LESSON 12

CAUSALITY AND STABILITY SURVEY IN Z DOMAIN

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❑ CONTENT

1. Investigate the causality of the system on the Z domain.
2. Examine the stability of the system on the Z domain.
3. Schur-Cohn Stability Criterion
4. Investigate the stable causality of the second order recursion system

Lesson Objectives

After completing this lesson, you will be able to understand the following topics:

- Methods to investigate the causality and stability of the system on the Z domain.
- Investigate the stable causality of the second order recursion system

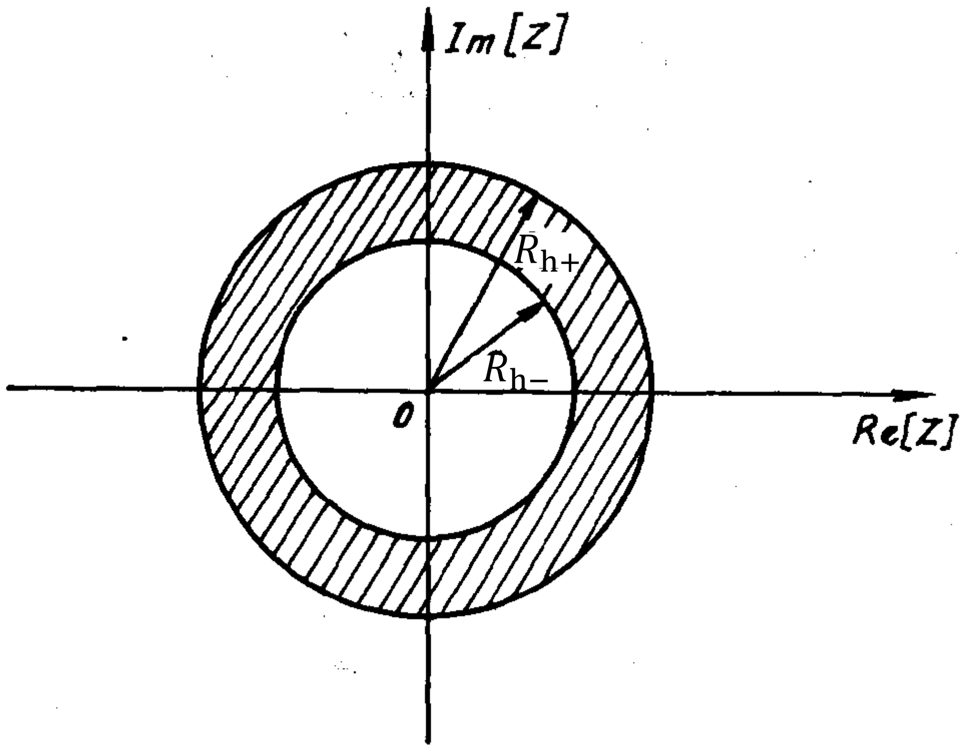
1. Characteristics of the transfer function H(Z)

- Transfer function H(Z):

$$h(n) \xrightarrow{ZT} H(Z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \quad \text{với} \quad R_{h-} < |z| < R_{h+}$$

