Bifurcations in the Kuramoto model with external forcing and higher-order interactions

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Introduction

- The emergence of synchronization is of particular interest to scientists from several areas, such as physics, social sciences and biology. Kuramoto [1] proposed a model that became a paradigm in the field. This model undergoes a second order phase transition from disordered motion to synchronization as the coupling between oscillators k increases.
- An important extension of the original model is the inclusion of external forcing, introducing a competition between spontaneous mutual synchronization and forced entrainment.
- The circustances of collective behavior under external drives draw attention to synchronization for describing numerous phenomena, such as circadian rhythms, injection locking of lasers, artificial pacemaker devices and epileptic seizures.
- In many complex systems interactions go beyond pairwise relations, involving the collective action of groups of three or more agents that cannot be decomposed into a sum of pairs of interactions.
- In this work we combined these two features by considering a Kuramoto system in which the oscillators are forced by an external periodic drive as in [3] and whose internal dynamics contains higher-order interactions such as those in [4].

Methodology

Higher order + Forced Oscillators

- Dynamical equation:

$$\dot{\theta}_i = \omega_i + \frac{K_1}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i) + \frac{K_2}{N^2}\sum_{j,k=1}^{N}\sin(2\theta_j-\theta_k-\theta_i) + \frac{K_3}{N^3}\sum_{j,k,m=1}^{N}\sin(\theta_j+\theta_k-\theta_m-\theta_i) + F\sin(\sigma t-\theta_i), \quad (1)$$

- Order parameters:

$$z = re^{i\psi} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \quad (2)$$

$$z_2 \equiv r_2 e^{i\psi_2} = \frac{1}{N}\sum_{j} e^{2i\theta_j}. \quad (3)$$

- Changing to $\theta_j' = \theta_j - \sigma t$ and dropping the primes, we obtain:

$$\dot{\theta}_j = (\omega_j - \sigma) + K_1 r\sin(\psi-\theta_j) + K_2 r r_2 \sin(\psi_2-\theta_j-\psi) + K_3 r^3 \sin(\psi-\theta_j) - F\sin\theta_j. \quad (4)$$

- $f(\omega,\theta,t) \to$ density of oscillators with natural frequency ω at position θ at time t.
- Expanding f in Fourier series:

$$f(\omega,\theta,t) = \frac{g(\omega)}{2\pi}\left(1+\sum_{n=1}^{\infty} f_n e^{in\theta} + c.c.\right). \quad (5)$$

- **OA ansatz [2]:** by choosing coeficients f_n as $f_n = [\alpha(\omega,t)]^n$, it is possible to obtain dynamical equations for the modulus and phase of the order paramenter. Taking $g(\omega)$ as a Lorentzian distribution centered in ω_0:

$$\dot{r} = -r + \frac{r}{2}(1-r^2)(K_1+K_{23}r^2) + \frac{F}{2}(1-r^2)\cos\psi \quad (6)$$

$$\dot{\psi} = -\Omega - \frac{F}{2}\left(r+\frac{1}{r}\right)\sin\psi \quad (7)$$

in which $K_{23} = K_2 + K_3$ and $\Omega = \sigma - \omega_0$.

- By imposing the appropriate conditions for the equilibrium solutions of Equations 6 and 7 along with the system's Jacobian J, Saddle-Node and Hopf manifolds were analitically calculated while more complex bifurcations were identified using numerical continuation.
- Saddle-Node: $\dot{r} = \dot{\psi} = \det[J] = 0$.
- Hopf: $\dot{r} = \dot{\psi} = \mathrm{Tr}[J] = 0$ and $\det[J] > 0$

