HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF ENGINEERING PHYSICS
DEPARTMENT OF GENERAL PHYSICS

PH1110E

ALSO PH1016, ETC.

GENERAL PHYSICS — PH1110E

THEORY SUMMARY

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Contents

Preface

To all of my friends who are using this document,

This document is written to summarize all the theories and give a recommended solution for the exercise of the subject General Physics (generally known as Physics 1 - Mechanics and Thermodynamics), a course in the curriculum of students of ELiTECH Program at Hanoi University of Science and Technology. Although I studied the course with code PH1110E, this document is also applicable to students who study PH1016, and so on. I hope you all use it as a reference and support while studying, and do not use it as a tool in order not to focus on the lecturers.

To write this document, I would like to give many thanks to professor Le Ba Nam, for his instructions and course materials, and many other professors at the School of Engineering Physics, Hanoi University of Science and Technology about their public documents about Statistics and Probability. Thanks to the authors of *Physics: principles with applications*, D.C. Giancoli and *Fundamentals of physics*, Halliday & Resnick for their books as reference. Moreover, I also want to say thanks to several universities in the world such as Indiana University–Purdue University Indianapolis, Massachusetts Institute of Technology, the University of Toronto, etc. because of their helpful public documents.

Because this document is written in such a short time and all the summaries are made by me, it is so hard to make sure it is completely correct. I would love to receive all of your contributions to my document via **minh.tn214918@sis.hust.edu.vn**.

Thank you for using this document.

Ngoc-Minh Ta

Part I

MECHANICS

Chapter 1

Preliminaries.

1.1 International System of Units (SI).

- There are seven primary unit in SI, which are:

Quantity	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	Ampere	A
Thermodynamic temperature	Kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

- **Consistency of Equations:** If we have $A = B$, so A and B have the same combination of units. There are some units that do not belong to SI system, but they have their own names.

Example

Unit of force – **Newton** (N): Following the Newton's second law, we have $\vec{F} = m\vec{a}$, so, the unit of force is Newton and $1N = 1kgms^{-1}$.

- **Multiples of units:** 10^6 – mega (M); 10^3 – kilo (k); 10^2 – deci (d); 10^1 – centi (c); ...

1.2 Significant figures.

- Significant figures of a number in positional notation are digits in the number that are reliable and necessary to indicate the quantity of something.
- There are five rules of significant figures:

The Five Rule

1. All non-zero digits are significant.
2. Zeros appear anywhere between two non-zero digits are significant.
3. Leading zeros are not significant. (e.g., 0.012 has two significant figure)
4. Trailing zeros in a number containing a decimal point are significant.
5. The significant of trailing zeros in a number not containing a decimal point can be ambiguous.

Example

- 3 has one significant figure.
- 3.0 has two significant figures.
- 3.0010 has three significant figures.

1.3 Uncertainties in indirect measurements.

- There are two methods of measuring a physics value: direct and indirect. While we can read the result of measurement directly in the measuring equipment by using direct measurement, we must calculate the demanding value using a formula of several known variables in indirect measurements.
- The result we get always has many uncertainties and errors. They can be divided into two types of uncertainties and errors:
  - **Systematic error** is consistent, repeatable error associated with faulty equipment or a flawed experiment design.
  - **Random error** (also called **unsystematic error**, **system noise** or **random variation**) has no pattern. They are unpredictable and can't be replicated by repeating the experiment again.
- When measuring a physics value by direct measurement, we can calculate the uncertainty by following these steps:

Calculate the uncertainty

1. We measure the object n times, get the n values of each time are $a_1, a_2, ..., a_n$. Let a is the exact value of the object. The average value of measurement is $\bar{a} = \frac{1}{n}. \sum_{i=1} n a_i$.
2. Calculate the uncertainties in each measuring time are $\delta a_i = |a_i - a|, \forall i = \overline{1,n}$. So, the average uncertainty of measuring process is $\Delta\bar{a} = \frac{1}{n}. \sum_{i=1} n \Delta a_i$. Denote $a_e q$ be the uncertainty of the equipment.
3. The absolute uncertainty of measurement is calculated by $\Delta a = \Delta\bar{a} + \Delta a_e q$. The relative uncertainty of measurement is denoted by $\delta = \frac{\Delta a}{a}$ and written in percentage form.
4. The result of measured value is written as $A = \bar{a} \pm \Delta a$ or $A = \bar{a} \pm \delta a$.

- In indirect measurement, we cannot calculate the uncertainty directly. The way how to calculate the uncertainty is divided into two cases:

Case 1: The demanding value is given by the formula $A = xX + yY + zZ$

1. To calculate the uncertainty, we apply the derivative of both sides: $\mathrm{d}A = x\mathrm{d}X + y\mathrm{d}Y + z\mathrm{d}Z$.
2. Denote the uncertainty is ΔA, so if we replace d by Δ and – by +, we get: $\Delta A = \Delta X + \Delta Y + \Delta Z$.

Case 2: The demanding value is given by the formula $B = \frac{XY^2}{Z}$

1. Apply the nature logarithm in both sides of the function and apply the derivative of them, we get:

$$\ln B = \ln X + 2\ln Y - \ln Z \Rightarrow \frac{\mathrm{d}B}{B} = \frac{\mathrm{d}X}{X} + 2\frac{\mathrm{d}Y}{Y} - \frac{\mathrm{d}Z}{Z}$$

2. Denote the uncertainty is ΔA, so if we replace d by Δ and – by +, we get: $\delta B = \delta X + 2\delta Y + \delta Z$.

1.4 Round up, reading and write the result.

- In reading the value of an object using the equipment, if the scale is not too fine, you can estimate the value of that object. But in a fine scale, just read the lower and nearest number and DO NOT estimate that value.
- The result of measurement is written as $a = \bar{a} + \Delta a$ and it follows two rules of round-up:
    1. Keep just two significant figures. The removal is less than or equal 10% of the measured value.
    2. The average value must be written in the form that ended at the same position with the ending of the uncertainty.

Example

- $D = \bar{D} \pm = (21.48 \pm 0.16)\mathrm{mm}$.
- $\rho = (8876.15 \pm 119.23)\mathrm{kgm}^3 \Rightarrow$ Result: $\rho = (8880 \pm 120)\mathrm{kgm}^3$ or $\rho = (8900 \pm 100)\mathrm{kgm}^3$

Chapter 2

Kinematics.

2.1 Starting points.

- Definition: A reference frame (or frame of reference) includes an origin, a coordinate system, and a clock.
- Position of a point in 3D space is completely determined by three coordinates.
  - In Cartesian coordinate system: (x, y, z)
  - In spherical coordinate system: (r, θ, φ)
  - In cylindrical coordinate system: (r, φ, z)
- The position can be determined by the position vector from the origin to the point. An object can be considered as a point particle if its size is very small as compared to the size of the space of motion.
- **Definition:** The orbit (or trajectory) of a motion is the path in which the object travels.

2.2 Speed and velocity.

- **Definition:** Consider the motion of an object from point M to point N.
  - Distance moved Δs is the length of the path MN.
  - Displacement is $\overrightarrow{MN} = \vec{r}_M - \vec{r}_N = \Delta\vec{r}$.
- **Average speed and average velocity**
  - Average speed $= \dfrac{\text{Distance moved}}{\text{Time taken}} \Leftrightarrow \bar{v} = \frac{\Delta s}{\Delta t}$. So, speed is a scalar unit.
  - Average velocity $= \dfrac{\text{Displacement}}{\text{Time taken}} \Leftrightarrow \vec{v} = \frac{\Delta\vec{r}}{\Delta t}$. So, velocity is a vector unit.
- Generally, $\Delta s \neq |\Delta r|$. Therefore, the average speed is not the same as the magnitude of the average velocity.
- **Instantaneous speed and instantaneous velocity**
  - Instantaneous speed: $v = \displaystyle\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \dot{s}$
  - Instantaneous velocity: $\vec{v} = \displaystyle\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \dot{\vec{r}}$
  - Since $\mathrm{d}s = |\mathrm{d}\vec{r}|$, the instantaneous speed is equal to the magnitude of the instantaneous velocity $\Rightarrow v = \dfrac{|\mathrm{d}\vec{r}|}{\mathrm{d}t} = \dfrac{\mathrm{d}s}{\mathrm{d}t}$.
- **Notice that:** $|\mathrm{d}\vec{r}| = \mathrm{d}r$.

- In Cartesian coordinates: $\begin{cases} \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{cases}$.

  Since $\vec{v} = \dot{\vec{r}}$, so $v_x = \dot{x} = \dfrac{dx}{dt}$, $v_y = \dot{y} = \dfrac{dy}{dt}$ and $v_z = \dot{z} = \dfrac{dz}{dt}$, we have $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

2.3 Acceleration.

- Average acceleration = $\dfrac{\text{Change in velocity}}{\text{Time taken}} \Leftrightarrow \vec{a}_a = \frac{\Delta\vec{v}}{\Delta t}$
- Instantaneous acceleration: $\vec{a} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\vec{v}}{\Delta t} = \dot{\vec{v}}$.
- In Cartesian coordinates: $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$. Since $a_x = \dfrac{dv_x}{dt} = \dfrac{d^2x}{dt}$, $a_y = \dot{v}_y = \ddot{y}$, $a_z = \dot{v}_z = \ddot{z}$, so we have $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$.

Note

- $|d\vec{v}| \neq dv \Rightarrow a \neq \dfrac{dv}{dt}$ (In general).
- Velocity is always tangential to the orbit, but the acceleration can be not tangential.
- In general case, $\vec{a}$ has both tangential component $\vec{a}_t$, and normal component $\vec{a}_n$:

$$\vec{a} = \vec{a}_t + \vec{a}_n \Rightarrow a = \sqrt{a_t^2 + a_n^2}$$

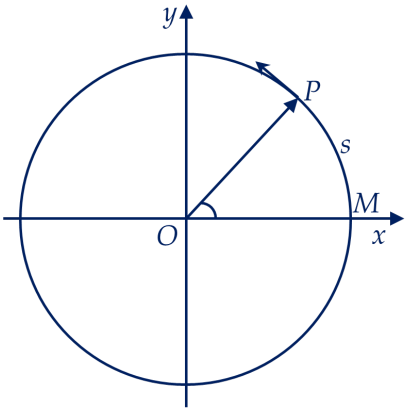
Special case of acceleration

- In a **linear motion**, there is a tangential acceleration only: $a_t = \dot{v}$
- In a **uniform circular motion**, $dv = 0$ and $a_t = 0$, there is normal acceleration only: $a_n = \dfrac{v^2}{R}$ with R is the radius of the circular orbit.
- Any planar motion can be decomposed into linear motion and a uniform circular motion. Hence, its $\vec{a}_t$ and $\vec{a}_n$ have the same formulae as shown above. Therefore, $a = \sqrt{v^2 + \left(\dfrac{v^2}{R}\right)^2}$

2.4 Uniformly accelerated linear motion (UALM).

- Uniformly accelerated linear motion (UALM) is motion in a straight line with constant acceleration.
- We have $\begin{cases} v = v_0 + at \\ x = x_0 + v_0 t + \dfrac{1}{2}at^2 \end{cases}$
  - Accelerating motion: $av > 0$.
  - Decelerating motion: $av < 0$.
  - Relation without time t: $v^2 - v_0^2 = 2a(x - x_0)$.

