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**Part IA**  
Cambridge Mathematical Tripos

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## Preface

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These are what I studied during the 2024 long vacation for transfer into part IB mathematical tripos. I guess nobody can tell if it was valuable, or if this was a right decision. Nevertheless, I do not want to study beam bending. Since it is more mathematics-oriented document, I will use the usual conventions, such as definition, lemma, etc.



# CHAPTER 1

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## Numbers and Sets

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Lecture given by Professor Julia Wolf, Michaelmas Term 2023.<sup>12</sup>

### 1.1 Sets, Functions, and Relations

As they contain concepts that we have not yet formally defined, one may want to revisit the examples on this section after studying them in detail later.

#### 1.1.1 Sets

A *set* is a collection of mathematical objects, e.g.

$$\mathbb{R}, \mathbb{N}, \{1, 5, 9\}, (-2, 3]$$

The order of elements in the set is immaterial, and elements are counted only once. For instance,  $\{1, 7, 3\} = \{1, 3, 7\}$  and  $\{3, 4, 4, 8\} = \{3, 4, 8\}$ . We write  $x \in A$  if  $x$  is an element of the set  $A$ , and  $x \notin A$  if not. Two sets are equal if they have the same elements. That is,  $A = B$  if and only if for all  $x$ ,

$$x \in A \Leftrightarrow x \in B$$

In particular, there is only one empty set  $\emptyset$ .

A set  $B$  is a *subset* of  $A$ , written  $B \subseteq A$ , if every element of  $B$  is an element of  $A$ .  $B$  is said to be a *proper subset* of  $A$  if  $B \subseteq A$  and  $B \neq A$ , and we write  $B \subset A$ .<sup>3</sup> Note that if  $A = B$ , then

$$A \subseteq B \text{ and } B \subseteq A$$

If  $A$  is a set and  $P$  is a property of (some) elements of  $A$ , we can write

$$\{x \in A : P(x)\}$$

---

<sup>1</sup>Subsection about sequences and series are omitted since they are also thoroughly covered in the ANALYSIS I course.

<sup>2</sup>Although I recommend listening lectures covering those parts to check if there are any distinctions.

<sup>3</sup>Some authors may write both  $\subset$  and  $\subseteq$  for subset, and only  $\subsetneq$  for proper subset.

or

$$\{x \in A \mid P(x)\}$$

for the subset of  $A$  comprising those elements for which  $P(x)$  holds. If  $A$  and  $B$  are sets, then their *union*,  $A \cup B$ , is

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Their *intersection*,  $A \cap B$ , is defined to be

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

and we say  $A$  and  $B$  are *disjoint* if  $A \cap B = \emptyset$ . Note that we can view intersection as a special case of subset selection:

$$A \cap B = \{x \in A : x \in B\}$$

Similarly, have *set differences*

$$A \setminus B = \{x \in A : x \notin B\}$$

Note that  $\cup$  and  $\cap$  are *commutative* and *associative*, i.e.

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

and

$$A \cup (B \cap C) = (A \cup B) \cap C, \quad A \cap (B \cup C) = (A \cap B) \cup C$$

Also,  $\cup$  is *distributive* over  $\cap$ , i.e.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

and  $\cap$  is distributive over  $\cup$  too, i.e.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Let's prove the last statement. If  $x \in A \cap (B \cup C)$ , then  $x \in A$  and  $x \in B \cup C$ , so  $x \in A$  and ( $x \in B$  or  $x \in C$ ). If  $x \in B$ , then  $x \in A \cap B$ , and if  $x \in C$ , then  $x \in A \cap C$ . In any case,  $x \in (A \cap B) \cup (A \cap C)$ . Conversely, if  $x \in (A \cap B) \cup (A \cap C)$ , then  $x \in A \cap B$  or  $x \in A \cap C$ . Either cases,  $x \in A$  and  $x \in B \cup C$ . Hence  $x \in A \cap (B \cup C)$ . This implies that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Given an index set  $I$  and a collection of sets  $A_i$  indexed by  $i \in I$ , we write

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \quad \forall i \in I\}$$

and

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

Given sets  $A$  and  $B$ , we can form their *cartesian product*

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

which is the set of *ordered pairs*  $(a, b)$  with  $a \in A$ ,  $b \in B$ . N.b. we can define  $(a, b) = \{\{a, b\}, a\}$ ; and also we can extend the definition to ordered triples  $(a, b, c)$  and so on.

For any set  $X$ , we can form the *power set*  $\mathcal{P}(X)$  consisting of all subsets of  $X$ , i.e.

$$\mathcal{P}(X) = \{Y : Y \subseteq X\}$$

For example, if  $X = \{1, 2\}$ ,  $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ .

Finally, observe that we cannot form a set such that

$$\{x : P(x)\}$$

for some property  $P$ . Suppose

$$X = \{x : x \text{ is a set and } x \notin x\}$$

were a set. Then  $X \in X$  implies  $X \notin X$ , which is a contradiction. On the other hand,  $X \notin X$  implies  $X \in X$ . This is known as *Russell's paradox*. Similarly, there is no 'universal' set  $Y$ , meaning a set  $Y$  such that  $\forall x, x \in Y$ . Otherwise, we could form  $X$  above by subset selection:

$$X = \{x \in Y : x \notin x\}$$

### 1.1.2 Functions

Given sets  $A$  and  $B$ , a function  $f$  from  $A$  to  $B$  is a *rule* that assigns to every  $x \in A$  a unique element  $f(x) \in B$ . More formally, a *function* from  $A$  to  $B$  is a subset

$$f \subseteq A \times B$$

such that for all  $x \in A$ , there is a unique  $y \in B$  such that  $(x, y) \in f$ . If  $f$  is a function from  $A$  to  $B$ , we write

$$f : A \rightarrow B$$

If  $(x, y) \in f$ , we can write

$$f(x) = y$$

or

$$x \mapsto y$$

We say  $f : A \rightarrow B$  is *injective* if  $\forall a, a' \in A$ ,

$$a \neq a' \Rightarrow f(a) \neq f(a')$$

Equivalently,  $f$  is injective if  $f(a) = f(a') \Rightarrow a = a'$ . We say  $f$  is *surjective* if  $\forall b \in B$ ,  $\exists a \in A$  such that  $f(a) = b$ . If  $f$  is both injective and surjective, we say  $f$  is *bijective*.

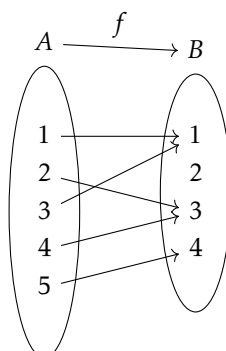
#### Example 1.1.1.

- (1)  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$  is a function.
- (2)  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 1/x$  is not a function.
- (3)  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \pm\sqrt{|x|}$  is not a function.
- (4)  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

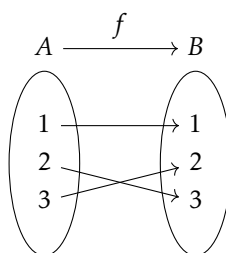
is a function.

(5)  $f : A \rightarrow B$  given by following



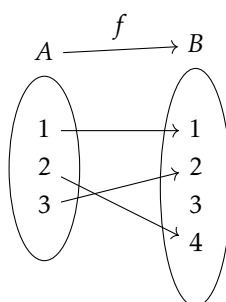
is a function, but it is neither injective nor surjective.

(6)  $f : A \rightarrow B$  given by following



is a bijection.

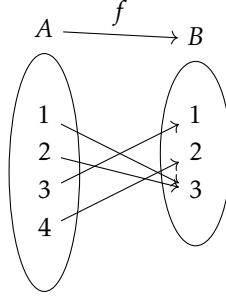
(7)  $f : A \rightarrow B$  given by following



is a injection, but not surjective.

(8)  $f : A \rightarrow B$  given by following





is a surjection, but not injective.

Given  $f : A \rightarrow B$ , we say  $A$  is the *domain* of  $f$  and  $B$  is its *codomain* (or *range*). The *image* of  $f$  is the set

$$\text{im}(f) = f(A) = \{f(a) : a \in A\} = \{b \in B : f(a) = b \text{ for some } a \in A\}$$

For example, image of a function  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$  is

$$\text{im}(f) = \{y \in \mathbb{R} : y \geq 0\}$$

Note that we must specify the domain and codomain of a function when specifying it. Observe that  $f$  is surjective if and only if  $f(A) = B$ . In particular, if  $|B| > |A|$ , then there can be no surjective function from  $A$  to  $B$ . Also, there is no injection from  $A$  to  $B$  if  $|A| > |B|$ .

For finite sets, if  $f : A \rightarrow A$ , then  $f$  is injective  $\Leftrightarrow f$  is surjective; and there is no bijection from  $A$  to any subset of  $A$ . N.b. they might not hold for infinite sets, e.g.  $f : \mathbb{N} \rightarrow \mathbb{N}, x \mapsto x + 1$  is injective but not surjective,  $g : \mathbb{N} \rightarrow \mathbb{N}$

$$x \mapsto \begin{cases} x - 1 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

is surjective but not injective, and  $h : \mathbb{N} \rightarrow \mathbb{N} \setminus \{1\}, x \mapsto x + 1$  is a bijection from  $\mathbb{N}$  to a proper subset.

**Example 1.1.2.**

- (1) For any set  $X$ , we say  $\text{id}_X : X \rightarrow X, x \mapsto x$  is the *identity function* on  $X$ .
- (2) Given a set  $X$  and  $A \subseteq X$ , we have the *indicator function* or *characteristic function* of  $A$ ,  $1_A : X \rightarrow \{0, 1\}$ ,

$$x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

- (3) A sequence of reals  $x_1, x_2, \dots$  is a function  $\mathbb{N} \rightarrow \mathbb{R}, n \mapsto x_n$ .
- (4) The operation  $+$  on  $\mathbb{N}$  is a function  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}, (a, b) \mapsto a + b$ .
- (5) A set  $X$  has size  $n$  if and only if there is a bijection

$$\{1, 2, \dots, n\} \rightarrow X = \{a_1, a_2, \dots, a_n\}, \quad i \mapsto a_i$$

Given  $f : A \rightarrow B$  and  $g : B \rightarrow C$ , the *composition*

$$g \circ f : A \rightarrow C$$

is defined by

$$a \mapsto g(f(a))$$

In general,  $\circ$  is not commutative. However,  $\circ$  is associative, i.e. given  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ ,  $h : C \rightarrow D$ , we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Indeed, for every  $x \in A$ ,

$$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$$

and

$$((h \circ g) \circ f)(x) = h \circ g(f(x)) = h(g(f(x)))$$

We may therefore drop the brackets and write  $h \circ g \circ f$  without ambiguity. We say  $f : A \rightarrow B$  is *invertible* if there exists  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$  and  $f \circ g = \text{id}_B$ , and we write  $g = f^{-1}$ .

**Proposition 1.1.1.**  $f$  is invertible if and only if it is bijective.

*Proof.* Given  $f : A \rightarrow B$ , suppose there is a map  $g : B \rightarrow A$  such that  $g \circ f = \text{id}_A$ . Then for  $a, a' \in A$  such that  $f(a) = f(a')$ ,  $g(f(a)) = g(f(a'))$  so  $a = a'$ . Thus  $f$  must be injective. Conversely, if  $f$  is injective, we can find  $g$  such that  $g \circ f = \text{id}_A$  if  $b \in f(A)$ . If  $b \in f(A)$ , We let  $g(b) = a$  where  $a$  is the unique element of  $A$  with  $f(a) = b$ ; otherwise, we may choose  $g(b)$  freely.

Meanwhile, for a map  $g : B \rightarrow A$  such that  $f \circ g = \text{id}_B$  to exist, we need  $f(g(B)) = B$ , hence  $f$  must be surjective. Conversely, if  $f$  is surjective, we can find  $g : B \rightarrow A$  with  $f \circ g = \text{id}_B$ : for each  $b \in B$ , pick some  $a \in A$  with  $f(a) = b$  and put  $g(b) = a$ . ■

N.b. given  $f : A \rightarrow B$  and  $U \subseteq B$ , we might use notation

$$f^{-1}(U) = \{a \in A : f(a) \in U\}$$

to denote the *preimage* of  $U$ , even when  $f$  does not have an inverse.

### 1.1.3 Relations

A *relation* on a set  $X$  is a subset

$$R \subseteq X \times X$$

and we write

$$aRb$$

for  $(a, b) \in R$  to denote that  $a$  is related to  $b$ . For example, " $aRb$  if  $a|b$ " is a relation on  $\mathbb{N}$ . There are three properties a relation might have are of special interest:

- $R$  is *reflective* if  $\forall x \in X, xRx$ .
- $R$  is *symmetric* if  $\forall x, y \in X, xRy \Rightarrow yRx$ .
- $R$  is *transitive* if  $\forall x, y, z \in X, xRy$  and  $yRz \Rightarrow xRz$ .

A relation  $R$  is an *equivalence relation* if it is reflective, symmetric, and transitive; and we can write  $a \sim b$  for  $aRb$  if  $R$  is an equivalence relation. For example, “ $a \sim b$  if  $a \equiv b \pmod{5}$ ” is a equivalence relation on  $\mathbb{Z}$ . This equivalence relation partitions  $\mathbb{Z}$  into pieces consisting of related elements,<sup>4</sup> namely

$$\{x \in \mathbb{Z} : x \equiv 0 \pmod{5}\}, \{x \in \mathbb{Z} : x \equiv 1 \pmod{5}\}, \dots, \{x \in \mathbb{Z} : x \equiv 4 \pmod{5}\}$$

Given a set  $X$ , a *partition* of  $X$  is a collection of pairwise disjoint subsets, i.e. *parts*, whose union is  $X$ . If  $\sim$  is an equivalent relation on  $X$ , then the *equivalence class* of  $x \in X$  is denoted by

$$[x] = \{y \in X : y \sim x\}$$

E.g. for the equivalence relation above,

$$[12] = \{y : y \equiv 2 \pmod{5}\}$$

**Theorem 1.1.2.** Let  $\sim$  be any equivalence relation on  $X$ . Then the equivalence classes form a partition of  $X$ .

*Proof.* Since  $\sim$  is reflective, we have  $x \in [x]$  for all  $x \in X$ . Thus

$$\bigcup_{x \in X} [x] = X$$

It remains to show that  $\forall x, y \in X$ , either  $[x] \cap [y] = \emptyset$ , or  $[x] = [y]$ . Suppose  $[x] \cap [y] \neq \emptyset$ , and let  $z \in [x] \cap [y]$ . Then  $z \sim x$ , so by symmetry,  $x \sim z$ . This with  $z \sim y$ , we have  $x \sim y$  by transitivity. Let now  $w \in [y]$ , so  $y \sim w$ . Since  $x \sim y$ , by transitivity,  $x \sim w$ . Thus,  $w \in [x]$ . Hence if  $[x] \cap [y] \neq \emptyset$ , then  $[y] \subseteq [x]$ . Similarly,  $[x] \subseteq [y]$ .

So  $[x] = [y]$ . ■

Conversely, given any partition of  $X$ , there is an equivalence relation  $R$  whose equivalence classes are precisely the parts of the partition: just define  $aRb$  if  $a$  and  $b$  lie in the same part.

Given an equivalence relation  $R$  on a set  $X$ , the *quotient of  $X$  by  $R$*  is

$$X/R = \{[x] : x \in X\}$$

The map  $q : X \rightarrow X/R$ ,  $x \mapsto [x]$  is the *quotient map* or *projection map*.

**Example 1.1.3.** On  $\mathbb{Z} \times \mathbb{N}$ , define  $(a, b)R(c, d)$  if  $ad = bc$ , where it turns out that  $R$  is an equivalence relation. E.g. one of the equivalence classes will be

$$[(1, 2)] = \{(1, 2), (2, 4), (3, 6), \dots\}$$

Observe that we may regard  $(\mathbb{Z} \times \mathbb{N})/R$  as a copy of  $\mathbb{Q}$ , by identifying  $[(a, b)]$  with  $a/b \in \mathbb{Q}$ . The quotient map is  $q : \mathbb{Z} \times \mathbb{N} \rightarrow (\mathbb{Z} \times \mathbb{N})/R$ ,  $(a, b) \mapsto a/b$ .

## 1.2 Numbers and Counting

### 1.2.1 Natural Number

Intuitively speaking, the natural numbers consists of

$$1, 1 + 1, 1 + 1 + 1, \dots$$

---

<sup>4</sup>Elements in each set are related.

But we cannot be sure if the list above contains all of the natural numbers, or if there are no duplicate numbers. Hence, for more rigorous understanding, we assume the natural numbers,  $\mathbb{N}$ , is a set containing “1”, with an operation “+” satisfying

- (1)  $\forall n \in \mathbb{N}, n + 1 \neq 1$ .
- (2)  $\forall m, n \in \mathbb{N}$ , if  $m \neq n$ , then  $m + 1 \neq n + 1$ .
- (3) For any property  $P(n)$ , if  $P(1)$  is true and  $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n + 1)$ , then  $P(n)$  is true for all natural numbers.

(1), (2), and (3) are known as the *Peano axioms*, and (3) is called the *induction axiom*. (1) and (2) capture the idea that any two natural numbers are distinct; (3) captures our intuition that the list is complete.<sup>5</sup> Now we can write 2 for  $1 + 1$ , 3 for  $1 + 1 + 1$  etc. and we can define “+ $k$ ” for any natural number  $k$ :

**Definition 1.2.1.** For every natural number  $n$ ,

$$n + (k + 1) = (n + k) + 1$$

It can be thought from induction, taking  $P(k) = “+k \text{ is defined}”$ . Similarly, we can define multiplication, powers etc. satisfying

- (1)  $\forall a, b, a + b = b + a$  (+ is commutative).
- (2)  $\forall a, b, ab = ba$  ( $\cdot$  is commutative).
- (3)  $\forall a, b, c, a + (b + c) = (a + b) + c$  (+ is associative).
- (4)  $\forall a, b, c, a(bc) = (ab)c$  ( $\cdot$  is associative).
- (5)  $\forall a, b, c, a(b + c) = ab + ac$  (multiplication is distributive over addition).

We define “ $a < b$ ” if  $a + c = b$  for some  $c \in \mathbb{N}$ , satisfying

- (6)  $\forall a, b, c, a < b \Rightarrow a + c < b + c$ .
- (7)  $\forall a, b, c, a < b \Rightarrow ac < bc$ .
- (8)  $\forall a, b, c, a < b \text{ and } b < c \Rightarrow a < c$ .
- (9)  $\forall a, \neg(a < a)$ .

The induction axiom is also known as the *weak principle of induction* (WPI). Equivalent form of this is the *strong principle of induction* (SPI):

**Theorem 1.2.1.** (*Strong Principle of Induction*) If

- (1)  $P(1)$  holds, and
- (2)  $\forall n \in \mathbb{N}$ , we have  $(P(m) \forall m \leq n) \Rightarrow P(n + 1)$ ,

then  $P(n)$  holds  $\forall n \in \mathbb{N}$ .

Clearly,  $\text{SPI} \Rightarrow \text{WPI}$ . To see that  $\text{WPI} \Rightarrow \text{SPI}$ , apply the former to

$$Q(n) = “P(m) \text{ holds } \forall m \leq n”$$

---

<sup>5</sup>Take  $P(n) = “n \text{ is on this list}”$ .

**Theorem 1.2.2.** (*Well-Ordering Principle (WOP)*) If  $P(n)$  holds for some  $n \in \mathbb{N}$ , then there is a least  $n \in \mathbb{N}$  such that  $P(n)$  holds. “Every non-empty subset of  $\mathbb{N}$  has a minimal element.”

**Theorem 1.2.3.** SPI is equivalent to WOP.

*Proof.* First we show that WOP implies SPI. We assume (1) and (2) of SPI, and show that  $P(n)$  holds  $\forall n \in \mathbb{N}$ , using WOP. Suppose, on the contrary, that  $P(n)$  is not true for all  $n \in \mathbb{N}$ . Then,

$$C = \{n \in \mathbb{N} \mid P(n) \text{ is false}\} \neq \emptyset$$

By well-ordering principle,  $C$  has a minimal element, say  $m$ . Now  $\forall k < m$ ,  $k \notin C$  (by minimality of  $m$ ), so  $P(k)$  holds  $\forall k < m$ . But by (2) of strong principle of induction,  $P(m)$  holds, contradicting  $m \in C$ . Hence SPI holds.

To show that SPI implies WOP, suppose there is no least  $n \in \mathbb{N}$  such that  $P(n)$  holds. We want to show that  $P(n)$  does not hold for any  $n \in \mathbb{N}$ , using SPI. Consider

$$Q(n) = \neg P(n)$$

Certainly  $P(1)$  is false,<sup>6</sup> so  $Q(1)$  holds. Given  $n \in \mathbb{N}$ , suppose that  $Q(k)$  is true  $\forall k < n$ . Then  $P(k)$  is false  $\forall k < n$ . So  $P(n)$  is false as otherwise  $n$  would be the minimal element for which  $P$  holds. Hence  $Q(n)$  is true, and (2) of SPI is satisfied, so  $Q(n)$  holds for all  $n \in \mathbb{N}$ . Thus  $P(n)$  is false  $\forall n \in \mathbb{N}$ . ■

Well-ordering principle enables us to prove  $P(n)$  is true  $\forall n \in \mathbb{N}$  as follows: if not, there is a minimal counterexample, and we may try and derive a contradiction.

### 1.2.2 Integer

The integer set,  $\mathbb{Z}$ , consist of all symbols  $n$ ,  $-n$ , where  $n \in \mathbb{N}$ , along with 0. We can also define  $+$  and  $\cdots$  etc. on  $\mathbb{Z}$  from  $\mathbb{N}$ , and check that the usual rules (1) - (5) of arithmetic hold. We also have

$$(10) \quad \forall a \in \mathbb{Z}, a + 0 = a \quad (\text{identity for } +).$$

$$(11) \quad \forall a \in \mathbb{Z}, \exists b \in \mathbb{Z} \text{ such that } a + b = 0 \quad (\text{inverses for } +). \text{ Here, } b = -a.$$

Similarly define  $a < b$  if  $a + c = b$  for some  $c \in \mathbb{N}$ . Rules (6), (8), and (9) continue to hold, but (7) must be modified:

$$(7) \quad \forall a, b, c \in \mathbb{Z}, a < b \text{ and } c > 0 \Rightarrow ac < bc.$$

### 1.2.3 Rational Number

The rational number set,  $\mathbb{Q}$ , consist of all expressions  $a/b$ , where  $a, b$  are integers,  $b \neq 0$ , and  $a/b$  and  $c/d$  are regarded as the same if  $ad = bc$ . Define

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and one can check it does not matters how we wrote  $a/b$  or  $c/d$  (commutivity). We similarly define multiplication, and

$$\frac{a}{b} < \frac{c}{d}$$

where  $b, d > 0$  if  $ad < bc$ . All the previous rules apply to rational numbers, but, furthermore,

<sup>6</sup>Else 1 would be the minimal element.

(12)  $\forall a \in \mathbb{Q}, a \neq 0, \exists b$  such that  $ab = 1$  (inverse for  $\cdot$ ).

Finally, note that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

### 1.2.4 Finite and Infinite Sets

#### \* Binomial coefficients

We write

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$$

to include 0 in the natural number set. Given  $n \in \mathbb{N}_0$ , we say a set  $A$  has *size*  $n$  if we can write

$$A = \{a_1, a_2, \dots, a_n\}$$

with the elements  $a_i$  distinct. For example,  $\{1, 3, 7\}$  has size 3,  $\emptyset$  has size 0. We say  $A$  is *finite* if  $\exists n \in \mathbb{N}_0$  such that  $A$  has size  $n$ , and  $A$  is *infinite* otherwise.

**Proposition 1.2.4.** A set of size  $n$  has exactly  $2^n$  subsets.

*Proof. (Method 1)* We may assume that our set is  $\{1, 2, \dots, n\}$ . To specify a subset  $S$  of  $\{1, 2, \dots, n\}$ , we must specify if  $1 \in S$  or  $1 \notin S$ , then  $2 \in S$  or  $2 \notin S$ , and so on. Hence the number of choices for  $S$  is

$$2 \cdot 2 \cdot 2 \cdots 2 = 2^n$$

■

*Proof. (Method 2)* We prove by induction on  $n$ . Clearly true for  $n = 0$ . Given  $n > 0$ , and  $T \subseteq \{1, 2, \dots, n-1\}$ , how many  $S \subseteq \{1, 2, \dots, n\}$  are there such that

$$S \cap \{1, 2, \dots, n-1\} = T$$

There are exactly 2, namely  $T$  and  $T \cup \{n\}$ . Hence the number of subsets of  $\{1, 2, \dots, n\}$  is  $2(2^{n-1}) = 2^n$  by the induction hypothesis. ■

If a set  $A$  has size  $n$ , we write

$$|A| = n$$

or  $\#A = n$ . So Proposition 1.2.4 states that

$$|A| = n \Rightarrow |\mathcal{P}(A)| = 2^n$$

Given  $n \in \mathbb{N}_0$  and  $0 \leq k \leq n$ , we write  $\binom{n}{k}$  for the number of subsets of an  $n$ -element set that are of size  $k$ . That is,

$$\binom{n}{k} = |\{S \subseteq \{1, 2, \dots, n\} : |S| = k\}|$$

$\binom{n}{k}$  is called a *binomial coefficient*. For example, the subsets of size 2 of  $\{1, 2, 3, 4\}$  are precisely  $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$  so  $\binom{4}{2} = 6$ . Note that by definition  $\binom{n}{0} = \binom{n}{n} = 1$ ,  $\binom{n}{1} = n$  (for  $n > 0$ ), and

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Also, we have

$$\binom{n}{k} = \binom{n}{n-k}$$

for all  $n \in \mathbb{N}_0$ ,  $0 \leq k \leq n$ . Moreover,

$$\binom{n}{k} = \binom{n-1}{k-1} \binom{n-1}{k}$$

for all  $n \in \mathbb{N}$ ,  $1 \leq k \leq n-1$ . N.b. we can use this property to obtain the *Pascal's triangle*.

**Proposition 1.2.5.** We have

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1} = \frac{n!}{(n-k)!k!}$$

*Proof.* Given a set of size  $n$ , there are  $n(n-1) \cdots (n-k+1)$  ways to pick  $k$  elements, in order, one by one. But each subset of size  $k$  is picked in  $k(k-1) \cdots 1$  ways by this method. Hence the number of subsets of size  $k$  in  $\{1, 2, \dots, n\}$  is

$$\frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1}$$

■

Note that the formula tells us, for example, that

$$\binom{n}{2} = \frac{n(n-1)}{2} \sim \frac{n^2}{2}$$

and

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} \sim \frac{n^3}{6}$$

for large  $n$ .

**Theorem 1.2.6.** (*Binomial Theorem*) For all  $a, b \in \mathbb{R}$ ,  $n \in \mathbb{N}$ ,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

*Proof.* When we expand

$$(a+b)^n = (a+b)(a+b) \cdots (a+b)$$

we obtain terms of the form  $a^{n-k}b^k$ ,  $0 \leq k \leq n$ , and the number of terms of the form  $a^{n-k}b^k$  is  $\binom{n}{k}$  as we must specify  $k$  brackets from which to pick  $b$ . Hence

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

■

✱ Sizes of sets

**Theorem 1.2.7.** (*Inclusion-Exclusion Principle*) Let  $S_1, S_2, \dots, S_n$  be finite sets. Then

$$|S_1 \cup S_2 \cup \dots \cup S_n| = \sum_{|A|=1} |S_A| - \sum_{|A|=2} |S_A| + \dots + (-1)^{n+1} \sum_{|A|=n} |S_A|$$

where  $S_A = \bigcap_{i \in A} S_i$  and  $\sum_{|A|=k}$  is taken over all  $A \subseteq \{1, 2, \dots, n\}$  of size  $k$ . Equivalently,

$$\left| \bigcup_{i=1}^n S_i \right| = \sum_{r=1}^n (-1)^{r+1} \sum_{|A|=r} \left| \bigcap_{i \in A} S_i \right|$$

Let  $x \in S_1 \cup S_2 \cup \dots \cup S_n$ , say  $x \in S_i$  for  $k$  of the  $S_i$ . We want  $x$  to be counted exactly once in the right hand side. Indeed,

$$\#\{A : |A| = 1 \text{ with } x \in S_A\} = k$$

$$\#\{A : |A| = 1 \text{ with } x \in S_A\} = \binom{k}{1}$$

and in general,

$$\#\{A : |A| = r \text{ with } x \in S_A\} = \begin{cases} \binom{k}{r} & r \leq k \\ 0 & r > k \end{cases}$$

Thus the number of times  $x$  is counted on the right hand side is

$$\begin{aligned} k - \binom{k}{2} + \binom{k}{3} - \dots + (-1)^{k+1} \binom{k}{k} &= 1 - \left( 1 - k + \binom{k}{2} - \binom{k}{3} + \dots + (-1)^k \binom{k}{k} \right) \\ &= (1 + (-1))^k + 1 = 1 \end{aligned}$$

for  $k \geq 1$  by the binomial theorem.

## 1.3 Elementary Number Theory

### 1.3.1 Primes

**Definition 1.3.1.** A natural number  $n \geq 2$  is *prime* if its only factors are  $\pm 1$  and  $\pm n$ . If  $n \geq 2$  is not prime, then it is *composite*.

**Proposition 1.3.1.** Every natural number  $n \geq 2$  can be written as a product of primes.

*Proof.* It is true for  $n = 2$ . Let  $n > 2$  and suppose that claim holds up to and including  $n - 1$ . If  $n$  is prime, done. If  $n$  is composite, then  $n = ab$  for some  $1 < a, b < n$ .

By the induction hypothesis, we have

$$\begin{aligned} a &= p_1 p_2 \dots p_k \\ b &= q_1 q_2 \dots q_l \end{aligned}$$

for some primes  $p_1, \dots, p_k, q_1, \dots, q_l$ . Hence

$$n = ab = p_1 \dots p_k q_1 \dots q_l$$

is a product of primes. Proof is complete by induction. ■



**Theorem 1.3.2.** There are infinitely many primes.

*Proof.* (Euclid, 300 B.C.) Suppose there are finitely many primes, say  $p_1, \dots, p_k$ . Let  $N = p_1 \cdots p_k + 1$ . Then  $p_1 \nmid N$ , else

$$p_1 \mid N - p_1 \cdots p_k = 1$$

Likewise, none of  $p_2, p_3, \dots, p_k$  divide  $N$ , contradicting the fact that  $N$  can be written as a product of primes (Proposition 1.3.1). ■

Can a number have more than one representation as a product of primes? Our proof of Proposition 1.3.1 does not give uniqueness.

**Definition 1.3.2.** Given  $a, b \in \mathbb{N}$ , a natural number  $c$  is the *highest common factor* (hcf) or *greatest common divisor* (gcd) of  $a$  and  $b$  if

- (1)  $c \mid a$  and  $c \mid b$ ;
- (2)  $d \mid a$  and  $d \mid b \Rightarrow d \mid c$ .

We write  $c = \text{hcf}(a, b)$ , or  $c = \text{gcd}(a, b)$ , or  $c = (a, b)$ .

For example, the factors of 12 are 1, 2, 3, 4, 6, 12, and those of 18 are 1, 2, 3, 6, 9, 18. So the common factors are 1, 2, 3, 6, hence  $\text{gcd}(12, 18) = 6$ . But observe that if  $a$  and  $b$  had common factors 1, 2, 3, 4, 6, then  $a$  and  $b$  would have no gcd according to Definition 1.3.2 (2). We will need to show that  $\text{gcd}(a, b)$  always exists.

**Proposition 1.3.3.** (*Division Algorithm*) Let  $n, k \in \mathbb{N}$ . Then we can write

$$n = qk + r$$

where  $q$  and  $r$  are integers with  $0 \leq r < k$ .

*Proof.* It is true for  $n = 1$ . Suppose  $n \geq 2$  and statement holds for  $n - 1$ , i.e.

$$n - 1 = qk + r$$

for some  $q, r \in \mathbb{Z}$ ,  $0 \leq r < k$ .

- If  $r < k - 1$ , then  $n = (n - 1) + 1 = qk + (r + 1)$ .
- If  $r = k - 1$ , then  $n = (n - 1) + 1 = qk + (k - 1) + 1 = (q + 1)k$ .

■

Note that  $q$  and  $r$  obtained by the division algorithm are unique: if  $n = qk + r = q'k + r'$ , then

$$(q - q')k = r' - r$$

is an integer smaller than  $k$  and larger than  $-k$  so  $q = q'$  and  $r = r'$ .

We now introduce the *Euclid's Algorithm*. We break down  $a$  and  $b$  until  $r_{n+1} = 0$  as follows;

$$\begin{aligned} a &= q_1 b + r_1 \\ b &= q_2 r_1 + r_2 \\ &\vdots \\ r_{n-2} &= q_n r_{n-1} + r_n \\ r_{n-1} &= q_{n+1} r_n + r_{n+1} \end{aligned}$$

and the algorithm returns  $r_n$ . N.b. the algorithm terminates in  $n < b$  steps, since

$$b > r_1 > r_2 > \dots > r_n > 0$$

**Theorem 1.3.4.** The output of Euclid's algorithm with input  $a, b$  is  $\gcd(a, b)$ .

*Proof.* We show (1) and (2) from Definition 1.3.2.

- (1) Have  $r_n/r_{n-1}$  (as  $r_{n+1} = 0$ ), so  $r_n/r_{n-2}$  and  $r_n/r_i \forall i = 1, 2, \dots, n-1$  by induction. Hence  $r_n/b$  and  $r_n/a$ .
- (2) Given  $d$  such that  $d|a$  and  $d|b$ , have  $d|r_1$ , so  $d|r_2$  and  $d|r_i \forall i = 1, 2, \dots, n$  by induction.