Results

Simplicial parameters as fixed

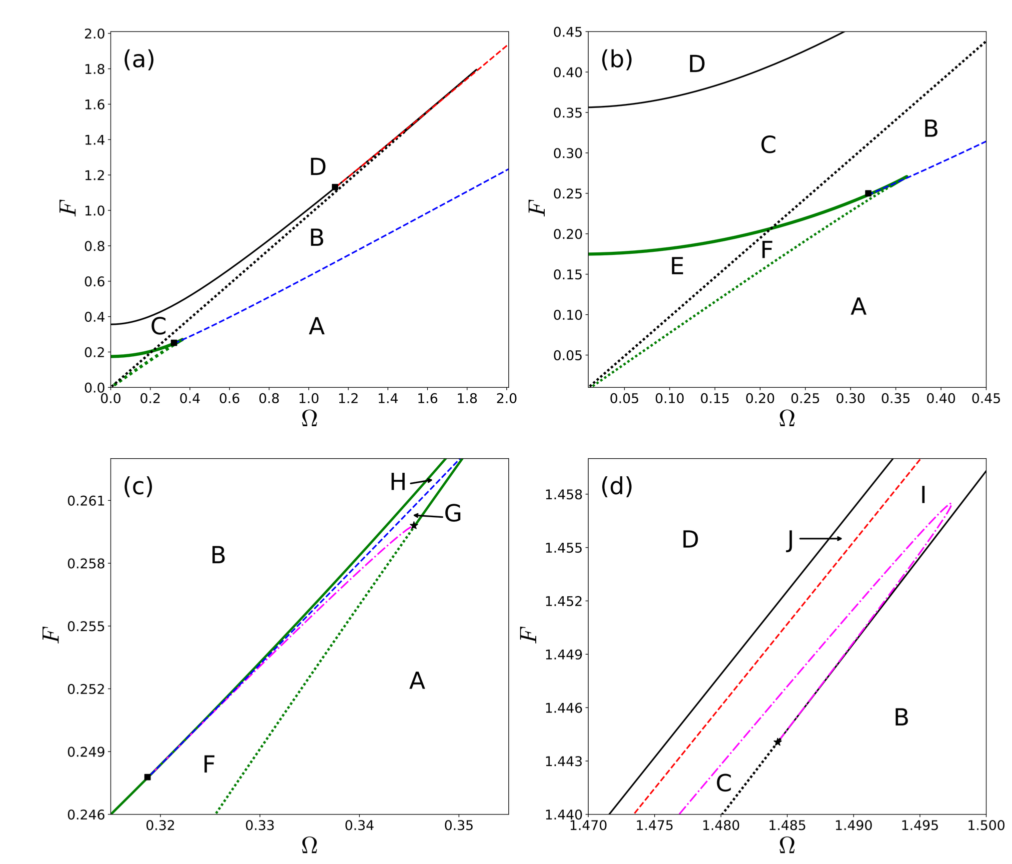

Fig. 1: Bifurcation curves in the (F,Ω) plane for $K_1 = 1$ and $K_{23} = 7$, dividing the plane into ten regions, indicated by capital letters. Bifurcation diagram overview: full lines are Saddle-Node bifurcations, dotted lines are SNIPERs and dashed lines are the Hopf curves. Black squares represent Takens-Bogdanov points, where the Hopf and SN bifurcations meet. Homoclinic bifurcation curves are shown in the dot-dashed pink lines.

