PHƯƠNG TRÌNH VI PHÂN VÀ LÍ THUYẾT CHUỖI

CHƯƠNG I. LÝ THUYẾT CHUỖI

§ 1. Chuỗi số

1(K64). $\sum_{n=1}^{\infty} \tan \frac{n^2+1}{n^2-n+1}$ (PK)

Giải

+) $\lim\limits_{n\to\infty} \tan\dfrac{n^2+1}{n^2-n+1} = \tan 1 \neq 0$

+) Do đó chuỗi phân kỳ (Vi phạm ĐK cần)

2) (K64) $\displaystyle\sum_{n=1}^{\infty} \frac{1-\cos\frac{1}{\sqrt{n}}}{\sqrt{n}}$ (HT)

+) $0 < a_n = \dfrac{1-\cos\dfrac{1}{\sqrt{n}}}{\sqrt{n}} = \dfrac{2\sin^2\dfrac{1}{2\sqrt{n}}}{\sqrt{n}} \sim \dfrac{1}{2n^{\frac{3}{2}}}, n \to \infty$

+) $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n^{\frac{3}{2}}}$ HT, nên chuỗi đã cho HT.

3 (BK61). 1) $\sum\limits_{n=2}^{\infty} \dfrac{1}{3^n} \dfrac{(2n+1)!}{n^2-1}$ (PK)

Giải

+) $a_n > 0$

+) $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{3^{n+1}}\frac{(2n+3)!}{(n+1)^2-1} : \frac{1}{3^n}\frac{(2n+1)!}{n^2-1} = \infty$$

$$\Rightarrow a_{n+1} > a_n > a_2 = \frac{40}{9} > 1 \Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow PK$$

4 (BK63).

$$\sum_{n=1}^{\infty}\left(\frac{n+2}{n}\right)^{n^2} \quad \text{(PK)}$$

Giải

+) $0 < a_n = \left(\dfrac{n+2}{n}\right)^{n^2} \Rightarrow \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^{n} = e^2 > 1$

+) Nên chuỗi phân kỳ.

5(K60) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

+) $f(x) = \dfrac{1}{x\ln x}$ dương, giảm với $x \geq 2$ và có $\lim\limits_{x\to+\infty} f(x) = 0$

+) $\displaystyle\int_2^\infty f(x)dx = \int_2^\infty \frac{d(\ln x)}{\ln x}$ PK

+) $\displaystyle\Rightarrow \sum_{n=2}^\infty \frac{1}{n\ln n}$ phân kỳ

Tổng quát có thể xét $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$ hội tụ chỉ khi $p > 1$.

6 (K64) $\displaystyle\sum_{n=1}^\infty \frac{\cos(n^2+1)}{\sqrt{n^3+1}}$ (HTTĐ)

+) $\left|\dfrac{\cos(n^2+1)}{\sqrt{n^3+1}}\right| \le \dfrac{1}{\sqrt{n^3+1}} \le \dfrac{1}{\sqrt{n^3}}, \forall n$

+) $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n^3}}$ hội tụ nên chuỗi ban đầu hội tụ.

7)(K64) $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\sqrt{n+1}\ln n}$ (HT)

Giải

+) Là chuỗi đan dấu, có $\left\{\frac{1}{\sqrt{n+1}\ln n}\right\}$ đơn điệu giảm,

$$\frac{1}{\sqrt{n+1}\ln n} > 0, \lim_{n\to\infty}\frac{1}{\sqrt{n+1}\ln n} = 0$$

+) Chuỗi đã cho hội tụ (ĐL. Leibnitz).

§ 2. Chuỗi hàm số

1)(K61) Tìm MHT $\sum_{n=1}^{\infty}\frac{e^{nx}}{n^2+n+1}$ $((-\infty;0])$

+) $\sum_{n=1}^{\infty} \frac{e^{nx_0}}{n^2+n+1}$ (2) là chuỗi số dương, có

$$\frac{a_{n+1}}{a_n} = \frac{e^{(n+1)x_0}}{(n+1)^2+n+2} \frac{n^2+n+1}{e^{nx_0}} = e^{x_0} \frac{n^2+n+1}{(n+1)^2+n+2}$$

$\sim e^{x_0}, n \to \infty.$

$e^{x_0} < 1 \Leftrightarrow x_0 < 0 \Rightarrow$ (2) hội tụ.

$e^{x_0} > 1 \Leftrightarrow x_0 > 0 \Rightarrow$ (2) phân kỳ.

$e^{x_0} = 1 \Leftrightarrow x_0 = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ là chuỗi số dương

hội tụ do : $\frac{1}{n^2+n+1} \sim \frac{1}{n^2}, n \to \infty; \sum_{n=1}^{\infty} \frac{1}{n^2}$ HT.

+) Kết luận : MHT $(-\infty;0]$.

