LESSON 15

SPECTRUM ANALYSIS OF DISCRETE SIGNALS

PhD. Nguyen Hong Quang
Assoc. Prof. Trinh Van Loan
PhD. Doan Phong Tung
Computer Engineering Department

❑ CONTENT

1. Analyze the spectrum of a discrete non-periodic signal.
2. Discrete Fourier Transform DFT.
3. Fast Fourier transform FFT.

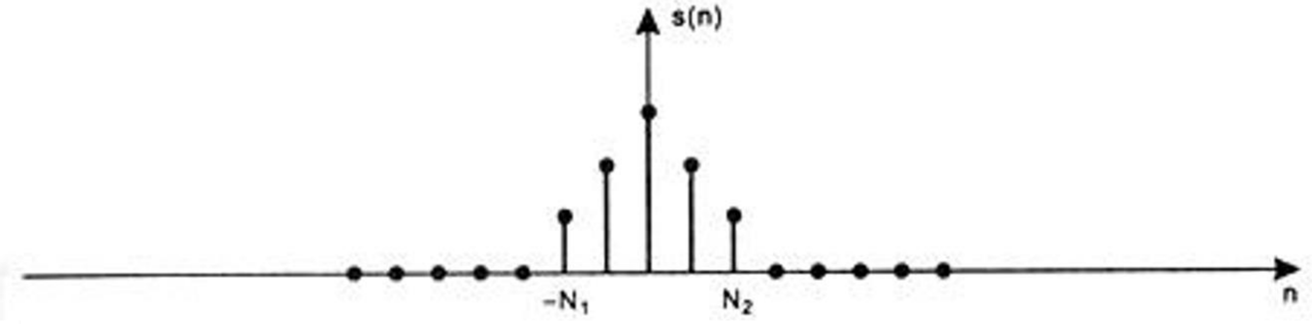
❑ Lesson Objectives

After completing this lesson, you will be able to understand the following topics:

- Spectral analysis of discrete non-periodic signals.
- Discrete Fourier transform method.
- Fast Fourier transform method.

1. Spectral analysis of discrete non-periodic signals

- The Fourier transform of a discrete non-periodic signal:



DTFT: $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

IDTFT: $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{-j\omega n}d\omega$$

