MIDTERM MOCK TEST - MI1016 - SEMESTER 20241

Questions with only one correct answer

Question 1. Which of the following functions is odd?

By definition !

A. $y = \arccos x$.

C. $y = \cos x$.

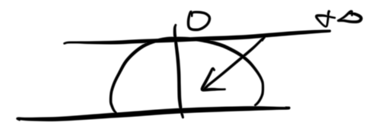
B. $y = \arcsin x$.

D. $y = \sin x^2$.

Question 2. Determine the range of the function $y = \text{arccot}(\tan^2 x)$.

$\tan^2 x \in [0, +\infty)$

$\Rightarrow \text{arccot}(\tan^2 x) \in (0, \frac{\pi}{2}]$



A. $(0, \pi)$.

C. $\left[\frac{\pi}{2}, \pi\right)$.

B. $\left(0, \frac{\pi}{2}\right]$.

D. $\mathbb{R}$.

Question 3. Determine the value $a \in \mathbb{R}$ such that the function $y = \begin{cases} 2^{\frac{1}{\arcsin x}}, x \neq 0, \\ a, x = 0 \end{cases}$ is continuous from the left.

y is cont from the left $\Leftrightarrow \lim_{x\to 0^-} 2^{\frac{1}{\arcsin x}} = a$

Since $\lim_{x\to 0^-} \arcsin x = 0^-$ $\Rightarrow \lim_{x\to 0^-} 2^{\frac{1}{\arcsin x}} = 0$

A. $a = -1$.

C. $a = 1$.

B. $a = 0$.

D. $a = 2$.

Question 4. Compute the following indefinite integral $\int \frac{dx}{(x+1)\ln(x+1)}, x > -1$.

$\int \frac{dx}{(x+1)\ln(x+1)} =$

$u = \ln(x+1)$

$du = \frac{dx}{x+1}$

$\int \frac{du}{u} = \ln|u| + C$

$= \ln|\ln(x+1)| + C$

A. $\ln(x+1) + C, C \in \mathbb{R}$.

C. $\ln^2(x+1) + C, C \in \mathbb{R}$.

B. $\ln|\ln(x+1)| + C, C \in \mathbb{R}$.

D. $\frac{1}{\ln(x+1)} + C, C \in \mathbb{R}$.

Question 5. Consider $f\colon \mathbb{R} \to \mathbb{R}, f(x) = \arctan x - \frac{\pi}{4} + x^3$ and let $g(x) = f^{-1}(x) + x^2$. Compute $g'(1)$.

$g'(1) = (f^{-1})'(1) + 2. = \frac{2}{7} + 2 = \frac{16}{7}$

$f(x) = 1 \Leftrightarrow x = 1 \Rightarrow (f^{-1})'(1) = (f^{-1})'(f(1)) = \frac{1}{f'(1)}$

$f'(x) = \frac{1}{1+x^2} + 3x^2 \Rightarrow f'(1) = \frac{1}{2} + 3 = \frac{7}{2}$

$\Rightarrow (f^{-1})'(1) = \frac{2}{7}$

A. $g'(1) = \frac{7}{2}$.

C. $g'(1) = \frac{2}{7}$.

B. $g'(1) = \frac{11}{2}$.

D. $g'(1) = \frac{16}{7}$.

Question 6. Suppose that the function $y = \begin{cases} \frac{mx - \sin(2x)}{x^2}, x \neq 0 \\ 0 \end{cases}$ is differentiable at $x = 0$ and $f'(0) = n$. Compute $\lambda = m \cdot n$?

y is diff $\Rightarrow$ y is cont $\Rightarrow \lim_{x\to 0} \frac{mx - \sin 2x}{x^2} = 0$ $\Leftrightarrow m = 2.$

Since $mx - \sin 2x = mx - 2x + \frac{(2x)^3}{3!} + o(x^3)$

With $m=2$, $f'(0) = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{2x - \sin 2x}{x^3} = \lim_{x\to 0} \frac{\frac{8x^3}{3!} + o(x^3)}{x^3}$

$= \frac{8}{6} = n$

$\Rightarrow m.n = \frac{8}{3}$

A. $\lambda = \frac{4}{3}$.

C. $\lambda = \frac{8}{3}$.

B. $\lambda = 2$.

D. $\lambda = 0$.

Question 7. Consider the sequence $u_n = \frac{\cos n}{n!}, n \geq 1$. Which of the following statements is true?

A. (u_n) is increasing.

B. (u_n) is bounded.

C. $\lim\limits_{n\to\infty} u_n$ does not exists. $\lim u_n = 0$

D. (u_n) is decreasing.

Question 8. Which of the following functions is bounded over its domain of definition?

A. $y = \cancel{e^{x^2}}.\ e^{x^2}$

B. $y = \arctan \frac{1}{x}$. $|\arctan| \leq \frac{\pi}{2}$

C. $y = \tan x$.

D. $y = e^{\frac{1}{x^2}}$.

Questions with multiple correct answers

Question 9. Which of the following functions is an infinitesimal as $x \to 0^+$.