2)(K58) $$\sum_{n=1}^{\infty}\left(\int_0^{\frac{1}{n}}\frac{\sqrt[3]{t}}{\sqrt[4]{1+\sin^2 t}}dt\right)\cos nx$$ (HTĐ)

Giải

+) $$\left|u_n(x)\right| = \left|\left(\int_0^{\frac{1}{n}}\frac{\sqrt[3]{t}}{\sqrt[4]{1+\sin^2 t}}dt\right)\cos(nx)\right|$$

$$= \left| \left(\int_0^{\frac{1}{n}} \frac{\sqrt[3]{t}}{\sqrt[4]{1+\sin^2 t}} dt \right) \right| \left| \cos(nx) \right| \le \int_0^{\frac{1}{n}} \sqrt[3]{t}\, dt = \frac{3}{4} t^{\frac{4}{3}} \Bigg|_0^{\frac{1}{n}} = \frac{3}{4} \frac{1}{n^{\frac{4}{3}}},$$

$\forall x \in \mathbb{R}$

+) $\sum_{n=1}^{\infty} \frac{3}{4} \frac{1}{n^{\frac{4}{3}}}$ HT, nên (1) HT đều

trên $\mathbb{R}$ (Weierstrass)

3 (K64) Tìm bán kính hội tụ

2) $\sum_{n=2}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ (e^2) $\sum_{n=2}^{\infty} \left(1-\frac{2}{n}\right)^{n^2} x^n$

Tìm miền hội tụ

3) $\sum_{n=2}^{\infty} \frac{n+1}{n(n-1)} \left(\frac{2x+1}{1-x}\right)^n$ ($-2 \le x < 0$)

$$\sum_{n=2}^{\infty} \frac{n+1}{n^2+5} (x-1)^{3n}$$

Giải p2 : Tìm bán kính hội tụ $\sum_{n=2}^{\infty} \left(1-\frac{2}{n}\right)^{n^2} x^n$

+) $R = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[n]{|a_n|}} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[n]{\left(1-\dfrac{2}{n}\right)^{n^2}}} = \lim\limits_{n\to\infty} \dfrac{1}{\left(1-\dfrac{2}{n}\right)^{n}}$

+) $= \dfrac{1}{e^{-2}} = e^2.$

Giải p3 : Tìm miền hội tụ $\sum\limits_{n=2}^{\infty} \dfrac{n+1}{n(n-1)}\left(\dfrac{2x+1}{1-x}\right)^n$

+) Đặt $X = \frac{2x+1}{1-x} \Rightarrow \sum_{n=2}^{\infty} \frac{n+1}{n(n-1)} X^n$ (2)

+) (2) có $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n(n-1)} \frac{(n+1)n}{n+2} = 1$

+) $X = 1 \Rightarrow \sum_{n=2}^{\infty} \frac{n+1}{n(n-1)}$ (3) phân kỳ, do

$0 < \frac{n+1}{n(n-1)} \sim \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ phân kỳ

+) $X = -1 \Rightarrow \sum_{n=2}^{\infty} (-1)^n \frac{n+1}{n(n-1)}$ (4) hội tụ theo ĐL Leibnitz , do

$$0 < \frac{n+1}{n(n-1)}, \quad \lim_{n\to\infty} \frac{n+1}{n(n-1)} = 0, \quad \left\{ \frac{n+1}{n(n-1)} \right\} \text{ giảm.}$$

+) MHT của (2)

$$-1 \le \frac{2x+1}{1-x} < 1 \Leftrightarrow \begin{cases} \left|\dfrac{2x+1}{1-x}\right| \le 1 \\ \dfrac{2x+1}{1-x} \ne 1 \end{cases} \Leftrightarrow \begin{cases} (2x+1)^2 \le (1-x)^2 \\ \dfrac{2x+1}{1-x} \ne 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 3x(x+2) \le 0 \\ \dfrac{2x+1}{1-x} \ne 1 \end{cases} \Leftrightarrow \begin{cases} -2 \le x \le 0 \\ \dfrac{2x+1}{1-x} \ne 1 \end{cases} \Leftrightarrow -2 \le x < 0.$$

4 (K63) Khai triển thành chuỗi Maclaurin

2) $f(x) = \dfrac{4}{x^2 - 6x + 5}$

$$\left(\sum_{n=0}^{\infty}\left(1 - \frac{1}{5^{n+1}}\right)x^n, |x| < 1\right)$$

Giải 2 $f(x) = \dfrac{4}{x^2 - 6x + 5}$

+) $f(x) = \dfrac{4}{(x-1)(x-5)} = \dfrac{1}{x-5} - \dfrac{1}{x-1}$, có

$$-\frac{1}{x-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1.$$

+)

$$\frac{1}{x-5} = -\frac{1}{5}\frac{1}{1-\dfrac{x}{5}} = -\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}, \qquad \left|\frac{x}{5}\right| < 1,$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{5^{n+1}}\right)x^n, \qquad |x| < 1.$$

