Swing-up and balancing of an inverted pendulum on a 2-D plannar quadrotor

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Abstract— **In this paper, we presents a hybrid controller for swing-up and balancing of an inverted single and double pendulum on a flying 2-D quadrotor platform. The objective of swing-up and balancing of an inverted pendulum on a quadrotor was chosen in order to demonstrate our understanding and capability of linear and nonlinear control on general dynamic systems. The dynamics of the models for both single and double pendulum cases are analyzed based on Lagrangian dynamics analysis method. The analyzed dynamics models were utilized and verified on our dynamic simulator based on *PyDrake* environment. For swing-up and stabilization maneuvers, we developed two separate controllers for each purpose. For stabilization, we implement LQR controller based on the linearized dynamics at the upright state. For swing-up controller, we developed a custom trajectory planning controller based on the intuition from a a non-linear behavior of a quadrotor dynamics. In result, we successfully swung up and stabilized a single and double inverted pendulum on a quadrotor using both the proposed hybrid controller and the sole LQR controller. We compared the efficacy of the proposed hybrid controller to the sole LQR controller, and argued the proposed controller showed improved response of the system.**

I. INTRODUCTION

In recent, control of a modern quadrotor has been an eminent research field. With advancements of electric motor and microcontroller technologies, maneuverability of a quadrotor has been rapidly expanding. However, although it gains more thrust and computing power, advanced and high-speed maneuver of a quadrotor still requires advanced control algorithm to effectively navigate under the nonlinear dynamics and disturbances. Regarding this, many researchers have been working on trajectory planning and stabilization methodologies in extreme quadrotor manuevers. For instance, [1] suggested simple yet powerful paradigm of generating smooth trajectory for a high-speed quadrotor manuever using differential flatness theory. On top of these approaches, several researchers have been working on a quadrotor maneuver with dynamic constraints or conditions. To describe, [2] presented a methodology to optimize quadrotor maneuver with mathematical programming while a quadrotor is cable-suspended with payload. For demonstrating control with extreme dynamic constraints or conditions, [3], [4] studied an approach to stabilize inverted pole around upright state with linear control theory and designed throw and catch maneuver using mathematical optimization. [5] also demonstrated another approach to stabilize inverted pendulum with bilinear system approximation. These researches collectively push the boundary to improve an efficacy of a quadrotor maneuver with the advancement of nonlinear control theory. Along with this trend, we also wanted to develop and implement a special purpose quadrotor controller with a specific dynamic condition. Regarding selection of a dynamic condition that we want to apply on a quadrotor, we wanted to demonstrate a concept of non-linear control theory through our controller.

Meanwhile, unlike a quadrotor control research field (where is an eminent research field), a control of inverted pendulum is one of the most classical and representative example in a research field of the non-linear control theory. In general, control of an inverted pendulum can be divided into two sub-topics: 1) Stabilization, and 2) Swing-up. Stabilization maneuver of a pendulum around upright state is a representative measure for demonstrating controller's robustness, while Swing-up maneuver of a pendulum shows efficacy of non-linear trajectory planning capability. Many researchers have been understood, developed, and demonstrated non-linear control theories on this example from several decades ago to even nowadays. To be specific, [6], [7] demonstrated swing-up and and balancing of a single inverted pendulum while analyzing and utilizing non-linear characteristics and behavior. Similarly [8] showed swing-up and balancing of a double pendulum with non-linear analysis. [9] demonstrated a double pendulum control on a cart with an experiment, and [10] applied model predictive control paradigm on swing-up and balancing of a double inverted pendulum. Also, diverged from inverted pendulum, [11] showed interesting applications of a moving cube driven by an inverted active

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inertial wheel pendulum. Since an inverted pendulum has been widely used to demonstrate the efficacy of the non-linear control theory with the underactuated system, we decided to combine the idea of a quadrotor maneuver control and an inverted pendulum. We ranged our inverted pendulum problems not only stabilizing but also swing-up. We also decided to implement both single and double inverted pendulum on a quadrotor since there exists several works that stabilized a single inverted pendulum.

To summarize, we defined our goal as swing-up and stabilization of single and double pendulum on a flying 2-D quadrotor. The figure 1 and the figure 2 shows the goal of the project with simplified models of the systems.

