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1.1

1.1

$$\ddot{x}+x+x^3=0$$

$$x(0)=A$$

$$\dot{x}(0)=0$$

Multiplying by $\dot{x}$ on both sides and integrating we get

$$\frac{(\dot{x})^2}{2}+\frac{x^2}{2}+\frac{x^4}{4}=E$$

Apply ICs.

$$\hookrightarrow \quad E=\frac{A^2}{2}+\frac{A^4}{4}$$

$$\Rightarrow \frac{(\dot{x})^2}{2}=\frac{A^2-x^2}{2}+\frac{A^4-x^4}{4}$$

$$2(\dot{x})^2=2(A^2-x^2)+(A^4-x^4)$$

$$\dot{x}=\pm\frac{1}{\sqrt{2}}\sqrt{2(A^2-x^2)+(A^4-x^4)}$$

$$\dot{x}=\pm\frac{1}{\sqrt{2}}\sqrt{(A^2-x^2)}\sqrt{A^2+x^2+2}$$

$$\Rightarrow \int_0^A \frac{dx}{\sqrt{A^2-x^2}\sqrt{A^2+x^2+2}}=\int_0^{T/4}\frac{dt}{\sqrt{2}}$$

$$\Rightarrow \frac{T}{4\sqrt{2}}=I$$

We can make this claim that in T/4 time P. the sys goes from 0 → A as the force is conservative and restoring.

where $I = \int_0^A \frac{dx}{\sqrt{A^2-x^2}\sqrt{A^2+x^2+2}}$

$x = Au$

$dx = A\,du$

$$I = \int_0^1 \frac{A\,du}{\sqrt{1-u^2}\sqrt{A^2(1+u^2)+2}}$$

using ~~MP~~ MAPLE

$$I = \frac{57\sqrt{2}\pi A^4}{1024} - \frac{3A^2\sqrt{2}\pi}{32} + \frac{\sqrt{2}\pi}{4}$$

$$\Rightarrow T = 4\sqrt{2}\,I$$

$$= \frac{57\pi A^4}{128} - \frac{3A^2\pi}{4} + 2\pi$$

$$= 2\pi\left(1 - \frac{3}{8}A^2 + \frac{57}{256}A^4\right)$$

Now if $A \to 0$

$T \to 2\pi$

1.2

1.2.

$$\ddot{x} + x^3 = 0$$

$$x(0) = A$$

$$\dot{x}(0) = 0$$

Multiplying by $\dot{x}$ on both sides & integrating we get

$$\frac{(\dot{x})^2}{2} + \frac{x^4}{4} = E$$

Applying IC

$$\Rightarrow \quad E = \frac{A^4}{4}$$

$$\Rightarrow \quad \frac{(\dot{x})^2}{2} + \cancel{\frac{A^4}{4}}\ \frac{x^4}{4} = \frac{A^4}{4}$$

$$\frac{(\dot{x})^2}{2} = \frac{A^4 - x^4}{4}$$

$$\Rightarrow \dot{x} = \pm \frac{1}{\sqrt{2}} \sqrt{A^4 - x^4}$$

$\Rightarrow$ Again as the force is restoring ~~and~~ in nature and conservative.

We can claim that as x ~~goes~~ goes from $0 \to A$ in $T/4$ time period.

$$\therefore \int_0^A \frac{dx}{\sqrt{A^4 - x^4}} = \frac{T}{4\sqrt{2}}$$

Putting $x = Au$ $\quad dx = A\,du$

$$\Rightarrow \frac{T}{4\sqrt{2}} = \int_0^1 \frac{A\,du}{\sqrt{A^4 - A^4u^4}} = \frac{1}{A}\int_0^1 \frac{du}{\sqrt{1-u^4}}$$

Now using maple we get the value of $I = \frac{1}{A}\int_0^1 \frac{du}{\sqrt{1-u^4}}$

$$I = \frac{1.311}{A}$$

$$\boxed{T = 4\sqrt{2}\, I = \frac{7.4162}{A}}$$

1.3

1.3

$$A_{ij} = \sin(i)\cos(j)$$

$\therefore$ ~~A_{ij}~~ ~~$A_{11} = \sin$~~

First column of A

$$\Rightarrow \begin{matrix} \sin(1)\cos(1) \\ \sin(2)\cos(1) \\ \sin(3)\cos(1) \\ \sin(4)\cos(1) \\ \sin(5)\cos(1) \end{matrix} \Rightarrow \begin{bmatrix} \sin(1) \\ \sin(2) \\ \sin(3) \\ \sin(4) \\ \sin(5) \end{bmatrix} \cos(1)$$

Similarly other columns are scaled versions of this column.