5)(K64) Khai triển $y = \dfrac{x^2 - 1}{x + 2}$ thành chuỗi lũy thừa (x-1)

$$\left(-1 + (x-1) + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}(x-1)^n, |x-1| < 3\right)$$

Giải

+) $y=\dfrac{x^2-4+3}{x+2}=x-2+\dfrac{3}{3+x-1}$

$$=-1+(x-1)+\frac{1}{1-(-\frac{x-1}{3})}$$

+) $\dfrac{1}{1-(-\frac{x-1}{3})}=\displaystyle\sum_{n=0}^{\infty}\left(-\frac{x-1}{3}\right)^n,\quad \left|\frac{x-1}{3}\right|<1,$

$$\Rightarrow f(x)=-1+(x-1)+\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(x-1)^n, |x-1|<3.$$

6 (K63) 1) Khai triển hàm $y = x, -2 \le x < 2,$ tuần hoàn $T = 4$ thành chuỗi Fourier.

$$\left(\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}\right)$$

Giải

+) $2\ell = 4 \Rightarrow \ell = 2$; $f(x)$ lẻ $\Rightarrow a_n = 0,\ n = 0, 1, 2, \ldots$

+) $$b_n = \frac{2}{2}\int_0^2 x\sin\left(n\frac{\pi x}{2}\right)dx = \int_0^2 x\, d\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right)$$

$$= -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx = -\frac{4}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\left.\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right|_0^2$$

$$= \frac{4}{n\pi}\cos n\pi = (-1)^n\frac{4}{n\pi}$$

+) f(x) liên tục $\Rightarrow f(x) = \sum_{n=1}^{\infty}(-1)^n\frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$

CHƯƠNG II. PHƯƠNG TRINH VI PHÂN

1) (K58) Tìm $h(y)$ để phương trình sau là toàn phần và giải

$$2xh(y)\tan y + h(y)(x^2 - 2\sin y)dy = 0$$

$$\left(h = C\cos y,\ x^2\sin y + \frac{\cos 2y}{2} = C\right)$$

GIẢI

+) $Q'_x = 2x; P'_y = \dfrac{2x}{\cos^2 y} \Rightarrow \dfrac{Q'_x - P'_y}{P} = \dfrac{2x - \dfrac{2x}{\cos^2 y}}{2x\tan y}$

$= -\tan y \neq 0 \Rightarrow h(y) = e^{-\int \tan y dy} = e^{\ln(\cos y)} = \cos y \Rightarrow$

$(1) \Leftrightarrow 2x\cos y\tan y + (x^2 - 2\sin y)\cos y dy = 0, \cos y \neq 0$ (2)

có

$$\frac{\partial}{\partial x}\left((x^2 - 2\sin y)\cos y\right) = 2x\cos y = \frac{\partial}{\partial y}\left(2x\cos y\tan y\right)$$

Do đó (2) là PTVP toàn phần. Giải (2) ta được nghiệm của (1).

2) (K64) Tìm thừa số tích phân có dạng $h(xy)$ để phương trình

$$(x^2y + 2x + y^2)dx + (x^3 + xy + 1)dy = 0$$

trở thành phương trình vi phân toàn phần. Giải phương trình vi phân đó với h(xy) tìm được.

$(h(xy) = Ce^{xy}, C \neq 0; e^{xy}(x^2 + y) = C)$

GIẢI

+)

$$\frac{\partial}{\partial x}\Big(h(xy)(x^3+xy+1)\Big)=\frac{\partial}{\partial y}\Big(h(xy)(x^2y+2x+y^2)\Big)\Leftrightarrow$$

$$h'(xy)-h(xy)=0; t=xy\Rightarrow h'(t)-h(t)=0\Leftrightarrow$$

$$\frac{dh(t)}{dt}=h(t)\Rightarrow\frac{dh(t)}{h(t)}=dt\Rightarrow\ln|h(t)|=t+\ln|C|\Rightarrow$$

$$h(t)=Ce^t; C\neq 0\Rightarrow h(xy)=Ce^{xy}\Rightarrow$$

+)

$$(1)\Leftrightarrow e^{xy}(x^2y+2x+y^2)dx+e^{xy}(x^3+xy+1)dy=0 \qquad (2)$$

có

$$\frac{\partial}{\partial x}\Big(e^{xy}(x^3+xy+1)\Big)=e^{xy}[y(x^3+xy+1)+3x^2+y]=$$

$\frac{\partial}{\partial y}[e^{xy}(x^2y+2x+y^2)] \Rightarrow$ (2) là PTVP toàn phần. Giải (2) ta được nghiệm của (1).