$$R_{\mathrm{h}-} = \lim_{n\to\infty} |\mathrm{h}(n)|^{1/n}$$

$$R_{\mathrm{h}+} = \frac{1}{\lim_{n\to\infty} |\mathrm{h}(-n)|^{1/n}}$$

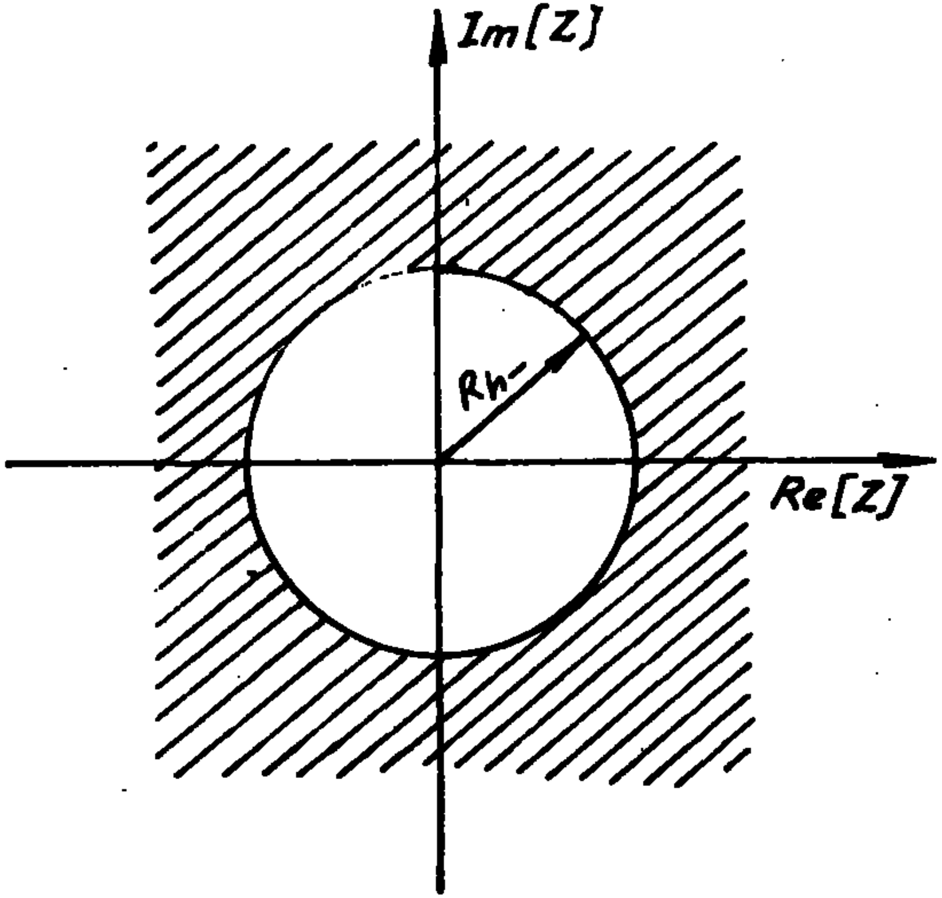


2. Causality

- $h(n) = 0, \forall n < 0$

$$R_{\mathrm{h}+} = \frac{1}{\lim_{n\to\infty} |\mathrm{h}(-n)|^{1/n}}$$

- $\Longrightarrow R_{h+} = +\infty$
- The system is causal $\Leftrightarrow$ the region of convergence of H(Z) lies outside the circle $R_{\mathrm{h}-}$



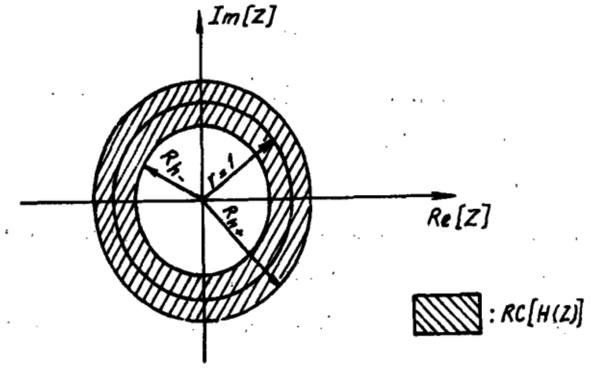
3. Stability

- $S = \sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \Rightarrow |H(z)| = \left|\sum_{n=-\infty}^{\infty} h(n)z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} |h(n)z^{-n}|$$

