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Chapter 4: Solving Nonlinear Equation

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Introduction

Problem

Given the non-linear function $f(x)$, we need to find x satisfying

$$f(x) = 0.$$

The solution x is the solution of the equation and is also called the (zero point) solution of the function $f(x)$. The problem of finding x is called the root finding problem.

Solve nonlinear equations

Examples of problems finding solutions of nonlinear equations

1. $1 + 4x - 16x^2 + 3x^3 - 3x^4 = 0$
2. $\frac{x\sqrt{(2.1-0.5x)}}{(1-x)\sqrt{(1.1-0.5x)}} - 369 = 0$ with $(0 < x < 1)$
3. $tg(x) - tanh(x) = 0$ where $tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Problem

If the equations $f(x)$ are nonlinear then

- it usually does not have an explicit formula solution
- numerical methods that allow us to find solutions based on *iterative procedure*

Solution interval

For function $f : \mathbb{R} \mapsto \mathbb{R}$ the interval $[a, b]$ is called **solution interval** if function f has opposite signs at both ends a, b, i.e. $f(a)f(b) < 0$.

Existence of solutions

If f is a continuous function on the interval $[a, b]$ and $f(a)(f(b) < 0$ then there exists $x^* \in [a, b]$ such that $f(x^*) = 0$.

Examples of solutions to nonlinear equations

1. $e^x + 1 = 0$ has no solution
2. $e^{-x} - x = 0$ has a solution
3. $x^2 - 4sin(x) = 0$ has two solutions
4. $x^3 - 6x^2 + 11x - 6 = 0$ has three solutions
5. $cos(x) = 0$ has infinitely many solutions

Conditions for solving equations

- The absolute value of the Condition number x^* of the function $f : \mathbb{R} \mapsto \mathbb{R}$ is $\frac{1}{|f'(x^*)|}$.
- A solution of x^* is said to be *ill-conditioned* (well) if the tangent line to the graph of the function f(x) atx^* is almost horizontal (vertical).

Iterative procedure

Nonlinear equations often do not have an explicit solution. Therefore, to find solution we often have to use numerical methods based on iterative procedures.

- **Stop condition**: $|f(x_k)| < \epsilon$ or $|x^* - x_k| < \epsilon$ where ϵ is the given *precision* and x_k is the approximate solution obtained at step k
- **Convergence rate**: We denote *error* at iteration k as: $e_k = x_k - x^*$. The sequence $\{e_k\}$ is said to converge to the degree r if

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^r} = C$$

where C is a non-zero constant

Solve nonlinear equations

Iterative procedure

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Name of the speed of convergence in some cases

- $r = 1$ - linear convergence rate
- $r > 1$ - linear convergence rate
- $r = 2$ - convergence rate squared

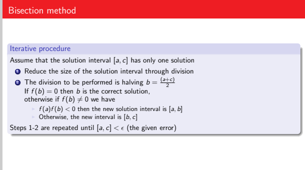
Question

Iterative procedure

Assume that the solution interval $[a, c]$ has only one solution

1. Reduce the size of the solution interval through division
2. The division to be performed is halving $b = \frac{(a+c)}{2}$
   If $f(b) = 0$ then b is the correct solution,
   otherwise if $f(b) \neq 0$ we have
   - $f(a)f(b) < 0$ then the new solution interval is $[a, b]$
   - Otherwise, the new interval is $[b, c]$

Steps 1-2 are repeated until $[a, c] < \epsilon$ (the given error)



If precision ϵ is given, the number of iterations is an integer n satisfying

$$n \geq \log_2 \frac{ca}{\epsilon}$$

because of

$$\frac{ca}{2^n} < \epsilon$$

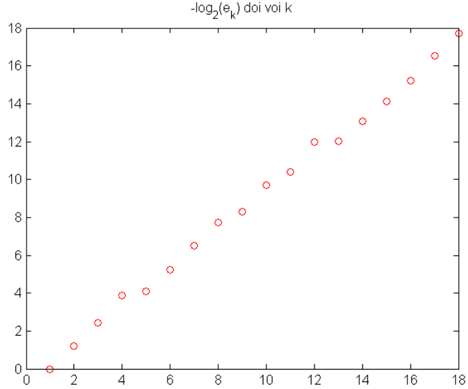
Example 1:

Find the solution of the equation $e^x - 2 = 0$, given a solution interval [0.2] with precision $\epsilon = 0.01$

Loops	a	b	c	f(a)	f(b)	f(c)	error
1	0.0000	1.0000	20000	-1.0000	0.7183	5.0389	20000
2	0.0000	0.5000	1,00000	-1.0000	-0.3513	0.7183	1,0000
3	0.5000	0.7500	1,0000	-0.3513	0.1170	0.7183	0.5000
4	0.5000	0.6250	0.7500	-0.3513	-0.1318	0.1170	0.2500
5	0.6250	0.6875	0.7500	-0.1318	-0.0113	0.1170	0.1250
6	0.6875	0.7188	0.7500	-0.0113	0.0519	0.1170	0.0625
7	0.6875	0.7031	0.7188	-0.0113	0.0201	0.0519	0.0313
8	0.6875	0.6953	0.7031	-0.0113	0.0043	0.0201	0.0156
9	0.6875	0.6914	0.6953	-0.0113	-0.00349	0.0043	0.0078

Example 2

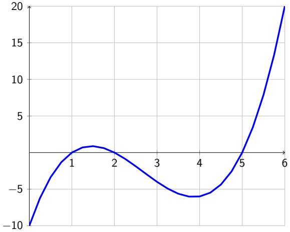
Consider the solution of the equation $f(x) = 1/(xe^{-x})$, given a solution interval $[0, 1]$ with precision $\epsilon = 0.00001$



Error: $e_k = \max\{x^* - a_k, c_k - x^*\}$
Horizontal axis: number of iterations k
Vertical axis: $-\log_2(e_k)$
Apparently $e_k \approx 2^{-k}$

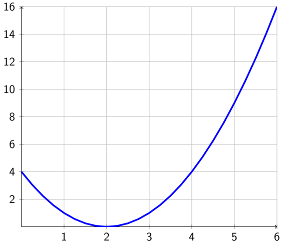
Comment on the Bisection method

- Strengths: Works even with non-analytic functions.
- Weaknesses:
  - Need to determine a solution interval and find only one solution.
  - Could not find a double solution.
  - When the function f has singularities, the bisection method can treat them as solutions.



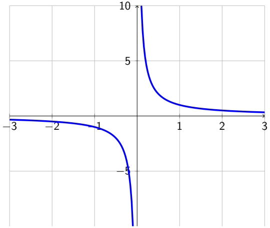
a)

Many solutions



b)

Double root



c)

Singularity

Bisection method

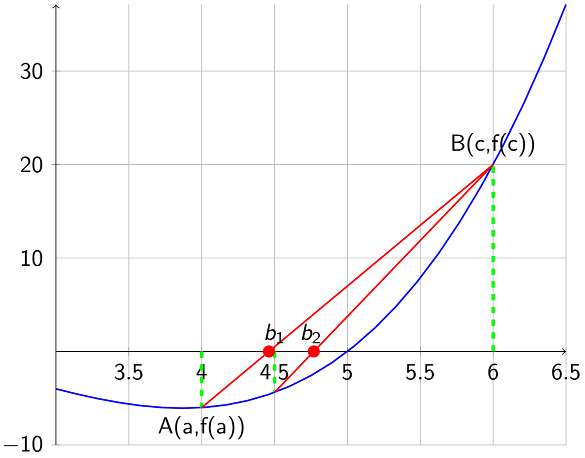
Iterative procedure

Assume that the solution interval $[a, c]$ has only one solution

1. Reduce the size of the solution interval through division
2. The division to be performed is $b = a - \frac{ca}{f(c)-f(a)}f(a) = \frac{af(c)-cf(a)}{f(c)-f(a)}$ If $f(b) = 0$ then b is the solution, Otherwise, if $f(b) \neq 0$, we have:
   - If $f(a)f(b) < 0$ then the new interval is $[a, b]$
   - Otherwise, the new interval is $[b, c]$

