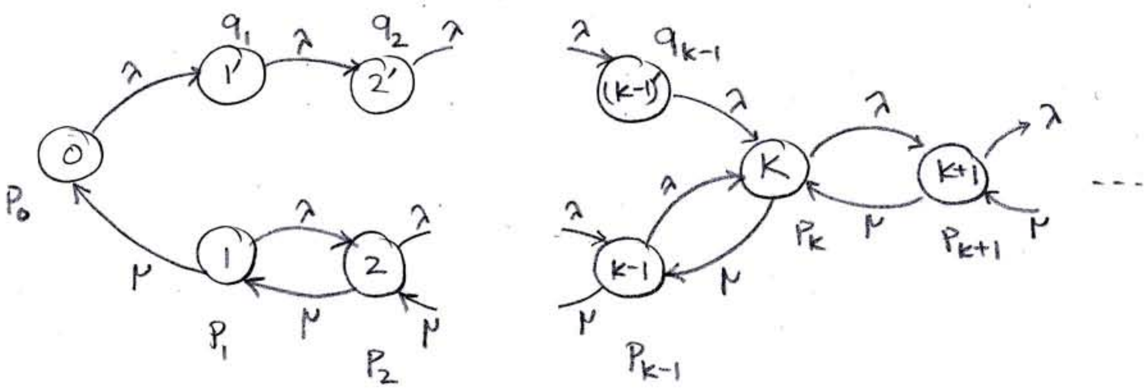
3.15 the system is modeled by the following Markov chain



Global balance equations

$$0: \quad \lambda P_0 = \mu P_1$$

$$1: \quad (\mu+\lambda) P_1 = \mu P_2$$

$$2: \quad (\mu+\lambda) P_2 = \mu P_3 + \lambda P_1$$

$\vdots$

$$k-1: \quad (\mu+\lambda) P_{k-1} = \mu P_k + \lambda P_{k-2}$$

$$k: \quad (\mu+\lambda) P_K = \mu P_{K+1} + \lambda P_{K-1} + \lambda q_{k-1}$$

$$1': \quad \lambda q_1 = \lambda P_0$$

$$2': \quad \lambda q_2 = \lambda q_1$$

$\vdots$

$$(k-1)': \quad \lambda q_{k-1} = \lambda q_{k-2}$$

$$\Rightarrow P_0 = q_1 = q_2 = \dots = q_{k-1}$$

define $\rho = \frac{\lambda}{\mu}$

$$0 \Rightarrow \boxed{P_1 = \rho P_0}$$

$$1 \Rightarrow \mu P_2 = \mu P_1 + \lambda P_1 = \lambda P_0 + \lambda P_1 \quad \Rightarrow \boxed{P_2 = \rho P_0 (1+\rho)}$$

$$2 \Rightarrow \mu P_3 = \mu P_2 + \lambda P_2 - \lambda P_1 = (\lambda P_0 + \lambda P_1) + \lambda P_2 - \lambda P_1$$

$$= \lambda P_0 + \lambda P_2$$

$$P_3 = \rho P_0 + \rho(\rho P_0 (1+\rho)) = \rho P_0 (1+\rho+\rho^2) \quad \Rightarrow \boxed{P_3 = \rho P_0 (1+\rho+\rho^2)}$$

$\vdots$

$$k-1 \Rightarrow \mu P_K = \lambda P_0 + \lambda P_{K-1} \quad \Rightarrow \boxed{P_K = \rho P_0 (1+\rho+\dots+\rho^{K-1})}$$

$$k \Rightarrow \mu P_{K+1} = \mu P_K + \lambda P_K - \lambda P_{K-1} - \lambda q_{K-1}$$

$$= (\lambda P_0 + \lambda P_{K-1}) + \lambda P_K - \lambda P_{K-1} - \lambda P_0$$

$$= \lambda P_K \quad \Rightarrow \boxed{P_{K+1} = \rho P_K}$$

$$\dots \quad \boxed{P_{i+1} = \rho P_i \quad \forall i \geq K}$$

In summary

Note
$$1+\rho+\cdots+\rho^{i-1} = \frac{1-\rho^i}{1-\rho}$$

$$P_i = \rho P_0 (1+\rho+\cdots+\rho^{i-1}) \qquad 1 \le i \le K$$

$$P_i = \rho^{i-K} \times \rho P_0 (1+\rho+\cdots+\rho^{K-1}) \qquad i > K$$

$$q_1 = q_2 = \cdots = q_{K-1} = P_0$$

To find P_0, use: $\displaystyle\sum_{j=1}^{K-1} q_j + \sum_{i=0}^{\infty} P_i = 1$

$$\underbrace{\sum_{j=1}^{K-1} P_0}_{(K-1)P_0} + P_0 + \sum_{i=1}^{K} \rho P_0 \frac{(1-\rho^i)}{1-\rho} + \underbrace{\sum_{i=K+1}^{\infty} \rho^{i-K} \times \rho P_0 \frac{(1-\rho^K)}{1-\rho}} = 1$$