$$X(e^{j\omega}) = R(\omega).\,e^{j.\varphi(\omega)}$$

$|X(\omega)|$: magnitude spectrum

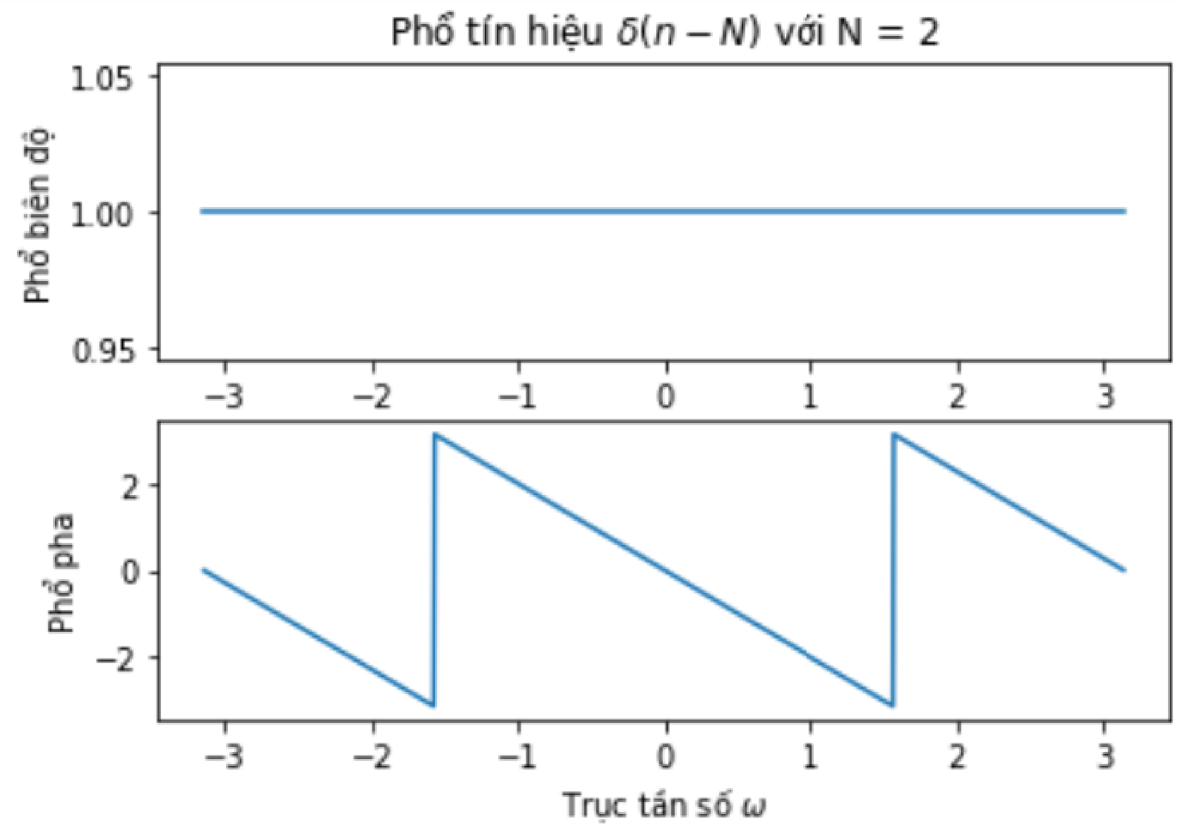
$$R(\omega) = |X(e^{j\omega})| \geq 0$$

$\Theta(\omega) = \measuredangle X(\omega)$: phase spectrum

$$-\pi \leq \varphi(\omega) = \arg[X(e^{j\omega})] \leq \pi$$

Example: $x(n) = \delta(n-2)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad \Longrightarrow \quad X(\omega) = e^{-j2\omega}$$

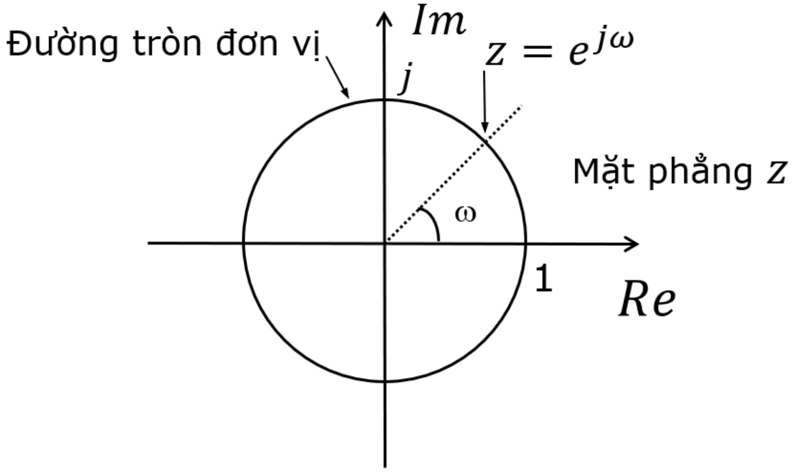


Relationship between the Fourier transform and the Z transform?

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$\text{ROC: } r_2 < |z| < r_1$$

$$X(z)\Big|_{z=e^{j\omega}} \equiv X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$



- The z transform becomes the Fourier transform when the amplitude of the z variable is 1, i.e. on a circle with radius 1 in the z-plane.
- This circle is called the unit circle.