2.5 Circular motion.



- Consider the motion of a point particle P in the circle (O, r):
  - The circle intersects with x-axis at M.
  - The position vector $\vec{r} = \overrightarrow{OP}$, forms an angle θ to the x-axis.
  - Let s be the length of the arc OP: $s = r\theta$.
- We use Cartesian and polar coordinates:

$$\vec{r} = r.\hat{r} = x\hat{x} + y\hat{y} = r.\cos\theta\hat{x} + r.\sin\theta\hat{y}$$

- Unit vector in radial direction: $\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$. So $\vec{v} = -r.\dot{\theta}\sin\theta\hat{x} + r.\theta\cos\theta\hat{y}$.
- Therefore, unit vector in tangential direction: $\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}$
  $\Rightarrow \vec{a} = -r.\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta}$.
- **Angular velocity** $\vec{\omega}$ is defined as a vector which
  - has the size is equal to the angular speed,
  - lies on the axis of rotation, and
  - has direction determined by the **right-hand rule**.

Right-hand rule

Wrap tour right hand around the rotation axis, so that your fingers point in the direction of motion. The angular velocity is in the direction pointed by your thumb.

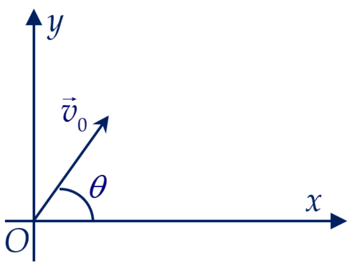
- Since the three vectors $\vec{v}, \vec{\omega}, \vec{r}$ are perpendicular to each other, and $v = r\omega \Rightarrow \vec{v} = \vec{\omega} \times \vec{r}$ or $\vec{\omega} = \dfrac{\vec{r} \times \vec{v}}{r^2}$.
- **The angular acceleration vector** is $\vec{\beta} = \dfrac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} = \dfrac{\vec{r} \times \vec{a}_t}{r^2} = \dfrac{\hat{r} \times \vec{a}_t}{r}$. So, $\beta = \dfrac{a_t}{r}$.

2.6 Projectile motion.

Example

- Motion of a mass in a uniform gravitational field.
- Motion of a charge in a uniform electric field.

- Projectile motion is the motion whose acceleration is constant, in both magnitude and direction, and is not always in the direction of motion.



- According to Galileo Galilei, any planar motion can be decomposed into two motions in two perpendicular directions. We consider the motion of a mass in the uniform gravitational field of the Earth. The initial velocity v_0 forms an angle θ to the horizontal.
  - The motion can be decomposed into a horizontal motion in the x-axis and a vertical motion in the y-axis:

$$\begin{cases} a_x = 0,\ v_{0x} = v_0 \cos\theta,\ x_0 = 0 & \rightarrow \text{uniform motion} \\ a_y = -g,\ v_{0y} = v_0 \sin\theta,\ y_0 = 0 & \rightarrow \text{uniformly accelerated motion} \end{cases}$$

  - Components of velocity: $\begin{cases} v_x = v_0 \cos\theta \\ v_y = v_0 \sin\theta - gt \end{cases}$.
  - Equation of motion: $\begin{cases} x = (v_0 \cos\theta)t \\ y = (v_0 \sin\theta)t - \dfrac{1}{2}gt^2 \end{cases}$.
  - Orbital equation: $y = (\tan\theta)x - \dfrac{g}{2v_0^2 \cos^2\theta}x^2$.
  - From these equations, we can determine, e.g., the maximum height of the motion.

Chapter 3

Dynamics.

3.1 Newton's laws of motion.

Newton's first law of motion

Every object continues in its state of rest, or uniform velocity, unless acted on by a net force. The tendency of an object to resist changes in its velocity is called the inertia. In other words, inertia is the resistance to change in motion.

- As a result, the first law is also called **law of inertia**.
- We consider an event: A school bus comes to a sudden stop, a backpack on the floor starts to slide forward. By Newton's first law, the backpack continues its state of motion, maintaining its velocity with respect to the Earth.
  However, the backpack does move with respect to the bus although there was no force pushing it forward. It means that the law of inertia does not hold if your frame was the bus, an accelerating frame of reference. Such a frame is called a non-inertial frame.
- In conclusion, Newton's first law **only hold in inertial frames of reference**.
- **Inertial frame and non-inertial frame.**
  - In classic mechanics, we use the Newton's first law to define that: An inertial reference frame is a frame in which bodies, whose net force acting upon them is zero, are not accelerated. In fact, all the three Newton's laws of motion only hold in inertial frames of reference.

    Note

    * The Earth can approximately be considered as an inertial frame.
    * A frame which moves at constant velocity on Earth is an inertial frame.
    * An accelerating frame on Earth is a non-inertial frame.

  - **Definition: Mass** is a measure of **the inertial of an object**. In other words, mass represents for the resistance to change in motion of the object.

Newton's second law of motion

The acceleration of an object is directly proportional to the net force acting on it, and is inversely proportional to its mass. The direction of the acceleration is in the direction of the net force acting on the object.

The formula of the law is $\vec{F} = m\vec{a}$ or $\vec{a} = \dfrac{\vec{F}}{m}$. It is **valid only in inertial frames of reference**.

- $\vec{F}$ is the resultant force. In case the object is acted by several forces: $\vec{F} = \vec{F}_1 + \vec{F}_2 + ...$

Newton's third law of motion

Whenever one body exerts a force on the another, the second body exerts an equal and opposite force on the first.
That means, if we denote $\vec{F}_{AB}$ be the force that A exerts on B, we have $\vec{F}_{AB} = -\vec{F}_{BA}$. If we call $\vec{F}_{AB}$ be an action force, $\vec{F}_{BA}$ will be a reaction force.

- **Convention:** The first index is for the object which gives the force and the second for the object which is acted by the force. The point of action is on the object which is acted. $\vec{F}_{AB}$ and $\vec{F}_{BA}$ form a pair of interacting forces. They do not balance each other.

3.2 Galilean relativity.

Definition

A coordinate transformation relates the coordinates of the same event on different coordinates systems. Each event happens at some point in space and some instant in time. Therefore, **coordinates = special coordinates + time**.

- A single arbitrary event has different coordinates in different coordinate systems:
  - Given (x, y, z, t) in the system S, and (x', y', z', t') in the system S′.
  - In case S and S′ are in uniform relative motion, the Galilean transformation can be used to transform between (x, y, z, t) and (x', y', z', t').
  - For simplicity, suppose that S′ initially coincides with S. S′ moves with respect to S at constant velocity V in x-axis. We have: $\begin{cases} x' = x - Vt \\ y' = y,\ z' = z,\ t' = t \end{cases}$.
  - Here, time interval and special distance are absolute.
- **Velocity addition**
  - Let O be the origin of S and O′ be the origin of S′. We have $\vec{R} = \overrightarrow{OO'}, \vec{V} = \dfrac{d\vec{R}}{dt}$.
  - Let $\vec{r}$ and $\vec{r}'$ be the position vectors of the same particle P in the two systems S and S′.
  - **Galilean transformations:** From $\vec{r} = \vec{r}' + \vec{R}, t = t'$, we have:

$$\boxed{\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{R}}{dt} \Rightarrow \frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt'} + \frac{d\vec{R}}{dt} \Rightarrow \vec{v} = \vec{v}' + \vec{V}}$$

  - So, the Galilean transformation gives rise to the **law of velocity addition**.
- Suppose that S is an inertial frame, and S′ moves at constant velocity with respect to S. Then S′ is another inertial frame.
  - Newton's second law in S: $\vec{F} = m\vec{a}$.
  - $\vec{V} = \text{const} \Rightarrow \vec{A} = 0 \Rightarrow \vec{a} = \vec{a}'$.
  - Therefore: $\vec{F} = m\vec{a} \rightarrow$ Newton's second law in S′.

- As Newton's second law is the same in S and S′, all Newton's laws of motion are the same in the two systems.

- **Inertial force**:
  - If S is an inertial frame and S′ moves at acceleration $\vec{A}$ with respect to S, then S′ is a **non-inertial frame**.
    * Newton's second law in S: $\vec{F} = m\vec{a}$
      Here, $\vec{F}$ is the resultant force of all the **real forces** acting on the particle.
    * Since $\vec{a} = \vec{a}' + \vec{A}$, the equation in S′ is $\vec{F} = m\left(\vec{a}' + \vec{A}\right)$.
    * Recall that the form of Newton's second law is net force $=$ mass $\times$ acceleration. Clearly, Newton's second law does not hold in S′.
    * In order to write the equation in S′ in the form of Newton's second law, $\vec{F}' = m\vec{a}'$, we have to accept that the net force is $\vec{F}' = \vec{F} - m\vec{A}$.
    * So, in S′, beside being acted by the real forces, the particle is acted by the force $\vec{F}_i \equiv -m\vec{A}$, called the **inertial force**.

> An **inertial force** is a force that appears to act on a mass whose motion is described using a non-inertial frame of reference.

    * An inertial force is a **fictitious force** or a **pseudo force**, since it has no reaction force.