$\therefore$ ~~Rank~~ Rank(A) = 1

e.vals (0.159208284526419,
0,
0,
0,
0)

e.vectors for the corresponding e.vals.

eval = 0.1592.. evec = $\begin{pmatrix} 0.482054438978975 \\ 0.520910249868625 \\ 0.080843579329759 \\ -0.433550305215620 \\ -0.599340038565430 \end{pmatrix}$

other eigen vectors will be 0
as eigen values are 0.
But in MATLAB we get some e.vectors.
This is due to the truncation error introduced
in the calculation of the e.vals because of
which the evals are not exactly zero, due to
which they give rise to some eigen vectors.
But as rank(A) = 1 therefore there will
be 1 eigen vector of A.

MATLAB Code

assl_3.m | assl_4.m

```
clc;
clear;
format long;
digits(30);
A = eye(5);
for i = 1:5
    for j=1:5
        A(i,j)=sin(i)*cos(j);
    end
end

[V,d]=eig(A);
```

Computed Eigenvalues and Eigenvectors

```
V =

  Columns 1 through 3

   0.482054438978975 + 0.000000000000000i   0.336662033939936 + 0.000000000000000i   0.276064968493918 - 0.075620366464841i
   0.520910249868625 + 0.000000000000000i   0.559843964884533 + 0.000000000000000i   0.683212892662701 + 0.000000000000000i
   0.080843579329759 + 0.000000000000000i   0.086885965478399 + 0.000000000000000i   0.061295453135051 - 0.039272877355287i
  -0.433550305215620 + 0.000000000000000i  -0.465954589893416 + 0.000000000000000i  -0.468145566977027 - 0.112300279024072i
  -0.549340038565430 + 0.000000000000000i  -0.590398644176912 + 0.000000000000000i  -0.388349189750716 - 0.251800769350853i

  Columns 4 through 5

   0.276064968493918 + 0.075620366464841i   0.307215794831590 + 0.000000000000000i
   0.683212892662701 + 0.000000000000000i   0.395849377083625 + 0.000000000000000i
   0.061295453135051 + 0.039272877355287i  -0.005019444046615 + 0.000000000000000i
  -0.468145566977027 + 0.112300279024072i  -0.336476131092328 + 0.000000000000000i
  -0.388349189750716 + 0.251800769350853i  -0.797295644322168 + 0.000000000000000i


d =

  Columns 1 through 3

   0.159208284526419 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i
   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i
   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i  -0.000000000000000 + 0.000000000000000i
   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i
   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i

  Columns 4 through 5

   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i
   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i
   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i
  -0.000000000000000 - 0.000000000000000i   0.000000000000000 + 0.000000000000000i
   0.000000000000000 + 0.000000000000000i   0.000000000000000 + 0.000000000000000i
```

1.4

1.4.) $\dot{x}_1 = -\sin(x_1)(1+x_2)$

$\dot{x}_2 = x_1 + x_2$

Fixed Points

$\therefore f(x_1, x_2) = 0$

$\Rightarrow \quad x_1 + x_2 = 0 \quad (1) \Rightarrow x_2 = -x_1$

$-\sin(x_1)(1+x_2) = 0 \quad (2)$

$\hookrightarrow \quad x_1 = n\pi$ or $x_2 = -1$

Now if $x_2 = -1 \Rightarrow x_1 = 1$

f.p. 1 $\rightarrow \boxed{(1,-1)}$

if $x_1 = \pi \quad x_2 = -\pi$

f.p. 2 $\rightarrow \boxed{(\pi, \pi)}$

$x_1 = -\pi \quad x_2 = \pi$

f.p. 3 $\rightarrow \boxed{(-\pi, \pi)}$

$$D_{(x_1,x_2)} = \begin{bmatrix} -(\cos x_1)(1+x_2) & -\sin(x_1) \\ 1 & 1 \end{bmatrix}$$

