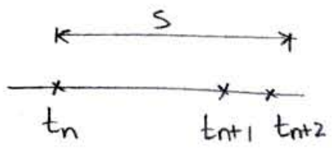
3.10

(a) $\tau_n = t_{n+1} - t_n$

t_{n+1}: (n+1) arrival, t_n: n^{th} arrival



$$\mathbb{P}(\tau_n \leq s) = \mathbb{P}(t_{n+1} - t_n \leq s)$$

$$= \mathbb{P}(\text{at least one arrival occurs in } [t_n, t_n+s])$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(k \text{ arrivals occurs in } [t_n, t_n+s])$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(A(t_n+s) - A(t_n) = k)$$

$$= \sum_{k=1}^{\infty} e^{-\lambda s} \frac{(\lambda s)^k}{k!}$$

$$= 1 - e^{-\lambda s} \qquad (3.11)$$

$$\mathbb{P}(A(t+\delta) - A(t) = 0) = e^{-\lambda\delta} = 1 - \lambda\delta + \sum_{k=2}^{\infty} \frac{(-\lambda\delta)^k}{k!} \qquad \left(\text{Taylor expansion}\right)$$

$$\lim_{\delta \to 0} \frac{1}{\delta} \sum_{k=2}^{\infty} \frac{(-\lambda\delta)^k}{k!} = \lim_{\delta \to 0} \left(\frac{\lambda^2}{2}\delta - \frac{\lambda^3}{3!}\delta^2 + \frac{\lambda^4}{4!}\delta^3 + \cdots \right) = 0$$

$$\therefore \sum_{k=2}^{\infty} \frac{(\lambda\delta)^k}{k!} \text{ is } o(\delta)$$

Hence, $\mathbb{P}(A(t+\delta) - A(t) = 0) = 1 - \lambda\delta + o(\delta) \qquad (3.12)$

$$\mathbb{P}(A(t+\delta) - A(t) = 1) = e^{-\lambda\delta}\lambda\delta = \lambda\delta \underbrace{- (\lambda\delta)^2 + \frac{(\lambda\delta)^3}{2!} - \frac{(\lambda\delta)^4}{3!} + \cdots}_{o(\delta)}$$

$$\therefore \mathbb{P}(A(t+\delta) - A(t) = 1) = \lambda\delta + o(\delta) \qquad (3.13)$$

$$\mathbb{P}(A(t+\delta) - A(t) \geq 2) = 1 - \mathbb{P}(A(t+\delta) - A(t) < 2)$$

$$= 1 - [e^{-\lambda\delta} + e^{-\lambda\delta}\lambda\delta]$$

$$= 1 - (1+\lambda\delta)e^{-\lambda\delta}$$

$$= 1 - (1+\lambda\delta)\left(1 - \lambda\delta + \frac{(\lambda\delta)^2}{2!} - \frac{(\lambda\delta)^3}{3!} + \cdots\right)$$

$$= 1 - \left[1 + \left(\tfrac{1}{2!} - 1\right)(\lambda\delta)^2 + \left(\tfrac{1}{3!} - \tfrac{1}{2!}\right)(\lambda\delta)^3 + \cdots\right]$$

$$= \left(1 - \tfrac{1}{2!}\right)(\lambda\delta)^2 + \left(\tfrac{1}{2!} - \tfrac{1}{3!}\right)(\lambda\delta)^3 + \cdots \qquad (3.14)$$

$$= o(\delta)$$

(b) Let $[t_1, t_1+\tau_1]$ and $[t_2, t_2+\tau_2]$ be two disjoint intervals

let N be the number of arrivals in $[t_1, t_1+\tau_1] \cup [t_2, t_2+\tau_2]$

Let N_1 be the number of arrivals in $[t_1, t_1+\tau_1]$

N_2 be the number of arrivals in $[t_2, t_2+\tau_2]$

$\Rightarrow$ $N_1 = A(t_1+\tau_1) - A(t_1)$

$N_2 = A(t_2+\tau_2) - A(t_2)$

Since $[t_1, t_1+\tau_1]$ and $[t_1, t_2+\tau_2]$ are disjoint, then $N = N_1 + N_2$

$$\mathbb{P}(N=n) = \sum_{k=0}^{n} \mathbb{P}(N_1 = k)\, \mathbb{P}(N_2 = n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda\tau_1} \frac{(\lambda\tau_1)^k}{k!} \cdot e^{-\lambda\tau_2} \frac{(\lambda\tau_2)^{n-k}}{(n-k)!}$$

$$= \frac{e^{-\lambda(\tau_1+\tau_2)}}{n!} (\lambda\tau_2)^n \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \left(\frac{\tau_1}{\tau_2}\right)^k$$

$$= \frac{e^{-\lambda(\tau_1+\tau_2)}}{n!} (\lambda\tau_2)^n \left[1 + \frac{\tau_1}{\tau_2}\right]^n$$

$$= \frac{e^{-\lambda(\tau_1+\tau_2)}}{n!} \left[\lambda(\tau_1+\tau_2)\right]^n$$

