(1) Lecture [redacted]

Direct N-Body Simulations [redacted]

Sep. 27-29, 2021, ARI/ZAH via zoom

R. Spurzem

Topics: 1) General Features 2) Data Structure
3) Program Structure I 4) Run code I
5) Output Data 6) Progr. Structure II
7) Run the code II

First important Keywords:

- collisional — collisionless stellar system

Relaxation time $t_{rx} \propto \frac{v^3}{m\rho}$; v = r.m.s. velocity dispersion, ρ = density, m = average star mass

+ low v, high ρ, m $\Rightarrow$ short t_{rx}

- gravothermal stellar system

$\Downarrow$ if $t_{rx} \lesssim$ age of system

collisional (this word does not mean physical collision, it mean grav. encounter, 2-body)

Examples:

collisional: globular / young dense / nuclear / open star clusters

collisionless: galaxies (bulge / disk / halo) dark matter particles in cosmological N-body models

Example of time scales (very approximate)

	open $*$-cl.	glob. $*$-cl	gal. disk
t_{rx}	$10^4 - 10^6$	10^{8-9}	$10^{>10}$
t_{dyn} (orbital time of star in system)	10^4	10^6	10^8

Notice: open $*$-cl : $t_{rx} \approx t_{dyn}$; glob. $*$-cl.: $t_{rx} \gg t_{dyn}$

Examples of floating point operations (flop) needed:

globular cluster: $N = 10^6$, 100 steps per orbit, 10^4 orbits*

$\sim 10^{12}$ steps

$\sim$ 100 flop per grav. force comp. (Hermite scheme use $\vec{a}, \dot{\vec{a}}$) $\sim 10^{14} \cdot N$ flop

$\boxed{\sim 10^{20} \text{ flop}}$

cosmological N-body: $N = 10^{10}$, 100 steps p. orbit, 100 orbits

$\sim 10^{14}$ steps

$\sim$ 20 flop per grav. force comp. (use only $\vec{a}$) $\sim 2 \cdot 10^{15} \sqrt{N}$ flop**

$\boxed{\sim 2 \cdot 10^{20} \text{ flop}}$

SIMILAR

*Number of orbits $\propto t_{rx} / t_{dyn}$

**Use $\sqrt{N}$, because approximate codes (like Tree code) do not use full N loops; very rough!

The issue about accuracy

- Simulation uses $10^6 - 10^{10}$ steps; N force calculations per step; $N = 10^6 \sim 10^{12} - 10^{16}$ force calculations (for pairs of particles)
- Double Precision accuracy $\sim 10^{-13}$

If all errors sum up in a bad way, final error ~ 1 (unity)

In real system not so serious:

1) errors are often uncorrelated, grow much slower
2) controlled by check of globally conserved quantities, like E energy, L angular mom.
3) Miller, R. 19 64: exponential divergence of orbits with small initial differences N-body meaningless? See also Quinlan+ 1992 Tremaine-
   In fact it is shown that this effect simulates the known deterministic chaos in N-body somehow (no strict proof)
4) comparison with statistical physics usually very good $f = f(\vec{r}, \vec{v}, t)$, Boltzmann + Fokker-Planck Equations, see Cohn + Kulsrud 78, Cohn 1980, Einsel + Spurzem 99, Takahashi + Lee 2000 (but no s.ev., no binaries)

General Features in the code

Hermite Scheme
+ hierarchically blocked
time steps
(4th order, 2 time points)

Makino + Aarseth 92
(earlier: McMillan 87)

NBODY 4, 6, 6++, 7

Ahmad- Cohen Neighbour
Scheme

Ahmad + Cohen 73

NBODY, 2, 4, 5, 6, 6++, 7

Regularizations
(2-body KS, Chain)