■

By the division algorithm and Euclid's algorithm, we have found out that the  $\gcd$  of any two natural numbers always exists, and is unique. For example, consider Euclid's algorithm with 87 and 52 as inputs:

$$\begin{aligned} 87 &= 1 \cdot 52 + 35 \\ 52 &= 1 \cdot 35 + 17 \\ 35 &= 2 \cdot 17 + 1 \\ 17 &= 17 \cdot 1 \end{aligned}$$

so  $\gcd(87, 52) = 1$ .

**Definition 1.3.3.** When  $\gcd(a, b) = 1$ , we say that  $a$  and  $b$  are *coprime*.

Observe that we can reverse the Euclid's algorithm:

$$\begin{aligned} 1 &= 35 - 2 \cdot 17 \\ &= 35 - 2(52 - 1 \cdot 35) \\ &= -2 \cdot 52 + 3 \cdot 35 \\ &= -2 \cdot 52 + 3(87 - 1 \cdot 52) \\ &= -5 \cdot 52 + 3 \cdot 87 \end{aligned}$$

**Theorem 1.3.5.**  $\forall a, b \in \mathbb{N}, \exists x, y \in \mathbb{Z}$  such that

$$xa + yb = \gcd(a, b)$$

"We can write  $\gcd(a, b)$  as a linear combination of  $a$  and  $b$ ."

*Proof.* (Method 1) Run Euclid's algorithm with input  $a, b$  to obtain an output  $r_n$ . At step  $n$ , have

$$r_n = xr_{n-1} + yr_{n-2}$$

for some  $x, y \in \mathbb{Z}$ . But from step  $n-1$ , we see that  $r_{n-1}$  is expressible as

$$r_{n-1} = xr_{n-2} + yr_{n-3}$$

for some  $x, y \in \mathbb{Z}$ , whence

$$r_n = xr_{n-2} + yr_{n-3}$$

for some  $x, y \in \mathbb{Z}$ . Continuing inductively, we have  $\forall i = 2, \dots, n-1$ ,

$$r_n = xr_i + yr_{i-1}$$

for some  $x, y \in \mathbb{Z}$ . Thus

$$r_n = xa + yb$$

for some  $x, y \in \mathbb{Z}$ , by steps one and two.

■

*Remark 1.3.1.* Euclid's algorithm not only proves the existence of  $x, y \in \mathbb{Z}$ , but gives a quick way to find them.

*Proof. (Method 2)* Let  $g$  be the least positive linear combination of  $a$  and  $b$ , i.e. the least positive integer of the form  $xa + yb$  for some  $x, y \in \mathbb{Z}$ . We shall show that  $g = \gcd(a, b)$  (Definition 1.3.2).

To see (2), observe that given  $d$  such that  $d|a$  and  $d|b$ , we have

$$d|ax + by \quad \forall x, y \in \mathbb{Z}$$

so in particular,  $d|g$ .

To see (1), suppose that  $g \nmid a$ . Then we can write

$$a = qg + r$$

for some  $q, r \in \mathbb{Z}$  with  $0 < r < g$ . Hence

$$r = a - qg = a - q(xa + yb)$$

is also a positive linear combination of  $a$  and  $b$ , and strictly smaller than  $g$ , contradicting the definition of  $g$ . Therefore,  $g|a$  and by the same argument,  $g|b$ . ■

*Remark 1.3.2.* (Method 2) tells us that  $\gcd(a, b)$  exists and is a linear combination of  $a$  and  $b$ , but gives no way to find  $\gcd(a, b)$  or the coefficients  $x, y \in \mathbb{Z}$ .

With help of Theorem 1.3.5, we can tell

$$160x + 72y = 33$$

does not have any integer solution, but

$$87x + 52y = 33$$

does. Let's formalise this.

**Corollary 1.3.5.1.** (*Bézout's Theorem*) Let  $a, b \in \mathbb{Z}$ . Then the equation

$$ax + by = c$$

has a solution in integers  $x, y$  if and only if

$$\gcd(a, b) | c$$

*Proof.* Let  $g = \gcd(a, b)$ . Suppose there are  $x, y \in \mathbb{Z}$  such that

$$ax + by = c$$

Then, since  $g|a$  and  $g|b$ ,  $g|c$ .

Conversely, suppose  $g|c$ . But Theorem 1.3.5 implies that there exist  $x, y \in \mathbb{Z}$  such that

$$g = ax + by$$

But then

$$c = \frac{c}{g}g = \frac{c}{g}(ax + by) = a\left(x\frac{c}{g}\right) + b\left(y\frac{c}{g}\right)$$

■

**Proposition 1.3.6.** If  $p$  is a prime and  $p|ab$ , then  $p|a$  or  $p|b$ .

*Proof.* Suppose  $p|ab$  but  $p \nmid a$ . We claim  $p|b$ .

Since  $p$  is prime and  $p \nmid a$ ,  $\gcd(a, p) = 1$ . Thus by Theorem 1.3.5, there exists  $x, y \in \mathbb{Z}$  such that  $xp + ya = 1$ . It follows that  $xpb + yab = b$ , whence  $b$  is a multiple of  $p$  (as each of  $p$  and  $ab$  is). ■

*Remark 1.3.3.*

(1) Similarly,

$$p|a_1 a_2 \cdots a_n \Rightarrow p|a_i$$

for some  $i = 1, 2, \dots, n$ . Indeed, Proposition 1.3.1 tells us that if  $p|a_1 a_2 \cdots a_n$ , the  $p|a_1$  or  $p|a_2 \cdots a_n$ , so we may conclude by induction on the number of terms in the product.

(2) We do need  $p$  to be prime.

**Theorem 1.3.7.** (*Fundamental Theorem of Arithmetic*) Every natural number  $n \geq 2$  is expressible as a product primes, uniquely up to reordering.

*Proof.* Existence of factorisation follows from Proposition 1.3.1. We can show uniqueness by induction. Clearly it is unique for  $n = 2$ . Given  $n \geq 2$ , suppose

$$n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_l$$

where  $p_i, q_j$  are all prime. Want to show that  $k = l$ , and, after reordering,  $p_i = q_i$   $\forall i = 1, \dots, k$ . We have

$$p_1 | n = q_1 q_2 \cdots q_l$$

so by Proposition 1.3.6,

$$p_1 | q_i$$

for some  $i$ . Relabelling the  $q_i$ , we may assume that  $p_1 | q_1$ . Since  $q_1$  is prime,  $p_1 = q_1$ , so

$$\frac{n}{p_1} = p_2 \cdots p_k = q_2 \cdots q_l < n$$

By the induction hypothesis,  $k = l$ , and, after reordering,  $p_2 = q_2, \dots, p_k = q_k$ . ■

*Remark 1.3.4.* There are arithmetical systems (permitting addition and multiplication) in which factorisation is not unique.

For example, consider  $\mathbb{Z}[\sqrt{-3}]$ , meaning all complex numbers of the form  $x + y\sqrt{-3} = x + y\sqrt{3}i$  where  $x, y \in \mathbb{Z}$ . We can add and multiply two elements of  $\mathbb{Z}[\sqrt{-3}]$  to get another element of  $\mathbb{Z}[\sqrt{-3}]$ , e.g.

$$(1 + \sqrt{-3}) + (q - \sqrt{-3}) = 2$$

$$(1 + \sqrt{-3})(q - \sqrt{-3}) = 4$$

In  $\mathbb{Z}[\sqrt{-3}]$ , we can define what it means to be a prime, and both  $1 + \sqrt{-3}$  and  $1 - \sqrt{-3}$  happen to be prime in this sense. But, we can also write  $4 = 2 \cdot 2$ , so factorisation is not unique.

Let's take a look at some applications of the fundamental theorem of arithmetic.

- (1) What are the factors of  $n = 2^3 3^7 11$ ? – All numbers of the form  $2^a 3^b 11^c$  where  $0 \leq a \leq 3, 0 \leq b \leq 7, 0 \leq c \leq 1$ . There are no others: if, for example,  $7|n$ , then we would have a factorisation of  $n$  involving 7, contradicting uniqueness. More generally, the factors of  $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$  are precisely the numbers of the form

$$p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$$

with  $0 \leq b_i \leq a_i \forall i = 1, 2, \dots, k$ .

- (2) What are common factors of  $2^3 \cdot 3^7 \cdot 5 \cdot 11^3$  and  $2^4 \cdot 3^2 \cdot 11 \cdot 13$ ? – All numbers of the form  $2^a 3^b 11^c$  where  $0 \leq a \leq 3, 0 \leq b \leq 2, 0 \leq c \leq 1$ . Thus the gcd is  $2^3 \cdot 3^2 \cdot 11$ . In general, the gcd of  $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$  and  $p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ , where  $a_i, b_i \geq 0$ , is

$$p_1^{\min\{a_1, b_1\}} \cdots p_k^{\min\{a_k, b_k\}}$$

- (3) What are the common multiples of the two numbers in (2)? – All numbers of the form  $2^a 3^b 5^c 11^d 13^e$  where  $a \geq 4, b \geq 7, c \geq 1, d \geq 3, e \geq 1$ , times any integer. Hence  $2^4 \cdot 3^7 \cdot 5 \cdot 11^3 \cdot 13$  is a common multiple, and any other common multiple is a multiple of it. We say that it is the *least common multiple* (lcm) of the two numbers. In general, the lcm of  $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$  and  $p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ , where  $a_i, b_i \geq 0$ , is

$$p_1^{\max\{a_1, b_1\}} \cdots p_k^{\max\{a_k, b_k\}}$$

Since  $\min\{a_i, b_i\} + \max\{a_i, b_i\} = a_i + b_i$ , we have

$$\gcd(x, y) \cdot \text{lcm}(x, y) = xy$$

for any  $x, y$ .

- (4) Another proof of Theorem 1.3.2, due Erdős (1930): let  $p_1, p_2, \dots, p_k$  be all the primes. Any number which is a product of just these primes is of the form (\*):

$$p_1^{j_1} p_2^{j_2} \cdots p_k^{j_k} = m^2 p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$$

where  $i_l = 0$  or  $1$ . Let  $M \in \mathbb{N}$ . If a number is equal or less than  $M$  is of the form (\*), then  $m \leq \sqrt{M}$ . So there are at most  $\sqrt{M} 2^k$  numbers of the form (\*) that are equal or less than  $M$ . If  $M > \sqrt{M} 2^k$ , i.e. if  $M > 4^k$ , then there must be a number equal or less than  $M$  which is not of the form (\*). But this number must have a prime factor not amongst the  $p_1, \dots, p_k$ . ■

Proof by Euclid tells us that  $k^{\text{th}}$  prime is less than  $2^{2^k}$ , while the proof by Erdős tells us  $k^{\text{th}}$  prime is less than  $4^k$ . In fact, we know  $k^{\text{th}}$  prime is  $\sim k \ln k$  (*Prime Number Theorem*).

### 1.3.2 Modular Arithmetic

Let  $n \geq 2$  be a natural number. Then the *integer modulo  $n$* , written as  $\mathbb{Z}_n$  or  $\mathbb{Z}/n\mathbb{Z}$  consist of the integers, with two regarded as the same if they differ by a multiple of  $n$ . If  $x$  and  $y$  are the same in  $\mathbb{Z}_n$ , we write

$$x \equiv y \pmod{n}$$

or  $x \equiv y \pmod{n}$  or  $x = y$  in  $\mathbb{Z}_n$ . Thus

$$\begin{aligned} x \equiv y \pmod{n} &\Leftrightarrow n \mid x - y \\ &\Leftrightarrow x = y + kn \text{ for some } k \in \mathbb{Z} \end{aligned}$$

N.b. if  $a \equiv a' \pmod{n}$  and  $b \equiv b' \pmod{n}$ , then

$$n \mid (a - a') + (b - b') = (a + b) - (a' + b')$$

so  $a + b \equiv a' + b' \pmod{n}$ . Similarly,

$$n \mid (a - a')b + a'(b - b') = ab - a'b'$$

so  $ab \equiv a'b' \pmod{n}$ . Hence we can do arithmetic on modulo  $n$  with inheriting the usual rules of arithmetic in  $\mathbb{Z}$ .

**Example 1.3.1.** Does  $2a^2 + 3b^3 = 1$  have a solution with  $a, b \in \mathbb{Z}$ ?

If there is a solution, then  $2a^2 \equiv 1 \pmod{3}$ , but  $2 \cdot 0^2 \equiv 0$ ,  $2 \cdot 1^2 \equiv 2$ ,  $2 \cdot 2^2 \equiv 2 \pmod{3}$  so there is no solution.

#### \* Solution of congruences

**Example 1.3.2.** Solve  $7n \equiv 2 \pmod{10}$ .

We note  $3 \cdot 7 \equiv 1 \pmod{10}$  so  $3 \cdot 7n \equiv 3 \cdot 2 \pmod{10}$  and  $n \equiv 6 \pmod{10}$ .

As shown in the example above, given  $a, b \in \mathbb{Z}$ , we say that  $b$  is an *inverse of  $a$  modulo  $n$*  if

$$ab \equiv 1 \pmod{n}$$

We say that  $a$  is *invertible modulo  $n$*  or that  $a$  is a *unit modulo  $n$* , if it has an inverse. For example, in  $\mathbb{Z}_{10}$ , 3 is an inverse of 7, and both 3 and 7 are units modulo 10; but 4 is not a unit modulo 10 since  $4n \not\equiv 1 \pmod{10}$  for all  $n \in \mathbb{Z}$ .

**Remark 1.3.5.** If  $a$  is a unit modulo  $n$ , then

- (1) its inverse is unique. Suppose  $\exists b, b'$  such that

$$ab \equiv ab' \equiv 1 \pmod{n}$$

then

$$b \equiv bab \equiv bab' \equiv b' \pmod{n}$$

- (2) we can write  $a^{-1}$  for its inverse.  
 (3) if  $ab \equiv ac \pmod{n}$ , then  $b \equiv c \pmod{n}$ .

However, this is not true in general, e.g.

$$4 \cdot 3 \equiv 4 \cdot 8 \pmod{10}$$

but  $3 \not\equiv 8 \pmod{10}$ .

**Proposition 1.3.8.** Let  $p$  be prime. Then every  $a \not\equiv 0 \pmod{p}$  is a unit modulo  $p$ .

*Proof.* From  $\gcd(a, p) = 1$ ,  $\exists x, y \in \mathbb{Z}$  such that  $ax + py = 1$  (Corollary 1.3.5.1). Hence  $ax \equiv 1 - py$ , so

$$ax \equiv 1 \pmod{p}$$

for some  $x \in \mathbb{Z}$ . ■

**Proposition 1.3.9.** Let  $n \geq 2$ . Then  $a$  is a unit modulo  $n$  if and only if  $\gcd(a, n) = 1$ .

*Proof.*

$$\begin{aligned}\gcd(a, n) = 1 &\Leftrightarrow ax + ny = 1 \text{ for some } x, y \in \mathbb{Z} \\ &\Leftrightarrow ax = 1 - ny : \text{ for some } x, y \in \mathbb{Z} \\ &\Leftrightarrow ax \equiv 1 \pmod{n} : \text{ for some } x, y \in \mathbb{Z}\end{aligned}$$

**Corollary 1.3.9.1.** If  $\gcd(a, n) = 1$ , then the congruence

$$ax \equiv b \pmod{n}$$

has a unique solution. In particular, if  $\gcd(a, n) = 1$ , then there is a unique inverse of  $a$  modulo  $n$ .

For cases like  $ax \equiv b \pmod{n}$  with  $\gcd(a, n) = d \neq 1$ , the solution may not exist.  $n|ax - b$  so  $d|ax - b$  and thus  $d|b$  for solution to exist. If  $d|b$ , write  $n = dn'$ ,  $a = da'$ ,  $b = db'$ . Then,

$$\begin{aligned}ax \equiv b \pmod{n} &\Leftrightarrow ax - b = kn \text{ for some } k \in \mathbb{Z} \\ &\Leftrightarrow da'x - db' = kdn' : \text{ for some } k \in \mathbb{Z} \\ &\Leftrightarrow a'x - b' = kn' : \text{ for some } k \in \mathbb{Z} \\ &\Leftrightarrow a'x \equiv b' \pmod{n'}\end{aligned}$$

Since  $\gcd(a', n') = 1$ ,  $a'x \equiv b' \pmod{n'}$  has a unique solution.

**Example 1.3.3.** Solve  $7x \equiv 4 \pmod{30}$ .

We have  $\gcd(7, 30) = 1$ , so by Euclid's algorithm,  $13 \cdot 7 - 3 \cdot 30 = 1$ . Hence  $13 \cdot 7 \equiv 1 \pmod{30}$ , whence

$$x \equiv 13 \cdot 4 \equiv 22 \pmod{30}$$

**Example 1.3.4.** Solve  $10x \equiv 12 \pmod{34}$ .

This is equivalent with  $5x \equiv 6 \pmod{17}$ , and we can now solve as the example above to obtain  $x \equiv 6 \cdot 7 \equiv 8 \pmod{17}$ .

### \* Simultaneous congruences

Note

$$x \equiv 5 \pmod{12} \Rightarrow \begin{cases} x \equiv 2 \pmod{3} \\ x \equiv 1 \pmod{4} \end{cases}$$

However, we cannot be sure if the converse is true, i.e. does  $x \equiv 2 \pmod{3}$  and  $x \equiv 1 \pmod{4}$  imply that  $x \equiv 5 \pmod{12}$ ? It turns out that it is true for the case above, but it is not generally true – see  $x \equiv 1 \pmod{4}$  and  $x \equiv 2 \pmod{6}$ .

**Theorem 1.3.10.** (*The Chinese Remainder Theorem*) Let  $m, n$  be coprime, and  $a, b \in \mathbb{Z}$ . Then there is a unique solution modulo  $mn$  to the simultaneous congruences

$$x \equiv a \pmod{m} \text{ and } x \equiv b \pmod{n}$$

That is, there is a solution  $x$  to

$$\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases}$$

and  $y$  is a solution if and only if  $x \equiv y \pmod{mn}$ .

*Proof.* We first prove existence. Since  $\gcd(m, n) = 1$ ,  $\exists s, t \in \mathbb{Z}$  with  $sm + tn = 1$ . Note

$$sm \equiv 1 \pmod{n}, \quad sm \equiv 0 \pmod{m}$$

and

$$tn \equiv 1 \pmod{m}, \quad tn \equiv 0 \pmod{n}$$

Hence

$$\begin{aligned} x = a(tn) + b(sm) &\equiv a \pmod{m} \\ &\equiv b \pmod{n} \end{aligned}$$

Next, for uniqueness, suppose  $y$  is also a solution, i.e.  $y \equiv a \pmod{m}$  and  $y \equiv b \pmod{n}$ . This implies

$$\begin{aligned} y \equiv x \pmod{m} \text{ and } y \equiv x \pmod{n} &\Leftrightarrow m|y-x \text{ and } n|y-x \\ &\Leftrightarrow mn|y-x \text{ since } \gcd(m, n) = 1 \\ &\Leftrightarrow y \equiv x \pmod{mn} \end{aligned}$$

■

*Remark 1.3.6.* Theorem 1.3.10 can be extended, by induction, to more than two moduli:

if  $m_1, m_2, \dots, m_k$  are pairwise coprime, then  $\forall a_1, a_2, \dots, a_k \in \mathbb{Z}$ ,  $\exists x \in \mathbb{Z}$  such that

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_k \pmod{m_k} \end{aligned}$$

**Definition 1.3.4.** We define the *Euler totient function*,  $\varphi(m)$ , as the number of integers  $a$  with  $1 \leq a \leq m$  such that  $\gcd(a, m) = 1$ , i.e.  $\varphi(m)$  is the number of units modulo  $m$ . Additionally, we define  $\varphi(1) = 1$ .

For example, when  $p$  is prime,  $\varphi(p) = p - 1$ , and  $\varphi(p^2) = p^2 - p$ . When  $p, q$  are distinct primes,

$$\varphi(p, q) = pq - p - q + 1$$

Next, let's take a look at behavior of a power of an integer modulo  $p$ .

**Theorem 1.3.11.** (*Fermat's Little Theorem*) Let  $p$  be a prime. Then  $a^p \equiv a \pmod{p}$  for all  $a \in \mathbb{Z}$ . Equivalently,

$$a^{p-1} \equiv 1 \pmod{p}$$

for all  $a \not\equiv 0 \pmod{p}$ .

*Proof.* If  $a \not\equiv 0 \pmod{p}$ , then  $a$  is a unit mod  $p$ . Thus  $ax \equiv ay \pmod{p}$  if and only if  $x \equiv y \pmod{p}$ . Hence the numbers  $a, 2a, \dots, (p-1)a$  are pairwise incongruent (distinct) modulo  $p$ , and  $\not\equiv 0 \pmod{p}$ , so they are  $1, 2, 3, \dots, p-1$  in some order. Hence

$$a \cdot 2a \cdots (p-1)a \equiv 1 \cdot 2 \cdots (p-1) \pmod{p}$$

or

$$a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}$$

But  $(p-1)!$  is a unit mod  $p$  (since it is a product of units), so we can cancel it to obtain  $a^{p-1} \equiv 1 \pmod{p}$ . ■



**Theorem 1.3.12.** (*Fermat-Euler Theorem*) Let  $\gcd(a, m) = 1$ . Then  $a^{\varphi(m)} \equiv 1 \pmod{m}$ .

*Proof.* Let

$$U = \{x \in \mathbb{Z} \mid 0 < x < m, \gcd(x, m) = 1\}$$

be the  $\varphi(m)$  numbers that are coprime to  $m$ . Label them  $u_1, u_2, \dots, u_{\varphi(m)}$ . Then  $au_1, au_2, \dots, au_{\varphi(m)}$  are all distinct and invertible modulo  $m$  (since  $a$  is a unit), and hence they are  $u_1, u_2, \dots, u_{\varphi(m)}$  in some order. It follows that

$$au_1 \cdot au_2 \cdots au_{\varphi(m)} \equiv u_1 u_2 \cdots u_{\varphi(m)} \pmod{m}$$

i.e.

$$a^{\varphi(m)} z \equiv z \pmod{m}$$

where  $z = u_1 u_2 \cdots u_{\varphi(m)}$  is a product of unit modulo  $m$ , whence itself a unit. Thus we may cancel it to obtain the theorem. ■

**Lemma 1.3.13.** Let  $p$  be prime. Then

$$x^2 \equiv 1 \pmod{p} \iff x \equiv 1 \text{ or } -1 \pmod{p}$$

*Proof.*

$$\begin{aligned} x^2 \equiv 1 \pmod{p} &\iff x^2 - 1 \equiv 0 \pmod{p} \\ &\iff (x+1)(x-1) \equiv 0 \pmod{p} \end{aligned}$$

But by Proposition 1.3.6,  $x+1 \equiv 0 \pmod{p}$  or  $x-1 \equiv 0 \pmod{p}$ . Therefore,  $x \equiv -1$  or  $1 \pmod{p}$ . ■

More generally, a non-zero polynomial of degree  $k$  over  $\mathbb{Z}_p$  has at most  $k$  roots in  $\mathbb{Z}_p$ .

**Theorem 1.3.14.** (*Wilson's Theorem*) Let  $p$  be prime. Then  $(p-1)! \equiv -1 \pmod{p}$ .

*Proof.* It is true for  $p = 2$ . Now assume  $p > 2$ . Note that the units modulo  $p$  comes in pairs whose product is 1, together with some elements that are self-inverse, i.e. elements  $x$  such that  $x^2 \equiv 1 \pmod{p}$ . By Lemma 1.3.13, the elements that are self-inverse are 1 and  $-1$ , so the remaining  $p-3$  units of  $\mathbb{Z}_p$  come in inverse pairs. Hence  $(p-1)!$  is the product of  $(p-3)/2$  pairs of inverses together with 1 and  $-1$ , giving  $(p-1)! \equiv -1 \pmod{p}$ . ■

**Proposition 1.3.15.** Let  $p$  be an odd prime. Then  $-1$  is a square modulo  $p$ , i.e.  $\exists x \in \mathbb{Z}$  such that  $x^2 \equiv -1 \pmod{p}$ , if and only if  $p \equiv 1 \pmod{4}$ .

*Proof.* Suppose  $p \equiv 1 \pmod{4}$ . By Wilson's theorem,

$$\begin{aligned} -1 &\equiv (p-1)! \equiv 1 \cdot 2 \cdots \left(\frac{p-1}{2}\right) \left(\frac{p+1}{2}\right) \cdots (p-3)(p-2)(p-1) \\ &\equiv 1 \cdot 2 \cdots \left(\frac{p-1}{2}\right) \left(-\frac{p-1}{2}\right) \cdots (-3)(-2)(-1) \\ &= (-1)^{(p-1)/2} \left(\left(\frac{p-1}{2}\right)!\right)^2 \pmod{p} \end{aligned}$$

But if  $p = 4k + 1$  for some  $k \in \mathbb{Z}$ , then

$$-1 \equiv (-1)^{2k} \left( \left( \frac{p-1}{2} \right)! \right)^2 \pmod{p}$$

so  $-1$  is a square modulo  $p$ .

Suppose, on the other hand, that  $p \equiv -1 \pmod{4}$ , i.e.  $p = 4k + 3$  for some  $k \in \mathbb{Z}$ . If  $-1$  were a square modulo  $p$ , i.e. if there were  $z \in \mathbb{Z}$  such that  $z^2 \equiv -1 \pmod{p}$ , then by Fermat's little theorem,

$$1 \equiv z^{p-1} \equiv z^{4k+2} \equiv z^{2(2k+1)} \equiv (-1)^{2k+1} \equiv -1 \pmod{p}$$

which is a contradiction. ■

*Remark 1.3.7.* When  $p \equiv 1 \pmod{4}$ , Wilson's theorem tells us a solution to

$$x^2 \equiv -1 \pmod{p}$$

For example, when  $p = 29 = 4 \cdot 7 + 1$ ,  $x = (2 \cdot 7)!$  works.

### 1.3.3 Public Key Cryptography

In this subsection, we look into an example of practical application of the number theory, *public key cryptography*. Let us agree to write message as sequences of numbers. To send the message as encrypted form in such a way that only the reciever can decrypt them easily, we use the RSA (Rivest, Shamir, Adleman) scheme. To apply the RSA scheme, reciever do as follows.

- Think of two large primes  $p, q$ .
- Let  $n = pq$ , and pick an *encoding exponent*  $e$ , coprime to  $\varphi(n) = (p-1)(q-1)$ .
- Publish the pair  $(n, e)$ , *encryption scheme* to public.

Now, to send an encrypted message,

- Sender chops the message into pieces/numbers  $M < n$ ;
- and send  $M^e \pmod{n}$ , computed quickly by repeated squaring  $\pmod{n}$ .

Then, to decrypt it,

- Reciever work out the *decoding exponent*  $d$ , such that  $ed \equiv 1 \pmod{\varphi(n)}$ ;
- and compute

$$(M^e)^d = M^{k\varphi(n)+1} \equiv M \pmod{n}$$

(consider both cases  $M \equiv 0 \pmod{p}$  and  $M \not\equiv 0 \pmod{p}$ ).

Note that in order to decrypt in this way, we require  $n$  and  $d$ , or  $n, e$  and  $\varphi(n)$ . Finding  $\varphi(n)$  is as difficult as finding the prime factors of  $n$ , which is believed to be computationally hard.

## 1.4 The Real Numbers

### 1.4.1 Real Numbers

We have seen  $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ . Now it is time to extend the boundaries of numbers once again.

**Proposition 1.4.1.** There is no rational  $x$  with  $x^2 = 2$ .

*Proof. (Method 1)* Suppose  $x^2 = 2$ . Note we may assume  $x > 0$  since  $(-x)^2 = x^2$ . If  $x$  is rational, then  $x = a/b$  for some  $a, b \in \mathbb{N}$ . Thus  $a^2/b^2 = 2$ , or

$$a^2 = 2b^2$$

But the exponent of 2 in the prime factorisation of  $a^2$  is even while the exponent of 2 in the prime factorisation of  $2b^2$  is odd, contradicting the Fundamental Theorem of Arithmetic. Thus  $x \notin \mathbb{Q}$ . ■

N.b. the same proof shows that if  $\exists x \in \mathbb{Q}$  with  $x^2 = n$  for some  $n \in \mathbb{N}$ , then  $n$  must be a square.

*Proof. (Method 2)* Suppose  $x^2 = 2$  for some  $x = a/b$  with  $a, b \in \mathbb{N}$ . Then for any  $c, d \in \mathbb{Z}$ ,  $cx + d$  is of the form  $e/b$  for some  $e \in \mathbb{Z}$ . Thus if  $cx + d > 0$ , then

$$cx + d \geq \frac{1}{b}$$

But  $0 < x - 1 < 1$  as  $1 < x < 2$  so if  $n$  is sufficiently large,

$$0 < (x - 1)^n < \frac{1}{b}$$

But for any  $n \in \mathbb{N}$ ,  $(x - 1)^n$  is of the form  $cx + d$  for some  $c, d \in \mathbb{Z}$ , since  $x^2 = 2$ ; and this is a contradiction. ■

So one may say  $\mathbb{Q}$  has a gap. For example, consider set  $A = \{x \mid x \in \mathbb{Q}, x^2 < 2\}$ . 2 is an upper bound for the set  $A$  but so is 1.5, 1.42, etc. We observe that in  $\mathbb{Q}$ , there is no *least upper bound*. Then, what will be the set without any gaps?

The *real numbers*, written  $\mathbb{R}$ , is a set with elements 0 and 1 ( $0 \neq 1$ ), equipped with operations  $+$  and  $\cdot$ , and an ordering  $<$  satisfying the following.

- (1)  $+$  is commutative and associative with identity 0, and every  $x$  has an inverse under  $+$ .
- (2)  $\cdot$  is commutative and associative with identity 1, and every  $x \neq 0$  has an inverse under  $\cdot$ .
- (3)  $\cdot$  is distributive over addition, i.e.  $\forall a, b, c$ ,

$$a(b + c) = ab + ac$$

- (4) For all  $a, b$ , exactly one of  $a < b$  or  $a = b$  or  $a > b$  holds, and  $\forall a, b, c$ ,

$$a < b \text{ and } a < c \Rightarrow a < c$$

(5)  $\forall a, b, c,$

$$a < b \Rightarrow a + c < b + c$$

and

$$a < b \Rightarrow ac < bc \text{ if } c > 0$$

(6) Given any set  $S$  of reals that is non-empty and bounded above,  $S$  has a *least upper bound* (Theorem 3.1.2).

We say that a set  $S$  is *bounded above* if  $\exists x \in \mathbb{R}$  such that  $x \geq y$  for all  $y \in S$ . Such an  $x$  is an upper bound for  $S$ . For the definition of least upper bound (supremum), see Definition 3.1.2.

*Remark 1.4.1.*

(1) From (1)-(5), we can check, for example,  $0 < 1$ . If not, then  $1 < 0$  so

$$0 = 1 - 1 < 0 - 1 = -1$$

and hence

$$0 = 0(-1) < (-1)(-1) = 1$$

which is a contradiction.

- (2) We may consider  $\mathbb{Q}$  as contained in  $\mathbb{R}$ , by identifying  $a/b \in \mathbb{Q}$  with  $ab^{-1} \in \mathbb{R}$ , where  $b^{-1}$  is the multiplicative inverse of  $b$ .
- (3)  $\mathbb{Q}$  does not satisfy (6) as e.g. the set of  $x$  such that  $x^2 < 2$  does not have a supremum.
- (4) In (6), the conditions non-empty and bounded above are crucial: if  $S$  is empty, then every  $x \in \mathbb{R}$  is an upper bound for  $S$ , so there is no least upper bound. If  $S$  is not bounded above, then it has no upper bound, and certainly no least upper bound.
- (5) It is possible to construct  $\mathbb{R}$  out of  $\mathbb{Q}$  and check that (1)-(6) hold, but it takes effort.
- (6) A structure obeying rules (1)-(3) is called a *field*. For example,  $\mathbb{R}, \mathbb{Q}, \mathbb{Z}_p$  with  $p$  prime are all fields, but  $\mathbb{Z}$  is not.

**Example 1.4.1.** Consider

$$S = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$$

Is 2 an upper bound for  $S$ ? – yes, since  $\forall x \in S, x \leq 2$ .

Is  $3/4$  an upper bound for  $S$ ? – no, since  $7/8 \in S$  but  $7/8 > 3/4$ .

The least upper bound of  $S$  is 1 because 1 is an upper bound and every other upper bound is equal or greater than 1. Hence  $\sup(S) = 1$ .

**Example 1.4.2.** Consider

$$S = \{x \in \mathbb{R} : 0 < x < 1\} = (0, 1)$$

Is 2 an upper bound for  $S$ ? – yes, since  $\forall x \in S, x \leq 2$ .

Is  $3/4$  an upper bound for  $S$ ? – no, since  $7/8 \in S$  but  $7/8 > 3/4$ .

We still have  $\sup(S) = 1$  because 1 is an upper bound, and there is no upper bound  $c$  such that  $c < 1$ ; indeed,  $c$  is certainly greater than 0 so if  $c < 1$ , then  $0 < c < 1$ , so

$$\frac{c+1}{2} \in S$$

with  $(c+1)/2 > c$ .

