APS/ICTP-SAIFR Satellite March Meeting

Dynamic of frustrated Kuramoto oscillators with modular connections

Guilherme Henrique da Silva Costa

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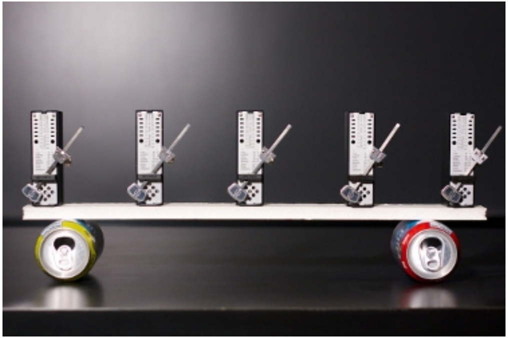
Synchronization



Synchronization

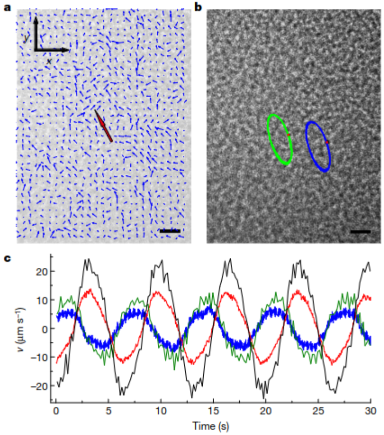
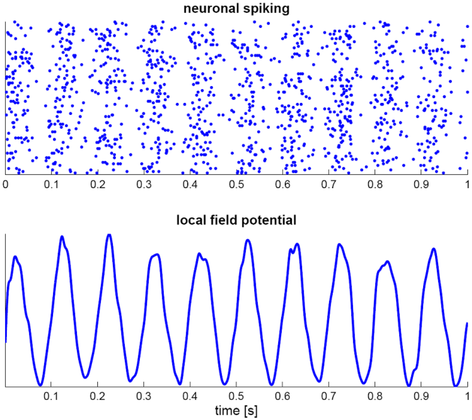


Figure 1 | Collective oscillation in dense suspensions of *E. coli*. [1]



[1] C. Cheng, Nature 542, 210–214 (2017)

Kuramoto model[2]

- Oscillators described by phases θ_i with natural frequency ω_i:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin\left(\theta_j - \theta_i\right)$$

- Complex order parameter:

$$z = pe^{i\psi} \equiv \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i}$$

- $p \approx 0 \rightarrow$ disordered motion
- $p \approx 1 \rightarrow$ synchronization

[2] Y. Kuramoto, International Symposium on Mathematical Problems in Theoretical Physics, Springer Berlin Heidelberg, 1975.

Matrix coupling [3]

- Vectorial representation: $\theta_i \to \vec{\sigma}_i = (\cos\theta_i, \sin\theta_i)$:

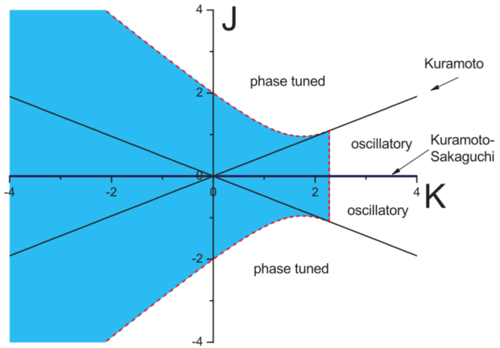
$$\frac{d\vec{\sigma}_i}{dt} = \mathbf{W}_i\vec{\sigma}_i + \frac{1}{N}\sum_{j=1}^{N}[\mathbf{K}\vec{\sigma}_j - (\vec{\sigma}_i \cdot \mathbf{K}\vec{\sigma}_j)\vec{\sigma}_i]$$

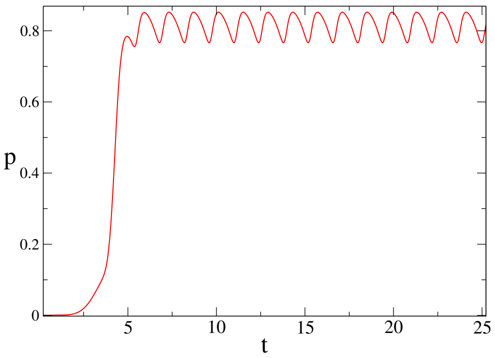
- Matrix coupling **K**:

$$\mathbf{K} \equiv \mathbf{K}_R + \mathbf{K}_S = K\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} + J\begin{pmatrix} -\cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

