ĐẠI HỌC BÁCH KHOA HÀ NỘI
VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

Discrete Mathematics

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Kind of problems solved by discrete mathematics

- How many ways are there to choose a computer password?
- What is the probability of winning a lottery?
- Is there a link between two users in a social network?
- What is the shortest path between two cities using a transportation system?
- How can a list of integers sorted in increasing order? How many steps are required to do such a sorting?

Discrete Mathematics

Discrete Mathematics deals with

- "Separated" or discrete sets of objects (rather than continuous sets)
- Processes with a sequence of individual steps (rather than continuously changing processes)

Importance of Discrete Mathematics

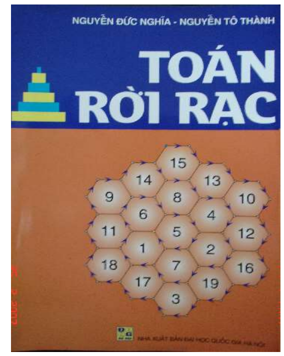
- Information is stored and manipulated by computers in a discrete fashion
- Applications in many different areas
- Discrete mathematics is a gateway to more advanced courses
- Develops mathematical reasoning skills
- Emphasizes the new role of mathematics

Goals of this course

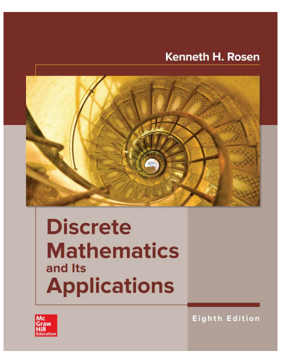
- Study of standard facts of discrete mathematics
- Development of mathematical reasoning skills (emphasis on modeling, logic, efficiency)
- Discussion of applications

Text book

Nguyễn Đức Nghĩa,
Nguyễn Tô Thành
TOÁN RỜI RẠC
(in lần thứ ba)
Nhà xuất bản Đại học Quốc gia Hà nội, 2003, 290 trang



Text book



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Use lecture notes as study guide.

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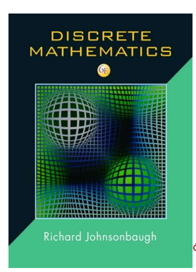

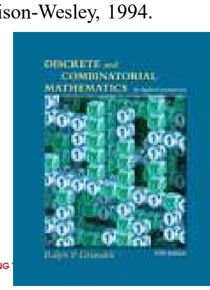

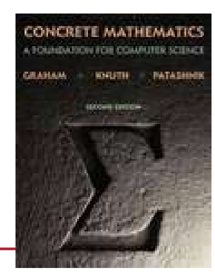

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PART 1

COMBINATORIAL THEORY

(Lý thuyết tổ hợp)

PART 2

GRAPH THEORY

(Lý thuyết đồ thị)

PART 1

COMBINATORIAL THEORY

(Lý thuyết tổ hợp)

PART 2

GRAPH THEORY

(Lý thuyết đồ thị)

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Contents of Part 1

Chapter 0: Sets, Relations

Chapter 1: Counting problem

Chapter 2: Existence problem

Chapter 3: Enumeration problem

Chapter 4: Combinatorial optimization problem

Contents

1. Definitions

2. Set operations

3. The algebra of sets

4. Computer representation of sets

5. Relations

6. Functions

1.Definitions

We have already implicitly dealt with sets

Integers (Z), rationals (Q), naturals (N), reals (R), etc.

We will develop more fully

The definitions of sets

The properties of sets

The operations on sets

1. Definitions

1.1 Set and element

1.2. Specification of set

1.1 Set and element

Definition:

A set is an unordered collection of (unique) objects

The objects in a set are called elements or members of a set.

A set is said to contain its elements.

Notation, for a set A:

$x \in A$: x is an element of A

$x \notin A$: x is not an element of A

Example:

V={a, e, i, o, u} (vowels in English)

C = all students subscribed to IT3020E in Winter 2020

Note:

We often denote sets with capitals

Brackets are used to define the set. {.}

1.1 Set and element

Definition: A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where

Element m_1 occurs a_1 times

Element m_2 occurs a_2 times

…

Element m_r occurs a_r times

1. Definitions

1.1 Set and element

1.2. Specification of set

1.2 Specification of set

Our first concern will be how to describe a set; that is, how do we most conveniently describe a set and the elements that are in it? Sets can be defined in various ways.