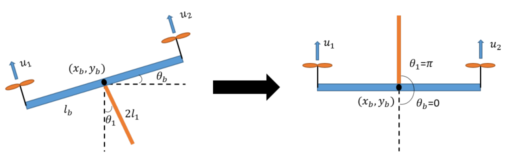

Fig. 1. Single inverted pendulum on a flying 2-D planar quadrotor

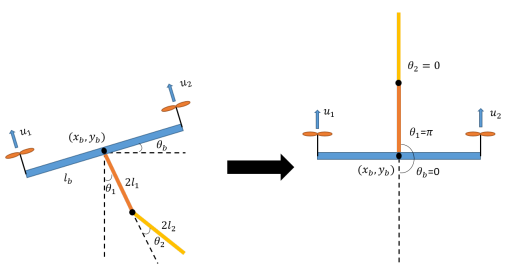

Fig. 2. Double inverted pendulum on a flying a 2-D planar quadrotor

II. SYSTEM MODELING

A. Assumption

For modeling of the proposed system, we applied several assumptions to the quadrotor with single and double pendulum in order to simplify the problem while maintaining realistic condition:

- the quadrotor is only maneuvering on 2-D plane
- the quadrotor body is vertically symmetric
- two vertical thrusts located at each end of the body has no hysteresis and bi-directional
- body is allowed to be flipped
- A maximum thrust level of each rotor is limited to its 1x to 1.5x of the total weights
- there exists no friction or contact between masses

B. Dynamics Analysis

Based on the model shown in the figure 1 and the figure 2, Lagrangian method was used to derive the manipulator equations of these system. In the case of single pendulum, position and angle of body, and angle of pendulum are used as a state $(x_b, y_b, \theta_b, \theta_1)$. Similarly, for the case of double pendulum, we added one more state, angle of another pendulum $(x_b, y_b, \theta_b, \theta_1, \theta_2)$. Both system only have two control inputs (u_1, u_2) which correspond to the two thrusts of the rotors.

Due to these systems' unique geometry, angular acceleration of body only depends on the body torque which is the difference between two control inputs. The remaining state variables are coupled with each other and controlled by the total thrust which is the sum of two control inputs. This intuition allows us to design the swing-up controller by building two separate feedback loops for the body torque control and the total thrust control.

A detailed derivation of the manipulator equations is attached on the Appendix.

C. Simulation Environment

Before designing the control scheme of the system, it is essential to build a simulation framework that we can apply the control scheme in an iterative manner. Thus, based on the manipulator equations in the previous subsection II-B, we developed the simulation environment with the proposed physical system. We transformed these equations into generalized matrix form of dynamics and implement them on the *PyDrake* environment. Also visualization environment of the system was implemented with the *PyDrake* PyPlotVisualizer. The figure 3 and the figure 4 show snap shots of a simulation environment.

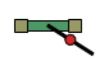

Fig. 3. Simulation environment of a single pendulum on a flying 2-D quadrotor based on *PyDrake* environment.

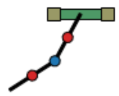

Fig. 4. Simulation environment of a double pendulum on a flying 2-D quadrotor based on *PyDrake* environment.

III. CONTROLLER

A. Stabilization Controller

We proposed the hybrid controller for swing-up and stabilization of the inverted pendulum. For stabilization, we used the Linear Quadratic Regulator(LQR). The linearized dynamics around the arbitrary state was derived from the Taylor expansion of the equation of motion. The symbolic differentiation in *Matlab* is utilized to get first order coefficients of Taylor expansion. Not only for LQR formulation, this general linearization of system dynamics also enabled implementation of other trajectory optimization approach such as iterative LQR. The detailed equations of the dynamics linearization is attached on Appendix.

Utilizing the linearized dynamics, we formulated the Riccati equation around the upright state where the pendulum is inverted. *PyDrake* LQR solver was used to solve these equations for both single and double inverted pendulum.