Steps 1-2 are repeated until $[a, c] < \epsilon$ (given error).
So b is the intersection of the horizontal axis with the line segment connecting A(a,f(a)) to B(c, f(c))

Chord method



Comment

- Advantages: like bisection, we do not need the analytic form of the equation f
- Disadvantages:
  - Need to know the solution interval
  - Single-sided convergence is slow, especially when the solution interval is large
  - Can be improved by using together with the bisection method

Chord method

Description

The basic idea of the method is to replace the nonlinear equation $f(x) = 0$ with an approximate, linear equation for x based on Taylor expansion.
Assuming $f(x)$ is continuously differentiable to the order $n+1$ then there exists $\xi \in (a, b)$

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \frac{(b-a)^{(n+1)}}{(n+1)!}f^{(n+1)}(\xi)$$

Description (continued)

Taylor expansion of $f(x)$ at the neighborhood of the original approximate solution x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + O(h^2)$$

where $h = x - x_0$.
Solve the approximate equation for x:

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

Obtained: $x = x_0 - \frac{f(x_0)}{f'(x_0)}$
x is an incorrect solution, but this solution will be closer to the correct solution than the initial value x_0.

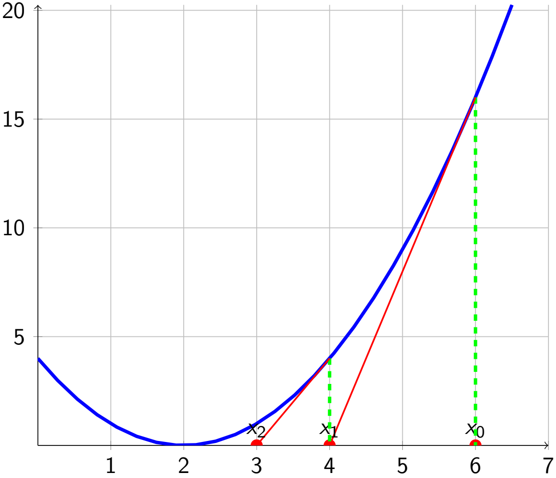
Newton's method

Iterative procedure

1. Initialize with x_0
2. Calculates with $k > 0$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

3. Repeat step 2 until $|f(x_k)| < \epsilon$ where ϵ is the given precision


20
15
10
5
1
2
3
4
5
6
7
x_2
x_1
x_0

Comment

- Advantages:
  - For a smooth enough function and we start from the point near the solution, the convergence rate of the method is squared or $r = 2$
  - No need to know the solution dissociation, just the initial point x_0
- Disdvantages:
  - Need to calculate the first derivative $f'(x_k)$, we can also approximate it with the formula $f'(x_k) = \frac{f(x_k+h)-f(x_k-h)}{2h}$ where h is a very small value eg $h = 0.001$
  - Not always converges

Example 1

Use Newton's method to find the solution of the equation

$$f(x) = x^2 - 4\sin(x) = 0$$

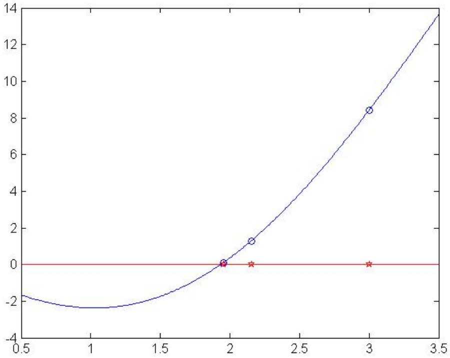
First derivative of $f(x)$

$$f'(x) = 2x - 4\cos(x)$$

The iterative formula of Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 4\sin(x_k)}{2x_k - 4\cos(x_k)}$$

Approximate starting point $x_0 = 3$



k	x_k	$f(x_k)$
0	3.000000	8.435520
1	2.153058	1.294773
2	1.954039	0.108439

Example 2

Solve the equation $f(x) = x^2 - 2 = 0$ because $f'(x) = 2x$ so the iterative formula will be $x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k}$ error $e_k = x_k - x^* = x_k - \sqrt{2}$

k	x_k	e_k
0	4,000000000	2.5857864376
1	2.250000000	0.8357864376
2	1.569444444	0.1552308821
3	1.421890364	0.0076768014
4	1.414234286	0.0000207236
5	1.414213563	0.0000000002