Kustaanheimo + Stiefel 65
Mikkola + Aarseth 90, 93, 96

NBODY 3, 4, 5, 6, 6++, 7

Main Authors: Sverre Aarseth, Seppo Mikkola

Competing Codes: Kira Starlab, φGRAPE / GPU
HiGPU, new code Japan

More authors: Jarrod Hurley, Rainer Spurzem, Long WangMo

(5) **Hermite Scheme – how it works**

acceleration $\vec{a}_i$; force $\vec{F}_i = m_i \vec{a}_i$

particle i; all other particles j

$\dot{} = d/dt \quad \vec{v}_i = \dot{\vec{r}}_i$

Newton's Law:

$$\vec{a}_i = -G \sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3} \; ; \quad \dot{\vec{a}}_i = -G \sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \left[\frac{\vec{v}_i - \vec{v}_j}{|\vec{r}_i - \vec{r}_j|^3} - 3 \frac{\{(\vec{r}_i - \vec{r}_j)(\vec{v}_i - \vec{v}_j)\}(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^5} \right]$$

Note: $|\vec{r}_i - \vec{r}_j|^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$; $\vec{r} = (x, y, z)$

Low Order Taylor series for prediction

$$\vec{r}_p(t_1) = \vec{r}_0 + \vec{v}_0 \Delta t + \frac{1}{2} \vec{a}_0 \Delta t^2 + \frac{1}{6} \dot{\vec{a}}_0 \Delta t^3 + \mathcal{O}(\Delta t^4)$$

$$\vec{v}_p(t_1) = \vec{v}_0 + \vec{a}_0 \Delta t + \frac{1}{2} \dot{\vec{a}}_0 \Delta t^2 + \mathcal{O}(\Delta t^3)$$

$t_1 = t_0 + \Delta t$, $\vec{r}_0 = \vec{r}_i(t_0)$, $\vec{v}_0 = \vec{v}_i(t_0)$, ... $\vec{a}_0 = \vec{a}_i(t_0)$, $\dot{\vec{a}}_0 = \dot{\vec{a}}_i(t_0)$

In the following: index i dropped for better notation.

(6) Hermite Scheme cont'd

Compute $\vec{a}_1 = \vec{a}(t_1)$; $\mathring{\vec{a}}_1 = \mathring{\vec{a}}(t_1)$ in two ways:

1) Newton's Law: $\vec{a}_1 = -G \sum_{\substack{j=1 \\ j \neq i}}^{N} m_j \frac{\vec{r}_p - \vec{r}_{pj}}{|\vec{r}_p - \vec{r}_{pj}|^3}$ and $\mathring{\vec{a}}_1 = \cdots$ use $\vec{r}_p, \vec{v}_p, \vec{r}_{pj}, \vec{v}_{pj}$

2) Another Taylor Series for acceleration:

$$\left.\begin{aligned} \vec{a}_1 &= \vec{a}_0 + \mathring{\vec{a}}_0 \Delta t + \frac{1}{2} a_0^{(2)} \Delta t^2 + \frac{1}{6} a_0^{(3)} \Delta t^3 + \mathcal{O}(\Delta t^4) \\ \mathring{\vec{a}}_1 &= \mathring{\vec{a}}_0 + a_0^{(2)} \Delta t + \frac{1}{2} a_0^{(3)} \Delta t^2 + \mathcal{O}(\Delta t^3) \end{aligned}\right\} (xx)$$

(xx) is: two equations, linear, two unknowns $a_0^{(2)}, a_0^{(3)}$ $\Longrightarrow$

$$a_0^{(2)} = \left[-3 (\vec{a}_0 - \vec{a}_1) - (2 \mathring{a}_0 + \mathring{a}_1) \Delta t \right] \frac{2}{\Delta t^2}$$

$$a_0^{(3)} = \left[2 (\vec{a}_0 - \vec{a}_1) + (\mathring{a}_0 + \mathring{a}_1) \Delta t \right] \frac{6}{\Delta t^3}$$

Just check it $\Longleftarrow$ two linear eqs. (xx)!!

7

Hermite Scheme cont'd

Correction Step:

$$\vec{r}_c(t_1) = \vec{r}_p(t_1) + \frac{1}{24}\vec{a}_0^{(2)}\Delta t^4 + \frac{1}{120}\vec{a}_0^{(3)}\Delta t^5 + O(\Delta t^6)$$

$$\vec{v}_c(t_1) = \vec{v}_p(t_1) + \frac{1}{6}a_0^{(2)}\Delta t^3 + \frac{1}{24}a_0^{(3)}\Delta t^4 + O(\Delta t^5)$$

Error decreases with $1/\Delta t^5$!