**Proposition 1.4.2.** (*Axiom of Archimedes*)  $\mathbb{N}$  is not bounded above in  $\mathbb{R}$ .

*Proof.* Suppose on the contrary that  $\mathbb{N}$  is bounded above. Let  $c = \sup \mathbb{N}$ . By definition,  $c - 1$  is not an upper bound for  $\mathbb{N}$ , so  $\exists n \in \mathbb{N}$  such that  $n > c - 1$ . But then  $n + 1 \in \mathbb{N}$  with  $n + 1 > c$ , contradicting the fact that  $c$  was an upper bound for  $\mathbb{N}$ . ■

**Corollary 1.4.2.1.** For any  $t \in \mathbb{R}$  with  $t > 0$ , there exists  $n \in \mathbb{N}$  with  $1/n < t$ .

*Proof.* Given  $t > 0$ , by Proposition 1.4.2,  $\exists n \in \mathbb{N}$  such that  $n > 1/t$ . Hence  $1/n < t$ . ■

A set  $S$  is said to be *bounded below* if  $\exists x$  such that  $x \leq y$  for all  $y \in S$ . Such an  $x$  is called a lower bound for  $S$ . If  $S$  is non-empty and bounded below, then  $-S = \{-y : y \in S\}$  is non-empty and bounded above, so it has a least upper bound, say  $c$ . Hence  $-c$  is the greatest lower bound, i.e. *infimum* for  $S$  and we write  $\inf(S)$ . Corollary 1.4.2.1 immediately implies that

$$\inf(\{1/n : n \in \mathbb{N}\}) = 0$$

**Example 1.4.3.** Consider

$$S = \{0, 1/2, 2/3, 3/4, \dots\} = \{1 - 1/n : n \in \mathbb{N}\}$$

Clearly, 1 is an upper bound. Suppose  $c < 1$  is an upper bound for  $S$ . Then

$$1 - \frac{1}{n} \leq c \quad \forall n \in \mathbb{N}$$

so  $0 < 1 - c \leq 1/n$  for all  $n \in \mathbb{N}$ , contradicting Corollary 1.4.2.1. Thus  $\sup(S) = 1$ .

**Theorem 1.4.3.** There exists  $x \in \mathbb{R}$  with  $x^2 = 2$ .

*Proof.* Let

$$S = \{x \in \mathbb{R} : x^2 < 2\}$$

Note that  $S$  is non-empty since e.g.  $1 \in S$ . It is also bounded above, e.g. by 2. Hence we may let  $\sup(S) = c$ . Observe that  $1 < c < 2$ . We claim that  $c^2 = 2$ . First, suppose  $c^2 < 2$ . For  $0 < t < 1$ ,

$$(c+t)^2 = c^2 + 2ct + t^2 < c^2 + 4t + t = c^2 + 5t \leq 2$$

for sufficiently small  $t$ , e.g.  $t \leq (2 - c^2)/5$ . But this contradicts the assumption that  $c$  was an upper bound for  $S$ . Now suppose  $c^2 > 2$ . For  $0 < t < 1$  we have

$$(c-t)^2 = c^2 - 2ct + t^2 > c^2 - 4t + 0 \geq 2$$

for sufficiently small  $t$ , e.g.  $t \leq (c^2 - 2)/4$ . This contradicts the assumption that  $c$  is the least upper bound for  $S$ . Hence  $c^2 = 2$  by construction of real numbers. ■

N.b. the same proof shows that  $\sqrt[n]{x}$  exists  $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, n > 0$ .

A real that is not rational is called *irrational*. For example,  $\sqrt{2}, \sqrt{3}, \sqrt{4}$  are all irrational. Also, observe that  $2 + 3\sqrt{5}$  is irrational, since if  $2 + 3\sqrt{5} = a/b$  with  $a, b \in \mathbb{N}$ , then

$$\sqrt{5} = \frac{a - 2b}{3b} \in \mathbb{Q}$$

The rationals are *dense* in  $\mathbb{R}$ , in a sense that  $\forall a < b \in \mathbb{R}, \exists c \in \mathbb{Q}$  with  $a < c < b$ . To show the assertion above, assume that  $a \geq 0$ . By Corollary 1.4.2.1,  $\exists n \in \mathbb{N}$  with  $1/n < b - a$ . Let

$$T = \{k \in \mathbb{N} : k/n \geq b\}$$

By the Axiom of Archimedes,  $\exists N \in \mathbb{N}$  such that  $N > b$ . Hence  $Nn \in T$ , so  $T \neq \emptyset$ . By the well-ordering principle (Theorem 1.2.2),  $T$  has a least element  $m$ . Let's set  $c = (m - 1)/n$ . Since  $m - 1 \notin T$ ,  $c < b$ . If  $c \leq a$ , then

$$\frac{m}{n} = c + \frac{1}{n} < a + (b - a) = b$$

which is a contradiction. Therefore,  $a < c < b$ . Similarly, the irrationals are also dense in  $\mathbb{R}$ , i.e.  $\forall a < b \in \mathbb{R}, \exists c \in \mathbb{R} \setminus \mathbb{Q}$  with  $a < c < b$ . For example, this can be constructed by taking a rational  $c$  with  $a\sqrt{2} < c < b\sqrt{2}$  and then

$$a < \frac{c}{\sqrt{2}} < b$$

### 1.4.2 Decimal Expansions

We assume basic knowledge of sequences and series. Let  $(d_n)$  be a sequence with  $d_n \in \{0, 1, \dots, 9\}$ . Then

$$\sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

converges to some limit  $x$ , where  $0 \leq x \leq 1$ , because (by Theorem 3.1.1) the partial sums are increasing and bounded by

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = 1$$

We say that

$$0.d_1d_2d_3\dots$$

is the *decimal expansion* of  $x$ . To show that any  $x \in [0, 1)$  has a decimal expansion, pick  $d_1 \in \mathbb{Z}$  maximal such that

$$\frac{d_1}{10} \leq x < 1$$

Then  $0 \leq d_1 \leq 9$  because  $0 \leq x < 1$ , and

$$0 \leq x - \frac{d_1}{10} < \frac{1}{10}$$

because  $d_1$  maximal. Now pick  $d_2 \in \mathbb{Z}$  maximal such that

$$\frac{d_2}{100} \leq x - \frac{d_1}{10}$$

Then  $0 \leq d_2 \leq 9$  because

$$0 \leq x - \frac{d_1}{10} < \frac{1}{10}$$

and

$$0 \leq x - \frac{d_1}{10} - \frac{d_2}{100} < \frac{1}{100}$$

because  $d_2$  maximal. Inductively, pick  $d_n \in \mathbb{Z}$  maximal such that

$$\frac{d_n}{10^n} \leq x - \sum_{j=1}^{n-1} \frac{d_j}{10^j}$$

so

$$0 \leq x - \sum_{j=1}^n \frac{d_j}{10^j} < \frac{1}{10^n}$$

Since  $1/10^n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $x - \sum_{j=1}^n (d_j/10^j) \rightarrow 0$ , i.e.

$$x = \sum_{j=1}^{\infty} \frac{d_j}{10^j} = 0.d_1d_2d_3\dots$$

*Remark 1.4.2.*

- (1) Decimal expansions need not be unique, e.g.  $0.47999\dots = 0.4800\dots$
- (2) A decimal expansion is *periodic* if, after a finite number of items, it repeats in blocks, of length  $k$  say. I.e.  $\exists l, k$  such that  $d_{n+k} = d_n$  for all  $n > l$ .

### 1.4.3 Complex Numbers

Observe that some polynomials have no real roots, e.g.  $x^2 + 1$ . We will try and define  $x$  with  $x^2 + 1 = 0$  into existence.

The *complex numbers*, written  $\mathbb{C}$ , consist of  $\mathbb{R}^2$  (the set of all ordered pairs  $(a, b)$  with  $a, b \in \mathbb{R}$ ) together with operations  $+$  and  $\cdot$  defined by

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

We can view  $\mathbb{R}$  as contained in  $\mathbb{C}$  by identifying  $a \in \mathbb{R}$  with  $(a, 0) \in \mathbb{C}$ . Note that

$$(a, 0) + (b, 0) = (a + b, 0)$$

and

$$(a, 0) \cdot (b, 0) = (ab, 0)$$

Now let  $i = (0, 1)$ , with  $i^2 = (-1, 0)$ , corresponding to  $-1 \in \mathbb{R}$ . Then, for every  $z \in \mathbb{C}$ , we can write

$$z = (a, b) = a(1, 0) + b(0, 1) = a + bi$$

for some  $a, b \in \mathbb{R}$ .

*Remark 1.4.3.*

- (1)  $\mathbb{C}$  is a field, i.e. it obeys (1)-(3) from the definition of  $\mathbb{R}$ , including that for some  $z \in \mathbb{C}$  ( $z \neq 0$ ),  $\exists w$  such that  $zw = 1$ . Indeed, given  $z = a + bi$ , since

$$(a + bi)(a - bi) = a^2 + b^2$$

we have

$$(a + bi) \frac{a - bi}{a^2 + b^2} = 1$$

- (2) Every non-zero polynomial (even allowing complex coefficients) has a root in  $\mathbb{C}$ . This is known as the *Fundamental Theorem of Algebra*.

## 1.5 Countability

We introduce the concept countability to obtain a deeper understanding of sizes of sets – in particular, infinite sets. We say a set  $X$  is *countable* if  $X$  is finite or there is a bijection  $X \rightarrow \mathbb{N}$ . That is,  $X$  is countable if and only if we can list the elements of  $X$  as  $x_1, x_2, \dots$

### Example 1.5.1.

- (1) Any finite set is countable.
- (2)  $\mathbb{N}$  is countable.
- (3)  $\mathbb{Z}$  is countable, as we may list the elements of  $\mathbb{Z}$  as  $0, 1, -1, 2, -2, \dots$ , i.e.

$$x_n = \begin{cases} n/2 & 2|n \\ -(n-1)/2 & 2 \nmid n \end{cases}$$

**Lemma 1.5.1.** Any subset of  $\mathbb{N}$  is countable.

*Proof.* If  $S \subseteq \mathbb{N}$  is non-empty, by well-ordering principle, there is a least element  $s_1 \in S$ . If  $S \setminus \{s_1\} \neq \emptyset$ , by well-ordering principle, there is a least element  $s_2 \in S \setminus \{s_1\}$ . And if  $S \setminus \{s_1, s_2\} \neq \emptyset$ , we can repeat the process. If at some point this process ends, then

$$S = \{s_1, s_2, \dots, s_m\}$$

is finite. If it goes on forever, then the map

$$g : \mathbb{N} \rightarrow S, \quad n \mapsto s_n$$

is well-defined, and is injective. It is also surjective because if  $k \in S$ , then  $k \in \mathbb{N}$ , and there are less than  $k$  elements of  $S$  less than  $k$ , so  $k = s_n$  for some  $n \leq k$ . ■

Note that the idea used in the proof of Lemma 1.5.1 may not work for other sets. For example, in  $\mathbb{R}$ , let

$$S = \{1/2, 2/3, 3/4, \dots\} \cup \{1\}$$

then  $S$  is countable as we can list it as

$$1, 1/2, 2/3, 3/4, \dots$$

but if we had tried to list the elements in increasing order, then 1 would not be on the list.



**Theorem 1.5.2.** The following statements are equivalent:

- (1)  $X$  is countable;
- (2) there is an injection  $X \rightarrow \mathbb{N}$ ;
- (3)  $X = \emptyset$  or there is surjection  $\mathbb{N} \rightarrow X$ .

*Proof.* ((1) $\Rightarrow$ (2)) If  $X$  finite, it obviously injects into  $\mathbb{N}$ . Otherwise, if  $X$  infinite,  $X$  bijects with  $\mathbb{N}$ , and it injects into  $\mathbb{N}$ .

((2) $\Rightarrow$ (1)) If there is an injection  $f : X \rightarrow \mathbb{N}$ , then  $f$  is a bijection between  $X$  and  $S = f(X)$ . If  $S$  is finite, then so is  $X$ . If  $S$  is infinite, then by Lemma 1.5.1, there is a bijection  $g : S \rightarrow \mathbb{N}$ , and thus

$$X \xrightarrow{f} S = f(X) \xrightarrow{g} \mathbb{N}$$

is a bijection.

((3) $\Rightarrow$ (2)) Suppose that  $X \neq \emptyset$  and there is a surjection  $f : \mathbb{N} \rightarrow X$ . Now define  $g$  by

$$g(a) = \min f^{-1}(\{a\})$$

which exists by well-ordering principle. But by construction,  $g$  is injective. Hence (3) implies (2).

Finally, it is clear that (1) implies (3). Therefore, three statements are equivalent. ■

**Corollary 1.5.2.1.** Any subset of a countable set is countable.

*Proof.* If  $Y \subseteq X$  and  $X$  is countable, then take the injection  $X \rightarrow \mathbb{N}$  restricted to  $Y$ . ■

We may thus view “countable” as saying that a set is “at most as big as  $\mathbb{N}$ ”.<sup>7</sup>

**Theorem 1.5.3.**  $\mathbb{N} \times \mathbb{N}$  is countable.

*Proof.* (Method 1) Define  $a_1 = (1, 1)$ , and  $a_n$  inductively, given  $a_{n-1} = (p, q)$ , by writing

$$a_n = \begin{cases} (p-1, q+1) & p \neq 1 \\ (p+q, 1) & p = 1 \end{cases}$$

This list includes every point  $(x, y) \in \mathbb{N} \times \mathbb{N}$ , hence there exists a bijection  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . ■

*Proof.* (Method 2) Define  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ ,  $(x, y) \mapsto 2^x 3^y$ . Then  $f$  is injective by the fundamental theorem of arithmetic (Theorem 1.3.7). Thus  $\mathbb{N} \times \mathbb{N}$  is countable by Theorem 1.5.2. ■

**Corollary 1.5.3.1.**  $\mathbb{Z} \times \mathbb{Z}$  is countable.

*Proof.* Since  $\mathbb{Z}$  is countable, there is an injection  $f : \mathbb{Z} \rightarrow \mathbb{N}$ , and since  $\mathbb{N} \times \mathbb{N}$  is countable, there is an injection  $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . Hence we have an injection

$$\mathbb{Z} \times \mathbb{Z} \xrightarrow{(f,f)} \mathbb{N} \times \mathbb{N} \xrightarrow{g} \mathbb{N}$$

■

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<sup>7</sup>Review definition if confused.

By induction, we further can tell that  $\mathbb{Z}^k$  is countable for any  $k \in \mathbb{N}$ .

**Theorem 1.5.4.** A countable union of countable sets is countable.

*Proof. (Method 1)* May assume that our countable sets are indexed by  $\mathbb{N}$ , so given countable sets  $A_1, A_2, \dots$ , we wish to show

$$\bigcup_{n \in \mathbb{N}} A_n$$

is countable. For each  $i \in \mathbb{N}$ , since  $A_i$  is countable, we may list its elements as  $a_1^{(i)}, a_2^{(i)}, \dots$ . Define

$$f : \bigcup_{n \in \mathbb{N}} A_n \rightarrow \mathbb{N}, \quad x \mapsto 2^i 3^j$$

where  $x = a_j^{(i)}$  for the least  $i$  such that  $x \in A_i$ . Then  $f$  is injective by the fundamental theorem of arithmetic (Theorem 1.3.7). ■

*Proof. (Method 2)* Let  $I$  be a countable index set, and for each  $i \in I$ ,  $A_i$  is a countable set. Since  $I$  is countable, there is an injection  $f : I \rightarrow \mathbb{N}$ , and for each  $i \in I$ , since  $A_i$  is countable, there is an injection  $g_i : A_i \rightarrow \mathbb{N}$ . We construct an injection

$$h : \bigcup_{i \in I} A_i \rightarrow \mathbb{N} \times \mathbb{N}$$

as follows: for each  $x \in \bigcup_{i \in I} A_i$ , pick

$$m_x = \min\{j \in \mathbb{N} : x \in A_j, f(j) = j\}$$

which exists by well-ordering principle. Let  $i_x \in I$  be such that  $f(i_x) = m_x$  ( $i_x$  is unique because  $f$  is injective). Then

$$h(x) = (m_x, g_{i_x}(x))$$

defines an injection. ■

**Example 1.5.2.**

$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \frac{1}{n} \mathbb{Z} = \bigcup_{n \in \mathbb{N}} \left\{ \frac{m}{n} : m \in \mathbb{Z} \right\}$$

so  $\mathbb{Q}$  is a countable union of countable sets, hence countable.

**Theorem 1.5.5.** The set  $\mathbb{A}$  of algebraic numbers is countable.

*Proof.* It suffices to show that the set of all polynomials with integer coefficients is countable, as then  $\mathbb{A}$  is a countable union of finite sets, so by Theorem 1.5.4, is countable. In fact, it suffices to show that for each  $d \in \mathbb{N}$ , the set  $P_d$  of all integer polynomials of degree  $d$  is countable, again by Theorem 1.5.4. But the map  $P_d \rightarrow \mathbb{Z}^{d+1}$ ,

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0 \mapsto (a_d, a_{d-1}, \dots, a_1, a_0)$$

is an injection, so since  $\mathbb{Z}^{d+1}$  is countable,  $P_d$  is. ■

But certainly, not all sets are countable.

**Theorem 1.5.6.**  $\mathbb{R}$  is uncountable.

*Proof. (Diagonal argument, Cantor)* If  $\mathbb{R}$  were countable, we would be able to list all the reals as  $r_1, r_2, r_3, \dots$ . Write each  $r_n$  in decimal form in some way:

$$\begin{aligned} r_1 &= n_1.d_{11}d_{12}d_{13}d_{14}\dots \\ r_2 &= n_2.d_{21}d_{22}d_{23}d_{24}\dots \\ r_3 &= n_3.d_{31}d_{32}d_{33}d_{34}\dots \\ &\vdots \end{aligned}$$

Now define

$$r = 0.d_1d_2d_3\dots$$

by

$$d_n = \begin{cases} 1 & d_{nn} \neq 1 \\ 2 & d_{nn} = 1 \end{cases}$$

Note that  $r$  has only one decimal expansion, and is not on the list, since  $\forall n \in \mathbb{N}, r \neq r_n$ . This contradicts the assumption that  $\mathbb{R}$  is countable. ■

N.b. it in fact shows that the interval  $(0, 1)$  is uncountable.

**Corollary 1.5.6.1.** There are uncountably many transcendental numbers.

*Proof.* If  $\mathbb{R} \setminus \mathbb{A}$  were countable, then since  $\mathbb{A}$  is countable by Theorem 1.5.5,  $\mathbb{R} = (\mathbb{R} \setminus \mathbb{A}) \cup \mathbb{A}$  would also be countable, leading to contradiction. ■

**Theorem 1.5.7.**  $\mathcal{P}(\mathbb{N})$  is uncountable.

*Proof. (Method 1)* If  $\mathcal{P}(\mathbb{N})$  were countable, we could just list it as  $S_1, S_2, \dots$ . Let

$$S = \{n \in \mathbb{N} : n \notin S_n\}$$

Then  $S$  is not on our list since  $\forall n \in \mathbb{N}, S \neq S_n$  (as  $S$  and  $S_n$  differ in the element  $n$ ). This is a contradiction, and thus  $\mathcal{P}(\mathbb{N})$  is uncountable. ■

Note that the proof above is similar as the diagonal argument by Cantor.

*Proof. (Method 2)* Note that there is an injection from  $(0, 1)$  into  $\mathcal{P}(\mathbb{N})$ : write  $x \in (0, 1)$  in binary

$$0.x_1x_2x_3\dots$$

with  $x_i \in \{0, 1\}$ , and set  $f(x) = \{n \in \mathbb{N} : x_n = 1\}$ . Then,  $f$  is an injection. ■

**Theorem 1.5.8.** For any set  $X$ , there is no bijection between  $X$  and its power set  $\mathcal{P}(X)$ .

*Proof.* Given any  $f : X \rightarrow \mathcal{P}(X)$ , we shall show it is not a surjection. Let

$$S = \{x \in X : x \notin f(x)\}$$

Then  $S$  does not belong to the image of  $f$ , since  $\forall x \in X, S$  and  $f(x)$  differ in the element  $x$ . Thus  $S \neq f(x)$  for any  $x \in X$ . ■

*Remark 1.5.1.*

- (1) This is reminiscent of Russell's paradox.

- (2) In fact, it gives another proof that there is no universal set. Suppose we had such a universal set  $V$ , then we would have  $\mathcal{V} \subseteq V$ , in which case there would certainly be a surjection from  $V$  to  $\mathcal{P}(V)$ .

**Example 1.5.3.** Let  $\{A_i : i \in I\}$  be a family of open pairwise disjoint intervals of  $\mathbb{R}$ . Such family is countable.

Each interval  $A_i$  contains a rational and  $\mathbb{Q}$  is countable, so since the intervals are disjoint, we have an injection from  $I$  into  $\mathbb{Q}$ .

To summerise, to show that a set  $X$  is countable, we

- (1) list its elements;
- (2) inject it into  $\mathbb{N}$ ;
- (3) use the fact that a countable union of countable sets is countable;
- (4) if the set is in or near  $\mathbb{R}$ , consider  $\mathbb{Q}$ .

To show that  $X$  is uncountable,

- (1) run a diagonal argument on  $X$ ;
- (2) inject some uncountable set into  $X$ .

Intuitively, we think of “ $A$  bijects with  $B$ ” as saying that “ $A$  and  $B$  are of the same size”, “ $A$  injects into  $B$ ” as saying that “ $A$  is at most as big as  $B$ ”, and “ $A$  surjects onto  $B$ ” as saying that “ $A$  is at least as big as  $B$ ” (for  $B \neq \emptyset$ ). However, for these interpretations to make sense, we need that if “ $A$  is at most as big as  $B$ ”, then “ $B$  is at least as big as  $A$ ”, and conversely.

**Lemma 1.5.9.** Given non-empty sets  $A$  and  $B$ ,

$$\exists \text{ injection } f : A \rightarrow B \Leftrightarrow \exists \text{ surjection } g : B \rightarrow A$$

*Proof.* Suppose  $f : A \rightarrow B$  is injective. Fix  $a_0 \in A$ . Define  $g : B \rightarrow A$  by

$$b \mapsto \begin{cases} \text{unique } a \in A \text{ such that } f(a) = b & \text{if it exists} \\ a_0 & \text{otherwise} \end{cases}$$

Then  $g$  is surjective. Conversely, suppose  $g : B \rightarrow A$  is surjective. Let  $f : A \rightarrow B$ ,

$$a \mapsto \text{some } b \in B \text{ such that } g(b) = a$$

Then  $f$  is injective. ■

We also need that if “ $A$  is at most as big as  $B$ ” and “ $B$  is at most as big as  $A$ ”, then “ $A$  and  $B$  have the same size”.

**Theorem 1.5.10.** (*Schröder-Bernstein Theorem*) If  $f : A \rightarrow B$  and  $g : B \rightarrow A$  are injections, then there exists bijection  $h : A \rightarrow B$ .

*Proof.* For  $a \in A$ , write  $g^{-1}(a)$  for the  $b \in B$  (if it exists) such that  $g(b) = a$ . Similarly for  $B$ , write  $f^{-1}(b)$  for the  $a \in A$  (if it exists) such that  $f(a) = b$ . We call  $g^{-1}(a), f^{-1}(g^{-1}(a)), g^{-1}(f^{-1}(g^{-1}(a))), \dots$  the *ancestor sequence* of  $a \in A$  (might terminate). Similarly, define the ancestor sequence of  $b \in B$ . Let

$$A_0 = \{a \in A : \text{ancestor sequence of } a \text{ stops at an even time, i.e. the last point is in } A\}$$

$A_1 = \{a \in A : \text{ancestor sequence of } a \text{ stops at an odd time, i.e. the last point is in } B\}$

and

$$A_\infty = \{a \in A : \text{ancestor sequence of } a \text{ does not stop}\}$$

Similarly, define  $B_0, B_1, B_\infty$ . Note that  $f$  bijects  $A_0$  with  $B_1$  (observing that every  $b \in B_1$  has at least one ancestor, so is  $f(a)$  for some  $a \in A_0$ ), and similarly,  $g$  bijects  $B_0$  with  $A_1$ . And  $f$  (or  $g$ ) bijects  $A_\infty$  with  $B_\infty$ . Then the function  $h : A \rightarrow B$ ,

$$a \mapsto \begin{cases} f(a) & a \in A_0 \\ g^{-1}(a) & a \in A_1 \\ f(a) & a \in A_\infty \end{cases}$$

is a bijection. ■

**Example 1.5.4.** Is there a bijection from  $[0, 1]$  to  $[0, 1] \cup [2, 3]$ ? Observe that we have an injection

$$f : [0, 1] \rightarrow [0, 1] \cup [2, 3], \quad x \mapsto x$$

and an injection

$$g : [0, 1] \cup [2, 3] \rightarrow [0, 1], \quad x \mapsto x/3$$

So by the Schröder-Bernstein theorem, there is a bijection between  $[0, 1]$  and  $[0, 1] \cup [2, 3]$ .

It would be nice to be able to say that for any two sets  $A$  and  $B$ , either  $A$  injects into  $B$  or  $B$  injects into  $A$ . This is true, but proof is beyond the scope of this course.<sup>8</sup>

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<sup>8</sup>See Part II Logic and Set Theory.

## CHAPTER 2

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### Groups

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Lecture by Professor Henry Wilton. Michaelmas term 2023.

### 2.1 Introduction

**Definition 2.1.1.** *A binary operation*

Lecture by Professor Claude Warnick. Lent term 2024.

### 3.1 Limits and Convergence

#### 3.1.1 Sequences

Consider a sequence of real numbers

$$a_1, a_2, a_3, \dots$$

$(a_n)$  where  $a_n \in \mathbb{R}$ . We start by defining *limit*.

**Definition 3.1.1.** We say  $a_n \rightarrow a$  as  $n \rightarrow \infty$  for some  $a \in \mathbb{R}$  if: given  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|a_n - a| < \epsilon \quad \forall n \geq N$$

If  $a_n \leq a_{n+1} \quad \forall n$  we say  $(a_n)$  is *increasing*; and *decreasing*, *strictly increasing*, *strictly decreasing* for  $a_n \geq a_{n+1}$ ,  $a_n > a_{n+1}$ ,  $a_n < a_{n+1}$  respectively. In all above cases,  $(a_n)$  is *monotone*.

**Theorem 3.1.1.** (*Monotone Convergence Theorem*) An increasing sequence of real numbers which is bounded above converges (i.e. it has a limit).

In other words, monotone convergence theorem says if  $a_n \in \mathbb{R}$  ( $n \geq 1$ ),  $A \in \mathbb{R}$  with

$$a_1 \geq a_2 \geq a_3 \geq \dots \text{ and } a_n \geq A \quad \forall n$$

there exists  $a \in \mathbb{R}$  such that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ . Equivalently, decreasing sequence bounded below has a limit. Theorem 3.1.1 is also equivalent with Theorem 3.1.2

**Theorem 3.1.2.** (*Least Upper Bound Axiom*) Every non-empty set bounded above has a *supremum*.

**Definition 3.1.2.** (Supremum) If  $S \subset \mathbb{R}, S \neq \emptyset$  we say that

$$\sup S = K$$

if

- (1)  $x \leq K \quad \forall x \in S$  ( $K$  is upper bound for  $S$ ).
- (2) Given  $\epsilon > 0$ ,  $\exists x \in S$  such that  $x > K - \epsilon$  ( $K$  is least upper bound).

Note that similar definition can be made for greatest lower bound and infimum. If  $\sup S \in S$ , we say  $\sup S$  is the *maximum* of  $S$ , i.e.  $\sup S = \max S$ .

**Lemma 3.1.3.**

- (1) The limit is unique, i.e. if  $a_n \rightarrow a$  and  $a_n \rightarrow b$  as  $n \rightarrow \infty$  then  $a = b$ .
- (2) Subsequences converge to the same limit, i.e. if  $a_n \rightarrow a$  as  $n \rightarrow \infty$  and  $n_1 < n_2 < \dots$ , then  $a_{n_j} \rightarrow a$  as  $j \rightarrow \infty$ .
- (3) If  $a_n = c \quad \forall n$ , then  $a_n \rightarrow c$  as  $n \rightarrow \infty$ .
- (4) If  $a_n \rightarrow a$  and  $b_n \rightarrow b$  then  $a_n + b_n \rightarrow a + b$ .
- (5) If  $a_n \rightarrow a$  and  $b_n \rightarrow b$  then  $a_n b_n \rightarrow ab$ .
- (6) If  $a_n \rightarrow a$ ,  $a_n \neq 0$  and  $a \neq 0$ , then  $1/a_n \rightarrow 1/a$ .
- (7) If  $a_n \leq A \quad \forall n$  and  $a_n \rightarrow a$ , then  $a \leq A$ .
- (8) If  $a_n \rightarrow a$  and  $c_n \rightarrow a$  as  $n \rightarrow \infty$  and  $a_n \leq b_n \leq c_n$ , then  $b_n \rightarrow a$ .

*Proof.* We will only prove 1, 2, and 5 here.

- (1) For any  $\epsilon > 0$ , we can find  $N_1(\epsilon)$  and  $N_2(\epsilon)$  such that

$$n \geq N_1(\epsilon) \Rightarrow |a_n - a| < \epsilon$$

$$n \geq N_2(\epsilon) \Rightarrow |a_n - b| < \epsilon$$

If  $n \geq \max\{N_1(\epsilon), N_2(\epsilon)\}$ , then

$$\begin{aligned} 0 \leq |b - a| &= |b - a_n + a_n - a| \\ &\leq |b - a_n| + |a_n - a| < 2\epsilon \end{aligned}$$

This implies  $|b - a| = 0$ , otherwise we can set  $\epsilon = |b - a|/3$  to find

$$0 \leq |b - a| < \frac{2}{3}|b - a|$$

■

- (2) Since  $n_j < n_{j+1} \Rightarrow n_{j+1} \geq n_j + 1$ , by induction, we have  $n_j \geq j$ . As  $a_n \rightarrow a$  as  $n \rightarrow \infty$ , given  $\epsilon > 0$  there exists  $N = N(\epsilon)$  such that

$$n \geq N(\epsilon) \Rightarrow |a_n - a| < \epsilon$$

So if  $j \geq N(\epsilon)$ , then  $n_j \geq j \geq N(\epsilon)$  and

$$|a_{n_j} - a| < \epsilon$$

■



- (3) As  $a_n \rightarrow a$ ,  $b_n \rightarrow b$  as  $n \rightarrow \infty$ , for any  $\epsilon > 0$ , we can find  $N_1(\epsilon)$  and  $N_2(\epsilon)$  such that

$$n \geq N_1(\epsilon) \Rightarrow |a_n - a| < \epsilon$$

$$n \geq N_2(\epsilon) \Rightarrow |b_n - b| < \epsilon$$

Now

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n| |b_n - b| + |b| |a_n - a| \end{aligned}$$

If  $n \geq N_1(1)$  then  $|a_n - a| \leq 1$  and hence

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a|$$

Thus if  $n \geq N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}$  then

$$|a_n b_n - ab| < (1 + |a|)\epsilon + |b|\epsilon = (1 + |a| + |b|)\epsilon$$

which can be made as small as we like. ■

**Lemma 3.1.4.**

- (1)  $1/n \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.*  $(1/n)$  is a decreasing sequence, bounded below by 0, so by monotone convergence theorem it has a limit  $L$ . Now,

$$\frac{1}{2n} = \left(\frac{1}{2}\right) \left(\frac{1}{n}\right) \rightarrow \frac{1}{2}L$$

(Lemma 3.1.3 (3), (5)). But  $(1/2n)$  is a subsequence of  $(1/n)$ , so by Lemma 3.1.3 (2),

$$\frac{1}{2n} \rightarrow L$$

Then, by Lemma 3.1.3 (1),

$$\frac{1}{2}L = L$$

and  $L = 0$ . ■

- (2) If  $|x| < 1$ ,  $x^n \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* Suppose  $0 \leq x < 1$ . Then  $(x^n)$  is a decreasing sequence bounded below by 0, and it converges by monotone convergence theorem to a limit  $L$ . Now,

$$x^{n+1} = x x^n \rightarrow xL$$

(Lemma 3.1.3 (5)). However,  $(x^{n+1})$  is a subsequence of  $(x^n)$  so by Lemma 3.1.3 (2),  $x^{n+1} \rightarrow L$ ; and consequently by Lemma 3.1.3 (1),  $L = xL \Rightarrow L = 0$ .

Suppose  $-1 < x < 1$ . Then,

$$-|x|^n \leq x^n \leq |x|^n$$

We know  $|x|^n \rightarrow 0$  and  $-|x|^n \rightarrow 0$  from above, so  $x^n \rightarrow 0$  by Lemma 3.1.3 (8). ■

Note that when we talk about a sequence converging, we mean to a *finite* limit, while there is a notion of *tending to infinity*:  $a_n \rightarrow \infty$  if  $\forall M \in \mathbb{R} \exists N = N(M)$  such that  $n \geq N \Rightarrow a_n > M$ .

Meanwhile, we can see that our definition of convergence still works if  $a_n, a \in \mathbb{C}$  (Definition 3.1.1). Furthermore, Lemma 3.1.3 (1) to (6) all apply for complex sequences, but (7), (8), and Theorem 3.1.1 use the *order relation* so do not carry over directly.

**Lemma 3.1.5.** If  $(z_n)$  is a complex sequence, then  $z_n \rightarrow z$  if and only if  $\operatorname{Re}(z_n) \rightarrow \operatorname{Re}(z)$  and  $\operatorname{Im}(z_n) \rightarrow \operatorname{Im}(z)$ .

*Proof.* Note if  $\omega \in \mathbb{C}$ ,

$$\max\{|\operatorname{Re}(\omega)|, |\operatorname{Im}(\omega)|\} \leq |\omega| \leq |\operatorname{Re}(\omega)| + |\operatorname{Im}(\omega)|$$

( $\Rightarrow$ ) Suppose  $z_n \rightarrow z$ . Then,  $\forall \epsilon > 0$  there exists  $N = N(\epsilon)$  such that

$$\begin{aligned} n \geq N \Rightarrow |z_n - z| < \epsilon &\Rightarrow |\operatorname{Re}(z_n) - \operatorname{Re}(z)| < \epsilon \\ &\text{and } |\operatorname{Im}(z_n) - \operatorname{Im}(z)| < \epsilon \end{aligned}$$

Therefore  $\operatorname{Re}(z_n) \rightarrow \operatorname{Re}(z)$  and  $\operatorname{Im}(z_n) \rightarrow \operatorname{Im}(z)$ .

