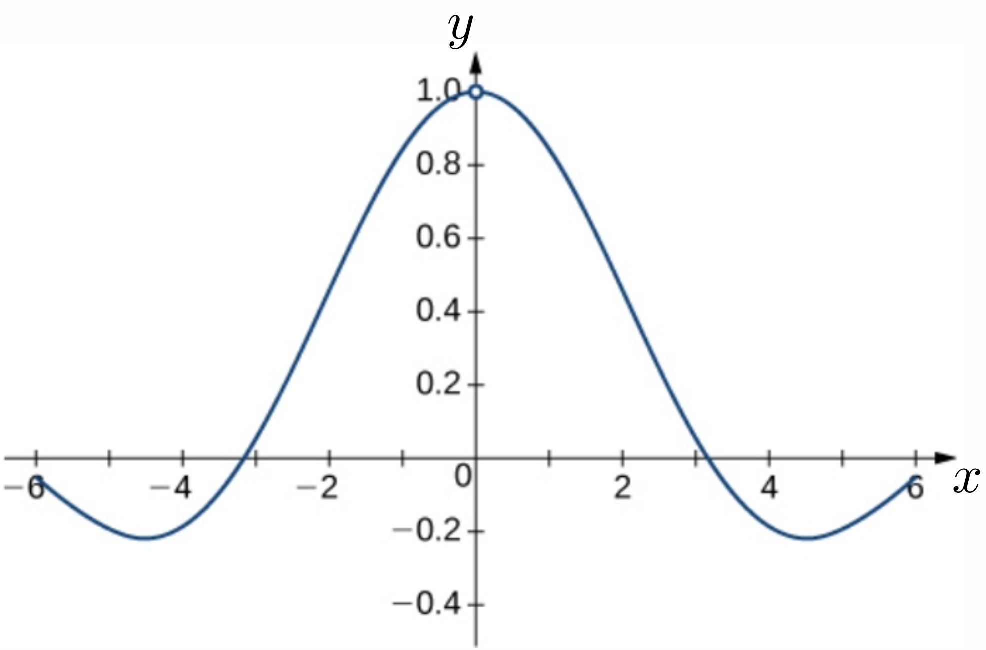
$n = 1, 3, 5, \ldots$

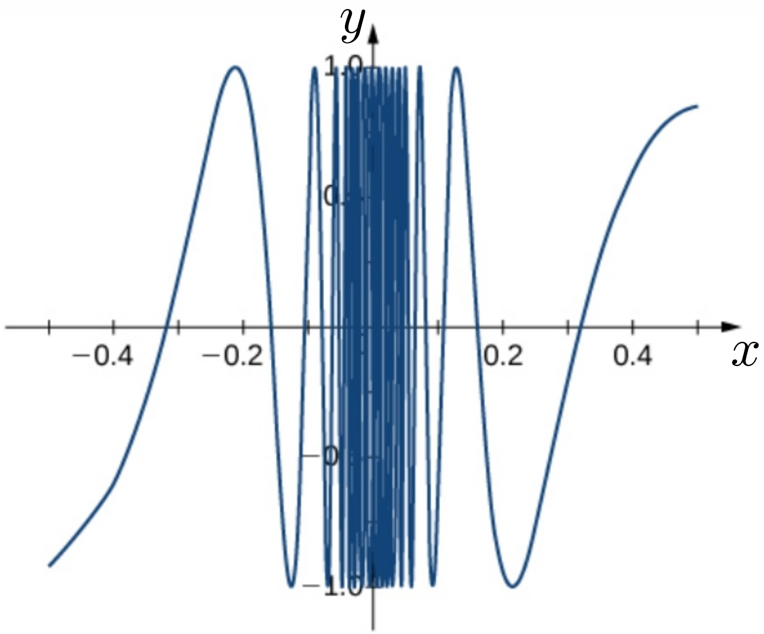
http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_intuitive_one_side.html

