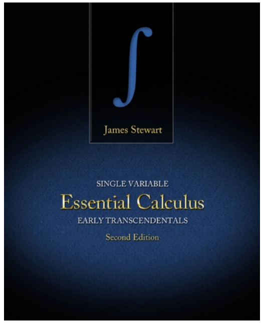
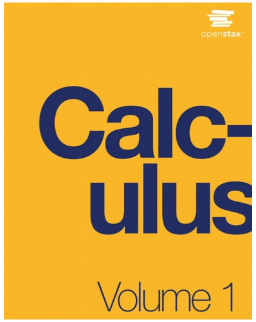
Chapter 6: Techniques of Integration





6.1 Integration by Parts

6.2 Trigonometric Integrals and Substitutions

6.3 Partial Fractions

6.4 Integration with Tables and Computer Algebra Systems

6.5 Approximate Integration

6.6 Improper Integrals

The pictures are taken from the books:

[1) James Stewart, Essential Calculus, Early Transcendentals, Cengage Learning, 2nd Edition, 2012, ISBN-13: 978-1133112280
2) G. Strang and E. J. Herman, Calculus 1, https://openstax.org/details/books/calculus-volume-1]

6.1 Integration by Parts

- Formula for Integration by parts

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

- Let $u = f(x)$ and $v = g(x)$, then

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

6.1 Examples

- Calculate the following integrals

1. $\displaystyle\int x\sin(x)\mathrm{d}x$

2. $\displaystyle\int \ln(x)\mathrm{d}x$

3. $\displaystyle\int e^x\sin(x)\mathrm{d}x$

4. $\displaystyle\int t^2e^t\mathrm{d}t$

6.1 Integration by Parts

- Formula for definite Integration by parts

$$\int_a^b f(x)g'(x)\mathrm{d}x = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\mathrm{d}x$$

Examples

1. $\displaystyle\int_0^1 \tan^{-1}(x)\mathrm{d}x$

2. $\displaystyle\int \sin^n(x)\mathrm{d}x = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\mathrm{d}x$, where $n \geq 2$

6.2 Trigonometric Integrals and Substitutions

1. $\int \cos^3(x)\mathrm{d}x$

2. $\int \sin^5(x)\cos^2(x)\mathrm{d}x$

3. $\int_0^{\pi} \sin^2(x)\mathrm{d}x$

4. $\int \sin^4(x)\mathrm{d}x$

5. $\int \tan^6(x)\sec^4(x)\mathrm{d}x$

6. $\int \tan(x)\mathrm{d}x$

7. $\int \sec(x)\mathrm{d}x$

8. $\int \tan^3(x)\mathrm{d}x$

6.2 Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

6.2 Examples

1. $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,\mathrm{d}x$

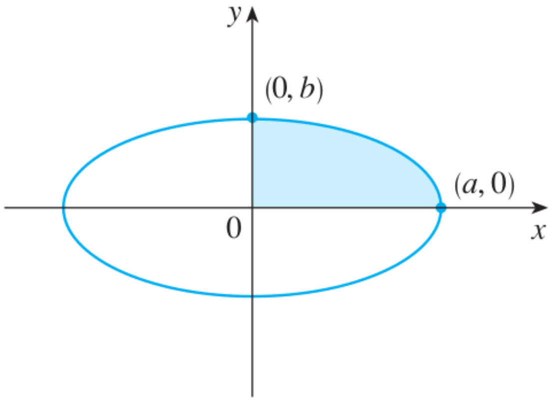
2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x$

3. $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\,\mathrm{d}x$

4. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x$

5. $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,\mathrm{d}x$

6. Find the area enclosed by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$



6.3 Partial Fractions

• Integrate any rational function (a ratio of polynomials) by expressing it as a sum of simpler fractions, called partial fractions, that we already know how to integrate.