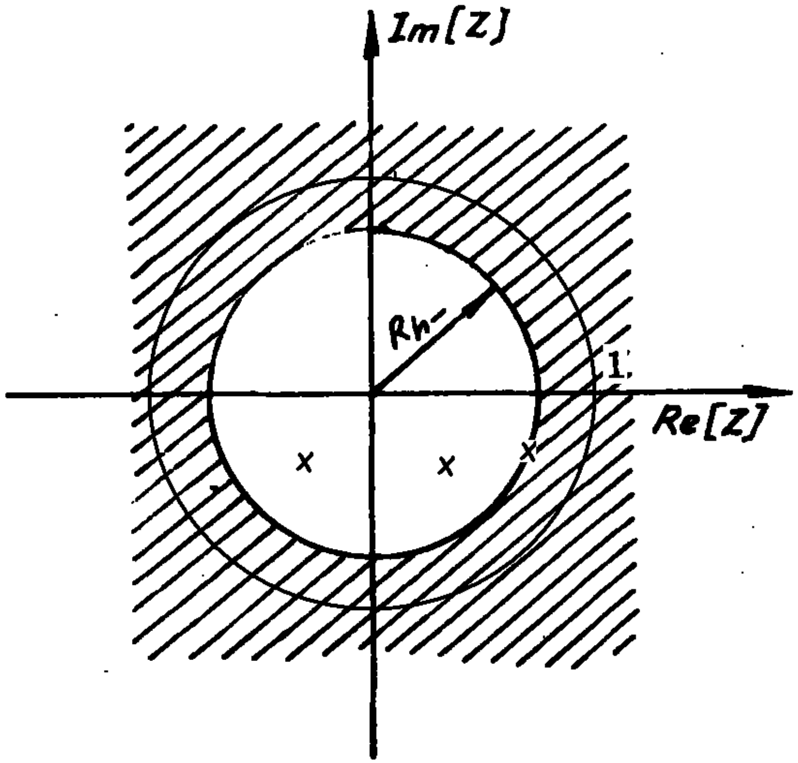
$$\Rightarrow |H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|.|z|^{-n}$$

- So for the points z satisfying $|z| = 1$ then $|H(z)| \leq \sum_{n=-\infty}^{\infty}|h(n)| < \infty$
- The system is stable $\Leftrightarrow$ the unit circle lies in the region of convergence of the transfer function H(Z) of the system



System satisfying causality, stable

- Causality: $R_{h+} = +\infty$
- Stability: $R_{h-} < 1 < R_{h+}$
- Causal and stable system: $R_{h-} < 1$ và $R_{h+} = \infty$
- Causal and stable system $\Leftrightarrow$ the poles of H(Z) lie within the unit circle.



4. Investigate the stability of the causal LTI system by the poles

- Differential Equation : $y(n) + a_1.y(n-1) + \cdots + a_N.y(n-N) = x(n)$
- Algorithm:
  - Step 1: Find the transfer function H(z)

$$H(z) = 1/\left(1 + a_1.z^{-1} + \cdots + a_N.z^{-N}\right) = z^N/(z^N + a_1.z^{N-1} + \cdots + a_N)$$

  - Step 2: Find the poles Z_{pk}

    Solve the equation : $z^N + a_1 z^{N-1} + \cdots + a_N = 0$

  - Step 3: Compare the poles with the unit circle
    - If all the poles are inside the unit circle, then the system is stable
    - If there is a pole on or outside the unit circle, the system is unstable

Disadvantages

- For large order systems there will be many poles Z_{pk}
- Determining Z_{pk} will have great complexity
- Problem:
  - $H(z) = 1/\left(1 + a_1.z^{-1} + \cdots + a_N.z^{-N}\right)$
  - Need to check if the poles (the roots of the denominator) are inside the unit circle?
- A method that doesn't need to be solved directly
  - Investigate the function ⟶ standard stability Jury, Schur-Cohn, …
  - Survey on function $A(z) = 1 + a_1.z^{-1} + \cdots + a_N.z^{-N}$

5. Schur-Cohn Stability Criterion

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

$$A_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k} \ \text{với} \ a_m(0) = 1$$

$$B_m(z) = z^{-m} A_m(z^{-1})$$

$$= \sum_{k=0}^{m} a_m(m-k)\, z^{-k}$$

- The coefficients of $\mathrm{B_m(z)}$ are the same as those of $\mathrm{A_m(z)}$, but in reverse order