Now f.p. 1 $D(1,-1) = \begin{bmatrix} 0 & -\sin(1) \\ 1 & 1 \end{bmatrix}$

e. val $= \dfrac{1 \pm \sqrt{1 - 4\sin(1)}}{2}$

$\Rightarrow$ As Real part > 0 the flow has exponential growth about this point. (Unstable Eq. pt).

e.vec.

$$e.v_1 = \begin{pmatrix} -0.368+0.566i \\ 0.736 \end{pmatrix} \qquad e.val_1 = 0.5+0.769i$$

$$e.v_2 = \begin{pmatrix} -0.368-0.56i \\ 0.736 \end{pmatrix} \qquad e.val_2 = 0.5-0.769i$$

<u>f.p2</u> $(\pi,-\pi)$

$$D_{(\pi,\pi)} = \begin{bmatrix} 1-\pi & 0 \\ 1 & 1 \end{bmatrix}$$

$$eval_1 = (1-\pi) = -2.141 \qquad e.vec_1 = \begin{pmatrix} 0.952 \\ -0.303 \end{pmatrix}$$

$$eval_2 = 1 \qquad e.vec_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>f.p.3</u> $(-\pi,\pi)$

$$D(-\pi,\pi) = \begin{bmatrix} 1+\pi & 0 \\ 1 & 1 \end{bmatrix}$$

$$e.val_1 = (1+\pi) = 4.141 \qquad evec_1 = \begin{pmatrix} 0.952 \\ 0.303 \end{pmatrix}$$

$$eval_2 = 1 \qquad evec_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Taking x^* as the f.p.

linearizing about x^*, we get

$$\dot{x} = f(x^*) + f'(x^*)(x - x^*)$$

Taking $x = x^* + \xi$ we get

$$\underbrace{\dot{x}^*}_{0} + \dot{\xi} = \underbrace{f(x^*)}_{0} + f'(x^*)(\xi)$$

$$\Rightarrow \dot{\xi} = f'(x^*)\,\xi$$

$\therefore$ for fp $= (-\pi, \pi)$

$$\begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} = \begin{bmatrix} 1+\pi & 0 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

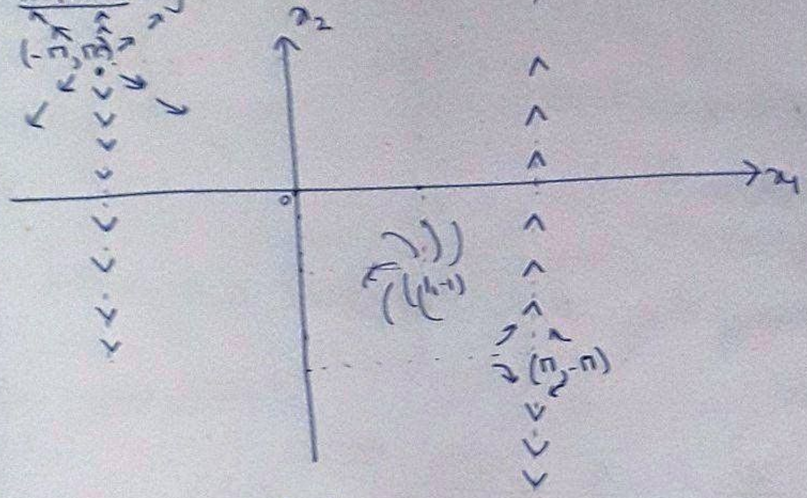
for fp $= (\pi, -\pi)$

$$\begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} = \begin{bmatrix} 1-\pi & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

for fp $= (-1, 1)$

$$\begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} = \begin{bmatrix} 0 & -\sin(1) \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