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x+2)(x-1)} = \frac{x+5}{x^2+x-2}$$

• Reverse the procedure.

$$\int \frac{x+5}{x^2+x-2}\mathrm{d}x = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)\mathrm{d}x = 2\ln|x-1| - \ln|x+2| + C$$

6.3 Partial Fractions

- Consider the polynomials

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
$$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

with $\deg(P) > \deg(Q) \;\Leftrightarrow\; n > m$, and the integrand

$$f(x) = \frac{P(x)}{Q(x)}\,.$$

Then, the general procedure for f is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}\,, \qquad \deg(R) < \deg(Q)\,,$$

where S and R are also polynomials

6.3 Partial Fractions

- There are 4 cases to be considered:

Case I: $Q(x)$ is a product of linear factors

$$
\begin{aligned}
& Q(x) = (a_1x + b_1)(a_2x + b_2) + \cdots + (a_kx + b_k) \\
\Rightarrow \quad & \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}
\end{aligned}
$$

Example: $$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\mathrm{d}x = \int \frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)}\mathrm{d}x$$

6.3 Partial Fractions

- There are 4 cases to be considered:

Case II: $Q(x)$ is a product of linear factors, some of which are repeated.

$$Q(x) = (a_1x + b_1)^2(a_2x + b_2)^3$$

$$\Rightarrow \quad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \frac{B_1}{a_2x + b_2} + \frac{B_2}{(a_2x + b_2)^2} + \frac{B_3}{(a_2x + b_2)^3}$$

Example: $$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\mathrm{d}x = \int \frac{x^4 - 2x^2 + 4x + 1}{(x-1)^2(x+1)}\mathrm{d}x$$

6.3 Partial Fractions

- There are 4 cases to be considered:

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

$$Q(x) = (a_1x + b_1)(a_2x^2 + b_2)(a_3x^2 + b_3)$$

$$\Rightarrow \quad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{B_1x + B_2}{a_2x^2 + b_2} + \frac{C_1x + C_2}{a_3x^2 + b_3}$$

Example:

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} \mathrm{d}x = \int \frac{2x^2 - x + 4}{x(x^2 + 4)} \mathrm{d}x$$

6.3 Partial Fractions

- There are 4 cases to be considered:

Case IV: $Q(x)$ contains a repeated irreducible quadratic factor.

$$Q(x) = (a_1x + b_1)(a_2x^2 + b_2x + c_2)(a_3x^2 + b_3)^3$$

$$\Rightarrow \quad \frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{B_1x + B_2}{a_2x^2 + b_2x + c_2} + \frac{C_1x + C_2}{a_3x^2 + b_3} + \frac{D_1x + D_2}{(a_3x^2 + b_3)^2} + \frac{E_1x + E_2}{(a_3x^2 + b_3)^3}$$

Example:

$$\int \frac{1 - x + 2x^2 - x^3}{x + 2x^3 + x^5}\mathrm{d}x = \int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2}\mathrm{d}x$$

6.4 Integration with Tables and Computer Algebra Systems

• There are extensive **tables** with **hunderts of pages** of indefinite integrals. It should be remembered, however, that integrals do not often occur in exactly the form listed in a table.

Examples:

1. $I_1 = \displaystyle\int \frac{x^2}{\sqrt{5-4x^2}}\mathrm{d}x$

2. $I_2 = \displaystyle\int x^3 \sin(x)\mathrm{d}x$

3. $I_3 = \displaystyle\int x\sqrt{x^2+2x+4}\mathrm{d}x$

6.4 Integration with Tables and Computer Algebra Systems

- Use of software: Derive, Mathematica, Maple

Caution! A hand computation sometimes produces an indefinite integral in a form that is more convenient than a machine answer.