( $\Leftarrow$ ) Suppose  $\operatorname{Re}(z_n) \rightarrow \operatorname{Re}(z)$  and  $\operatorname{Im}(z_n) \rightarrow \operatorname{Im}(z)$ . Then  $\forall \epsilon > 0 \exists N_1 = N_1(\epsilon), N_2 = N_2(\epsilon)$  such that

$$\begin{aligned} n \geq N_1 &\Rightarrow |\operatorname{Re}(z_n) - \operatorname{Re}(z)| < \epsilon \\ n \geq N_2 &\Rightarrow |\operatorname{Im}(z_n) - \operatorname{Im}(z)| < \epsilon \end{aligned}$$

So if  $n \geq \max\{N_1(\epsilon), N_2(\epsilon)\} = N_3(\epsilon)$ , then

$$|z_n - z| \leq |\operatorname{Re}(z_n) - \operatorname{Re}(z)| + |\operatorname{Im}(z_n) - \operatorname{Im}(z)| < 2\epsilon$$

and therefore  $z_n \rightarrow z$ . ■

**Theorem 3.1.6. (Bolzano-Weierstrass Theorem)** Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence, i.e. if  $x_n \in \mathbb{R}$  and there exists  $K > 0$  such that  $|x_n| \leq K \quad \forall n$ , we can find  $n_1 < n_2 < n_3 < \dots$  and  $x \in \mathbb{R}$  such that  $x_{n_j} \rightarrow x$  as  $j \rightarrow \infty$ . N.b. we do not assert uniqueness, e.g. if  $x_n = (-1)^n$ ,  $x_{2n} \rightarrow 1$  but  $x_{2n+1} \rightarrow -1$ .

*Proof.* Set  $a_1 = -K, b_1 = K$  so that all terms of  $(x_n)$  lie in  $[a_1, b_1]$ . Let  $c = (a_1 + b_1)/2$ . Either

- (1) There are infinitely many terms of  $(x_n)$  in  $[a_1, c]$  or
- (2) There are infinitely many terms of  $(x_n)$  in  $[c, b_1]$ .

If (1) holds, we set  $a_2 = a_1, b_2 = c$ ; or if (1) does not hold, we set  $a_2 = c, b_2 = b_1$ . Continue inductively to construct  $a_k, b_k$  such that infinitely many terms of  $(x_n)$  lie in  $[a_k, b_k]$ , with

$$b_{k+1} - a_{k+1} = \frac{1}{2}(b_k - a_k)$$

and

$$a_k \leq a_{k+1} < b_{k+1} \leq b_k$$

By construction,  $(a_k)$  is increasing and bounded above by  $b_1$ . So, from the monotone convergence theorem,

$$a_k \rightarrow a \in [a_1, b_1] \text{ as } k \rightarrow \infty$$

Similarly, since  $(b_k)$  is decreasing and bounded below by  $a_1$ ,

$$b_k \rightarrow b \in [a_1, b_1] \text{ as } k \rightarrow \infty$$

But

$$b_{k+1} - a_{k+1} = \frac{1}{2}(b_k - a_k) \Rightarrow b - a = \frac{1}{2}(b - a)$$

and  $b = a$ . Now construct  $n_j$  as follows.

- Pick  $n_1 = 1$ .
- Pick  $n_j$  to be the smallest integer greater than  $n_{j-1}$  such that  $x_{n_j} \in [a_j, b_j]$ .

We can do this as  $[a_j, b_j]$  contains infinitely many terms of  $(x_n)$ . Viz,  $a_j \leq x_{n_j} \leq b_j$  and by Lemma 3.1.3 (8),  $x_{n_j} \rightarrow a$ . ■

**Definition 3.1.3.** A sequence  $(a_n)$  of real numbers is a *Cauchy sequence* (Cauchy) if for all  $\epsilon > 0$  there exists  $N = N(\epsilon)$  such that  $n, m \geq N \Rightarrow |a_n - a_m| < \epsilon$ .

**Lemma 3.1.7.** Every convergent sequence is a Cauchy sequence.

*Proof.* If  $a_n \rightarrow a$ , then given  $\epsilon > 0 \exists N = N(\epsilon)$  such that if  $n \geq N$ ,  $|a_n - a| < \epsilon$ . But if  $m, n \geq N$ ,

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon$$

■

**Theorem 3.1.8.** Every Cauchy sequence is convergent.

*Proof.* Suppose  $(a_n)$  is Cauchy. First we show  $(a_n)$  is bounded. Since  $(a_n)$  is Cauchy, given  $\epsilon > 0 \exists N = N(\epsilon)$  such that

$$|a_n - a_m| < \epsilon \quad \forall n, m \geq N(\epsilon)$$

So if  $n, m \geq N(1)$ , then  $|a_n - a_m| < 1$ . Set  $m = N(1)$  to see

$$|a_n| = |a_n - a_{N(1)} + a_{N(1)}| \leq |a_n - a_{N(1)}| + |a_{N(1)}| \leq 1 + |a_{N(1)}| \quad \forall n \geq N(1)$$

Thus if  $K = 1 + \max\{|a_1|, \dots, |a_{N(1)}|\}$ ,

$$|a_n| \leq K \quad \forall n$$

Now, by Bolzano-Weierstrass theorem (Theorem 3.1.6), there must be  $a \in \mathbb{R}$  and  $n_1 < n_2 < n_3 < \dots$  such that  $a_{n_j} \rightarrow a$  as  $j \rightarrow \infty$ . We claim, in fact,  $a_n \rightarrow a$  as  $n \rightarrow \infty$ . To see this, write

$$|a_n - a| = |a_n - a_{n_j} + a_{n_j} - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a|$$

and fix  $\epsilon > 0$ . Since  $(a_n)$  is Cauchy,  $\exists N_1(\epsilon)$  such that if  $n, n_j \geq N_1(\epsilon)$ ,  $|a_n - a_{n_j}| < \epsilon$ .

Since  $a_{n_j} \rightarrow a$ ,  $\exists N_2(\epsilon)$  such that if  $j \geq N_2(\epsilon)$ ,  $|a_{n_j} - a| < \epsilon$ . Fix  $j$  such that  $j \geq N_2(\epsilon)$  and  $n_j \geq N_1(\epsilon)$ . Then we have, for any  $n \geq N_1(\epsilon)$ ,

$$|a_n - a| < 2\epsilon$$

i.e.  $a_n \rightarrow a$ . ■

We have shown a real sequence converges if and only if it is Cauchy (Lemma 3.1.7 and Theorem 3.1.8). This is called the *Cauchy's general principle of convergence*.

**Corollary 3.1.8.1.** A complex sequence converges if and only if it is Cauchy.

*Proof.* From estimate at start of proof of Lemma 3.1.5, it follows that  $(z_n)$  is Cauchy if and only if  $(\operatorname{Re}(z_n))$  and  $(\operatorname{Im}(z_n))$  are both Cauchy. ■

### 3.1.2 Series and Convergence Tests

**Definition 3.1.4.** Suppose  $a_j \in \mathbb{R}$  (or  $\mathbb{C}$ ). Define sequence of *partial sums*  $(S_N)$  where

$$S_N = \sum_{j=1}^N a_j$$

Then, if  $S_N \rightarrow S$  as  $N \rightarrow \infty$ , we write

$$S = \sum_{j=1}^{\infty} a_j$$

and say  $\sum_{j=1}^{\infty} a_j$  converges. Conversely, if  $S_N$  does not converge, we say  $\sum_{j=1}^{\infty} a_j$  diverges.

*Remark 3.1.1.* Any problem on series is a problem about sequences, and vice versa.

**Lemma 3.1.9.**

- (1) If  $\sum_{j=1}^{\infty} a_j$  and  $\sum_{j=1}^{\infty} b_j$  converge, then so does

$$\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j)$$

for  $\lambda, \mu \in \mathbb{R}$  (or  $\mathbb{C}$ ); and

$$\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j) = \lambda \sum_{j=1}^{\infty} a_j + \mu \sum_{j=1}^{\infty} b_j$$

- (2) Suppose there exists  $N$  such that  $a_j = b_j \forall j \geq N$ . Then either  $\sum_{j=1}^{\infty} a_j$  and  $\sum_{j=1}^{\infty} b_j$  both converge, or both diverge.

*Proof.* Suppose  $n \geq N$ . Then,

$$S_n = \sum_{j=1}^{N-1} a_j + \sum_{j=N}^n a_j$$

and

$$\begin{aligned} D_n &= \sum_{j=1}^{N-1} b_j + \sum_{j=N}^n b_j \\ &= S_n + \left( \sum_{j=1}^{N-1} b_j - \sum_{j=1}^{N-1} a_j \right) \\ &= S_n + k \end{aligned}$$

where  $k$  finite constant. Thus,  $D_n$  converges if and only if  $S_n$  converges. ■

An important example is the *geometric series*. Suppose  $x \in \mathbb{R}$  and let  $a_j = x^{j-1}$  for  $j \geq 1$ . Then,

$$S_n = \sum_{j=1}^n x^{j-1} = 1 + x + x^2 + \cdots + x^{n-1}$$

and

$$S_n = \begin{cases} \frac{1-x^n}{1-x} & x \neq 1 \\ n & x = 1 \end{cases}$$

We can check the convergence of  $S_n$  for different values of  $x$ :

- if  $|x| < 1$ ,  $x^n \rightarrow 0$  and  $S_n \rightarrow 1/(1-x)$ .
- if  $x \geq 1$ , write  $x = 1 + \delta$  where  $\delta \geq 0$ . By expansion,

$$x^n = 1 + \delta n + \cdots + \delta^n \geq 1 + \delta n$$

Hence,  $x^n \rightarrow \infty$  and  $S_n \rightarrow \infty$  for  $x > 1$ .

- if  $x = 1$ , trivially  $S_n = n \rightarrow \infty$ .
- if  $x < -1$ ,  $x^n = (-1)^n |x|^n$  does not converge as  $|x|^n \rightarrow \infty$ .  $|S_n| \rightarrow \infty$  but  $S_n$  may alternate in sign.
- if  $x = -1$ ,

$$S_n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

and hence it does not converge.

Therefore, we may conclude that  $\sum_{j=1}^{\infty} x^{j-1}$  converges if and only if  $|x| < 1$ .

Let's take a further look into convergence tests. A simple, but useful result is the  $n^{\text{th}}$  term test.

**Lemma 3.1.10.** If  $\sum_{j=1}^{\infty} a_j$  converges, then  $a_j \rightarrow 0$  as  $j \rightarrow \infty$ .

*Proof.* If  $S_n = \sum_{j=1}^n a_j$ ,  $a_n = S_n - S_{n-1}$  but

$$S_n \rightarrow S \Rightarrow S_{n-1} \rightarrow S$$

so  $a_n \rightarrow 0$ . ■

Note that converse of Lemma 3.1.10 is false. For example, consider  $\sum_{j=1}^{\infty} 1/j$ , i.e. *harmonic series*. If we let  $S_n = \sum_{j=1}^n 1/j$ ,

$$\begin{aligned} S_{2n} &= S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\ &\geq S_n + \frac{1}{2n} + \frac{1}{2n+1} + \cdots + \frac{1}{2n} \end{aligned}$$

Thus  $S_{2n} \geq S_n + 1/2$  and  $S_{2n} - S_n \geq 1/2$ . If  $S_n \rightarrow S$ , then  $S_{2n} \rightarrow S$  and  $0 \geq 1/2$ , which is a contradiction. Therefore,  $S_n = \sum_{j=1}^n 1/j$  diverges.

Moreover, Lemma 3.1.10 is often used to show a series does not converge.

✱ **Series with non-negative terms**

We consider, for the time being, series whose terms satisfy

$$a_j \geq 0 \quad \forall j$$

**Theorem 3.1.11.** (*Comparison Test*) Suppose  $0 \leq b_n \leq a_n \quad \forall n$ . Then if  $\sum_{n=1}^{\infty} a_n$  converges, so does  $\sum_{n=1}^{\infty} b_n$ .

*Proof.* Let  $S_N = \sum_{n=1}^N a_n$  and  $D_N = \sum_{n=1}^N b_n$ . Both  $S_N$  and  $D_N$  are monotonically increasing and  $S_N \rightarrow S$ . But

$$b_n \leq a_n \Rightarrow D_N \leq S_N \leq S$$

So, since  $(D_N)$  is increasing and bounded above,  $D_N$  converges (Theorem 3.1.1). ■

For example of the comparison test, consider  $\sum_{n=1}^{\infty} 1/n^2$ . We have

$$0 \leq \frac{1}{n^2} < \frac{1}{n(n-1)} = \underbrace{\frac{1}{n-1} - \frac{1}{n}}_{a_n}$$

for  $n \geq 2$ . Then,

$$\sum_{n=2}^N a_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) = 1 - \frac{1}{N} \rightarrow 1$$

as  $N \rightarrow \infty$ . Hence, since  $\sum_{n=2}^{\infty} 1/(n-1)n$  converges,  $\sum_{n=2}^{\infty} 1/n^2$  and consequently  $\sum_{n=1}^{\infty} 1/n^2$  converges by comparison test.

**Theorem 3.1.12.** (*Root Test*) Suppose  $a_n \geq 0$  and  $a_n^{1/n} \rightarrow a$  as  $n \rightarrow \infty$ . Then if  $a < 1$ ,  $\sum a_n$  converges; else if  $a > 1$ ,  $\sum a_n$  diverges. If  $a = 1$ , test is inconclusive.

*Proof.* If  $a < 1$ , choose  $r$  such that  $a < r < 1$ . From Definition 3.1.1,  $\exists N$  such that  $\forall n \geq N^1$

$$a_n^{1/n} < r \Rightarrow a_n < r^n$$

Since  $r < 1$ , the series  $\sum r^n$  converges, so by comparison test and Lemma 3.1.9 (2), we have that  $\sum a_n$  converges.

If  $a > 1$ ,  $\exists N$  such that  $\forall n \geq N$   $a_n^{1/n} > 1 \Rightarrow a_n > 1$ . Hence,  $\sum a_n$  diverges (Lemma 3.1.10). ■

**Theorem 3.1.13.** (*Ratio Test*) Suppose  $a_n > 0$  and  $a_{n+1}/a_n \rightarrow l$ . If  $l < 1$   $\sum a_n$  converge; else if  $l > 1$   $\sum a_n$  diverge. If  $l = 1$ , test is inconclusive.

*Proof.* Suppose  $l < 1$  and choose  $r$  with  $l < r < 1$ . Then from Definition 3.1.1,  $\exists N$  such that  $\forall n \geq N$   $a_{n+1}/a_n < r$ . Hence, for  $n > N$ ,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N} \Rightarrow a_n < k r^n$$

where  $k$  is independent of  $n$ . But since  $\sum k r^n$  converges, so does  $\sum a_n$  (Lemma 3.1.9 (2)).

---

<sup>1</sup>Choose  $\epsilon = r - a > 0$ .

If  $l > 1$ , pick  $r > 1$  such that  $1 < r < l$ . Then,  $\exists N$  such that  $\forall n \geq N$   $a_{n+1}/a_n > r$  and

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}$$

But  $r^{n-N} \rightarrow \infty$  as  $n \rightarrow \infty$ . Thus  $a_n \rightarrow \infty$  and  $\sum a_n$  diverges. ■

Here are some examples for the tests above.

- $\sum (n/2^n)$ . Can show the convergence by both ratio and root test.
- $\sum 1/n$  diverges but ratio test and root test both yields 1 (inconclusive).
- $\sum 1/n^2$  converges but ratio test and root test both yields 1 (inconclusive).

*Remark 3.1.2.* To show  $n^{1/n} \rightarrow 1$ , write  $n^{1/n} = 1 + \delta_n$  ( $\delta_n > 0$ ). Then, binomial expansion gives

$$n = (1 + \delta_n)^n > \frac{n(n-1)}{2} \delta_n^2$$

and

$$0 < \delta_n^2 < \frac{2}{n+1} \Rightarrow \delta_n = 0$$

**Theorem 3.1.14.** (*Cauchy's Condensation Test*) Let  $(a_n)$  be a decreasing sequence of positive terms. Then  $\sum_{n=1}^{\infty} a_n$  converges if and only if

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

converges.

*Proof.* Since  $a_n$  is decreasing, note that

$$a_{2^k} \leq a_{2^{k-1}+i} \leq a_{2^{k-1}}$$

for  $1 \leq i \leq 2^{k-1}$ . Let us prove each direction separately.

( $\Rightarrow$ ) Suppose  $\sum a_n$  converges. But

$$\begin{aligned} 2^{k-1} a_{2^k} &= a_{2^k} + \cdots + a_{2^k} \\ &\leq a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^{k-1}+2^{k-1}} \\ &= \sum_{n=2^{k-1}+1}^{2^k} a_n \end{aligned}$$

Hence,

$$\sum_{k=1}^K 2^{k-1} a_{2^k} \leq \sum_{k=1}^K \left( \sum_{n=2^{k-1}+1}^{2^k} a_n \right) = \sum_{n=2}^{2^K} a_n \leq \sum_{n=1}^{\infty} a_n$$

and  $\sum_{k=1}^K 2^k a_{2^k}$  converges since it is increasing in  $K$  and bounded above.

( $\Leftarrow$ ) Suppose  $\sum 2^n a_{2^n}$  converges. Then,

$$\begin{aligned} \sum_{n=2^{k-1}+1}^{2^k} a_n &= a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^{k-1}+2^{k-1}} \\ &\leq a_{2^{k-1}} + \cdots + a_{2^{k-1}} \\ &= 2^{k-1} a_{2^{k-1}} \end{aligned}$$

This implies

$$\sum_{n=2}^{2^K} a_n = \sum_{k=1}^K \left( \sum_{n=2^{k-1}+1}^{2^k} a_n \right) \leq \sum_{k=1}^K 2^{k-1} a_{2^{k-1}} \leq \sum_{n=1}^{\infty} 2^{k-1} a_{2^{k-1}}$$

Thus for any  $N$ , if we pick  $K$  such that  $2^K > N$ ,

$$\sum_{n=2}^N a_n \leq \sum_{n=2}^{2^K} a_n \leq \sum_{n=1}^{\infty} 2^{k-1} a_{2^{k-1}}$$

Similarly, by monotone convergence theorem,  $\sum_{n=2}^N a_n$  converges. ■

**Corollary 3.1.14.1.** (*p-series test*)  $\sum_{n \geq 1} (1/n^p)$  (for  $p > 0$ ) converges if and only if  $p > 1$ .

*Proof.*  $a_n = 1/n^p$  is a decreasing sequence of positive numbers, since

$$\frac{n}{n+1} < 1 \Rightarrow \left( \frac{n}{n+1} \right)^p < 1 \Rightarrow \frac{1}{(n+1)^p} < \frac{1}{n^p}$$

Meanwhile,

$$2^n a_{2^n} = 2^n \left( \frac{1}{2^n} \right)^p = 2^{n-np} = \left( 2^{1-p} \right)^n = r^n$$

We know that  $\sum r^n$  converges if and only if  $r < 1$ , which is equivalent to  $p > 1$ . Hence, by Cauchy's condensation test,  $\sum_{n \geq 1} (1/n^p)$  (for  $p > 0$ ) converges if and only if  $p > 1$ . ■

### \* Alternating series

**Theorem 3.1.15.** (*Alternating Series Test*) Suppose  $(a_n)$  is a decreasing sequence, tending to 0 as  $n \rightarrow \infty$ . Then, the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

*Proof.* Let

$$S_n = \sum_{k=1}^n (-1)^{k+1} a_k$$

Then we can find  $S_{2n}$  is increasing, since

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n})$$

Similarly,

$$S_{2n+1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n} - a_{2n+1})$$

gives

$$S_{2n+1} \leq S_{2n-1} \leq \cdots \leq S_3 \leq S_1$$

Furthermore,  $S_{2n+1} = S_{2n} + a_{2n+1} \leq S_{2n}$ , so

$$S_2 \leq S_4 \leq \cdots \leq S_{2n} \leq S_{2n+1} \leq S_{2n-1} \leq \cdots \leq S_3 \leq S_1$$



See that  $(S_{2n})$  is increasing, bounded above by  $S_1$ ; so  $S_{2n} \rightarrow S$ ; and  $(S_{2n+1})$  is decreasing, bounded below by  $S_2$ , so  $S_{2n+1} \rightarrow \tilde{S}$ . But

$$S_{2n+1} = S_{2n} + a_{2n+1}$$

implies  $\tilde{S} = S$ . Thus, given  $\epsilon > 0$  there exists  $N_1, N_2$  such that

$$|S_{2n} - S| < \epsilon \quad \forall n \geq N_1$$

$$|S_{2n+1} - S| < \epsilon \quad \forall n \geq N_2$$

Therefore, if we choose  $n \geq 2 \max\{N_1, N_2\} + 1 = N$ ,  $|S_n - S| < \epsilon \Rightarrow S_n \rightarrow S$ . ■

One may easily see that  $\sum (-1)^{n+1}/n$  converges using the alternating series test.

#### \* Absolute and conditional convergence

**Definition 3.1.5.** Let  $a_n \in \mathbb{C}$ . If  $\sum_{n=1}^{\infty} |a_n|$  converges, we say  $\sum_{n=1}^{\infty} a_n$  is *absolutely convergent*.

Note that since  $|a_n| \geq 0$ , previous tests can be applied to show absolute convergence.

**Theorem 3.1.16.** If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then it is convergent.

*Proof.* Let

$$S_n = \sum_{k=1}^n a_k, \quad \tilde{S}_n = \sum_{k=1}^n |a_k|$$

Suppose that  $n \leq m$ . Then

$$|S_m - S_n| = \left| \sum_{k=n+1}^m a_k \right| \leq \sum_{k=n+1}^m |a_k| = |\tilde{S}_m - \tilde{S}_n|$$

$(\tilde{S}_n)$  is convergent, hence Cauchy (Lemma 3.1.7), so given  $\epsilon > 0 \exists N$  such that  $\forall m \geq n \geq N$  we have

$$|\tilde{S}_m - \tilde{S}_n| < \epsilon \Rightarrow |S_m - S_n| < \epsilon$$

Thus  $(S_n)$  is Cauchy and it converges (Theorem 3.1.8). ■

For example of absolute convergence, consider  $\sum_{n=1}^{\infty} a_n$  where  $z \in \mathbb{C}$  and  $a_n = (z^n/2^n)$ .  $|a_n| = (|z|/2)^n$  so  $\sum |a_n|$  converges if and only if  $|z|/2 < 1$ . This gives absolute convergence for  $|z| < 2$ . If  $|z| \geq 2$ ,  $|a_n| \geq 1$  and there is no absolute convergence. Finally, note that converse of Theorem 3.1.16 is false.

**Definition 3.1.6.** If  $\sum a_n$  converges, but  $\sum |a_n|$  does not, we say series is *conditionally convergent*.

N.b. order of terms matters. If rearranged, the sum may change. Consider the rearranged sequences below.

$$(1) \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$(2) \quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{7} + \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

If  $S_n$  is partial sum of (1),  $T_n$  partial sum of (2), then

$$\begin{aligned} S_n &\rightarrow S > 0 \\ T_n &\rightarrow \frac{3S}{2} \neq S \end{aligned}$$

**Definition 3.1.7.** Let  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  be a bijection, then  $(a'_n)$  with  $a'_n = a_{\sigma(n)}$  is a *rearrangement* of  $(a_n)$ .

**Theorem 3.1.17.** If  $a'_n = a_{\sigma(n)}$  is a rearrangement of  $(a_n)$ , and  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a'_n$$

*Proof.* Fix  $\epsilon > 0$ . Since  $\sum_{n=1}^{\infty} |a_n|$  converges,  $\exists N$  such that

$$\sum_{n=N+1}^{\infty} |a_n| < \epsilon$$

Pick  $M$  such that  $\sigma^{-1}(k) < M$  for  $k = 1, \dots, N$ . Then if  $m \geq M$ ,

$$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^m a_{\sigma(n)} = \sum_{n \in K_m} a_n$$

where  $K_m = \{N+1, N+2, \dots\} \setminus \{\text{finitely many points}\}$ . So

$$\left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^m a'_n \right| \leq \sum_{n \in K_m} |a_n| \leq \sum_{n \geq N+1} |a_n| < \epsilon$$

i.e.  $\sum_{n=1}^m a'_n \rightarrow \sum_{n=1}^{\infty} a_n$  as  $m \rightarrow \infty$ . ■

## 3.2 Continuity

### 3.2.1 Continuity of a Function

Suppose  $E \subseteq \mathbb{C}$  is non-empty, and  $f : E \rightarrow \mathbb{C}$  is any function (includes case where  $E \subseteq \mathbb{R}$ ,  $f : E \rightarrow \mathbb{R}$ ).

**Definition 3.2.1.**  $f$  is *continuous* at  $a \in E$  if: given  $\epsilon > 0$ , there exists  $\delta = \delta(\epsilon) > 0$  such that

$$|f(z) - f(a)| < \epsilon$$

for all  $z \in E$  with  $|z - a| < \delta$ , i.e. “Points close to  $a$  in the domain are mapped close to  $f(a)$  in range.”

We say  $f$  is continuous on  $E$  if  $f$  is continuous at  $a$  for all  $a \in E$ .

**Theorem 3.2.1.**  $f : E \rightarrow \mathbb{C}$  is continuous at  $a \in E$  if and only if

$$f(z_n) \rightarrow f(a)$$

for all sequences  $(z_n)$  with  $z_n \in E$ ,  $z_n \rightarrow a$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $f$  is continuous at  $a$ , and  $(z_n)$  is a sequence  $z_n \in E$  with  $z_n \rightarrow a$ . Let  $\epsilon > 0$ . Then  $\exists \delta > 0$  such that

$$z \in E, |z - a| < \delta \Rightarrow |f(z) - f(a)| < \epsilon$$

Since  $z_n \rightarrow a$ , there exists  $N$  such that

$$n \geq N \Rightarrow |z_n - a| < \delta \Rightarrow |f(z_n) - f(a)| < \epsilon$$

Therefore  $f(z_n) \rightarrow f(a)$ .

( $\Leftarrow$ ) Suppose  $f$  is not continuous at  $a$ . Then,  $\exists \epsilon > 0$  such that  $\forall \delta > 0, \exists z \in E$  with

$$|z - a| < \delta \text{ and } |f(z) - f(a)| \geq \epsilon$$

apply this with

$$\delta = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$$

to find for each  $n$  and  $z_n \in E$  with

$$|z_n - a| < \frac{1}{n} \text{ and } |f(z_n) - f(a)| \geq \epsilon > 0$$

Then  $z_n \rightarrow a$  as  $n \rightarrow \infty$  but  $f(z_n) \not\rightarrow f(a)$ , which is a contradiction. Hence  $f$  is continuous at  $a$ . ■

This means we could alternatively take our definition of continuity as:

$$f(z_n) \rightarrow f(a)$$

for all sequences  $(z_n)$  with  $z_n \in E, z_n \rightarrow a$ .

**Lemma 3.2.2.** Suppose  $f, g : E \rightarrow \mathbb{C}$  are continuous at  $a \in E$ . Then so are the functions

- (1)  $f(z) + g(z)$ ,
- (2)  $f(z)g(z)$ ,
- (3)  $\lambda g(z)$  (constant  $\lambda \in \mathbb{C}$ ),
- (4)  $1/f(z)$  provided  $f(z) \neq 0 \forall z \in E$ .

*Proof.* Using the sequential characterisation of continuity, this follows easily from analogous results for sequences (Lemma 3.1.3). E.g. if  $z_n \rightarrow a$ ,

$$f(z_n) \rightarrow f(a) \text{ and } g(z_n) \rightarrow g(a) \Rightarrow f(z_n) + g(z_n) \rightarrow f(a) + g(a)$$

etc. ■

Note that Lemma 3.2.2 can also be directly proved from Definition 3.2.1. We now show that composition of continuous functions is continuous.

**Theorem 3.2.3.** Suppose  $A, B \subseteq \mathbb{C}, f : A \rightarrow B, g : B \rightarrow \mathbb{C}$  and that  $f$  is continuous at  $a \in A$ ,  $g$  is continuous at  $f(a) \in B$ . Then  $g \circ f : A \rightarrow \mathbb{C}$  is continuous at  $a$ .

*Proof.* Suppose  $(z_n)$  is any sequence with  $z_n \in A$ ,  $z_n \rightarrow a$ . Then by continuity of  $f$  at  $a$ ,

$$f(z_n) \rightarrow f(a)$$

By continuity of  $g$  at  $f(a)$ ,

$$g(f(z_n)) \rightarrow g(f(a)) \Leftrightarrow g \circ f(z_n) \rightarrow g \circ f(a)$$

Hence  $g \circ f$  is continuous at  $a$ . ■

Here are some examples of continuous (and discontinuous) functions.

- (1)  $f(z) = z$  is continuous at all points of  $\mathbb{C}$ .
- (2) By Lemma 3.2.2 and Example (1) above, any polynomial in  $z$  is continuous at all points of  $\mathbb{C}$ .
- (3)  $f(z) = |z|$  is continuous at all points of  $\mathbb{C}$ .<sup>2</sup>
- (4)  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$x \mapsto \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

is not continuous at  $x = 0$ .

- (5)  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\sin x$  is continuous,<sup>3</sup> so if  $x \neq 0$  then Theorem 3.2.3 and Lemma 3.2.2 imply  $f$  is continuous at  $x$ . However,  $f$  is discontinuous at  $x = 0$ . Choose  $x_n$  such that

$$\frac{1}{x_n} = \left(2n + \frac{1}{2}\right)\pi$$

which imply  $f(x_n) = 1$ ,  $x_n \rightarrow 0$ ,  $f(0) = 0 \neq \lim_{n \rightarrow \infty} f(x_n)$ .

- (6)  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f$  is continuous at  $x \neq 0$  by same reasons as (5). But  $f$  is also continuous at 0 this time. Suppose  $x_n \rightarrow 0$ . Then

$$|f(x_n)| = |x_n| \left| \sin \frac{1}{x_n} \right| \leq |x_n|$$

So  $|f(x_n)| \leq |x_n|$  and  $f(x_n) \rightarrow 0 = f(0)$ .

---

<sup>2</sup>See reverse triangle inequality.

<sup>3</sup>See later sections for proof.

(7) (*Dirichlet Function*)  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is discontinuous at every point. If  $x \in \mathbb{Q}$ , take sequence  $x_n \notin \mathbb{Q}$ , then  $x_n \rightarrow x$  but

$$f(x_n) = 0 \not\rightarrow f(x) = 1$$

If  $x \notin \mathbb{Q}$ , take  $x_n \in \mathbb{Q}$ ,  $x_n \rightarrow x$  but

$$f(x_n) = 1 \not\rightarrow f(x) = 0$$

### 3.2.2 Limit of a Function

Suppose  $E \subseteq \mathbb{C}$ ,  $f : E \rightarrow \mathbb{C}$ . We want to define

$$\lim_{z \rightarrow a} f(z)$$

even when  $a$  may not belong to  $E$ . For instance, consider  $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ ,

$$z \mapsto \sin z / z$$

What is  $\lim_{z \rightarrow 0} f(z)$ ? – However, this does not always make sense. If  $E = \{0\} \cup [1, 2]$ , for  $f : E \rightarrow \mathbb{R}$  it is impossible to consider  $\lim_{x \rightarrow 0} f(x)$ , because there are no point near 0 except 0 itself.

**Definition 3.2.2.** Suppose  $E \subseteq \mathbb{C}$ ,  $a \in \mathbb{C}$ . We say that  $a$  is a *limit point* of  $E$  if for any  $\delta > 0$  there exists  $z \in E$  such that

$$0 < |z - a| < \delta$$

N.b.  $a$  is a limit point of  $E$  if and only if there exists a sequence  $(z_n)$  such that  $z_n \in E$ ,  $z_n \neq a$ , and

$$z_n \rightarrow a$$

**Definition 3.2.3.** Suppose  $E \subseteq \mathbb{C}$ ,  $f : E \rightarrow \mathbb{C}$  and let  $a \in \mathbb{C}$  be a limit point of  $E$ . We say

$$\lim_{z \rightarrow a} f(z) = l$$

if: given  $\epsilon > 0$ ,  $\exists \delta > 0$  such that  $\forall z \in E$ ,

$$0 < |z - a| < \delta \Rightarrow |f(z) - l| < \epsilon$$

**Lemma 3.2.4.** If  $f, E, a$  as above, then

$$\lim_{z \rightarrow a} f(z) = l$$

if and only if  $f(z_n) \rightarrow l$  for all sequences  $z_n \in E$ ,  $z_n \neq a$ ,  $z_n \rightarrow a$ .

Observe that the definitions immediately tell us that if  $a \in E$  is a limit point then  $f$  is continuous at  $a$  if and only if

$$\lim_{z \rightarrow a} f(z) = f(a)$$

If  $a \in E$  is *isolated*, i.e. not a limit point, then  $f$  is always continuous at  $a$ .

Note that limit of functions behave similarly to limits of sequences.

**Lemma 3.2.5.** Suppose  $E \subseteq \mathbb{C}$ ,  $a \in \mathbb{C}$  is a limit point, and  $f, g : E \rightarrow \mathbb{C}$

- (1) The limit is unique, i.e. if  $f(z) \rightarrow A$  and  $f(z) \rightarrow B$  as  $z \rightarrow a$ , then  $A = B$ .
- (2) If  $f(z) \rightarrow l$ ,  $g(z) \rightarrow k$ , then  $f(z) + g(z) \rightarrow l + k$  as  $z \rightarrow a$ ;
- (3)  $f(z)g(z) \rightarrow lk$  as  $z \rightarrow a$ ;
- (4)  $f(z)/g(z) \rightarrow l/k$  as  $z \rightarrow a$  (if  $k \neq 0$ ).