3.3 Some types of force

- **Centripetal / centrifugal force.**
  - In a uniform circular motion, the speed v of the particle is constant, but the velocity $\vec{v}$ always changes in direction.
    * Centripetal acceleration: $\vec{a}_c = -\dfrac{v^2}{r}\hat{r}$.
    * Centripetal force: $\vec{F}_c = m\vec{a}_c = -\dfrac{mv^2}{r}\hat{r}$.
  - The reaction of the centripetal force is the force acting on the center of the circle by the particle.
    It is called the **centrifugal force**: $\vec{F}_{cf} = \dfrac{mv^2}{r}\hat{r}$
- **Centrifugal inertial force.**
  - Consider a particle in a frame of reference with respect to a uniform circular motion (with respect to the Earth):
    * Acceleration of the frame: $\vec{a}_c = -\dfrac{v^2}{r}\hat{r}$.
    * Inertial force acting on the particle: $\vec{F}_c = m\vec{a}_c = -\dfrac{mv^2}{r}\hat{r}$.
  - This is called the **centrifugal inertial force**.
    It has the same formula as that for the centrifugal foce. However, **the two fores are different**.
- **Frictional force.**

- Static friction: $\boxed{F_{fr} = F_{ll} \text{ and } Ffr \leq \mu_s N}$
- Kinetic friction: $\boxed{\vec{F}_{fr} = \mu_k N \hat{v}}$
- Rolling friction: $\boxed{\vec{F}_{fr} = \mu_r N \hat{v}}$
- Here:
  * N is the normal force.
  * μ is the **coefficient of friction**.

3.4 Momentum

- Newton's second law for a particle of constant mass:

$$\vec{F} = m\vec{a} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\vec{v})$$

The quantity $\vec{p} = m\vec{v}$ is called the **momentum** of the particle.

Second form of Newton's second law

The net force acting on a particle is equal to the rate of change of the particle's momentum $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$.

- It is more convenient to apply the second form for an object with a changing mass.
- $\vec{F}\mathrm{d}t$ is called the **impulse** of $\vec{F}$ in time $\mathrm{d}t$.
- Impulse of $\vec{F}$ in time $\Delta t = t_2 - t_1$: $\int\limits_{t_1}^{t_2} \vec{F}\mathrm{d}t$.
- Change in momentum in time Δt: $\Delta\vec{p} \equiv \vec{p}(t_2) - \vec{p}(t_1) = \int\limits_{t_1}^{t_2} \vec{F}\mathrm{d}t$
- Momentum of a system is sum of all particles' momentum: $\vec{p} = \sum \vec{p}_i$.

Theorem

Change in momentum of a particle is equal to the impulse of the net force acting on it.

$$\Delta\vec{p} \equiv \vec{p}(t_2) - \vec{p}(t_1) = \int\limits_{t_1}^{t_2} \vec{F}\mathrm{d}t$$

- $\vec{F}$ is the sum of external forces acting on all particles of the system and $\vec{p}$ is the total momentum of the system.
- If $\vec{F} = \vec{0}$, the system is said to be **isolated** or **closed**: $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = 0 \Rightarrow \vec{p} \equiv \sum\limits_{i=1}^{n} \vec{p}_i = \text{const.}$

Law of conservation of momentum

An isolated system is the system that the external net force acting on it is $\vec{0}$.
The total momentum of an isolated system is conserved.

3.5 Torque and angular momentum.

- Let $\vec{r}$ be the position vector of a particle with respect to some point O. Newton's second law:

$$\boxed{\vec{r} \times \vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \vec{p})}$$

Definition

- $\vec{\tau} = \vec{r} \times \vec{F}$ is the **torque** of the force $\vec{F}$ about the point O.
- $\vec{L} = \vec{r} \times \vec{p}$ is the **angular momentum** of the particle about O.

- Since $\vec{r} \times \vec{F} = \frac{\mathrm{d}}{\mathrm{d}t} \Rightarrow \vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \equiv \dot{\vec{L}}$ (for a point particle).
- For a system: $\sum \overrightarrow{\tau}_i = \frac{\mathrm{d}}{\mathrm{d}t} \sum \overrightarrow{L}_i$ or $\vec{\tau} = \dot{\vec{L}}$
  - τ is the net torque of all external forces. (The net torque of the internal forces is zero).
  - $\vec{L}$ is the total angular momentum of all particles.

Theorem

The net torque of external forces acting on a system about a point is equal to the rate of change of the total angular momentum of the system about the point.

- When $\tau = \vec{0} \quad \Rightarrow \quad \dot{\vec{L}} = \vec{0} \quad \Rightarrow \quad \vec{L} = \text{const}.$

Law of conservation of angular momentum

When the net torque of external forces acting on a system about a point is zero, the total angular momentum of the system about the point is conserved.

- For a particle moving around a point: $\vec{L}\vec{r} \times \vec{p} = m\vec{r} \times \vec{v} = mr^2 \frac{\vec{r} \times \vec{v}}{r^2} \Rightarrow \vec{L} = I\vec{\omega}$.
  - $I = mr^2$ called the **moment of inertia** of the particle about the point (r is the distance from the particle to the point).
  - If the force acting on the particle is always towards the point: $\tau = \vec{0} \Rightarrow \vec{L} = \text{const} \Rightarrow \vec{\omega} = \text{const}$.
  - $\vec{\omega}$ is perpendicular to the orbital plane. This explains why the orbital plane of the Earth around the Sun is unchanged.

Chapter 4

Energy and Conservative force.

4.1 Work and power.

- Work done by force $\vec{F}$ in displacement $\mathrm{d}\vec{r}$ is defined as $\boxed{\mathrm{d}W = \vec{F}.\mathrm{d}\vec{t}}$
- Let $\alpha = \left(\vec{F}, \mathrm{d}\vec{r}\right)$ and $\mathrm{d}s = |\mathrm{d}\vec{r}|$, we have $\boxed{\mathrm{d}W = F.\mathrm{d}s.\cos\alpha}$
- Work done by force $\vec{F}$ in path AB: $\boxed{W = \int_{AB} \mathrm{d}W = \int_{AB} \vec{F}.\mathrm{d}\vec{t}}$
- Work done is dependent of the process of doing work.
- Differentiation of a work is a partial differentiation.
- **Power** is defined as work per unit time: $\boxed{p = \dfrac{\mathrm{d}W}{\mathrm{d}t}}$. From that, we have $p = \vec{F}.\vec{v}$
- Work and power are scalar quantities.
- Work is measured in Joule: $1\mathrm{J} = 1\mathrm{N.m} = 1\mathrm{kg.m^2.s^{-2}}$.
- Power is measured in Watt: $1\mathrm{W} = 1\mathrm{N.m.s^{-1}} = 1\mathrm{kg.m^2.s^{-3}}$
- Kilowatt-hour is a unit of energy: $1\mathrm{kWh} = 3.6 \times 10^6\mathrm{J}$

4.2 Energy.

- **Energy** is the ability to do work.
- Energy of a system depends on the state of the system. Differentiation of energy is a total differentiation.
- Energy can exist in many types (forms).

Conservation of energy

Energy can be transferred or transformed but cannot be destroyed and created.

- A perpetual motion machine of first kind is the machine which produce work without the input of energy.
  $\rightarrow$ **Consequence:** It is impossible to ... perpetual motion machines of the first kind.

4.3 Kinetic energy.

- A particle is accelerate by force $\vec{F}$: $\vec{F} = m\vec{a}$
- By conservation of energy, work done by $\vec{F}$ is equal to the change of kinetic energy of the particle: $\mathrm{d}E_k = \mathrm{d}W = \vec{F}\mathrm{d}\vec{r} = [...] = m.\vec{v}\mathrm{d}\vec{v} \Rightarrow E_k = \dfrac{1}{2}mv^2$.

Theorem
Change in kinetic energy is equal to work done **by external force**.

$$\boxed{W = \Delta E_k = E_{k2} - E_{k1}}$$

- **Conservative force** is a force with the property that the total work done in moving a particle between two points is independent of the taken path.
- Each conservative force is associated with a potential energy.
- Work done by a conservative force is equal to the decrease in potential energy associated with the force: $\boxed{W = E_{P1} - E_{P2}}$ or $\boxed{\mathrm{d}W = -\mathrm{d}E_P}$

4.4 Uniform gravitational field.

- $\vec{F} = -mg\hat{z} \Rightarrow W = mgz_1 - mgz_2$
- The work done is independent of the taken path. So, the filed is a field of conservative force: $E_P(z) = mgz + c$. Take the potential reference at $z = 0$: $E_P = mgz$.

4.5 Field strength and potential.

- **Definition:**

$$\text{field strength} = \frac{\text{force}}{\text{mass}} \Leftrightarrow \vec{g} = \frac{\vec{F}}{m}$$

$$\text{potential} = \frac{\text{potential energy}}{\text{mass}} \Leftrightarrow \Phi = \frac{E_P}{m}$$

- These quantities depend on the property of filed only. They can be used to represent for the field.
  - field strength $\rightarrow$ dynamic aspect.
  - potential $\rightarrow$ energetic aspect.
- For a uniform gravitational filed:

$$\vec{g} = -g\hat{z},\ \Phi = gz; \quad \nabla\Phi = \frac{\delta\Phi}{\delta x}.\hat{x} + ... = g\hat{z} = -\vec{g} \Rightarrow \vec{g} = -\nabla\Phi,\ \vec{F} = -\nabla E_P$$

- For a point of conservative force, the filed strength is equal and opposite to the gradient of the object's potential energy.
- Work done by the conservative force is equal to the decrease in potential energy and is equal to change in kinetic energy: $W = E_k + E_p = \text{const}$.
- This is applied for a motion under the action of several conservative forces (and no external forces): $E_k + \sum E_P = \text{const}$.
- For the motion of an object under the action of conservative forces only, the sum of kinetic and potential ...
- In the case of gravitational force, the sum of the kinetic and potential energies is called the mechanical energy: $E_m = E_k + E_P$
- For a motion of a mass in a uniform gravitational field: $E_m = \dfrac{1}{2}mv^2 + mgz = \text{const}$.