Example:

For the function $f(x) = x^2 + 2x + 4$ we have

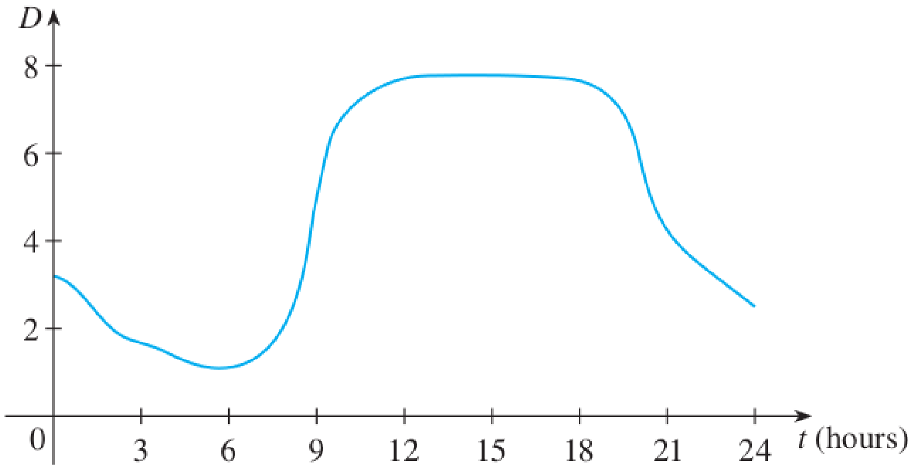
$$\begin{aligned}
\int x f(x) \mathrm{d}x &= \frac{1}{3}[f(x)]^{3/2} - \frac{1+x}{2}\sqrt{f(x)} - \frac{3}{2}\ln\left(x + 1 + f(x)\right) + C \quad \text{(by hand)} \\
&= \frac{1}{3}[f(x)]^{3/2} - \frac{2x+2}{4}\sqrt{f(x)} - \frac{3}{2}\sinh^{-1}\left[\frac{\sqrt{3}}{3}(1+x)\right] \quad \text{(Maple)} \\
&= \left(\frac{5}{6} + \frac{x}{6} + \frac{x^2}{3}\right)\sqrt{f(x)} - \frac{3}{2}\sinh^{-1}\left(\frac{1+x}{\sqrt{3}}\right) \quad \text{(Mathematica)}
\end{aligned}$$

6.5 Approximate Integration

• There are two situations in which it is impossible to find the exact value of a definite integral:

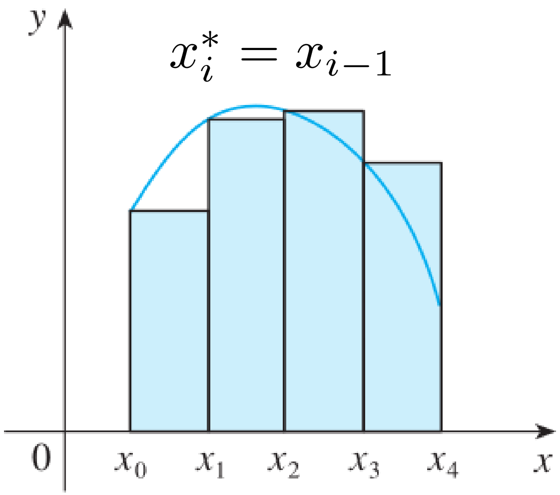
i. Unknown Antiderivative, $f(x) = \int_0^1 e^{x^2} \mathrm{d}x\,,$ $g(x) = \int_{-1}^1 \sqrt{1+x^3}\mathrm{d}x$