*Proof.* (of (1))

$$|A - B| = |A - f(z) + f(z) - B| \leq |A - f(z)| + |f(z) - B|$$

for all  $z \in E$ ,  $z \neq a$ . Given  $\epsilon > 0$ ,  $\exists \delta_1 > 0$  such that

$$0 < |z - a| < \delta_1, z \in E \Rightarrow |f(z) - A| < \epsilon, z \in E$$

Also  $\exists \delta_2 > 0$  such that

$$0 < |z - a| < \delta_2, z \in E \Rightarrow |f(z) - B| < \epsilon, z \in E$$

Since  $a$  is a limit point,  $\exists z \in E$  such that

$$0 < |z - a| < \min\{\delta_1, \delta_2\} \Rightarrow |A - B| < 2\epsilon$$

Because  $\epsilon$  is arbitrary,  $A = B$ . ■

**Theorem 3.2.6.** (*Intermediate Value Theorem*) Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f(a) < f(b)$ . Then for any  $\eta$  with  $f(a) < \eta < f(b)$  there exists  $c \in [a, b]$  such that  $f(c) = \eta$ .

*Proof.* Let

$$S = \{x \in [a, b] \mid f(x) < \eta\}$$

Clearly  $S$  is not empty as  $a \in S$ .  $S$  is bounded above by  $b$ . So  $S$  has a supremum,  $c$  (Theorem 3.1.2). We claim  $f(c) = \eta$ . From the definition of a supremum (Definition 3.1.2), for each  $n \in \mathbb{N}$  there exists  $x_n \in S$  with

$$c - \frac{1}{n} \leq x_n \leq c$$

Hence  $x_n \rightarrow c$ . But  $f$  is continuous so  $f(x_n) \rightarrow f(c)$  and

$$f(x_n) < \eta \Rightarrow f(c) \leq \eta$$

In particular,  $c \neq b$ . Now let

$$\tilde{x}_n = c + \frac{1}{n}$$

For  $n$  large enough,  $\tilde{x}_n \in [a, b]$ , and  $\tilde{x}_n \rightarrow c$ . Furthermore,  $f(\tilde{x}_n) \geq \eta$  since

$$\tilde{x}_n > c \Rightarrow \tilde{x}_n \notin S$$

Hence

$$f(c) = \lim_{n \rightarrow \infty} f(\tilde{x}_n) \geq \eta$$

and thus  $f(c) = \eta$ . ■

This theorem is useful to find roots of functions. For example, we can check existence of  $N^{\text{th}}$  roots. Suppose  $y > 0$ . Consider for  $N \in \mathbb{N}$  the function  $f : [0, 1+y] \rightarrow \mathbb{R}$ ,

$$x \mapsto x^N$$

Then since

$$(a+y)^N \geq 1 + Ny > y$$

we have

$$f(0) < y < f(1+y)$$

and there exists  $c \in (0, 1+y)$  such that  $f(c) = y$  by intermediate value theorem (IVT).  $c$  is a *positive  $N^{\text{th}}$  root* of  $y$ . In fact, since

$$y_1 < y_2 \Rightarrow f(y_1) < f(y_2)$$

$c$  is unique.

**Lemma 3.2.7.** (*Bounds on Continuous Function*) Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then there exist  $K$  such that

$$|f(x)| \leq K \quad \forall x \in [a, b]$$

*Proof.* Suppose not. Then for each  $n = 1, 2, \dots$  we can find  $x_n \in [a, b]$  with  $|f(x_n)| > n$ . By Bolzano-Weierstrass theorem (Theorem 3.1.6),  $(x_n)$  is a bounded sequence, and hence has a subsequence  $x_{n_j} \rightarrow x$  for some  $x$ . Moreover, since

$$a \leq x_{n_j} \leq b \Rightarrow a \leq x \leq b$$

so  $x \in [a, b]$ . But

$$|f(x_{n_j})| > n_j \geq j$$

so  $f(x_{n_j})$  cannot converge. This contradicts the assumption that  $f$  is continuous as  $f(x_{n_j}) \not\rightarrow f(x)$ . ■

**Theorem 3.2.8.** (*Extreme Value Theorem*) Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then there exist  $y, Y \in [a, b]$  such that

$$f(y) \leq f(x) \leq f(Y) \quad \forall x \in [a, b]$$

*Proof.* The set

$$A = \{f(x) \mid x \in [a, b]\}$$

is bounded by Lemma 3.2.7, so has a supremum  $M = \sup A$ . From Definition 3.1.2,  $M - 1/n$  is not an upper bound of  $A$  for  $n \in \mathbb{N}$ . So  $\exists y_n \in A$  with

$$M - \frac{1}{n} \leq y_n \leq M$$

From definition of  $A$ , there exists  $x_n \in [a, b]$  with

$$y_n = f(x_n) \Rightarrow M - \frac{1}{n} \leq f(x_n) \leq M$$

$(x_n)$  is a bounded sequence. So by Bolzano-Weierstrass theorem we can pick a subsequence  $(x_{n_j})$  such that

$$x_{n_j} \rightarrow y$$

for some  $y \in [a, b]$ . Also  $f(x_{n_j}) \rightarrow M$ , so by continuity of  $f$ ,

$$f(y) = \lim_{j \rightarrow \infty} f(x_{n_j}) = M$$

It follows immediately that

$$f(x) \leq f(y) \quad \forall x \in [a, b]$$

For lower bound, we can either consider  $\inf A$  and argue similarly, or apply result already established to  $-f$ . ■

Note that, for Lemma 3.2.7 and Theorem 3.2.8, it is crucial that  $f$  is defined on a closed bounded interval: e.g. consider

- $f : (0, 1] \rightarrow \mathbb{R}, x \mapsto 1/x$ , continuous, but unbounded.
- $f : [0, \infty) \rightarrow \mathbb{R}, x \mapsto x$ , continuous, but unbounded.

### 3.2.3 Inverse Function Theorem

**Definition 3.2.4.** Function  $f : [a, b] \rightarrow \mathbb{R}$  is

- *increasing* if  $a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) \leq f(x_2)$ ;
- *strictly increasing* if  $a \leq x_1 < x_2 \leq b \Rightarrow f(x_1) < f(x_2)$ .

Decreasing and strictly decreasing are defined similarly.

**Theorem 3.2.9.** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and strictly increasing. Let  $c = f(a)$ ,  $d = f(b)$ . Then

$$f : [a, b] \rightarrow [c, d]$$

is a bijection and

$$f^{-1} : [c, d] \rightarrow [a, b]$$

is continuous and strictly increasing.



*Proof.* First we note that  $f$  is injective. If  $x \neq y$ , then without loss of generality

$$a \leq x < y \leq b \Rightarrow f(x) < f(y) \Rightarrow f(x) \neq f(y)$$

Next,  $f$  maps into  $[c, d]$ . If  $a \leq x \leq b$ , then

$$c = f(a) \leq f(x) \leq f(b) = d$$

Also,  $f$  is surjective by intermediate value theorem: if  $\eta \in [c, d]$  then  $\exists x \in [a, b]$  with  $f(x) = \eta$ .

To show  $f^{-1}$  is continuous, fix  $z \in (c, d)$  and let  $y = f^{-1}(z) \in (a, b)$ . Let  $\epsilon > 0$  to be sufficiently small that

$$a \leq y - \epsilon < y < y + \epsilon \leq b \Rightarrow c \leq f(y - \epsilon) < f(y) < f(y + \epsilon) \leq d$$

Pick  $\delta > 0$  such that  $(z - \delta, z + \delta) \subset (f(y - \epsilon), f(y + \epsilon))$ . Since  $f : (y - \epsilon, y + \epsilon) \rightarrow (f(y - \epsilon), f(y + \epsilon))$  is bijective, if  $|z' - z| < \delta$ , then

$$\begin{aligned} z' \in (f(y - \epsilon), f(y + \epsilon)) &\Rightarrow f^{-1}(z') \in (y - \epsilon, y + \epsilon) \\ &\Rightarrow |f^{-1}(z') - f^{-1}(z)| < \epsilon \end{aligned}$$

If  $z = c$  or  $d$  similar argument works. Therefore  $f^{-1}$  is continuous.

To see  $f^{-1}$  is increasing, suppose not. Then  $\exists c \leq z_1 < z_2 \leq d$  with  $f^{-1}(z_1) \geq f^{-1}(z_2)$ . But  $f$  is strictly increasing; this implies

$$f(f^{-1}(z_1)) \geq f(f^{-1}(z_2)) \Rightarrow z_1 \geq z_2$$

which is a contradiction. Thus  $f^{-1}$  is strictly increasing. ■

### 3.3 Differentiability

In this section, let

$$f : E \rightarrow \mathbb{C}$$

where  $E \subseteq \mathbb{C}$ , while mostly  $E$  will be a real interval.

**Definition 3.3.1.** Let  $x \in E$  be a limit point.  $f$  is said to be *differentiable* at  $x$ , with derivative  $f'(x)$  if

$$\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = f'(x)$$

exists.

If  $f$  is differentiable at each  $x \in E$ , we say  $f$  is differentiable on  $E$ . Also, we will assume from now on  $E$  has no isolated points (set is either interval or disc).

*Remark 3.3.1.*

(1) Other common notations are

$$\frac{df}{dx}, \frac{dy}{dx}$$

etc.

(2) We can equivalently write

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- (3) Another ways to phrase definition: let  $\epsilon(h) = f(x+h) - f(x) - hf'(x)$ ; then

$$\lim_{h \rightarrow 0} \frac{\epsilon(h)}{h} = 0$$

We say,  $f$  has a *best affine approximation* near  $x$ .

- (4) If  $f$  is differentiable at  $x$ , then  $f$  is continuous at  $x$ , since  $\epsilon(h) \rightarrow 0$  as  $h \rightarrow 0$  giving  $f(x+h) \rightarrow f(x)$  as  $h \rightarrow 0$ .

Let's look at some examples.

- (1)  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x$  is differentiable at each  $x$  and  $f'(x) = 1$ , i.e.

$$f(x+h) = x+h = f(x) + h \cdot 1 + 0$$

where  $1 = f'(x)$  and  $0 = \epsilon(h)$ .

- (2)  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|$  is differentiable for  $x \neq 0$ . At  $x = 0$  consider

$$\frac{f(h) - f(0)}{h} = \frac{|h|}{h}$$

We have

$$\lim_{h_n \rightarrow 0^+} \frac{f(h) - f(0)}{h} = 1$$

but

$$\lim_{h_n \rightarrow 0^-} \frac{f(h) - f(0)}{h} = -1$$

hence the limit does not exist and  $f$  is not differentiable at  $x = 0$ .

### 3.3.1 Differentiation of Sums and Products

**Proposition 3.3.1.**

- (1) If  $f(x) = c \forall x \in E$  then  $f$  is differentiable on  $E$  with  $f'(x) = 0$ .  
 (2) If  $f, g$  are differentiable at  $x$ , so is  $f + g$  and

$$(f + g)'(x) = f'(x) + g'(x)$$

- (3) If  $f, g$  are differentiable at  $x$ , so is  $fg$  and

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

- (4) If  $f$  is differentiable at  $x$  and  $f(y) \neq 0 \forall y \in E$  then  $1/f$  is differentiable at  $x$  and

$$\left(\frac{1}{f}\right)' = -\frac{f'(x)}{f(x)^2}$$

*Proof.*

- (1) Can easily see

$$\lim_{h \rightarrow 0} \frac{c - c}{h} = 0$$

■

(2) By Lemma 3.2.5, and from

$$\frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

we have

$$\lim_{h \rightarrow 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = f'(x) + g'(x)$$

■

(3) Let  $\phi(x) = f(x)g(x)$ . Then for  $h \neq 0$

$$\begin{aligned} \frac{\phi(x+h) - \phi(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= f(x+h) \left( \frac{g(x+h) - g(x)}{h} \right) + g(x) \left( \frac{f(x+h) - f(x)}{h} \right) \end{aligned}$$

implying

$$\frac{\phi(x+h) - \phi(x)}{h} \rightarrow f(x)g'(x) + f'(x)g(x)$$

■

(4) Let  $\phi(x) = 1/f(x)$ . Then,

$$\frac{\phi(x+h) - \phi(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \frac{f(x) - f(x+h)}{h} \frac{1}{f(x)f(x+h)}$$

Again, by Lemma 3.2.5,

$$\frac{\phi(x+h) - \phi(x)}{h} \rightarrow -\frac{f'(x)}{f(x)^2}$$

■

Note that (3) and (4) give the quotient rule:

$$\left( \frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

For example, consider  $f_n(x) = x^n$  where  $n \in \mathbb{N}$ . Shown that  $f_1(x)$  is differentiable and  $f_1'(x) = 1$ , we claim  $f_n(x)$  is differentiable with  $f_n'(x) = nx^{n-1}$  for all  $n$ . If the claim is true for  $f_{n-1}$ , observe that

$$f_n(x) = x^{n-1}x = f_{n-1}(x)f_1(x)$$

By Proposition 3.3.1 (3),  $f_n$  is differentiable, and

$$\begin{aligned} f_n'(x) &= f_1(x)f_{n-1}'(x) + f_1'(x)f_{n-1}(x) \\ &= x(n-1)x^{n-2} + x^{n-1} = nx^{n-1} \end{aligned}$$

Hence, by induction,  $f_n'(x) = nx^{n-1}$  for all  $n \in \mathbb{N}$ .

Now consider  $f_{-n}(x) = x^{-n}$  where  $n \in \mathbb{N}$ . If  $x \neq 0$   $f_{-n}(x) = 1/f_n(x)$ . So by Proposition 3.3.1 (4),  $f_{-n}(x)$  is differentiable and

$$f'_{-n}(x) = -\frac{f'_n(x)}{f_n(x)^2} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

Therefore  $f'_n(x) = nx^{n-1}$  holds for all  $n \in \mathbb{Z}$  (with condition  $x \neq 0$  if  $n < 0$ ).

Combining results above with Proposition 3.3.1, we see all polynomial functions are differentiable everywhere, and rational functions  $f(x) = p(x)/q(x)$  ( $p, q$  coprime polynomials) are differentiable everywhere except the roots of  $q$ .

**Theorem 3.3.2.** (*Chain Rule*) Suppose  $U, V \subset \mathbb{C}$  and  $f : U \rightarrow V, g : V \rightarrow \mathbb{C}$  are such that  $f$  is differentiable at  $a \in U$  and  $g$  is differentiable at  $f(a) \in V$ . Then,  $g \circ f : U \rightarrow \mathbb{C}$  is differentiable at  $a$  and

$$(g \circ f)'(a) = g'(f(a))f'(a)$$

*Proof.* Let  $f(a) = b$ . We can restate the differentiability of  $f$  at  $a$ ,  $g$  at  $b$  as:

$$f(x) = f(a) + (x - a)f'(a) + \epsilon_f(x)(x - a)$$

$$g(y) = g(b) + (y - b)g'(b) + \epsilon_g(y)(y - b)$$

with  $\lim_{x \rightarrow a} \epsilon_f(x) = \lim_{y \rightarrow b} \epsilon_g(y) = 0$ . Notice we can set  $\epsilon_f(a) = 0, \epsilon_g(b) = 0$  to make  $\epsilon_f, \epsilon_g$  continuous at  $a, b$  respectively. Now set  $y = f(x)$  and write

$$\begin{aligned} g(f(x)) &= g(f(a)) + (f(x) - f(a))g'(f(a)) + \epsilon_g(f(x))(f(x) - f(a)) \\ &= g(f(a)) + ((x - a)f'(a) + \epsilon_f(x - a))g'(f(a)) \\ &\quad + \epsilon_g(f(x))((x - a)f'(a) + \epsilon_f(x)(x - a)) \end{aligned}$$

Thus

$$\begin{aligned} g(f(x)) &= g(f(a)) + (x - a)f'(a)g'(f(a)) + (x - a)[g'(f(a))\epsilon_f(x) + f'(a)\epsilon_g(f(x)) + \epsilon_g(f(x))\epsilon_f(x)] \\ &= g(f(a)) + (x - a)f'(a)g'(f(a)) + (x - a)\sigma(x) \end{aligned}$$

where

$$\sigma(x) = g'(f(a))\epsilon_f(x) + f'(a)\epsilon_g(f(x)) + \epsilon_g(f(x))\epsilon_f(x)$$

Note that  $\epsilon_f(x) \rightarrow 0$  as  $x \rightarrow a$ , and  $\epsilon_g(f(x))$  is composition of continuous functions, so  $\epsilon_g(f(x)) \rightarrow 0$  as  $x \rightarrow a$ . Hence  $\sigma(x) \rightarrow 0$  as  $x \rightarrow a$  and therefore  $(g \circ f)'(a) = g'(f(a))f'(a)$  from definition of differentiation. ■

*Remark 3.3.2.* Chain rule is often written as

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

where  $y(t(x))$ .

### 3.3.2 The Mean Value Theorem

**Theorem 3.3.3.** (*Rolle's Theorem*) Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$  then  $\exists c \in (a, b)$  such that  $f'(c) = 0$ .

*Proof.* By Theorem 3.2.8 (EVT),  $\exists y, Y \in [a, b]$  such that

$$m = f(y) \leq f(x) \leq f(Y) = M$$

$\forall x \in [a, b]$ . If  $m = M = f(a)$  then  $f(x) = M$  and  $f$  constant, so we can take any  $c \in (a, b)$ .

Otherwise, either  $M > f(a)$  or  $m < f(a)$ . If  $M > f(a)$ , then  $Y \in (a, b)$ . We claim  $f'(Y) = 0$ . To see this, suppose  $h_n \rightarrow 0+$ . Then

$$\frac{f(Y + h_n) - f(Y)}{h_n} \leq 0 \Rightarrow \lim_{h \rightarrow 0} \frac{f(Y + h) - f(Y)}{h} \leq 0$$

If  $h_n \rightarrow 0-$ ,

$$\frac{f(Y + h_n) - f(Y)}{h_n} \geq 0 \Rightarrow \lim_{h \rightarrow 0} \frac{f(Y + h) - f(Y)}{h} \geq 0$$

Hence by uniqueness of limit,  $f'(Y) = 0$ . If  $m < f(a)$  a similar argument shows  $y \in (a, b)$  and  $f'(y) = 0$ . Regardless  $\exists c \in (a, b)$  such that  $f'(c) = 0$ . ■

**Theorem 3.3.4.** (*Mean Value Theorem*) Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function, differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  such that

$$f(b) - f(a) = f'(c)(b - a) \Leftrightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

*Proof.* Consider

$$\phi(x) = (x - a)(f(b) - f(a)) - (b - a)(f(x) - f(a))$$

Clearly,  $\phi$  satisfies conditions of Rolle's theorem, so  $\exists c \in (a, b)$  such that

$$\phi'(c) = 0 \Rightarrow 0 = (f(b) - f(a)) - (b - a)f'(c)$$

■

If  $f$  is differentiable on an open interval  $I$  containing  $a$ , another way to state mean value theorem (MVT) is: given  $h$  such that  $a + h \in I$ ,  $\exists \Theta = \Theta(h) \in (0, 1)$  such that

$$f(a + h) = f(a) + hf'(a + \Theta h)$$

The mean value theorem allows us to export local information about the derivative to global properties of the function.

**Corollary 3.3.4.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous, and differentiable on  $(a, b)$ . Then,

- (1) if  $f'(x) > 0 \forall x \in (a, b)$ ,  $f$  is strictly increasing.
- (2) if  $f'(x) \geq 0 \forall x \in (a, b)$ ,  $f$  is increasing.
- (3) if  $f'(x) = 0 \forall x \in (a, b)$ ,  $f$  is constant on  $[a, b]$

*Proof.*

- (1) Let  $a \leq x < y \leq b$ . Hypotheses of mean value theorem apply to  $f : [x, y] \rightarrow \mathbb{R}$  so  $\exists c \in (x, y)$  such that

$$f(y) - f(x) = f'(c)(y - x) > 0$$

- (2) Same as above, but use  $f'(c) \geq 0$ .

- (3) Pick  $x \in (a, b]$ . Then, by applying mean value theorem on  $[a, x]$ , we deduce that there exists  $c \in (a, x)$  such that

$$f(x) - f(a) = f'(c)(x - a) = 0$$

and  $f$  is constant. ■

Meanwhile, note that the mean value theorem is not necessarily true on general sets: e.g.  $f : \mathbb{Q} \rightarrow \mathbb{Q}$ ,

$$x \mapsto \begin{cases} 0 & x^2 < 2 \\ 1 & x^2 > 2 \end{cases}$$

Now we revisit the inverse function theorem.

**Theorem 3.3.5.** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and differentiable on  $(a, b)$ , with  $f'(x) > 0 \forall x \in (a, b)$ . Let  $f(a) = c$ ,  $f(b) = d$ . Then,

$$f : [a, b] \rightarrow [c, d]$$

is bijective and  $f^{-1}$  is differentiable on  $(c, d)$  with

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

*Proof.* By Corollary 3.3.4.1  $f$  is strictly increasing on  $[a, b]$ . By Theorem 3.2.9,

$$f : [a, b] \rightarrow [c, d]$$

is bijective and

$$f^{-1} : [c, d] \rightarrow [a, b]$$

is continuous and strictly increasing. However, it remains to show  $f^{-1}$  is differentiable on  $(c, d)$ .

Let  $y \in (c, d)$  and set  $x = f^{-1}(y)$ . Given  $h$  such that  $y + h \in (c, d)$ , let  $k$  be such that  $y + h = f(x + k)$ . Then,

$$\frac{f^{-1}(y + h) - f^{-1}(y)}{h} = \frac{x + k - x}{f(x + k) - y} = \frac{k}{f(x + k) - f(x)}$$

Fix  $\epsilon > 0$ ; then by differentiability of  $f$  and facts about limits,  $\exists \delta > 0$  such that for all  $0 < |k| < \delta$  we have

$$\left| \frac{k}{f(x + k) - f(x)} - \frac{1}{f'(x)} \right| < \epsilon$$

Since  $f^{-1}$  is continuous, there exists  $\delta'$  such that

$$\begin{aligned} 0 < |h| < \delta' &\Rightarrow 0 < |f^{-1}(y+h) - x| < \delta \\ &\Rightarrow 0 < k < \delta \\ &\Rightarrow \left| \frac{f^{-1}(y+h) - f^{-1}(y)}{h} - \frac{1}{f'(x)} \right| < \epsilon \end{aligned}$$

Therefore  $f^{-1}$  is differentiable at  $y$  and

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

■

With fixed  $R > 0$ , consider  $f : [0, R] \rightarrow \mathbb{R}$ ,  $x \mapsto x^n$  for  $n \in \mathbb{N}$ .  $f$  is continuous on  $[0, R]$ , differentiable on  $(0, R)$  with  $f'(x) = nx^{n-1} > 0$  ( $x \in (0, R)$ ). Hence by Theorem 3.3.5,  $f$  maps  $[0, R]$  bijectively onto  $[0, R^n]$  and the inverse  $g(y) = f^{-1}(y) = y^{1/n}$  is differentiable with

$$g'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n} y^{\frac{1}{n}-1}$$

More generally, let  $h(x) = x^r$  where  $r \in \mathbb{Q}$ . Writing  $r = m/n$  for  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$  and defining

$$x^r = \left(x^{1/n}\right)^m$$

$h$  is differentiable on  $(0, \infty)$  and

$$h'(x) = m \left(x^{1/n}\right)^{m-1} \left(\frac{1}{n} x^{\frac{1}{n}-1}\right) = \frac{m}{n} x^{\frac{m}{n}-1} = rx^{r-1}$$

by chain rule.

**Theorem 3.3.6.** (*Cauchy's Mean Value Theorem*) Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  such that

$$g'(f(a) - f(b)) = f'(c)(g(a) - g(b))$$

which can be also written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

if both sides are well-defined.

*Proof.* Let

$$\phi(x) = (g(x) - g(a))(f(b) - f(a)) - (g(b) - g(a))(f(x) - f(a))$$

$\phi$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$  and  $\phi(a) = \phi(b) = 0$ . Hence by Rolle's theorem, there exists  $c \in (a, b)$  such that

$$\phi'(c) = 0 \Rightarrow g'(c)(f(a) - f(b)) - f'(c)(g(a) - g(b)) = 0$$

■

A famous application of Cauchy's mean value theorem is the *L'Hôpital's rule*. Consider

$$\frac{e^x - 1}{\sin x}$$

as  $x \rightarrow 0$ .<sup>4</sup> With noting that

$$\frac{e^x - 1}{\sin x} = \frac{e^x - e^0}{\sin x - \sin 0}$$

apply Cauchy's mean value theorem with  $f(x) = e^x$ ,  $g(x) = \sin x$  on interval  $[x, 0]$  or  $[0, x]$  to deduce that  $\exists \theta_x \in (0, 1)$  such that

$$\frac{e^x - 1}{\sin x} = \frac{e^{\theta_x x}}{\cos(\theta_x x)}$$

Now  $0 < |\theta_x x| < |x|$  so as  $x \rightarrow 0$   $\theta_x x \rightarrow 0$ . Further, because  $e^y / \cos y$  is continuous, we have

$$\frac{e^{\theta_x x}}{\cos(\theta_x x)} \rightarrow 1$$

as  $x \rightarrow 0$ . Thus

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x} = 1$$

### 3.3.3 Higher Derivatives and Taylor's Theorem

Suppose  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ . Then we can consider the function

$$f' : (a, b) \rightarrow \mathbb{R}, \quad x \mapsto f'(x)$$

If this is differentiable at  $x \in (a, b)$  we say  $f$  is twice differentiable at  $x$  and write

$$(f')'(x) = f''(x) = f^{(2)}(x)$$

We can iterate to define  $k$ -times differentiability and write

$$f^{(k)}(x) = \left(f^{(k-1)}\right)'(x)$$

Furthermore, function is said to be *smooth* if it is  $k$ -times differentiable on  $(a, b)$  for every  $k$ . We would like to extend mean value theorem for a function which is more differentiable to incorporate higher derivatives.

**Theorem 3.3.7.** (*Taylor's Theorem with Lagrange Remainder*) Suppose  $f$  and its derivatives to order  $n - 1$  are continuous on  $[a, a + h]$  and  $f$  is  $n$  times differentiable on  $(a, a + h)$ . Then there exists  $\theta \in (0, 1)$  such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a + \theta h)$$

<sup>4</sup>We will formally define  $e$  and  $\sin$  later.



N.b. for  $n = 1$  Theorem 3.3.7 is mean value theorem, so Theorem 3.3.7 is a  $n^{\text{th}}$  order mean value theorem. We say

$$R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h)$$

is *Lagrange's form of the remainder*.

Also, considering  $g(x) = -f(x)$  and applying Theorem 3.3.7 at  $x = -a$ , we can show the result also holds if  $h < 0$ , provided conditions hold on  $[a + h, a]$ ,  $(a + h, a)$  etc. – which is able to be assured by stating  $f$  is  $n$  times differentiable on some interval  $(c, d)$  with  $[a, a + h] \subset (c, d)$  (or  $[a + h, a]$ ).

*Proof. (Method 1)* Observe we can choose  $a = 0$  without loss of generality; if we prove the result for  $a = 0$ , applying it to  $g(x) = f(a + x)$  gives general case.

For  $0 \leq t \leq h$  let

$$\phi(t) = f(t) - f(0) - tf'(0) - \dots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(0) - \frac{t^n}{n!} B$$

where we choose  $B$  such that  $\phi(h) = 0$ , i.e.

$$\frac{h^n}{n!} B = f(h) - f(0) - hf'(0) - \dots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) \quad (*)$$

Now repeatedly apply Rolle's theorem (Theorem 3.3.3) to find another expression for  $B$ . First observe  $\phi$  and its first  $n-1$  derivatives are continuous on  $[0, h]$ , and  $\phi^{(n)}$  exists on  $(0, h)$ . Also note that  $\phi^{(k)}(0) = 0$  for  $0 \leq k \leq n-1$ . Then

$$\phi(0) = \phi(h) = 0 \Rightarrow \exists \theta_1 \in (0, 1) \text{ such that } \phi'(\theta_1 h) = 0$$

$$\phi'(0) = \phi'(\theta_1 h) = 0 \Rightarrow \exists \theta_2 \in (0, 1) \text{ such that } \phi''(\theta_2 \theta_1 h) = 0$$

$\vdots$

$$\phi^{(n-1)}(0) = \phi^{(n-1)}(\theta_{n-1} \dots \theta_1 h) = 0 \Rightarrow \exists \theta_n \in (0, 1) \text{ such that } \phi^{(n)}(\theta_n \theta_{n-1} \dots \theta_1 h) = 0$$

Let  $\theta = \theta_1 \theta_2 \dots \theta_n \in (0, 1)$ . Then

$$\phi^{(n)}(\theta h) = 0 \Rightarrow f^{(n)}(\theta h) - B = 0$$

i.e.  $B = f^{(n)}(\theta h)$  for some  $\theta \in (0, 1)$ . Rearranging equation (\*) gives us the desired result. ■

*Proof. (Method 2)* Again, assume  $a = 0$  without loss of generality. This time, for  $0 \leq t \leq h$ , let

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \frac{(h-t)^2}{2!} f''(t) - \dots - \frac{(h-t)^{n-1}}{(n-1)!} f^{(n-1)}(t)$$

$F$  is continuous on  $[0, h]$  and differentiable on  $(0, h)$ , and

$$\begin{aligned} F'(t) &= -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \frac{(h-t)^2}{2!} f^{(3)}(t) + \dots - \frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t) \\ &= -\frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t) \end{aligned}$$

Now set

$$\phi(t) = F(t) - \left(\frac{h-t}{h}\right)^p F(0)$$

with  $1 \leq p \leq n$ ,  $p \in \mathbb{N}$ . Then  $\phi(0) = 0 = \phi(h)$ . So by Rolle's theorem there exists  $\theta \in (0, 1)$  such that  $\phi'(\theta h) = 0$ . But

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h} F(0)$$

Hence

$$\begin{aligned} 0 &= -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta h) \\ &\quad + \frac{p(1-\theta)^{p-1}}{h} \left( f(h) - f(0) - hf'(0) - \frac{h^2}{2!} f''(0) - \cdots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) \right) \end{aligned}$$

If  $p = n$ , we can rearrange to find Taylor's theorem with Lagrange remainder. Otherwise, if  $p = 1$ , we have proved the *Taylor's theorem with Cauchy remainder*. ■

**Theorem 3.3.8.** (*Taylor's Theorem with Cauchy Remainder*) With the same hypotheses as Theorem 3.3.7, we have

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \tilde{R}_n$$

where

$$\tilde{R}_n = \frac{(1-\theta)^{n-1} h^n}{(n-1)!} f^{(n)}(a + \theta h)$$

for some  $\theta \in (0, 1)$ .

Both versions of Taylor's theorem assert that

$$f(h) = P_{n-1}(h) + R_n(h)$$

where

$$P_{n-1}(h) = \sum_{i=0}^{n-1} \frac{h^i}{i!} f^{(i)}(0)$$

the *Taylor polynomial*, and

$$R_n(h) = \begin{cases} \frac{h^n}{n!} f^{(n)}(a + \theta h) & \text{(Lagrange)} \\ \frac{(1-\tilde{\theta})^{n-1} h^n}{(n-1)!} f^{(n)}(a + \tilde{\theta} h) & \text{(Cauchy)} \end{cases}$$

for some  $\theta, \tilde{\theta} \in (0, 1)$ . Finally, be aware that Taylor's theorem does not say

$$f(h) = \sum_{i=0}^{\infty} \frac{h^i}{i!} f^{(i)}(0)$$

where the right hand side is called the *Taylor series*. In fact this is not always true, even if  $f$  is smooth and even if the series converges. To show a function is equal to its Taylor series, we need to show  $R_n(h) \rightarrow 0$  as  $n \rightarrow \infty$  for fixed  $h$ . For example, consider the binomial series  $f(x) = (1+x)^r$  where  $r \in \mathbb{Q}$ . Claim that if  $|x| < 1$ , then

$$(1+x)^r = 1 + \binom{r}{1}x + \cdots + \binom{r}{n}x^n + \cdots$$

where we define

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$$

where the series converges absolutely.

*Proof.* Clearly,

$$f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n}$$

By Theorem 3.3.7, for  $|x| < 1$  and  $n \geq r$ ,

$$(1+x)^r = 1 + \binom{r}{1}x + \cdots + \binom{r}{n-1}x^{n-1} + \binom{r}{n} \frac{x^n}{(1+\theta x)^{n-r}}$$

for some  $\theta \in (0, 1)$ , which can depend on both  $n$  and  $x$ . But if  $x \geq 0$ ,  $(1+\theta x)^{n-r} \geq 1$  so

$$0 \leq \frac{1}{(1+\theta x)^{n-r}} \leq 1 \Rightarrow |R_n(x)| = \left| \binom{r}{n} \frac{x^n}{(1+\theta x)^{n-r}} \right| \leq \left| \binom{r}{n} x^n \right|$$

Now observe that  $\sum_{n \geq 0} \binom{r}{n} x^n$  converges absolutely for  $|x| < 1$ :

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{r-n}{n+1} \right| |x| \rightarrow |x| < 1 \quad \text{as } n \rightarrow \infty$$

So by ratio test and Theorem 3.1.16,  $\sum_{n \geq 0} \binom{r}{n} x^n$  converges. This implies

$$\binom{r}{n} x^n \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Thus,  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  for  $x \in [0, 1)$ .

However, if  $-1 < x < 0$ , the above approach does not work as  $(1+\theta x)^{n-1} < 1$ . Instead, we can use Cauchy's remainder

$$R_n = \frac{(1-\theta)^{n-1} x^n}{(n-1)!} r(r-1)\cdots(r-n+1)(1+\theta x)^{r-n} = r \binom{r-1}{n-1} \left( \frac{1-\theta}{1+\theta x} \right)^{n-1} (1+\theta x)^{r-1} x^n$$

for some  $\theta \in (0, 1)$ . But  $(1-\theta)/(1+\theta x) < 1$  for all  $x \in (-1, 1)$  as  $1+\theta x = 1-\theta + \theta(x+1) \geq 1-\theta$ . Hence

$$|R_n| \leq r|x| \left| \binom{r-1}{n-1} x^{n-1} \right| (1+\theta x)^{r-1}$$

Moreover,  $(1+\theta x)^{r-1} \leq \max\{(1-|x|)^{r-1}, (1+|x|)^{r-1}\}$ , and

$$|R_n| \leq r|x| \max\{(1-|x|)^{r-1}, (1+|x|)^{r-1}\} \left| \binom{r-1}{n-1} x^{n-1} \right| = K_{r,x} \left| \binom{r-1}{n-1} x^{n-1} \right| \rightarrow 0$$

as  $n \rightarrow \infty$  since  $K_{r,x}$  is independent of  $n$ . Therefore  $R_n \rightarrow 0$  as  $n \rightarrow \infty$  for  $x$  fixed in  $(-1, 1)$ . ■

### 3.4 Power Series

We want to consider functions defined by power series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where  $a_n, z, z_0 \in \mathbb{C}$ . Obviously the series may not converge for every  $z \in \mathbb{C}$ . First thing to do is to find out the set of points for which it does converge.<sup>5</sup> By translation, we can assume  $z_0 = 0$ .