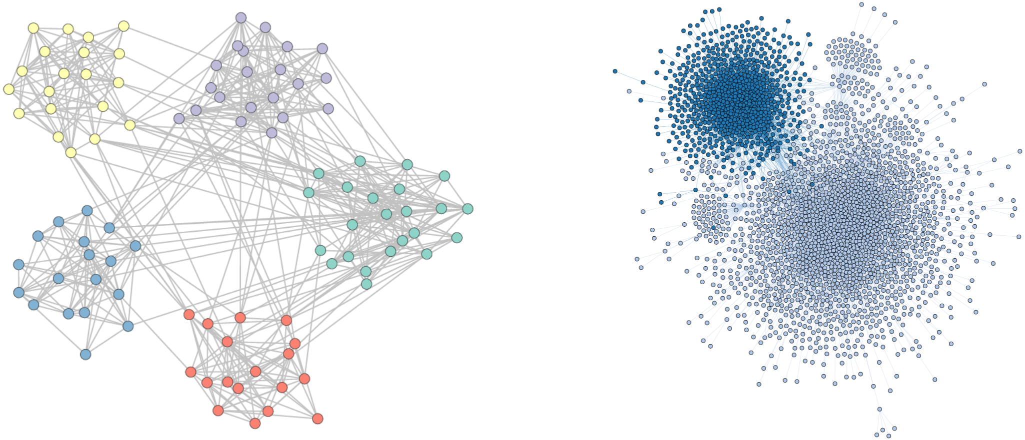
[3] G. L. Buzanello, A. E. D. Barioni, and M. A. M. de Aguiar, Chaos 32, 093130 (2022)

Phase diagram





Modular connections



Ott-Antonsen ansatz [4]

$$\dot{p}_1 = -\Delta_1 p_1 + \frac{p_1}{2}(1 - p_1^2)\left[K_1 \cos\alpha_1 - J_1 \cos(2\psi_1)\right] + \frac{p_2}{2}(1 - p_1^2)K_{12}\cos\xi$$

$$\dot{p}_2 = -\Delta_2 p_2 + \frac{p_2}{2}(1 - p_2^2)\left[K_2 \cos\alpha_2 - J_2 \cos(2\psi_2)\right] + \frac{p_1}{2}(1 - p_2^2)K_{12}\cos\xi$$

$$p_1\dot{\psi}_1 = +\omega_1 p_1 - \frac{p_1}{2}(1 + p_1^2)\left[K_1 \sin\alpha_1 - J_1 \sin(2\psi_1)\right] - \frac{p_2}{2}(1 + p_1^2)K_{12}\sin\xi$$

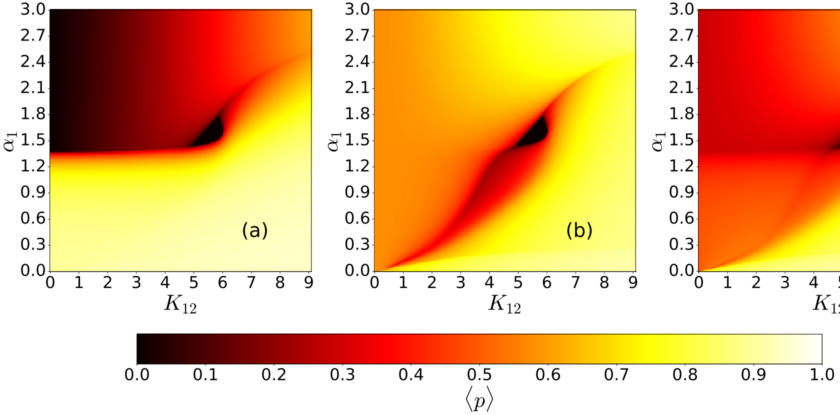
$$p_2\dot{\psi}_2 = +\omega_2 p_2 - \frac{p_2}{2}(1 + p_2^2)\left[K_2 \sin\alpha_2 - J_2 \sin(2\psi_2)\right] + \frac{p_1}{2}(1 + p_2^2)K_{12}\sin\xi$$

[4] E. Ott and T. M. Antonsen, Chaos 18, 037113, 2008.