4.6 Elastic and inelastic collisions.

- In a collision of two objects, we consider:
  - Total momentum = const (no external force).
  - Total energy = const.
  - Total kinetic energy = const, only if no heat is produced.
  - Elastic collision: $\sum E_k = \text{const}$.
  - Inelastic collision: $\sum E_k \neq \text{const}$.
    $\rightarrow$ If after collision, the two objects move together as one $\rightarrow$ completely inelastic collision.
- For elastic collision in 1D space:
  - $u_1,\ u_2 \rightarrow$ before, $v_1,\ v_2 \rightarrow$ after.
    $$\begin{cases} m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \\ \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \end{cases} \Rightarrow \begin{cases} v_1 = \dfrac{(m_1 - m_2)u_1 + 2m_2u_2}{m_1 + m_2} \\ v_2 = \dfrac{(m_2 - m_1)u_2 + 2m_1u_1}{m_1 + m_2} \end{cases}$$
  - From that, we have $u_2 - u_1 = v_1 - v_2 \Leftrightarrow u_{21} = -v_{21}$
  - If $m_1 = m_2 \Rightarrow v_1 = u_2$ and $v_2 = u_1$
  - If $m_1 \ll m_2$ and $u_2 = 0 \Rightarrow v_1 = -u_1,\ v_2 = 0$
- For completely inelastic collision in 1D space:
  - $u_1,\ u_2 \rightarrow$ before, $v_1,\ v_2 \rightarrow$ after.
  - $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \Rightarrow v = \dfrac{m_1u_1 + m_2v_2}{m_1 + m_2}$
    $\Rightarrow Q = (\sum E_k)_{\text{before}} - (E_k)_{\text{after}} = \dfrac{m_1m_2(u_1 - u_2)^2}{2(m_1 + m_2)}$
- For a hammer-nail system:
  - $m_h \gg m_n,\ u_n = 0$
  - $Q = \dfrac{m_hm_nu_h^2}{2(m_h + m_n)} \approx \dfrac{1}{2}m_hv_h^2 \approx 0 \rightarrow$ Most of energy becomes kinetic energy.
- For a hammer-anvil system:
  - $m_h \ll m_n,\ u_n = 0$
  - $Q = \dfrac{m_hm_nu_h^2}{2(m_h + m_n)} \approx \dfrac{1}{2}m_hv_h^2 \rightarrow$ Most of energy of hammer becomes heat.

4.7 Gravitational field.

Newton's law of universal gravitation

Two point particles attract each other with a force that is proportional to the product of their masses and inversely proportional to the square of their separation.

$$\begin{cases} F_G \propto m_1,\ m_2 \\ F_G \propto \dfrac{1}{r^2} \end{cases} \Rightarrow \boxed{F_G = G\frac{m_1m_2}{r^2}} \quad \text{and} \quad \boxed{\vec{F}_{ij} = -G\frac{m_1m_2}{r^2}\hat{r}_{ij}}$$

where $G \approx 6.67 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}$. The law is applied for two point masses. **The law can be applied for two spheres which have mass distributed symmetrically**. In this case, the separation is the distance between the centers of them.

- Each mass generates a special medium around it, called the **gravitational field**. A gravitational filed is a region of space where a mass experiences a force.
- Consider the gravitational field of a point mass M: $\mathrm{d}W = \vec{F}\mathrm{d}\vec{r}$ and $\vec{F} = -G\dfrac{m_1 m_2}{r^2}.\hat{r}$

$$\Rightarrow \mathrm{d}W = -G\frac{m_1 m_2}{r^2}\mathrm{d}\vec{r} = -G\frac{m_1 m_2}{r^2}\mathrm{d}r \quad \Rightarrow \quad \boxed{W = G\frac{m_1 m_2}{r}}$$

$\rightarrow$ This implies that **the gravitational field of a point mass is conservative**.

> A general gravitational field is a combination of fields of point masses and, therefore, is a field of conservative force too.

- Back to the case of gravitational field of a point mass M:
  - For the test mass m in the field: $\mathrm{d}W = \mathrm{d}\left(G\dfrac{mM}{r}\right) = -\mathrm{d}E_P \Rightarrow E_P = -G\dfrac{mM}{r} + c$.
  - The potential reference is often chosen at infinity: $0 = -G\dfrac{mM}{r} + c \Rightarrow c = 0$.
- We are considering the point mass m at position $\vec{r}$ with respect to the point mass M.
  - Gravitational force acting on m by M: $\vec{F} = -G\dfrac{mM}{r}\hat{r}$
  - Gravitational potential energy of m: $F_P = -G\dfrac{mM}{r}$
- Now, consider the gravitational filed of M at position of $\vec{r}$:
  - Gravitational field strength: $\vec{g} = -G\dfrac{M}{r^2}\hat{r}$.
  - Gravitational potential: $\Phi = -G\dfrac{M}{r}$.

> ### Gravitational field strength
> **Gravitational field strength** $\vec{g}$ at a point in a gravitational field is defined as the force acting per unit mass on a small test mass m placed at the point.

> ### Gravitational potential
> **Gravitational potential** Φ at a point in a gravitational field is defined as the work done per unit mass in bringing a small test mass m from infinity to the point.

  - Note that both $\vec{g}$ and Φ are independent to m.
  - Force acting on m: $\boxed{\vec{F} = m\vec{g}}$
  - Potential energy: $\boxed{E_P = m\Phi}$
- **Orbital and escape velocities**

> ### Orbital velocity
> **Orbital velocity** is the velocity that is sufficient for a body to remain in orbit around a gravitational center.

Escape velocity

Escape velocity is the velocity that is sufficient for a body to escape from a gravitational center.

- If the velocity reaches some value v_O, the object travels in a circular orbit around the Earth. This called **orbital velocity**.
- For orbital velocity: **gravitational force = centripetal force**.

$$G\frac{mM}{r^2} = m\frac{v_O^2}{r} \Rightarrow \boxed{v_O = \sqrt{\frac{GM}{r}}}$$

- For escape velocity: **sum of potential and kinetic energies is zero**

$$-G\frac{mM}{r} + \frac{1}{2}mv_e^2 = 0 \Rightarrow \boxed{v_e = \sqrt{\frac{2GM}{r}}}$$

- **Properties**
  - Both velocities depend on the mass at the center of attraction, but not the mass of the object in motion.
  - They also depend on the distance from the gravitational center, or the altitude above the surface of the center body.
  - Relation at the same distance or altitude: $v_e = \sqrt{2}v_O$

Example

Suppose that air resistance is negligible, the orbital and escape velocities on the surface of the Earth is: $v_O = 7,910 \text{ ms}^{-1}$; $v_e = 11,180 \text{ ms}^{-1}$.
At the height 35.79 km above the sea level: $v_O = 7,880 \text{ ms}^{-1}$; $v_e = 11,150 \text{ ms}^{-1}$.

Chapter 5

Dynamics of a Rigid Body.

5.1 Rigid body.

Rigid body

A rigid body is a solid body in which deformation is zero or negligibly small.

- A rigid body can be considered as a system of a point particles in which the distance between any two point particles is always a constant
- On the other hand, a rigid body can be considered as a continuous distribution of mass:
  - The first idea is opposite to the second. But both are useful.
  - Some result studied in this chapter can be applied for some solid bodies that are not rigid.

5.2 Center of mass (CM).

The center of gravity

The center of gravity (CG) of a body is the point through which the force of gravity is assumed to act on the body.

- Generally, CG of a body is different from its CM. However, in a uniform gravitational field, CG coincides with CM.
- Consider a system of two point particles, m_1 and m_2 in a uniform gravitational field:

$$m_1 d_1 = m_2 d_2 \Rightarrow 0 = m_1 \vec{r}_1' + m_2 \vec{r}_2' = \sum_{i=1}^{2} m_i \vec{r}_i'$$

  - $\vec{r}_i'$ = position vector of the mass m_i relative to the CM.
  - $m_i \vec{r}_i'$ = weighted position relative to the CM of the mass m_i.
- For a system of n point particles: $\sum_{i=1}^{n} m_i \vec{r}_i' = 0$

The CM of a system

The CM of a system is the point where the weighted relative position of the distributed mass sums to zero.

- Since $\vec{r}_i' = \vec{r}_i - \vec{r}_{CM} \Rightarrow \sum_{i=1}^{n} m_i(\vec{r}_i - \vec{r}_{CM}) = 0;\ \ m \equiv \sum_{i=1}^{n} m_i$

$$\vec{r}_{CM} = \frac{1}{m}\sum_{i=1}^{n} m_i \vec{r}_i'$$

- **Continuous distribution of mass**: $\sum m_i \to \int \mathrm{d}m_i \quad \sum_{i=1}^{n} m_i\vec{r}_i \to \int \vec{r}\mathrm{d}m$
  - In 1D: $\mathrm{d}m = \lambda \mathrm{d}x$ (λ = mass per unit length).
  - In 2D: $\mathrm{d}m = \sigma \mathrm{d}A$ (σ = mass per area).
  - In 3D: $\mathrm{d}m = \rho \mathrm{d}V$ (ρ = mass per volume = density).
    * Cartesian: $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$.
    * Spherical $\mathrm{d}V = r^2 \sin\theta \mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$.
    * Cylindrical: $\mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z$.
- **The CM does not need to be on the object.**
  - For a homogeneous point-symmetric object, the CM coincides with the center of symmetry.
  - For a non-symmetric object, it is better to find the CM experimentally.

The general motion of a solid body

The general motion of a solid body can be considered as the sum of the translational motion of the CM, plus rotational, vibrational or other types of motion about the CM.

- For a rigid body*, the general motion is the combination of the translation and rotational motions.

5.3 Translation and rotation.

Translational motion

Translational motion is the motion such that, at any instant, all the particles of the body move parallel to each other.

- A translational motion can be **rectilinear** or **curvilinear**. The body is shifted from a point to point without rotation.
- Momentarily, all the particles (of the body) move in the same direction.
- All the particles have the same velocity and acceleration.
  $\to$ Therefore, in studying a pure translational motion, we only need to study **the motion of the CM of the body**.

***Rigid body** is the object that cannot have deformation.

Rotational motion

In a pure rotational motion of a rigid body, the body rotates about an axis which is fixed in an inertial reference frame.