**Lemma 3.4.1.** If  $\sum_{n=0}^{\infty} a_n z^n$  converges, and  $|w| < |z|$ , then  $\sum_{n=0}^{\infty} a_n w^n$  converges absolutely.

*Proof.* Since  $\sum_{n=0}^{\infty} a_n z^n$  converges,  $a_n z^n \rightarrow 0$ . Thus  $\exists K > 0$  such that<sup>6</sup>

$$|a_n z^n| \leq K \quad \forall n$$

Now consider (note that  $z > 0$  to define  $w$ )

$$|a_n w^n| = |a_n z^n| \left| \frac{w}{z} \right|^n \leq K \rho^n$$

where  $\rho = |w/z| < 1$  by assumption. Thus  $\sum_{n=0}^{\infty} |a_n w^n|$  converges by comparison to the geometric series  $\sum_{n=0}^{\infty} K \rho^n$ , which converges. ■

We now use Lemma 3.4.1 to show every power series has a well-defined *radius of convergence*.

**Theorem 3.4.2.** Let  $\sum_{n=0}^{\infty} a_n z^n$  be a power series. Then there exists  $R \in [0, \infty]$ , the radius of convergence, such that the series converges absolutely for  $|z| < R$  and diverges for  $|z| > R$ .

*Proof.* Let

$$A = \left\{ r \geq 0 \mid \exists z \in \mathbb{C} \text{ with } |z| = r \text{ such that } \sum_{n=0}^{\infty} a_n z^n \text{ converges} \right\}$$

Clearly,  $0 \in A$ , and  $A$  is non-empty. So let

$$R = \sup A$$

with noting that  $\sup A = \infty$  meaning  $A$  is unbounded above. From definition of  $A$ ,  $\sum a_n z^n$  diverges for  $|z| > R$ . Suppose  $|w| < R$ . Then there exists  $r \in \mathbb{A}$  with  $|w| < r$  so  $\exists z \in \mathbb{C}$  with  $|z| = r$  and  $\sum a_n z^n$  converging.  $|w| < |z|$  so by Lemma 3.4.1  $\sum a_n w^n$  converges absolutely. ■

Note that  $R = 0$  means  $\sum a_n z^n$  converges only for  $z = 0$ , and  $R = \infty$  means  $\sum a_n z^n$  converges absolutely for all  $z \in \mathbb{C}$ . When  $0 < R < \infty$  theorem tells us nothing about convergence for  $|z| = R$ . Now we introduce a lemma that is useful at finding  $R$ :

**Lemma 3.4.3.** If  $|a_{n+1}/a_n| \rightarrow l$  as  $n \rightarrow \infty$ , then  $R = 1/l$ .

<sup>5</sup>Note that set is always non-empty as it contains  $z_0$ .

<sup>6</sup>Recall that convergent series is bounded.

*Proof.* We use ratio test. Consider

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}z^{n+1}}{a_n z^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |z| = l|z|$$

so if  $l|z| < 1$  we get absolute convergence, and if  $l|z| > 1$  we get divergence. ■

Let's take a look at some examples.

- $\sum_{n=0}^{\infty} z^n/n!$  converges absolutely for all  $z \in \mathbb{C}$ , since

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \rightarrow 0 = l$$

and  $R = \infty$ .

- $\sum_{n=0}^{\infty} n!z^n$  converges only for  $z = 0$ , since  $|a_{n+1}/a_n| = n+1 \rightarrow \infty$  for  $R = 0$ .
- $\sum_{n=0}^{\infty} z^n$  geometric series converges if and only if  $|z| < 1$ .
- $\sum_{n=0}^{\infty} z^n/n^2$  has radius of convergence  $R = 1$ ; specifically, absolute convergence for  $|z| \leq 1$  and divergence for  $|z| > 1$ .
- $\sum_{n=0}^{\infty} z^n/n$  has radius of convergence  $R = 1$ . For  $z = 1$ , it diverges (harmonic series). For  $|z| = 1, z \neq 1$ , consider

$$\begin{aligned} (1-z) \sum_{n=1}^N \frac{z^n}{n} &= \sum_{n=1}^N \left( \frac{z^n}{n} - \frac{z^{n+1}}{n} \right) \\ &= \sum_{n=1}^N \left( \frac{z^{n+1}}{n+1} - \frac{z^{n+1}}{n} \right) + z - \frac{z^{N+1}}{N+1} \\ &= z - z \sum_{n=1}^N \frac{1}{n(n+1)} z^n - \frac{z^{N+1}}{N+1} \end{aligned}$$

If  $|z| \leq 1$ ,

$$\sum_{n=1}^N \frac{1}{n(n+1)} z^n$$

converges absolutely by comparison to  $\sum 1/n(n+1)$ , and

$$\left| \frac{z^{N+1}}{N+1} \right| \leq \frac{1}{N+1} \rightarrow 0$$

Therefore  $\sum_{n=0}^{\infty} z^n/n$  converges for  $|z| \leq 1, z \neq 1$ .

We conclude that nothing can be said in general about convergence on  $|z| = R$ , and need to analyse case by case. We shall see that inside the radius of convergence, power series are very well behaved, and “can treated like polynomials.”

**Theorem 3.4.4.** Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence  $R > 0$ . Then  $f$  is differentiable at all points  $z$  with  $|z| < R$ , and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

*Proof.* We start by stating two auxillary lemmas:

**Lemma 3.4.5.** If  $\sum_{n=0}^{\infty} a_n z^n$  has radius of convergence  $R$ , then so do  $\sum_{n=1}^{\infty} n a_n z^{n-1}$  and  $\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$

**Lemma 3.4.6.**

(1) Let  $n, r \in \mathbb{N}$ . For all  $2 \leq r \leq n$ ,

$$\binom{n}{r} \leq n(n-1) \binom{n-2}{r-2}$$

(2) Let  $n \in \mathbb{N}$ . For all  $z, h \in \mathbb{C}$ ,

$$|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z|+|h|)^{n-2}|h|^2$$

Assume, for now, that these results hold. By Lemma 3.4.5, we may define

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

where  $|z| < R$ . We need to show

$$I = \frac{f(z+h) - f(z) - hg(z)}{h} \rightarrow 0$$

as  $h \rightarrow 0$ . Fix  $z$  with  $|z| < R$ , and assume  $|z| + |h| < r < R$  for some  $r$ . All sums in  $I$  converge by Lemma 3.4.5, and

$$I = \frac{1}{h} \sum_{n=0}^{\infty} a_n [(z+h)^n - z^n - nhz^{n-1}]$$

Then by continuity of absolute value function,

$$\begin{aligned} |I| &= \frac{1}{|h|} \left| \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n [(z+h)^n - z^n - nhz^{n-1}] \right| \\ &= \lim_{N \rightarrow \infty} \frac{1}{|h|} \underbrace{\left| \sum_{n=0}^N a_n [(z+h)^n - z^n - nhz^{n-1}] \right|}_{I_N} \end{aligned}$$

Now, using the triangle inequality and Lemma 3.4.6 (2),

$$\begin{aligned} |I_N| &\leq \frac{1}{|h|} \sum_{n=0}^N |a_n| |(z+h)^n - z^n - nhz^{n-1}| \\ &\leq \frac{1}{|h|} \sum_{n=0}^N |a_n| n(n-1)(|z|+|h|)^{n-2}|h|^2 \\ &\leq |h| \sum_{n=0}^N n(n-1)|a_n|r^{n-2} \\ &\leq |h| \sum_{n=0}^{\infty} n(n-1)|a_n|r^{n-2} = |h|A_r \end{aligned}$$

where  $A_r$  converges by Lemma 3.4.5 plus the fact that  $r < R$ . We have shown that

$$|I| = \lim_{N \rightarrow \infty} |I_N| \leq |h|A_r \rightarrow 0$$

as  $h \rightarrow 0$ . ■

Here we proof two auxillary lemmas:

*Proof.* (Lemma 3.4.5) Suppose  $0 < |z| < R$ , then  $\exists r$  such that  $|z| < r < R$ . We know  $\sum_{n=0}^{\infty} a_n r^n$  converges so  $a_n r^n \rightarrow 0$  as  $n \rightarrow \infty$  and hence  $\exists K$  such that

$$|a_n r^n| \leq K \quad \forall n \geq 0$$

Thus

$$|na_n z^{n-1}| = \frac{|a_n r^n|}{|z|} n \left| \frac{z}{r} \right|^n \leq \frac{K}{|z|} n \rho^n$$

where  $K$  is independent of  $n$  and  $\rho = |z|/r < 1$ . But  $\sum_{n=1}^{\infty} n \rho^n$  converges by ratio test:

$$\left| \frac{(n+1)\rho^{n+1}}{n\rho^n} \right| = \rho \left( 1 + \frac{1}{n} \right) \rightarrow \rho < 1$$

Therefore,  $\sum_{n=1}^{\infty} na_n z^{n-1}$  converges absolutely by comparison.

If  $|z| > R$ , then

$$|na_n z^{n-1}| \geq \frac{1}{|z|} |a_n z^n|$$

If  $\sum_{n=1}^{\infty} na_n w^{n-1}$  converges for some  $|w| > R$ , then taking  $|w| > |z| > R$  we have  $\sum_{n=1}^{\infty} na_n z^{n-1}$  converging absolutely by Lemma 3.4.1. This implies that  $\sum a_n z^n$  converges absolutely (by comparison), contradicting  $R$  being radius of convergence of  $\sum a_n z^n$ . Therefore  $\sum_{n=1}^{\infty} na_n w^{n-1}$  diverges, and  $\sum_{n=1}^{\infty} na_n z^{n-1}$  has radius of convergence  $R$ .

For the series  $\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}$ , apply result above, starting with  $\sum_{n=1}^{\infty} na_n z^{n-1}$ . ■

*Proof.* (Lemma 3.4.6)

(1) Simple manipulation gives

$$\binom{n}{r} / \binom{n-2}{r-2} = \frac{n!}{(n-r)!r!} \frac{(n-r)!(r-2)!}{(n-2)!} = \frac{n(n-1)}{r(r-1)} \geq n(n-1)$$

since  $r \in \mathbb{N}$ ,  $r \geq 2$ .

(2) By binomial expansion, (1), and triangle inequality, we have

$$\begin{aligned} |(z+h)^n - z^n - nhz^{n-1}| &= \left| \sum_{r=2}^n \binom{n}{r} z^{n-r} h^r \right| \\ &\leq \sum_{r=2}^n n(n-1) \binom{n-2}{r-2} |z|^{n-r} |h|^r \\ &= n(n-1) |h|^2 \sum_{r=2}^n \binom{n-2}{r-2} |z|^{n-r} \\ &= n(n-1) |h|^2 (|z| + |h|)^{n-2} \end{aligned}$$
■

N.b. Theorem 3.4.4 can be iterated: power series are smooth inside the radius of convergence.

### 3.4.1 The Standard Functions

#### \* Exponentials and logarithms

We saw in previous example that  $\sum_{n=0}^{\infty} z^n/n!$  has  $R = \infty$ , i.e. it converges at all  $z \in \mathbb{C}$ . With this fact, we define the *exponential function* as follows.

**Definition 3.4.1.** (*Exponential Function*) Let  $\exp : \mathbb{C} \rightarrow \mathbb{C}$ ,

$$z \mapsto \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

We immediately have from Theorem 3.4.4 that  $\exp$  is differentiable and

$$\exp'(z) = \exp(z)$$

Nextly, we claim  $\exp(a+b) = \exp(a)\exp(b)$ . To show this, we need the following fact: if  $F : \mathbb{C} \rightarrow \mathbb{C}$  satisfies  $F'(z) = 0$  for all  $z \in \mathbb{C}$ , then  $F$  is constant.<sup>7</sup>

*Proof.* Consider  $g : \mathbb{R} \rightarrow \mathbb{C}$ ,  $g(t) = F(tz)$  for some fixed  $z \in \mathbb{C}$ . By chain rule,

$$g'(t) = zF'(tz) = 0$$

We can write

$$g(t) = u(t) + iv(t)$$

with  $u, v$  real, and then

$$g'(t) = u'(t) + iv'(t)$$

giving

$$u'(t) = v'(t) = 0 \Rightarrow u, v \text{ constant}$$

by Corollary 3.3.4.1. Therefore  $F(z) = F(0)$  (put  $t = 0$  and  $t = 1$ ). But since  $z$  is arbitrary,  $F$  is constant. ■

Now for  $a, b \in \mathbb{C}$  consider

$$F(z) = \exp(a+b-z)\exp(z)$$

We compute

$$F'(z) = -\exp'(a+b-z)\exp(z) + \exp(a+b-z)\exp'(z) = 0$$

to find out that  $F(0) = F(b)$ . Hence  $\exp(a+b)\exp(0) = \exp(a)\exp(b)$ , but

$$\exp(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = 1$$

Therefore,  $\exp(a+b) = \exp(a)\exp(b)$  for all  $a, b \in \mathbb{C}$ .

From now on we restrict to  $\mathbb{R}$ .

**Theorem 3.4.7.** Consider  $\exp : \mathbb{R} \rightarrow \mathbb{R}$ .

(1)  $\exp$  is everywhere differentiable and  $\exp'(x) = \exp(x)$ .

<sup>7</sup>Note the difference with Corollary 3.3.4.1 – range is now  $\mathbb{C}$ .



- (2)  $\exp(x + y) = \exp(x)\exp(y)$  for all  $x, y \in \mathbb{R}$ .
- (3)  $\exp(x) > 0$  for all  $x \in \mathbb{R}$ .
- (4)  $\exp$  is strictly increasing.
- (5)  $\exp(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ; and  $\exp \rightarrow 0$  as  $x \rightarrow -\infty$ .
- (6)  $\exp : \mathbb{R} \rightarrow (0, \infty)$  is a bijection.

*Proof.* (1) and (2) are done. We prove the remainings.

- (3) If  $x > 0$ , clearly

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq 1 > 0$$

Also,  $\exp(0) = 1$  and  $\exp(x - x) = \exp(x)\exp(-x) = 1$ . Thus  $\exp(-x) > 0$  for all  $x > 0$ .

- (4)  $\exp'(x) = \exp(x) > 0$ . Hence  $\exp$  is strictly increasing by Corollary 3.3.4.1.
- (5)  $\exp(x) \geq 1 + x$  for  $x \geq 0$  so if  $x \rightarrow \infty$ ,  $\exp(x) \rightarrow \infty$ . Meanwhile, for  $x \geq 0$ ,

$$\exp(-x) = \frac{1}{\exp(x)}$$

so  $\exp(-x) \rightarrow 0$  as  $x \rightarrow \infty$ , i.e.  $\exp(x) \rightarrow 0$  as  $x \rightarrow -\infty$ .

- (6) Injectivity is immediate from being strictly increasing. For surjectivity, suppose  $y \in (0, \infty)$ . Then from (4), there exists  $a, b \in \mathbb{R}$  such that

$$\exp(a) < y < \exp(b)$$

Applying intermediate value theorem (Theorem 3.2.6) to  $\exp : [a, b] \rightarrow \mathbb{R}$  gives  $\exists x \in \mathbb{R}$  such that  $\exp(x) = y$ . Hence  $\exp$  is surjective. ■

Notice that  $\exp : (\mathbb{R}, +) \rightarrow ((0, \infty), \times)$  is a group isomorphism. And since  $\exp$  is a bijection, it has an inverse function

$$\ln : (0, \infty) \rightarrow \mathbb{R}$$

which is called the *logarithmic function*.

**Theorem 3.4.8.**

- (1)  $\ln : (0, \infty) \rightarrow \mathbb{R}$  is a bijection, and

$$\ln(\exp(x)) = x \quad \forall x \in \mathbb{R},$$

$$\exp(\ln(t)) = t \quad \forall t \in (0, \infty)$$

- (2)  $\ln$  is differentiable and monotone, with

$$\ln'(t) = \frac{1}{t}$$

- (3)  $\ln(st) = \ln(s) + \ln(t)$  where  $s, t > 0$ .  
 (4)  $\ln(x) \rightarrow \infty$  as  $x \rightarrow \infty$ ; and  $\ln(x) \rightarrow -\infty$  as  $x \rightarrow 0$ .

*Proof.*

- (1) Trivial from construction ( $\ln$  is the inverse of  $\exp$ ).  
 (2) Inverse function theorem (Theorem 3.3.5) gives that  $\ln$  is differentiable and

$$\ln'(t) = \frac{1}{\exp'(\ln(t))} = \frac{1}{\exp(\ln t)} = \frac{1}{t}$$

for all  $t > 0$ .

- (3) Because  $e$  is an isomorphism, so is its inverse. ■

Now define for  $\alpha \in \mathbb{R}$  and  $x > 0$ ,

$$\Gamma_\alpha(x) = \exp(\alpha \ln(x))$$

**Theorem 3.4.9.** Suppose  $x, y > 0$  and  $\alpha, \beta \in \mathbb{R}$ . Then

- (1)  $\Gamma_\alpha(xy) = \Gamma_\alpha(x)\Gamma_\alpha(y)$ .  
 (2)  $\Gamma_{\alpha+\beta}(x) = \Gamma_\alpha(x)\Gamma_\beta(x)$ .  
 (3)  $\Gamma_\alpha(\Gamma_\beta(x)) = \Gamma_{\alpha\beta}(x)$ .  
 (4)  $\Gamma_1(x) = x, \Gamma_0(x) = 1$ .

*Proof.*

- (1)  $\Gamma_\alpha(xy) = \exp(\alpha \ln(xy)) = \exp(\alpha \ln(x) + \alpha \ln(y)) = \exp(\alpha \ln(x))\exp(\alpha \ln(y)) = \Gamma_\alpha(x)\Gamma_\alpha(y)$ .  
 (2)  $\Gamma_{\alpha+\beta}(x) = \exp((\alpha + \beta) \ln(x)) = \exp(\alpha \ln(x) + \beta \ln(x)) = \exp(\alpha \ln(x))\exp(\beta \ln(x)) = \Gamma_\alpha(x)\Gamma_\beta(x)$ .  
 (3)  $\Gamma_\alpha(\Gamma_\beta(x)) = \exp(\alpha \ln(\exp(\beta \ln(x)))) = \exp(\alpha\beta \ln(x)) = \Gamma_{\alpha\beta}(x)$ .  
 (4)  $\Gamma_1(x) = \exp(\ln(x)) = x, \Gamma_0(x) = \exp(0) = 1$ . ■

Now suppose  $p, q \in \mathbb{Z}, p, q \geq 1$ . Then,

$$\Gamma_p(x) = \Gamma_{1+1+\dots+1}(x) = \Gamma_1(x)\Gamma_1(x) \cdots \Gamma_1(x) = x^p$$

and

$$\Gamma_p(x)\Gamma_{-p}(x) = \Gamma_0(x) = 1 \Rightarrow \Gamma_{-p}(x) = \frac{1}{x^p}$$

Further,

$$(\Gamma_{1/q}(x))^q = \Gamma_{1/q}(x) \cdots \Gamma_{1/q}(x) = \Gamma_{1/q+\dots+1/q}(x) = \Gamma_1(x) = x$$

gives  $\Gamma_{1/q}(x) = x^{1/q}$ . Finally,

$$\Gamma_{p/q}(x) = \Gamma_{\underbrace{1/q + \dots + 1/q}_{p \text{ times}}}(x) = (\Gamma_{1/q}(x))^p = (x^{1/q})^p = x^{p/q}$$

. Thus  $\Gamma_\alpha(x)$  agrees with  $x^\alpha$  for  $x \in \mathbb{Q}$  as previously defined. Hence we can extend the definition of  $x^\alpha$  into  $x \in \mathbb{R}$  and write

$$x^\alpha = \Gamma_\alpha(x)$$

for  $\alpha \in \mathbb{R}$ ,  $x \in (0, \infty)$ . We also introduce *Euler's number*

$$e = \exp(1) = \sum_{n=1}^{\infty} \frac{1}{n!}$$

and write  $\exp$  as a power:

$$\exp(x) = \exp(x \ln e) = \Gamma_x(e) = e^x$$

Now we can generalise as follows.

$$(x^\alpha)' = (e^{\alpha \ln x})' = \frac{\alpha}{x} e^{\alpha \ln x} = \alpha x^{\alpha-1}$$

for all  $x > 0$ ,  $\alpha \in \mathbb{R}$ . Meanwhile, for  $f(x) = a^x$  ( $a > 0$ ,  $x \in \mathbb{R}$ ),

$$f'(x) = (e^{x \ln a})' = e^{x \ln a} \ln a = a^x \ln a$$

In particular,  $f'(x) > 0$  for  $a > 1$  and  $f'(x) < 0$  for  $a < 1$ .

**Lemma 3.4.10.** For any  $r > 0$  we have

- (1)  $x^r e^{-x} \rightarrow 0$  as  $x \rightarrow \infty$ .
- (2)  $x^{-r} \ln x \rightarrow 0$  as  $x \rightarrow \infty$ .
- (3)  $x^r \ln x \rightarrow 0$  as  $x \rightarrow 0+$ .

*Proof.*

- (1) For  $x > 0$ ,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots > \frac{x^n}{n!}$$

for any  $n \in \mathbb{N}$ . Pick  $n$  such that  $n - r \geq 1$ . Then,

$$0 \leq \frac{x^r}{e^x} \leq \frac{n!}{x^{n-r}} \leq \frac{n!}{x}$$

so  $x^r e^{-x} \rightarrow 0$  as  $x \rightarrow \infty$ .

- (2) Pick  $m$  such that  $m \geq 2/r$ . For  $x > 1$ ,

$$0 \leq \frac{x^{1/r}}{e^x} \leq \frac{m!}{x^{m-1/r}} \leq \frac{m!}{x^{1/r}}$$

Let  $x = \ln y$  and write

$$0 \leq \frac{(\ln y)^{1/r}}{r} \leq \frac{m!}{(\ln y)^{1/r}}$$

giving

$$0 \leq \frac{\ln y}{y^r} \leq \frac{(m!)^r}{\ln y}$$

But  $\ln y \rightarrow \infty$  as  $y \rightarrow \infty$ ; thus  $\ln y / y^r \rightarrow 0$ .

(3) Set  $z = 1/y$  where  $y$  is from above. We have

$$0 \leq -z^r \ln z \leq \frac{(m!)^r}{-\ln z}$$

Since  $\ln z \rightarrow -\infty$  as  $z \rightarrow 0+$ ,  $z^r \ln z \rightarrow 0$ . ■

### \* Trigonometric functions

**Definition 3.4.2.** (*Trigonometric Functions*) We define

$$\begin{aligned}\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\end{aligned}$$

We can check that, like the exponential function, both series have infinite radius of convergence by ratio test. Hence,

$$\cos' z = -\sin z \text{ and } \sin' z = \cos z$$

by Theorem 3.4.4. Also observe that

$$\begin{aligned}e^{iz} &= \lim_{N \rightarrow \infty} \left( \sum_{k=0}^{2N+1} \frac{(iz)^k}{k!} \right) \\ &= \lim_{N \rightarrow \infty} \left( \sum_{k=0}^N \frac{(iz)^{2k}}{(2k)!} + \sum_{k=0}^N \frac{(iz)^{2k+1}}{(2k+1)!} \right) \\ &= \lim_{N \rightarrow \infty} \left( \sum_{k=0}^N (-1)^k \frac{z^{2k}}{(2k)!} \right) + i \sum_{k=0}^N (-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= \cos z + i \sin z\end{aligned}$$

Similarly,  $e^{-iz} = \cos z - i \sin z$ . Hence,

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

and

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

These formulae give many trigonometric identities, e.g.  $\cos z = \cos(-z)$ ,  $\sin(z) = -\sin(-z)$ ,  $\cos(0) = 1$ ,  $\sin(0) = 0$  etc. Employing  $e^{a+b} = e^a e^b$ , we also find

$$\sin(z+w) = \sin z \cos w + \cos z \sin w$$

and

$$\cos(z+w) = \cos z \cos w - \sin z \sin w$$

for all  $z, w \in \mathbb{C}$ . Furthermore, set  $w = -z$  to deduce

$$\cos^2 + \sin^2 = \cos(0) = 1 \tag{*}$$

Now, if  $x \in \mathbb{R}$  then  $\sin x, \cos x \in \mathbb{R}$  and (\*) implies  $|\cos x|, |\sin x| \leq 1$ . N.b. this need not be true away from real axis, e.g. if  $y \in \mathbb{R}$ ,

$$\cos iy = \frac{1}{2}(e^{-y} + e^y) \rightarrow \infty$$

as  $y \rightarrow \pm\infty$ .

✱ **Periodicity of trigonometric functions**

**Proposition 3.4.11.** There is a smallest positive number  $\omega$  (where  $\sqrt{2} < \omega/2 < \sqrt{3}$ ) such that

$$\cos \frac{\omega}{2} = 0$$

*Proof.* Suppose  $0 \leq x \leq 2$ .

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots + \frac{x^{2n-1}}{(2n-1)!} \left(1 - \frac{x^2}{2n(2n+1)}\right) + \cdots$$

Since each term in parenthesis are positive,  $\sin x \geq 0$ , and if  $0 < x \leq 2$ ,  $\sin x > 0$ . So  $\cos' x = -\sin x < 0$  for  $0 < x < 2$ . Hence  $\cos x$  is strictly decreasing in this interval, hence it has at most one root in  $[0, 2]$ . To complete the proof we show  $\cos \sqrt{2} > 0 > \cos \sqrt{3}$ , then intermediate valuable theorem implies a root exists in  $[\sqrt{2}, \sqrt{3}]$ . But

$$\cos \sqrt{2} = 1 - \underbrace{\frac{(\sqrt{2})^2}{2!}}_{=0} + \underbrace{\frac{(\sqrt{2})^4}{4!} - \frac{(\sqrt{2})^6}{6!}}_{>0} + \cdots + \underbrace{\frac{(\sqrt{2})^{2n}}{(2n)!} \left(1 - \frac{2}{(2n+1)(2n+2)}\right)}_{>0} + \cdots$$

so  $\cos \sqrt{2} > 0$ , and

$$\cos \sqrt{3} = 1 - \underbrace{\frac{(\sqrt{3})^2}{2!} + \frac{(\sqrt{3})^4}{4!}}_{-1/8} - \underbrace{\left(\frac{(\sqrt{3})^6}{6!} - \frac{(\sqrt{3})^8}{8!}\right)}_{>0} - \cdots - \frac{(\sqrt{3})^{2n}}{(2n)!} \underbrace{\left(1 - \frac{3}{(2n+1)(2n+2)}\right)}_{>0} - \cdots$$

implies  $\cos \sqrt{3}$ , completing the proof. ■

**Corollary 3.4.11.1.**  $\sin \omega/2 = 1$ .

*Proof.* From

$$\sin^2 \frac{\omega}{2} + \cos^2 \frac{\omega}{2} = 1$$

we have  $\sin^2 \omega/2 = 1 \Rightarrow \sin \omega/2 = \pm 1$ . But  $\omega/2 \in (0, 2)$  so  $\sin \omega/2 > 0$ . Therefore  $\sin \omega/2 = 1$ . ■

Since we established some properties, we now define  $\pi = \omega$ . Periodic properties of the trigonometric functions follows.

**Theorem 3.4.12.**

- (1)  $\sin(z + \pi/2) = \cos z$ ;  $\cos(z + \pi/2) = -\sin z$ .
- (2)  $\sin(z + \pi) = -\sin z$ ;  $\cos(z + \pi) = -\cos z$ .

$$(3) \sin(z + 2\pi) = \sin z; \cos(z + 2\pi) = \cos z.$$

*Proof.* (1) Follows from addition formulae plus  $\cos(\pi/2) = 0$ ,  $\sin(\pi/2) = 1$ . (2) and (3) follow by iterating. ■

It follows that

$$e^{iz+2\pi i} = \cos(z + 2\pi) + i \sin(z + 2\pi) = \cos z + i \sin z = e^{iz}$$

and hence  $e^{i\omega}$  is periodic with period  $2\pi i$ . Also

$$e^{i\pi} = \cos \pi + i \sin \pi = -\cos 0 + i(-\sin 0) = -1$$

giving the *Euler's identity*:

$$e^{i\pi} + 1 = 0$$

#### \* Trigonometric functions and geometry

Given  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ , define  $\mathbf{x} \cdot \mathbf{y}$  as usual. We set  $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ . Cauchy-Schwarz shows that

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

Hence for  $\mathbf{x} \neq 0, \mathbf{y} \neq 0$ ,

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|} \leq 1$$

Thus we can define the angle between  $\mathbf{x}$  and  $\mathbf{y}$  as the unique  $\theta \in [0, \pi]$  with

$$\cos \theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

#### \* Hyperbolic functions

**Definition 3.4.3.** (*Hyperbolic Functions*) Define

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$

where  $z \in \mathbb{C}$ .

It follows that

$$\cosh z = \cos(iz) \text{ and } \sinh z = -i \sin(iz)$$

Also,

$$\cosh' z = \sinh z, \quad \sinh' z = \cosh z$$

and

$$\cosh^2 z - \sinh^2 z = 1$$

## 3.5 Integration

### 3.5.1 Riemann Integral

Informally, we define

$$\int_a^b f(x)dx$$

as the (signed) area under the graph of  $f(x)$ . To find the area, we approximate graph from above and below by rectangles. If  $f$  is ‘nice’, estimates from above and below will be close for a suitably fine dissection of  $[a, b]$ . We assume  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, i.e.  $\exists K$  such that  $|f(x)| \leq K$  for all  $x \in [a, b]$ .

**Definition 3.5.1.** A *dissection*  $\mathcal{D}$  of  $[a, b]$  is a finite subset of  $[a, b]$  containing the end points  $a, b$ . We write

$$\mathcal{D} = \{x_0, x_1, \dots, x_n\}$$

with

$$a = x_0 < x_1 < \dots < x_n = b$$

Associated to a dissection are the *upper* and *lower sums*, given by

$$U(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x)$$

and

$$L(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x)$$

respectively. Clearly, for any dissection,  $L(f, \mathcal{D}) \leq U(f, \mathcal{D})$ .

**Lemma 3.5.1.** Suppose  $\mathcal{D}', \mathcal{D}$  are dissections with  $\mathcal{D}' \supseteq \mathcal{D}$ . Then,

$$L(f, \mathcal{D}) \leq L(f, \mathcal{D}') \leq U(f, \mathcal{D}') \leq U(f, \mathcal{D})$$

We say  $\mathcal{D}'$  is a *refinement* of  $\mathcal{D}$ .

*Proof.* Let  $\mathcal{D} = \{x_0, \dots, x_n\}$  with  $x_0 < x_1 < \dots < x_n$ . First, consider the case where  $\mathcal{D}' = \mathcal{D} \cup \{y\}$ . Then,  $y \in (x_{r-1}, x_r)$  for some  $r$ . Clearly,

$$\sup_{[x_{r-1}, y]} f(x) \leq \sup_{[x_{r-1}, x_r]} f(x)$$

and

$$\sup_{[y, x_r]} f(x) \leq \sup_{[x_{r-1}, x_r]} f(x)$$

Combining these,

$$\begin{aligned} (y - x_{r-1}) \sup_{[x_{r-1}, y]} f(x) + (x_r - y) \sup_{[y, x_r]} f(x) &\leq (y - x_{r-1}) \sup_{[x_{r-1}, x_r]} f(x) + (x_r - y) \sup_{[x_{r-1}, x_r]} f(x) \\ &= (x_r - x_{r-1}) \sup_{[x_{r-1}, x_r]} f(x) \end{aligned}$$

Hence  $U(f, \mathcal{D}') \leq U(f, \mathcal{D})$ . Similarly,

$$\inf_{[x_{r-1}, y]} f, \inf_{[y, x_r]} f \geq \inf_{[x_{r-1}, x_r]} f$$

and  $L(f, \mathcal{D}') \geq L(f, \mathcal{D})$ . Now if  $\mathcal{D}'$  has more extra points, we add them one at a time and use the result recursively. ■

**Lemma 3.5.2.** Suppose  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are two dissections. Then,

$$L(f, \mathcal{D}_1) \leq U(f, \mathcal{D}_2)$$

*Proof.* Since  $\mathcal{D}_1 \subseteq \mathcal{D}_1 \cup \mathcal{D}_2$  and  $\mathcal{D}_2 \subseteq \mathcal{D}_1 \cup \mathcal{D}_2$ , we have, from Lemma 3.5.1,

$$L(f, \mathcal{D}_1) \leq L(f, \mathcal{D}_1 \cup \mathcal{D}_2) \leq U(f, \mathcal{D}_1 \cup \mathcal{D}_2) \leq U(f, \mathcal{D}_2)$$

■

Note that if  $\mathcal{D}_0 = \{a, b\}$ ,

$$L(f, \mathcal{D}_0) = (b - a) \inf_{[a, b]} f \geq -(b - a)K$$

and

$$U(f, \mathcal{D}_0) = (b - a) \sup_{[a, b]} f \leq (b - a)K$$

Since  $\mathcal{D}_0 \subseteq \mathcal{D}$  for any  $\mathcal{D}$ , we deduce

$$\{U(f, \mathcal{D}) \mid \mathcal{D} \text{ dissections}\}$$

and

$$\{L(f, \mathcal{D}) \mid \mathcal{D} \text{ dissections}\}$$

are bounded above by  $(b - a)K$ , and below by  $-(b - a)K$ ; and both sets are non-empty. We can define the *upper* and *lower integral* now.