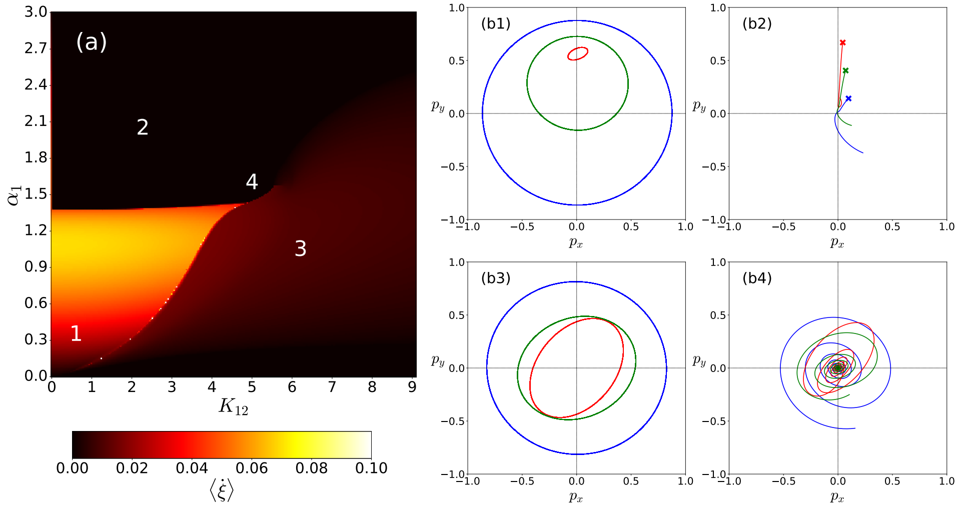
Considerations

- $\mathbf{K}_{12} = \mathbf{K}_{21} = K_{12}$
- $\beta_1 = \beta_2 = 0$
- Module 1 $\rightarrow (K_1 \neq 0; \alpha_1 \neq 0; J_1 = 0)$
- Module 2 $\rightarrow (K_2 = 0; \alpha_2 = 0; J_2 \neq 0)$
- Strong synchronization on uncoupled dynamic

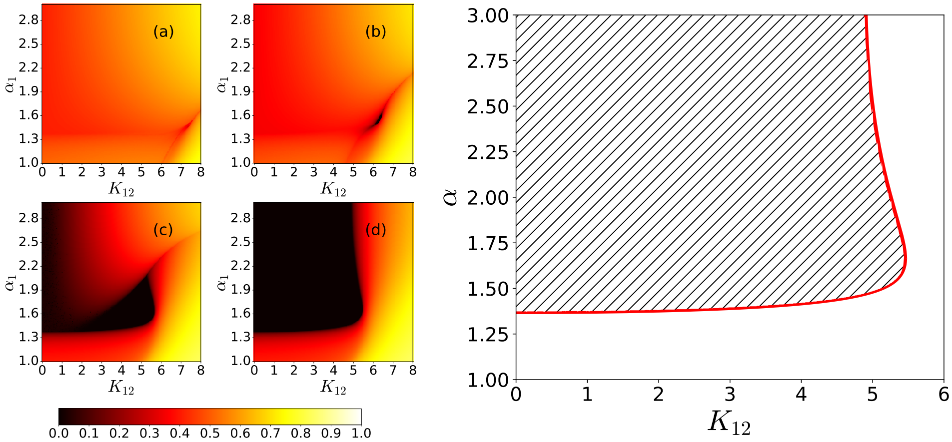
Heatmaps in $\alpha_1 \times K_{12}$ space



Detuning in $\alpha_1 \times K_{12}$ space

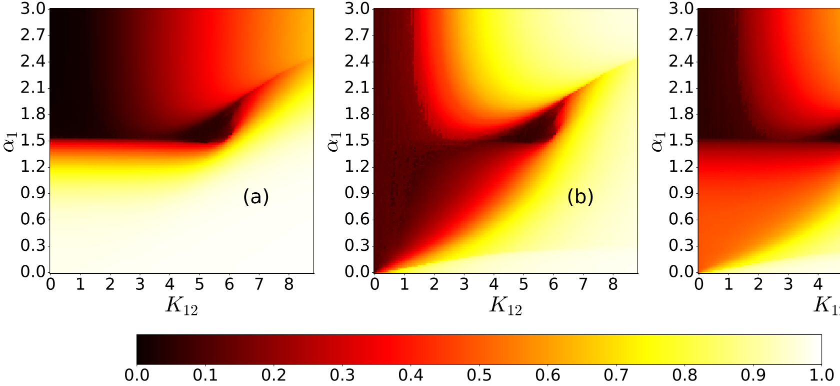


Varying J

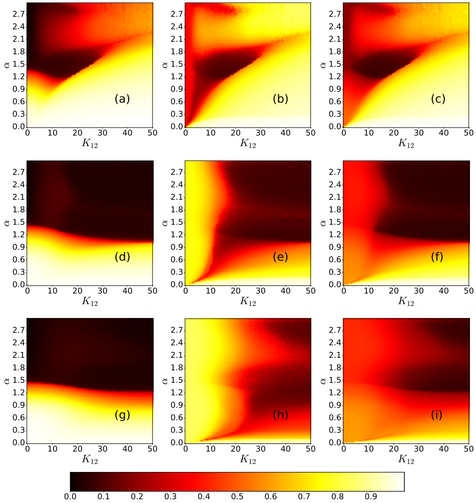


(a)
(b)
(c)
(d)
α_1
K_{12}
2.8
2.5
2.2
1.9
1.6
1.3
1.0
0 1 2 3 4 5 6 7 8
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
3.00
2.75
2.50
2.25
2.00
1.75
1.50
1.25
1.00
α
0 1 2 3 4 5 6

Gaussian frequencies ($N = 10000$)



Random partitions ($N = 1000$; $\langle k \rangle = 10$)



Concluding remarks

- The system exhibits a rich and complex dynamics with the existence of integration and segregation regions:
  - Independent modules
  - Global phase tuning
  - Global oscillations
  - Asynchronous region
- The size of the asynchronous region depends on the parameters and vanishes if J is large enough.
- Similar dynamics were obtained for simulations on complete and random partition graphs, if they are sufficiently connected.

Thank you!

FAPESP
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