Line Integrals

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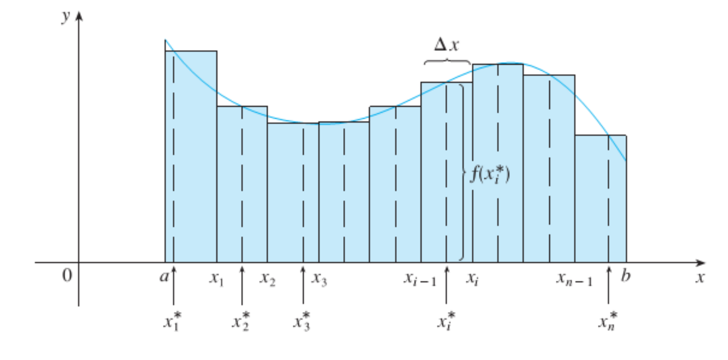
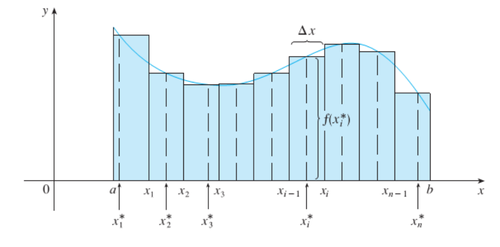
Review of the Definite Integral



$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Line Integrals of scalar Fields

Review of the Definite Integral



1. divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$
2. choose sample points x_i^* in these subintervals,
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4. take the limit $\int\limits_a^b f(x)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$

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- Line integrals are integrated over a curve C instead of over an interval $[a, b]$.
- "curve integrals" would be better terminology.

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Definition

Let C be a curve given by $r(t) = x(t)\vec{i} + y(t)\vec{j}, a \le t \le b$.

1. *divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$*
2. *choose sample points $P_i^*(x_i^*, y_i^*) \in P_{i-1}P_i$,*
3. *form the Riemann sum $\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$*
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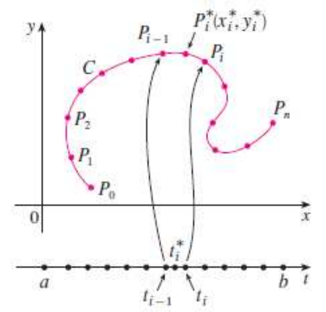
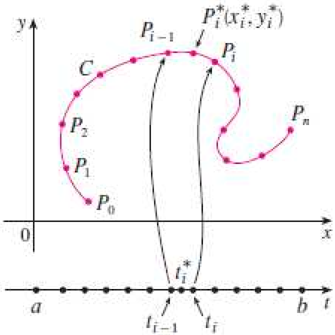


Line Integrals of scalar Fields



$$\int_C f(x,y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

$$= \int_a^b f(x(t), y(t))\sqrt{(x_t')^2 + (y_t')^2}dt = \int_a^b f(r(t))|r'(t)|dt.$$

Formulations

1. If C is given by $x = x(t), y = y(t), a \leq t \leq b$, then

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{x'^2(t)+y'^2(t)}dt.$$

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$$\int_C f(x,y)\,ds = \int_a^b f(x, y(x))\sqrt{1 + y'^2(x)}dx.$$

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Example

Evaluate $\int\limits_C (x - y)\,ds$, where C is the circle $x^2 + y^2 = 2x$.

Line Integrals

Properties

- Line Integrals of scalar Fields do not depend on the direction of C.

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- Linearity and additivity.

Line Integrals

line integrals of f along C with respect to x and y

1. $$\int\limits_C f(x,y)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

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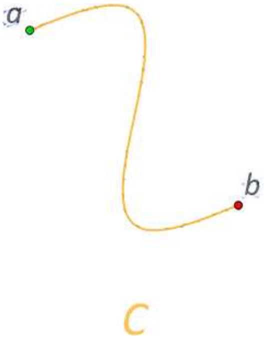
3. $$\int_{-C} f(x,y)dx = -\int_C f(x,y)dx, \quad \int_{-C} f(x,y)dy = -\int_C f(x,y)dy.$$

Line Integrals

Line Integrals of vector Fields

Suppose that $F = P\vec{i} + Q\vec{j}$ is a continuous force field. Compute the work done by this force in moving a particle along a smooth curve C.



