ĐẠI HỌC BÁCH KHOA HÀ NỘI
VIỆN CÔNG NGHỆ THÔNG TIN VÀ TRUYỀN THÔNG

Discrete Mathematics

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Content of Part 2

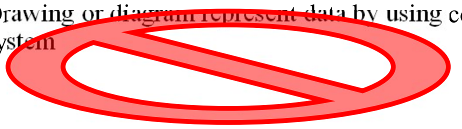
PART 1
COMBINATORIAL THEORY
(Lý thuyết tổ hợp)

PART 2
GRAPH THEORY
(Lý thuyết đồ thị)

Content

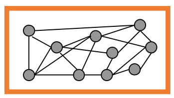
What is graph?

- In Maths:
  Drawing or diagram represent data by using coordinate system



Not what we want to mention

- In discrete mathematics:
  This is discrete structure with highly intuitive, very useful for expressing relationships.



Content

Applications of graph in practice

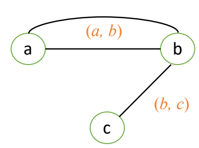
Has the potential to be applied in many fields:

- Internet
- Traffic network
- Electrical network
- Water supply network
- Scheduling
- Flow optimization, circuit design
- DNA gene analysis
- Computer games
- Object oriented design
- ….

Undirected Graphs

Definition. Undirected simple (multi) graph $G = (V,E)$ consists of 2 sets:

- Vertex set V is a finite set, each element is called vertex
- Edge set E is the set (family) of unordered pairs
  (u, v), where $u, v \in V, u \neq v$



Undirected Multigraph

Undirected (simple) graph

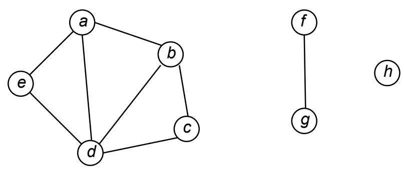
(a, b): multiple edges (parallel edges)

Undirected Simple Graph

- **Example:** Simple graph $G_1 = (V_1, E_1)$, where

  $V_1=\{a, b, c, d, e, f, g, h\}$,

  $E_1=\{(a,b), (b,c), (c,d), (a,d), (d,e), (a,e), (d,b), (f,g)\}$.



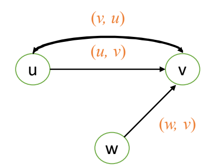
Graph G_1

Directed Graph

Definition. Directed simple (multi) graph $G = (V,E)$ consists of 2 sets:

- Vertex set V is finite element, each element is called vertex
- Edge set E is set (family) of ordered pairs

  (u, v), where $u, v \in V, u \neq v$



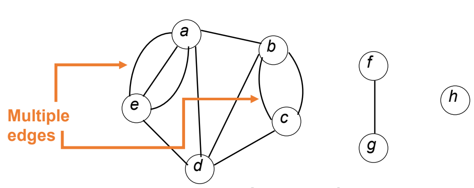
Directed multigraph

(Simple) Directed graph
~ Digraph

Undirected MultiGraph

- **Example:** Multigraph $G_2 = (V_2, E_2)$, where $V_2=\{a, b, c, d, e, f, g, h\}$,

  $E_2=\{(a,b), (b,c), (b,c), (c,d), (a,d), (d,e), (a,e), (a,e), (a, e), (d,b), (f,g)\}$.