B. Swing-up Controller

The swing-up controller was formulated based on the intuition attained from observation of the behavior of the LQR controller and the analysis of dynamics model. To describe, before the development of swing-up controller, the LQR controller's swing-up capability was tested and evaluated. Unexpectedly, the LQR controller was capable of swing-up and stabilize pendulum from the resting state with initial unstable behavior. As one peculiar behavior of the LQR controller in swing-up phase, the controller accelerated the quadrotor to downward. As the quadrotor accelerated downward, the pendulum gets the reverse torque to stand it upright. Under this negative gravity, a small perturbation on the angle of pendulum resulted the swing-up of the pendulum. With this mechanism, the sudden drop of the quadrotor at the beginning of LQR controller based swing-up can be explained as a the procedure towards the local minimum. This phenomena was also observed when executing other trajectory optimization techniques which converge to this local minimum and keep dropping the quadrotor.

Also, on top of the downward accelerating phenomena, the LQR controller also presented one other critical behavior. As the pendulum rised, the LQR controller quickly tried to set the body perpendicular to the rising pendulum and started stabilization routine while maintaining a perpendicular state.

With these observations, while preserving the core nature of swing-up maneuver, the new swing-up controller was designed in order to prevent sudden drop of the body. As mentioned in the subsection II-B, the intuition from the geometry of the system dynamics were used to separate control process into two main component; the total thrust control and the torque control.

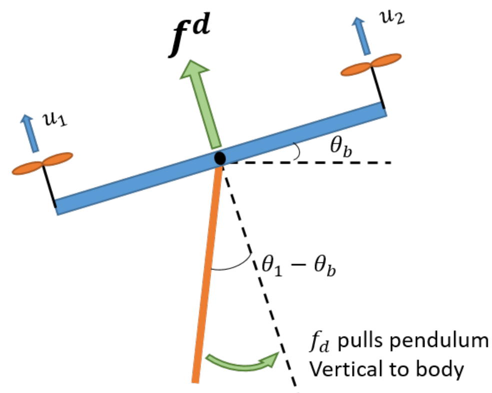

Fig. 5. Linear acceleration control to guide the system towards the desired state while keeping the pendulum perpendicular to the body

$$
\begin{aligned}
f^d &= (\ddot{x}_b^d \hat{x} + \ddot{y}_b^d \hat{y}) \\
&= -K_1(\theta_1 - \theta_{1f}) - K_2\dot{\theta}_1 \\
f^d \cos\theta_b &= \ddot{y}_b \\
&= (u_1 + u_2)\cos\theta_b - g \\
u_1 + u_2 &= f^d + \frac{g}{\cos\theta_b}
\end{aligned}
\tag{1}
$$

The first component of the controller controls the total thrust and a magnitude of the desired body center-of-mass(COM) acceleration. The figure 5 and the equation 2 explains the design details of this controller component. The desired acceleration of the body was derived based on the PD feedback along the he perturbation of θ_1. Based on the derived desired acceleration, the desired total thrust was calculated with consideration of the gravity compensation. This feedback process provides robustness between the position of the body and the pendulum being perpendicular while in maneuvering the body from initial state to desired state.

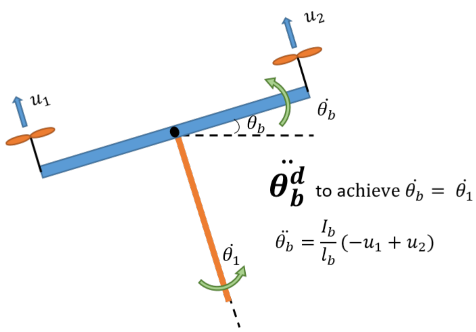

Fig. 6. Torque control to stabilize relative angular velocity

$$
\begin{aligned}
\ddot{\theta}_b^d &= -K_3(\dot{\theta}_b - \dot{\theta}_1) \\
\ddot{\theta}_b^d &= \ddot{\theta}_b = \frac{l_b}{I_b}(-u_1 + u_2) && (2) \\
-u_1 + u_2 &= \frac{I_b}{l_b}\ddot{\theta}_b^d && (3)
\end{aligned}
$$

The second component of the controller controls a torque applied to the body and its resultant angular acceleration. The figure 6 and the equation 3 show the conceptual idea of the body torque control. To explain, the desired acceleration was designed based on the feedback loop over the relative angular velocity between the body and the pendulum. The desired torque was then back-calculated based on the desired acceleration. Intuitively, the feedback loop in this controller component keeps the relative angular velocity between the pendulum and body as zero to stabilize the system. As a result, this process provided a smooth transition between the swing-up controller and the LQR controller while pulling towards the state space region that LQR controller functions more effectively.

