UNIT 8

PROPERTIES OF THE Z-TRANSFORM

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❑ Contents

❑ Learning Objectives

After completing this lesson, you will have a grasp of the following concepts:

- Properties of the Z-transform
- Application of these properties in efficiently calculating Z-transform of complex signals

Linearity

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$$

$$x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

Then

$$x(n) = ax_1(n) + bx_2(n) \xrightarrow{z} X(z) = aX_1(z) + bX_2(z)$$

- ROC of $X(z)$ is the intersection of the 2 ROCs of $X_1(z)$ and $X_2(z)$

$$R_{x-} = \max[R_{x1-}, R_{x2-}]$$

$$R_{x+} = \min[R_{x1+}, R_{x2+}]$$

Example

- Calculate Z-transform and its ROC of the following signal

$$x(n) = [3(2^n) + 4(3^n)]u(n)$$

- Applying the linearity

$$\alpha^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} = \frac{z}{Z-\alpha} \quad \text{và} \quad \text{ROC: } |z| > \alpha$$

- Results

$$X(z) = \frac{3}{1-2z^{-1}} + \frac{4}{1-3z^{-1}} \quad \text{ROC: } |z| > 3$$

$$X(z) = \frac{3z}{z-2} + \frac{4z}{z-3} = \frac{7z^2 - 17z}{z^2 - 5z + 6}$$

2. Z-transform of delayed signals

$$x(n) \xrightarrow{z} X(z) \quad \Longrightarrow \quad x(n-n_0) \xrightarrow{z} z^{-n_0}X(z)$$

- Example

$$x(n) = rect_N(n) = u(n) - u(n-N)$$

$$X(z) = Z\{u(n)\} - Z\{u(n-N)\} = \left(1 - z^{-N}\right)Z\{u(n)\}$$

$$Z\{u(n)\} = \frac{1}{1-z^{-1}} \quad \text{and} \quad \text{ROC: } |z| > 1$$

$$\Longrightarrow \quad X(z) = \begin{cases} N & if\ z = 1 \\ \dfrac{1-z^{-N}}{1-z^{-1}} & if\ z \neq 1 \end{cases}$$

3. Scaling in the Z domain

$$x(n) \xleftrightarrow{z} X(z) \qquad \text{ROC: } r_1 < |z| < r_2$$

$$a^n x(n) \xleftrightarrow{z} X(a^{-1}z) \qquad \text{ROC: } |a|r_1 < |z| < |a|r_2$$

$$\begin{matrix} a = r_0 e^{j\omega_0} \\ z = re^{j\omega} \end{matrix} \quad \Longrightarrow \quad \omega = a^{-1}z = \left(\frac{r}{r_0}\right) e^{j(\omega-\omega_0)}$$

- The meaning of the scaling properties
  - Contraction of ROC (nếu $r_0 > 1$) on the complex plane
  - Expansion of ROC (nếu $r_0 < 1$) on the complex plane
  - Combination with the rotation property (nếu $\omega_0 \neq 2k\pi$) on the complex plane

4. Time reversal property

$$x(n) \xleftrightarrow{z} X(z) \qquad \text{ROC: } r_1 < |z| < r_2$$

$$x(-n) \xleftrightarrow{z} X\left(\frac{1}{z}\right) \qquad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

- Example: Calculate the Z-transform of the signal $x(n) = u(-n)$

$$u(n) \xleftrightarrow{z} \frac{1}{1 - z^{-1}} \qquad \text{ROC: } |z| > 1$$

$$\Rightarrow \quad u(-n) \xleftrightarrow{z} \frac{1}{1 - z} \qquad \text{ROC: } |z| < 1$$

5. Differentiation of the Z-transform

$$-z\frac{\partial X(z)}{\partial z} = \sum_{n=-\infty}^{\infty} [nx(n)]z^{-n} = Z\{nx(n)\}$$

- Example: Calculate the Z-transform of the signal $x(n) = n\alpha^n u(n)$

$$\alpha^n u(n) \overset{z}{\longleftrightarrow} X_1(z) = \frac{1}{1-\alpha z^{-1}} \qquad \text{ROC: } |z| > |\alpha|$$

$$\Rightarrow \qquad n\alpha^n u(n) \overset{z}{\longleftrightarrow} X(z) = -z\frac{\partial X_1(z)}{\partial z} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \qquad \text{ROC: } |z| > |\alpha|$$

6. Z-transform of convolution

$$y(n) = x(n) * h(n) \Longrightarrow Y(z) = X(z).H(z)$$

- Calculating the convolution of two signals using Z-transform:
  - Step 1. Compute the Z-transform of each signal..
  - Step 2. Multiply the two Z-transforms.
  - Step 3. Find the inverse Z-transform
- Note: This method can be easier to perform in many cases compared to directly computing the convolution sum.

Example

- Compute the convolution $x(n) = x_1(n) * x_2(n)$ where

$$x_1(n) = \{\mathbf{1}, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

$$x_1(n) = \{1, -2, 1\} \longrightarrow X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases} \longrightarrow X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$\Rightarrow \quad X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

$$\Rightarrow \quad x(n) = \{\mathbf{1}, -1, 0, 0, 0, 0, -1, 1\}$$

4. Summary

- The Z-transform has several properties such as linearity, time delay, time reversal, and differentiation in the Z-domain. These properties make it more convenient to calculate the Z-transform of complex signals..
- In particular, the property of the Z-transform with convolution allows for easier computation of the convolution sum of two signals in many cases.

5. Assignment

- Exercise 1
  - ❑ Calculate the Z-transform and the corresponding ROC of the following signals:
    a. $x_1(n) = (\cos \omega_0 n)u(n)$
    b. $x_2(n) = (\sin \omega_0 n)u(n)$
    c. $x_3(n) = (3^{n+1} - 1)u(n)$
    d. $x_4(n) = 2^{-n}u(n) + 3^{n+1}u(n)$

Homework

- Exercise 2
  - Calculate the Z-transform and the corresponding ROC of the following signals. The give comments on the change of ROC:

a. $x(n) = 2^n u(n)$

b. $y_1(n) = 3^n x(n)$

c. $y_2(n) = \left(\frac{1}{3}\right)^n x(n)$

d. $y_3(n) = e^{j\pi n/2} x(n)$

Homework

- Exercise 3
  - Calculate the Z-transform and the corresponding ROC of the following signals :

    a. $x(n) = a^n(\cos \omega_0 n)u(n)$

    b. $x(n) = a^n(\sin \omega_0 n)u(n)$

    c. Ramp signal $u_r(n)$

The next unit 9

INVERSE Z-TRANSFORM

References:

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wishing you all the best in your studies!