Schur-Cohn Stability Criterion

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

- Step 1. $A_N(z) = A(z) \quad K_N = a_N(N)$
- Step 2 (iteration step). Calculate lower degree polynomials $A_m(z)$, $m = N, N-1, N-2, \dots, 1$:

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2} \quad \text{với} \quad K_m = a_m(m)$$

- Step 3. Polynomial A(z) has poles in the unit circle if and only if: The coefficients K_m satisfy the condition $|K_m| < 1 \ \forall\, m = 1, 2, \dots, N$.

Example

$$H(z) = \frac{1}{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}$$

- Step 1: $A_2(z) = 1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2} \quad \Rightarrow \quad K_2 = -\frac{1}{2}$
- Step 2 (iteration step):

$$\Rightarrow B_2\,(z) = -\frac{1}{2} - \frac{7}{4}z^{-1} + z^{-2}$$

$$\Rightarrow A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2} = 1 - \frac{7}{2}z^{-1}$$

$$\Rightarrow K_1 = -\frac{7}{2} \;\Rightarrow \mathit{Unstable\ system}$$

Programming Algorithms

- Initialization step :

$$a_N(k) = a_k \qquad k = 1, 2, \dots, N$$

$$K_N = a_N(N)$$

- Repeat step with $m = N, N-1, \dots, 1$:

$$K_m = a_m(m) \qquad a_{m-1}(0) = 1$$

$$a_{m-1}(k) = \frac{a_m(k) - K_m b_m(k)}{1 - K_m^2} \qquad k = 1, 2, \dots, m-1$$

$$b_m(k) = a_m(m-k) \qquad k = 0,1, \dots, m$$

6. Investigation of the stability of the 2-order IIR system

- The second order system has the differential equation:

$$y(n) + a_1.y(n-2) + a_2.y(n-2) = x(n)$$

- Transfer function $H(z)$:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{z^2}{z^2 + a_1 z + a_2}$$

- Solve the equation: $z^2 + a_1.z + a_2 = 0$

$$\Delta = a_1^2 - 4a_2 \geq 0$$

- Two real roots: $z_{1,2} = -a_1/2 \pm \mathrm{sqrt}(\Delta)/2$

$$\Delta = a_1^2 - 4a_2 < 0$$

- Two complex solutions: $z_{1,2} = -a_1/2 \pm \ j.\mathrm{sqrt}(-\Delta)/2$

Schur-Cohn Stability Criterion

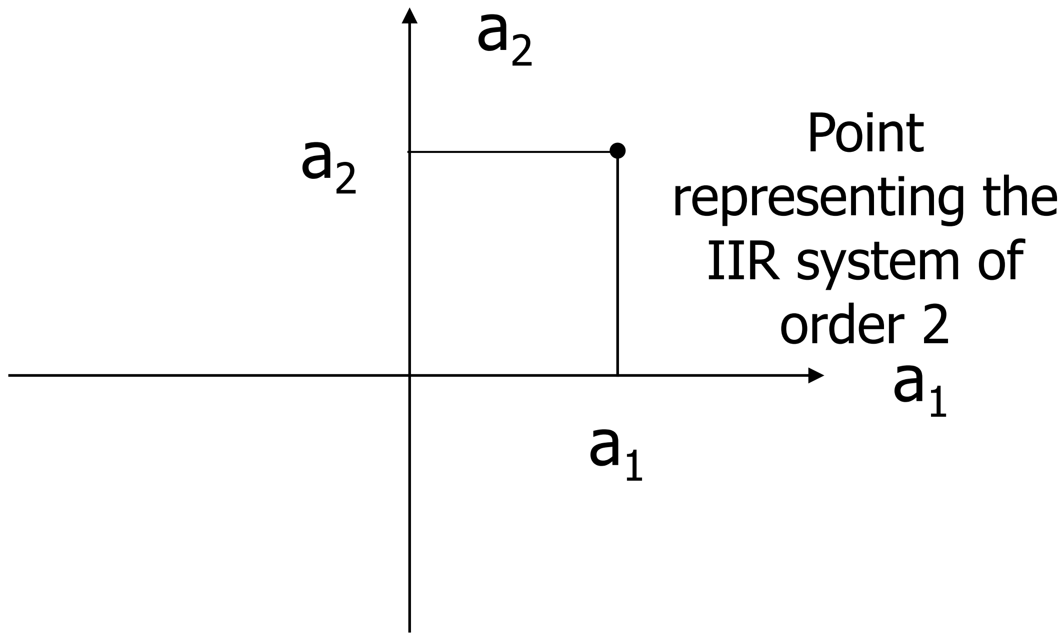
- Set $A_2(z) = 1 + a_1.z^{-1} + a_2.z^{-2} \Longrightarrow K_2 = a_2$
- $B_2(z) = z^{-2} + a_1.z^{-1} + a_2$
- $A_1(z) = [A_2(z) - K_2.B_2(z)]/\left(1 - K_2^2\right)$
- $A_1(z) = 1 + a_1.\frac{z^{-1}}{1+a_2} \quad \Longrightarrow \quad K_1 = \frac{a_1}{1+a_2}$
- The system is stable if and only if $|K_1| < 1$ và $|K_2| < 1$