A. $y = x \ln x$.

B. $y = \frac{\ln x}{x}$. $\lim\limits_{x\to 0^+} \frac{\ln x}{x} = -\infty$

C. $y = \frac{x}{\ln x}$. $\lim\limits_{x\to 0^+} \frac{x}{\ln x} = 0$ since $\lim\limits_{x\to 0^+} \frac{1}{\ln x} = 0$

D. $y = x^{\ln x}$. $\lim\limits_{x\to 0^+} x^{\ln x} = \lim\limits_{x\to 0^+} e^{\ln x . \ln x} = +\infty$ since $\lim\limits_{x\to 0^+} \ln^2 x = +\infty$

Question 10. Given $f\colon [0;2] \to \mathbb{R}$ be a continuously differentiable function. Which of the following statements is always correct?

A. If $f(2)f(0) < 0$ then $\exists c \in (0;2)$ such that $f'(c) = 0$. Wrong if $f(x) = x-1$.

B. If $f(2)f(0) < 0$ then $\exists c \in (0;2)$ such that $f(c) = 0$.

C. If $f(0) = 0$ then $\exists c \in (0;2)$ such that $f(2) = 2f'(c)$. $\to$ Correct. Consider $g(x) = xf(2) - 2f(x)$. Then $g(0) = g(2) = 0 \Rightarrow \exists c \in (0,2) \mid g'(c) = 0$ $\Rightarrow f(2) = 2f'(c)$.

D. Function f cannot attain its maximum in $[0;2]$. f is continuous then it attains its maximum

Question 11. Which of the following functions is an infinitesimal of higher order than $\alpha(x) = e^{\sqrt{x}} - 1$ as $x \to 0^+$. $\alpha \sim \sqrt{x}$

A. $y = \sqrt[3]{1+x} - 1$. $\sim \frac{1}{3}x$

B. $y = \arctan \sqrt{x}$. $\sim \sqrt{x}$

C. $y = \sin x$. $\sim x$

D. $y = \cos \sqrt{x}$. not infinitesimal

E. $y = 1 - \cos \sqrt{x}$. $\sim \frac{x}{2}$

F. $y = \sqrt{1+\sqrt{x}} - \cos x$. $\underbrace{\sqrt{1+\sqrt{x}} - 1}_{\alpha_1} + \underbrace{1 - \cos x}_{\alpha_2}$ $\left.\begin{array}{l}\alpha_1 \sim \frac{1}{2}\sqrt{x} \\ \alpha_2 \sim \frac{x^2}{2}\end{array}\right\} \Rightarrow \alpha_1 + \alpha_2 \sim \alpha_1 \sim \frac{1}{2}\sqrt{x}$

Question 12. Which of the following functions is convex over $(0, +\infty)$?

A. $y = \ln x$. $y'' = \frac{-1}{x^2} < 0$

B. $y = e^x$. $y'' = e^x > 0$

C. $y = \sin^2 x$. $y' = 2\sin x . \cos x = \sin 2x$ $y'' = 2\cos 2x$ not > 0

D. $y = -\ln(1 + x^2)$. $y' = \frac{-2x}{1+x^2}$, $y'' = \frac{-2(x^2-1)}{(1+x^2)^2} > 0 \iff |x| < 1$

E. $y = \arctan x$. $y' = \frac{1}{1+x^2}$ $y'' = \frac{-2x}{(1+x^2)^2} < 0$ $\forall x > 0$

F. $y = \operatorname{arccot} x$. $y' = \frac{-1}{1+x^2}$ $y'' = \frac{2x}{1+x^2} > 0$ $\forall x > 0$

Constructed-repsonse questions

Question 13. Compute the Maclaurin polynomial of order 6 of $\frac{1}{1+x^2}$.

Question 14. Determine the local extremes of $y = \sin x + \cos x$.

Question 15. Show that if n is odd then the equation $x^n + x - 10 = 0$ has at least one solution.

(13) $$\frac{1}{1+u} = 1 - u + u^2 - u^3 + o(u^3)$$

$$\Rightarrow \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + o(x^6)$$

(14) $$y = \sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$$

$\Rightarrow$ Local maximum at $x_k + \frac{\pi}{4} = \frac{\pi}{2} + k2\pi \Leftrightarrow x_k = \frac{\pi}{4} + k2\pi$, $y(x_k) = \sqrt{2}$

—— minimum at $t_k + \frac{\pi}{4} = \frac{-\pi}{2} + k2\pi \Leftrightarrow t_k = \frac{-3\pi}{4} + k2\pi$, $y(t_k) = -\sqrt{2}$

(15) If n is odd, $\lim_{x \to +\infty} (x^n + x - 10) = +\infty \Rightarrow \exists a > 0$, $f(a) > 0$

Set $f(x) = x^n + x - 10$

and $f(0) = -10 < 0$ $\Rightarrow$ By continuity $f(a)f(0) < 0 \Rightarrow \exists c \in (0,a) \mid f(c) = 0$

$\Rightarrow f(x) = 0$ has at least one solution