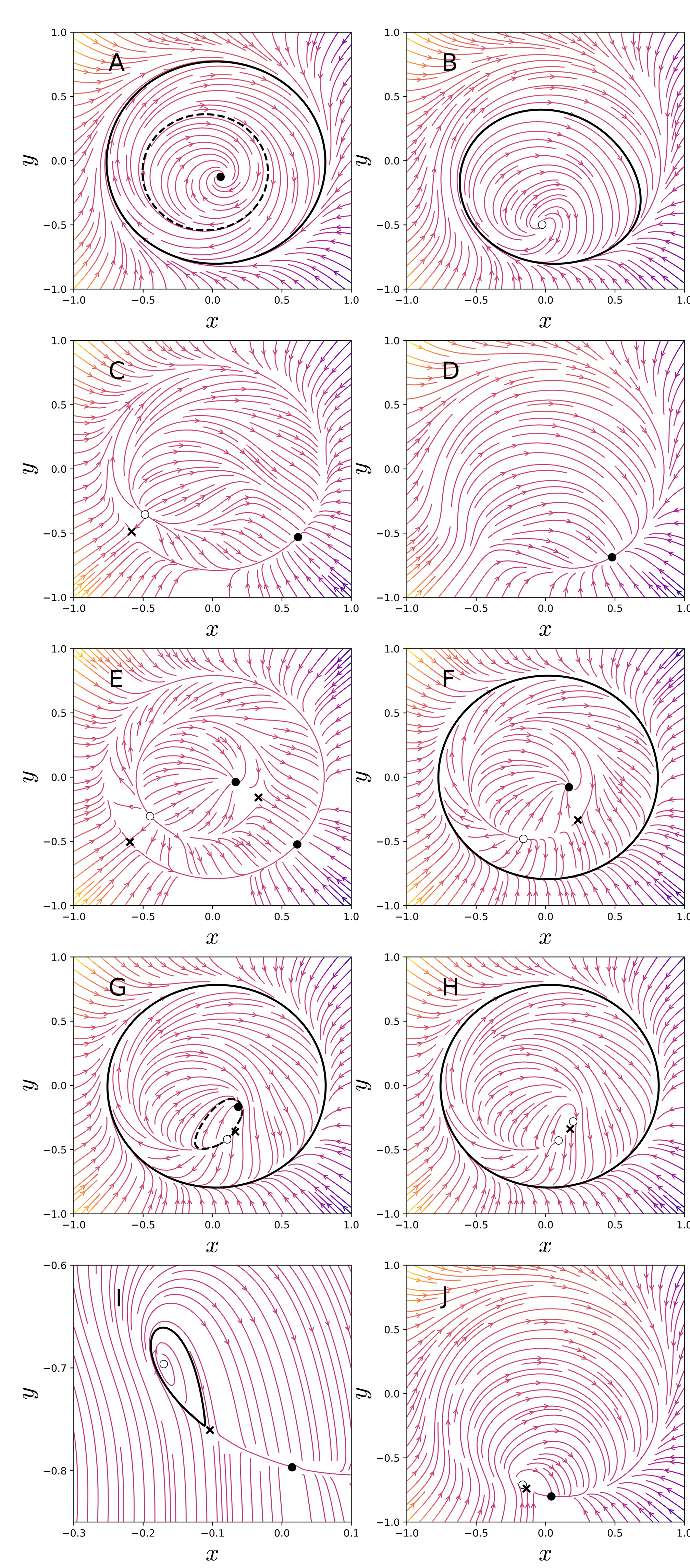

Fig. 2: Vector fields, fixed points and periodic orbits of regions A-J shown in Figure 1. Full (empty) circles are stable (unstable) nodes and spirals, and crosses are saddle points. Full and dashed lines are stable and unstable orbits, respectively.

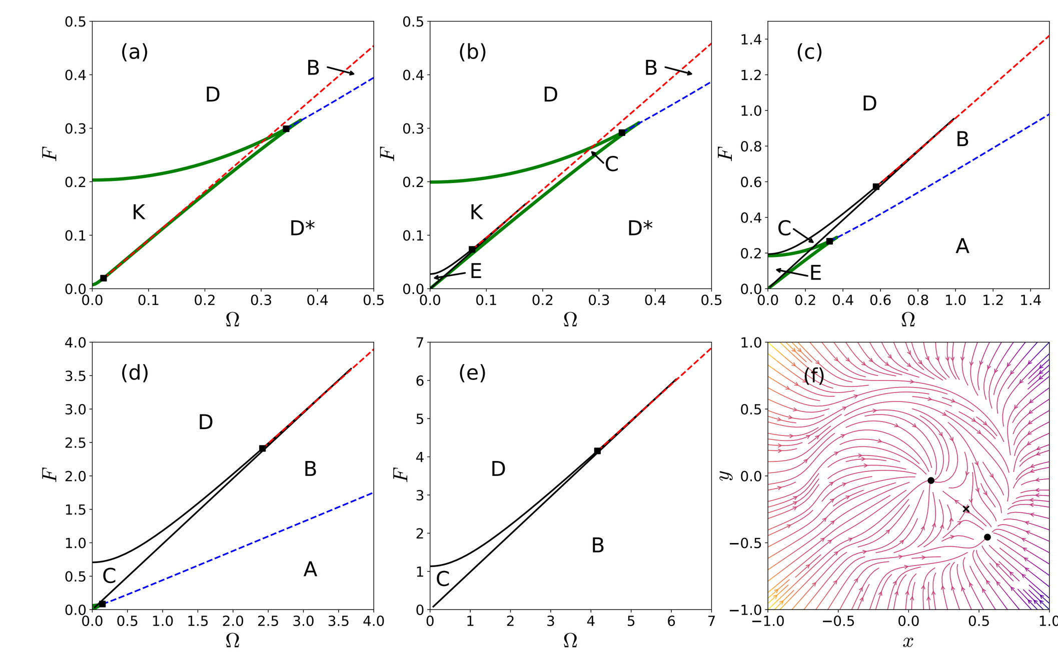

Fig. 3: Bifurcation curves in the (F,Ω) plane for different values of K_1 and K_{23}. In panels (a)-(c) $K_1 = 1$ and K_{23} is 5.80, 5.93 and 6.5 respectively. In (d)-(e) $K_{23} = 7$ and K_1 is 1.5 and 2.1 respectively.

