3.38

(a) Let N_{Qk} = number in queue for priority k

W_k = queueing (waiting) time for priority k

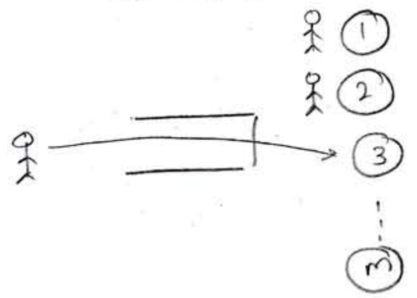
R = residual service time

ρ_k = $\frac{\lambda_k}{\mu}$ = utilization for priority k

N = number of customers in the system

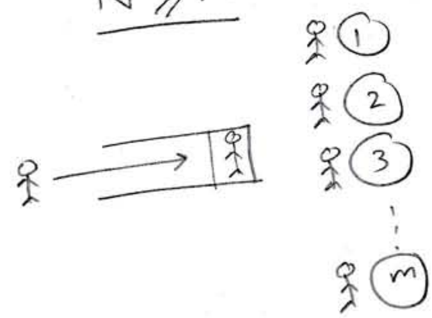
* First, we evaluate $E[R]$. Consider the two following cases:

$N < m$



an arriving customer goes immediately to service i.e. $R = 0$

$N \geqslant m$



- in this case, the residual service time is the time until one server empties
- Let $\tilde{T}_i$, $i = 1, \dots, m$, be the service time of customer at server i

$\tilde{T}_i$: exponential, mean $= \frac{1}{\mu}$

$R = \min\{\tilde{T}_1, \dots, \tilde{T}_m\}$

$$
\begin{aligned}
\mathbb{P}(R \leqslant t) &= \mathbb{P}(\min\{\tilde{T}_i\} \leqslant t) \\
&= 1 - \mathbb{P}(\min\{\tilde{T}_i\} > t) \\
&= 1 - \prod_{i=1}^{m} \mathbb{P}(\tilde{T}_i > t) \\
&= 1 - (e^{-\mu t})^m \\
&= 1 - e^{-m\mu t}
\end{aligned}
$$

more accurately, this should be written as $\mathbb{P}(R \leqslant t \mid N \geqslant m)$

$E[R \mid N < m] = 0$

$E[R \mid N \geqslant m] = \frac{1}{m\mu}$

$$E[R] = \underbrace{\mathbb{P}(N<m)}_{1-P_Q} \underbrace{E[R \mid N<m]}_{0} + \underbrace{\mathbb{P}(N \geq m)}_{P_Q} \underbrace{E[R \mid N \geq m]}_{\frac{1}{m\mu}}$$

$$\therefore \; E[R] = \frac{P_Q}{m\mu}$$

* Next, we evaluate $E[W_k]$

$\boxed{\text{priority class 1}}$ $$E[W_1] = E[R] + \frac{1}{m\mu} E[N_{Q1}]$$

Intuitively speaking,

$$\text{avg waiting time} = \text{avg residual time} + \underbrace{\text{avg time it takes to serve the customers of priority 1 in the queue}}_{= \frac{\text{avg \# of customers} \times \text{avg service time for each}}{m}}$$

(we divide by m because the servers work in parallel)

From Little's law, we have

$$E[N_{Q1}] = \lambda_1 E[W_1]$$

Therefore, $$E[W_1] = \frac{E[R]}{1-\rho_1}, \quad \rho_1 = \frac{\lambda_1}{m\mu}$$

$\boxed{\text{priority class 2}}$ $$E[W_2] = E[R] + \frac{1}{m\mu} E[N_{Q1}] + \frac{1}{m\mu} E[N_{Q2}] + \frac{1}{m\mu} \lambda_1 E[W_2]$$

Using Little's theorem, we have $E[N_{Q1}] = \lambda_1 E[W_1]$ and $E[N_{Q2}] = \lambda_2 E[W_2]$

$$\therefore \; E[W_2] = E[R] + \rho_1 E[W_1] + \rho_2 E[W_2] + \rho_1 E[W_2]$$

$$\therefore \; E[W_2] = \frac{E[R] + \rho_1 E[W_1]}{1-\rho_1-\rho_2}, \quad \rho_1 = \frac{\lambda_1}{m\mu} \text{ and } \rho_2 = \frac{\lambda_2}{m\mu}$$

$$\therefore \; E[W_2] = \frac{E[R]}{(1-\rho_1)(1-\rho_1-\rho_2)}$$

$\boxed{\text{priority class } k}$ $$E[W_k] = \frac{E[R]}{(1-\rho_1-\cdots-\rho_{k-1})(1-\rho_1-\cdots-\rho_k)} \quad \text{where } \rho_k = \frac{\lambda_k}{m\mu}$$

and $E[R] = \frac{P_Q}{m\mu}$ (P_Q as defined in sec. 3.4.1 with $\rho = \rho_1 + \rho_2 + \cdots + \rho_n$)

<u>Check</u>: substitute $m=1$, [illegible]

$$\rho_k = \frac{\lambda_k}{\mu}$$

$$E[W_k] = \frac{E[R]}{(1-\rho_1 - \cdots - \rho_{k-1})(1-\rho_1-\cdots-\rho_k)}$$

$$P_Q = 1 - P_0 = 1-(1-\rho) = \rho = \rho_1 + \rho_2 + \cdots + \rho_n = \sum_{i=1}^{n} \frac{\lambda_i}{\mu}$$

$$E[R] = \frac{P_Q}{1 \times \mu} = \sum_{i=1}^{n} \lambda_i \frac{1}{\mu^2}$$

Compare that with what we get by considering exponential service time in the M/G/1 expressions

$$E[W_k] = \frac{E[R]}{(1-\rho_1 - \cdots - \rho_{k-1})(1-\rho_1 - \cdots - \rho_k)}$$

$$E[R] = \frac{1}{2} \sum_{i=1}^{n} \lambda_i \overline{X^2}$$

$X \sim$ exponential with mean $= \frac{1}{\mu}$

$$\Rightarrow E[X^2] = \frac{2}{\mu^2}$$

$$\therefore E[R] = \sum_{i=1}^{n} \lambda_i \frac{1}{\mu^2} \qquad \checkmark \qquad \text{(expressions match)}$$