ii. No formula for experimental data,

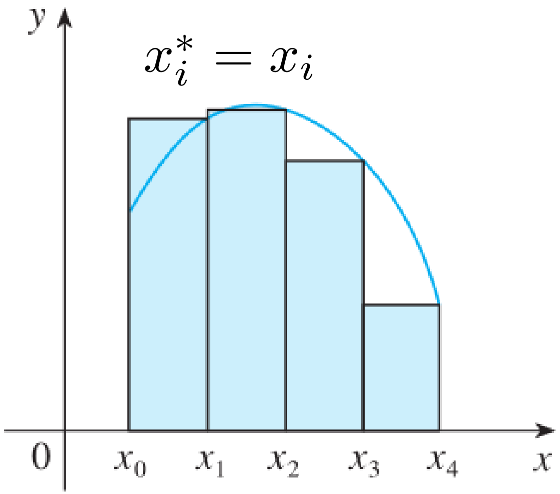


6.5 Approximate Integration

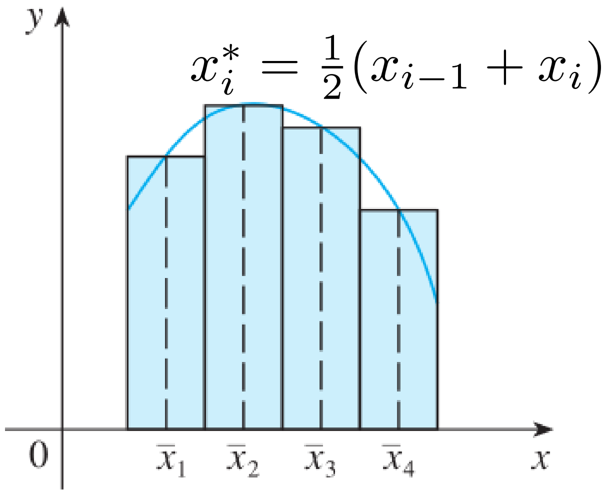
• Numerical Solution: Any **Riemann sum** could be used as an **approximation** to the integral, $\displaystyle\int_a^b f(x)\mathrm{d}x \approx \sum_{i=1}^{n} f(x_i^*)\Delta x,$ with $\Delta x = \dfrac{b-a}{n}$



Left Endpoint



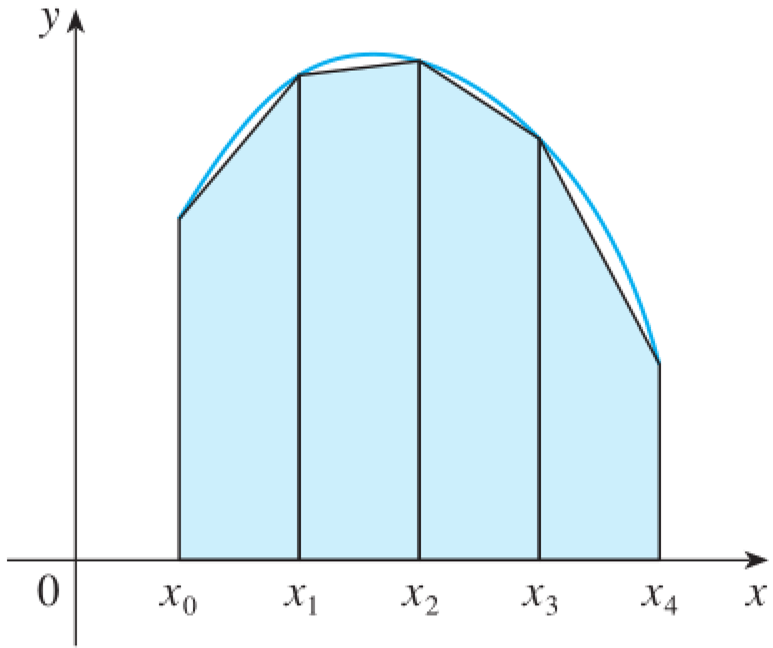
Right Endpoint



Midpoint

6.5 Approximate Integration

- Trapezoidal Rule: $\displaystyle\int_a^b f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \frac{1}{2}\Big[f(x_{i-1}) + f(x_i)\Big]\Delta x$