Chapter 1.3: The Limit of a Function

$$\lim_{x \to 0} \frac{\sin(x)}{x} = ?$$

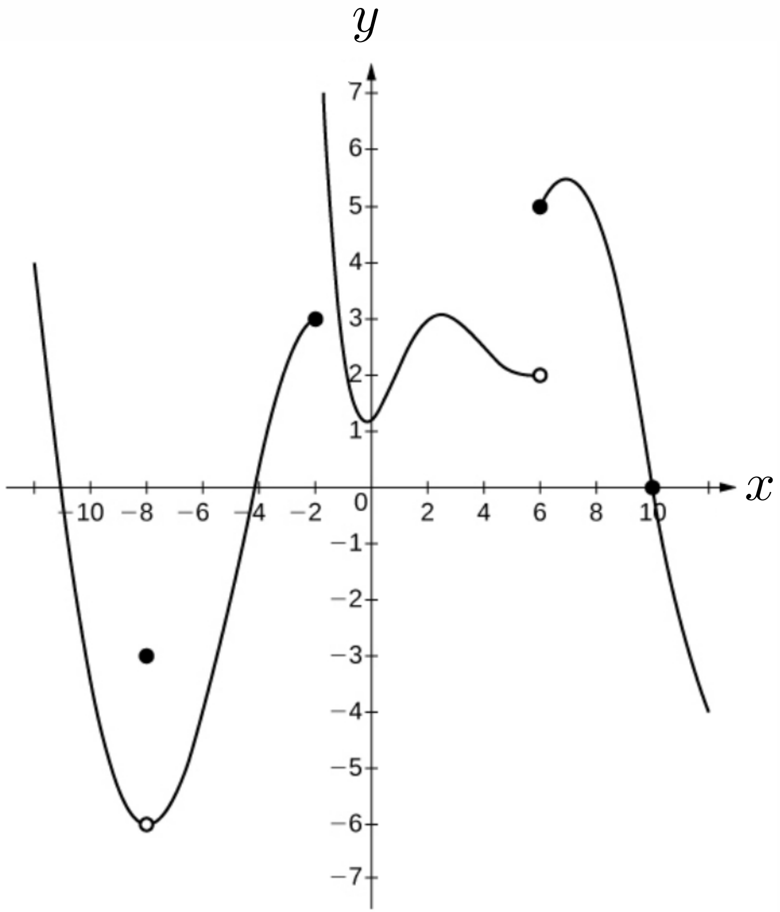
$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = ?$$





http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_trig_sin.html

Chapter 1.3: The Limit of a Function



1. $\lim_{x \to 10} f(x) = ?$

2. $\lim_{x \to -2^+} f(x) = ?$

3. $\lim_{x \to -8} f(x) = ?$

4. $\lim_{x \to 6} f(x) = ?$

Chapter 1.3: Intuitive Definition

- Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L \qquad \Leftrightarrow \qquad \lim_{h \to 0} f(h+a) = L$$

- Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

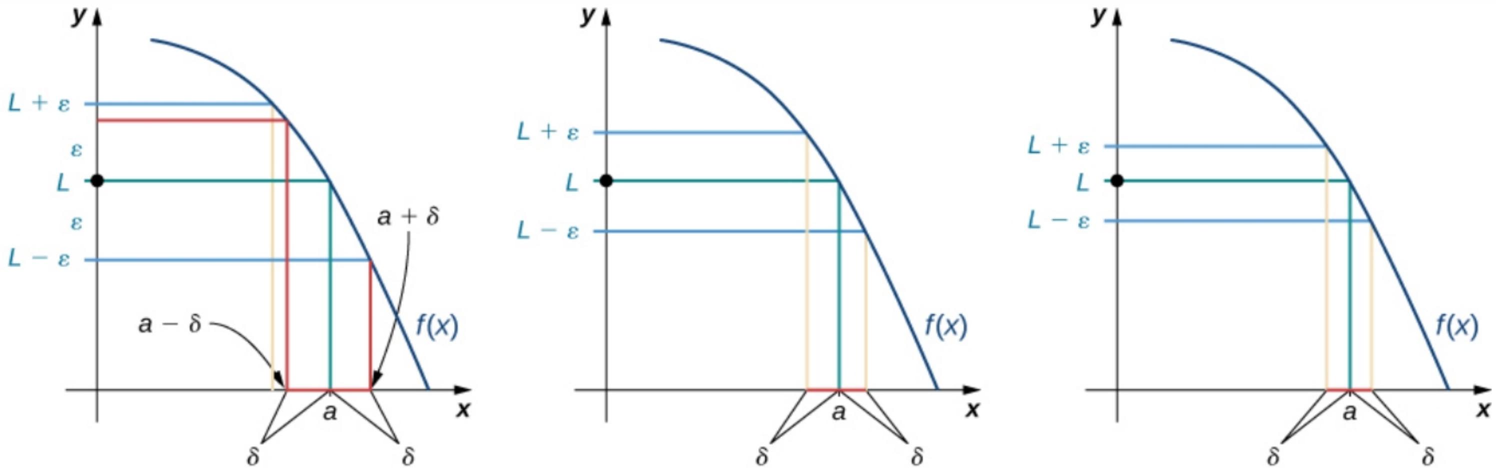
$$\exists \lim_{x \to a} f(x) = L \qquad \text{iff} \qquad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Chapter 1.3: Precice Definition

- Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.