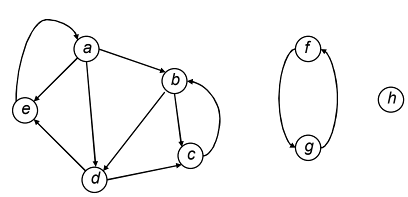
Graph G_2

Simple digraph

Example: Simple digraph $G_3= (V_3, E_3)$, where

$V_3=\{a, b, c, d, e, f, g, h\}$,

$E_3=\{(a,b), (b,c), (c,b), (d,c), (a,d), (b, d), (a,e), (d,e), (e,a), (f,g), (g,f)\}$



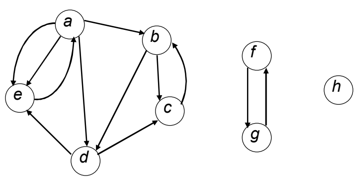
Graph G_3

Directed MultiGraph

Example: Directed multigraph $G_4 = (V_4, E_4)$, where

$V_4=\{a, b, c, d, e, f, g, h\}$,

$E_4=\{(a,b), (b,c), (c,b), (d,c), (a,d), (b, d), (a,e), (a,e), (d,e), (e,a), (f,g), (g,f)\}$



Graph G_4

Content

1. Graph in practice
2. Graph types
3. **Degree of vertex**
4. Subgraph
5. Isomorphism of Graphs
6. Path and cycle
7. Connectedness
8. Special graphs
9. Graph Coloring problem

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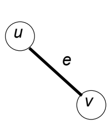
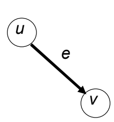
Graph Terminology

We have graph terminology related to relationship between vertices and edges:

- *Adjacency, connect, degree, start, end, indegree, outdegree,…*



Undirected edge $e=(u,v)$



Directed edge $e=(u,v)$

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Degree of a vertex in undirected graph

Assume G is undirected graph, $v \in V$ is a vertex.

- *Degree* of vertex v, $\deg(v)$, the number of edges incident on a vertex.
- Vertex with degree 0 is called isolated.
- Vertex with degree 1 is called pendant.
- Symbol often used:

$$\delta(G) = \min \{\deg(v): v \in V\},$$
$$\Delta(G) = \max \{\deg(v): v \in V\}.$$

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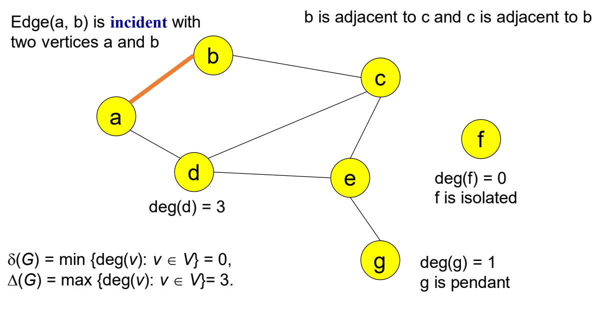
Example



Content

Degree of a vertex in directed graph

Given G directed graph, v is a vertex of G:

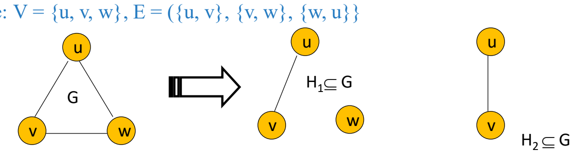
- *In-degree* of v, $deg^-(v)$, number of edges for which v is terminal vertex (goes into v).
- *Out-degree* of v, $deg^+(v)$, number of edges for which v is initial vertex (goes out of v).
- *Degree* of v, $deg(v)=deg^-(v)+deg^+(v)$

4. Subgraph

A subgraph of a graph G = (V, E) is a graph H =(V', E') where V' is a subset of V and E' is a subset of E.

Denoted by: H⊆G

Example: V = {u, v, w}, E = ({u, v}, {v, w}, {w, u}}



Spanning Subgraph

Definition.

Subgraph $H \subseteq G$ is called spanning subgraph of G if vertex set of H is vertex set of G: $V(H) = V(G)$.

Definition.

We write $H = G + \{(u,v), (u,w)\}$ to mean

$E(H) = E(G) \cup \{(u,v), (u,w)\}$, where $(u,v), (u,w) \notin E(G)$.