- The orbit of any particle (in the body) is a circle which is perpendicular to the axis and centered at the axis.
- All particles rotate the same angle in the same time interval, and have the same angular velocity and angular acceleration.
- **If the axis of rotation is the z-axis** (cylindrical coordinates):

$$\boxed{\vec{\omega} = \dot{\varphi}\hat{z}} \quad \boxed{\vec{\beta} = \ddot{\varphi}\hat{z}} \quad \boxed{\vec{v} = \rho\dot{\varphi}\hat{\varphi}} \quad \boxed{\vec{a}_n = -\rho\dot{\varphi}^2\hat{\rho}} \quad \boxed{\vec{a}_t = \rho\ddot{\varphi}\hat{\varphi}}$$

The vectors $\vec{\omega}$ and $\vec{\beta}$ lie on the z-axis, direction of $\vec{\omega}$ depends on the sign of $\dot{\varphi}$.
- The unit vectors $(\vec{\rho}, \vec{\varphi}, \vec{z})$ form a **right-handed triad**:

$$\hat{\rho} \times \hat{\varphi} = \hat{z} \quad ; \quad \hat{\varphi} \times \hat{z} = \hat{\rho} \quad ; \quad \hat{z} \times \hat{\rho} = \hat{\varphi} \quad ; \quad \hat{\rho} \times \hat{\rho} = 0 \quad ; \quad \hat{\varphi} \times \hat{\varphi} = 0 \quad ; \quad \hat{z} \times \hat{z} = 0$$

  - Let $\vec{\rho} = \rho\hat{\rho}$. Now, you can easily prove the following relations:

$$\boxed{\vec{v} = \vec{\omega} \times \vec{\rho}} \quad \boxed{\vec{a}_t = \vec{\beta} \times \vec{\rho}} \quad \boxed{\vec{a}_n = \vec{v} \times \vec{\omega}} \quad \boxed{\vec{\omega} = \frac{\vec{\rho} \times \vec{v}}{\rho^2}} \quad \boxed{\vec{\beta} = \frac{\vec{\rho} \times \vec{a}}{\rho^2}}$$

- **Rotation with constant acceleration:**

$$\beta = \text{const} \quad ; \quad \omega = \omega_0 + \beta t \quad ; \quad \varphi = \varphi_0 + \omega_0 + \frac{1}{2}\beta t^2$$

$$\bar{\omega} = \frac{\omega + \omega_0}{2} \quad ; \quad \varphi - \varphi_0 = \frac{1}{2}(\omega + \omega_0)t = \bar{\omega}t \quad ; \quad \omega_0^2 - \omega = 2\beta(\varphi_0 - \varphi)$$

5.4 Rotational energy and moment of inertia.

- **Rotation of a system about a fixed axis** (the z-axis)
  - Kinetic energy: $E_k = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\rho_i \omega)^2 = \frac{1}{2}\left(\sum m_i \rho_i^2\right)\omega^2$

$$\boxed{E_k = \frac{1}{2}I\omega^2} \quad \boxed{I = \sum m_i \rho_i^2}$$

    Here, ρ_i is the distance of m_i from the axis.
  - The quantity I called the moment of intertia of the system. For a continuous distribution of mass:

$$\boxed{I = \int \rho^2 \mathrm{d}m}$$

Parallel-axis theorem (Huygens-Steiner theorem)

The moment of inertial I of a system about a given axis is calculated as $I = I_{CM} + md^2$, where I_{CM} is the moment of inertia of the system about the axis which passes through the center of mass of the system, is parallel to and is at distance d from the given axis.

- Thin rod about axis through center perpendicular to length L: $I = \frac{1}{12}ML^2$

  Proof: Since $\lambda = \frac{M}{L} \Rightarrow \mathrm{d}M = \frac{M}{L}\mathrm{d}x \Rightarrow I_0 = \int x^2 \mathrm{d}M = \int\limits_{-L/2}^{L/2} x^2 \frac{M}{L}\mathrm{d}x$

  $I = \int\limits_{0}^{L} x^2 \frac{M}{L}\mathrm{d}x = \frac{ML^2}{3} \Rightarrow I = I_0 + M\left(\frac{L}{2}\right)^2$
- Slab (a $\times$ b) about perpendicular axis through center: $I = \frac{1}{12}M(a^2 + b^2)$.
- Hoop (radius R) about central axis: $I = MR^2$.
- Solid cylinder (or disk) about central axis $I = \frac{1}{2}MR^2$.
- Solid cylinder (or disk) about central diameter $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$.

• **Rolling without slipping**

(a) A wheel is rolling without slipping. Point P is instantaneously at rest and point C moves with velocity $\vec{v}$.

(b) If the wheel is seen from a reference frame where C is at rest then point P is moving with velocity $-\vec{v}$.

- Condition for rolling without slipping is $v = R\omega$ (R = radius of the wheel).

$$E_k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2$$

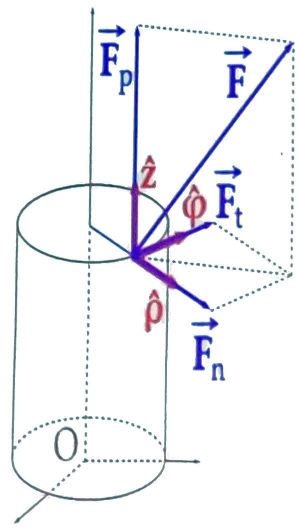
5.5 Torque of a force.

• For a body rotation about point O, if it is acted by a force $\vec{F}$ at position $\vec{r}$ from O, it is also said to be acted by a torque $\vec{\tau}$:

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

- $\vec{\tau} \perp \vec{r}$ and $\vec{\tau} \perp \vec{F}$.
- $(\vec{r}, \vec{F}, \vec{\tau})$ is a right-hand triad.
- $\tau = rF\sin\alpha$ where α is the angle between $\vec{r}$ and $\vec{F}$.

• For a body rotating about a fixed axis, chosen as the z-axis (in cylindrical coordinates):



$$\vec{F} = \vec{F}_n + \vec{F}_t + \vec{F}_p = F_\rho\hat{\rho} + F_\varphi\hat{\varphi} + F_z\hat{z}$$

- $\vec{F}_n$ is perpendicular to the axis and $\vec{F}_p$ is parallel to the axis. These components give no effect to the rotational motion. They are canceled out by the reaction forces from the axis.
- The tangential component $\vec{F}_t$ is the only one that gives effect to the rotational motion about the z-axis.

$$\vec{F}_t = F_\varphi\hat{\varphi} \quad ; \quad F_\varphi = -F_x\sin\varphi + F_y\sin\varphi$$

- Therefore, in rotational motion about a fixed axis, the torque of a force is defined as the torque of its tangential component.

$$\vec{\tau} = \vec{\rho} \times \vec{F}_t = (\rho\hat{\rho}) \times (F_\varphi \hat{\varphi}) \quad \rightarrow \quad \vec{\tau} = \rho F_\varphi \hat{z}$$

> * The torque is along the axis of rotation.
> * The torque is equal to the product of the perpendicular distance and the tangential component of the force.

- Generally, for rotational motion about the origin,

$$\vec{\tau}_O = \vec{r} \times \vec{F} = (x\hat{x} + y\hat{y} + z\hat{z}) \times (F_x x + \hat{F_y}\hat{y}6F_z\hat{z})$$

$$\boxed{\vec{\tau}_O = (yF_z - zF_y)\hat{x} + (zF_x - xF_z)\hat{y} + (xF_y - yF_x)\hat{z}}$$

- The z-component: $\tau_z = xF_y - yF_x = \rho\cos\varphi F_y - \rho\sin\varphi F_x$

> The torque of a force about an axis is equal to the component on the axis of the torque of that force about a point.

5.6 Angular momentum.

- For a system rotating about the origin:

$$\vec{L} = \sum \vec{L}_i = \sum \vec{r}_i \times \vec{p}_i = \sum m_i (\vec{r}_i \times \vec{v}_i)$$

  - Cylindrical coordinates (unit vectors): $\vec{r}_i = \rho_i\hat{\rho} + z_i\hat{z}$.

- For a system rotating about the z-axis:

$$\vec{v}_i = v_i\hat{\varphi} = \rho_i\omega\hat{\varphi} \Rightarrow \vec{r}_i \times \vec{v}_i = (\rho_i\hat{\rho} + z_i\hat{z}) \times (\rho_i\omega\hat{\varphi}) = \rho_i^2\omega\hat{z} - z_i\rho_i\omega\hat{\rho}.$$

$$\vec{L} = \left(\sum m_i\rho_i^2\right)\omega\hat{z} - \left(\sum m_i z_i\rho_i\hat{\rho}\right)\omega = I\vec{\omega} - \left(\sum m_i z_i \rho_i\hat{\rho}\right)\omega$$

- If the mass of the system is distributed symmetrically about the z-axis, the second term is zero.

> For a symmetric homogeneous body in pure rotation about its symmetric axis, the total angular momentum is $\boxed{\vec{L} = I\vec{\omega}}$
> Recall that $\vec{\omega} = \omega\hat{z} \rightarrow \vec{L}$ lies on the axis of rotation.

5.7 Torque and angular acceleration.

- Net external torque acting on a body rotating about the z-axis:

$$\vec{\tau} = \sum \rho_i F_{\varphi i}\hat{z} = \sum \rho_i m_i a_i \hat{z} = \sum \rho_i^2 m_i \beta\hat{z} = \left(\sum \rho_i^2 m_i\right)\beta\hat{z}$$

$$\Rightarrow \vec{\tau} = I\vec{\beta} \qquad \text{or} \qquad \vec{\beta} = \frac{\vec{\tau}}{I}$$

> In a pure rotation of a body about a fixed axis, its angular acceleration is proportional to the net external torque and is inversely proportional to the moment of inertia.

$\Rightarrow$ This is applied to any rigid object, symmetric or not.

- Now, let's consider a symmetric homogeneous rigid body in pure rotation about its symmetric axis.

$$\vec{L} = I\vec{\omega} \Rightarrow \dot{\vec{L}} = I\dot{\vec{\omega}} = I\vec{\beta} \Rightarrow \vec{\tau} = \dot{\vec{L}} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$

> For a symmetric homogeneous rigid body in pure rotation about its symmetric axis, the net external torque acing on the body is equal to the rate of change of the total angular momentum of the body.

$\Rightarrow$ For a non-symmetric body rotating about z-axis: $\tau = \dot{L}_z$

Chapter 6

Mechanical Oscillations and Waves.

6.1 Oscillating motions.

Definition

An oscillation is a repetitive variation in time of some quantity about a central value.

- Oscillations occur not only in mechanical systems but also in dynamic systems in virtually every area of science.
- In this chapter, we concentrate on mechanical oscillations.

Definition of mechanical oscillation

A mechanical oscillation is a repeated back and forth movement of an object about a point of equilibrium.

- An oscillation can be a **periodic motion** that repeats itself in a regular cycle.
- A typical example of periodic oscillation is the **harmonic motion**, which is described by a sinusoidal function.
- Three conditions for a mechanical oscillation:
  - There is a point of equilibrium at which the net force acting on the object is zero.
  - There is a restoring force acting on the object whenever it is out of the point of equilibrium.
  - When the object goes to the point of equilibrium, in order for it to pass this position, it need to have an inertia.