Chapter 1.3: Conceptual Calculation of a Limit

- For a function $f : D \to R$ calculate the limit $\lim_{x \to x_0} f(x)$

1. Check if x_0 is a point at which f changes type, and/or a point which does not belong to the domain of f.

2a. If yes, then calculate $\lim_{x \to x_0^{\pm}} f(x)$ and compare them

2b. If no, then $\lim_{x \to x_0} f(x) = f(x_0)$

Chapter 1.4: Calculating Limits

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1.) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

2.) $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

3.) $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

4.) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

5.) $\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)|$

6.) $\lim_{x\to a} \sqrt[\nu]{f(x)} = \sqrt[\nu]{\lim_{x\to a} f(x)}$, if $f(x) \geq 0$ around a

7.) $\lim_{x\to a} f^{\nu}(x) = [\lim_{x\to a} f(x)]^{\nu}$, $\nu \in \mathbb{N}$

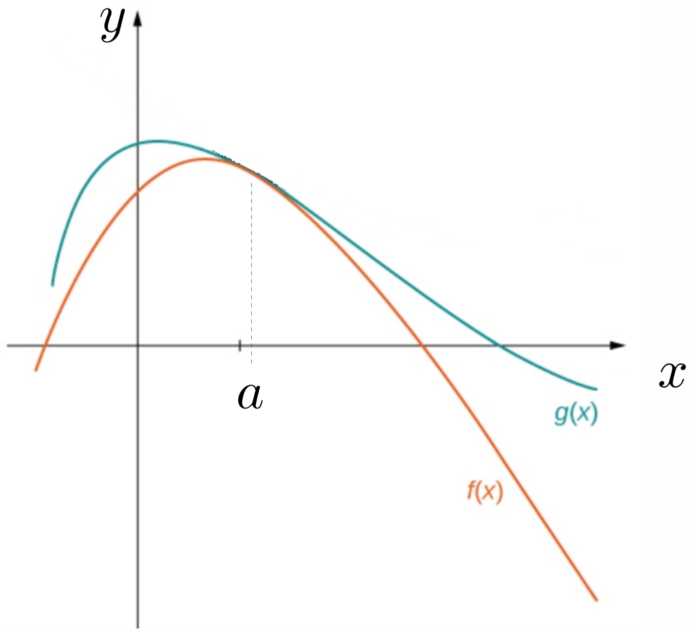
http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_laws.html

Chapter 1.4: Calculating Limits

Theorem:

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$



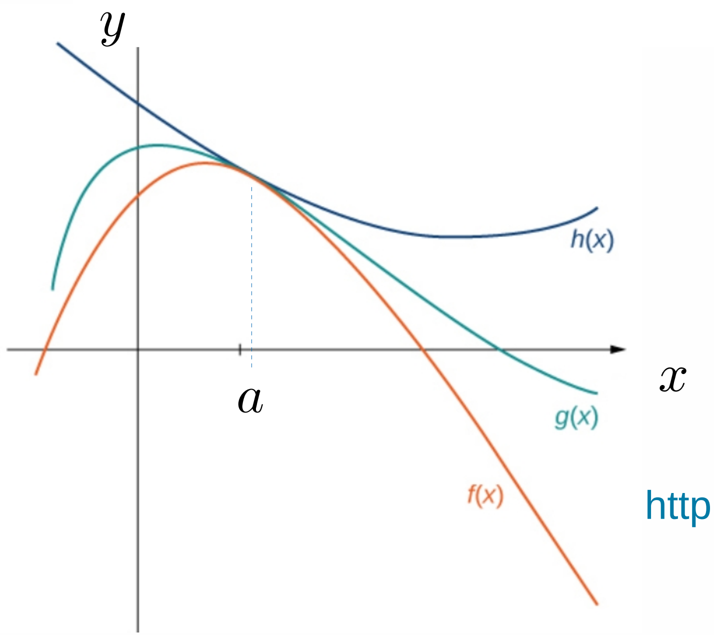
Chapter 1.4: Calculating Limits

Theorem:

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then

(Sandwich Theorem)

$$\lim_{x \to a} g(x) = L$$



Example: Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_trig_sin.html

Chapter 1.4: Calculating Limits

1. $\displaystyle \lim_{x\to 2} \frac{x^2-4}{x^2+x-6} = \frac{4}{5}$

2. $\displaystyle \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$

3. $\displaystyle \lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta} = 0$

4. $\displaystyle \lim_{x\to 2} \frac{2x^2-3x+1}{5x+4} = \frac{3}{14}$

5. $\displaystyle \lim_{x\to 3} \left(\frac{1}{x-3} - \frac{4}{x^2-2x-3}\right) = \frac{1}{4}$

