HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF INFORMATION AND COMMUNICATION TECHNOLOGY

Scientific Computing

Project

Two Dimension Heat Equation

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- Trần Duy Khánh : Coding and testing

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- Nguyễn Hữu Phúc : Report content and using Latex
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Making slides Task

- Hoàng Tuấn Kỳ : Making slides
- Trịnh Tiến Dũng : Making slides

Contents

1 Introduction

There is plate with thermal diffusivity D. Initially, temperature of the plate is held at 200 degrees Celcius. At time t = 0 and onwards, assume that edges of the plate are kept at 20 degrees Celsius. Give the temperature profile at various time.

2 Inputs/Outputs

2.1 Input:

- Simulation time
- Thermal diffusivity D
- Initial condition: The initial temperature distribution across the plate at t = 0
- Boundary Conditions: The temperature boundary conditions at each edge of the plate

2.2 Output:

- Temperature profile over time: The variation of temperature at specific points on the plate over time

2.3 Numerical Method

2.3.1 Finite Difference Method

We use Variable Separation Method with respect to 2 dimension space and time. In a 2D space, we divide objects along the Ox and Oy axes. We divide the Ox axis into M - 1 segments of length dx, and divide the Oy axis into N - 1 segments of length dy.

With respect to time, we divide the time into P-1 steps of dt. We also use Backward difference method to approximate the derivatives of temperature over the time and space :

$$\frac{\partial T_{i,j}^t}{\partial t} = \frac{T_{i,j}^t - T_{i,j}^{t-1}}{dt} \tag{1}$$

$$\frac{\partial^2 T_{i,j}}{\partial x^2} = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{dx^2} \tag{2}$$

$$\frac{\partial^2 T_{i,j}}{\partial y^2} = \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{dy^2} \tag{3}$$

2.3.2 Finite Volume Method

In a 2D space, we divide objects along the Ox and Oy axes. We divide the Ox axis into Nx segments of length Δx, and divide the Oy axis into Ny segments of length Δy. With respect to time, we divide the time into Nt steps of Δt.

By integrating the heat equation governing the evolution of temperature with respect to space and time we receive:

$$\int_t^{t+\Delta t} \int_s^n \int_w^e \frac{\partial T}{\partial t} dxdydt = D(\int\int\int \frac{\partial}{\partial x}\frac{\partial T}{\partial x} dxdydt + \int\int\int \frac{\partial}{\partial y}\frac{\partial T}{\partial y} dxdydt) \tag{4}$$

$$(T_p^{n+1} - T_p^n)\Delta x \Delta y = D[(\frac{\partial T}{\partial x})_e - (\frac{\partial T}{\partial x})_w]\Delta y \Delta t + D[(\frac{\partial T}{\partial y})_n - (\frac{\partial T}{\partial y})_s]\Delta x \Delta t \tag{5}$$

where

- *T(P,n)* : the temperature at position P and time t
- D : thermal diffusivity coefficient
- $n,\ s,\ e\ ,w$: point adjacent to the north, south, east, west of point P
- t : time
- x,y : 2 dimension space

After simplifying the above equation, we get:

$$T_P^{n+1} = T_P^n + X_{right} + X_{left} + Y_{up} + Y_{down} \tag{6}$$

$$X_{rigth} = D\frac{\Delta t}{\Delta x}\frac{T_E^n - T_P^n}{\delta x} \tag{7}$$

$$X_{left} = D\frac{\Delta t}{\Delta x}\frac{T_W^n - T_P^n}{\delta x} \tag{8}$$

$$Y_{up} = D\frac{\Delta t}{\Delta y}\frac{T_N^n - T_P^n}{\delta y} \tag{9}$$

$$Y_{down} = D\frac{\Delta t}{\Delta y}\frac{T_S^n - T_P^n}{\delta y} \tag{10}$$

where

- T_E, T_W, T_N, T_S : the temperature of the point adjacent to the north, south, east, west of point P

2.3.3 Finite Element Method

We use the strong form of the heat conduction equation, which is given by:

$$\rho Cp\frac{\partial T}{\partial t} - \nabla.(k\nabla T) = \quad Q \tag{11}$$

where

- ρ : the density
- Cp : the heat capacity
- k : the thermal conductivity
- T : the temperature
- Q : heat source/sink term

In this problem ,we assumed the heat source/sink term to be zero.

