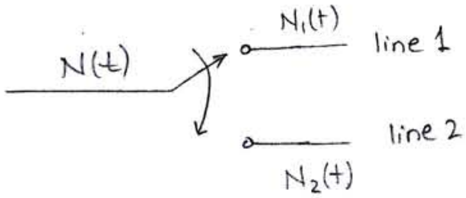
3.11 Let $N(t)$ be the number of arrivals in $[0,t]$.

(a) $N_1(t)$ be the number of arrivals in $[0,t]$ in line 1

$N_2(t)$ be the number of arrivals in $[0,t]$ in line 2



route to line 1 w.p. p
and to line 2 w.p. $1-p$

Note that $N(t) = N_1(t) + N_2(t)$.

$$\mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \mathbb{P}(N_1(t)=n,\ N_2(t)=m,\ N(t)=n+m)$$
$$= \mathbb{P}(N(t)=n+m)\ \mathbb{P}(N_1(t)=n,\ N_2(t)=m \mid N(t)=n+m)$$

$\because$ $N(t)$ is a Poisson process with rate $=\lambda$

$\therefore$ $\mathbb{P}(N(t)=n+m) = \dfrac{(\lambda t)^{n+m}}{(n+m)!} e^{-\lambda t}$

Define $X_i = \begin{cases} 1 & \text{if the } i\text{th arrival is routed to line 1} \\ 0 & \text{if the } i\text{th arrival is routed to line 2} \end{cases}$

Bernoulli R.V.

$\Rightarrow \quad X_i = \begin{cases} 1 & \text{with prob. } p \\ 0 & \text{with prob } 1-p \end{cases}$

$$\mathbb{P}(N_1(t)=n,\ N_2(t)=m \mid N(t)=n+m) = \mathbb{P}\left(\sum_{i=1}^{n+m} X_i = n\right)$$

$\because$ each packet is routed independently

$\therefore$ $X_1, X_2, X_3, \dots$ are independent R.V.s

$\therefore$ $\sum_{i=1}^{n+m} X_i$ has a binomial distribution

i.e. $\mathbb{P}\left(\sum_{i=1}^{n+m} X_i = m\right) = \binom{n+m}{n} p^n (1-p)^m = \dfrac{(n+m)!}{n!\, m!} p^n (1-p)^m$

Therefore

$$\mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \frac{(\lambda t)^{n+m}}{(n+m)!} e^{-\lambda t} \cdot \frac{(n+m)!}{n!\, m!} p^n (1-p)^m$$
$$= \frac{(p\lambda t)^n e^{-p\lambda t}}{n!} \cdot \frac{((1-p)\lambda t)^m e^{-(1-p)\lambda t}}{m!}$$

$$\mathbb{P}(N_1(t)=n) = \sum_{m=0}^{\infty} \mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \frac{(p\lambda t)^n e^{-p\lambda t}}{n!} \rightarrow \text{Poisson (rate} = p\lambda)$$

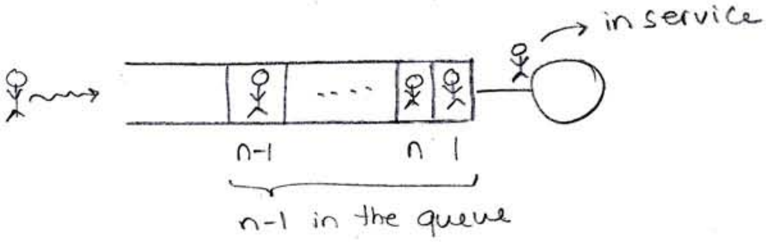
$$\mathbb{P}(N_2(t)=m) = \sum_{n=0}^{\infty} \mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \frac{((1-p)\lambda t)^m e^{-(1-p)\lambda t}}{m!} \rightarrow \text{Poisson (rate} = (1-p)\lambda)$$

$N_1(t)$ and $N_2(t)$ are independent.

(b) "upon" arrival of a customer, the probability of n customers in the system is $p_n = (1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n$

"upon" $\rightarrow$ righ before

n customers in the system $(n \geq 1)$ $\rightarrow$ 1 in service ; $n-1$ in the queue



Let $T_1, T_2, \dots, T_{n-1}$ be the service time of the customers that were in the queue when the new customer arrived

$$T_i : f_{T_i}(t) = \mu e^{-\mu t} \quad t \geq 0$$

Let T_0 be the residual service time of the customer that was in service when the new customer arrived

$$T_0 : f_{T_0}(t) = \mu e^{-\mu t} \quad t \geq 0$$ because of the memoryless property of the exponential distrib.

Let T_W be the waiting time of the newly arriving customer

Given that there are n customers in the system

$$T_W = \sum_{i=0}^{n-1} T_i$$

$T_0, T_1, T_2, \dots, T_{n-1}$ are independent and identically distributed

$\left(\sum_{i=0}^{n-1} T_i\right)$ follows a Gamma distribution $(\alpha = n, \beta = \frac{1}{\mu})$

pdf : $\frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!}$

Note

- $\mu e^{-\mu x} \longleftrightarrow (1 - \frac{j\omega}{\mu})^{-1}$
- $\frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} \longleftrightarrow (1 - j\omega\beta)^{-\alpha}$
- $\Gamma(n+1) = n!$

Note that if there are no customers "upon" arrival, the waiting time is 0 (deterministic)

The system time seen by the newly arriving customer is

$T = T_W + T_S$

$T_S \rightarrow$ service time of the newly arriving customer $(T_S : f_{T_S}(t) = \mu e^{-\mu x} \; x \geq 0)$

Given that there are n customers in the system

$T \sim \text{Gamma}(\alpha = n+1, \beta = \frac{1}{\mu})$ pdf : $\frac{\mu^{n+1} x^n e^{-\mu x}}{n!}$

Let N be the number of customers in the system (right before the new arrival)

$$f_{T_W}(t) = \sum_{n=0}^{\infty} f_{T_W|N}(t|n)\; P\{N=n\} \qquad \text{(total probability)}$$

$$P\{N=n\} = P_n = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$f_{T_W|N}(t|n) = \begin{cases} \delta(t) & n=0 \\ \dfrac{\mu^n t^{n-1} e^{-\mu t}}{(n-1)!} & n \geq 1 \end{cases}$$

$$f_{T_W}(t) = \left(1-\frac{\lambda}{\mu}\right)\delta(t) + \sum_{n=1}^{\infty} \frac{\mu^n t^{n-1} e^{-\mu t}}{(n-1)!} \cdot \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1-\frac{\lambda}{\mu}\right)\delta(t) + \left(1-\frac{\lambda}{\mu}\right) e^{-\mu t}\lambda \underbrace{\sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}}_{e^{\lambda t}}$$

$$\therefore\; f_{T_W}(t) = \left(1-\frac{\lambda}{\mu}\right)\delta(t) + \left(\frac{\lambda}{\mu}\right)(\mu-\lambda)\, e^{-(\mu-\lambda)t} \qquad t \geq 0$$

$$f_T(t) = \sum_{n=0}^{\infty} f_{T|N}(t|n)\; P\{N=n\}$$

$$f_{T|N}(t|n) = \frac{\mu^{n+1} t^n e^{-\mu t}}{n!} \qquad n=0,1,2,\ldots$$

$$f_T(t) = \sum_{n=0}^{\infty} \frac{\mu^{n+1} t^n e^{-\mu t}}{n!} \cdot \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= (\mu-\lambda)\, e^{-\mu t} \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}}_{e^{\lambda t}}$$

$$\therefore\; f_T(t) = (\mu-\lambda)\, e^{-(\mu-\lambda)t} \qquad t \geq 0$$