3 (K64)

$$y''+9y=2\cos^2 x$$

$$(C_1\cos 3x + C_2\sin 3x + \frac{1}{5}\cos 2x + \frac{1}{9})$$

GIẢI

+) $y'' + 9y = 0 \qquad (2), k^2 + 9 = 0 \qquad (3)$

$(3) \Leftrightarrow k^2 = -9 \Leftrightarrow k_{1,2} = \pm 3i \Rightarrow \overline{Y} = C_1\cos(3x) + C_2\sin(3x$

là NTQ của (2).

+) $(1) \Leftrightarrow y'' + 9y = 1 + \cos(2x) \Rightarrow$ Dạng NR (1) :

$Y = A + B\cos(2x) + C\sin(2x) \Rightarrow$

$Y' = 2C\cos(2x) - 2B\sin(2x) \Rightarrow$

$Y'' = -4B\cos(2x) - 4C\sin(2x) \Rightarrow$

$\Rightarrow Y'' + 9Y = 5B\cos(2x) + 5C\sin(2x) + 9A = 1 + \cos(2x)$

$, \forall x \Rightarrow A = \dfrac{1}{9}; B = \dfrac{1}{5}; C = 0 \Rightarrow Y = \dfrac{1}{9} + \dfrac{\cos(2x)}{5} \Rightarrow$

$y = \overline{y} + Y$ là NTQ của (1).

4(K51) $y'' - 6y' + 9y = 3x - 8e^{3x}$

$$\left(y = (C_1 + C_2 x - 4x^2)e^{3x} + \frac{x}{3} + \frac{2}{9}\right)$$

GIẢI

+) $y'' - 6y' + 9y = 0 \quad (2), k^2 - 6k + 9 = 0 \quad (3)$

$(3) \Leftrightarrow (k-3)^2 = 0 \Leftrightarrow k_1 = k_2 = 3 \Rightarrow \overline{Y} = e^{3x}(C_1x + C_2)$

là NTQ của (2).

+) Dạng NR (1) : $Y = Ax + B + Cx^2e^{3x}$

$\Rightarrow Y' = A + Ce^{3x}(3x^2 + 2x) \Rightarrow Y'' = Ce^{3x}(9x^2 + 12x + 2)$

$\Rightarrow Y'' - 6Y' + 9Y = 9Ax + 9B - 6A + Cx^2e^{3x} = 3x - 8e^{3x}$,

$\forall x \Rightarrow A = \dfrac{1}{3}; B = \dfrac{2}{9}; C = -4 \Rightarrow Y = \dfrac{x}{3} + \dfrac{2}{9} - 4x^2e^{3x} \Rightarrow$

$y = \overline{y} + Y$ là NTQ của (1).

5(K64)

$$x^2y'' - 4xy' + 6y = 2x^2\ln x.$$

$$\left(C_1x^2 + C_2x^3 - x^2(\ln^2 x + 2\ln x)\right)$$

GIẢI

+) $x = e^t \Rightarrow xy' = \dfrac{dy}{dt}; x^2y'' = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} \Rightarrow$

$(1) \Leftrightarrow \dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 6y = 2te^{2t}\ (2) \quad \Rightarrow k^2 - 5k + 6 = 0.$

$\Leftrightarrow k_1 = 2; k_2 = 3 \Rightarrow \overline{y} = C_1e^{2t} + C_2e^{3t}$

+) NR (2) có dạng $Y = t(At + B)e^{2t} = (At^2 + Bt)e^{2t} \Rightarrow$

$Y' = e^{2t}[2At^2 + (2A + 2B)t + B] \Rightarrow$

$Y'' = e^{2t}[4At^2 + (8A + 4B)t + 2A + 4B] \Rightarrow$

$Y'' - 5Y' + 6Y = e^{2t}[-2At + 2A - B] = 2te^{2t}, \forall t \Rightarrow$

$A = -1; B = -2 \Rightarrow Y = (-t^2 - 2t)e^{2t} \Rightarrow$

$y = C_1e^{2t} + C_2e^{3t} + (-t^2 - 2t)e^{2t}.$

$$= C_1x^2 + C_2x^3 - x^2(\ln^2 x + 2\ln x).$$

CHƯƠNG III. PHƯƠNG PHÁP TOÁN TỬ LAPLACE

1) (K63) $F(s) = \dfrac{1}{s^3 + s}$

GIẢI

+) $F(s) = \dfrac{1}{s^3 + s} = \dfrac{1}{s} - \dfrac{s}{s^2 + 1}$

+) $L^{-1}\{F(s)\}(t) = L^{-1}\left\{\dfrac{1}{s}\right\}(t) - L^{-1}\left\{\dfrac{s}{s^2+1}\right\}(t)$

$= 1 - \cos t$

2) (K63) $F(s) = \dfrac{1}{s^4 - 1}$ $\left(f(t) = \dfrac{1}{2}(\sinh t - \sin t)\right)$