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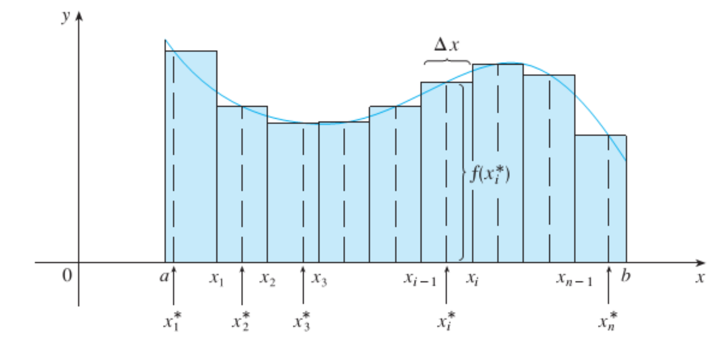

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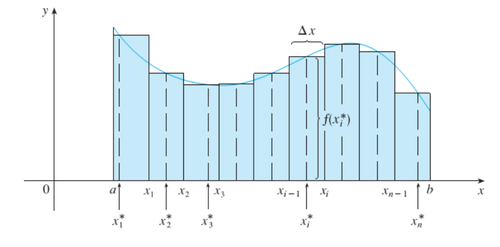

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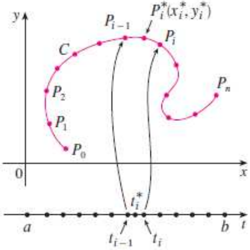

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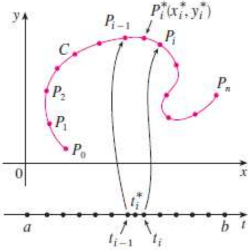

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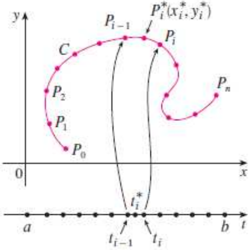

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The work done by the force F in moving the particle from P_{i-1} to P_i is approximately

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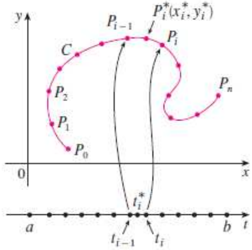

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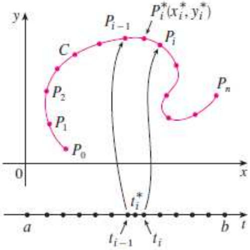

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3. $S_n = \sum_{i=1}^{n} [F(x_i^*, y_i^*) \cdot T(t_i^*)]\Delta s_i.$
4. $W = \lim_{n\to\infty} S_n = \int_C F(x, y) \cdot T(x, y) ds.$

Line Integrals of vector Fields

Definition

Let F be a continuous vector field defined on a smooth curve C given by a vector function $r(t), a \leq t \leq b$. Then the line integral of F along C is

$$\int_C F \cdot T ds$$

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It is sometimes denoted by

$$\int_C F \cdot T ds = \int_C P(x, y)dx + Q(x, y)dy.$$

Line Integrals of vector Fields

Properties

- Line integrals of vector fields depend on the direction of the curve, i.e., $\int\limits_{-C} P(x,y)\,dx + Q(x,y)\,dy = -\int\limits_{C} P(x,y)\,dx + Q(x,y)\,dy$.
- Linearity and additivity.

Formulations

1. If $C = \widetilde{AB}$ is given by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, A, B correspond to $t = a, t = b$, then

$$\int\limits_{C} Pdx + Qdy = \int\limits_{a}^{b} \left[P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)\right] dt.$$

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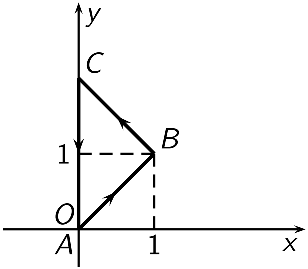
Line Integrals of vector Fields

Example

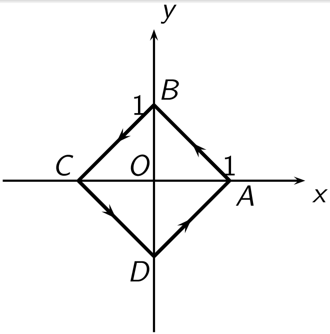
Evaluate $\int\limits_{ABCA} 2\left(x^2+y^2\right)dx + x\left(4y+3\right)dy$, where $ABCA$ is the quadrangular curve, $A(0,0)$, $B(1,1)$, $C(0,2)$.



Line Integrals of vector Fields

Example

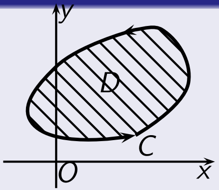
Evaluate $\int\limits_{ABCDA} \frac{dx+dy}{|x|+|y|}$, where $ABCDA$ is the triangular curve, $A(1,0), B(0,1), C(-1,0), D(0,-1)$.



Green's Theorem

Closed Curve Orientation

We use the convention that the positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.