At first we consider two ways:

1. Set extension
2. Set intension

1.2 Specification of set

A set is defined in **extension** when you enumerate all the elements:

O={0,2,4,6,8}

The **set-builder** notation

A = {*x* | conditions(*x*)}.

this could be read as "all *x* such that the conditions hold true".

Example: O={ x | ($x \in Z$) $\wedge$ (x=2k) for some $k \in$Z}

reads: O is the set that contains all *x* such that *x* is an integer and *x* is even

A set is defined in **intension** when you give its set-builder notation

O={ x | ($x \in Z$) $\wedge$ ($0 \le x \le 8$) $\wedge$ (x=2k) for some $k \in$ Z }

1.2 Specification of set

If there are exactly *n* distinct elements in a set S, with *n* is a nonnegative integer, we say that:

S is a finite set, and

The cardinality of S is *n*. Notation: |S| = *n*.

Definition. A set is a ***finite set*** if it has a finite number of elements. A set that is not finite is an *infinite set*.

Let *A* be a finite set. The number of different elements in *A* is called its *cardinality* and is denoted by |*A*|. Other notations commonly used for the cardinality of *A* are *N*(*A*), #*A*.

If *A* be a infinite set, then we write |*A*| = ∞.

1.2 Specification of set

Well-known sets in math:

N = {0,1,2,3,...}

Z = {...,-2,-1,0,1,2,...}

Z^+ = {1,2,3,...}

Q = {p/q | p in Z, q in Z, q is not 0}

R = {x | x is a real number}.

{,...} is used to indicate the the rest of the sequence once it's clear how to proceed

Example: {1,2,3,4,...}

Contents

2. Set operations

2.1 Set comparison

2.2 Venn diagram

2.3 Set operations

2.4 Partition and cover

2.1. Set comparison

Definition: Two sets, A and B, are <u>equal</u> if they contain the same elements. We write A=B.

Example:

$\{2,3,5,7\}=\{3,2,7,5\}$, because a set is <u>unordered</u>

Also, $\{2,3,5,7\}=\{2,2,3,5,3,7\}$ because a set contains <u>unique</u> elements

However, $\{2,3,5,7\} \neq \{2,3\}$

2.1. Set comparison

if $P(x)$ and $Q(x)$ are propositional functions which are true for the same objects x, then the sets they define are equal, i.e.

$$\{x : P(x)\} = \{x : Q(x)\}.$$

Example: there are 2 sets

$A = \{x: (x-4)^2 = 25\}$

$B = \{x: (x+1)(x-9) = 0\}$

Question: A = B ?

Yes: A = B, since the two propositional functions $P(x)$: $(x-4)^2 = 25$ and $Q(x)$: $(x+1)(x-9) = 0$ are true for the same values of x, namely -1 and 5.

2.1. Set comparison

Definition: A is said to be a **subset** of B, if and only if every element of A is also an element of B

that is: $\forall x\ (x \in A \Rightarrow x \in B)$

Denote: $A \subseteq B$ or $B \supseteq A$,

Example: $S = \{ 1, 2, 3, \ldots, 11, 12 \}$ and $T = \{ 1, 2, 3, 6 \}$ then $T \subseteq S$.

Theorem: For any set S

$\varnothing \subseteq S$ and

$S \subseteq S$

2.1. Set comparison

•**Definition**: If $A \subseteq B$ and $A \neq B$ then set A is called a **proper subset** of set B.

(that is there is an element $x \in B$ such that $x \notin A$)

Denote: $A \subset B$

Example 1: $A = \{ 1, 2, 3 \}$, $B = \{ 2, 3, 1 \}$, $C = \{ 3 \}$. Then:

$$B = A,\ C \subset A,\ C \subset B.$$

2.1. Set comparison

We shall sometimes use E and O to denote the sets of even and odd integers respectively:

$E = \{2n : n \in Z\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

$O = \{2n+1 : n \in Z\} = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$

Universal set **(U)** : contains as subsets all sets relevant to the current task or study. Anything outside the universal set is simply not considered. The universal set is not something fixed for all time -we can change it to suit different contexts.