where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

6.5 Approximate Integration

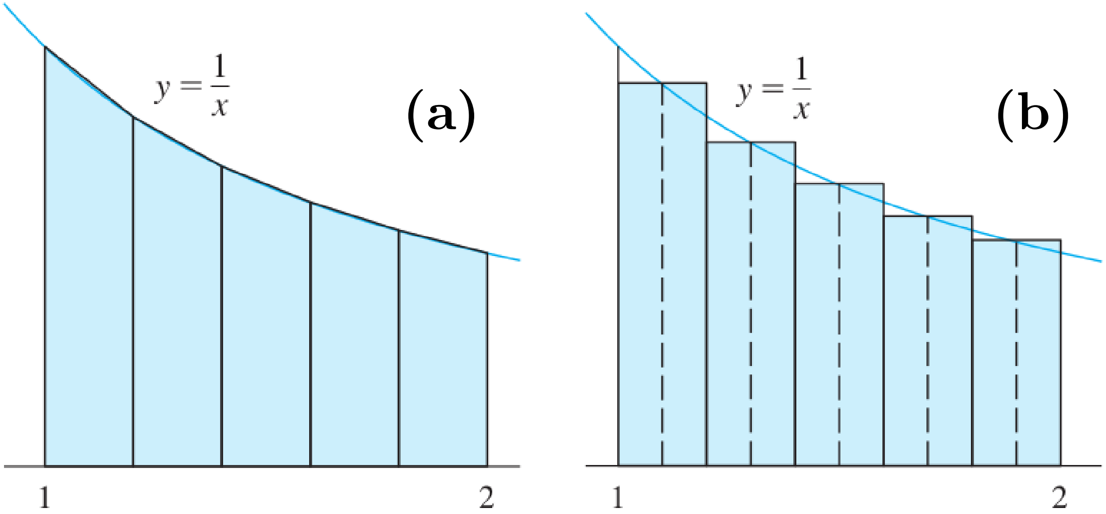
- The **Error** : $\displaystyle\int_a^b f(x)\mathrm{d}x = \text{Approximation} + \text{Error}$

Error Bounds: Suppose $|f''(x)| < K$ for $a \leq x \leq b$. If E_T and E_M are the errors in the **Trapezoidal** and **Midpoint Rules**, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

6.5 Example

- Use **(a)** the Trapezoidal Rule and **(b)** the Midpoint Rule with $n = 5$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\mathrm{d}x$.



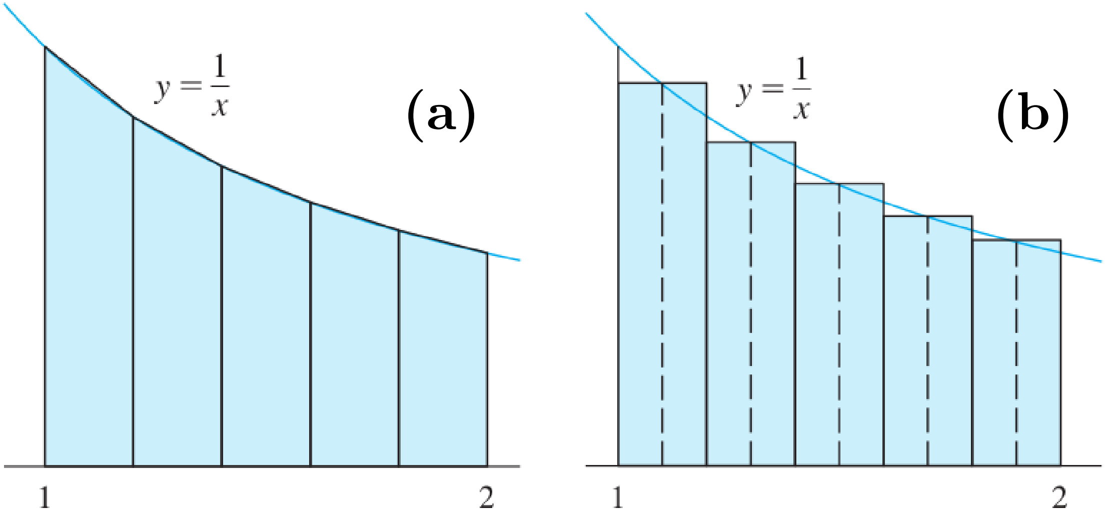
$$\int_1^2 \frac{1}{x}\mathrm{d}x = \ln(2) = 0.693147\ldots$$

n	L_n	R_n	T_n	M_n
5	0.745635	0.645635	0.695635	0.691908
10	0.718771	0.668771	0.693771	0.692835
20	0.705803	0.680803	0.693303	0.693069

