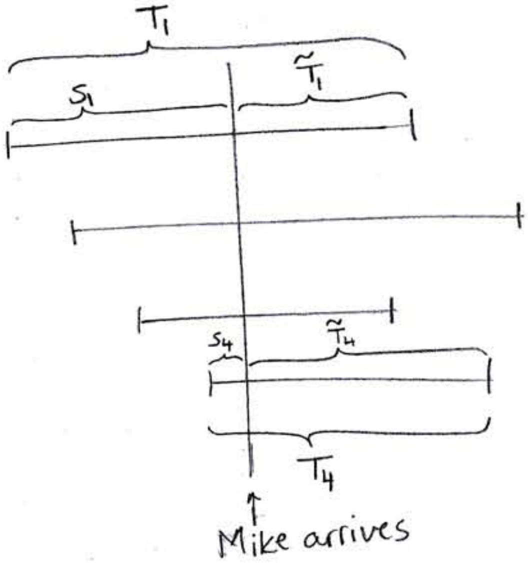
$\boxed{3.6}$



(a) $\mathbb{P}(\text{Mike leaves last}) = ?$

$$p = \mathbb{P}(\text{Mike leaves last})$$
$$= \mathbb{P}\left(\bigcap_{i\in C\setminus\{i^*\}} \{\tilde{T}_i < \tilde{T}_{i^*} + T\}\right)$$

where

$\tilde{T}_i$ = service time after arrival of Mike for user i

T = service time of Mike

$C = \{1,2,3,4\}$

$i^* = \arg\min_{i\in C} T_i$ (i^* is a R.V.)

$$p = \sum_{k\in C} \underbrace{\mathbb{P}(i^*=k)}_{\substack{=\frac{1}{4} \\ \text{(from symmetry)}}} \underbrace{\mathbb{P}\left(\bigcap_{i\in C\setminus\{i^*\}} \{\tilde{T}_i < \tilde{T}_{i^*}+T\} \,\middle|\, i^*=k\right)}_{\substack{\text{same for } k=1,2,3,4 \\ \text{(bec. of symmetry)}}}$$

$$\therefore\; p = \mathbb{P}\left(\bigcap_{i\in C\setminus\{i^*\}} \{\tilde{T}_i < \tilde{T}_{i^*}+T\} \,\middle|\, i^*=1\right)$$
$$= \mathbb{P}\left(\bigcap_{i=2}^{4} \{\tilde{T}_i < \tilde{T}_1+T\} \,\middle|\, i^*=1\right)$$

but $\{i^*=1\} = \bigcap_{i=2}^{4} \{\tilde{T}_1 < \tilde{T}_i\}$ ($i^*=1 \Leftrightarrow$ person #1 finished first)

i.e. $$p = \mathbb{P}\left(\bigcap_{i=2}^{4}\{\tilde{T}_i < \tilde{T}_1+T\} \,\middle|\, \bigcap_{i=2}^{4}\{\tilde{T}_i > \tilde{T}_1\}\right)$$

$$p = \int_{t=0}^{\infty}\int_{t_1=0}^{\infty} f_{\tilde{T}_1 T}(t_1,t)\, \mathbb{P}\left(\bigcap_{i=2}^{4}\{\tilde{T}_i < \tilde{T}_1+T\} \,\middle|\, \bigcap_{i=2}^{4}\{\tilde{T}_i > \tilde{T}_1\},\, \tilde{T}_1=t_1, T=t\right) dt_1\, dt$$

$$p = \int_{t=0}^{\infty}\int_{t_1=0}^{\infty} f_{\tilde{T}_1}(t_1)\, f_T(t)\, \mathbb{P}\left(\bigcap_{i=2}^{4}\{\tilde{T}_i < t_1+t\} \,\middle|\, \bigcap_{i=2}^{4}\{\tilde{T}_i > t_1\}\right) dt_1\, dt$$

$$\mathbb{P}\left(\bigcap_{i=2}^{4}\{\tilde{T}_i < t_1+t\} \,\Big|\, \bigcap_{i=2}^{4}\{\tilde{T}_i > t_1\}\right)$$

$$= \frac{\mathbb{P}\left(\bigcap_{i=2}^{4}\{t_1 < \tilde{T}_i < t_1+t\}\right)}{\mathbb{P}\left(\bigcap_{i=1}^{4}\{\tilde{T}_i > t_1\}\right)}$$

$$= \prod_{i=2}^{4}\left(\frac{e^{-\mu t_1} - e^{-\mu(t_1+t)}}{e^{-\mu t_1}}\right)$$

$$= (1-e^{-\mu t})^3$$

$$\therefore\ p = \int_{t=0}^{\infty} \underbrace{\left(\int_{t_1=0}^{\infty} f_{\tilde{T}_1}(t_1)\,dt_1\right)}_{=1} \cdot \underbrace{f_T(t)}_{\mu e^{-\mu t}} \times (1-e^{-\mu t})^3\,dt$$

$$\therefore\ p = \int_{t=0}^{\infty} \mu e^{-\mu t}(1-e^{-\mu t})^3\,dt$$

$$= \left.\frac{(1-e^{-\mu t})^4}{4}\right|_{t=0}^{\infty}$$

$$= \frac{1}{4}$$

$\Rightarrow$ $\mathbb{P}$(Mike leaves last) = $1/4$

(b) Let T_W = time Mike wait until he gets to a clerk

T_{BANK} = time Mike spends in the bank

$E[T_{BANK}] = ?$

$$E[T_{BANK}] = E[T_W + T] = E[T_W] + \underbrace{E[T]}_{=\text{mean service time}=1}$$

$$T_W = \min\{T_1, T_2, T_3, T_4\}$$

$$\mathbb{P}(T_W < t) = \mathbb{P}(\min\{T_1,T_2,T_3,T_4\} < t) = 1 - \mathbb{P}(\min\{T_1,T_2,T_3,T_4\} > t)$$

$$= 1 - \mathbb{P}\left(\bigcap_{i=1}^{4}\{T_i > t\}\right) = 1 - \prod_{i=1}^{4}\mathbb{P}(T_i > t) = 1 - e^{-4t}$$

$E[T_W] = 1/4 \quad \Rightarrow \quad E[T_{BANK}] = 1.25$ min

Note

$$\mathbb{P}\left(\bigcap_{i=1}^{4}\{\tilde{T}_i < t_i\} \,\Big|\, \bigcap_{i=1}^{4}\{T_i > s_i\}\right) \qquad t_i \geq 0 \quad i=1,2,3,4$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{4}\{T_i - s_i < t_i\} \,\Big|\, \bigcap_{i=1}^{4}\{T_i > s_i\}\right)$$

$$= \frac{\mathbb{P}\left(\bigcap_{i=1}^{4}\{T_i < s_i + t_i\}\right)}{\mathbb{P}\left(\bigcap_{i=1}^{4}\{T_i > s_i\}\right)}$$

$$= \frac{\prod_{i=1}^{4} \mathbb{P}(s_i < T_i < s_i + t_i)}{\prod_{i=1}^{4} \mathbb{P}(T_i > s_i)}$$

$$= \prod_{i=1}^{4} \frac{e^{-\mu s_i} - e^{-\mu(s_i+t_i)}}{e^{-\mu s_i}}$$

$$= \prod_{i=1}^{4} \left(1 - e^{-\mu t_i}\right)$$

$\therefore$ $\{\tilde{T}_i\}_{i=1}^{4}$ are exponential with mean $\frac{1}{\mu}$ and are independent