2. Set operations

2.1 Set comparison

2.2 Venn diagram

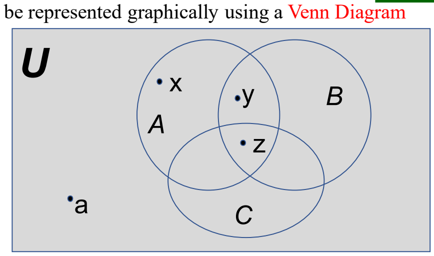
2.3 Set operations

2.4 Partition and cover

2.2. Venn diagram

John Venn
1834-1923

A set can be represented graphically using a Venn Diagram



The universal set U is represented by the interior of a rectangle

The sets by region inside the rectangle and elements which belong to a given set are placed inside the region representing it.

If an element belongs to more than one set in the diagram, the two regions representing the sets concerned must overlap and the element is placed in the overlapping region.

In this way the picture represents the relationships between the sets concerned.

2. Set operations

2.1 Set comparison

2.2 Venn diagram

2.3 Set operations

2.4 Partition and cover

2.3. Set operations

Arithmetic operators (+,-, × ,÷) can be used on pairs of numbers to give us new numbers

Similarly, set operators exist and act on two sets to give us new sets:

1. Union
2. Generalized union
3. Intersection
4. Generalized intersection
5. Set difference
6. Set complement

2. Set operations

2.1 Set comparison

2.2 Venn diagram

2.3 Set operations

2.4 Partition and cover

2.4. Partition and cover

Let $\mathcal{E} = \{E_i\}_{i \in I}$ be a collection of subsets of the set M, $E_i \subseteq M$. Collection $\mathcal{E}$ will be called a cover of M if each element of M must be an element of at least one of the sets of $\mathcal{E}$:

$$M \subset \bigcup_{i \in I} E_i \Leftrightarrow \forall x \in M\ \exists i \in I\ x \in E_i .$$

The disjoint cover $\mathcal{E}$ of M is called a partition of M, i.e.

$$\mathcal{E} \text{ is a partition of } M \Leftrightarrow M = \cup_{i \in I} E_i,\ E_i \cap E_j = \varnothing,\ i \neq j.$$

Example: M = {1, 2, 3, 4}

$\mathcal{E}_1$ = {{1, 2}, {3, 4}} is a partition of M

$\mathcal{E}_2$ = {{1, 2, 3}, {3, 4}} is not a partition of M

Contents

3.1. Power Set

- **Definition**: The power set of a set A, denoted P(A), is the set of all subsets of A.

Examples

Let A = {∅ } → P (A) = {∅}
Let A = {a} → P (A) = {∅, {a}}
Let A = {a, b} → P(A)= {∅, {a}, {b}, {a, b}}
Let A = {a, b, c} → P(A) = ?

{
∅,
{a}, {b}, {c},
{a, b}, {a, c}, {b, c},
{a, b, c}
}

- Note: the empty set ∅ and the set itself are always elements of the power set.

Contents

3.1. Power Set

- **Theorem**: Let A be a set such that |A|=n, then

$$|P(A)| = 2^n$$

Proof: Let A be the set $\{a_1, a_2, ..., a_n\}$.

- We can form a subset of A by considering each element a_i in turn and either including it or not in the subset.
- For each element there are two choices (either include it or don't) and the choice for each element is independent of the choices for the other elements, so there are 2^n choices altogether.
- Each of these 2^n choices gives a different subset and every subset of A can be obtained in this way.

3.1. Power Set

Theorem.

For all sets A and B:

1. $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.
2. $P(A) \cap P(B) = P(A \cap B)$.
3. $P(A) \cup P(B) \subseteq P(A \cup B)$.

3.2 Properties of set operations

Let A, B and C be any sets. The following laws hold

Equality	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \varnothing$ $\bar{\varnothing} = U$ $\bar{U} = \varnothing$	Complementation laws
$\overline{(\bar{A})} = A$	Involution laws

Contents

3.2 Properties of set operations

Equality	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$	De Morgan's laws
$A \cap (A \cup B) = A$ $A \cup (A \cap B) = A.$	Absorption laws

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5. Relations

5.1. Ordered pair

An *ordered pair* is a set of a pair of objects with an order associated with them.

In general (x, y) is different from (y, x).