6.5 Example

- Use **(a)** the Trapezoidal Rule and **(b)** the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 \frac{1}{x}\mathrm{d}x$.

$$\int_1^2 \frac{1}{x}\mathrm{d}x = \ln(2) = 0.693147\ldots$$



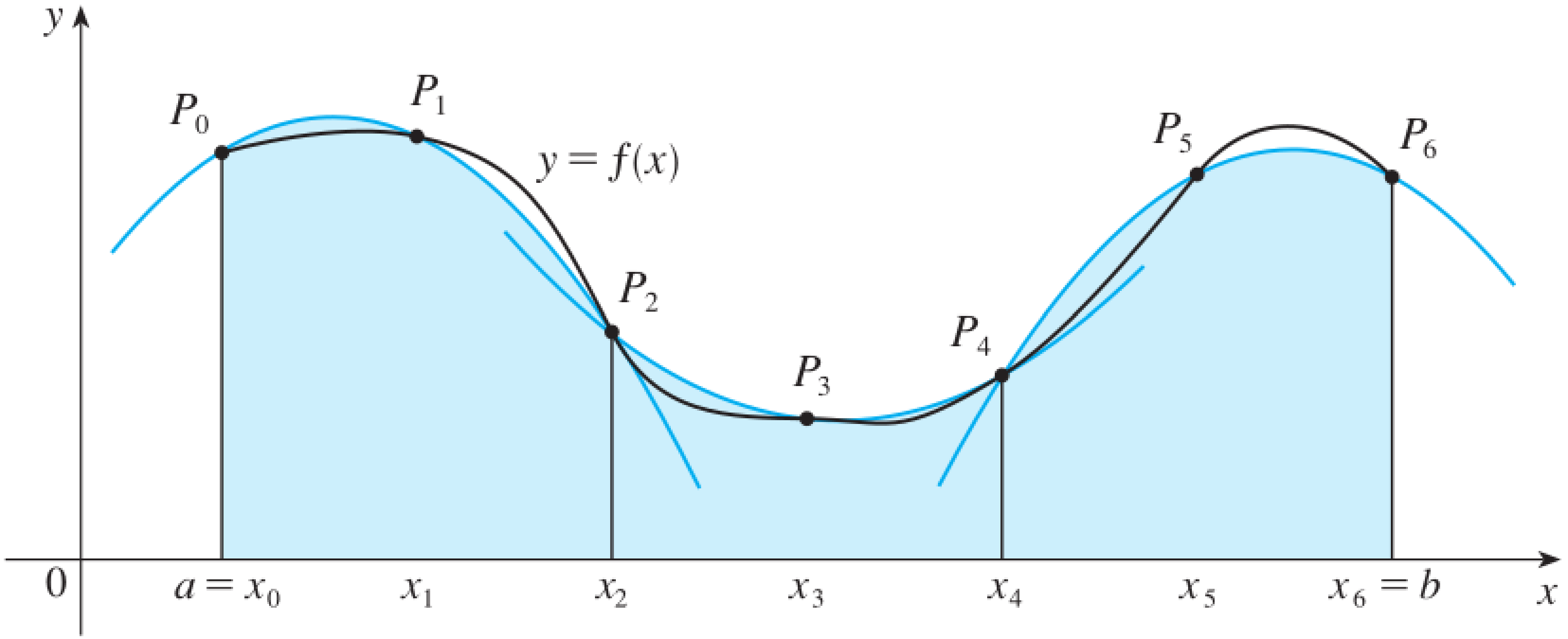
n	E_L	E_R	E_T	E_M
5	-0.052488	0.047512	-0.002488	0.001239
10	-0.025624	0.024376	-0.000624	0.000312
20	-0.012656	0.012344	-0.000156	0.000078

6.5 Example

- How large should we take n in order to guarantee that the **Trapezoidal** and **Midpoint Rule** approximations for $\displaystyle\int_1^2 \frac{1}{x}\mathrm{d}x$ are accurate to within 0.0001?

