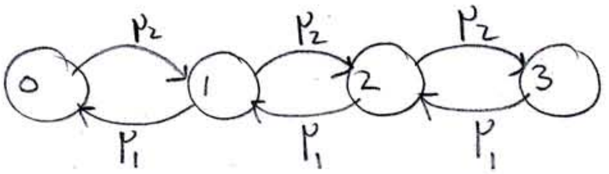
$\boxed{3.64}$ * exponential service times $\Rightarrow$ evolution of the system is Markovian

* Let N_i be the number of customers in queue i, $i=1,2$.

* (N_1, N_2) defines the state of the system

* Since the system is closed with $N_1+N_2=3$, then N_1 completely defines the state of the system. (The same can be said about N_2)

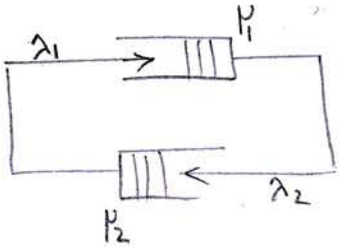
(a) Therefore the system can be described by a Markov chain whose state is N_1



(b) $$P_{N_1}(n_1) = \left(\frac{1-\rho}{1-\rho^4}\right)\rho^{n_1} = \frac{\rho^{n_1}}{1+\rho+\rho^2+\rho^3} \qquad n_1 = 0,1,2,3$$ (Recall Problem 3.21)

where $\rho = \frac{\mu_2}{\mu_1}$

(c)



the termination rate at queue 2 is $\begin{cases} \mu_2 & \text{when queue 2 is busy} \\ 0 & \text{when queue 2 is empty} \end{cases}$

the average arrival rate at queue 1
= average termination (departure) rate at queue 2

i.e. $$\lambda_1 = \mu_2 \times \underbrace{(1-P_{N_2}(0))}_{\text{prob. queue 2 busy}} + 0 \times \underbrace{P_{N_2}(0)}_{\text{prob. queue 2 empty}}$$

<u>Note:</u> $P_{N_2}(n_2) = P_{N_1}(3-n_1)$ $\quad \therefore P_{N_2}(0) = P_{N_1}(3)$ $\quad$ #

$$\lambda_1 = \frac{\mu_2(1+\rho+\rho^2)}{1+\rho+\rho^2+\rho^3}$$

(d)

$\lambda_1 \times$ avg time of a cycle = avg # of customer = 3

avg time of a cycle $= \frac{3}{\lambda_1}$

rate at which a user cycles through the system $= \frac{\lambda_1}{3}$

(e) The Markov chain is a birth-death process.
Birth-death processes are reversible

Departure from queue 1 in the forward process corresponds to arrival to queue 1 in the reversed process

Aside

$$P_{N_1}(n_1) = \frac{\rho^{n_1}}{1+\rho+\rho^2+\rho^3} = \frac{\rho^{n_1}}{(1+\rho)(1+\rho^2)} \quad \text{where } \rho = \frac{\mu_2}{\mu_1}$$

One can also obtain the steady state distrib using:

$$P_{N_1,N_2}(n_1,n_2) = \frac{\rho_1^{n_1}\rho_2^{n_2}}{G}, \quad n_1+n_2=3 \quad \text{where } \rho_1 = \frac{\lambda_1}{\mu_1} \text{ \& } \rho_2 = \frac{\lambda_2}{\mu_2}$$

$$G = \rho_2^3 + \rho_1\rho_2^2 + \rho_1^2\rho_2 + \rho_1^3 = (\rho_1+\rho_2)(\rho_1^2+\rho_2^2)$$

$$P_{N_1,N_2}(n_1, 3-n_1) = \frac{(\rho_1/\rho_2)^{n_1}\rho_2^3}{(\rho_1+\rho_2)(\rho_1^2+\rho_2^2)} = \frac{(\rho_1/\rho_2)^{n_1}}{(\rho_1/\rho_2+1)((\rho_1/\rho_2)^2+1)}$$

$$\frac{\rho_1}{\rho_2} = \frac{\lambda_1/\mu_1}{\lambda_2/\mu_2} = \frac{\mu_2}{\mu_1}\times\frac{\lambda_1}{\lambda_2} = \frac{\mu_2}{\mu_1} = \rho \quad (\text{bec } \lambda_1=\lambda_2=\lambda)$$

$$\therefore P_{N_1,N_2}(n_1, 3-n_1) = \frac{\rho^{n_1}}{(\rho+1)(\rho^2+1)}$$

✓ #