GIẢI

+) $F(s)=\dfrac{1}{(s^2-1)(s^2+1)}=\dfrac{1}{2}\left(\dfrac{1}{s^2-1}-\dfrac{1}{s^2+1}\right)$

+) $L^{-1}\{F(s)\}(t)=\dfrac{1}{2}[L^{-1}\left\{\dfrac{1}{s^2-1}\right\}(t)-L^{-1}\left\{\dfrac{1}{s^2+1}\right\}(t)]$

$=\dfrac{1}{2}(\sinh t-\sin t)$

3 (K64) 1) $L\left\{e^t[\sin(2t)+3\cos(2t)]\right\}$

$\left(\dfrac{3s-1}{s^2-2s+5}, s>1\right)$

2) Tính $L^{-1}\left\{\dfrac{s+1}{s^2-6s+13}\right\}(t)$

$\left(e^{3t}\cos(2t) + 2e^{3t}\sin(2t)\right)$

3) Tính $L^{-1}\left\{\dfrac{13s+14}{(s+2)^2(s-1)}\right\}(t)$ $\left(-3e^{2t}+3e^{t}\right)$

GIẢI 1

+) $L\left\{e^t[\sin(2t)+3\cos(2t)]\right\}(s)=$

$=L\left\{e^t\sin(2t)\right\}(s)+3L\left\{e^t\cos(2t)\right\}(s)$

$=L\{\sin(2t)\}(s-1)+3L\{\cos(2t)\}(s-1)$

+) $=\dfrac{2}{(s-1)^2+4}+3\dfrac{s-1}{(s-1)^2+4}=\dfrac{3s-1}{(s-1)^2+4}, s>1$

2 .

+) $L^{-1}\left\{\dfrac{s+1}{s^2-6s+13}\right\}(t)=L^{-1}\left\{\dfrac{s-3+4}{(s-3)^2+4}\right\}(t)$

$=L^{-1}\left\{\dfrac{s-3}{(s-3)^2+4}\right\}(t)+2L^{-1}\left\{\dfrac{2}{(s-3)^2+4}\right\}(t)$

+) $=e^{3t}L^{-1}\left\{\dfrac{s}{s^2+4}\right\}(t)+2e^{3t}L^{-1}\left\{\dfrac{2}{s^2+4}\right\}(t)$

$=e^{3t}[\cos(2t)+2\sin(2t)].$

4 (K63)

$x^{(3)}-2x''+16x=0,\ \ x(0)=x'(0)=0, x''(0)=20.$

$(e^{-2t}-e^{2t}[\cos(2t)-2\sin(2t)])$

GIẢI

+)

$$L\{x(t)\}(s)=X(s); L\{x'(t)\}(s)=sX(s)-x(0)=sX(s)$$

$$L\{x''(t)\}(s)=s^2X(s)-sx(0)-x'(0)=s^2X(s)$$

$$L\{x'''(t)\}(s)=s^3X(s)-s^2x(0)-sx'(0)-x''(0)=$$

$$=s^3X(s)-20\Rightarrow(s^3-2s^2+16)X(s)=20$$

$$\Rightarrow X(s)=\frac{20}{s^3-2s^2+16}$$

+) $x(t)=L^{-1}\{X(s)\}(t)=L^{-1}\left\{\dfrac{20}{(s+2)[(s-2)^2+4]}\right\}(t)$

$$=L^{-1}\left\{\frac{1}{s+2}\right\}(t)-L^{-1}\left\{\frac{s-6}{(s-2)^2+4}\right\}(t)$$

$$= e^{-2t} - L^{-1}\left\{\frac{s-2}{(s-2)^2+4}\right\}(t) + 2L^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}(t)$$

$$= e^{-2t} - e^{2t}L^{-1}\left\{\frac{s}{s^2+4}\right\}(t) + 2e^{2t}L^{-1}\left\{\frac{2}{s^2+4}\right\}(t)$$

$$= e^{-2t} - e^{2t}[\cos(2t) - 2\sin(2t)]$$

5 (K64)

1)

$$x^{(4)} + 4x'' + 4x = 0,\ \ x(0) = x'(0) = 0, x''(0) = 1, x'''(0) = 2.$$

$$\left(\frac{\sqrt{2}}{4}t\sin(t\sqrt{2}) + \frac{\sqrt{2}}{4}\left(\sin(\sqrt{2}t) - t\sqrt{2}\cos(t\sqrt{2})\right)\right)$$

2) $y^{(3)} + y'' = e^t,\ \ y(0) = y'(0) = y'' = 0.$

$$\left(-1+\frac{e^t}{2}+\frac{\cos t-\sin t}{2}\right)$$

3) $x^{(3)}+x''-6x'=0,\ \ x(0)=0, x'(0)=x''(0)=2.$

$$\left(-\frac{2}{3}+\frac{4}{5}e^{2t}-\frac{2}{15}e^{-3t}\right)$$

4) $y^{(4)}-4y^{(3)}+6y''-4y'+y=0,\ \ y(0)=0=y''(0),$

$y'(0)=1=y^{(3)}(0).$ $\left(\frac{1}{3}e^t(3t-3t^2+2t^3)\right)$

GIẢI 1)