6.5 Approximate Integration

- Simpson's Rule: Approximate a curve by a **parabola**



6.5 Approximate Integration

- Simpson's Rule: Approximate a curve by a **parabola**

$$\int_a^b f(x)\mathrm{d}x \approx \frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3) + \cdots + 2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]$$

Error Bound: Suppose $|f^{(4)}(x)| < K$ for $a \le x \le b$. If E_S is the errors in the **Simpson's Rule**, then

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

6.5 Example

- How large should we take n in order to guarantee that the **Simpson's Rule** approximations for $\displaystyle\int_1^2 \frac{1}{x}\mathrm{d}x$ are accurate to within 0.0001?

6.6 Improper Integrals

Improper Integrals of TYPE I:

1. If $f(x)$ is continuous on $[a,\infty)$ then: $\displaystyle\int_a^\infty f(x)\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\mathrm{d}x$

2. If $f(x)$ is continuous on $(-\infty,b]$ then: $\displaystyle\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\mathrm{d}x$

3. If $f(x)$ is continuous on $(-\infty,\infty)$ then: $\displaystyle\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^c f(x)\mathrm{d}x + \int_c^\infty f(x)\mathrm{d}x$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral **diverges**.

6.6 Examples

1. For what values of p is the following integral convergent?

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$$

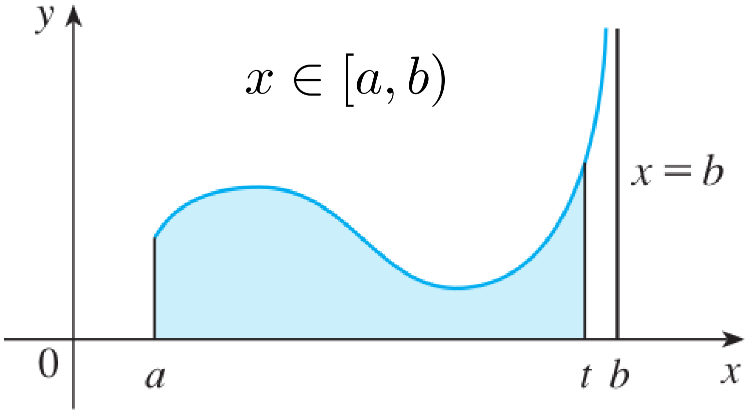
2. Evaluate $\displaystyle\int_{-\infty}^0 x\,e^x\,\mathrm{d}x$

3. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

6.6 Improper Integrals

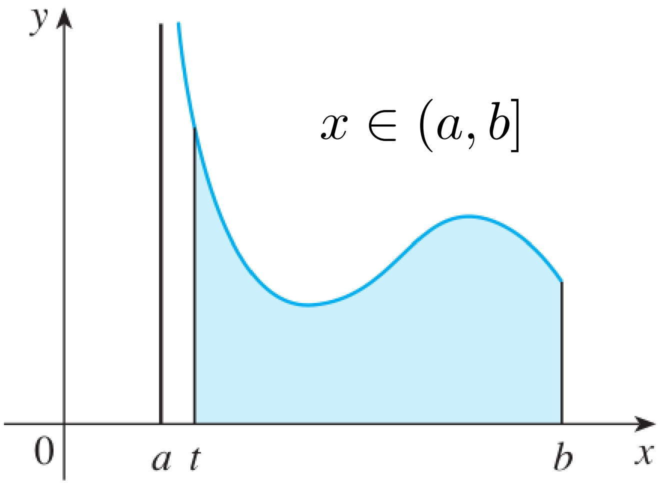
Improper Integrals of TYPE II: Discontinuous Integrands

$x \in [a, b)$



$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to b} \int_a^t f(x)\mathrm{d}x$$