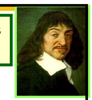
Definition (equality of ordered pairs): Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Example: if the ordered pair (a, b) is equal to $(1, 2)$, then a=1, and b=2. $(1, 2)$ is not equal to the ordered pair $(2, 1)$.

5. Relations

5.2. Cartesian product

René Descartes (1596-1650)



Let $A_1, A_2, \ldots, A_n$ be any sets, where $n \in \mathbb{Z}^+$ and $n \geq 3$.

Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is defined as follows:

$$A_1 \times A_2 \times \ldots \times A_n \equiv_{\text{def}} \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, 1 \leq i \leq n\}.$$

When $A_1 = A_2 = \ldots = A_n = A$, it is usually to denote $A \times A \times \ldots \times A$ by A^n

An element of $A_1 \times A_2 \times \ldots \times A_n$ is called an ordered n-tuple.

When n=3, we have a *triple*.

Example: $A = \{1, 2\}$, $B = \{a, b\}$ and $C = \{\alpha, \beta\}$ then

$A \times B \times C = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}$.

5. Relations

5.3. Binary Relation

Let A and B be sets:

Definition (binary relation): A binary relation from a set A to a set B is a set of ordered pairs (a, b) where a is an element of A and b is an element of B.

A binary relation from A to B is a subset $R \subseteq A \times B$

A relation on a set A is a relation from A to A, i.e., a subset $R \subseteq A \times A$

Notation: When an ordered pair (a, b) is in a relation R, we write $a\ R\ b$, or $(a, b) \in R$. It means that element a is related to element b in relation R. We will write $a\ \overline{R}\ b$ when a element a is not related to element b in relation R.

5. Relations

5.1. Ordered pair

5.2. Cartesian product

5.3. Binary relation

5.4. Relation representation

5.5. Operations on relations

5.6. Properties of relations

Relations *on* a Set

- A relation *on* a set A is a relation from A to A.

Examples of relations on Z^+: $R_<$, $R_\geq$, $R_>$:

- $R_< = \{(x, y) | x < y\}$ ($R_<$ is relation "strictly less than").
- $R_\geq = \{(x, y) | x \geq y\}$ ($R_\geq$ is relation "greater or equal").
- $R_> = \{(x, y) | x > y\}$ ($R_>$ is relation "strictly greater than").

5.4. Relation representation

1. Set of ordered pairs
2. Mapping
3. Table
4. Grid graph
5. Binary matrix
6. Directed graph

5. Relations

5.5. Operations on relations

A relation is a set. It is a set of ordered pairs if it is a binary relation. Thus all the set operations apply to relations such as union, intersection and complementing.

Example:

The union of the "less than" and "equality" relations on the set of integers is the "less than or equal to" relation on the set of integers.

The intersection of the "less than" and "less than or equal to" relations on the set of integers is the "less than" relation on the same set.

The complement of the "less than" relation on the set of integers is the "greater than or equal to" relation on the same set.

1. **Complementary Relations**
2. **Inverse Relations**
3. **Identity relation**
4. ***n*-ary Relations**
5. **Composite Relation**

5. Relations

5.6. Properties of relations

Six properties of relations we will study:

1. Reflexive
2. Irreflexive
3. Symmetric
4. Asymmetric
5. Anti-symmetric
6. Transitive

Contents

6. Functions

6. Functions

Definition: A function f from a set A to a set B, denote it by *f: $A \rightarrow B$*, is a relation from A to B that satisfies:

for each element a in A, there is an element b in B such that (a, b) is in the relation, and

if (a, b) and (a, c) are in the relation, then $b = c$.

➔ **1 to 1:**

For every input there is exactly one output.

A function is also called a *mapping* or a *transformation*.

6. Functions

Definition: A function f from a set A to a set B, denote it by *f: $A \rightarrow B$*, is a relation from A to B that satisfies:

for each element a in A, there is an element b in B such that (a, b) is in the relation, and

if (a, b) and (a, c) are in the relation, then $b = c$.

A function is also called a *mapping* or a *transformation*. ➔ **1 to 1**

The set A in the above definition is called the *domain* of the function and B its *codomain*.