+)

$$L\{x(t)\}(s) = X(s); L\{x'(t)\}(s) = sX(s) - x(0) = sX(s)$$

$$L\{x''(t)\}(s) = s^2X(s) - sx(0) - x'(0) = s^2X(s)$$

$$L\left\{x^{(4)}(t)\right\}(s) = s^4X(s) - s^3x(0) - s^2x'(0) - sx''(0) - x'''(0)$$

$$= X(s) - s - 2$$

$$\Rightarrow (s^4 + 4s^2 + 4)X(s) = s + 2 \Rightarrow X(s) = \frac{s+2}{s^4 + 4s^2 + 4}$$

+) $$\Rightarrow x(t) = L^{-1}\{X(s)\}(t) = L^{-1}\left\{\frac{s+2}{(s^2+2)^2}\right\}(t)$$

$$= L^{-1}\left\{\frac{s}{(s^2+2)^2}\right\}(t) + 2L^{-1}\left\{\frac{1}{(s^2+2)^2}\right\}(t)$$

$$= \frac{1}{2\sqrt{2}} t \sin(\sqrt{2}t) + \frac{2}{2(\sqrt{2})^3} [\sin(\sqrt{2}t) - \sqrt{2}t \cos(\sqrt{2}t)]$$

$$= \frac{1}{2\sqrt{2}} t \sin(\sqrt{2}t) + \frac{1}{2\sqrt{2}} [\sin(\sqrt{2}t) - \sqrt{2}t \cos(\sqrt{2}t)]$$

6 (K63)

1) $L\left\{t\sin^2 t\right\}(s)$ $\left(\frac{1}{2}\left[\frac{1}{s^2}+\frac{4-s^2}{(s^2+4)^2}\right]\right)$

2) $tx''+(t-3)x'+2x=0, x(0)=0.$

$\left(Ce^{-t}\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right), C\neq 0.\right)$

3) $L^{-1}\left\{arc\cot\frac{-1}{s}\right\}(t)$ $\left(\frac{\sin t}{t}\right)$

GIẢI 2)

+) Tác động phép biến đổi Laplace và sử dụng định lí 2 ta có

$$\mathcal{L}\{x(t)\}(s) = X(s); \mathcal{L}\{x'(t)\}(s) = sX(s) - x(0) = sX(s)$$

$$\mathcal{L}\{tx'\}(s) = -\frac{d}{ds}\mathcal{L}\{x'\}(s) = -\frac{d}{ds}(sX(s) - x(0))$$

$$= -\frac{d}{ds}(sX(s)) = -[X(s) + sX'(s)]$$

$$\mathcal{L}\{3x'\}(s) = 3\mathcal{L}\{x'\}(s) = 3(sX(s) - x(0)) = 3sX(s)$$

$$\mathcal{L}\{tx''\}(s) = -\frac{d}{ds}\mathcal{L}\{x''\}(s) = -\frac{d}{ds}\left[s^2X(s) - s\,x(0) - x'(0)\right]$$

$$= -\frac{d}{ds}\left[s^2X(s) - x'(0)\right] = -[2sX(s) + s^2X'(s)]$$

Thay vào phương trình ta có

$$-[2sX(s)+s^2X'(s)]-[X(s)+sX'(s)]-3sX(s)+2X(s)=0$$

$$\Leftrightarrow -\left(s^2+s\right)X'(s)+(-5s+1)X(s)=0.$$

+) $\Leftrightarrow \dfrac{X'(s)}{X(s)}=\dfrac{5s-1}{-\left(s^2+s\right)}, s^2+s\neq 0$

$$\Leftrightarrow \frac{d(X(s))}{X(s)}=\frac{5s-1}{-\left(s^2+s\right)}ds, s^2+s\neq 0$$

là phương trình vi phân phân li biến số, có nghiệm là

$$\Rightarrow \ln|X(s)|=\int\frac{5s-1}{-s(s+1)}ds=-\int[\frac{6}{s+1}-\frac{1}{s}]ds$$

$$\Rightarrow \ln|X(s)|=\ln|s+1|^{-6}+\ln|s|+\ln|C|, C\neq 0$$

$$\Rightarrow X(s)=\frac{Cs}{(s+1)^6}=\frac{C}{(s+1)^5}-\frac{C}{(s+1)^6},\ C\neq 0$$

$$\Rightarrow x(t) = L^{-1}\{X(s)\}(t) = L^{-1}\left\{\frac{C}{(s+1)^5} - \frac{C}{(s+1)^6}\right\}(t)$$

$$= Ce^{-t}\left[L^{-1}\left\{\frac{1}{s^5}\right\}(t) - L^{-1}\left\{\frac{1}{s^6}\right\}(t)\right] = Ce^{-t}\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right), C \neq 0.$$