Energy density spectrum

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$$

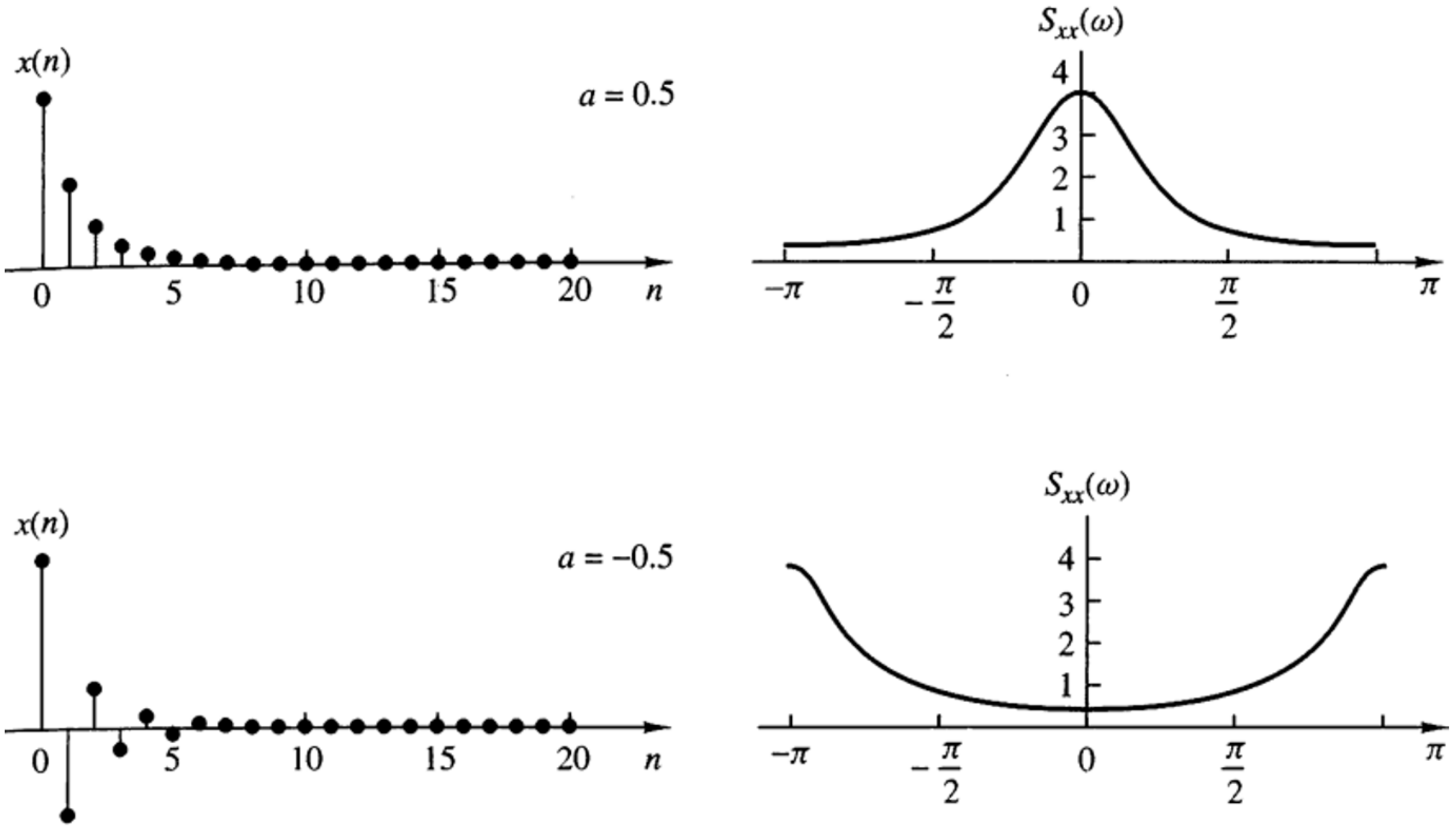
- Energy density spectrum: $S_{xx}(\omega) = |X(\omega)|^2$
- When x(n) is real : $S_{xx}(-\omega) = S_{xx}(\omega)$
- Example: Determine and plot the energy density spectrum of a signal:

$x(n) = a^n.u(n)$, $-1<a<1$, với $a = 0.5$ và $a = -0.5$

$$\Rightarrow \quad X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

$$\Rightarrow \quad S_{xx}(\omega) = \frac{1}{1 - 2a\cos(\omega) + a^2}$$

Example



Signal $0.5^n u(n), (-0,5)^n u(n)$ and energy density spectrum

Some basic properties of the Fourier transform

- Linearity: $ax_1(n) + bx_2(n) \xrightarrow{\mathcal{F}} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
- Periodicity: $X(e^{j\omega})$ periodic period 2π, $X(f)$ periodic period is 1
- Delay:

$$x(n) \xrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$x(n - n_0) \xrightarrow{\mathcal{F}} ?$$

$$\mathcal{F}\{x(n - n_0)\} = \sum_{n=-\infty}^{\infty} x(n - n_0)e^{-j\omega n}$$

$$= e^{-j\omega n_0}X(e^{j\omega})$$

- Comment: The delay signal has a constant amplitude spectrum, but the phase spectrum is shifted by an amount ωn_0
- Convolution: $y(n) = x(n) * h(n) \xrightarrow{\mathcal{F}} Y(e^{j\omega}) = X(e^{j\omega}).H(e^{j\omega})$

2. Discrete Fourier Transform DFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- DTFT: Frequency ω is continuous, $X(\omega)$ is periodic with a period of 2π.
- For x(n) of finite length： $n = 0, 1, 2, \dots, N-1$.
- Discrete N frequencies $\omega \to \omega_k = k2\pi/N$

$\Longrightarrow$ DFT (Discrete Fourier Transform): Fourier transform of a sequence of finite length with discrete frequency, called discrete Fourier transform for short

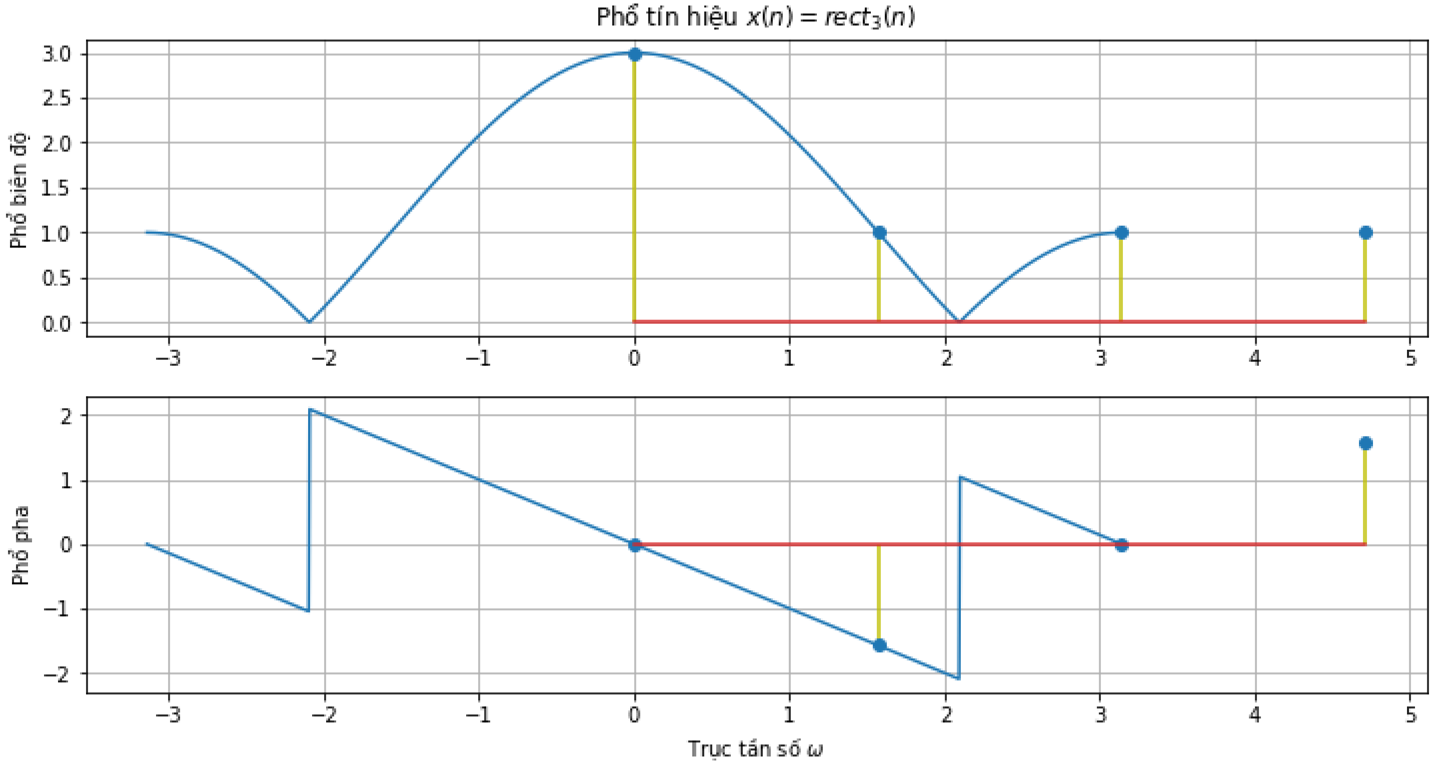
- Forward transform (analytical), reverse transform (synthetic)