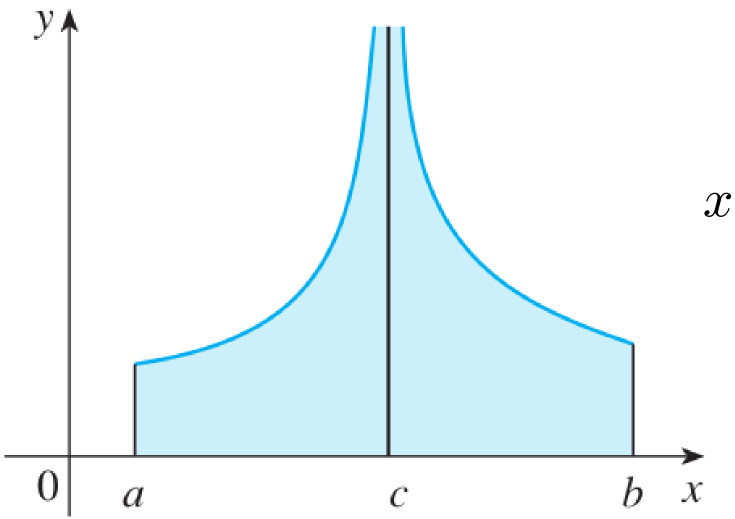
$x \in (a, b]$



$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to a} \int_t^b f(x)\mathrm{d}x$$

6.6 Improper Integrals

Improper Integrals of TYPE II: Discontinuous Integrands



$$x \in (a, c) \cup (c, b)$$

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{c} f(x)\mathrm{d}x + \int_{c}^{\infty} f(x)\mathrm{d}x$$

6.6 Examples

1. Find $\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$

2. Determine whether $\int_0^{\pi/2} \sec(x)\mathrm{d}x$ converges or diverges.

3. Evaluate $\int_0^3 \frac{\mathrm{d}x}{x-1}$ if possible

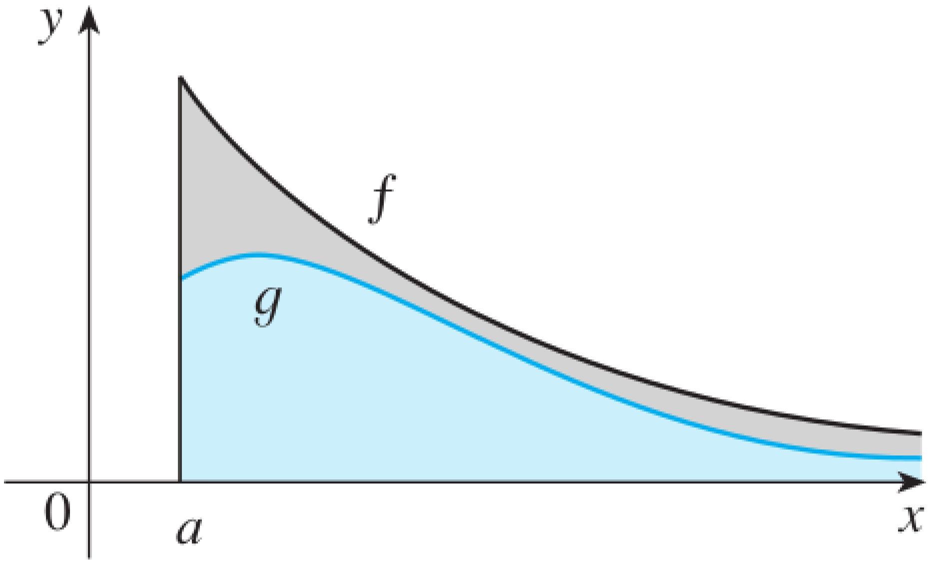
6.6 Improper Integrals

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

a. If $\int_a^\infty f(x)\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\mathrm{d}x$ is convergent.

b. If $\int_a^\infty g(x)\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\mathrm{d}x$ is divergent.



6.6 Examples

1. Show that $\displaystyle\int_0^\infty e^{-x^2}\,\mathrm{d}x$ is convergent.

2. Show that the integral $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,\mathrm{d}x$ diverges.