Tutorial 1 - Probability and Inference

COMP9418 – Advanced Topics in Statistical Machine Learning

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Lecture: Propositional Logic and Probability Calculus

Topic: Questions from lecture topics

Last revision: Tuesday 27th May, 2025 at 12:50

Question 1

For each of the following pairs of sentences, decide whether the first sentence implies the second. If the implication does not hold, identify a world in which the first sentence is true, but the second is not.

a. $(A \Rightarrow B) \land \neg B$ and A.
b. $(A \lor \neg B) \land B$ and A.
c. $(A \lor B) \land (A \lor \neg B)$ and A.

a.

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \land \neg B$	$[(A \Rightarrow B) \land \neg B] \models A$
T	T	T	F	T
T	F	F	F	T
F	T	T	F	T
F	F	T	T	F

World this is not true is when $A=F$ and $B=F$

b.

A	B	$A \lor \neg B$	$(A \lor \neg B) \land B$	$[(A \lor \neg B) \land B] \models A$
T	T	T	T	T
T	F	T	F	T
F	T	F	F	T
F	F	T	F	T

Question 2

never co-occur

one is always true
covers all cases

Which of the following pairs of sentences are mutually exclusive? Which are exhaustive? If a pair of sentences are not mutually exclusive, identify a world in which they both hold. If a pair of sentences are not exhaustive, identify a world neither holds.

a. $A \lor B$ and $\neg A \lor \neg B$.
b. $A \lor B$ and $\neg A \land \neg B$.
c. A and $(\neg A \lor B) \land (\neg A \lor \neg B)$.

a.

A	B	$A \lor B$	$\neg A \lor \neg B$	mutually exclusive	exhaustive
T	T	T	F		
T	F	T	T	no	yes
F	T	T	T		
F	F	F	T		

b.

A	B	$A \lor B$	$\neg A \land \neg B$	mutually exclusive	exhaustive
T	T	T	F		
T	F	T	F		
F	T	T	F	yes	yes
F	F	F	T		

Question 3

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Suppose that 24% of the population are smokers and that 5% of the population has cancer. Suppose further that 86% of the population with cancer are also smokers. What is the probability that a smoker will also have cancer?

$P(\text{smoker}) = 24\%$
$P(\text{cancer}) = 5\%$
$P(\text{smoker}|\text{cancer}) = 86\%$

$$P(\text{cancer}|\text{smoker}) = \frac{0.86 \times 0.05}{0.24} = 0.179 = 17.9\%$$

Question 4

(After Koller & Friedman) Suppose a tuberculosis (TB) skin test is 95% accurate. If the patient is TB-infected, the test will be positive with a probability of 0.95; if the patient is not infected, then the test will be negative with a probability of 0.95. Now, suppose that a person gets a positive test result. What is the probability that he is infected? Suppose that 1 in 1000 of the subjects who get tested is infected.

To answer this question, provide the following intermediate quantities:

$P(TB = +) =$

$P(Test = +|TB = +) =$

$P(Test = +|TB = -) =$

$P(Test = +) =$

Which equation provides a direct answer to the question posed in this problem?

Question 5

Consider the following distribution over three variables:

A	B	C	$P(A, B, C)$
1	1	1	.27
1	1	0	.18
1	0	1	.03
1	0	0	.02
0	1	1	.02
0	1	0	.03
0	0	1	.18
0	0	0	.27

For each pair of variables, state whether they are independent. State also whether they are independent given the third variable. Justify your answer.

$P(x,y)=P(x)P(y)$

$P(x,y|z)=P(x|z)P(y|z)$ $x \perp\!\!\!\perp y|z$

Question 6

We have three binary random variables: season (S), temperature (T) and weather (W). Let us suppose you are given the following conditional probability distributions (CPDs):

$P(W,S,T)=P(W|S,T)\,P(T|S)\,P(S)$

S	$P(S)$
summer	0.5
winter	0.5

S	T	$P(T|S)$
summer	hot	0.7

S	T	$P(T\|S)$
summer	cold	0.3
winter	hot	0.3
winter	cold	0.7

S	T	W	$P(W\|S,T)$
summer	hot	sun	0.86
summer	hot	rain	0.14
summer	cold	sun	0.67
summer	cold	rain	0.33
winter	hot	sun	0.67
winter	hot	rain	0.33
winter	cold	sun	0.43
winter	cold	rain	0.57

Calculate the joint probability distribution $P(S,T,W)$ using the chain rule.

S	T	W	$P(S,T,W)$
summer	hot	sun	$0.86 \times 0.7 \times 0.5 = 0.301$
summer	hot	rain	
summer	cold	sun	
summer	cold	rain	
winter	hot	sun	
winter	hot	rain	
winter	cold	sun	
winter	cold	rain	

Question 7

(From Ben Lambert's book "A Student's Guide to Bayesian Statistics") Suppose that, in an idealised world, the ultimate fate of a thrown coin - head or tails - is deterministically given by the angle at which you throw the coin and its height above the table. Also, in this ideal world, the heights and angles are discrete. However, the system is chaotic[1] (highly sensitive to initial conditions), and the results of throwing a coin at a given angle (in degrees) and height (in meters) are shown in the following table.

Angle (degree)	0.2	0.4	0.6	0.8	1
0	T	H	T	T	H
45	H	T	T	T	T
90	H	H	T	T	H
135	H	H	T	H	T
180	H	H	T	H	H
225	H	T	H	T	T
270	H	T	T	T	H

[1] The authors of the following paper experimentally tested this and found it to be the case, "The three-dimensional dynamics of the die throw", Chaos, Kapitaniak et al. (2012).

Angle (degree)	0.2	0.4	0.6	0.8	1
315	T	H	H	T	T

a. Suppose all combinations of angles and heights are equally likely to be chosen. What is the probability that the coin lands heads up?

b. Now suppose that some combinations of angles and heights are more likely to be chosen than others, with the probabilities shown in the following table. What are the new probabilities that the coin lands heads up?

Angle (degree)	0.2	0.4	0.6	0.8	1
0	0.05	0.03	0.02	0.04	0.04
45	0.03	0.02	0.01	0.05	0.02
90	0.05	0.03	0.01	0.03	0.02
135	0.02	0.03	0.04	0.00	0.04
180	0.03	0.02	0.02	0.00	0.03
225	0.00	0.01	0.04	0.03	0.02
270	0.03	0.00	0.03	0.01	0.04
315	0.02	0.03	0.03	0.02	0.01

c. We force the coin thrower to throw the coin at an angle of 45 degrees. What is the probability that the coin lands heads up?

d. We force the coin-thrower to throw the coin at a height of 0.2m. What is the probability that the coin lands heads up?

e. If we constrained the angle and height to be fixed, what would happen in repetitions of the same experiment?

Question 8

(After Koller & Friedman) An often useful rule in dealing with probabilities is known as reasoning by cases. Let X, Y , and Z be random variables, then $P(X|Y) = \sum_z P(X, z|Y)$.

Prove this equality using the basic properties of (conditional) distributions.

$$\sum_z P(X, z|Y) = \sum_z \frac{P(X,z,Y)}{P(Y)} = \frac{\sum_z P(X,z,Y)}{P(Y)}$$

$$= \sum_z \frac{P(z|X,Y)\,P(X,Y)}{P(Y)}$$

$$= \frac{P(X,Y)\sum_z (z|X,Y)}{P(Y)}$$