7 (K64)

$L\left\{t(e^{2t} + 3\cos t)\right\}(s)$ $\quad$ $\left(\frac{1}{(s-2)^2} - \frac{3(1-s^2)}{(s^2+1)^2}, s > 2\right)$

GIẢI

+) $L\left\{t(e^{2t}+3\cos t)\right\}(s)=-\dfrac{d}{ds}L\left\{e^{2t}+3\cos t\right\}(s)$

$$=-\frac{d}{ds}L\left\{e^{2t}\right\}(s)-\frac{d}{ds}L\left\{3\cos t\right\}(s)$$

$$=-\frac{d}{ds}\left(\frac{1}{s-2}+3\frac{s}{s^2+1}\right), s>2$$

+) $=\dfrac{1}{(s-2)^2}-3\dfrac{(s^2+1)-2s^2}{(s^2+1)^2}=\dfrac{1}{(s-2)^2}-\dfrac{3(1-s^2)}{(s^2+1)^2}.$

b) $L^{-1}\left\{\ln\dfrac{s^2+1}{s^2+4}\right\}$ $\left(\dfrac{2(\cos 2t-\cos t)}{t}\right)$

c) $L^{-1}\left\{\tan^{-1}\dfrac{3}{s+2}\right\}$ $\left(\dfrac{e^{-2t}\sin 3t}{t}\right)$

GIẢI c) $L^{-1}\left\{\tan^{-1}\dfrac{3}{s+2}\right\}$

+) $\mathcal{L}^{-1}\left\{\tan^{-1}\left(\dfrac{3}{s+2}\right)\right\}(t) = -\dfrac{1}{t}\mathcal{L}^{-1}\left\{\dfrac{d}{ds}\tan^{-1}\dfrac{3}{s+2}\right\}(t)$

•

$$= -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{d}{ds}\tan^{-1}\frac{3}{s+2}\right\}(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{-\dfrac{3}{(s+2)^2}}{1+\left(\dfrac{3}{s+2}\right)^2}\right\}(t)$$

+) $= -\dfrac{1}{t}\mathcal{L}^{-1}\left\{-\dfrac{3}{(s+2)^2+9}\right\}(t) = \dfrac{1}{t}e^{-2t}\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\}(t).$

$$= \frac{1}{t}e^{-2t}\sin(3t).$$

8) (K61) $f(t)=\begin{cases} 0 & 0<t<2 \\ \cos(\pi t), & 2\le t\le 4 \\ 0 & t>4 \end{cases}$

$$\left(F(s)=\frac{1}{s^2+\pi^2}[e^{-2s}-e^{-4s}]\right)$$

GIẢI. Cách 1

+) $L\{f(t)\}(s) = \int_2^4 e^{-st}\cos(\pi t)dt =$

$$= \frac{e^{-st}}{s^2+\pi^2}[-s\cos(\pi t)+\pi\sin(\pi t)]\Big|_2^4$$

$$= \frac{1}{s^2+\pi^2}\left\{e^{-4s}(-s) - e^{-2s}(-s)\right\} = \frac{s}{s^2+\pi^2}(e^{-2s} - e^{-4s}).$$

Cách 2

PGS. TS. Nguyễn Xuân Thảo Email: thao.nguyenxuan@.hust.edu.vn

+)

$$f(t)=\begin{cases} 0 & 0<t<2 \\ \cos(\pi t), & 2\le t\le 4 \\ 0 & t>4 \end{cases} = u(t-2)[1-u(t-4)]\cos(\pi t)$$

$$= u(t-2)\cos[\pi(t-2)] - u((t-4)+2)u(t-4)\cos[\pi(t-4)]$$

+) $\Rightarrow L\{f(t)\}(s) = L\{u(t-2)\cos[\pi(t-2)]\}(s)$

$$-L\{u(t-4)u((t-4)+2)\cos[\pi(t-4)]\}(s)$$

$$= e^{-2s}L\{\cos(\pi t)\}(s) - e^{-4s}L\{u(t+2)\cos(\pi t)\}(s)$$

$$= e^{-2s}\frac{s}{s^2+\pi^2} - e^{-4s}L\{\cos(\pi t)\}(s)$$

$$= \frac{s}{s^2+\pi^2}(e^{-2s}-e^{-4s}).$$

9(K64) 1) $x'' + 2x' + 5x = f(t),\ x(0) = x'(0) = 0,$

$$f(t) = \begin{cases} 20\cos t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

$$(4\cos t + 2\sin t - 4e^{-t}\cos(2t) - 3e^{-t}\sin(2t))$$

2) Cho $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ t, & t \ge 3 \end{cases}$

a) Tính $L\{f(t)\}(s)$. $\quad (e^{-3t}(\dfrac{1}{s^2} + \dfrac{3}{s}), s > 0)$

b) $y'' + 4y = f(t),\ y(0) = y'(0) = 0.$

$$(\frac{1}{4}u(t-3)[t - 3\cos 2(t-3) - \frac{1}{2}\sin 2(t-3)])$$

GIẢI 2) a)

