$\boxed{3.50}$

original M.C.: state space S, transition rates $\{q_{ij}\}$, stationary distrib $\{P_j\}$

truncated M.C.: state space $\bar{S}$, transition rates $\{q_{ij}\}$

(a) Given :

$$P_j \sum_{i \in \bar{S}} q_{ji} = \sum_{i \in \bar{S}} P_i q_{ij} \qquad \forall j \in \bar{S}$$

Divide both sides by $\sum_{k \in \bar{S}} P_k$

$$\left(\frac{P_j}{\sum_{k \in \bar{S}} P_k}\right) \sum_{i \in \bar{S}} q_{ji} = \sum_{i \in \bar{S}} \left(\frac{P_i}{\sum_{k \in \bar{S}} P_k}\right) q_{ij} \qquad \forall j \in \bar{S}$$

Define $\bar{P}_j = \dfrac{P_j}{\sum_{k \in \bar{S}} P_k}$.

We have $\bar{P}_j > 0$ (bec $P_j > 0$) and $\sum_{j \in \bar{S}} \bar{P}_j = 1$, and therefore $\{\bar{P}_j\}$ is a valid probability distribution. Furthermore, $\{\bar{P}_j\}$ satisfy the global balance equations of the truncated M.C.

Hence, $\{\bar{P}_j\}$ is the stationary distribution of the truncated M.C.

(b) If the original M.C. is time reversible, then

$$P_j q_{ji} = P_i q_{ij} \qquad \forall i, j \in S$$

detailed balance equations hold (for original M.C.)

It follows that

$$\sum_{i \in \bar{S}} P_j q_{ji} = \sum_{i \in \bar{S}} P_i q_{ij} \qquad \forall j \in S$$

Thus, we have

$$P_j \sum_{i \in \bar{S}} q_{ji} = \sum_{i \in \bar{S}} P_i q_{ij} \qquad \forall j \in \bar{S}$$

$\therefore$ if the original M.C. is time reversible, the condition in part (a) is satisfied.

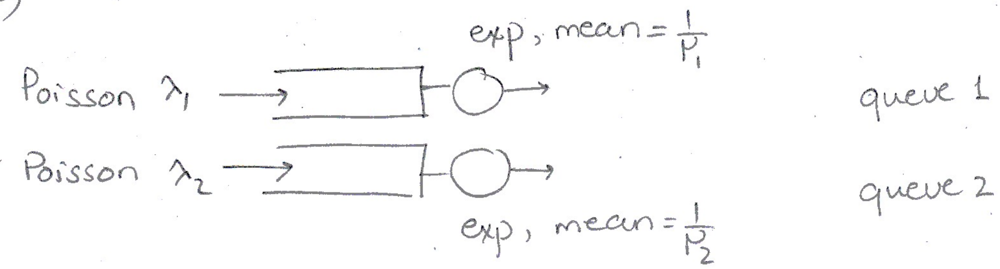
Moreover,

$$P_j q_{ji} = P_i q_{ij} \;\; \forall i,j \in S \Rightarrow P_j q_{ji} = P_i q_{ij} \qquad \forall i,j \in \bar{S} \subset S$$

$$\Rightarrow \bar{P}_j q_{ji} = \bar{P}_i q_{ij} \qquad \forall i,j \in \bar{S}$$

Therefore, the truncated M.C. is time reversible too.

(c)



The queues are independent.

Let N_k be the number of customers in "queue k", $k=1,2$.

Define $\rho_k = \frac{\lambda_k}{\mu_k}$ $\quad k=1,2$.

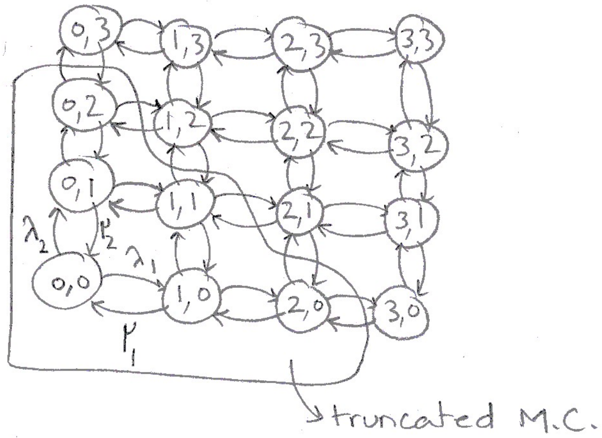
* Without a bound on the number of customers, we have

$$\mathbb{P}(N_1=m) = (1-\rho_1)\,\rho_1^m \qquad m=0,1,\dots$$
$$\mathbb{P}(N_2=n) = (1-\rho_2)\,\rho_2^n \qquad n=0,1,\dots$$

$\because$ the queues are independent

$\therefore$ $\mathbb{P}(N_1=m, N_2=n) = \mathbb{P}(N_1=m)\,\mathbb{P}(N_2=n) = (1-\rho_1)(1-\rho_2)\,\rho_1^m \rho_2^n$, $\quad m=0,1,\dots$; $n=0,1,\dots$

Define the state of the M.C. as (N_1, N_2)



* Now, consider the finite capacity case : $N_1+N_2 \leq B$

the original M.C. is time reversible

$\therefore$ the condition in part (a) is satisfied

$\therefore$ the stationary distribution of the truncated M.C. can be obtained from the stationary distribution of the original M.C. by renormalization. More precisely

$$\mathbb{P}(N_1=m, N_2=n) = \frac{(1-\rho_1)(1-\rho_2)}{K} \cdot \rho_1^m \rho_2^n \qquad \forall (m,n) \in \bar{S}$$

where $\bar{S} = \{(m,n) : m+n \leq B\}$

and $\quad K = \sum_{\substack{m,n:\\ m+n\leq B}} (1-\rho_1)(1-\rho_1)\,\rho_1^m \rho_2^n$