6. $f(x) = \begin{cases} 4x-3\,, & x<2 \\ (x-3)^2\,, & x\geq 2 \end{cases}$

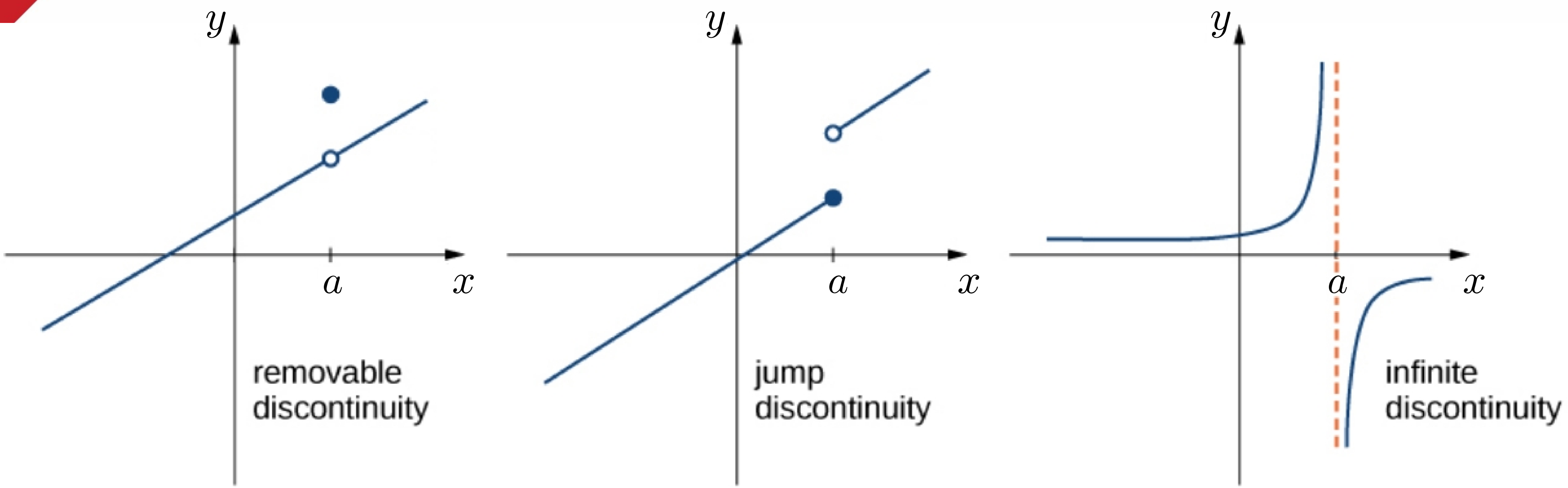
7. $\displaystyle \lim_{x\to 1} \frac{\sqrt{x-1}-\sqrt{x^2-x}}{\sqrt{x^2}-1} = 0$

8. $\displaystyle \lim_{x\to 0} \sin(x)\cos(1/x) = 0$

9. $\displaystyle \lim_{x\to 2} \frac{\sin(x-2)}{x^2-5x+6} = -1$

10. $\displaystyle \lim_{x\to 2} \frac{|x^3-x-1|-|x-7|}{x^2-4} = 3$

Chapter 1.5: Continuity



http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_intuitive_one_side.html

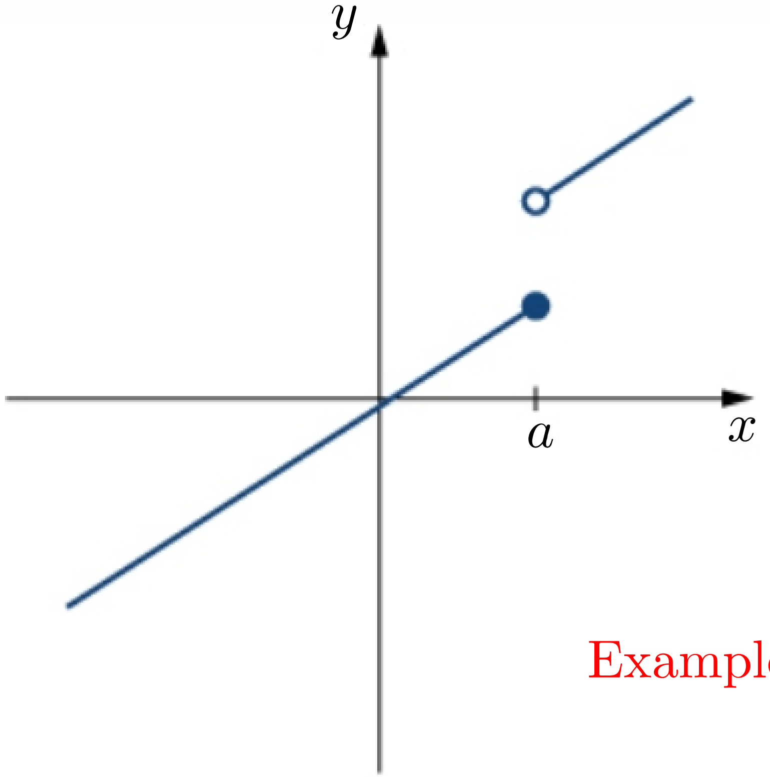
Definition:

A function f is **continuous at a number** a if and only if the following three conditions are satisfied:

$$\begin{array}{rl} \text{i.} & f(a) \quad \text{is defined} \\ \text{ii.} & \lim_{x \to a} f(x) \quad \text{exists} \\ \text{iii.} & \lim_{x \to a} f(x) = f(a) \end{array}$$

A function is **discontinuous at a point** a if it fails to be continuous at a.

Chapter 1.5: Continuity over an Interval



- A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$
- A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$

Example: In which intervals is $f(x) = \dfrac{x-1}{x^2+2x}$ continuous?

Chapter 1.5: Continuity

Theorems:

- If $f(x)$ is continuous at L and $\lim\limits_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f\big(g(x)\big) = f\Big(\lim_{x\to a} g(x)\Big) = f(L)$$

- If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f\big(g(x)\big)$ is continuous at a.