$$X(k) = \begin{cases} \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} & 0 \le k \le N-1 \\ 0 & \text{k còn lại} \end{cases} \qquad x(n) = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{-j\frac{2\pi}{N}nk} & 0 \le n \le N-1 \\ 0 & \text{n còn lại} \end{cases}$$

Example

$$X(k) = \begin{cases} \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} & 0 \le k \le N-1 \\ 0 & \text{k còn lại} \end{cases}$$

- DFT analysis with $N = 4$ of $x(n) = rect_3(n)$.
- $X(k) = [3, -j, 1, j]$
- Relationship with DTFT:



3. Fast Fourier Transform (FFT)

$$X(k) = \begin{cases} \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} & 0 \leq k \leq N-1 \\ 0 & k \text{ còn lại} \end{cases}$$

(FFT: Fast Fourier Transform)

- Directly calculating the DFT requires N^2 complex number multiplications and N(N−1) complex number addition.
- FFT algorithm: decompose the DFT of a sequence of N numbers into DFT of smaller sequences
- Conditions to apply the algorithm: $N = 2^M$
- The number of operations is reduced to $N\log_2 N$

Time division FFT

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-J\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}x(n)W_N^{kn}\ \text{ với }\ W_N=e^{-j\frac{2\pi}{N}}\ \text{ và }\ 0\le k\le N-1$$

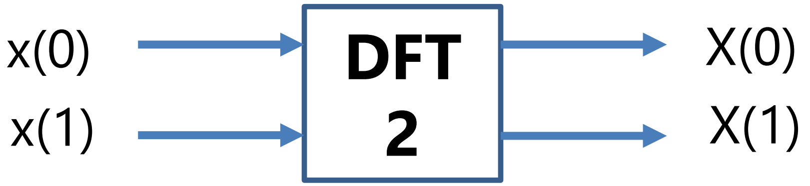
$$X(k)=\sum_{n=0}^{N-1}x(n)W_N^{kn}=\sum_{n=0,2,4\dots}^{N-1}x(n)W_N^{kn}+\sum_{n=1,3,5\dots}^{N-1}x(n)W_N^{kn}$$

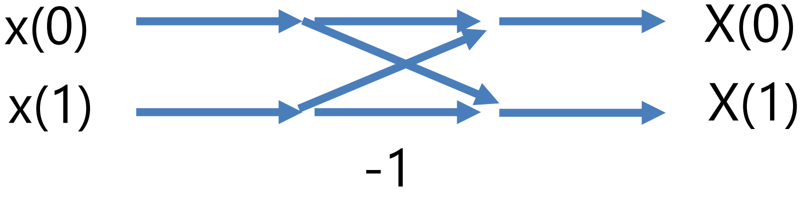
- Replace $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

$$X(k)=\sum_{r=0}^{\left(\frac{N}{2}\right)-1}x(2r)W_N^{2kr}+\sum_{r=0}^{\left(\frac{N}{2}\right)-1}x(2r+1)W_N^{k(2r+1)}$$