$$
\begin{aligned}
u_1 =& \frac{1}{2}\big(- K_1(\theta_1 - \theta_{1f}) - K_2\dot{\theta}_1 + \frac{g}{\cos\theta_b} \\
& + K_3\frac{I_b}{l_b}(\dot{\theta}_b - \dot{\theta}_1)\big) \\
u_2 =& \frac{1}{2}\big(- K_1(\theta_1 - \theta_{1f}) - K_2\dot{\theta}_1 + \frac{g}{\cos\theta_b} \\
& - K_3\frac{I_b}{l_b}(\dot{\theta}_b - \dot{\theta}_1)\big) && (4)
\end{aligned}
$$

By fusing these two controller components, the control inputs were generated as shown in the equation 4. The switching between the swing-up controller and the LQR based stabilization controller was formulated based on the value function of the LQR controller. In order to maximize the efficacy of the LQR based stabilization controller, we switched from the swing-up controller to the LQR controller when this value functions reached to its minimal value. Since the region of attraction estimated with S-procedure is too small, Lyapunov stability bounds were not chosen as a switching criteria.

IV. RESULTS

Based on the dynamics and the simulation environment, we verified the efficacy of the LQR controller and the proposed hybrid controller. The result of the simulation for swing-up and balancing of a single pendulum is shown in the figure 7, and the result for a double pendulum case is shown in the figure 8.

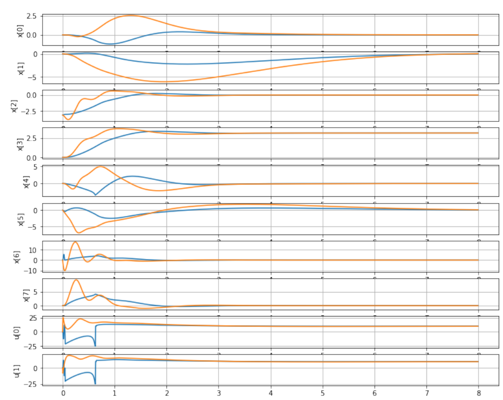

Fig. 7. Simulation result of a single pendulum swing-up and balancing on a quadrotor. $x = [x_b, y_b, \theta_b, \theta_1, \dot{x}_b, \dot{y}_b, \dot{\theta}_b, \dot{\theta}_1]^T$. Orange line represents the result from sole LQR controller. Blue line represents the result from the proposed hybrid controller.

In the figure 7, the trajectories of the state variables of the system with the proposed controller showed more smooth trajectories compared to the trajectories with the sole LQR controller. Also, the proposed hybrid controller generated less fluctuation in the first order derivative terms $(\dot{x_b}, \dot{y_b}, \dot{\theta_b}, \dot{\theta_1})$ and smaller total displacement of each position term $(x_b, y_b, \theta_b, \theta_1)$. In the figure 8, similar tendencies of the trajectories were also presented for a double pendulum swing-up and stabilization case.

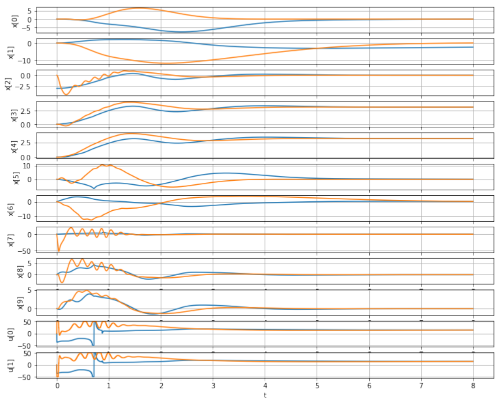

Fig. 8. Simulation result of a double pendulum swing-up and balancing on a quadrotor. $x = [x_b, y_b, \theta_b, \theta_1, \theta_2, \dot{x_b}, \dot{y_b}, \dot{\theta_b}, \dot{\theta_1}, \dot{\theta_2}]^T$ Orange line represents the result from sole LQR controller. Blue line represents the result from the proposed hybrid controller.