Intuitively



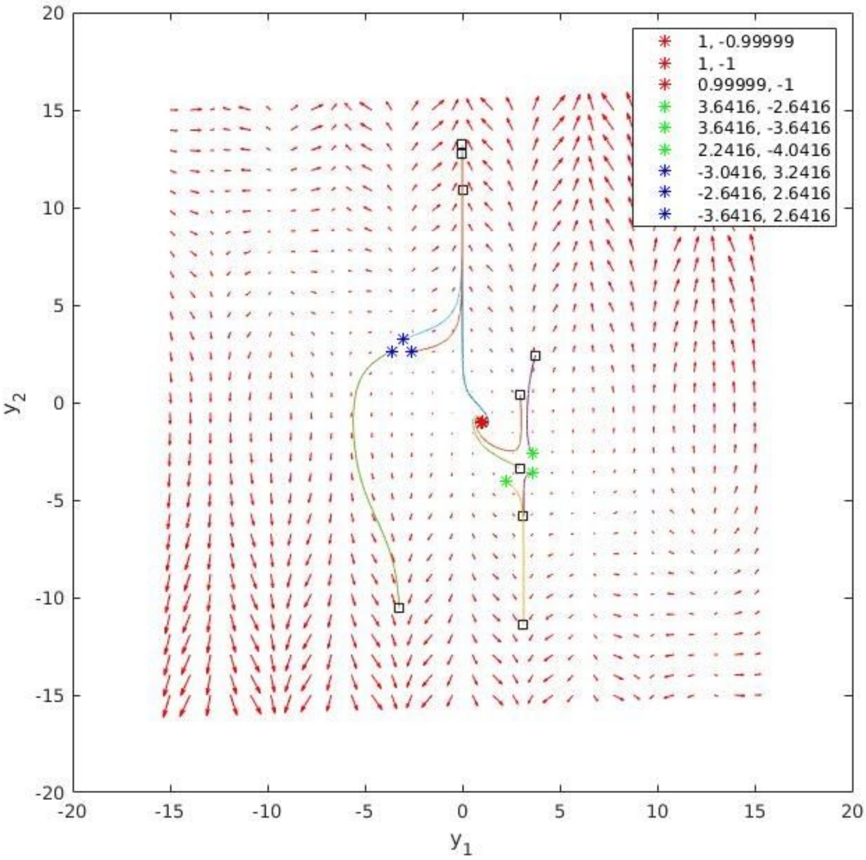
Now for fp.2 & fp.3

$\dot{x}_1 = 0$ & $\dot{x}_2 = 0$

If for these pts we check the derivatives on the $x_1 = \pi$ and $x_1 = -\pi$ line we see that $\dot{x}_1 = 0$ therefore if a pt is chosen on this line it stays on this line

At $\begin{pmatrix} 1 & -1 \\ 1 & \end{pmatrix}$ we should get a growing spiral as real part of e.val > 0 and e.val is complex.

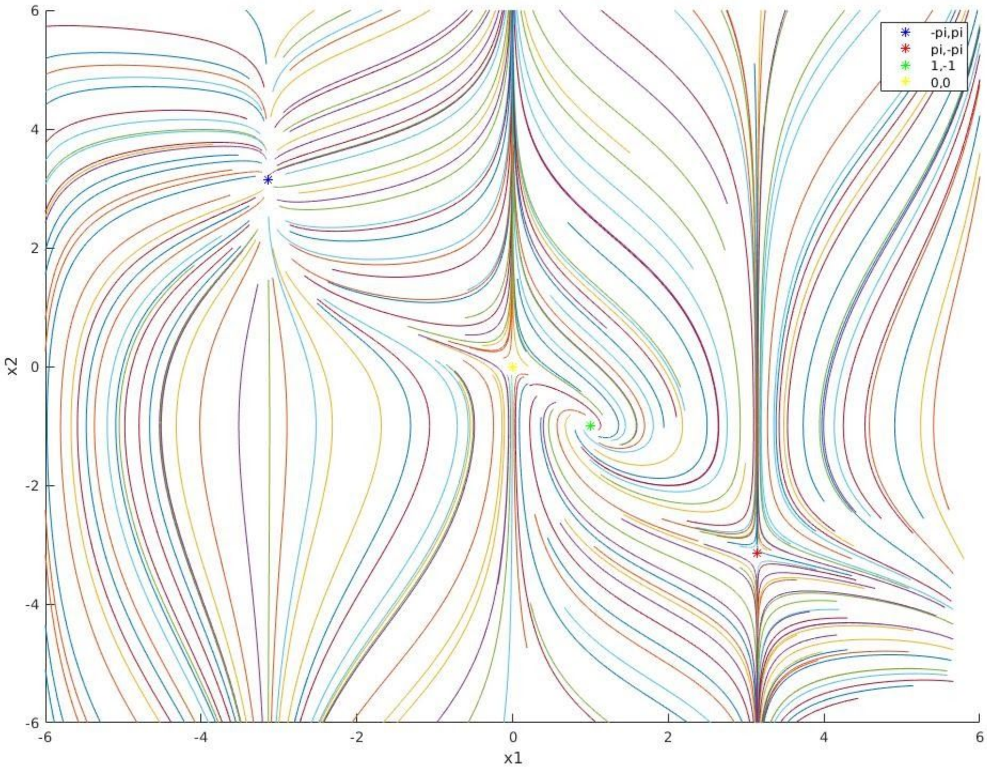
Generated Vector Field and some Trajectories