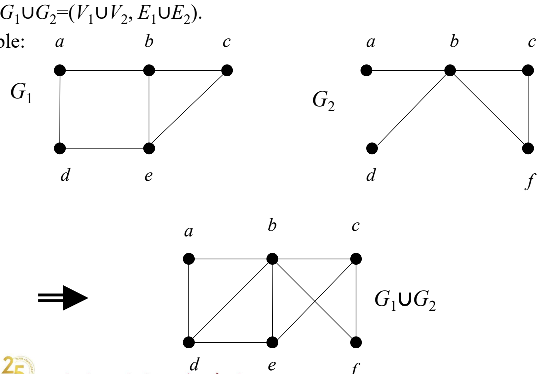
Content

1. Graph in practice
2. Graph types
3. Degree of vertex
4. Subgraph
5. **Isomorphism of Graphs**
6. Path and cycle
7. Connectedness
8. Special graphs
9. Graph Coloring problem

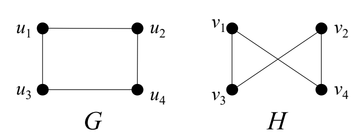
The union of graphs

The union of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph $G_1 \cup G_2=(V_1 \cup V_2, E_1 \cup E_2)$.

Example:



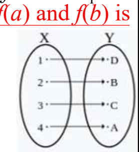
Isomorphism of Graphs



G is isomorphic to H

Definition. The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that a and b is adjacent in G_1 iff $f(a)$ and $f(b)$ is adjacent in G_2, $\forall a,b \in V_1$

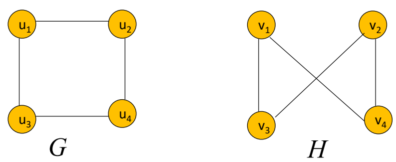
f is called an isomorphism.



Application Example:

In chemistry, to find if two compounds have the same structure

Example. Show that G and H are isomorphic.



Solution.

The function f with $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between $V(G)$ and $V(H)$.

Isomorphism graphs there will be:
(1) The same number of vertices
(2) The same number of edges
(3) The same number of degree

Path

Definition for Directed Graphs

A **Path** of length $n > 0$ from u to v in G is a sequence of n edges $e_1, e_2, e_3, \ldots, e_n$ of G such that $f(e_1) = (x_0, x_1), f(e_2) = (x_1, x_2), \ldots, f(e_n) = (x_{n-1}, x_n)$, where $x_0 = u$ and $x_n = v$.
A path is said to pass through $x_0, x_1, \ldots, x_n$ or traverse $e_1, e_2, e_3, \ldots, e_n$

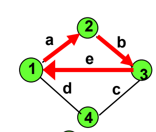
- A path is called *elementary* if all the edges are distinct.
- A path is called *simple* if all the vertices are distinct.
- A path is *closed* if $v_0 = v_n$.
- A closed elementary path is called a *cycle*. A cycle is called simple if all the vertices are distinct (except $v_0 = v_n$).

Content

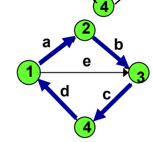
Cycle

- A closed elementary path is called a *cycle*.
  - A path is called *elementary* if all the edges are distinct.
  - A path is *closed* if $v_0 = v_n$.
- A cycle is called simple if all the vertices are distinct (except $v_0 = v_n$).

Simple cycle: (1, 2, 3, 1)



Simple cycle: (1, 2, 3, 4, 1)



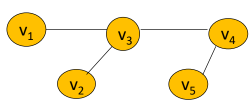
Content

Connectedness

Undirected Graph

An undirected graph is connected if there exists is a simple path between every pair of vertices

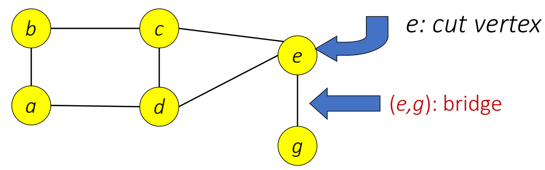
Example: G (V, E) is connected since for V = $\{v_1, v_2, v_3, v_4, v_5\}$, there exists a path between $\{v_i, v_j\}$, $1 \le i, j \le 5$