Forcing parameters as fixed

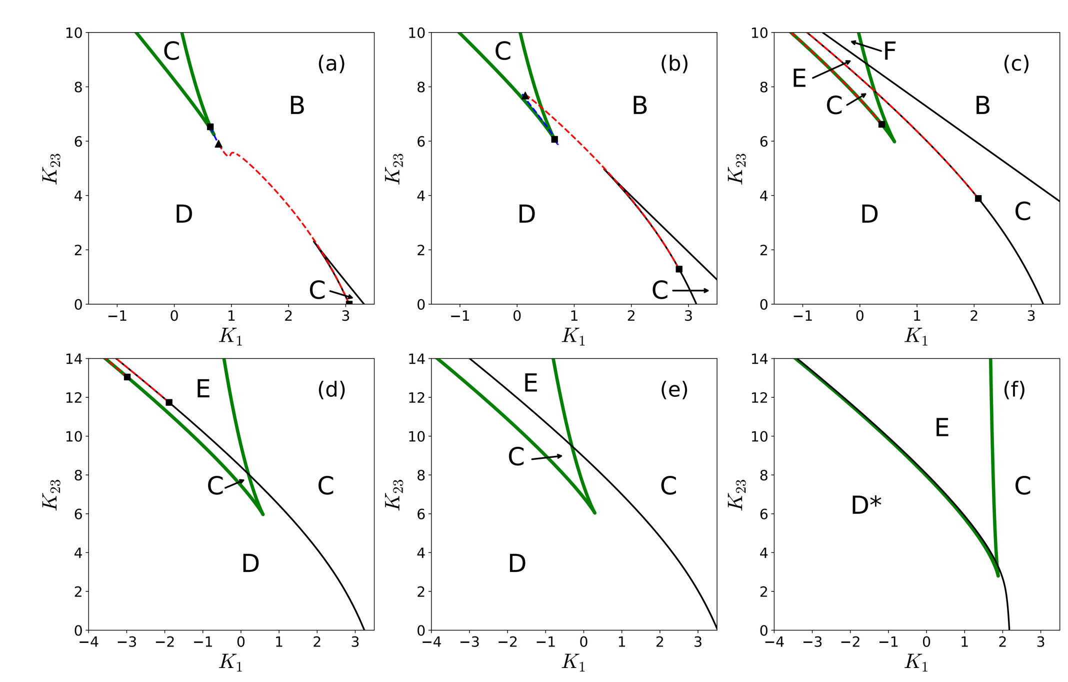

Fig. 4: Saddle-node and Hopf curves for different values of F and Ω. In panels (a)-(e) $\Omega = 0.5$ and F is 0.45, 0.47, 0.49, 0.499 and 0.59, respectively. In panel (f), $\Omega = 0.01$ and $F = 0.02$.

Discussions and Perspectives

- In this work we extended the bifurcation analysis of the forced Kuramoto model to include higher order interactions between the oscillators, describing in detail how simplicial and forcing parameters impact the bifurcation diagrams obtained in (F,Ω) and (K_1,K_{23}) planes.
- We have shown that two branches of the saddle-node and Hopf bifurcation manifolds can coexist over a range of parameters, reflecting the bi-stability of the unforced case. The combined effects of higher order interactions and external periodic force increases dramatically the complexity of the bifurcation diagrams, that is divided into 11 different topological regions.
- Finally, we note that higher order interactions, as derived from phase reduction methods , can have different functional forms, and a study of their combined effects has not yet been investigated in detail. The Kuramoto model with symmetric higher order terms, for example, exhibit multi-stability and anomalous transitions to synchrony. We leave the exploration of such cases for future work.

References

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