The stiffness matrix K and mass matrix M uses the element stiffness and mass matrices calculated for each triangular element in the mesh. The element matrices are computed using the area of the triangular element, the Jacobian of the element, and the inverse of the Jacobian.

$$\mathbf{A}ij = \int \Omega\nabla\varphi_i \cdot \nabla\varphi_j, dx \tag{12}$$

$$\mathbf{M}ij = \int \Omega\varphi_i\varphi_j, dx \tag{13}$$

The time-stepping scheme is implemented using the backward Euler method. The temperature distribution at each time step is obtained by solving a system of linear equations using the stiffness and mass matrices.

$$\frac{\partial T_{i,j}^{t}}{\partial t} = \frac{T_{i,j}^{t} - T_{i,j}^{t-1}}{dt} \tag{14}$$

The mesh is generated using n nodes, resulting in a triangular mesh with $4n^2$ triangular elements and $(n+1)^2 + n^2$ nodes.

3 Algorithm

3.1 Finite Difference Method

1. Input the parameters:
   - time: Total simulation time.
   - D: Thermal diffusivity coefficient.
   - TL, TR, TB, TT: Boundary temperatures for the left, right, bottom, and top sides of the plate.
   - Tinit: Initial temperature of the plate.
2. Define the grid:
   - Set the grid spacing values: dx for the x-direction and dy for the y-direction.
   - Create arrays X and Y to represent the x and y coordinates of the grid points.
   - Create an array E to represent the time steps.
3. Initialize the temperature matrix:
   - Set the size of the temperature matrix T based on the number of time steps (P), grid points in the x-direction (M), and grid points in the y-direction (N).
   - Set the initial temperature of the plate as Tinit for all grid points.
   - Apply the boundary conditions by assigning the respective temperatures TL, TR, TB, TT to the appropriate locations in the matrix.

- The values at the corner are taken as the average of the values from the surrounding edges

4. Using the backward difference method:

- Iterate over the time steps (E).
- Within each time step, iterate over the grid points in the x-direction (M) and y-direction (N).
- Calculate the second derivatives of the temperature in the x and y directions using formulas (11) and (3)
- Update the temperature at each grid point using the heat equation formula (**??**) and formula (1)

3.2 Finite Volume Method

1. Input the parameter

- Lx, Ly: Length and width of the plate
- time: Total simulation time
- D: Thermal diffusivity coefficient
- Tinit: Initial temperature of the plate
- Tt, Tb, Tl, Tr: Boundary temperatures for the left, right, bottom, and top sides of the plate

2. Define the grid

- Set the grid spacing values: dx for the x-direction and dy for the y-direction
- Create arrays X and Y to represent the x and y coordinates of the grid points
- Set the time spacing dt and calculate number of timesteps

3. Initialize the temperature matrix

- Set the size of the temperature matrix T based on the numbergrid points in the x-direction (Nx), and grid points in the y-direction (Ny)

- Set the initial temperature of the plate as Tinit for all grid points

4. Applying the define volume method
   - Iterate over the time steps
   - Within each time step, iterate over the grid points in the x-direction and y-direction
   - Update the value of temperature matrix using formula (6) to (10)