$$\boxed{-1 < a_2 < 1}$$

$$-1 < \frac{a_1}{1 + a_2} < 1 \Leftrightarrow \boxed{\begin{array}{c} a_1 < 1 + a_2 \\ a_1 > -1 - a_2 \end{array}}$$

Geometry representation of the 2 order IIR system

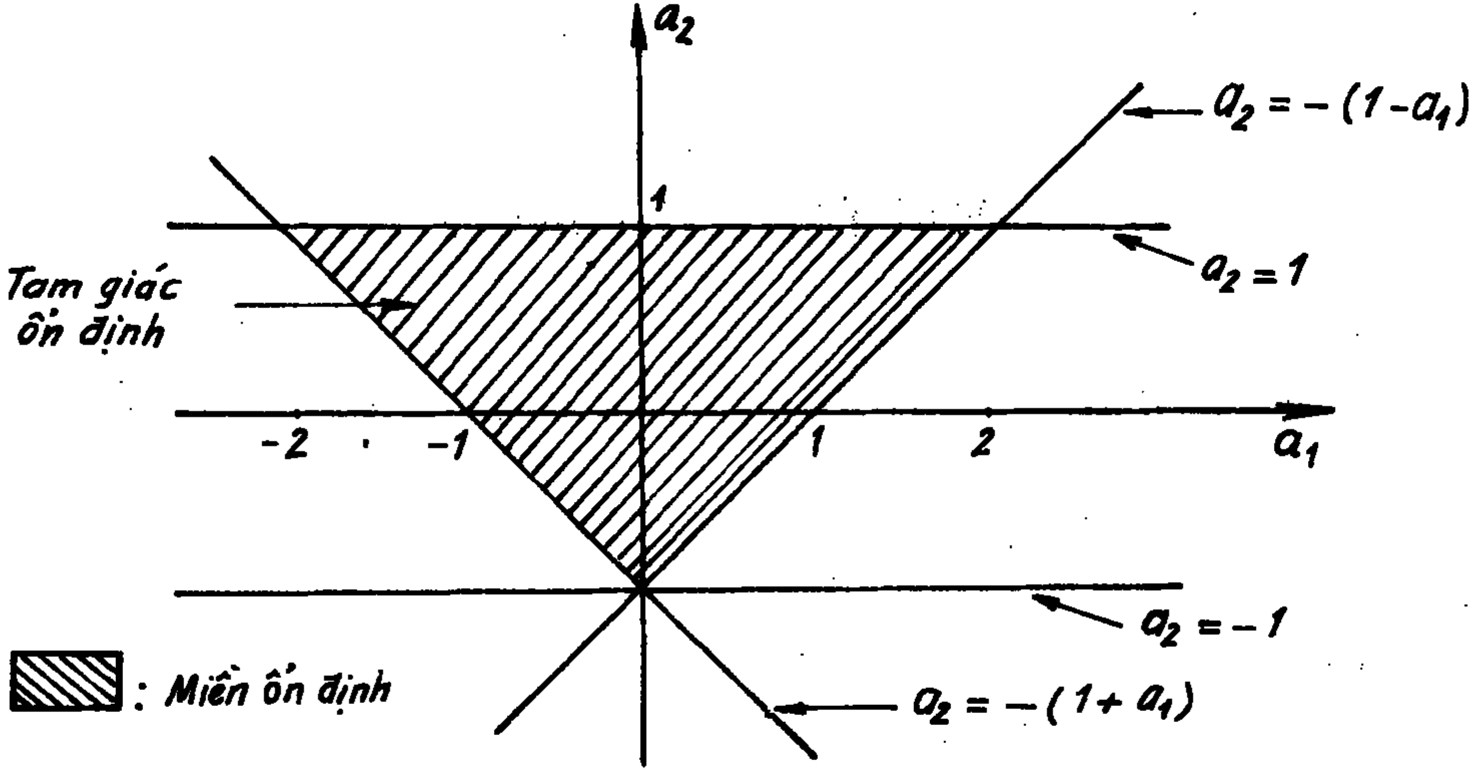
- 2-order IIR system
- Fully deterministic via (a_1, a_2)
- Represented by a point on the plane (a_1, a_2)