6.2 A math note in differential equations.

- In Mathematics, a **differential equations** is an equation that relates one or more functions and their derivatives.
- A linear differential equation of a function $y(x)$:

$$a_0 y + a_1 y' + ... + a_n y^{(n)} = b \quad \text{or} \quad \sum_{i=0}^{n} a_i y^{(i)} = b$$

  Here, we mostly consider the equation with a and b are constants.
- If $b = 0$, the equation is homogeneous.

- Consider a homogeneous linear differential equation (HLDE) which has constant coefficients: $a_0y + a_1y' + ... + a_ny^{(n)} = 0$.
  - Characteristic equation: $a_0y + a_1\lambda + ... + a_n\lambda^n = 0$
  - If this characteristic equation has n distinct roots, $\lambda_1, \lambda_2, ..., \lambda_n$, then the general solution of the HLDE is: $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + ... + c_ne^{\lambda_n x}$ where c_i is arbitrary constant.
- Now consider a non-homogeneous equation: $a_0y + a_1y' + ... + a_ny^{(n)} = f(x)$.
  - To solve this equation, firstly, we need to find the general solution $y_1(x)$ of the associated homogeneous equation.
  - Secondly, we base on the form of the function $f(x)$ to find an arbitrary solution $y_2(x)$ of the non-homogeneous equation. E.g., if $f(x)$ is a trigonometric function then $y_2(x)$ is another trigonometric function of the same period.
  - Finally, the solution of the non-homogeneous equation is $y(x) = y_1(x) + y_2(x)$.

6.3 Simple harmonic motions.

- Consider a mass-spring system, that is a mass m on a helical spring of stiffness k, on the x-axis: $\vec{F}_{\text{res}} = -kx$ or $a = -\omega^2 x$, where $\omega = \dfrac{k}{m}$
  - This applies the oscillation of thhe mass-spring system in moth horizontal and vertical motion.
  - Such an oscillation is called **a simple harmonic motion.**

> **Simple harmonic motion**
> A simple harmonic motion is the motion about a fixed point such that the acceleration is proportional to the displacement from, and is directed towards, the point.

  - Differential equation: $\boxed{\ddot{x} = -\omega^2 x}$ where $\omega = \text{const}$.

  ⇒ The solution for the above equation: $x = A\sin(\omega t + \varphi)$, where

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \quad , \quad \sin\varphi = \frac{x_0}{A} \quad , \quad \sin\varphi = \frac{v_0}{\omega A}$$

  - Here, x is called the displacement, A is the amplitude, ω is the angular frequency, and $(\omega t + \varphi)$ is the phase of the SHM.
  - Frequency and period:

$$f = \frac{\omega}{2\pi} \qquad \Leftrightarrow \qquad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- Energies:
  - Kinetic energy: $E_k = \dfrac{1}{2}mv^2 = \dfrac{1}{2}m\omega^2 A^2\cos^2\omega t$
  - Potential energy: $E_p = \dfrac{1}{2}kx^2 = \dfrac{1}{2}m\omega^2 A^2\sin^2\omega t$
  - Total energy: $E_{\text{tot}} = E_k + E_p = \dfrac{1}{2}m\omega^2 A^2$

$$\Rightarrow \boxed{E_k = \frac{1}{2}E(1 + \cos 2\omega t)} \quad \boxed{E_p = \frac{1}{2}E(1 - \cos 2\omega t)}$$

> * In a SHM, the kinetic and potential energies oscillate at frequency which is two times greater that the frequency of the motion, and the total energy is conserved.
> * The kinetic and potential energies vary in opposite phase with a period that is half of the period of oscillation.

- Physical pendulum (compound pendulum)

> **Definition**
> A physical pendulum is a rigid body which is free to rotate about a fixed horizontal axis.

  - Newton's second law for torques: $\vec{\tau} = I\vec{\beta} \rightarrow (-mg\sin\theta)L = I\ddot{\theta}$.
  - For small θ, $\sin\theta \approx \theta$: $\ddot{\theta} = -\omega^2\theta$ where $\omega = \sqrt{\frac{mgL}{I}}$.
  - Thus, the oscillation of a physical pendulum at small angle is a SHM.
  - Period and frequency: $T = 2\pi\sqrt{\frac{I}{mgL}}$; $f = \frac{1}{2\pi}\sqrt{\frac{mgL}{I}}$.
    Note that I is the moment of inertia of the body about the axis of rotation and L is the distance from its CM to the axis.

- A simple pendulum consists of a mass-less string of length L connecting with a point particle of mass m. The other end of the string is connected to a fixed point.
  - This is a special case of physical pendulum with $I = mL^2$.
  - It oscillates harmonically at small angle $\omega = \sqrt{\frac{g}{L}}$.

> **Example 1**
> A mass oscillating on a spring has an amplitude of 0.10m and period of 2.0s.
>
> (a) Deduce the equation for the displacement x if timing starts at the instant when the mass has its maximum displacement.
> (b) Calculate the time interval from $t = 0$ before the displacement is 0.08m.

Solution

(a) $x = 0.10\sin\left(\pi t + \frac{\pi}{2}\right)$, where x is in meters and t is in seconds.

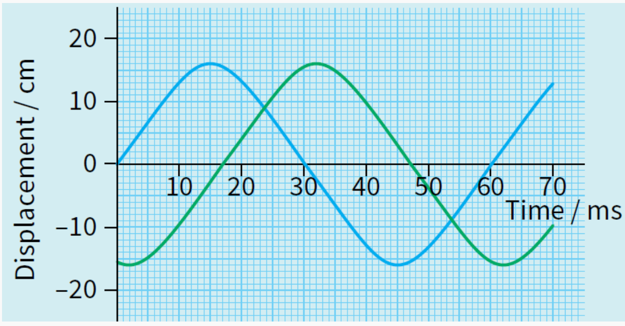
(b)
  - We have: $\sin\left(\pi t + \frac{\pi}{2}\right) = 0.80$
    $\Leftrightarrow \pi t + \frac{\pi}{2} = \arcsin 0.80 + n2\pi$ or $\pi t + \frac{\pi}{2} = \pi - \arcsin 0.80 + n2\pi,\ n \in \mathbb{Z}$.
  - Therefore, $t = \frac{\arcsin 0.8}{\pi} - \frac{1}{2} + 2n$ or $t = \frac{1}{2} - \frac{\arcsin 0.8}{\pi} + 2n$.
  - The time interval is the smallest positive value of t: $t = \frac{1}{2} - \frac{\arcsin 0.8}{\pi} = 0.2s$.

Example 2

State at what point in an oscillation the oscillator has zero velocity but positive acceleration.

Example 3

The figure shows displacement-time graphs for two identical oscillators. Calculate the phase difference (p.d.) between the two oscillations. Give your answer in degrees and in radians.



Solution

- Period: $T = 60$ms.
- Two corresponding points: $\Delta t = 17$ms.
- Phase difference $pd = \frac{t}{T} = 0.283$ oscillation.
- In other words, $pd = 0.283 \times 360° = 102°$ or $pd = 0.283 \times 2\pi = 1.78$ rad.

6.4 Damped oscillation.

Definition

Oscillation in which energy is dissipated and, therefore, amplitude is decreased due to frictional or resistive forces are called oscillation.

- Damped harmonic oscillator
  - The oscillator is acted by a harmonic restoring force and a viscous force which is proportional to the speed v of it:
  $$\boxed{F_{\text{viscous}} = -cv} \quad (c \text{ is the viscous coefficient}).$$
  - Newton's second law: $F_{\text{restoring}} + F_{\text{viscous}} = ma$
  - We denote $\omega_0 = \sqrt{\frac{k}{m}}$ is natural angular frequency, $\beta = \frac{c}{2m}$ is damping coefficient, and $\zeta = \frac{\beta}{\omega_0} = \frac{c}{2\sqrt{km}}$ is damping ratio.
  - The heavier the damping, the smaller the frequency and the greater the rate of decaying of the amplitude. The ratio ζ is used to determine the behavior of the system.
    * If $\zeta \geq 1 \Leftrightarrow \beta \geq \omega_0$, the displacement gradually decreases to zero with no oscillation.

  * For $\zeta = 1 \Leftrightarrow \beta = \omega_0$, the oscillator is said to be at **critical damping**. The displacement of the oscillator decreases to zero without oscillating in the shortest time.
  * For $\zeta > 1 \Leftrightarrow \beta > \omega_0$, the oscillator is **overdamped**. The oscillator returns to the equilibrium position without oscillating. The greater the damping ratio, the larger the returning time.
  * In case of underdamped oscillation, $\zeta < 1 \Leftrightarrow \beta > \omega_0$: The oscillator oscillates with decaying amplitude:

$$x = A\sin(\omega t + \varphi) \quad ; \quad A = A_0 e^{-\beta t} \quad ; \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

- Underdamped system:
  - The amplitude A decays in time at a rate which depends on the damping. To represent for this rate, we use a quantity called the **logarithmic decrement**.
  - The logarithmic decrement δ is defined as the natural logarithm of the ratio of any two successive amplitudes: $\boxed{\delta = \ln \dfrac{A(t)}{A(t+T)} = \dfrac{1}{n}\ln\dfrac{A(t)}{A(t+nT)}}$
  - In experiment, we determine the damping ratio by $\zeta = \dfrac{1}{\sqrt{1+\left(\dfrac{2\pi}{\delta}\right)^2}}$ (i.e. you don't need to care about this formula in exercises ☺.)

6.5 Forced oscillations and resonance.

Definition

A forced oscillation is an oscillation under the action of external periodic forces.

- Take the external force $F_{\text{external}} = F_0\sin(\Omega t)$, also denote $\beta = \dfrac{c}{2m}$ and $\omega_0 = \sqrt{\dfrac{k}{m}}$.
- The amplitude: $A = \dfrac{F_0/m}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\beta^2\Omega^2}}$. The amplitude of the forced oscillation depends not only on the magnitude but also on the frequency of the driving force.
- The amplitude is maximum when the frequency is $\Omega_r = \sqrt{\omega_0^2 - 2\beta^2} = \omega_0\sqrt{1-2\zeta^2}$:

$$A_{\max} = \frac{F_0}{2m\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{F_0}{2m\zeta\omega_0^2\sqrt{1-\zeta^2}}$$

- In a forced oscillation, **resonance** is the phenomenon that the amplitude is maximum at certain value of driving frequency. The resonant amplitude depends on the damping ratio.

Example

- When you push a swing, you need to match its timing. An in-sync gentle push helps it to oscillate at large amplitude due to resonance.
- When you pluck a string of an acoustic guitar, it vibrates and transmits the sound energy into the hollow wooden body of the guitar, making it resonate and amplifying the sound.