Thus, f is a function if it *covers* the domain (maps every element of the domain) and it is *single valued.*

Proposition: If $|A| = m$, $|B| = n$, then the number of possible functions from A to B is n^m

6. Functions

Thus, $f: A \to B$ is a function if it *covers* the domain (maps every element of the domain) and it is *single valued.*

- Single valued: each element in the domain is used only once
- Not allowed: 1 – many and 1 to empty

6. Functions

The image of the set S under function $f: A \to B$, denoted by $f(S)$ is:

$$f(S) = \{ f(a) \mid a \in S \}$$

The image of the domain under function $f: A \to B$, denoted by range f is:

$$\text{range } f = f(A)$$

(is also called the range of f)

In general case: range $f = f(A) \subseteq B$.

6. Functions

6.1. Definitions

6.2. Properties of function

6.3. Injective, surjective and bijective function

6.4. Function representation

6.2. Properties of function

$f: A \to B$ is a function from a set A to a set B, $S \subseteq A$, and $T \subseteq B$.

Property 1: $f(S \cup T) = f(S) \cup f(T)$

1. **Proof for $f(S \cup T) \subseteq f(S) \cup f(T)$:**

   Let y be an arbitrary element of $f(S \cup T)$. Then there is an element x in $S \cup T$ such that $y = f(x)$.

   If x is in S, then y is in $f(S)$. Hence y is in $f(S) \cup f(T)$.

   Similarly, if x is in T then y is in $f(S) \cup f(T)$.

   Hence if $y \in f(S \cup T)$, then $y \in f(S) \cup f(T)$.

2. **Proof for $f(S) \cup f(T) \subseteq f(S \cup T)$:**

   Let y be an arbitrary element of $f(S) \cup f(T)$. Then y is in $f(S)$ or in $f(T)$.

   If y is in $f(S)$, then there is an element x in S such that $y = f(x)$. Since $x \in S$ implies $x \in S \cup T$, $f(x) \in f(S \cup T)$. Hence $f(x) = y \in f(S \cup T)$.

   Similarly, if $y \in f(T)$ then $y \in f(S \cup T)$.

Property 1 has been proven.

6.2. Properties of function

$f: A \to B$ is a function from a set A to a set B, $S \subseteq A$, and $T \subseteq B$.

Property 2: $f(S \cap T) \subseteq f(S) \cap f(T)$

Proof.

Let y be an arbitrary element of $f(S \cap T)$. Then there is an element x in $S \cap T$ such that $y = f(x)$, that is there is an element x which is in S and in T, and for which $y = f(x)$ holds. Hence $y \in f(S)$ and $y \in f(T)$, that is $y \in f(S) \cap f(T)$.

Note here that the converse of Property 2 does not necessarily hold. For example let $S = \{1\}$, $T = \{2\}$, and $f(1) = f(2) = \{3\}$. Then $f(S \cap T) = f(\varnothing) = \varnothing$, while $f(S) \cap f(T) = \{3\}$. Hence $f(S) \cap f(T)$ cannot be a subset of $f(S \cap T)$ giving a counterexample to the converse of Property 2.

6. Functions

6.1. Definitions

6.2. Properties of function

6.3. Injective, surjective and bijective function

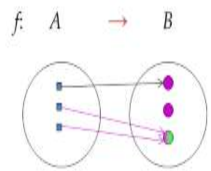
6.4. Function representation

6.3. Injective, surjective and bijective function

A function f from a set A to a set B is said to be *injective* (one-to-one) if and only if:

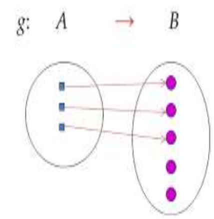
for all elements $a_1, a_2 \in A$

if $f(a_1) = f(a_2)$ then $a_1 = a_2$ no two inputs have the same output.



The function f is not injective

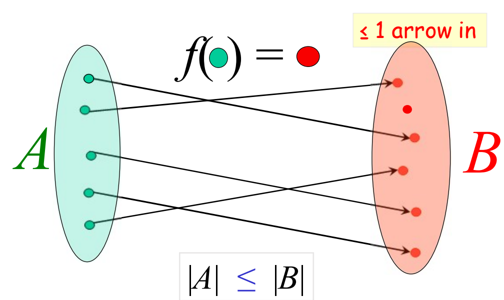
$a_1, a_2 \in A, a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$



The function g is injective

Injection

$f: A \to B$ is an *injection* iff no two inputs have the same output.