**Definition 3.5.2.** The *upper integral* of  $f$  is

$$I^*(f) = \int_a^b f(x) dx = \inf_{\mathcal{D}} U(f, \mathcal{D})$$

and the *lower integral* of  $f$  is

$$I_*(f) = \int_a^b f(x) dx = \sup_{\mathcal{D}} L(f, \mathcal{D})$$

Also notice that from Lemma 3.5.2,

$$L(f, \mathcal{D}_1) \leq U(f, \mathcal{D}_2) \Rightarrow L(f, \mathcal{D}_1) \leq \inf_{\mathcal{D}_2} U(f, \mathcal{D}_2) \Rightarrow \sup_{\mathcal{D}_1} L(f, \mathcal{D}_1) \leq \inf_{\mathcal{D}_2} U(f, \mathcal{D}_2) \Rightarrow I_*(f) \leq I^*(f)$$

Hence,

$$(b - a) \inf_{[a, b]} f(x) \leq I_*(f) \leq I^*(f) \leq (b - a) \sup_{[a, b]} f(x)$$

Now we formally define the integral as follows.

**Definition 3.5.3.** A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is *Riemann integrable* (integrable) if

$$I^*(f) = I_*(f)$$

Then we define

$$\int_a^b f(x) dx = I^*(f) = I_*(f) = \int_a^b f$$



**Example 3.5.1.** Consider function  $f : [0, 1] \rightarrow \mathbb{R}, x \mapsto x$ . Let

$$\mathcal{D}_k = \left\{ 0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1 \right\}$$

(uniform dissection). Then,

$$U(f, \mathcal{D}_k) = \sum_{j=1}^k \left( \frac{j}{k} - \frac{(j-1)}{k} \right) \sup_{\left[ \frac{j-1}{k}, \frac{j}{k} \right]} x = \sum_{j=1}^k \frac{1}{k} \frac{j}{k} = \frac{1}{k^2} \frac{1}{2} k(k+1) = \frac{1}{2} + \frac{1}{2k}$$

Similarly,

$$L(f, \mathcal{D}_k) = \frac{1}{2} - \frac{1}{2k}$$

Therefore,

$$I^*(f) = \inf_{\mathcal{D}} U(f, \mathcal{D}) \leq \inf_{\mathcal{D}_k} U(f, \mathcal{D}_k) = \frac{1}{2}$$

and

$$I_*(f) = \sup_{\mathcal{D}} L(f, \mathcal{D}) \geq \sup_{\mathcal{D}_k} L(f, \mathcal{D}_k) = \frac{1}{2}$$

But  $I_*(f) \leq I^*(f)$  so  $I_*(f) = I^*(f) = 1/2$ . Thus  $f$  is integrable and

$$\int_0^1 x dx = \frac{1}{2}$$

**Example 3.5.2.** Consider  $f : [0, 1] \rightarrow \mathbb{R}$ ,

$$x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This time,  $f$  is not Riemann integrable. For any dissection  $\mathcal{D}$ ,

$$U(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f = \sum_{j=1}^n (x_j - x_{j-1}) = 1$$

and

$$L(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f = 0$$

Thus,  $I_*(f) \neq I^*(f)$  and  $f$  is not Riemann integrable.

**Theorem 3.5.3.** (*Cauchy Criterion for Integrability*) A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable if and only if for all  $\epsilon > 0$  there exists a dissection  $\mathcal{D}$  with

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) < \epsilon$$

*Proof.* For any dissection, we have

$$0 \leq I^*(f) - I_*(f) \leq U(f, \mathcal{D}) - L(f, \mathcal{D}) < \epsilon$$

If the criterion holds, then

$$0 \leq I^*(f) - I_*(f) < \epsilon$$

for all  $\epsilon > 0$ . So  $I^*(f) = I_*(f)$  and  $f$  is integrable.

Conversely, suppose  $f$  is integrable and fix  $\epsilon > 0$ . It follows from definition of supremum and infimum that there exist  $\mathcal{D}_1, \mathcal{D}_2$  with

$$U(f, \mathcal{D}_1) \leq I^*(f) + \epsilon/2$$

and

$$L(f, \mathcal{D}_2) \geq I_*(f) - \epsilon/2$$

Let  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ . Then,

$$U(f, \mathcal{D}) \leq U(f, \mathcal{D}_1),$$

$$L(f, \mathcal{D}) \geq L(f, \mathcal{D}_2)$$

and we can find

$$\begin{aligned} 0 \leq U(f, \mathcal{D}) - L(f, \mathcal{D}) &\leq U(f, \mathcal{D}_1) - L(f, \mathcal{D}_2) \\ &\leq I^*(f) + \epsilon/2 - I_*(f) + \epsilon/2 = \epsilon \end{aligned}$$

■

This criterion (Theorem 3.5.3) can be used to show monotone functions and continuous functions are integrable.

Observe that if  $f : [a, b] \rightarrow \mathbb{R}$  is monotone, it is bounded by  $f(a)$  and  $f(b)$ .

**Theorem 3.5.4.** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is monotone. Then  $f$  is integrable.

*Proof.* Suppose  $f$  is increasing. Then  $\sup_{[x_{j-1}, x_j]} f = f(x_j)$  and  $\inf_{[x_{j-1}, x_j]} f = f(x_{j-1})$ . Thus for any dissection  $\mathcal{D}$ ,

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) [f(x_j) - f(x_{j-1})]$$

Now choose a uniform dissection into  $n$  pieces, i.e.

$$\mathcal{D}_n = \left\{ a, a + \frac{(b-a)}{n}, a + \frac{2(b-a)}{n}, \dots, a + \frac{n-1}{n}(b-a), b \right\}$$

Then,

$$\begin{aligned} U(f, \mathcal{D}_n) - L(f, \mathcal{D}_n) &= \sum_{j=1}^n \frac{(b-a)}{n} \left[ f\left(a + \frac{j(b-a)}{n}\right) - f\left(a + \frac{(j-1)}{n}(b-a)\right) \right] \\ &= \frac{(b-a)}{n} (f(b) - f(a)) \end{aligned}$$

Taking  $n$  sufficiently large, for any  $\epsilon > 0$ , we can find  $\mathcal{D} = \mathcal{D}_n$  such that

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) < \epsilon$$

Hence  $f$  is integrable by Theorem 3.5.3. ■

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  is bounded by the extreme value theorem (Theorem 3.2.8).

**Theorem 3.5.5.** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then  $f$  is integrable.

*Proof.* We prove the contrapositive, i.e. if  $f$  is not integrable, then  $f$  is not continuous. Suppose  $f$  is not integrable; then by Theorem 3.5.3, there exists  $\epsilon_0 > 0$  such that, for all dissections  $\mathcal{D}$ ,

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) > \epsilon_0$$

Since

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) [f(x_j) - f(x_{j-1})]$$

it follows that for any dissection there is a  $j$  with

$$\sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f > \frac{\epsilon_0}{b-a}$$

otherwise

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) \leq \sum_{j=1}^n (x_j - x_{j-1}) \frac{\epsilon_0}{b-a} = \epsilon_0$$

In particular, we can find  $y, z \in [x_{j-1}, x_j]$  with

$$f(y) - f(z) > \frac{\epsilon_0}{b-a}$$

Now let  $\mathcal{D}_n$  be the uniform dissection into  $n$  equal intervals:

$$\mathcal{D}_n = \left\{ a, a + \frac{(b-a)}{n}, a + \frac{2(b-a)}{n}, \dots, a + \frac{n-1}{n}(b-a), b \right\}$$

Applying the argument above to  $\mathcal{D}_n$ , there must exist  $y_n, z_n$  in the same subinterval so that  $|y_n - z_n| < (b-a)/n$ , with

$$f(y_n) - f(z_n) > \frac{\epsilon_0}{b-a}$$

$(y_n)$  is bounded by construction, so by Bolzano-Weierstrass theorem we can take a convergent subsequence  $y_{n_k} \rightarrow \eta$  for some  $\eta \in [a, b]$ . Moreover, since

$$|y_{n_k} - z_{n_k}| < \frac{b-a}{n_k} \leq \frac{b-a}{k}$$

we have  $z_{n_k} \rightarrow \eta$  also. But

$$f(y_{n_k}) - f(z_{n_k}) > \frac{\epsilon_0}{b-a} > 0$$

implies  $f(y_{n_k})$  and  $f(z_{n_k})$  cannot both converge to the same limit. So  $f$  is not continuous at  $\eta$  (Theorem 3.2.1). ■

However, a function not being continuous nor monotone does not suggest that function is not integrable. Between monotone and continuous functions, we have a large class of integrable functions, e.g. *Thomae's function*.<sup>8</sup>

<sup>8</sup>Moreover, this function is continuous at each irrational, and discontinuous at each rational.

### 3.5.2 Elementary Properties of the Integral

Before establishing various properties of the integral, we first give some properties of infimum and supremum.

**Lemma 3.5.6.** Suppose  $I = [a, b]$  and  $f, g : I \rightarrow \mathbb{R}$  are bounded. Then,

(1) If  $f(x) \leq g(x)$  for all  $x \in I$ ,

$$\sup_I f \leq \sup_I g \text{ and } \inf_I f \leq \inf_I g$$

(2)  $\sup_I (-f) = -\inf_I f$ .

(3) If  $\lambda > 0$ , then

$$\sup_I (\lambda f) = \lambda \sup_I f \text{ and } \inf_I (\lambda f) = \lambda \inf_I f$$

(4)

$$\begin{aligned} \sup_I (f + g) &\leq \sup_I f + \sup_I g \\ \inf_I (f + g) &\geq \inf_I f + \inf_I g \end{aligned}$$

(5)  $\sup_I |f| - \inf_I |f| \leq \sup_I f - \inf_I f$ .

(6)  $\sup_I f^2 - \inf_I f^2 \leq 2 \sup_I |f| (\sup_I f - \inf_I f)$ .

*Proof.* We will only prove (4)-(6).

(4) We have  $f(x) \leq \sup_I f, g(x) \leq \sup_I g$  for all  $x \in I$ . Hence

$$f(x) + g(x) \leq \sup_I f + \sup_I g \quad \forall x \in I$$

and

$$\sup_I (f + g) \leq \sup_I f + \sup_I g$$

Proof is similar for inf.

(5) • If  $f(x) \geq 0$  for all  $x$ , then  $|f| = f$  and

$$\sup_I |f| - \inf_I |f| = \sup_I f - \inf_I f$$

• If  $f(x) \leq 0$  for all  $x$ , then  $|f| = -f$  and

$$\sup_I |f| - \inf_I |f| = \sup_I (-f) - \inf_I (-f) = -\inf_I f + \sup_I f$$

by (2).

• If  $\inf_I f < 0 < \sup_I f$ , then

$$\begin{aligned} \sup_I |f| - \inf_I |f| &= \sup_I |f| = \max\{\sup_I f, \sup_I (-f)\} \\ &\leq \sup_I f + \sup_I (-f) \\ &= \sup_I f - \inf_I f \end{aligned}$$

(6) From

$$f(x)^2 - f(y)^2 = (f(x) + f(y))(f(x) - f(y)) \leq 2 \sup_I |f| (\sup_I f - \inf_I f)$$

we take supremum over  $x$  and  $y$  for the result. ■

**Corollary 3.5.6.1.** For any dissection  $\mathcal{D}$  of  $[a, b]$ , and bounded functions  $f, g : [a, b] \rightarrow \mathbb{R}$ ,

(1) If  $f(x) \leq g(x)$  for all  $x \in [a, b]$ ,

$$U(f, \mathcal{D}) \leq U(g, \mathcal{D}) \text{ and } L(f, \mathcal{D}) \leq L(g, \mathcal{D})$$

(2)  $L(-f, \mathcal{D}) = -U(f, \mathcal{D})$ .

(3) If  $\lambda > 0$ ,

$$U(\lambda f, \mathcal{D}) = \lambda U(f, \mathcal{D}) \text{ and } L(\lambda f, \mathcal{D}) = \lambda L(f, \mathcal{D})$$

(4)

$$U(f + g, \mathcal{D}) \leq U(f, \mathcal{D}) + U(g, \mathcal{D})$$

$$L(f + g, \mathcal{D}) \leq L(f, \mathcal{D}) + L(g, \mathcal{D})$$

(5)  $U(|f|, \mathcal{D}) - L(|f|, \mathcal{D}) \leq U(f, \mathcal{D}) - L(f, \mathcal{D})$ .

(6)  $U(f^2, \mathcal{D}) - L(f^2, \mathcal{D}) \leq 2 \sup_{[a, b]} |f| (U(f, \mathcal{D}) - L(f, \mathcal{D}))$ .

*Proof.* Recall the definition of upper/lower sum and results follow from Lemma 3.5.6. For (6) we additionally need that

$$\sup_{[x_{j-1}, x_j]} |f| \leq \sup_{[a, b]} |f|$$
■

**Theorem 3.5.7.** Let  $f, g$  be bounded and integrable on  $[a, b]$ . Then,

(1) If  $f(x) \leq g(x)$  for all  $x \in [a, b]$ ,

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

(2) If  $\lambda \in \mathbb{R}$ ,  $\lambda f$  is integrable, and

$$\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx$$

(3)  $f + g$  is integrable, and

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

(4)  $|f|$  is integrable, and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

(5) The product  $fg$  is integrable.

*Proof.*

(1) For any dissection  $\mathcal{D}$ , we have

$$L(f, \mathcal{D}) \leq L(g, \mathcal{D}) \leq I_*(g)$$

Hence  $I_*(f) \leq I_*(g)$ . But  $f, g$  integrable, so

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

(2) First consider  $\lambda > 0$ . Since  $f$  is integrable,  $\forall \epsilon > 0, \exists \mathcal{D}$  such that

$$U(f, \mathcal{D}) \leq \int_a^b f(x) dx + \epsilon, \quad L(f, \mathcal{D}) \geq \int_a^b f(x) dx - \epsilon$$

Then, by Corollary 3.5.6.1 (3),

$$\begin{aligned} \lambda \int_a^b f(x) dx - \lambda \epsilon &\leq \lambda L(f, \mathcal{D}) = L(\lambda f, \mathcal{D}) \leq I_*(\lambda f) \\ &\leq I^*(\lambda f) \leq U(\lambda f, \mathcal{D}) = \lambda U(f, \mathcal{D}) \leq \lambda \int_a^b f(x) dx + \lambda \epsilon \end{aligned}$$

But  $\epsilon$  is arbitrary so

$$I_*(\lambda f) = I^*(\lambda f) = \lambda \int_a^b f(x) dx$$

Now consider  $\lambda = -1$ ,  $\mathcal{D}$  as above. Then,

$$\int_a^b f(x) dx - \epsilon \leq L(f, \mathcal{D}) = -U(-f, \mathcal{D})$$

and

$$\int_a^b f(x) dx + \epsilon \geq U(f, \mathcal{D}) = -L(-f, \mathcal{D})$$

Therefore

$$-\int_a^b f(x) dx - \epsilon \leq L(-f, \mathcal{D}) \leq I_*(-f) \leq I^*(-f) \leq U(-f, \mathcal{D}) \leq -\int_a^b f(x) dx + \epsilon$$

and

$$I_*(-f) = I^*(-f) = -\int_a^b f(x) dx$$

since  $\epsilon$  arbitrary. And we can combine the two results above to get desired for all  $\lambda \in \mathbb{R}$ .

(3) Since  $f, g$  integrable,  $\forall \epsilon > 0$  there exist  $\mathcal{D}_1, \mathcal{D}_2$  such that

$$\begin{aligned} \int_a^b f(x)dx - \epsilon &\leq L(f, \mathcal{D}_1) \leq U(f, \mathcal{D}_1) \leq \int_a^b f(x)dx + \epsilon \\ \int_a^b g(x)dx - \epsilon &\leq L(g, \mathcal{D}_2) \leq U(g, \mathcal{D}_2) \leq \int_a^b g(x)dx + \epsilon \end{aligned}$$

Now let  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ . Then, by Corollary 3.5.6.1 (4),

$$\begin{aligned} \int_a^b f(x)dx + \int_a^b g(x)dx - 2\epsilon &\leq L(f, \mathcal{D}_1) + L(g, \mathcal{D}_2) \\ &\leq L(f, \mathcal{D}) + L(g, \mathcal{D}) \\ &\leq L(f + g, \mathcal{D}) \end{aligned}$$

Similarly,

$$\begin{aligned} \int_a^b f(x)dx + \int_a^b g(x)dx + 2\epsilon &\geq U(f, \mathcal{D}_1) + U(g, \mathcal{D}_2) \\ &\geq U(f, \mathcal{D}) + U(g, \mathcal{D}) \\ &\geq U(f + g, \mathcal{D}) \end{aligned}$$

Combining two, we obtain

$$\int_a^b f(x)dx + \int_a^b g(x)dx - 2\epsilon \leq I_*(f + g) \leq I^*(f + g) \leq \int_a^b f(x)dx + \int_a^b g(x)dx + 2\epsilon$$

Since  $\epsilon$  arbitrary,  $f + g$  is integrable and

$$\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b (f + g)(x)dx$$

(4) By Theorem 3.5.3 and Corollary 3.5.6.1 (5),  $\forall \epsilon > 0 \exists \mathcal{D}$  such that

$$\epsilon > U(f, \mathcal{D}) - L(f, \mathcal{D}) \geq U(|f|, \mathcal{D}) - L(|f|, \mathcal{D})$$

We immediately see that  $|f|$  is integrable by Theorem 3.5.3. Also,

$$-|f(x)| \leq f(x) \leq |f(x)| \quad \forall x$$

so by (1),

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

(5) Noting

$$fg = \frac{1}{4} \left[ (f + g)^2 - (f - g)^2 \right] \quad (\dagger)$$

it is sufficient to show  $f^2$  is integrable. Suppose  $|f(x)| \leq K \quad \forall x \in [a, b]$ . Then given  $\epsilon > 0$ , there exists  $\mathcal{D}$  such that

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) < \frac{\epsilon}{2K}$$

By Corollary 3.5.6.1 (6),

$$U(f^2, \mathcal{D}) - L(f^2, \mathcal{D}) \leq 2 \sup_{[a,b]} |f| (U(f, \mathcal{D}) - L(f, \mathcal{D})) < \epsilon$$

Hence  $f^2$  is integrable, and consequently  $fg$  is integrable by (†). ■

**Theorem 3.5.8.** (*Additivity of the Integral*) Suppose  $f : [a, b] \rightarrow \mathbb{R}$  and let  $c \in (a, b)$ . Then  $f$  is integrable if and only if  $f|_{[a,c]}$  and  $f|_{[c,b]}$  are integrable, and

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

*Proof.* First, suppose  $f : [a, b] \rightarrow \mathbb{R}$  is integrable. Let  $\epsilon > 0$ . By Theorem 3.5.3,  $\exists \mathcal{D}$ , a dissection of  $[a, b]$ , such that

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) < \epsilon$$

By considering  $\mathcal{D} \cup \{c\}$  if necessary, we can assume  $\mathcal{D} = \{x_0, \dots, x_n\}$  and  $x_l = c$  for some  $l$ . Let  $\mathcal{D}_L = \{x_0, \dots, x_l\}$  and  $\mathcal{D}_R = \{x_l, \dots, x_n\}$ , which are dissections of  $[a, c]$  and  $[c, b]$  respectively. Then, from

$$L(f, \mathcal{D}) = L(f|_{[a,c]}, \mathcal{D}_L) + L(f|_{[c,b]}, \mathcal{D}_R)$$

$$U(f, \mathcal{D}) = U(f|_{[a,c]}, \mathcal{D}_L) + U(f|_{[c,b]}, \mathcal{D}_R)$$

we have

$$\left[ U(f|_{[a,c]}, \mathcal{D}_R) - L(f|_{[a,c]}, \mathcal{D}_R) \right] + \left[ U(f|_{[c,b]}, \mathcal{D}_R) - L(f|_{[c,b]}, \mathcal{D}_R) \right] < \epsilon$$

Both terms in square brackets are non-negative, so

$$U(f|_{[a,c]}, \mathcal{D}_R) - L(f|_{[a,c]}, \mathcal{D}_R) < \epsilon$$

and

$$U(f|_{[c,b]}, \mathcal{D}_R) - L(f|_{[c,b]}, \mathcal{D}_R) < \epsilon$$

Therefore  $f|_{[a,c]}$  and  $f|_{[c,b]}$  are integrable.

Conversely, suppose  $f|_{[a,c]}$  and  $f|_{[c,b]}$  are integrable. Theorem 3.5.3 says that  $\exists \mathcal{D}_L, \mathcal{D}_R$  such that

$$U(f|_{[a,c]}, \mathcal{D}_R) - L(f|_{[a,c]}, \mathcal{D}_R) < \epsilon$$

$$U(f|_{[c,b]}, \mathcal{D}_R) - L(f|_{[c,b]}, \mathcal{D}_R) < \epsilon$$

But if we let  $\mathcal{D} = \mathcal{D}_L \cup \mathcal{D}_R$ , we have

$$U(f, \mathcal{D}) - L(f, \mathcal{D}) < 2\epsilon$$

Hence  $f$  is integrable.

Finally, we deduce that

$$\int_a^b f(x) dx \geq L(f, \mathcal{D}) = L(f|_{[a,c]}, \mathcal{D}_L) + L(f|_{[c,b]}, \mathcal{D}_R) \geq \int_a^c f(x) dx - \epsilon + \int_c^b f(x) dx - \epsilon$$



Similarly,

$$\int_a^b f(x)dx \leq U(f, \mathcal{D}) = U(f|_{[a,c]}, \mathcal{D}_L) + U(f|_{[c,b]}, \mathcal{D}_R) \leq \int_a^c f(x)dx + \epsilon + \int_c^b f(x)dx + \epsilon$$

Therefore,

$$\left| \int_a^b f(x)dx - \int_a^c f(x)dx - \int_c^b f(x)dx \right| \leq 2\epsilon$$

for arbitrary  $\epsilon$ , completing the proof. ■

Finally, we introduce a convention that if  $a > b$ ,

$$\int_a^b f(x)dx = - \int_b^a f(x)dx$$

and if  $a = b$

$$\int_a^a f = 0$$

With this convention, it follows that if  $|f| \leq K$ ,

$$\left| \int_a^b f(x)dx \right| \leq K|b - a|$$

### 3.5.3 Piecewise Continuous Functions

**Definition 3.5.4.** A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is *piecewise continuous* if there exists a dissection  $\mathcal{D}$  of  $[a, b]$  such that for each  $i = 1, 2, \dots, n$ , if we define  $f_i : (x_{i-1}, x_i) \rightarrow \mathbb{R}$ ,  $x \mapsto f(x)$ , then  $f_i$  is continuous and has a limit as  $x \rightarrow x_{i-1}$  and  $x \rightarrow x_i$ . I.e. the function  $\tilde{f}_i : [x_{i-1}, x_i] \rightarrow \mathbb{R}$  given by

$$\tilde{f}_i(x) = \begin{cases} \lim_{x \rightarrow x_{i-1}} f_i(x) & x = x_{i-1} \\ f_i(x) & x \in (x_{i-1}, x_i) \\ \lim_{x \rightarrow x_i} f_i(x) & x = x_i \end{cases}$$

is continuous (hence integrable).

Consequently, from the observation above, together with Theorem 3.5.8, piecewise continuous function is integrable. Furthermore,<sup>9</sup> if  $f : [a, b] \rightarrow \mathbb{R}$  is integrable, and  $g : [a, b] \rightarrow \mathbb{R}$  is bounded, with  $g(x) = f(x) \forall x \in [a, b] \setminus \{y_1, \dots, y_N\}$ , then  $g$  is integrable and

$$\int_a^b f(x)dx = \int_a^b g(x)dx$$

### 3.5.4 The Fundamental Theorem of Calculus

Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is integrable (hence bounded). Let

$$F(x) = \int_a^x f(t)dt$$

for  $x \in [a, b]$ .

---

<sup>9</sup>Left as an exercise.

**Theorem 3.5.9.**  $F$  is continuous.

*Proof.* Suppose  $|f| \leq K$  and assume  $x, x+h \in [a, b]$ . Then

$$F(x+h) - F(x) = \int_x^{x+h} f(t) dt$$

gives

$$|F(x+h) - F(x)| = \left| \int_x^{x+h} f(t) dt \right| \leq K|h| \rightarrow 0$$

as  $h \rightarrow 0$ . Therefore,  $\lim_{h \rightarrow 0} (F(x+h) - F(x)) = 0 \Rightarrow \lim_{h \rightarrow 0} F(x+h) = F(x)$  and  $F$  is continuous. ■

**Theorem 3.5.10.** (*Fundamental Theorem of Calculus I, FTC 1*) If in addition  $f$  is continuous at  $x$ , then  $F$  is differentiable at  $x$ , and

$$F'(x) = f(x)$$

*Proof.* We need to estimate

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} f(t) dt - f(x) \right| = \frac{1}{|h|} \left| \int_x^{x+h} (f(t) - f(x)) dt \right|$$

Now, given  $\epsilon > 0 \exists \delta > 0$  such that

$$|t - x| < \delta \Rightarrow |f(t) - f(x)| < \epsilon$$

since  $f$  is continuous. Suppose  $|h| < \delta$ . Then  $|f(t) - f(x)| < \epsilon$  for all  $t$  between  $x$  and  $x+h$ . Therefore,

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{|h|} \epsilon |h| = \epsilon$$

i.e.

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

■

Note that for Theorem 3.5.10 to hold, the condition  $f$  is continuous is necessary. For instance, consider

$$f(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in (0, 1] \end{cases}$$

Although  $f$  is montone and thus integrable,

$$F(x) = \begin{cases} -x - 1 & x \leq 0 \\ x - 1 & x \geq 0 \end{cases}$$

i.e.  $F(x) = -1 + |x|$  is not differentiable at  $x = 0$ .

**Corollary 3.5.10.1.** (*Integration is Inverse of Differentiation*) If  $f = g'$  is continuous on  $[a, b]$ , then

$$F(x) = \int_a^x f(t) dt = g(x) - g(a)$$

*Proof.*  $F - g$  has zero derivative on  $[a, b]$  by Theorem 3.5.10, and  $F(a) = 0$ . Hence  $(F - g)(a) = -g(a)$ . So by Corollary 3.3.4.1,

$$(F - g)(x) = -g(a) \quad \forall x$$

and  $F(x) = g(x) - g(a)$ . ■

Note that all continuous functions have an *indefinite integral*, or *anti-derivative*:

$$\int f(x)dx = \int_a^x f(t)dt$$

where  $a$  arbitrary.

**Theorem 3.5.11.** (*Fundamental Theorem of Calculus II, FTC 2*) Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$  and that  $f'$  is integrable. Then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

N.b. Theorem 3.5.11 is stronger than Corollary 3.5.10.1 since it does not require  $f'$  from Theorem 3.5.11 to be continuous.

*Proof.* Let  $\mathcal{D}$  be any dissection of  $[a, b]$ . Applying the mean value theorem (Theorem 3.3.4) to  $f$  on  $[x_{j-1}, x_j]$ ,  $\exists \xi_j \in (x_{j-1}, x_j)$  such that

$$f'(\xi_j)(x_j - x_{j-1}) = f(x_j) - f(x_{j-1})$$

This gives

$$\sum_{j=1}^n f'(\xi_j)(x_j - x_{j-1}) = f(b) - f(a)$$

But

$$\sum_{j=1}^n (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f' \leq \sum_{j=1}^n (x_j - x_{j-1}) f'(\xi_j) \leq \sum_{j=1}^n (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f'$$

So

$$L(f', \mathcal{D}) \leq f(b) - f(a) \leq U(f', \mathcal{D}) \Rightarrow I_*(f') \leq f(b) - f(a) \leq I^*(f')$$

But because  $f'$  is integrable,

$$I_*(f') = I^*(f') = \int_a^b f'(x)dx = f(b) - f(a)$$
■

Following are the important corollaries of fundamental theorem of calculus.

**Corollary 3.5.11.1.** (*Integration by Parts*) Suppose  $f'$  and  $g'$  exist and are continuous on  $[a, b]$ . Then

$$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx$$

*Proof.* By the product rule and Corollary 3.5.10.1,

$$\begin{aligned}\int_a^b f'(x)g(x)dx &= \int_a^b ((fg)'(x) - f(x)g'(x))dx = \int_a^b (fg)'(x)dx - \int_a^b f(x)g'(x)dx \\ &= f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx\end{aligned}$$

■

**Corollary 3.5.11.2.** (*Integration by Substitution*) Let  $g : [\alpha, \beta] \rightarrow [a, b]$  with  $g(\alpha) = a$ ,  $g(\beta) = b$  and suppose  $g'$  exists and is continuous on  $[\alpha, \beta]$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt$$

*Proof.* Set  $F(x) = \int_a^x f(t)dt$  as before. Let  $h(t) = F(g(t))$  which is well defined for  $t \in [\alpha, \beta]$  as  $g$  takes value in  $[a, b]$ . Then,

$$\begin{aligned}\int_\alpha^\beta f(g(t))g'(t)dt &= \int_\alpha^\beta F'(g(t))g'(t)dt \\ &= \int_\alpha^\beta h'(t)dt \\ &= h(\beta) - h(\alpha) \\ &= F(g(\beta)) - F(g(\alpha)) \\ &= F(b) - F(a) = \int_a^b f(t)dt\end{aligned}$$

■

**Theorem 3.5.12.** (*Taylor's Theorem with Integral Remainder*) Suppose  $f$  and its first  $n$  derivatives exist, and are continuous for  $x \in [0, h]$ . Then,

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th)dt$$

*Proof.* By Theorem 3.5.11,

$$\begin{aligned}f(h) &= f(0) + \int_0^h f'(u)du \\ &= f(0) - [(h-u)f'(u)]_0^h + \int_0^h (h-u)f''(u)du \\ &= f(0) + hf'(0) + \int_0^h \left( -\frac{d}{du} \frac{(h-u)^2}{2} \right) f''(u)du\end{aligned}$$

Keep integrating by parts to get

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} + \underbrace{\int_0^h \frac{(h-u)^{n-1}}{(n-1)!} f^{(n)}(u) du}_{R_n}$$

Substitution  $u = th$  gives the desired

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) dt$$

■

*Remark 3.5.1.* Note we assume  $f^{(n)}$  is continuous but in previous versions of the Taylor's theorem we just assumed  $f^{(n)}$  exists. If we make this additional assumption, we can recover Cauchy and Lagrange form of remainder.

**Theorem 3.5.13.** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous and  $g(x) \neq 0$  for all  $x \in (a, b)$ . Then  $\exists c \in (a, b)$  such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$$

N.b. if  $g(x) = 1$ , we get  $\int_a^b f(x)dx = f(c)(b-a)$ .

*Proof.* Apply Cauchy's mean value theorem (Theorem 3.3.6) to

$$F(x) = \int_a^x (fg)(x)dx \text{ and } G(x) = \int_a^x g(x)dx$$

to deduce that there exists  $c \in (a, b)$  such that

$$(F(b) - F(a))G'(c) = F'(c)(G(b) - G(a)) \Rightarrow \left( \int_a^b (fg)(x)dx \right) g(c) = f(c)g(c) \int_a^b g(x)dx$$

$g(c) \neq 0$  so we can cancel out  $g(c)$  to obtain the desired result. ■

Now consider

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) dt$$

To obtain the Cauchy remainder, apply Theorem 3.5.13 with  $\tilde{g} = 1$  and  $\tilde{f}(t) = (1-t)^{n-1} f^{(n)}(th)$  to get  $\exists \theta \in (0, 1)$  such that

$$\int_0^1 \tilde{f}(t) dt = \tilde{f}(\theta) \Rightarrow R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta h)$$

Alternatively, split integrands as

$$\underbrace{\left( (1-t)^{n-1} \right)}_{\tilde{g}} \underbrace{\left( f^{(n)}(th) \right)}_{\tilde{f}}$$

to find

$$\int_0^1 (1-t)^{n-1} f^{(n)}(th) dt = f^{(n)}(\theta h) \int_0^1 (1-t)^{n-1} dt$$

for some  $\theta \in (0, 1)$ , by theorem 3.5.13. But  $\int_0^1 (1-t)^{n-1} dt = 1/n$ , so we obtain the Lagrange remainder

$$R_n = \frac{h^n}{n!} f^{(n)}(\theta h)$$

Finally, we will point out that it is necessary to include the assumption that  $f'$  is integrable in the hypotheses of Theorem 3.5.11. For example, consider  $f : [-1, 1] \rightarrow \mathbb{R}$ , given by

$$f(x) = \begin{cases} |x|^{3/2} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies that  $f$  is continuous on  $[-1, 1]$ , differentiable on  $(-1, 1)$ , but  $f'$  is unbounded hence not integrable.

### 3.5.5 Improper Integrals

The theory of integration we have developed so far applies to bounded functions defined on bounded intervals. For a full treatment for functions whose domain or image is unbounded, we have to wait for the Lebesgue integral.<sup>10</sup> Note that content of this subsection, Riemann integral, is a partial resolution.

#### \* Integrals on an infinite domain

Suppose  $f : [a, \infty) \rightarrow \mathbb{R}$  is integrable on each interval  $[a, R]$  ( $R > a$ ) and

$$\int_a^R f(x) dx \rightarrow l$$

as  $R \rightarrow \infty$ . Then we say  $\int_a^\infty f(x) dx$  exists, or converges, and

$$\int_a^\infty f(x) dx = l$$

Otherwise, if  $\int_a^R f(x) dx$  has no limit, we say  $\int_a^\infty f(x) dx$  *diverges*. A similar definition applies to  $\int_{-\infty}^a f(x) dx$  for  $f : (-\infty, a] \rightarrow \mathbb{R}$ .