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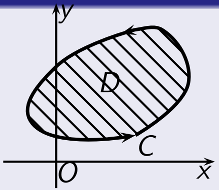


Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

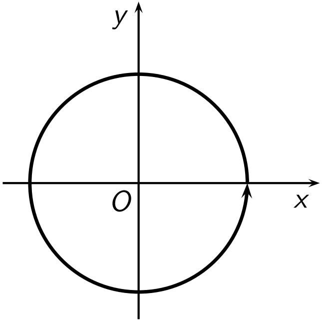
$$\int_C P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Green's Theorem

Example

Evaluate the integral $\int\limits_C (xy+x+y)\,dx + (xy+x-y)\,dy$, where C is the positively oriented circle $x^2+y^2=R^2$ by

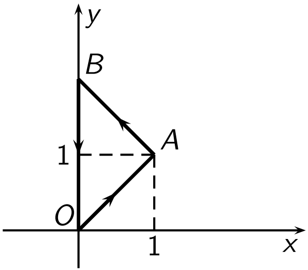
- computing it directly and
- Green's Theorem then compare the results,



Green's Theorem

Example

Evaluate $\oint\limits_{OABO} e^x\left[(1-\cos y)\,dx-(y-\sin y)\,dy\right]$, where $OABO$ is the triangle, $O(0,0), A(1,1), B(0,2)$.



Green's Theorem

- If ∂D is negatively oriented, then
$$\int_C Pdx + Qdy = -\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$
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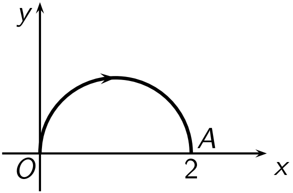
Example

Evaluate $\int\limits_C (xy + x + y)\, dx + (xy + x - y)\, dy$, where C is a half of the circle $x^2 + y^2 = 2x, y \geq 0$, traced from $O(0,0)$ to $A(2,0)$.



Applications of Line Integrals

Area of a Domain

If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, then $A(D) = \iint\limits_{D} 1dxdy = \int\limits_{\partial D} Pdx + Qdy$.

Applications of Line Integrals

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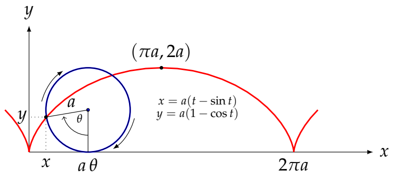
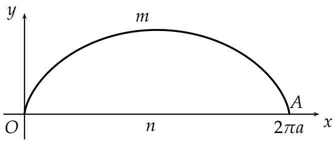
Example

Find the area of the domain bounded by an arch of the cycloid $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ and Ox $(a > 0)$.





Independence of Path

Assume that D is a simple domain, P, Q and their partial derivatives are continuous on $\overline{D}$. Then the following assertions are equivalent:

1. $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ for all $(x, y) \in D$.
2. $\int\limits_L Pdx + Qdy = 0$ for all closed curve L contained in D.
3. $\int\limits_C Pdx + Qdy$ is independent of path.
4. $F = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is conservative, i.e.,

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4. $F = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is conservative, i.e., $\exists u(x, y)$ s.t. $\nabla u = F$. The function u is computed by:

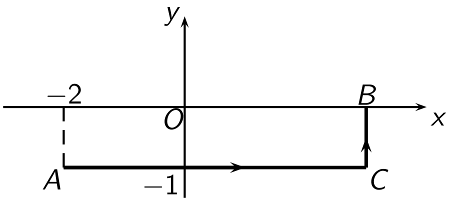
$$u(x, y) = \int_{x_0}^{x} P(x, y_0)dx + \int_{y_0}^{y} Q(x, y)dy$$
$$= \int_{x_0}^{x} P(x, y)dx + \int_{y_0}^{y} Q(x_0, y)dy.$$

Independence of Path

1. Check the condition $P'_y = Q'_x$.
2. Choose the path such that the integration is simplest. It might be the line segment $\overline{AB}$, or line segments parallel to axis.
3. $I = u(B) - u(A)$.

Example

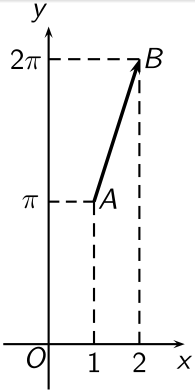
Evaluate $\displaystyle\int\limits_{(-2,-1)}^{(3,0)} (x^4 + 4xy^3)\,dx + (6x^2y^2 - 5y^4)\,dy$.



Independence of Path

Example

Evaluate $\int\limits_{(1,\pi)}^{(2,2\pi)} \left(1 - \frac{y^2}{x^2}\cos\frac{y}{x}\right) dx + \left(\sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x}\right) dy.$



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3. $\int\limits_{\widehat{AB}} Pdx + Qdy + Rdz$ is independent of path.
4. $\vec{F}$ is conservative, i.e. $\exists u(x, y, z)$ st. $\nabla u = \overrightarrow{F}$.

Independence of path

Equivalence

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2. $\int\limits_L Pdx + Qdy + Rdz = 0$ for all closed curve L contained in D.
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4. $\vec{F}$ is conservative, i.e. $\exists u(x, y, z)$ st. $\nabla u = \overrightarrow{F}$.

$$u(x, y, z) = \int_{x_0}^{x} P(x, y_0, z_0)dx + \int_{y_0}^{y} Q(x, y, z_0)dy + \int_{z_0}^{z} R(x, y, z)dz + C.$$