Connectedness

Articulation Point (Cut vertex): removal of a vertex produces a subgraph with more connected components than in the original graph. The removal of a cut vertex from a connected graph produces a graph that is not connected

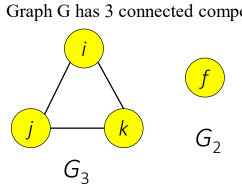
Bridge: An edge whose removal produces a subgraph with more connected components than in the original graph.

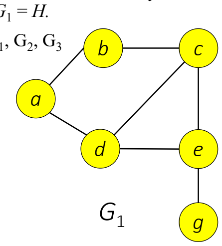


Connectedness

- If a graph is not connected then it splits up into a number of connected subgraphs, called its *connected components*.
- The connected components of G can be defined as its maximal connected subgraphs. This means that G_1 is a connected component of G if:
  - G_1 is a connected subgraph of G
  - G_1 is not itself a proper subgraph of any other *connected* subgraph of G. This second condition is what we mean by the term maximal; it says that if H is a connected subgraph such that $G_1 \subseteq H$, then $G_1 = H$.

Example: Graph G has 3 connected components: G_1, G_2, G_3



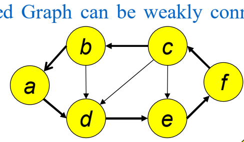


Connectedness

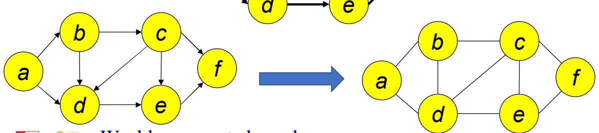
Directed Graph

- A directed graph is **strongly connected** if there is a path from u to v and from v to u whenever u and v are vertices in the graph
- A directed graph is **weakly connected** if its corresponding undirected graph is connected.

A strongly connected Graph can be weakly connected but the vice-versa is not true (why?)



Strongly connected graph



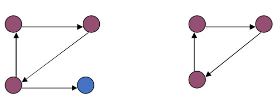
Weakly connected graph

Connectedness

Directed Graph

- **Strongly connected Components:** subgraphs of a Graph G that are strongly connected

Example: G1 is the strongly connected component in G



G G1

Content

Some special graphs

1. Null graph
2. Complete graphs K_n
3. Cycles C_n
4. Wheels W_n
5. n-Cubes Q_n
6. Bipartite graphs
7. Complete bipartite graphs $K_{m,n}$
8. r-regular graph
9. Planar graph
10. Euler graph and Hamilton graph

Some special graphs

1. Null graph
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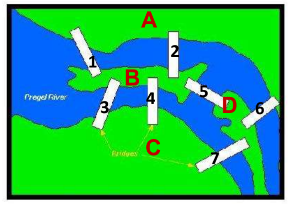
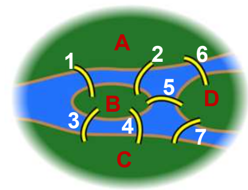
10. Euler graph and Hamilton graph

Seven Bridges of Königsberg

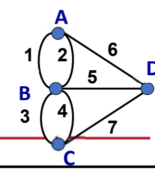
- To prove it, Euler reformulated the problem in graph terms:
  - Each land mass ~ a vertex
  - Each bridge ~ an edge





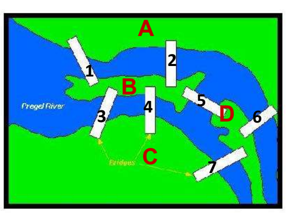
Is there a way to go through all 7 bridges, each exactly once, and then return to the starting position?

Whether or not there exists a cycle on a graph G that traverses through every edge of G exactly once.