+)

$$f(t)=\begin{cases}0, & 0\le t<3\\ t, & t\ge 3\end{cases}=tu(t-3)=(t-3)u(t-3)+3u(t-3)$$

+) $\Rightarrow L\{f(t)\}(s)=L\{[(t-3)+3]u(t-3)\}(s)$

$$=e^{-3s}L\{t+3)\}(s)=e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right).$$

b)

+) • Tác động phép biến đổi Laplace vào hai vế ta có

$$L\{y(t)\}(s) = Y(s); L\{y''(t)\}(s) = s^2 Y(s)$$

$$\Rightarrow (s^2 + 4)Y(s) = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right), s > 0$$

$$\Rightarrow Y(s) = e^{-3s}\frac{3s+1}{s^2(s^2+4)} = e^{-3s}\frac{3s+1}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right)$$

+) $y(t) = L^{-1}\{Y(s)\}(t) =$

$$= L^{-1}\left\{\frac{1}{4}e^{-3s}\left(\frac{3}{s} + \frac{1}{s^2} - \frac{3s}{s^2+4} - \frac{1}{s^2+4}\right)\right\}(t)$$

$$= \frac{1}{4}u(t-3)L^{-1}\left\{\frac{3}{s} + \frac{1}{s^2} - \frac{3s}{s^2+4} - \frac{1}{s^2+4}\right\}(t-3)$$

$$= \frac{1}{4}u(t-3)[3 + (t-3) - 3\cos 2(t-3) - \frac{1}{2}\sin 2(t-3)]$$

$$= \frac{1}{4}u(t-3)[t - 3\cos 2(t-3) - \frac{1}{2}\sin 2(t-3)].$$

Bảng 4. 1. 2.

Bảng các phép biến đổi Laplace

$f(t)$	$F(s)$
1	$\frac{1}{s}$ $(s > 0)$
t	$\frac{1}{s^2}$ $(s > 0)$
$t^n \quad (n \geq 0)$	$\frac{n!}{s^{n+1}}$ $(s > 0)$
$t^a \quad (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(s > 0)$, $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ $(\mathrm{Re}\, s > 0)$

e^{at}	$\dfrac{1}{s-a}$	$(s>a)$
$\cos kt$	$\dfrac{s}{s^2+k^2}$	$(s>0)$
$f(t)$		$F(s)$
$\sin kt$	$\dfrac{k}{s^2+k^2}$	$(s>0)$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	$(s>\|k\|)$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	$(s>\|k\|)$
$u(t-a)$	$\dfrac{e^{-as}}{s}$	$(s>0)$,a>0

Bảng 4. 1. 2. Bảng các phép biến đổi Laplace

BẢNG 2

	$f(t)$	$\mathcal{L}\{f(t)\}(s)$
1	$e^{at}f(t)$	$\mathcal{L}\{f(t)\}(s-a)$
2	$u(t-a)f(t-a$	$e^{-as}\mathcal{L}\{f(t)\}(s)$
3	$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s)$
4	$(f*g)(t)$	$\mathcal{L}\{f(t)\}(s).\mathcal{L}\{g(t)\}(s)$
5	$\dfrac{f(t)}{t}$	$\int_s^{\infty}\mathcal{L}\{f(t)\}(\tau)d\tau$
6	$f^{(n)}(t)$	$s^n\mathcal{L}\{f(t)\}(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-}$
7	$\int_0^t f(\tau)d\tau$	$\dfrac{1}{s}\mathcal{L}\{f(t)\}(s)$

BẢNG 3

	$F(s)$	$\mathcal{L}^{-1}\{F(s)\}(t)$
1	$F(s)$	$-\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}(t)$
2	$F(s)$	$t\mathcal{L}^{-1}\left\{\int_s^{\infty} F(\delta)\,d\delta\right\}(t)$
3	$F(s-a)$	$e^{at}\mathcal{L}^{-1}\{F(s)\}(t)$
4	$e^{-as}F(s)$	$u(t-a)\mathcal{L}^{-1}\{F(s)\}(t-a)$
5	$F(s)G(s)$	$\left(\mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}\right)(t)$
6	$\frac{F(s)}{s}$	$\int_0^t \mathcal{L}^{-1}\{F(s)\}(\tau)\,d\tau$

HAVE A GOOD UNDERSTANDING!