- Polynomials, rational functions, root functions and trigonometric functions are continuous at every number in their domains.

- If f and t are continuous at a and c is a constant, then the following functions are also continuous at a :

1. $f \pm g$, **2.** cf, **3.** fg, **4.** $\dfrac{f}{g}$ if $g(a) \neq 0$

Chapter 1.5: Continuity

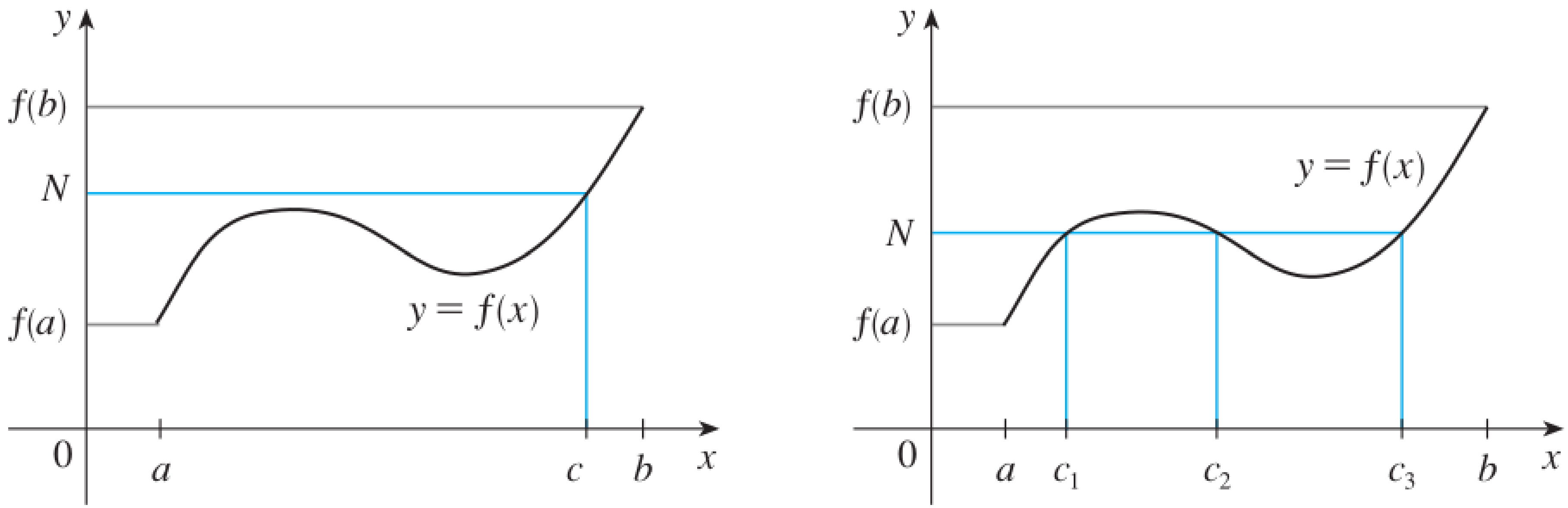
1. $$f(x) = \begin{cases} \ln(3x-2) + 3x^2 - 4\,, & x > 1 \\ \dfrac{\sqrt{|x|}}{x-2}\,, & x \leq 1 \end{cases}$$

2. $$g(x) = \begin{cases} \dfrac{ax^2 + bx - 5}{x-1}\,, & x \neq 1 \\ 8\,, & x = 1 \end{cases}$$

3. $$h(x) = \begin{cases} x^2 + 2x + a^2\,, & x \leq b \\ \sin(x-b) - 1\,, & x > b \end{cases}$$

Chapter 1.5: The Intermediate Value Theorem

- Let f be continuous over a closed, bounded interval $[a, b]$. If N is any real number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = N$.



Chapter 1.5: The Intermediate Value Theorem

1. Show that $f(x) = x - \cos(x)$ has at least one zero.

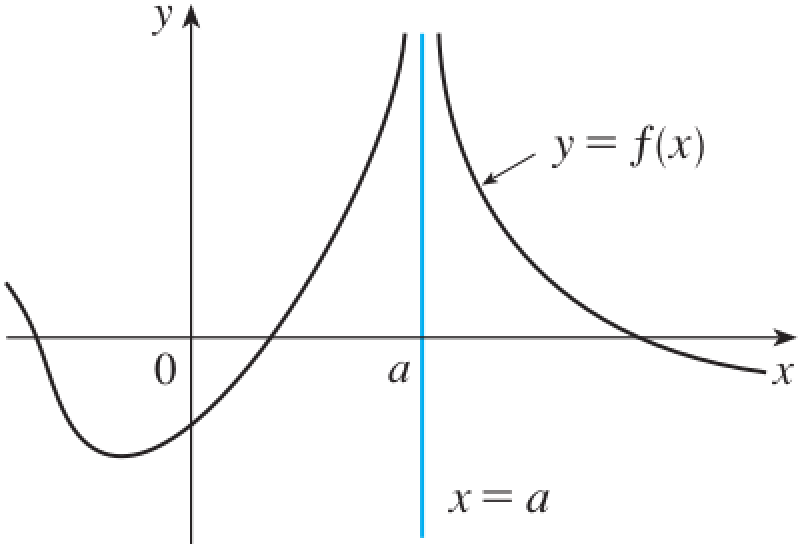
2. If $f(x)$ is continuous over $[1,2]$, $f(1) > 0$ and $f(2) > 0$, can we use I.V.T. to conclude that $f(x)$ has no zeros in the interval $[1,2]$?