Here y1 = x1 and y2 = x2.

Phase Portraits

With the fixed points (-pi,pi), (0,0), (1,-1) and (pi,-pi)



MATLAB Code to produce the graphs:

```
clear;
clc;

format long;

options = odeset('AbsTol',1e-13,'RelTol',1e-13);

f = @(t,y) [-sin(y(1))*(1+y(2)),y(1)+y(2)]';
y1 = linspace(-15,15,30);
y2 = linspace(-15,15,30);

[x,y]=meshgrid(y1,y2);
size(x)
size(y)

u = zeros(size(x));
v = zeros(size(x));

% we can use a single loop over each element to compute the derivatives at
% each point (y1, y2)
t=0; % we want the derivatives at each point at t=0, i.e. the starting time
for i = 1:numel(x)
   Yprime = f(t,[x(i); y(i)]);
   u(i) = Yprime(1);
   v(i) = Yprime(2);
end

quiver(x,y,u,v,'r','HandleVisibility','off'); figure(gcf)
xlabel('y_1')
ylabel('y_2')
axis square;

ics1 = [1+1e-5 -1+1e-5; 1+1e-5 -1-1e-5; 1-1e-5 -1-1e-5];
hold on
 for k = 1:3
   [ts,ys] = ode45(f,[0,25],[ics1(k,1);ics1(k,2)],options);
   plot(ys(:,1),ys(:,2),'HandleVisibility','off')
   plot(ys(1,1),ys(1,2),'r*','DisplayName',string(ys(1,1))+', '+string(ys(1,2))) % starting point
   plot(ys(end,1),ys(end,2),'ks','HandleVisibility','off') % ending point
 end
hold on
ics2 = [pi+5e-1 -pi+5e-1; pi+5e-1 -pi-5e-1; pi-9e-1 -pi-9e-1];
 for k = 1:3
   [ts,ys] = ode45(f,[0,2],[ics2(k,1);ics2(k,2)],options);
   plot(ys(:,1),ys(:,2),'HandleVisibility','off')
   plot(ys(1,1),ys(1,2),'g*','DisplayName',string(ys(1,1))+', '+string(ys(1,2))) % starting point
```

```
   plot(ys(end,1),ys(end,2),'ks','HandleVisibility','off') % ending point
 end
hold on
ics3 = [-pi+1e-1 pi+1e-1; -pi+5e-1 pi-5e-1; -pi-5e-1 pi-5e-1];
 for k = 1:3
   [ts,ys] = ode45(f,[0,2],[ics3(k,1);ics3(k,2)],options);
   plot(ys(:,1),ys(:,2),'HandleVisibility','off')
   plot(ys(1,1),ys(1,2),'b*','DisplayName',string(ys(1,1))+', '+string(ys(1,2))) % starting point
   plot(ys(end,1),ys(end,2),'ks','HandleVisibility','off') % ending point
 end
legend
hold off
figure
axis([-6,6,-6,6])
hold on
for k=1:200
   x=ginput(1);
   [t,y]=ode45(f,[0,5],x',options);
   plot(y(:,1),y(:,2),'HandleVisibility','off');
end
hold on
plot(-pi,pi,'b*','DisplayName','-pi,pi')
plot( pi,-pi,'r*','DisplayName','pi,-pi')
plot(1,-1,'g*','DisplayName','1,-1')
plot(0,0,'y*','DisplayName','0,0')
```