3.3 Finite Element Method

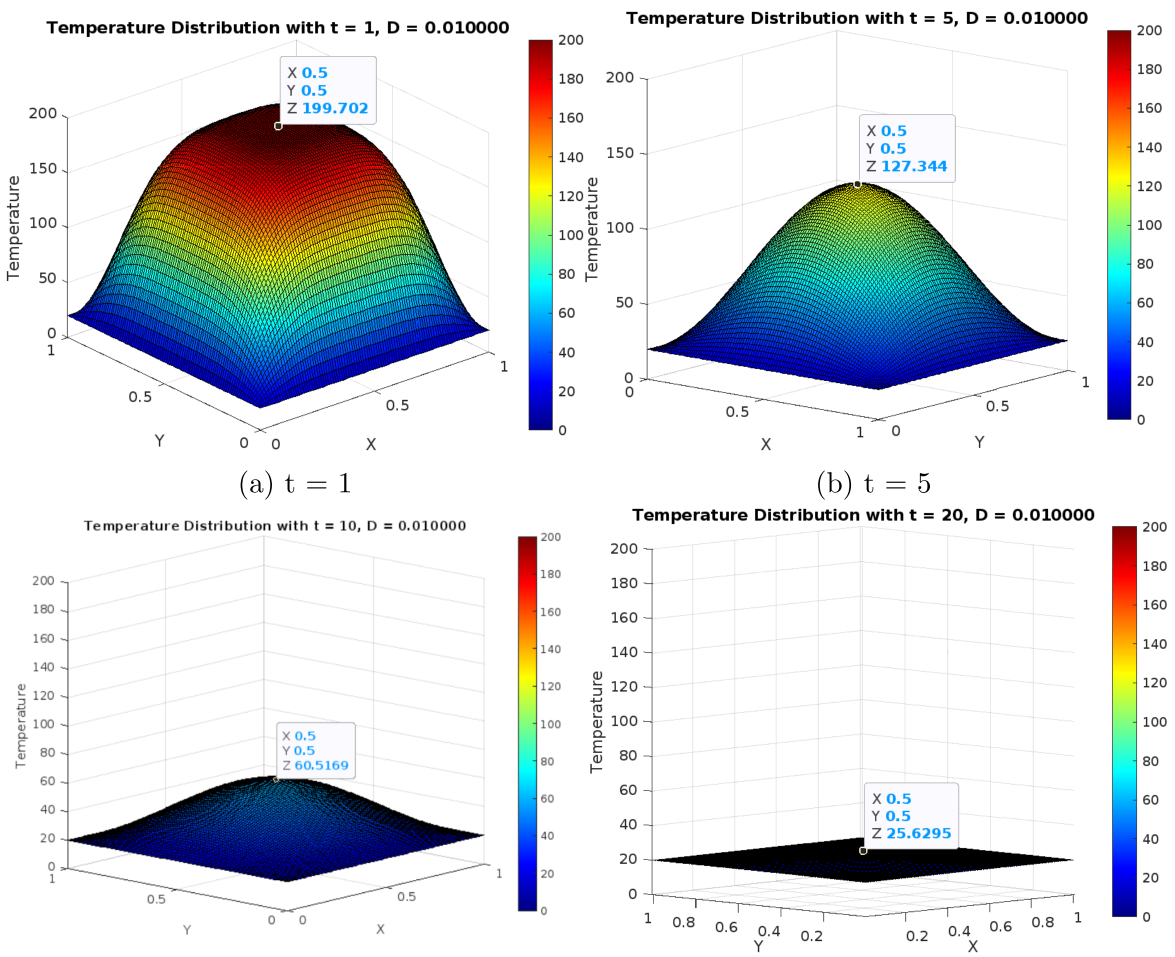
1. Input the parameters:
   - time: Total simulation time.
   - alfa : Thermal diffusivity coefficient.
   - T: Initial temperature of the plate.
2. Define the object and matrices
   - Create matrix UU to store the triangle mesh with its time steps.
   - Create matrix M for mass matrix and matrix K for stiffness matrix
   - Create an array t to represent the time steps.
3. Initialize the temperature matrix:
   - Set the size of the temperature matrix T based on the number of time steps and sizes of 2D space
   - Set the initial temperature of the plate as T with hoof-shape.
4. Using mass matrix , stiffness matrix to solve the heat equation:
   - Within each time step, update the value of mass matrix and stiffness matrix
   - Calculate the temperature using matrices and then update the UU matrix.