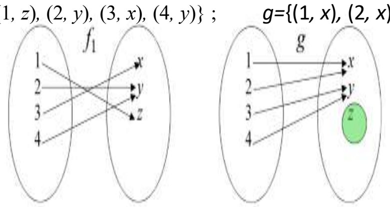
6.3. Injective, surjective and bijective function

A function f from a set A to a set B is said to be *surjective* (onto), if and only if: $\forall b \in B, \exists\, a \in A: b = f(a)$.

that is: f is onto if and only if $f(A) = B$. every output is possible.

Example: $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Then the functions:

$f_1 = \{(1, z), (2, y), (3, x), (4, y)\}$; $g=\{(1, x), (2, x), (3, y), (4, y)\}$

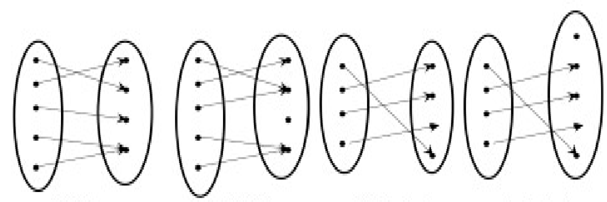


The function f_1 is onto

The function g is not onto because $g(A)=\{x,y\} \neq B$

6.3. Injective, surjective and bijective function

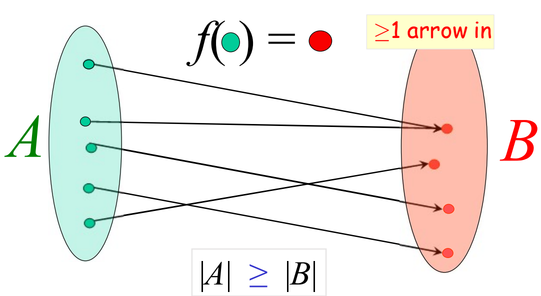
A function is called a *bijection*, if it is injective (1-1) and surjective (onto).



Onto (but not 1-1) | Not Onto (not 1-1) | Both 1-1 and onto | 1-1 but not onto

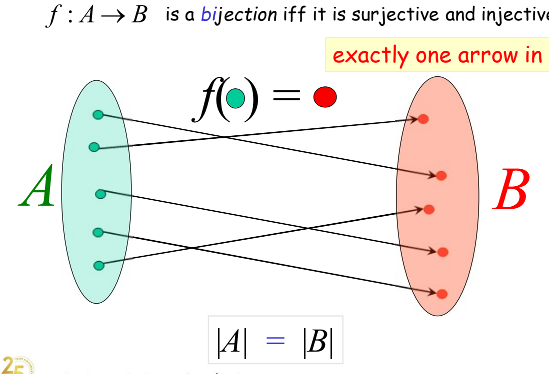
Surjection

$f : A \to B$ is a *surjection* iff every output is possible.



Bijection

$f : A \to B$ is a *bijection* iff it is surjective and injective.



6.3. Injective, surjective and bijective function

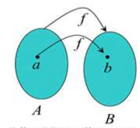
A function is called a *bijection*, if it is injective (1-1) and surjective (onto).

Examples:

1) Linear functions: f(x)=ax+b when a≠0
(with domain and co-domain **R**)
2) Exponential functions: $f(x)=b^x$ (b>0, b≠1)
(with domain **R** and co-domain $\mathbf{R}^+$)
3) Logarithmic functions: $f(x)=\log_b x$ (b>0, b≠1)
(with domain $\mathbf{R}^+$ and co-domain **R**)

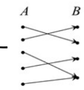
6.4. Function representation

Functions can be represented four different ways:

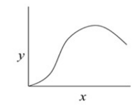


Like Venn diagrams

1. mapping
2. graph
3. table
4. matrix



Bipartite Graph



Plot

a	a_1	a_2	...	a_m
$f(a)$	$f(a_1)$	$f(a_2)$	...	$f(a_m)$

The function $f: A \to B$ can be defined by a matrix $A_f = \{a_{ij}\}$ size of $m \times n$ where

$$a_{ij} = \begin{cases} 1, \text{ if } b_j = f(a_i) \\ 0, \text{ otherwise} \end{cases}, i = 1,\dots,m;\ j = 1,\dots,n.$$

6. Functions

6.1. Definitions

6.2. Properties of function

6.3. Injective, surjective and bijective function

6.4. Function representation