$$\Rightarrow\quad X(k)=X_0(k)+W_N^k.X_1(k)$$

- Example with N = 2





$$X(0)=X_0(0)+W_2^0.X_1(0)=x(0)+x(1)$$

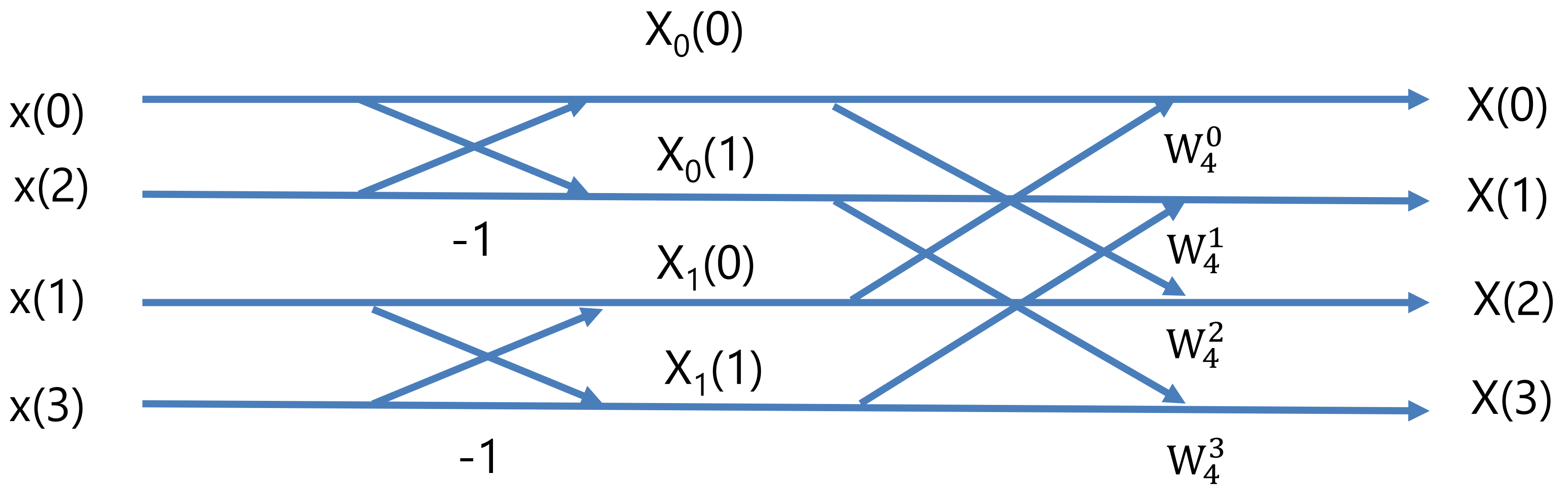
$$X(1)=X_0(1)+W_2^1.X_1(1)=x(0)-x(1)$$

Example with N = 4:

$$X(k) = X_0(k) + W_N^k.X_1(k)$$

Đảo bit

0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3



$$X_0(0) = x(0) + x(2)$$

$$X(0) = X_0(0) + W_4^0.X_1(0) = X_0(0) + X_1(0)$$

$$X_0(1) = x(0) - x(2)$$

$$X(1) = X_0(1) + W_4^1.X_1(1) = X_0(0) - j.X_1(0)$$

$$X_1(0) = x(1) + x(3)$$

$$X(2) = X_0(2) + W_4^2.X_1(2) = X_0(0) - X_1(0)$$

$$X_1(1) = x(1) - x(3)$$

$$X(3) = X_0(3) + W_4^3.X_1(3) = X_0(0) + j.X_1(0)$$

4. Summary

- The discrete-time Fourier transform converts a non-periodic discrete signal of finite energy into the frequency domain with a continuous frequency.
- Discrete Fourier transform is used to represent the frequency domain with discrete frequencies.
- The fast Fourier algorithm allows to perform the discrete Fourier transform quickly.

Homework

- Signal $x(n) = rect_3(n)$

a. Calculate and plot the discrete spectrum of this signal using the FFT algorithm with N=4

b. Calculate and plot the discrete spectrum of this signal using the FFT algorithm with N=8

c. Calculate and plot the spectrum of x(n) by DTFT transformation, then compare the results of sentences a and b with the results of sentences c and make comments on the relationship between these spectra.

Next lesson. Lesson 16

DISCRETE SYSTEM IN FREQUENCY DOMAIN

References :

- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

SCHOOL OF INFORMATION AND COMMUNICATION TECHNOLOGY

Wish you all good study!