3. For $f(x) = 1/x^3$, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1,1]$?

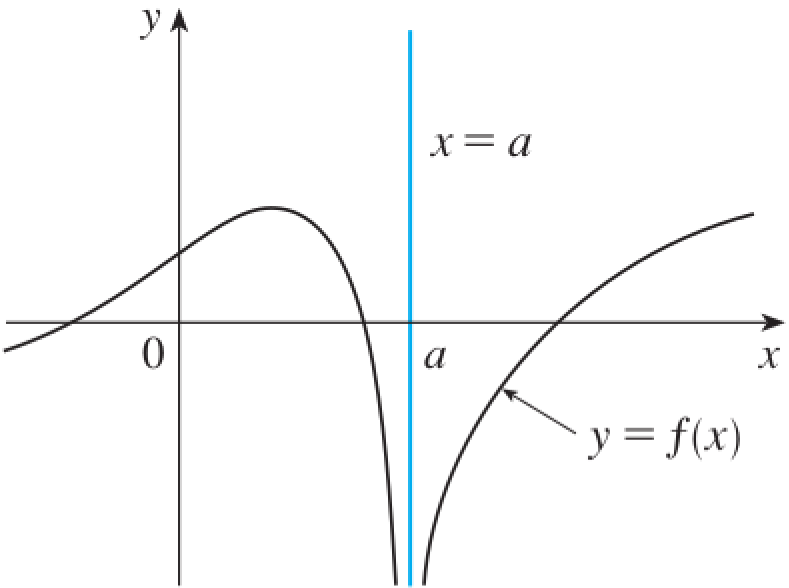
4. Show that there is at least one $x_0 \in (0,1)$, such that $x_0^2 + 3x_0 = e^{x_0} + 1$.

1.6 Limits involving Infinity

$$\lim_{x \to a} f(x) = \infty$$



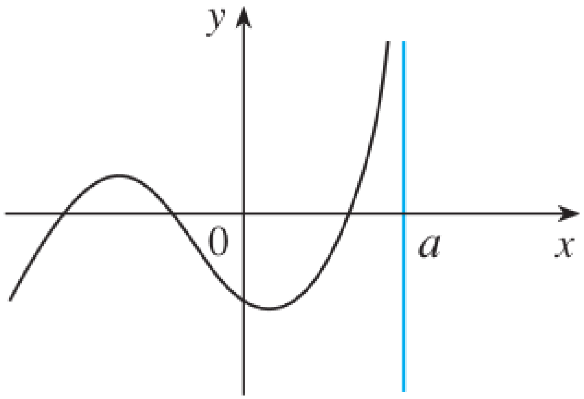
$$\lim_{x \to a} f(x) = -\infty$$



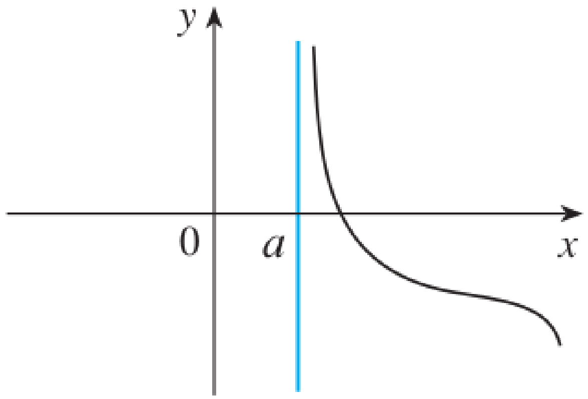
http://webspace.ship.edu/msrenault/GeoGebraCalculus/limit_intuitive_one_side.html