Example 3

Solve the equation

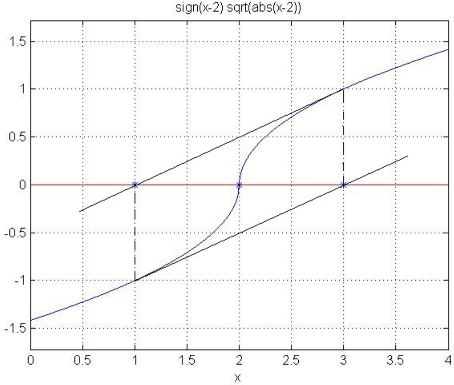
$$f(x) = \operatorname{sign}(xa)\sqrt{|xa|}$$

This equation satisfies:

$$xa - \frac{f(x)}{f'(x)} = -(xa)$$

The zero point of the function is $x^* = a$.
If we draw a tangent to the graph at any point, it always intersects the horizontal axis at the point of symmetry with the line $x = a$.
Newton's method infinitely iterative, neither convergent nor divergent.

Newton's method



Newton's method

Iterative procedure

An improvement of Newton's method, instead of using the derivative $f'(x)$, we use an approximate difference based on two successive iterations.

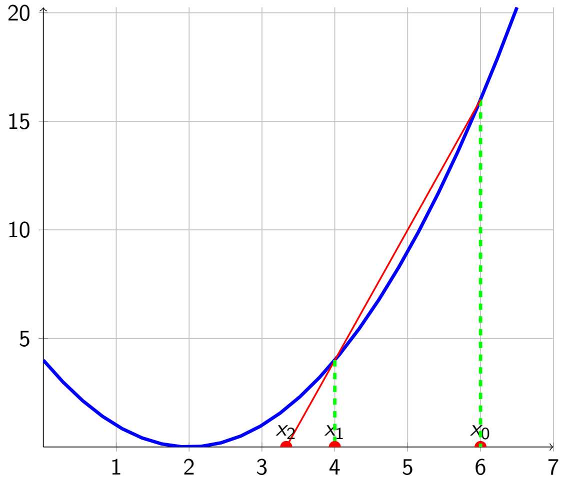
1. Starts with two starting points x_0 and x_1
2. With $k \geq 2$, we iterate by the formula

$$s_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{s_k}$$

3. Repeat step 2 until $|f(x_k)| < \epsilon$ (given error).

Secant method



Comment

- Advantages:
  - No need to know the solution interval, just two initial points x_0 and x_1
  - No need to calculate first derivative $f'(x_k)$
- Disdvantages:
  - Two initialization points are required
  - Convergence rate of method on linear $1 < r < 2$, specifically golden ratio $r \approx \frac{1+\sqrt{5}}{2} = 1.618$

Example 1

Solve the equation

$$f(x) = \text{sign}(x-2)\sqrt{|x-2|} = 0$$

with two starting points $x_0 = 4$, $x_1 = 3$ and $\epsilon = 0.001$

$$\begin{aligned} s_k &= \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \\ &= \frac{\text{sign}(x_k-2)\sqrt{|x_k-2|} - \text{sign}(x_{k-1}-2)\sqrt{|x_{k-1}-2|}}{x_k - x_{k-1}} \end{aligned}$$

$$x_{k+1} = x_k - \frac{f(x_n)}{s_n} = x_k - \frac{\text{sign}(x_k-2)\sqrt{|x_k-2|}}{s_k}$$

Secant method

k	x_k	e_k
0	4,000000000	2,000,000000000
1	3,000000000	1,000,000,000,000
2	0.585786438	1.4142135624
3	1.897220119	0.1027798813
$\vdots$	$\vdots$	$\vdots$
26	1.999989913	0.0000100868
27	1.999998528	0.0000014716
28	2,000003853	0.0000038528
29	2.000000562	0.0000005621