Seven Bridges of Königsberg

- The town of Königsberg, Russia was set on both sides (A and C) of the Pregel river, and included two large islands (Kneiphof and Lomse, denoted as B and D respectively) which were connected to each other by 7 bridges. The residents of Königsberg wondered if it was possible to take a walking tour of the town that crossed each of the seven bridges exactly once. Is it possible to start at some node and take a walk that uses each edge exactly once, and ends at the starting node?
- In 1736, Euler proved that the problem has no solution.



Leonhard Euler

1707-1783

Euler graph

- Definition
- Recognize Euler graph

Euler graph

- ***Euler path*** (***Eulerian trail***, ***Euler walk***) in graph is a path that traverses through every edge exactly once.
- ***Euler cycle*** (***Eulerian circuit***, ***Euler tour***) in graph is a Euler path begins and ends at the same vertex (is a cycle that traverses through every edge exactly once).

- Graph consisting of Euler cycle is called as ***Euler graph***.
- Graph consisting of Euler path is called as ***Half Euler graph.***
- Apparently, all Euler graphs are also half-Euler graphs.

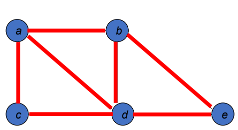
Euler's 1st theorem

- If a graph has any vertices of odd degree, then it can not have any Euler cycle.
- If a graph is connected and every vertex has an even degree, then it has at least one Euler cycle.

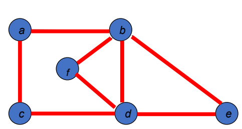
Proof:

- If a node has an odd degree, and the cycle starts at this node, then it must end elsewhere. This is because after we leave the node the first time the node has even degree, and every time we return to the node we must leave it. (On the paired arc.)
- If a node has an odd degree, and the cycle begins else where, then it must end at the node. This is a contradiction, since a cycle must end where it began.

Example



Half Euler graph



Euler graph

Euler path: a, c, d, b, e, d, a, b

Euler cycle: a, c, d, e, b, d, f, b, a

Euler path in graph is a path that traverses through every edge exactly once.
Euler cycle in graph is a cycle that traverses through every edge exactly once.

Half Euler graph: graph consists of Euler path
Euler graph: graph consists of Euler cycle

If a graph has all even degree nodes, then an Euler Circuit exists.

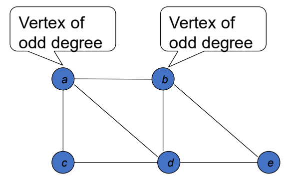
Algorithm:

- Step One: Randomly move from node to node, until stuck. Since all nodes had even degree, the circuit must have stopped at its starting point. (It is a circuit.)
- Step Two: If any of the arcs have not been included in our circuit, find an arc that touches our partial circuit, and add in a new circuit.
  - Each time we add a new circuit, we have included more nodes.
  - Since there are only a finite number of nodes, eventually the whole graph is included.

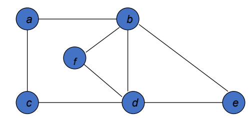
Euler's 2nd theorem

- If a graph has more than two vertices of odd degree, then it cannot have an Euler path.
- If a graph is connected and has exactly two vertices of odd degree, then is has at least one Euler path. Any such path must start at one of the odd degree vertices and must end at the other odd degree vertex.

Example



Half Euler graph



Euler graph

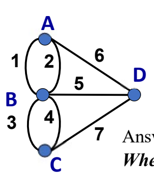
Euler's theorems

If a graph is connected and if the number of odd degree vertices

- = 0, then Euler cycle (Theorem 1)
- = 2, then Euler path (Theorem 2)
  - This Euler path must start at one of the odd degree vertices and must end at the other odd degree vertex.

Is there a way to go through all 7 bridges, each exactly once, and then return to the starting position?

Whether or not there exists a cycle on a graph G that traverses through every edge of G exactly once.



