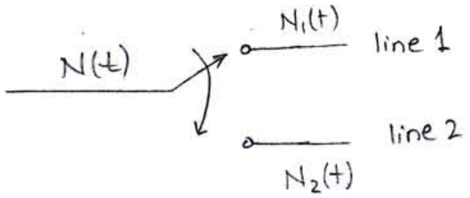
$\boxed{3.11}$ Let $N(t)$ be the number of arrivals in $[0,t]$.

(a) $N_1(t)$ be the number of arrivals in $[0,t]$ in line 1

$N_2(t)$ be the number of arrivals in $[0,t]$ in line 2



route to line 1 w.p. p
and to line 2 w.p. $1-p$

Note that $N(t) = N_1(t) + N_2(t)$.

$$\mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \mathbb{P}(N_1(t)=n,\ N_2(t)=m,\ N(t)=n+m)$$
$$= \mathbb{P}(N(t)=n+m)\ \mathbb{P}(N_1(t)=n,\ N_2(t)=m \mid N(t)=n+m)$$

$\because$ $N(t)$ is a Poisson process with rate $=\lambda$

$\therefore$ $\mathbb{P}(N(t)=n+m) = \dfrac{(\lambda t)^{n+m}}{(n+m)!} e^{-\lambda t}$

Define $X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ arrival is routed to line 1} \\ 0 & \text{if the } i^{th} \text{ arrival is routed to line 2} \end{cases}$

Bernoulli R.V.

$\Rightarrow \quad X_i = \begin{cases} 1 & \text{with prob. } p \\ 0 & \text{with prob } 1-p \end{cases}$

$$\mathbb{P}(N_1(t)=n,\ N_2(t)=m \mid N(t)=n+m) = \mathbb{P}\left(\sum_{i=1}^{n+m} X_i = n\right)$$

$\because$ each packet is routed independently

$\therefore$ $X_1, X_2, X_3, \dots$ are independent R.V.s

$\therefore$ $\sum_{i=1}^{n+m} X_i$ has a binomial distribution

i.e. $\mathbb{P}\left(\sum_{i=1}^{n+m} X_i = m\right) = \binom{n+m}{n} p^n (1-p)^m = \dfrac{(n+m)!}{n!\, m!} p^n (1-p)^m$

Therefore

$$\mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \frac{(\lambda t)^{n+m}}{(n+m)!} e^{-\lambda t} \cdot \frac{(n+m)!}{n!\,m!} p^n (1-p)^m$$
$$= \frac{(p\lambda t)^n e^{-p\lambda t}}{n!} \cdot \frac{((1-p)\lambda t)^m e^{-(1-p)\lambda t}}{m!}$$

$$\mathbb{P}(N_1(t)=n) = \sum_{m=0}^{\infty} \mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \frac{(p\lambda t)^n e^{-p\lambda t}}{n!} \longrightarrow \text{Poisson (rate} = p\lambda)$$

$$\mathbb{P}(N_2(t)=m) = \sum_{n=0}^{\infty} \mathbb{P}(N_1(t)=n,\ N_2(t)=m) = \frac{((1-p)\lambda t)^m e^{-(1-p)\lambda t}}{m!} \longrightarrow \text{Poisson (rate} = (1-p)\lambda)$$

$N_1(t)$ and $N_2(t)$ are independent.