Example 2

Solve the equation

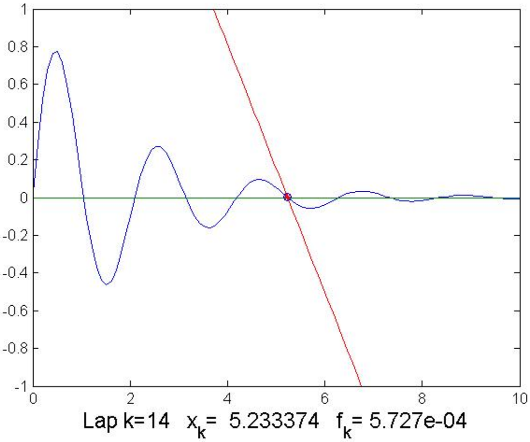
$$f(x) = e^{-x/2}\sin(3x) = 0$$

with two starting points x_0, x_1 and precision ϵ entered from the keyboard

$$s_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$= \frac{e^{-x_k/2}\sin(3x_k) - e^{-x_{k-1}/2}\sin(3x_{k-1})}{x_k - x_{k-first}}$$

$$x_{k+1} = x_k - \frac{f(x_n)}{s_n} = x_k - \frac{e^{-x_k/2}\sin(3x_k)}{s_k}$$



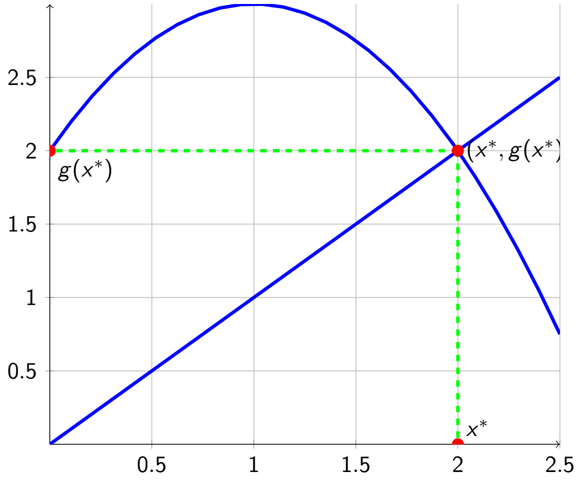
Two starting points $x_0 = 4$
$x_1 = 5$
Accuracy
$\epsilon = 0.001$

Fixed point

Instead of writing the equation as $f(x) = 0$, we rewrite it as a problem

$$\text{Find x satisfying } x = g(x)$$

The point x^* is called *fixed point* of the function $g(x)$ if $x^* = g(x^*)$, i.e. the point x^* is not transformed by g . mapping


2.5
2
1.5
1
0.5
0.5
1
1.5
2
2.5
$g(x^*)$
$(x^*, g(x^*))$
x^*

Examples

- Newton's method, according to the formula $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, the function g that we need to find the fixed point x^* would be $g(x) = x - f(x)/f'(x)$
- Find the solution $f(x) = x - e^{-x} \Rightarrow g(x) = e^{-x}$
- Find the solution $f(x) = x^2 - x - 2 \Rightarrow g(x) = \sqrt{x+2}$ or $g(x) = x^2 - 2$
- Find the solution $f(x) = 2x^2 - x - 1 \Rightarrow g(x) = 2x^2 - 1$

Iterative procedure

Approach to problem solving

$$x_{k+1} = g(x_k) \text{ with } k = 1, 2, \cdots$$

The above iterative procedure is often called an iterative **find fixed point** with a given starting point x_1

Iterative method

Comment

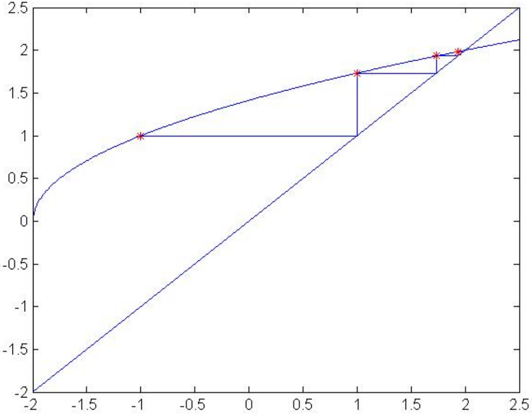
- Advantages:
  - No need to know the solution interval
- Disdvantages:
  - does not always converge