V. DISCUSSION

In this report, we presented a hybrid quadrotor controller for swing-up and balancing of a single and double inverted pendulum. We believe that our results implies that simple controller inspired by the intuition from dynamics analysis can effectively control highly nonlinear dynamic system without any complicated nonlinear trajectory optimization technique.

For the future works, this project can be extended to a real quadrotor in the 3-D environment. In order to extend the project to the real world, several assumptions have to be redefined, and the controller needs to be re-evaluated based on the refined assumptions. For instance, hysteresis on the rotor thrust with unidirectional capability could be critical assumptions for real world application deployment. Additionally, friction and contact model between the body and the pendulum has to be considered. We expected that the proposed hybrid controller paradigm for swing-up and balancing controller should be applicable even on the real world environment.

In the process of the project, other several approaches such as iterative LQR(iLQR) and LQR-Trees [12] were attempted to optimize swing-up trajectory. Nevertheless, our attempts of using these approaches were not successful for generating optimal trajectory due to the couple of rationales. We suspected that the first reason of failed attempts was that the dimension of state space of our system is higher than the dimension which an optimization tool can handle, which often refers as the curse of dimensionality phenomena. The second potential reason was that there exists discontinuities on the state space of the proposed system. The angle of body and the angle of pendulum is confined and winded on the limited range ($\theta \in [0, 2\pi]$). In order to utilize the trajectory optimization paradigm, methodologies to effectively unwind this limit or to replace the angle with triangular parameterization were required, but both approaches caused the inefficiency of the algorithm. To be specific, these methodologies were once implemented, but ineffciency caused by these methodologies ultimately caused numerically unstable and degraded convergence performance. This is due to the nature of the goal of the project requiring relatively long time horizon optimization with small time step discretization.

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APPENDIX

A. Linear Control Formulation

$$
\begin{aligned}
&\bar{q} = q - q_f, \quad \dot{\bar{q}} = \dot{q} - \dot{q}_f, \quad \ddot{\bar{q}} = \ddot{q} - \ddot{q}_f, \quad \bar{u} = u - u_f \\
&x = [\bar{q} \;\; \dot{\bar{q}}]^T \\
&\ddot{q} = \mathcal{M}(q)^{-1}\big(- \mathcal{C}(q, \dot{q}) + \mathcal{T}_g(q) + \mathcal{B}(q)u \big) \\
&\quad \approx \ddot{q}_f + \Psi_0(q_f, \dot{q}_f, u_f)\bar{q} + \Psi_1(q_f, \dot{q}_f, u_f)\dot{\bar{q}} + \Psi_2(q_f, \dot{q}_f, u_f)\bar{u} \\
&\ddot{\bar{q}} \approx \Psi_0 \bar{q} + \Psi_1 \dot{\bar{q}} + \Psi_2 \bar{u} \\
&\dot{x} \approx \begin{bmatrix} 0 & I \\ \Psi_0 & \Psi_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ \Psi_3 \end{bmatrix} \bar{u} \\
&\dot{x} \approx Ax + B\bar{u}
\end{aligned}
$$

B. Lagrangian formulation

$$\mathcal{L} = T - V$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = F_i$$

$$\mathcal{M}(q)\ddot{q} + \mathcal{C}(q, \dot{q}) = \mathcal{T}_g(q) + \mathcal{B}(q)u$$

C. Dynamics of single pendulum on the 2-D quadrotor

$$
\begin{aligned}
T &= \frac{1}{2}m_b(\dot{x_b}^2 + \dot{y_b}^2) + \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}I_b\dot{\theta_b}^2 + \frac{1}{2}I_1\dot{\theta_1}^2 \\
V &= m_b g y_b + m_1 g y_1 + m_2 g y_2 \\
x_1 &= x_b + l_1 \sin\theta_1, \quad y_1 = x_b - l_1 \cos\theta_1 \\
I_b &= \frac{1}{3}m_b l_b^2, \quad I_1 = \frac{1}{3}m_1 l_1^2 \\
q &= [x_b, y_b, \theta_b, \theta_1]^T
\end{aligned}
$$