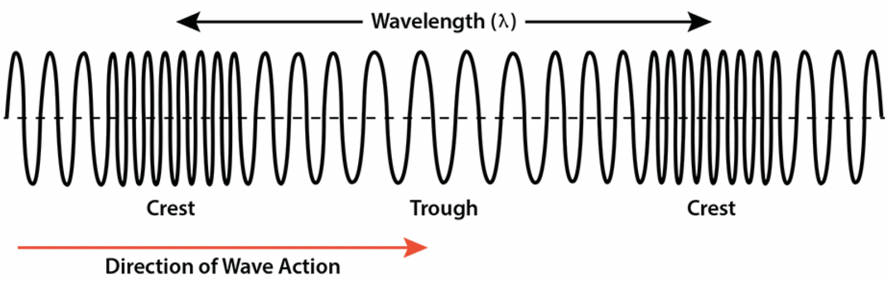
- A group of soldiers while marching on a bridge, if their footsteps synchronize with the natural frequency of the bridge, the bridge can vibrate strongly and break apart.
- More examples are Barton's Pendulum, tidal resonance, quartz clock, and orbital resonance.

6.6 Combination of oscillations.

Self-study part so I decided not to study ☺.

6.7 Wave equation and wave function.

- Consider the longitudinal wave on a long spring in the x-axis. The spring has a length l, mass m, and stiffness k.



- Speed of longitudinal wave: $c = \sqrt{\dfrac{E}{\rho}} = \sqrt{\dfrac{kl}{A\rho}} = \sqrt{\dfrac{kl^2}{m}}$, where ρ is the density of the medium.
- The wave equation (apply for any mechanical wave): $\dfrac{\partial^2 u(x,t)}{\partial t^2} = c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} \Leftrightarrow \ddot{u}(\vec{r},t) = c^2 \nabla^2 u(\vec{r},t)$
- Plane wave:
  - For a monochromatic plane wave traveling in the $\vec{r}$ direction: $\boxed{u(\vec{r},t) = Ae^{-i(\omega t - \vec{k}\vec{r})}}$ where $\vec{k} = \dfrac{\omega}{c}\hat{r} = \dfrac{2\pi}{\lambda}\hat{r}$.
  - For a monochromatic spherical wave travels in expanded spheres: with α is constant and A_0 is the amplitude at the origin, we have $\boxed{u(\vec{r},t) = \alpha \dfrac{A_0}{r} e^{-i(\omega t - \vec{k}\vec{r})}}$.

6.8 Wave energy.

- Consider a plane wave on a string: $u(x,t) = u_m \sin(\omega t - kx)$, we denote an element of mass $\mathrm{d}m = \mu \mathrm{d}x$.
- In an oscillating system, the average values of the kinetic energy and potential energy are the same: $\left(\dfrac{\mathrm{d}E_k}{\mathrm{d}t}\right)_{\text{avg}} = \left(\dfrac{\mathrm{d}E_p}{\mathrm{d}t}\right)_{\text{avg}} = \dfrac{1}{4}\mu c u_m^2 \omega^2$.
- The average power (the average rate at which energy of both kinds is transmitted by the wave): $P_{\text{avg}} = \left(\dfrac{\mathrm{d}E_k}{\mathrm{d}t}\right)_{\text{avg}} + \left(\dfrac{\mathrm{d}E_p}{\mathrm{d}t}\right)_{\text{avg}} = \dfrac{1}{2}\mu c u_m^2 \omega^2$
  - μ and c depend on the material and tension of the string.

  - ω and u_m depend on the process that generates the wave.
- Hence, we have:
  - For 2D: $dm = \sigma L dx \Rightarrow P_{\text{avg}} = \frac{1}{2}\sigma L c u_m^2 \omega^2$.
  - For 3D: $dm = \rho A dx \Rightarrow P_{\text{avg}} = \frac{1}{2}\rho A c u_m^2 \omega^2$.
  - It is true for all waves that:

> **Conclusion**
> The average power of a wave is proportional to the square of its amplitude and is proportional to the square of its frequency.
> $$P_{\text{avg}} \propto u_m^2 \quad ; \quad P_{\text{avg}} \propto \omega^2$$

- Energy flux:

> **Definition**
> The energy flux S of a wave is the energy transferred by the wave per unit time per unit area (normal to the direction of the wave): $S = \dfrac{P_{\text{avg}}}{A}$

  - For a wave in 3D: $S = \frac{1}{2}\rho c u_m^2 \omega^2$.
  - Energy density: $e = \dfrac{P_{\text{avg}}.t}{AL} = \dfrac{P_{\text{avg}}}{Ac} \Rightarrow e = \frac{1}{2}\rho u_m^2 \omega^2$.

> **Conclusion**
> Energy flux is equal to the product of the energy density and the speed of wave.
> $$S = ec$$

  - Umov-Poynting vector (energy flux vector) $\vec{S}$ is the vector in the direction of the wave with magnitude is the energy flux of the wave: $\boxed{\vec{S} = e\vec{c} = \frac{1}{2}\rho u_m^2 \omega^2 \vec{c}}$

Part II

THERMODYNAMICS

Chapter 7

Kinetic energy and laws of distribution.

7.1 Starting points.

- All matter consists of atoms or molecules (groups of atoms).
- Gases have no fixed volume and no fixed shape. The parameters of a gaseous state are volume, temperature and pressure (p, V, T).

7.2 Equation of state.

- Experimental laws for gases:

Boyle's law, (*a.k.a. Boyle-Marriotte's law*)

The volume of a gas is inversely proportional to its pressure, provided that the temperature is held constant: $p \propto \frac{1}{V} \Leftrightarrow pV = \text{const.}$

Charles's law

The volume of a gas is directly proportional to its thermodynamic temperature, provided that the pressure is held constant: $V \propto T \Leftrightarrow \frac{V}{T} = \text{const.}$

Gay-Lussac's law

The pressure of a gas is directly proportional to its thermodynamic temperature, provided that the volume is held constant: $p \propto T \Leftrightarrow \frac{p}{T} = \text{const.}$

- These laws can be combined as: $\frac{pV}{T} = \text{const.}$ The value of the constant is proportional to the number n of moles in the gas: $\frac{pV}{T} \propto n \Rightarrow \frac{pV}{T} = Rn$.
- $R = 8.31\ \text{JK}^{-1}\text{mol}^{-1}$ is the molar gas constant, hence, $k_B = \frac{R}{N_A} = 1.38 \times 10^{-23}\ \text{JK}^{-1}$ is called Boltzmann constant.

Equation of state (*a.k.a. Clapeyron-Mendeleev equation*)

$$pV = nRT = Nk_BT$$

Here, $N = n.N_A$ is the total number of particles in the gas.

- **Ideal gases:** A gas which obeys the equation of state, $pV = nRT$, for all pressures, volumes and temperatures is called an ideal gas.

7.3 Kinetic theory of gases.

Brownian motion

Brownian motion is the random motion of particles suspended in a fluid resulting from their collision with the fast-moving molecules in the fluid.

- The kinetic theory of an ideal gas:
  - A gas contains a very large number of particles (atoms or molecules).
  - The forces between particles are negligible, except during collisions.
  - The volume of particles is negligible compared to the volume occupied by gas.
  - The collisions of particles with each other and with container are perfectly elastic.
  - The particles mostly move in straight lines at constant velocities. The collision time is negligible compared with the time between collisions.
- **Molecular motion:** $pV = \frac{1}{3}Nm\langle c^2\rangle$, where m is the molecular mass and $\langle c^2\rangle$ is the mean-square speed of the molecules.
- **Average kinetic energy and thermodynamic temperature:**
  - The average value of kinetic energy: $\langle E_k\rangle = \frac{1}{2}m\langle c^2\rangle$.
  - The relation between kinetic energy and temperature for systems of mono-atomic molecules: $\langle E_k\rangle = \frac{3}{2}k_BT$. The position of each molecule is determined by three independent parameters.
  - **The number of degrees of freedom**, i, of a gas is the number of independent parameters used to determine the position of each molecule in the gas.
    - For mono-atomic gas: $i = 3$.
    - For diatomic gas: $i = 5$.
    - For poly-atomic gas (three or more atoms): $i = 6$.

Equipartition theorem

In thermal equilibrium, the average energy is shared equally to every degree of freedom. For gas of number of degrees of freedom i: $\langle E_k\rangle = \frac{i}{2}k_BT$

- **Root-mean-square speed:**
  - We have the square of mean speed is non-zero and not the same as the mean-square speed, that means $\langle c^2 \rangle \neq \langle c \rangle^2$.
  - The root-mean-square speed is defined as: $c_{rms} = \sqrt{\langle c^2 \rangle} = \sqrt{\dfrac{3k_B T}{m}}$.
  - Normally, $c_{rms} \approx 1.1\langle c \rangle$.
- **Internal energy:**

> **Definition**
> Internal energy of a thermodynamic system is determined by the state of the system and can be expressed as the sum of a random distribution of kinetic and potential energies associated with the molecules of the system.
>
> $$U = \sum E_k + \sum E_p, \quad \begin{cases} E_k \to \text{random motions of molecules.} \\ E_p \to \text{interactions between molecules.} \end{cases}$$

  - It does not include the kinetic energy in motion of the system as a whole, nor the potential energy due to external forces.
  - For ideal gases, since the molecules do not interact each other (except during collisions), the potential energy term is zero: $U = \sum E_k = \dfrac{m}{\mu}\dfrac{i}{2}RT$, where μ is the molar mass and m is the mass of the system.

7.4 Maxwell distribution.

- For a real gas system, the speed of the particles is continuously distributed (can have any value). The distribution depends on the temperature and properties of the system.
- The Maxwell distribution function depends on kinetic energy, i.e. depends on c^2.
- First part $\propto c^2$, second part decays rapidly: $F(c) = Ac^2.e^{-\frac{1}{2}\alpha c^2}$, where the constant α depends on temperature and the constant A is determined by the normalized condition: $A = \sqrt{\dfrac{2\alpha^3}{\pi}}$.
- After a bunch of integral calculations until die, we get the final results:
  - Mean-square speed: $\langle c^2 \rangle = \dfrac{3}{\alpha}$.
  - Average kinetic energy: $\langle E_k \rangle = \dfrac{1}{2}m\langle c^2 \rangle = \dfrac{3}{2}k_B T \Rightarrow \alpha = \dfrac{m}{k_B T}$.
  - Explicit formula: $F(c) = \sqrt{\left(\dfrac{m}{2\pi k_B T}\right)^3} 4\pi c^2 e^{-\frac{mc^2}{2k_B T}}$.
  - Mean speed: $\langle c \rangle = \sqrt{\dfrac{8k_B T}{\pi m}} = \sqrt{\dfrac{8RT}{\pi \mu}}$.
  - Root-mean-square speed: $c_{rms} = \sqrt{\dfrac{3k_B T}{m}} = \sqrt{\dfrac{3RT}{\mu}}$.
  - Most probable speed (at which $F(c)$ is maximum): $c_p = \sqrt{\dfrac{2k_B T}{m}} = \sqrt{\dfrac{2RT}{\mu}}$.