4 Implementation

4.1 Finite Difference Method

We use backward difference, which is suitable for decreasing function. We choose initial value of the object is 200. We choose the temperature of boundary is 20, 20, 20, 20 with respect to the top, bottom, left and right.

In MatLab, based on backward difference, we proposed the function NhietDo2D with the variables: time; thermal diffusivity D; thermal temperature of the left, right, bottom and top sides respectively as TL, TR, TB and TT; and the original temperature of the plate Tinit.



(a) t = 1 (b) t = 5

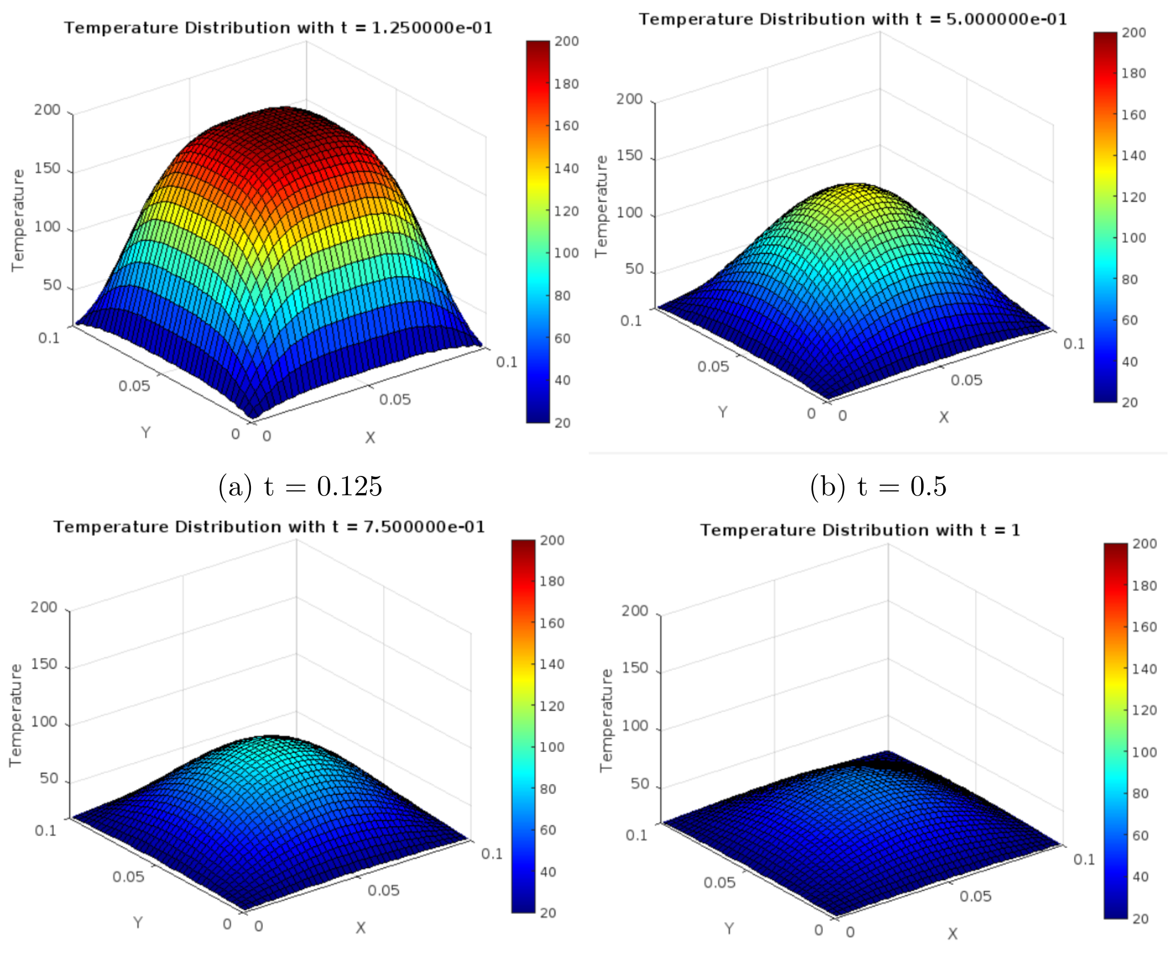
(c) t = 10 (d) t = 20

Figure 1: The result of Finite Difference Method

4.2 Finite Volume Method

We choose initial value of the object is 200, thermal diffusivity coefficient is 0.001. We choose the temperature of boundary is 20, 20, 20, 20 with respect to the top, bottom, left and right.