Geometric representation of the stability conditions of the 2-order IIR system

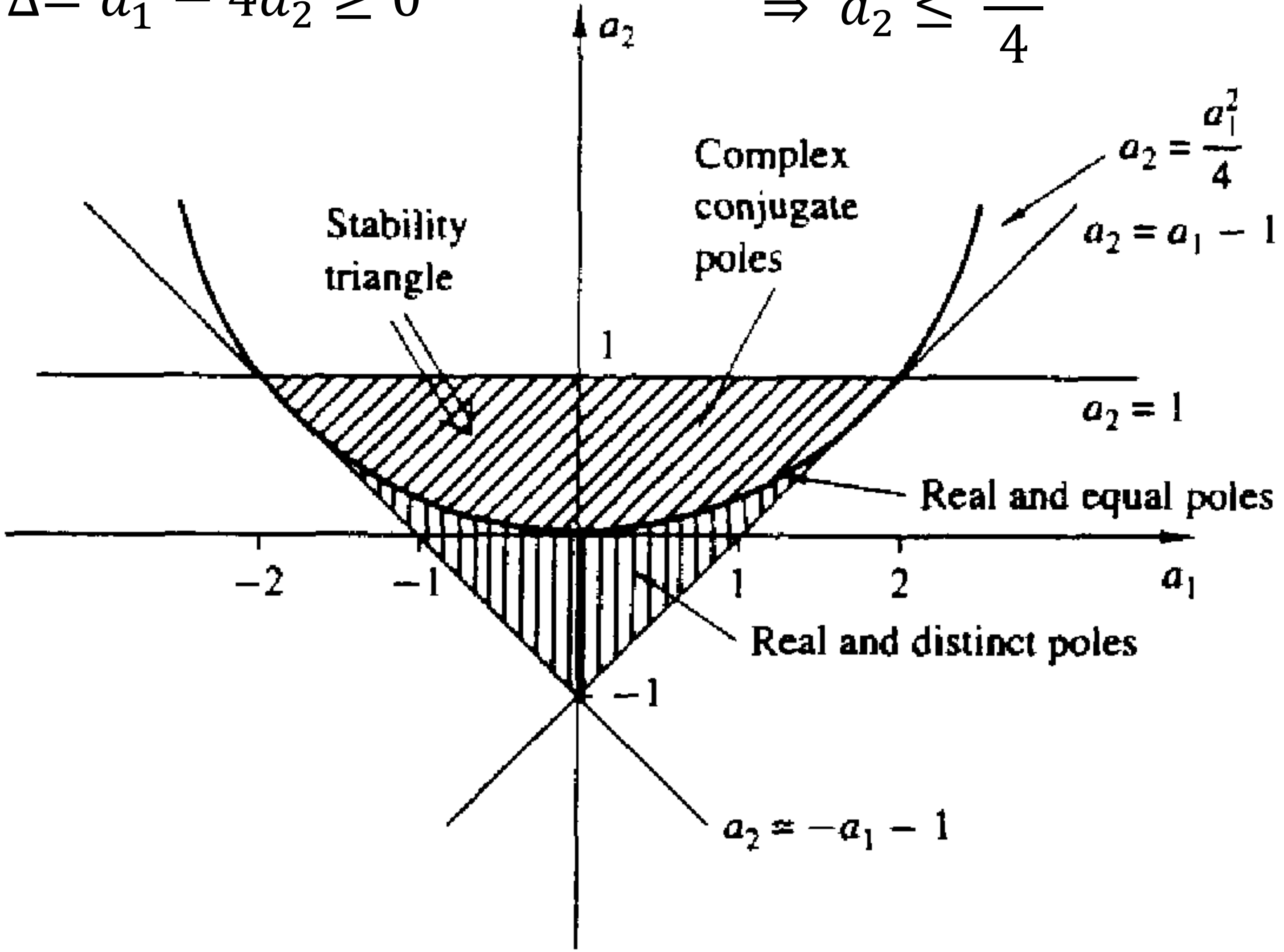
$$\boxed{\begin{array}{l} a_1 < 1 + a_2 \\ a_1 > -1 - a_2 \end{array}} \begin{array}{l} \Rightarrow\ a_2 > a_1 - 1 \\ \Rightarrow\ a_2 > -a_1 - 1 \end{array}$$