Furthermore, if  $\int_{-\infty}^a f(x) dx = l_1$  and  $\int_a^\infty f(x) dx = l_2$  we say  $\int_{-\infty}^\infty f(x) dx$  exists, and

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$$

independent of  $a$ .<sup>11</sup> Be careful that this is not the same as

$$\int_{-R}^R f(x) dx$$

converging. For instance,  $\int_{-R}^R x dx = 0$  but  $\int_0^\infty x dx$  does not converge.

<sup>10</sup>See Probability and Measure, Part II.

<sup>11</sup>This needs check.

✱ **Integrands with an isolated singularity**

Suppose  $f : (b, c] \rightarrow \mathbb{R}$  is integrable on each  $[b + \delta, c]$  for  $0 < \delta < c - b$ . Then if

$$\int_{b+\delta}^c f(x)dx \rightarrow l$$

as  $\delta \rightarrow 0$ , we say  $\int_b^c f(x)dx$  exists or converges and equals  $l$ ; and it is similarly defined for  $f : [a, b) \rightarrow \mathbb{R}$  integrable on  $[a, b - \delta]$  ( $0 < \delta < b - a$ ).

If  $f : [a, c] \setminus \{b\} \rightarrow \mathbb{R}$  and

$$\int_a^b f(x)dx, \quad \int_b^c f(x)dx$$

exist, we say  $\int_a^c f(x)dx$  exists, and

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

✱ **Properties and examples of improper integral**

For example,

$$\int_1^\infty \frac{dx}{x^k}$$

converges if and only if  $k > 1$ , and

$$\int_0^1 \frac{dx}{x^k}$$

converges if and only if  $k < 1$ , since ( $k \neq 1$ )

$$\int_0^R \frac{dx}{x^k} = \left[ \frac{x^{1-k}}{1-k} \right]_1^R = \frac{R^{1-k} - 1}{1-k}$$

- converges as  $R \rightarrow \infty$  if  $1 - k < 0$ , not if  $1 - k > 0$ ;
- converges as  $R \rightarrow 0$  if  $1 - k > 0$ , not if  $1 - k < 0$ ;

and because if  $k = 1$ ,

$$\int_1^R \frac{dx}{x} = [\ln x]_1^R = \ln R$$

does not converge as  $R \rightarrow \infty$  or  $R \rightarrow 0$ .

**Lemma 3.5.14.** Suppose  $f, g : [a, \infty) \rightarrow \mathbb{R}$  are integrable on each  $[a, R]$ ,  $R > a$ . If  $0 \leq f(x) \leq g(x)$  for all  $x \geq a$ , then

$$\int_a^\infty g(x)dx \text{ converges} \Rightarrow \int_a^\infty f(x)dx \text{ converges}$$

*Proof.* Note that

$$R \mapsto \int_a^R f(x)dx$$

is increasing as  $f(x) \geq 0$ , and

$$\int_a^R f(x)dx \leq \int_a^R g(x)dx \leq \int_a^\infty g(x)dx$$

Hence  $R \mapsto \int_a^R f(x)dx$  is increasing and bounded above. Let

$$l = \sup_{R \geq a} \int_a^R f(x)dx$$

and claim

$$l = \lim_{R \rightarrow \infty} \int_a^R f(x)dx$$

To see this, let  $\epsilon > 0$ . Then from definition of supremum,  $\exists R_0$  such that

$$\int_a^{R_0} f(x)dx \geq l - \epsilon$$

So if  $R \geq R_0$ ,

$$l - \epsilon \leq \int_a^{R_0} f(x)dx \leq \int_a^R f(x)dx \leq l$$

■

**Example 3.5.3.** Consider

$$\int_0^\infty e^{-x^2/2} dx$$

For  $x > 1$ ,  $e^{-x^2/2} \leq e^{-x/2}$ , and

$$\int_1^R e^{-x/2} dx = 2(e^{-1/2} - e^{-R/2}) \rightarrow 2e^{-1/2}$$

as  $R \rightarrow \infty$ . Hence  $\int_1^\infty e^{-x^2/2} dx$  converges, and therefore

$$\int_0^\infty e^{-x^2/2} dx$$

converges because from

$$\int_0^R e^{-x^2/2} dx = \int_0^1 e^{-x^2/2} dx + \int_1^R e^{-x^2/2} dx$$

left hand side converges as  $R \rightarrow \infty$  if and only if  $\int_1^R e^{-x^2/2} dx$  does.

However, be careful that, unlike the situation with series,

$$\int_0^\infty f(x)dx \text{ converges} \not\Rightarrow f(x) \rightarrow 0 \text{ as } x \rightarrow \infty$$



## \* Integral test

**Theorem 3.5.15.** (*Integral Test*) Suppose  $f : [1, \infty) \rightarrow \mathbb{R}$  is positive and decreasing. Then,

- (1) the integral  $\int_1^\infty f(x)dx$  converges if and only if the sum  $\sum_{n=1}^\infty f(n)$  converges.
- (2) as  $n \rightarrow \infty$ ,

$$\sum_{r=1}^n f(r) - \int_1^n f(x)dx \rightarrow l$$

for some  $l$  with  $0 \leq l \leq f(1)$ .

Note that (2) of Theorem 3.5.15 implies (1) of that. Also  $f$  decreasing from the hypotheses of that implies  $f$  integrable on  $[1, R]$  ( $R > 1$ ) by Theorem 3.5.4.

*Proof.* If  $n-1 \leq x \leq n$ ,

$$f(n-1) \geq f(x) \geq f(n) \Rightarrow f(n-1) \geq \int_{n-1}^n f(x)dx \geq f(n)$$

Adding, we have

$$\sum_{r=1}^{n-1} f(r) \geq \int_1^n f(x)dx \geq \sum_{r=2}^n f(r)$$

From here, claim (1) is clear since  $\int_1^R f(x)dx$  is monotonically increasing, as is  $\sum_{r=1}^n f(r)$ .

Let

$$\phi(n) = \sum_{r=1}^n f(r) - \int_1^n f(x)dx$$

Then,

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f(x)dx \leq 0$$

and

$$\phi(n) = f(n) + \sum_{r=1}^{n-1} f(r) - \int_1^n f(x)dx \geq f(n) \geq 0$$

So  $\phi(n)$  is decreasing and bounded below. Hence, from monotone convergence theorem,  $\phi(n) \rightarrow l$  with  $0 \leq l \leq \phi(1) = f(1)$ . ■

**Example 3.5.4.**

- (1)  $\sum_{n=1}^\infty 1/n^k$  converges if  $k > 1$ . We saw  $\int_1^\infty (dx/x^k)$  converges if  $k > 1$ . So integral test gives result.
- (2) Consider

$$\sum_{n=2}^\infty \frac{1}{n \ln n}$$

Let  $f(x) = 1/x \ln x$  for  $x \geq 2$ . Then,

$$\int_2^R \frac{dx}{x \ln x} = [\ln(\ln x)]_2^R = \ln(\ln R) - \ln(\ln 2) \rightarrow \infty$$

as  $R \rightarrow \infty$ , and the corresponding sum diverges by integral test.

**Corollary 3.5.15.1.** (*Euler-Mascheroni Constant*) As  $n \rightarrow \infty$ ,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \rightarrow \gamma$$

with  $0 \leq \gamma \leq 1$ .<sup>12</sup>

*Proof.* Set  $f(x) = 1/x$  in Theorem 3.5.15 (2). ■

### 3.5.6 Lebesgue's Criterion for Riemann Integrability

**Theorem 3.5.16.** (*Lebesgue's Criterion*) If  $f : [a, b] \rightarrow \mathbb{R}$  is bounded, then  $f$  is Riemann integral if and only if

$$\{x \mid f \text{ discontinuous at } x\}$$

has *measure zero*.

**Definition 3.5.5.** For an interval  $I$ , let  $|I|$  denote its length. A subset  $A \subset \mathbb{R}$  has *measure zero* if  $\forall \epsilon > 0$  there exists a countable collection of intervals  $I_j$  such that

$$A \subset \bigcup_{j=1}^{\infty} I_j \text{ and } \sum_{j=1}^{\infty} |I_j| < \epsilon$$

i.e. “can cover  $A$  with countable numbers of infinitesimal intervals.”

**Lemma 3.5.17.**

- (1) If  $B$  has measure zero and  $A \subset B$ , then  $A$  has measure zero.
- (2) If  $A_k$  has measure zero for each  $k \in \mathbb{N}$ , then  $\bigcup_{k \in \mathbb{N}} A_k$  has measure zero.

*Proof.*

- (1) Obvious from definition.
- (2) Fix  $\epsilon > 0$ . For each  $k$ , pick  $I_j^k$  ( $j = 1, 2, \dots$ ) such that

$$A_k \subset \bigcup_{j=1}^{\infty} I_j^k \text{ and } \sum_{j=1}^{\infty} |I_j^k| < \epsilon 2^{-k}$$

Then consider  $\{I_j^h\}_{j,h=1}^{\infty}$ . ■

**Will return.**

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<sup>12</sup>Question: is  $\gamma$  irrational?

## 3.6 Example Sheets

### 3.6.1 Sheet 1

1. Straightforward.
2. Converges for  $a_1 = 0, 1$ , diverges for  $a_1 = 2$ .
3.  $a_1 > b_1 > 0$  by construction. Let  $a_n > b_n > 0$ . Then

$$a_{n+1} > b_{n+1} \Leftrightarrow \frac{a_n + b_n}{2} > \frac{a_n b_n}{a_n + b_n} \Leftrightarrow (a_n + b_n)^2 > 2a_n b_n \Leftrightarrow a_n^2 + b_n^2 > 0$$

Thus  $a_{n+1} > b_{n+1}$ . Now since

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n} > b_n \Leftrightarrow a_n > b_n$$

we have  $b_{n+1} > 0$ . Therefore, by induction,  $a_n > b_n > 0$  for all  $n \geq 1$ . Finally, we also have

$$a_n > a_{n+1} = \frac{a_n + b_n}{2} \Leftrightarrow a_n > b_n$$

Combining all,  $a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ .

Now we find the common limit.  $(a_n)$  is decreasing and bounded by  $b_1$ , so  $a_n \rightarrow a$  by monotone convergence theorem. Similarly  $b_n \rightarrow b$ . Then, limit on both sides of  $a_{n+1} = (a_n + b_n)/2$  gives

$$a = \frac{a + b}{2}$$

i.e.  $a = b$  – the limit is common. Also note that

$$a_{n+1} b_{n+1} = a_n b_n = a_1 b_1$$

Therefore,  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \sqrt{a_1 b_1}$ .

4.  $a_n$  is not converging, hence it is not Cauchy, and there exists  $\epsilon > 0$  such that  $|a_n - a_m| \geq \epsilon$  for any  $n, m$ . Without loss of generality, let  $a_1 - a_2 = \epsilon$ . Now suppose

$$A = \{a_i \mid a_i \geq a_1\}$$

$$B = \{a_i \mid a_i \leq a_2\}$$

We may order members of set  $A$  to construct a increasing subsequence  $(a_{n_j})$ , which is bounded above. Thus, by monotone convergence theorem,  $a_{n_j} \rightarrow a < \infty$ . Similarly, we reorder members of  $B$  to construct a decreasing subsequence  $a_{m_j}$  with  $a_{m_j} \rightarrow b$ . But  $a \neq b$  since  $a \geq a_1 > a_2 \geq b$ .

5. Converges, converges for  $|z| < 5$ , converges absolutely, converges for  $|z| < 1$ . Boundaries are not checked.
6. Integral test.

7. We calculate as follows.

$$\begin{aligned} s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \overbrace{\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)}^{2n} - 2 \overbrace{\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right)}^n \\ &= h_{2n} - h_n \end{aligned}$$

Meanwhile,

$$\begin{aligned} t_{3n} &= \overbrace{\frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{4n-1}}^{2n} - \overbrace{\frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n}}^n \\ &= h_{4n} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{4n}\right) - \left(\frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n}\right) \\ &= h_{4n} - \frac{1}{2}h_{2n} - \frac{1}{2}h_n = s_{4n} + \frac{1}{2}s_{2n} \end{aligned}$$

By alternating series test,  $s_n$  converges to a finite value, say  $s$ . Then,  $t_{3n} = s_{4n} + s_{2n}/2 \rightarrow 3s/2$  so  $t_n \rightarrow 3s/2$  as  $n \rightarrow \infty$ .

8. Firstly,  $a_n > 0$  for all  $n$  since

$$a_n = \frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n} \geq \frac{1}{\sqrt{n}} - \frac{1}{n} > 0$$

for  $n \geq 2$  and  $a_1 = 2 > 0$ . Also, clearly  $\lim_{n \rightarrow \infty} a_n = 0$  since both  $1/\sqrt{n} \rightarrow 0$  and  $1/n \rightarrow 0$  as  $n \rightarrow \infty$ . Now consider

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} + \sum_{n=1}^{\infty} \frac{1}{n}$$

The first sum converges by alternating series test, but second sum diverges; hence,  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$  also diverges.

9. (*Dirichlet's test*) Simplify to obtain

$$\begin{aligned} S_n b_n - S_{m-1} b_m + \sum_{j=m}^{n-1} S_j b_j - \sum_{j=m}^{n-1} S_j b_{j+1} &= \sum_{j=m}^n S_j b_j - \sum_{j=m-1}^{n-1} S_j b_{j+1} \\ &= \sum_{j=m}^n S_j b_j - \sum_{j=m}^n S_{j-1} b_j \\ &= \sum_{j=m}^n (S_j - S_{j-1}) b_j = \sum_{j=m}^n a_j b_j \end{aligned}$$

Now since  $S_n$  is a bounded sequence, i.e.  $|S_n| \leq K$ , for each term

$$|S_j(b_j - b_{j-1})| \leq K(b_j - b_{j-1})$$

But series

$$\sum_{j=1}^{n-1} K(b_j - b_{j-1}) = K(b_1 - b_n) \rightarrow K b_1$$

as  $n \rightarrow \infty$  so series  $\sum_{j=1}^{n-1} S_j(b_j - b_{j+1})$  absolutely converges, and hence converges. Also, because  $b_n \rightarrow 0$  and  $S_n$  is bounded,

$$S_n b_n, S_{n-1} b_n \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore,  $\sum_{j=1}^{\infty} a_j b_j$  converges.

Alternating series test can be easily obtained by setting  $a_n = (-1)^n$ .

Now consider  $\sum_{n=1}^{\infty} \cos n/n$ . Let  $a_n = \cos n$  and  $b_n = 1/n$ . To apply the Dirichlet's test, we need to show  $\sum \cos n$  is bounded. Let  $S_n = \sum_{n=1}^{\infty} \cos n$  and  $S_0 = 0$ . Then,

$$\begin{aligned} S_n \sin 1 &= \cos 1 \sin 1 + \cos 2 \sin 1 + \cdots + \cos n \sin 1 \\ &= -\frac{1}{2}(\sin 2 - \sin 0 + \sin 3 - \sin 1 + \cdots + \sin(n+1) - \sin(n-1)) \\ &= \frac{\sin 1 - \sin(n+1)}{2} \end{aligned}$$

gives

$$S_n = \frac{\sin 1 - \sin(n+1)}{2 \sin 1}$$

which is bounded. Thus by Dirichlet's test,  $\sum_{n=1}^{\infty} \cos n/n$  converges.

10. Q10.

11. Note that

$$\frac{1}{1-z} - \frac{z}{1-z^2} = \frac{1+z-z}{1-z^2} = \frac{1}{1-z^2}$$

One can find out that by telescoping sums,

$$\sum_{j=1}^n \frac{z^{2^{j-1}}}{1-z^{2^j}} = \frac{1}{1-z} - \frac{1}{1-z^{2^n}}$$

Then, if  $|z| < 1$ ,  $z^{2^n} \rightarrow 0$  as  $n \rightarrow \infty$  so the series converges to

$$\frac{1}{1-z} - 1 = \frac{z}{1-z}$$

If  $|z| > 1$ ,  $z^{2^n} \rightarrow \infty$  as  $n \rightarrow \infty$  so series simply converges to  $1/1-z$ . Finally, if  $|z| = 1$ ,

$$\left| \frac{z^{2^{j-1}}}{1-z^{2^j}} \right| = \frac{1}{|1-z^{2^j}|} > \frac{1}{2}$$

so

$$\sum_{j=1}^n \frac{z^{2^{j-1}}}{1-z^{2^j}} > \frac{1}{2}n$$

diverges as  $n \rightarrow \infty$ .

12. Consider sequence  $(a_n)$ ,  $a_n \in \mathbb{R}$ . We can easily construct monotonic subsequence  $(a_{n_j})$  by choosing  $n_1 = 1$ , and then  $n_i$  as the least greatest  $j$  such that  $a_j \geq a_{n_{i-1}}$ ,  $j \geq n_{i-1}$  inductively.

Now assume the sequence is bounded. The subsequence we found above,  $(a_{n_j})$  will also be bounded and by the monotone convergence theorem,  $(a_{n_j})$  will converge. Thus we proved the Bolzano-Weierstrass theorem.

13. Since  $(x_n)$  converges, it is bounded. Now sequence  $(|x_n|)$ . Because it is bounded, we can rearrange terms to make  $(|x_n|)$  decreasing. Then, by alternating series test,  $\sum (-1)^n |x_n|$  converges.  $\epsilon_n$  will follow based on the original signs of  $x_n$ .
14. No. It is impossible.

### 3.6.2 Sheet 2

1. If  $x \notin \mathbb{Q}$ , take sequence  $x_n \in \mathbb{Q}$ . Then  $x_n \rightarrow x$  but

$$f(x_n) = x_n \rightarrow x \neq 1 - x = f(x)$$

so  $f$  is discontinuous. If  $x \in \mathbb{Q}$ , take sequence  $x_n \notin \mathbb{Q}$ . Then  $x_n \rightarrow x$ . Now

$$f(x_n) = 1 - x_n \rightarrow 1 - x \neq x = f(x)$$

for all  $x$  that is not  $1/2$ . Therefore,  $f$  is discontinuous for all  $x \in \mathbb{R}$ ,  $x \neq 1/2$ . However, for  $x = 1/2$ ,  $f$  is indeed continuous. Hence the required set will be  $\{1/2\}$ .

2. We say  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$  if for given  $M > 0$ , there exists  $\delta(M) > 0$  such that

$$\forall x \in \mathbb{R}, x > \delta \Rightarrow f(x) > M$$

Now let's solve the second part of the question.

( $\Rightarrow$ ) Suppose  $f \rightarrow \infty$  as  $x \rightarrow \infty$ , and  $x_n$  is a sequence with  $x_n \rightarrow \infty$ . Let  $M > 0$ . Then  $\exists \delta > 0$  such that

$$\forall x \in \mathbb{R}, x > \delta \Rightarrow f(x) > M$$

But since  $x_n \rightarrow \infty$ , there exists  $N$  such that

$$n \geq N \Rightarrow x_n > \delta \Rightarrow f(x_n) > M$$

Therefore,  $f(x_n) \rightarrow \infty$ .

( $\Leftarrow$ ) Suppose  $f$  is not continuous at  $a$ . Then,  $\exists M > 0$  such that for all  $\delta > 0$ ,  $\exists x \in \mathbb{R}$  with  $x > \delta$  and  $f(x) \leq M$ . Now consider

$$\delta = 1, 2, \dots, n, \dots$$

for  $x_1, x_2, \dots, x_n, \dots$  respectively. Then,  $x_n > n$  and  $f(x_n) \leq M$ . Then  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$  but  $f(x_n) \not\rightarrow \infty$ , which is a contradiction. Hence  $f \rightarrow \infty$  as  $x \rightarrow \infty$ .

3. No. For example, consider

$$f(x) = 0 \text{ for all } x \in \mathbb{R}$$

$$g(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then  $\lim_{x \rightarrow 0} f(x) = 0$  and  $\lim_{x \rightarrow 0} g(x) = 1$ . However,

$$\lim_{x \rightarrow 0} g(f(x)) = 0$$

instead of expected 1. Note that  $g$  is not continuous at  $x = 0$ .

4. We only need to check if  $h_n$  is continuous at a point  $a \in [0, 1]$  such that  $f_i(a) = f_j(a)$ , and  $f_i(x) - f_j(x)$  decreasing at  $a$  (without loss of generality) for some  $i, j \in \{1, 2, \dots, n\}$ . Then,

$$\lim_{x \rightarrow a-} h_n(x) = \lim_{x \rightarrow a-} f_i(x) = f_i(a)$$

$$\lim_{x \rightarrow a+} h_n(x) = \lim_{x \rightarrow a+} f_j(x) = f_j(a)$$

But  $f_i(a) = f_j(a)$  so the limit exists and  $\lim_{x \rightarrow a} h_n(x) = f_i(a) = f_j(a)$ . Hence  $h_n$  is continuous at  $a$ , and consequently at any  $x \in [0, 1]$ .

5. Let  $g(x) = f(x) - x$ , which is also continuous at  $[0, 1]$ . If  $g(0) = 0$ ,  $f(0) = 0$ , done. Now if  $g(0) > 0$ , consider  $g(1)$ . If  $g(1) = 0$ , done. Otherwise, we have  $g(1) < 0$ . Then, by intermediate value theorem, there exists  $c \in (0, 1)$  such that

$$g(c) = f(c) - c = 0 \Leftrightarrow f(c) = c$$

However, a continuous bijection  $h : (0, 1) \rightarrow (0, 1)$  need not to have a fixed point. E.g. consider  $h(x) = x^2$ . With such idea, we may construct a bijection of  $[0, 1]$  with no fixed point:

$$x \mapsto \begin{cases} 1 & x = 0 \\ x^2 & 0 < x < 1 \\ 0 & x = 1 \end{cases}$$

6. We assume the familiar features of sin and cos. For instance,  $f(x)$  is periodic with period  $2\pi$ . Thus we only consider  $f(x) : [0, 2\pi] \rightarrow [0, 2]$ . Since  $f$  continuous, there exists  $y \in [0, 2\pi]$  such that  $f(y) \leq f(x)$  for all  $x \in [0, 2\pi]$ , by the extreme value theorem. Now we only need to show that  $f(y) \neq 0$ , or furthermore,  $f(x) > 0 \forall x \in [0, 2\pi]$ . Note that  $f(x) = 0$  if and only if  $\sin^2 x = 0$  and  $\sin^2(x + \cos^7 x) = 0$ . Suppose there is a point  $c \in [0, 2\pi]$  with  $f(x) = 0$ . Then, we have

$$c = n\pi, \quad c + \cos^7 c = m\pi$$

for some  $n, m \in \mathbb{Z}$ . Simplification gives

$$(-1)^m = \cos^7(m\pi) = (n - m)\pi$$

which is a contradiction. Thus  $f(x) > 0$  for all  $x \in [0, 2\pi]$ .

7. Q7.

8. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = 2x^5 + 3x^4 + 2x + 16$ . Differentiate to find

$$f'(x) = 10x^4 + 12x^3 + 2$$

and

$$f''(x) = 40x^3 + 36x^2 = 4x^2(10x + 9)$$

Since  $f''(x) < 0$  for  $x < -9/10$ , we know that  $f'$  is strictly decreasing at  $[-2, -1]$ . Furthermore, because  $f'(-1) = 0$ ,  $f'(x) > 0$  for  $x \in [-2, -1)$ , and  $f$  is strictly increasing at  $[-2, -1)$ .

Now, since  $f(-2) = -4 < 0$  and  $f(-1) = 15 > 0$ , there exists  $c \in (-2, -1)$  such that  $f(c) = 0$  (by intermediate value theorem); and such point is unique since  $f$  is strictly increasing at  $(-2, -1)$ .

Moreover, there is no root for  $x \leq -2$  since  $f(-2) < 0$  and  $f'(x)$  is strictly decreasing for  $x \leq -2$ . Similarly, there is no root for  $x \geq 0$  since  $f(x) \geq f(0) = 16 > 0$ .

We only need to check for  $(-1, 0)$ . In this interval,

$$f(x) > 2x + 3x^4 + 2x + 16 = 3x^4 + 4x + 16 > 3x^4 + 12 > 12$$

so there is no root.

9. (2) (4) is true (see lecture note). (1) is also true. However, (3) is false, e.g. consider  $f(x) = x^3$  at  $x = 0$ .
10. Note that  $f(0) < \infty$  since  $f$  is differentiable for all  $x$ . By the mean value theorem, there exists  $c \in (0, x)$  such that

$$\frac{f(x) - f(0)}{x} = f'(c)$$

Taking limits of both sides,

$$\lim_{x \rightarrow \infty} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow \infty} f'(c) = l$$

and

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = l + \lim_{x \rightarrow \infty} \frac{f(0)}{x} = l$$

However, converse is not necessarily true, e.g. consider  $f(x) = \sin x$ .

11. If  $x \neq 0$ , simple calculation gives

$$f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x}$$

If  $x = 0$ ,

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \left( 1 + 2x \sin \frac{1}{x} \right) = 1$$

since  $h \rightarrow 0$  but  $|\sin(1/h)| \leq 1$ . Now consider an arbitrary interval  $[-\delta, \delta]$ . Let  $x = 2/(2n+1)\pi$  with  $n \in \mathbb{N}$ ,  $x < \delta$ . Then,

$$f(x) = x - 2x^2 < 0$$

since  $x < 1/2$ . Therefore, because  $f(x) < f(0) = 0$  while  $x > 0$ ,  $f(x)$  is not increasing in the interval  $[-\delta, \delta]$ . Since  $\delta$  was arbitrary, we are done.

### 3.6.3 Sheet 3

1. Put  $y = x + h$  where  $h \neq 0$ . Then,

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq |h|$$

Since it holds for all  $h$ ,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = 0$$

Thus  $f$  is constant.



2. Note that  $-1 \leq \sin(1/x) \leq 1$ . With this fact, claim that  $f_\alpha$  is bounded if and only if  $0 \leq \alpha \leq 1$ . One may easily proof this.

Meanwhile, It is clear that  $f_\alpha$  is continuous at all points except  $x = 0$ . We only need to check for  $x = 0$ :  $f_\alpha$  is continuous if and only if  $\alpha > 0$  and differentiable if and only if  $\alpha > 1$ . These give the table as follows.

$\alpha =$	$-1/2$	$0$	$1/2$	$1$	$3/2$	$2$	$5/2$	$3$	$7/2$
$f_\alpha$ bounded	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\times$	$\times$	$\times$	$\times$	$\times$
$f_\alpha$ cont.	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$f_\alpha$ diffable	$\times$	$\times$	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$f'_\alpha$ bounded	-	-	-	-	$\times$	$\checkmark$	$\times$	$\times$	$\times$
$f'_\alpha$ cont.	-	-	-	-	$\times$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$f'_\alpha$ diffable	-	-	-	-	$\times$	$\times$	$\times$	$\times$	$\checkmark$

3.  $\log(1+x)$  is continuous on  $[0, a/n]$  and differentiable on  $(0, a/n)$ . Hence, by mean value theorem, there exists  $c \in (0, a/n)$  such that

$$\frac{\ln(1+a/n)}{a/n} = \frac{1}{1+c}$$

Rearrange to find

$$\ln\left(\left(1+\frac{a}{n}\right)^n\right) = \frac{a}{1+c}$$

Since  $\ln$  is inverse of  $\exp$ ,

$$\left(1+\frac{a}{n}\right)^n = e^{a/(1+c)} \rightarrow e^a$$

as  $n \rightarrow \infty$  (Note that both limits exist).

4. Let  $1/n = m$ . Then

$$\lim_{n \rightarrow \infty} n(a^{1/n} - 1) = \lim_{m \rightarrow 0} \frac{a^m - 1}{m} = \lim_{m \rightarrow 0} \frac{e^{m \ln a} - 1}{m \ln a} \ln a = \ln a$$

5.  $f_{3/2}$  and  $f_2$  from Question 2 shows the argument is false at  $x = 0$ . Note that  $\lim_{h \rightarrow 0} f'(c + \theta h)$  may not exist.
6. It is clearly differentiable for  $x \neq 0$ . For  $x = 0$ , we can tell that  $f$  is infinitely differentiable since  $x^r e^{-x} \rightarrow 0$  as  $x \rightarrow \infty$  for any  $r > 0$ . Taylor's theorem states that

$$f(x) = \frac{x^n}{n!} f^{(n)}(\theta x)$$

However,

$$f(x) \neq \sum_{i=0}^{\infty} \frac{x^i}{i!} f^{(i)}(0) = 0$$

Taylor series, since

$$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x) \neq 0$$

as  $n \rightarrow \infty$ .

7.  $3/2, 2^{1/3}, 1/e, 1$ .

8. By Cauchy's mean value theorem,

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

for some  $c \in (a, x)$ . Hence

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(c)}{g'(c)} = l$$

Put  $g(x) = x - a$ . Then,

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} f'(x) = l$$

So  $f$  is also differentiable at  $a$  (with limit  $l$ ).

Meanwhile, consider  $g(x) = x$ ,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and  $a = 0$ . Then

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

and  $g'(x) = 1$ . We have

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

but  $\lim_{x \rightarrow 0} f'(x)/g'(x)$  is not defined.

For the final part, let  $f(x) = x - (n+1)x^{n+1} + nx^{n+2}$ , and  $g(x) = (1-x)^2$ . Then  $f'(x) = 1 - (n+1)^2x^n + n(n+2)x^{n+1}$ ,  $g'(x) = 2x - 2$ . Thus the limit will be

$$\frac{n(n+1)}{2}$$

9. We have  $\tan' x = \sec^2 x$  for  $x \in (-\pi/2, \pi/2)$ . Now consider  $\tan x : [-a, a] \rightarrow [-c, c]$  for some  $0 < a < \pi/2$ , with  $c = \tan a$ . Then by inverse function theorem,  $\tan^{-1}$  is differentiable on  $(-c, c)$  with

$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

Moreover, since  $c \rightarrow \infty$  as  $a \rightarrow \pi/2$ , it is differentiable for  $x \in \mathbb{R}$ .

By ratio test, we can check the power series

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

has a radius of convergence of 1. Hence  $g$  is differentiable for  $|x| < 1$  and

$$g'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} = (\tan^{-1} x)'$$

Integrate both sides to obtain that  $\tan^{-1} = g(x)$  ( $g(0) = \tan 0$ ).

10. Clearly,

$$p_n = (1 + a_1) \cdots (1 + a_n) = 1 + (a_1 + \cdots + a_n) + (\cdots) \geq s_n$$

Now since  $\ln$  is increasing,

$$\begin{aligned} p_n \leq e^{s_n} &\Leftrightarrow \ln p_n \leq s_n \\ &\Leftrightarrow \ln(1 + a_1) + \cdots + \ln(1 + a_n) \leq a_1 + \cdots + a_n \end{aligned}$$

Claim that  $\ln(1 + x) \leq x$  for  $x \geq 0$ . Let  $f(x) = \ln(1 + x) - x$ . But

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \geq 0$$

and  $f(0) = 0$ . So  $\ln(1 + x) \leq x$  for  $x \geq 0$ . Thus

$$\ln(1 + a_j) \leq a_j$$

for all  $j = 1, 2, \dots, n$  and

$$s_n \leq p_n \leq e^{s_n}$$

First suppose  $p_n \rightarrow p$  as  $n \rightarrow \infty$ . Then,  $s_n$  is bounded by  $p$  and it is increasing ( $a_n \geq 0$ ). Therefore it converges.

Suppose  $s_n \rightarrow s$  as  $n \rightarrow \infty$ . Then,  $p_n$  is bounded by  $e^s$  and it is increasing ( $(1 + a_n) \geq 1$ ). Therefore it converges. Thus  $\prod_{n=1}^{\infty} (1 + a_n)$  converges if and only if  $\sum_{n=1}^{\infty} a_n$  converges.

Let  $a_n = 1/(n^2 - 1)$ . Then,

$$s_n = \sum_{j=2}^n a_j = \sum_{j=2}^n \frac{1}{2} \left( \frac{1}{j-1} - \frac{1}{j+1} \right) = \frac{3}{4} - \frac{1}{2} \left( \frac{1}{n} + \frac{1}{n+1} \right) \rightarrow \frac{3}{4}$$

Thus by above,  $p_n$  converges. Now,

$$\begin{aligned} \ln p_n &= \sum_{j=2}^n \ln(1 + a_j) = \sum_{j=2}^n \ln \left( \frac{j^2}{j^2 - 1} \right) = \sum_{j=2}^n [\ln(j^2) - \ln(j^2 - 1)] \\ &= \sum_{j=2}^n [2 \ln j - \ln(j-1) - \ln(j+1)] \\ &= \sum_{j=2}^n [(\ln j - \ln(j-1)) - (\ln(j+1) - \ln j)] \\ &= \ln 2 + \ln \frac{n}{n+1} \rightarrow \ln 2 \end{aligned}$$

as  $n \rightarrow \infty$ . Thus  $p_n \rightarrow 2$  as  $n \rightarrow \infty$ .

11. Q11.

12. (Theorem of Darboux) Let  $d = (a + b)/2$ . For  $a \leq x \leq d$ , define  $\alpha(x) = a$  and  $\beta(x) = 2x - a$ , and for  $d \leq x \leq b$ , define  $\alpha(x) = 2x - b$  and  $\beta(x) = b$ . Then for  $x \in (a, b)$ , we have  $a \leq \alpha < \beta \leq b$ . Now, define

$$g(x) = \frac{(f \circ \beta)(x) - (f \circ \alpha)(x)}{\beta(x) - \alpha(x)}$$

with  $a < x < b$ . Furthermore,  $g(x) \rightarrow f'(a)$  as  $x \rightarrow a$  and  $g(x) \rightarrow f'(b)$  as  $x \rightarrow b$ . Hence, by the intermediate value theorem, there exists  $m \in (a, b)$  such that  $g(m) = y$ . Then, by the mean value theorem, there exists  $c \in (\alpha(m), \beta(m))$  such that

$$f'(c) = g(m) = y$$

13. We consider

$$f(x) = \exp\left(\frac{1}{(x-a)(x-b)}\right)$$

(See Rudin.)