In MatLab, based on backward difference, we proposed the function fvm with the variables: time; thermal diffusivity D; thermal temperature of the left, right, bottom and top sides respectively as Tl, Tr, Tb and Tt and the original temperature of the plate Tinit.



(a) t = 0.125 (b) t = 0.5 (c) t = 0.75 (d) t = 1

Figure 2: The result of Finite Volume Method

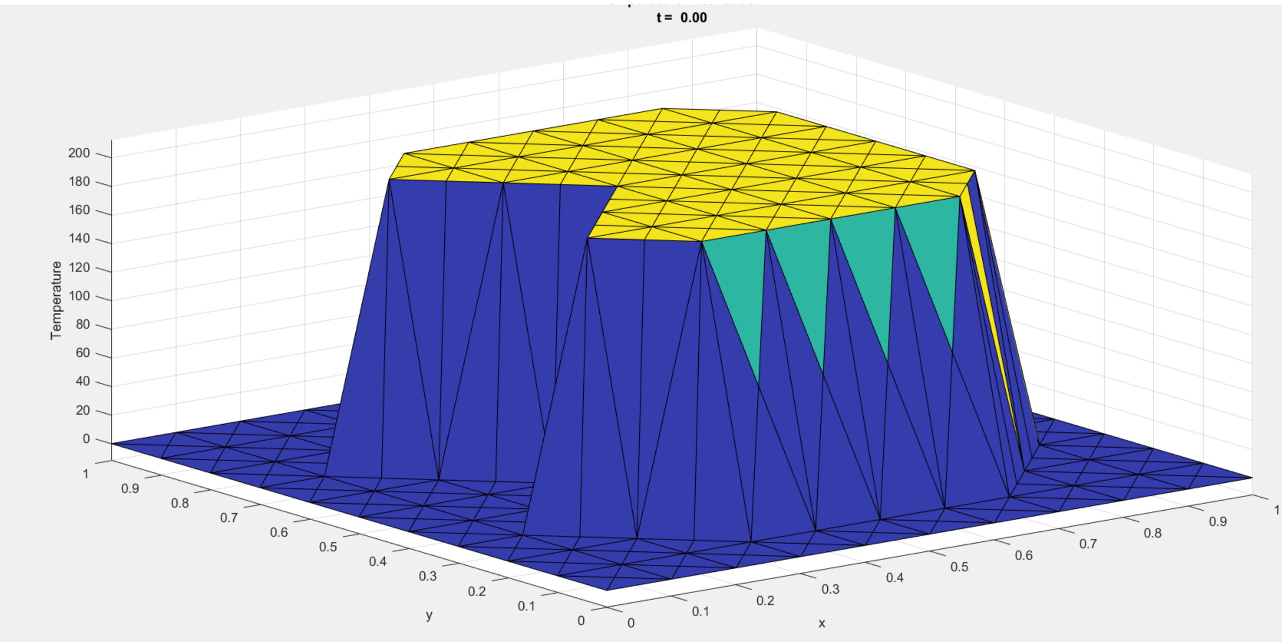
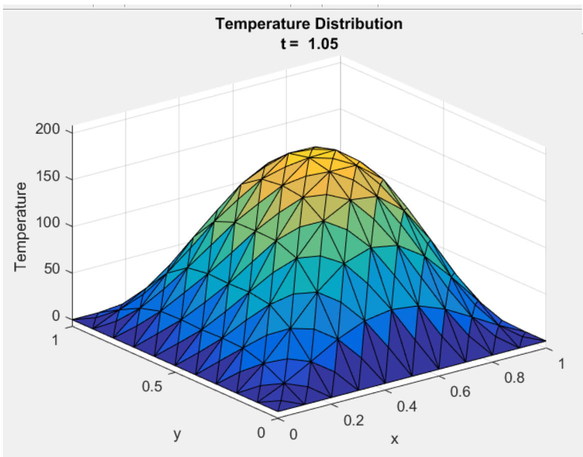