$$\boxed{-1 < a_2 < 1}$$



Real poles and imaginary poles on the stable triangle of a 2-order IIR system

$$\Delta = a_1^2 - 4a_2 \geq 0 \quad \Rightarrow \quad a_2 \leq \frac{a_1^2}{4}$$



4. Summary

- The causality and stability of a system can be investigated through its transfer function and convergence domain.
- The Schur-Cohn criterion allows to investigate the stability through the coefficients of the differential equation.
- The quadratic regression system is stable with the system representation points lying in the stability triangle on the plane A1-A2.

Exercise 1

- Examine the stability of the following causal systems :

a. $y(n) - 3.y(n-1) + 2.y(n-2) = 4.x(n) + 5.x(n-1)$

b. $y(n) - 3.y(n-1) + 2.y(n-2) = x(n)$

c. $y(n) - 5/8\,.\,y(n-1) + 1/8\,.\,y(n-2) = x(n)$

d. $y(n) - 4.y(n-1) + 5.y(n-2) = x(n)$

Exercise 2

- Examine the stability of the following causal systems :

a. $y(n) - 0.02\,y(n-1) - 0.1\,y(n-2) - 0.03\,y(n-3) - 0.25\,y(n-4) = 0.5\,x(n) + 0.3\,x(n-1)$

b. $y(n) - 0.15\,y(n-3) - 0.3\,y(n-4) - 0.2\,y(n-5) - 0.01\,y(n-6) = 0.1\,x(n) + 0.2\,x(n-1)$

Next lesson. Lesson 13

1-sided Z TRANSFORM

References:
- ***Nguyễn Quốc Trung (2008), Xử lý tín hiệu và lọc số, Tập 1, Nhà xuất bản Khoa học và Kỹ thuật, Chương 1 Tín hiệu và hệ thống rời rạc.***
- ***J.G. Proakis, D.G. Manolakis (2007), Digital Signal Processing, Principles, Algorithms, and Applications, 4th Ed, Prentice Hall, Chapter 1 Introduction.***

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Wish you all good study!