$$
\mathcal{M}(q) = \begin{bmatrix} m_b + m_1 & 0 & 0 & m_1 l_1 \cos\theta_1 \\ 0 & m_b + m_1 & 0 & m_1 l_1 \sin\theta_1 \\ 0 & 0 & I_b & 0 \\ m_1 l_1 \cos\theta_1 & m_1 l_1 \sin\theta_1 & 0 & I_1 + m_1 l_1^2 \end{bmatrix}
$$

$$
\mathcal{C}(q, \dot{q}) = \begin{bmatrix} -m_1 l_1 \sin\theta_1 \dot{\theta_1}^2 & m_1 l_1 \cos\theta_1 \dot{\theta_1}^2 & 0 & 0 \end{bmatrix}^T
$$

$$
\mathcal{T}_g(q) = \begin{bmatrix} 0 & -(m_b + m_1)g & 0 & -m_1 l_1 g \sin\theta_1 \end{bmatrix}^T
$$

$$
\mathcal{B}(q) = \begin{bmatrix} -\sin\theta_b & -\sin\theta_b \\ \cos\theta_b & \cos\theta_b \\ -l_b & l_b \\ 0 & 0 \end{bmatrix}
$$

D. Dynamics of double pendulum on the 2-D quadrotor

$$
\begin{aligned}
T &= \frac{1}{2}m_b(\dot{x_b}^2+\dot{y_b}^2)+\frac{1}{2}m_1(\dot{x_1}^2+\dot{y_1}^2)+\frac{1}{2}m_2(\dot{x_2}^2+\dot{y_2}^2)+\frac{1}{2}I_b\dot{\theta_b}^2+\frac{1}{2}I_1\dot{\theta_1}^2+\frac{1}{2}I_2\dot{\theta_2}^2\\
V &= m_b g y_b + m_1 g y_1 + m_2 g y_2\\
x_1 &= x_b + l_1\sin\theta_1, \quad y_1 = x_b - l_1\cos\theta_1\\
x_2 &= x_b + 2l_1\sin\theta_1 + l_2\sin\theta_2, \quad y_2 = x_b - 2l_1\cos\theta_1 - l_2\cos\theta_2\\
q &= [x_b, y_b, \theta_b, \theta_1, \theta_2]^T
\end{aligned}
$$

$$
\mathcal{M}(q) = \begin{bmatrix}
m_b+m_1+m_2 & 0 & 0 & (m_1+2m_2)l_1\cos\theta_1 & m_2l_2\cos\theta_2\\
0 & m_b+m_1+m_2 & 0 & (m_1+2m_2)l_1\sin\theta_1 & m_2l_2\sin\theta_2\\
0 & 0 & I_b & 0 & 0\\
(m_1+2m_2)l_1\cos\theta_1 & (m_1+2m_2)l_1\sin\theta_1 & 0 & I_1+m_1l_1^2+4m_2l_1^2 & 2m_2l_1l_2\cos(\theta_1-\theta_2)\\
m_2l_2\cos\theta_2 & m_2l_2\sin\theta_2 & 0 & 2m_2l_1l_2\cos(\theta_1-\theta_2) & I_2+m_2l_2^2
\end{bmatrix}
$$

$$
\mathcal{C}(q,\dot{q}) = \begin{bmatrix}
-(m_1+2m_2)l_1\sin\theta_1\dot{\theta_1}^2 - m_2l_2\sin\theta_2\dot{\theta_2}^2\\
(m_1+2m_2)l_1\cos\theta_1\dot{\theta_1}^2 + m_2l_2\cos\theta_2\dot{\theta_2}^2\\
0\\
2m_2l_1l_2\sin(\theta_1-\theta_2)\dot{\theta_2}^2\\
-2m_2l_1l_2\sin(\theta_1-\theta_2)\dot{\theta_1}^2
\end{bmatrix}
$$

$$
\mathcal{T}_g(q) = \begin{bmatrix}0 & -(m_b+m_1+m_2)g & 0 & -(m_1+2m_2)gl_1\sin\theta_1 & -m_2gl_2\sin\theta_2\end{bmatrix}^T
$$

$$
\mathcal{B}(q) = \begin{bmatrix}
-\sin\theta_b & -\sin\theta_b\\
\cos\theta_b & \cos\theta_b\\
-l_b & l_b\\
0 & 0\\
0 & 0
\end{bmatrix}
$$