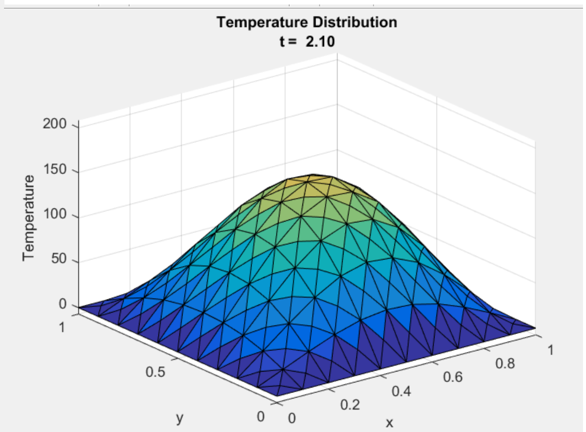
Figure 3: hoof-shaped heated region on the plate

4.3 Finite Element Method

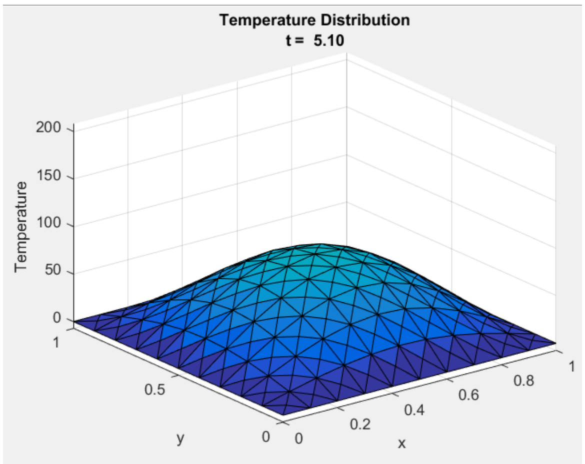
We implement Finite Element Method on a square plate heated at 200 , however , its heated area is hoof-shaped .



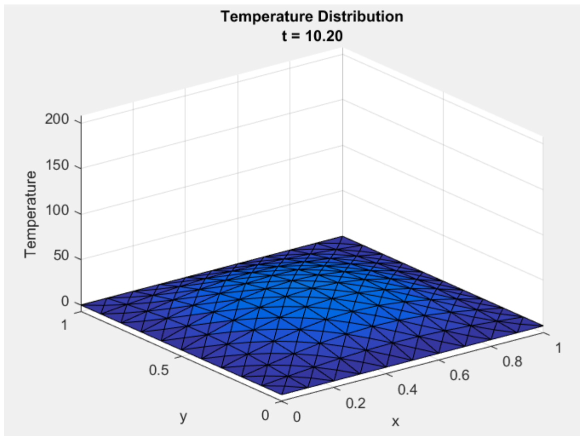
(a) t = 1.05



(b) t = 2.1



(c) t = 5.1



(d) t = 10.20

Figure 4: The result of Finite Element Method

5 Conclusion

In this report, we have investigated the heat transfer process on a rectangular plate with a given thermal diffusivity (D). Our report provided a comprehensive understanding of heat transfer on the plate with specific initial and boundary conditions. By analyzing the temperature profiles at different time intervals, we observed the evolution of heat distribution within the plate and gained insights into its transient behavior.

References

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[2] Pinghui Zhuang and Fawang Liu. Finite difference approximation for two-dimensional time fractional diffusion equation. *Journal of Algorithms & Computational Technology*, 1(1):1–16, 2007.