(c) $A(t) = A_1(t) + A_2(t) + \cdots + A_k(t)$ $\qquad$ $A_1, A_2, \ldots, A_k$ are independent

$$A(t+\tau) - A(t) = \sum_{i=1}^{k} [\underbrace{A_i(t+\tau) - A_i(t)}_{\sim \text{ Poisson R.V. with mean } (\lambda_i \tau)}]$$

As we saw in part (b), the sum of two independent Poisson R.V.s with means λ_a and λ_b is a Poisson R.V. with mean $\lambda_a + \lambda_b$

By induction, it follows that the sum of k independent Poisson R.V.s with mean $\lambda_1, \ldots, \lambda_k$ is also Poisson with mean $\lambda_1 + \ldots + \lambda_k$

Note: one can also prove that the sum of k independent Poisson R.V.s is also a Poisson R.V. using characteristic functions

$$X_i \sim \mathbb{P}(X_i = k) = \frac{\lambda_i^k e^{-\lambda_i}}{k!} \qquad k = 0, 1, \ldots$$

$j = \sqrt{-1}$

$$E[e^{jsX_i}] = \sum_{k=0}^{\infty} e^{jsk} \frac{\lambda_i^k e^{-\lambda_i}}{k!} = e^{-\lambda_i} \sum_{k=0}^{\infty} \frac{(\lambda_i e^{js})^k}{k!} = e^{-\lambda_i} e^{\lambda_i e^{js}}$$
$$= \exp(-\lambda_i(1 - e^{js}))$$

$X = X_1 + X_2 + \cdots + X_k$, $\{X_i\}$ independent

$$E[e^{jsX}] = E\left[\prod_{i=1}^{k} e^{jsX_i}\right] \underset{\text{independence}}{=} \prod_{i=1}^{k} E[e^{jsX_i}] = \prod_{i=1}^{k} \exp(-\lambda_i(1 - e^{js}))$$
$$= \exp\Big((\underbrace{\sum_{i=1}^{k} \lambda_i}_{\lambda})(1 - e^{js})\Big)$$

$$\therefore \quad \mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad k = 0, 1, 2, \ldots$$

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show $\mathbb{P}(\text{first arrival in } A \text{ comes from } A_1) = \frac{\lambda_1}{\lambda}$

$$\mathbb{P}(1 \text{ arrival from } A_1 \text{ in } [0,t] \mid 1 \text{ arrival from } A \text{ in } [0,t])$$

$$= \frac{\mathbb{P}(1 \text{ arrival from } A_1 \text{ in } [0,t],\ 1 \text{ arrival from } A \text{ in } [0,t])}{\mathbb{P}(1 \text{ arrival from } A \text{ in } [0,t])}$$

A_1 & A are not indep.

$$= \frac{\mathbb{P}(1 \text{ arrival from } A_1 \text{ in } [0,t],\ 0 \text{ arrival from } A_2 \text{ in } [0,t])}{\mathbb{P}(1 \text{ arrival from } A \text{ in } [0,t])}$$

A_1 & A_2 are indep

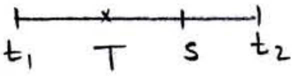
$$= \frac{\mathbb{P}(1 \text{ arrival from } A_1 \text{ in } [0,t])\ \mathbb{P}(0 \text{ arrival from } A_2 \text{ in } [0,t])}{\mathbb{P}(1 \text{ arrival from } A \text{ in } [0,t])}$$

$$= \frac{(\lambda_1 t) e^{-\lambda_1 t} \times e^{-\lambda_2 t}}{(\lambda_1 + \lambda_2) t\, e^{-(\lambda_1 + \lambda_2) t}} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

(d) Let T be the time of the arrival in the interval $[t_1, t_2]$

Let $s \in [t_1, t_2]$

$$\mathbb{P}(T < s \mid 1 \text{ arrival occured in } [t_1, t_2])$$



$$= \frac{\mathbb{P}(1 \text{ arrival in } [t_1, s),\ \text{no arrivals } [s, t_2])}{\mathbb{P}(1 \text{ arrival in } [t_1, t_2])}$$

$$= \frac{\mathbb{P}(1 \text{ arrival in } [t_1, s))\ \mathbb{P}(\text{no arrival in } [s, t_2])}{\mathbb{P}(1 \text{ arrival in } [t_1, t_2])}$$

$$= \frac{\lambda(s - t_2)\, e^{-\lambda(s - t_1)} \times e^{-\lambda(t_2 - s)}}{\lambda(t_2 - t_1)\, e^{-\lambda(t_2 - t_1)}}$$

$$= \frac{s - t_2}{t_2 - t_1}$$