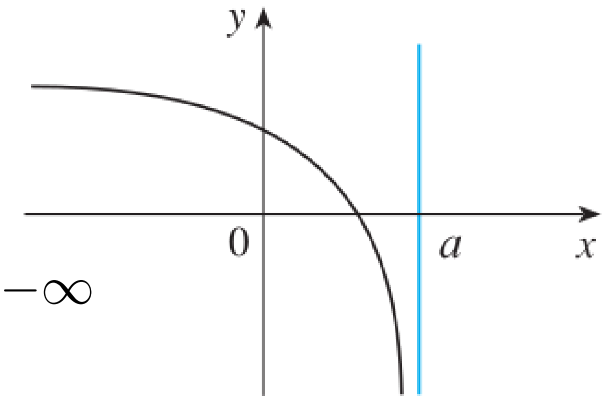
1.6 Limits involving Infinity

$$\lim_{x \to a^-} f(x) = \infty$$

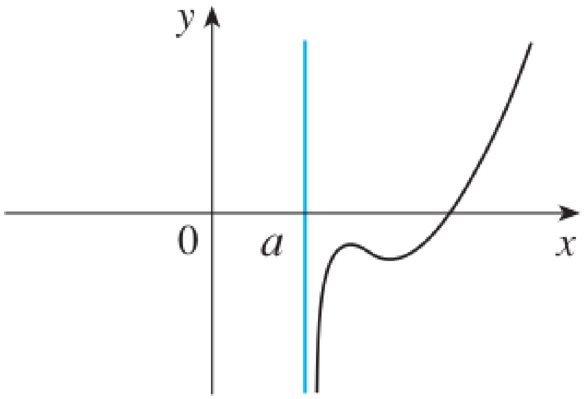


$$\lim_{x \to a^+} f(x) = \infty$$





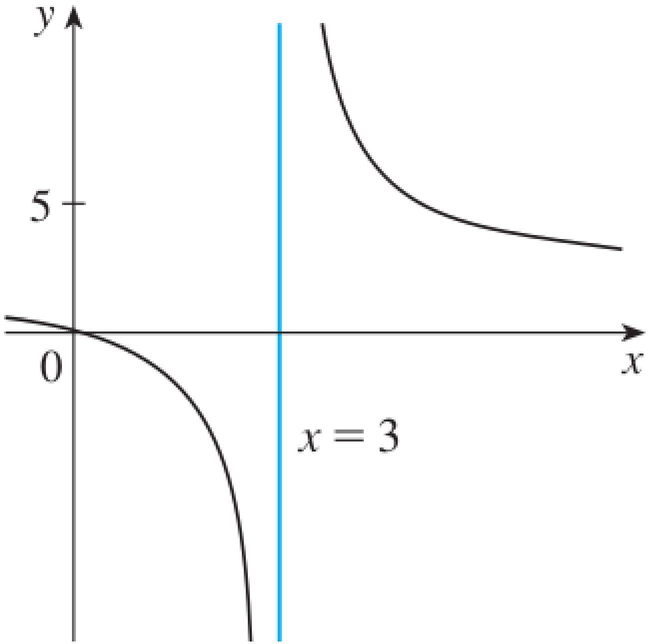
$$\lim_{x \to a^-} f(x) = -\infty$$



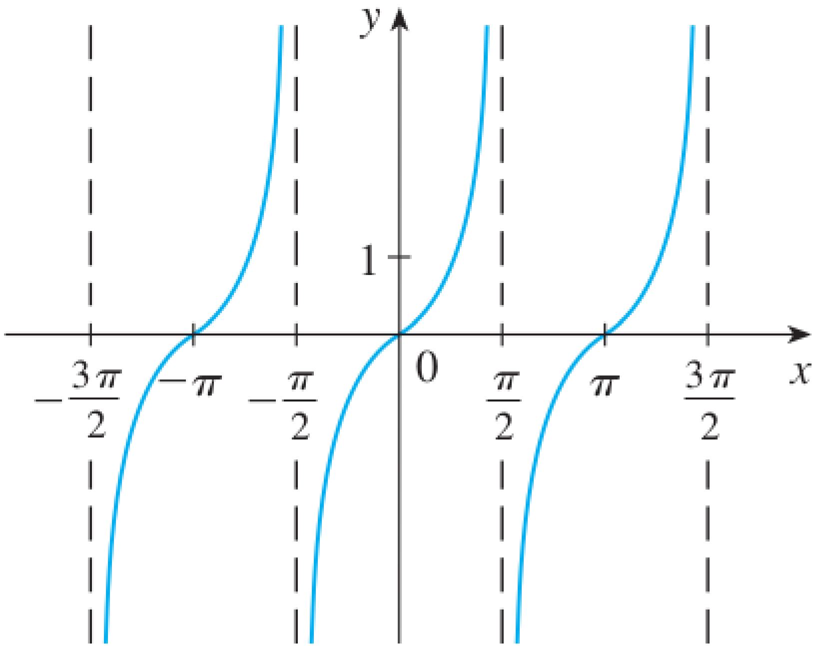
$$\lim_{x \to a^+} f(x) = -\infty$$

1.6 Limits involving Infinity

$$f(x) = \frac{2x}{x-3}$$



$$f(x) = \tan(x)$$



1.6 Examples

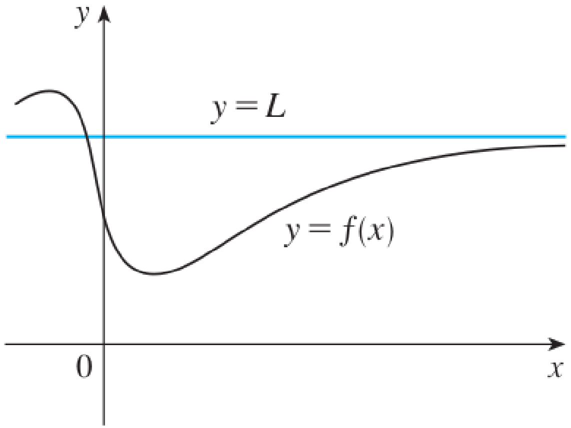
1. $$\lim_{x \to 1/2} \frac{4x-5}{(2x-1)^2} = -\infty$$