Answer*: There exists vertex of odd degree ➔ don't have Euler cycle*

Whether or not there exists Euler path in G*?*

Answer: *There are 3 vertices of degree 3, one vertex of degree 5 ➔ don't have Euler path*

Hamilton graph

- Definition
- Recognize Hamilton graph

Hamilton graph

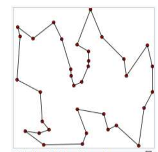
- ***Hamilton path*** in graph is a path that traverses every vertex exactly once.
- ***Hamilton cycle*** in graph is a cycle that traverses every vertex exactly once.
- Graph consisting of Hamilton cycle is called as ***Hamilton graph***.
- Graph consisting of Hamilton path is called as ***Half Hamilton graph.***
- Apparently, all Hamilton graphs are also half-Hamilton graphs.

Applications of Hamilton Cycles

- The famous **traveling salesman problem** or **TSP:**

The **travelling salesman problem** (**TSP**) asks the following question: "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?"

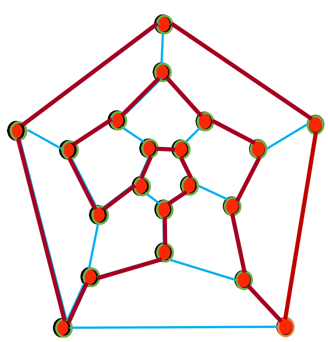
An equivalent formulation in terms of graph theory is: Find the Hamiltonian cycle with the least weight in a weighted graph.



Solution of a travelling salesman problem: the black line shows the shortest possible loop that connects every red dot

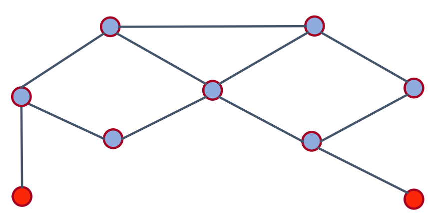
Example: Hamilton graph

- Is graph consisting of Hamilton cycle: traverse every vertex exactly once.



Hamilton graph

- Graph has 2 vertices of degree 1⇒ not Hamilton graph

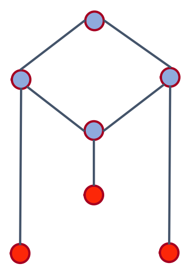


- The above graph is Half Hamilton graph

Half Hamilton graph

- Vertices of degree 1 must be either the starting or ending position of the Hamilton path.

Graph has 3 vertices of degree 1
⇒ Not Halft Hamilton graph



Content

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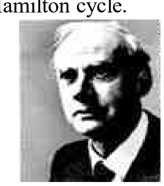
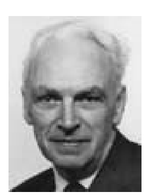
Theorem about existence of Hamilton path

- **Theorem Dirac**: If G is simple connected graph with $n \geq 3$ vertices, and $\forall v$ $\deg(v) \geq n/2$, then G has Hamilton cycle.

- **Theorem Ore:** If G is simple connected graph with $n \geq 3$ vertices, and $\deg(u) + \deg(v) \geq n$ for all vertices pair u, v not adjacent, then G has Hamilton cycle.



Paul Adrien Maurice Dirac
1902 - 1984
(USA)



Oystein Ore
1899 - 1968
(Norway)

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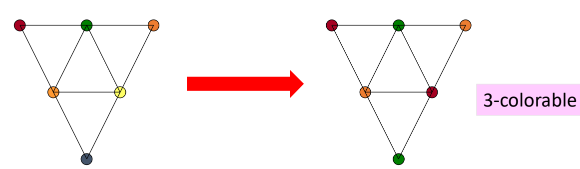
What is Coloring?

- Graph Coloring Problem is an assignment of colors to all vertices of a given graph such that two adjacent vertices get different colors.
- **Objective:** use minimum number of colors.



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