7.5 Boltzmann distribution.

- Atmospheric pressure:
  - Suppose that the gas in the uniform gravitational field of the Earth is an ideal gas. Let p be the pressure at height z and p_0 is the pressure at $z = 0$.
  - Atmospheric pressure formula: $p = p_0 e^{-\frac{\mu g z}{RT}}$.
- Boltzmann distribution (the distribution of particles in term of potential energy):
  - Let n be the molecular density (number of molecules per unit volume) $p \propto n \Rightarrow n = n_0 e^{-\frac{\mu g z}{RT}} = n_0 e^{-\frac{mgz}{k_B T}}$.
  - Here, μ is the molar mass, m is the molecular mass, and $E_p = mgz$ is the gravitational potential energy of the molecule. We have: $\boxed{n = n_0 e^{-\frac{E_p}{k_B T}}}$

Chapter 8

First law of thermodynamics.

8.1 Statement of the first law.

Definition

The change in internal energy of a system is equal to the heat added to the system plus the work done on the system.

$$\boxed{\Delta U = Q + W} \quad \text{or} \quad \boxed{\mathrm{d}U = \delta Q + \delta W}$$

- The change, $\Delta U = U_{\text{final}} - U_{\text{initial}}$, is the same as the increase in internal energy.
- If Q' is the heat transferred from the system and W' is the work done by the system: $\boxed{\Delta U + Q' + W' = 0}$

8.2 Work and heat in equilibrium processes.

- **Equilibrium states:** Each variable has a single value in an equilibrium state, and its change between any two equilibrium states is single-valued.
- **Equilibrium processes:** An equilibrium process is a series of equilibrium states.
  - Each equilibrium process is represented by a curve segment in the PV diagram. In practice, a quasi-static process (a sufficiently slow process) can be considered as an equilibrium process.
  - Work done in an equilibrium process: $\mathrm{d}W = -p\mathrm{d}V \Rightarrow W = -\int\limits_{\widehat{12}} p\mathrm{d}V$. This work is equal to the area under the pV curve.
  - Work done in an equilibrium cycle (closed process): $W = -\int\limits_{\widehat{1a2b1}} p\mathrm{d}V$ = area limited by the path.
  - Heat added to the system: $Q = \int\limits_{\widehat{12}} \frac{m}{\mu} C\mathrm{d}T$, where c is the specific heat capacity and $C = \frac{\mu}{m}\frac{\partial Q}{\partial T}$ is the molar heat capacity.
  - The heat Q is path dependent.

8.3 Equilibrium processes of ideal gases.

- **Isochoric process** (constant-volume process, V = constant):

- Equation: $\dfrac{p}{T} = \text{const.}$
- Work: $W = 0$.
- Heat: $Q = \dfrac{m}{\mu}\dfrac{i}{2}R\Delta T$.
- Molar heat capacity at constant volume: $C_V = \dfrac{i}{2}R$.

- **Isobaric process** (constant-pressure process, p = constant):
  - Equation: $\dfrac{V}{T} = \text{const.}$
  - Work: $W = -\dfrac{m}{\mu}R\Delta T$.
  - Heat: $Q = \dfrac{m}{\mu}\left(\dfrac{i}{2}+1\right)R\Delta T$.
  - Molar heat capacity at constant volume: $C_p = \dfrac{i+2}{2}R$.
- **Isothermal process** (constant-temperature, T = constant):
  - Equation: $pV = \text{const.}$
  - Work: $W = \dfrac{m}{\mu}RT\ln\dfrac{V_1}{V_2} = \dfrac{m}{\mu}RT\ln\dfrac{p_2}{p_1}$.
  - Heat: $Q = -A = \dfrac{m}{\mu}RT\ln\dfrac{V_2}{V_1} = \dfrac{m}{\mu}RT\ln\dfrac{p_1}{p_2}$.
- Heat capacity ratio (adiabatic index): $\gamma = \dfrac{C_p}{C_V}$. For ideal gases:

$$\begin{cases} C_V = \dfrac{i}{2}R,\ C_p = \dfrac{i+2}{2}R \Rightarrow \gamma = \dfrac{i+2}{i} \\ i = \dfrac{2}{\gamma-1} \Rightarrow C_V = \dfrac{R}{\gamma-1};\ C_p = \dfrac{\gamma R}{\gamma-1}. \end{cases}$$

- **Adiabatic process** (no heat transferred, $Q = 0$):
  - Equation: $pV^\gamma = \text{const.}$
  - The PV diagram of an adiabatic process is similar to that of an isothermal process, but the magnitude of its gradient is greater.
  - Work: $W = \dfrac{p_2V_2 - p_1V_1}{\gamma-1}$. In terms of change in temperature: $W = \dfrac{m}{\mu}C_V\Delta T$.

Chapter 9

Second law of thermodynamics.

9.1 Starting points.

- All natural processes obey the first law. However, there are processes that obey the first law but cannot occur naturally.
- The second law deals with these limitations using a new state function, the entropy S. It asserts that a natural process runs only in one direction, and is irreversible.
- For example, heat always flows spontaneously from a hotter body to a colder body.
- For an idealized reversible process, the entropy change obeys an equality and can be calculated. That is why it is important even though it is a hypothetical process.

9.2 Postulates of the second law.

Postulate of Clausius

A transformation whose only final result is to transfer heat from a body at a given temperature to a body at a higher temperature is impossible.

Postulate of Kelvin

A transformation whose only final result is to transfer into work heat extracted from a source which is at the same temperature throughout is impossible.

9.3 Heat engine.

Definition

Heat engine is a system that converts thermal energy to mechanical energy, which can then be used to do work.
A heat engine consists of a working substance, a heat source, and a colder sink.

- It works periodically. In each cycle, the working substance takes a heat Q_1 from the heat source, does a work W', and transfers a heat Q'_2 to the colder sink: $\Delta U = 0 \Rightarrow W' = Q_1 - Q'_2$.
- Its efficiency: $\eta = \dfrac{W'}{Q_1} = 1 - \dfrac{Q'_2}{Q_1}$.

- **Carnot heat engine** (a theoretical engine): The efficiency $\eta = 1 - \frac{T_2}{T_1}$. So, in the case of an ideal gas, the eÿciency of a reversible Carnot cycle depends only on the temperatures of the heat source and the colder sink. However, Carnot stated that this applies for any working substance.

> **Carnot's theorem**
> The efficiency of a Carnot engine depends solely on the temperatures of the hot and cold reservoirs. It is the maximum efficiency any heat engine can obtain.

Proof. See on the slides of professor Nam Le.

9.4 Entropy and statement of the second law.

> **Definition**
> The entropy S of a system is defined by $\mathrm{d}S = \frac{\delta Q}{T}$ in a reversible process.

- $\frac{\delta Q}{T} \leq \mathrm{d}S \quad \begin{cases} \text{"equal" for reversible cycles} \\ \text{"less than" for irreversible cycles} \end{cases}$
- For an isolated system: $\boxed{\mathrm{d}S \geq 0}$
  - $\mathrm{d}S > 0$ when there are irreversible processes in the system,that is when the system is evolving.
  - $\mathrm{d}S = 0$, or S reaches its maximum constant value, when all internal processes are reversible, that is when the system is at its thermodynamic equilibrium state.

> **Second law of thermodynamics**
> The total entropy of an isolated system can never decrease over time, and is constant if and only if all processes within the system are reversible. Isolated systems spontaneously evolve towards thermodynamic equilibrium, the state with maximum entropy.

9.5 Change in entropy of an ideal gas.

- In an adiabatic process: $\delta Q = 0 \Rightarrow S = \text{const}$.
- In an isothermal process: $T = \text{const}$. Hence, $\boxed{\Delta S = \frac{m}{\mu} R \ln \frac{V_2}{V_1} = \frac{m}{\mu} R \ln \frac{p_1}{p_2}}$
- For an arbitrary process: $\boxed{\Delta S = \frac{m}{\mu} C_V \ln \frac{p_2}{p_1} + \frac{m}{\mu} C_p \ln \frac{V_2}{V_1}}$

Chapter 10

Real gases.

10.1 Van der Waals equation of state.

- From ideal gases to real gases:
  - According to Van der Waals, the equation of state for real gases can be obtained from the equation of state for ideal gases by replacing:

$$V_m \to V_m - b \quad ; \quad p \to p + \frac{a}{V_m^2}$$

    where b is the volume occupied by molecules of one mole and a is a constant depending on the gas.
  - Hence, the EOS of real gases is: $\boxed{\left(p + \frac{a}{V_m^2}\right)(V_m - b) = RT}$
  - For an arbitrary amount of gas: $V_m \to \frac{\mu}{m} V \Rightarrow \boxed{\left(p + \frac{m^2}{\mu^2}\frac{a}{V^2}\right)\left(V - \frac{m}{\mu} b\right) = \frac{m}{\mu} RT}$
- Van der Waals equation for one mole ($V \equiv V_m$):
  - $\boxed{\left(p + \frac{a}{V_m^2}\right)(V_m - b) = RT} \Rightarrow p = \frac{RT}{V - b} - \frac{a}{V^2}$
  - At critical point: $V_c = 3b \quad ; \quad T_c = \frac{8a}{27bR} \quad ; \quad p_c = \frac{a}{27b^2}$.

10.2 Experimental isotherms.

- The critical temperature of CO_2 is $T_c = 31°C$.
- Above T_c, CO_2 can only exist in the gaseous state. The isotherms are similar to those of ideal gases.
- Below T_c, the isotherms is flat over a range of volume. Liquid and vapor coexist in this range, in dynamic equilibrium. On the left of this range is solid, and on the right is gas.
- The critical isotherm is flat at one point only.