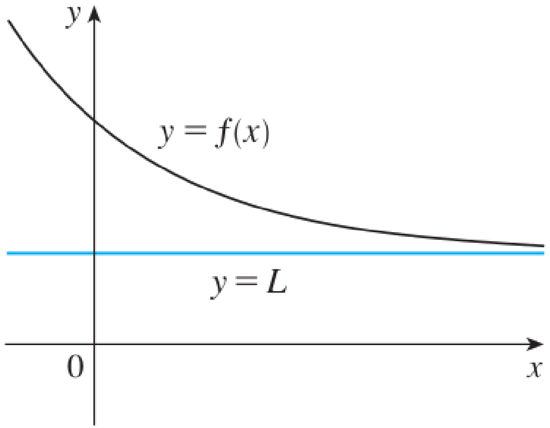
2. $$\lim_{x \to 2} \frac{6x-3}{\sqrt{x-2}} = \infty$$

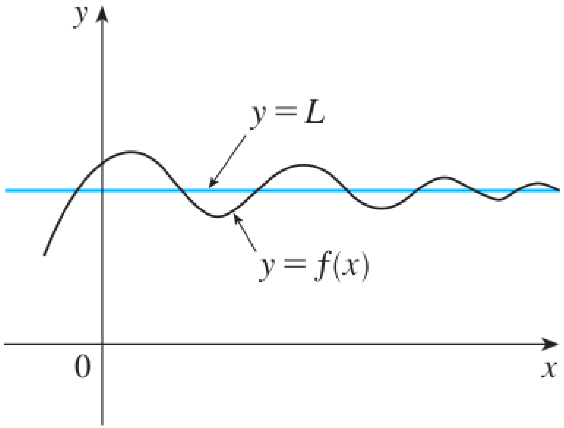
3. $$\lim_{x \to -1} \frac{4x+1}{|x+1|} = -\infty$$

4. $$\lim_{x \to 0} \left(\frac{1}{\sqrt{x}} - \frac{1}{|x|} \right) = -\infty$$

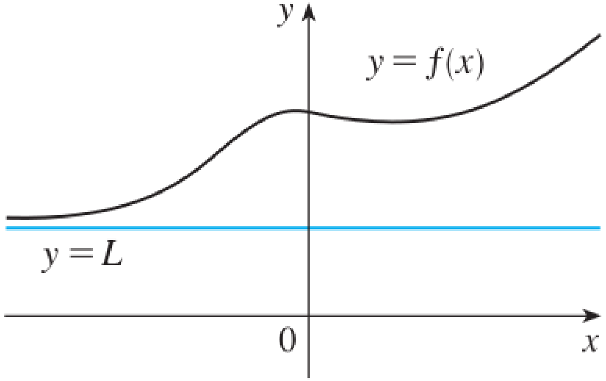
1.6 Limits involving Infinity

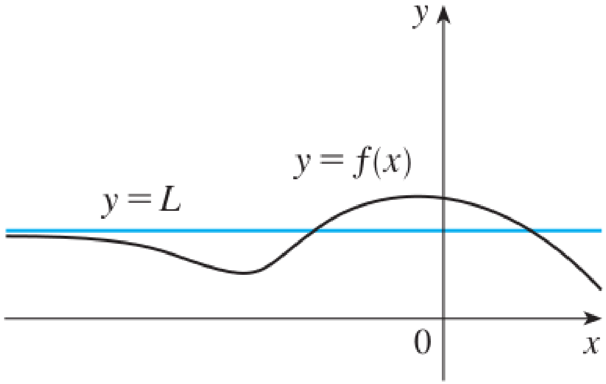






$$\boxed{\lim_{x\to\infty} f(x) = L}$$





$$\boxed{\lim_{x\to-\infty} f(x) = L}$$

1.6 Examples

1. $$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 5x - 3} = \frac{3}{5}$$

2. $$\lim_{x\to\infty} \left(\sqrt{x^2 + 2} - x\right) = 0$$

3. $$\lim_{x\to\infty} \frac{x^2 + 2x}{4 - x} = -\infty$$

4. $$\lim_{x\to\infty} \left(2x - |x^3 - x - 1|\right) = -\infty$$

5. $$\lim_{x\to\infty} \frac{e^x + 2^{x+1}}{2e^x + 2^x} = \frac{1}{2}$$

6. $$\lim_{x\to-\infty} x\big(\cos(1/x) - 1\big) = 0$$

7. $$\lim_{x\to-\infty} \frac{x^2\cos(1/x) - x^2}{2x - 1} = 0$$

8. $$\lim_{x\to\infty} \frac{x}{x^2 + 2}\cos(5x) = 0$$