Example 1

Finding the solution of the equation $f(x) = x^2 - x - 2 = 0$ can lead to finding a fixed point

$$g(x) = \sqrt{x+2}$$

Approximate starting point $x_1 = -1$



Starting point

$x_1 = -1$

Number of iterations

$n = 3$

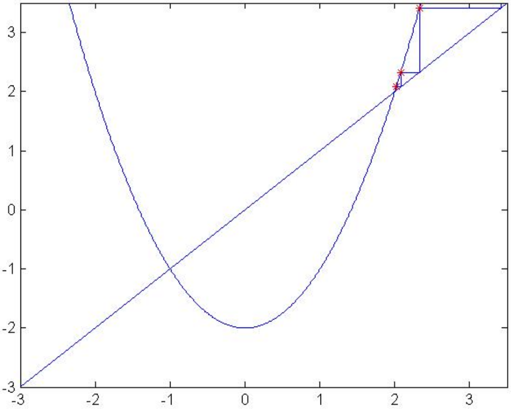
Example 2

Find the solution of the equation $f(x) = x^2 - x - 2$ by finding the point of disagreement of the function

$$g(x) = x^2 - 2$$

The starting point $x_1 = 2.02$ is very close to the solution

Iterative method



Starting point

$x_1 = 2.02$

Number of iterations

$n = 50$

Convergence theorem of iterative methods

Theorem 1: Assume the function $g(x)$ is continuous and the sequence repeats

$$x_{k+1} = g(x_k), k = 1, 2, \cdots$$

then if $x_k \to x^*$ when $k \to \infty$ then x^* is the fixed point of g.

Theorem 2: Suppose $g \in C^1$ and $|g'(x)| < 1$ in some interval containing the fixed point x^*. If x_0 is also in this range, the iterative sequence $\{x_k\}$ converges to x^*.

Theorem 3: If g is a **co function** then it has only one fixed point and the iterative sequence $\{x_k\}$ converges to x^* for all points starting x_0.

Note: The function g is called a co function if there is a constant $L < 1$ such that for every x, y we have: $|g(x) - g(y)| < L(xy))$.

Iterative method

Description

This is the method used to find the solution of a polynomial,:

$$y = a_0 + a_1 x + \cdots + a_N x_N$$

It can be rewritten as a quadratic factor plus a remainder

$$y = (x^2 + px + q)G(x) + R(x)$$

Where:

- p and q are arbitrary values.
- $G(x)$ is a polynomial of degree $N - 2$
- $R(x)$ is the remainder, usually a first degree polynomial.

Description (continued)

So the polynomial $G(x)$ and the remainder $R(x)$ have the form

$$G(x) = b_2 + b_3 x + b_4 x^2 + \cdots + b_N x^{N-2}$$
$$R(x) = b_0 + b_1 x$$

The value of b_0 and b_1 depends on the choice of p and q, the goal is to find $p = p^*$ and $q = q^*$ such that

- $b_0(p^*, q^*) = b_1(p^*, q^*) = 0 \Rightarrow R(x) = 0$
- $(x^2 + p^* x + q^*) \Rightarrow$ square factor of y

Iterative procedure

1. Initializes p and q and calculates b_0 and b_1 (see formular in the Ref book)
2. Calculates the values $(b_0)_p$,$(b_1)_p$ and $(b_0)_q$,$(b_1)_q$ (see formular in the Ref book)
3. Find Δp and Δq when solving equation (9)
4. Obtained p^* and q^* by the formula $p^* = p + \Delta p$ and $q^* = q + \Delta q$

Comment

- Advantages:
  - method that converges to the quadratic factor $(x^2 + px + q)$ regardless of the initialization value p, q
  - the coefficients of the polynomial $G(x)$ are also automatically obtained
- Disdvantages:
  - the accuracy of the solution is not high
  - to improve, you can use Newton's method to recalculate each solution