$$= \frac{\rho^2 P_0 (1-\rho^K)}{1-\rho} \left(\sum_{r=0}^{\infty} \rho^r \right)$$

$$= \frac{\rho^2 P_0 (1-\rho^K)}{(1-\rho)^2}$$

$$\sum_{i=1}^{K} \rho P_0 \frac{(1-\rho^i)}{1-\rho} = K \frac{\rho P_0}{1-\rho} - \frac{\rho^2 P_0}{1-\rho} \left(\sum_{r=0}^{K-1} \rho^r \right)$$

$$= K \frac{\rho P_0}{1-\rho} - \frac{\rho^2 P_0}{1-\rho} \left(\frac{1-\rho^K}{1-\rho} \right)$$

Therefore

$$K P_0 + K \frac{\rho P_0}{1-\rho} \cancel{- \frac{\rho^2 P_0}{1-\rho} \left(\frac{1-\rho^K}{1-\rho} \right)} + \cancel{\frac{\rho^2 P_0 (1-\rho^K)}{(1-\rho)^2}} = 1$$

$$P_0 \cdot K \left(1 + \frac{\rho}{1-\rho} \right) = P_0 \frac{K}{1-\rho} = 1 \implies \boxed{P_0 = \frac{1-\rho}{K}}$$

In conclusion

$$P_0 = q_1 = q_2 = \cdots = q_{K-1} = \frac{1-\rho}{K}$$

$$P_i = \frac{\rho}{K}(1-\rho^i) \qquad 1 \le i \le K$$

$$P_i = \rho^{i-K} \cdot \frac{\rho(1-\rho^K)}{K} \qquad i > K$$

The average number in the system:

$$N = \underbrace{\sum_{j=0}^{K-1} j \times \frac{1-\rho}{K}} + \sum_{i=1}^{K} i \, \frac{\rho(1-\rho^{i})}{K} + \underbrace{\sum_{i=K+1}^{\infty} i \rho^{i-K} \frac{\rho(1-\rho^{K})}{K}}$$

$$= \frac{K(K-1)}{2} \cdot \frac{1-\rho}{K}$$

$$= \frac{(K-1)}{2}(1-\rho)$$

$$= \sum_{r=0}^{\infty} (r+K+1)\rho^{r} \frac{\rho^{2}(1-\rho^{K})}{K}$$

$$= \frac{\rho^{2}(1-\rho^{K})}{K}\left[(K+1)\sum_{r=0}^{\infty}\rho^{r} + \sum_{r=0}^{\infty} r\rho^{r}\right]$$

$$= \frac{\rho^{2}(1-\rho^{K})}{K}\left[(K+1)\frac{1}{1-\rho} + \frac{\rho}{(1-\rho)^{2}}\right]$$

$$\sum_{i=1}^{K} i \, \frac{\rho(1-\rho^{i})}{K} = \frac{K(K+1)}{2} \times \frac{\rho}{K} - \frac{\rho}{K}\sum_{i=1}^{K} i \rho^{i}$$

$$= \frac{(K+1)}{2}\rho - \frac{\rho}{K} \times \rho\left[\frac{1-\rho^{K+1}-(K+1)\rho^{K}(1-\rho)}{(1-\rho)^{2}}\right]$$

$$= \frac{(K+1)}{2}\rho - \frac{\rho^{2}}{K}\left[\frac{1-\rho^{K}(\rho+(K+1)(1-\rho))}{(1-\rho)^{2}}\right]$$

$$N = \frac{(K-1)}{2}(1-\rho) + \frac{(K+1)}{2}\rho - \frac{\rho^{2}}{K(1-\rho)^{2}} + \frac{\rho^{2}}{K(1-\rho)^{2}}\left[(K+1)(1-\rho)+\rho\right]$$

$$= \frac{K-1}{2} + \rho + \frac{\rho^{2}}{K(1-\rho)^{2}}\left[-1 + K+1 - K\rho - \rho + \rho\right]$$

$$= \frac{K-1}{2} + \rho + \frac{\rho^{2}K(1-\rho)}{K(1-\rho)^{2}}$$

$$= \frac{K-1}{2} + \rho + \frac{\rho^{2}}{1-\rho}$$

$$= \frac{K-1}{2} + \frac{\rho}{1-\rho}$$

$$\boxed{N = \frac{K-1}{2} + \frac{\rho}{1-\rho}}$$

Note

$$\sum_{i=1}^{K} i\rho^{i} = \rho \frac{d}{d\rho}\left(\frac{1-\rho^{K+1}}{1-\rho}\right)$$

$$\boxed{T = \frac{N}{\lambda}}$$ (Little's thm)

average delay