The influence of external drives on the frustrated Kuramoto model

Guilherme S. Costa[1] and Marcus M. A. de Aguiar[1,2]

[1] *ICTP South American Institute for Fundamental Research & Instituto de Física Teórica, Universidade Estadual Paulista - UNESP, São Paulo, Brazil*
[2] *Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas - Unicamp, Campinas, São Paulo, Brazil*

guilherme.costa@ictp-saifr.org, aguiar@ifi.unicamp.br

Introduction

- The emergence of synchronization is of particular interest to scientists from several areas, such as physics, social sciences and biology. Kuramoto [1] proposed a model that became a paradigm in the field [2]. This model undergoes a second order phase transition from disordered motion to synchronization as the coupling between oscillators k increases.
- Several extensions of the original model were proposed in order to incorporate new ingredients and describe increasingly complex phenomena. A recent interesting one was to change the coupling from a scalar to a matrix, breaking the rotational symmetry and leading to generalized frustration [3].
- The circumstances of collective behavior under external drives draw attention to synchronization for describing numerous phenomena, such as circadian rhythms in the human body or the injections locking of lasers.
- Based on that, we investigated the dynamic of the frustrated Kuramoto model under the influence of external drives.

Methodology

Frustrated Kuramoto oscillators

- Dynamical equation:

$$\frac{d\vec{\sigma}_i}{dt} = \mathbf{W}_i\vec{\sigma}_i + \frac{1}{N}\sum_{j=1}^{N}[\mathbf{K}\vec{\sigma}_j - (\vec{\sigma}_i\cdot\mathbf{K}\vec{\sigma}_j)\vec{\sigma}_i], \quad (1)$$

- Natural frequencies:

$$\mathbf{W}_i = \begin{pmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{pmatrix}.$$

- Parametrization of the coupling matrix:

$$\mathbf{K} = K\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} + J\begin{pmatrix} -\cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

- Order parameter:

$$\vec{p} = \frac{1}{N}\sum_i \vec{\sigma}_i = (p\cos\psi, p\sin\psi),$$

Ott-Antonsen (OA) ansatz

- $f(\omega,\theta,t) \to$ density of oscillators with natural frequency ω at position θ in time t.
- f satisfies the continuity equation $\dfrac{\partial f}{\partial t} + \dfrac{\partial (f v_\theta)}{\partial \theta} = 0.$
- **OA ansatz [4]:** Coeficients of f Fourier expansion are chosen as $\rho^{|m|}e^{-im\langle\phi\rangle}$.

$$f(\omega,\theta,t) = \frac{g(\omega)}{2\pi}\sum_{m=-\infty}^{\infty}\rho^{|m|}e^{im(\theta-\phi)}. \quad (2)$$

- Useful for Lorentzian distributions $g(\omega) = \dfrac{1}{\pi}\dfrac{\Delta}{(\omega-\omega_0)^2+\Delta^2}$.

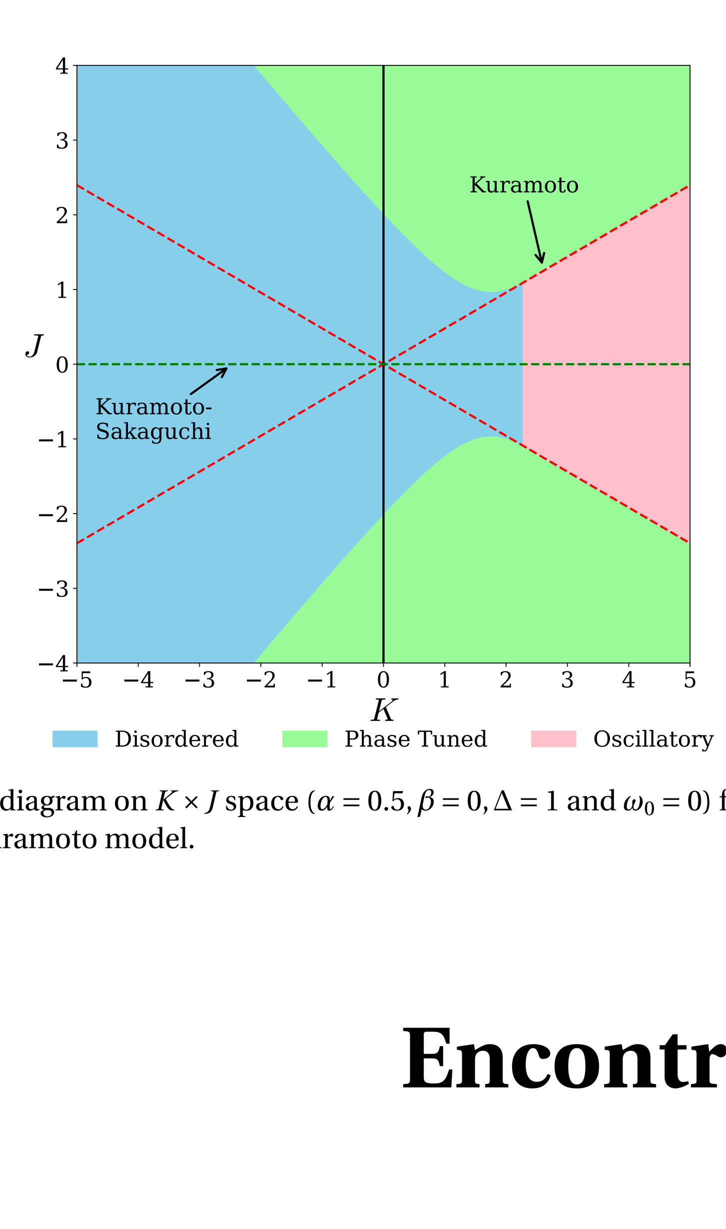

Fig. 1: Phase diagram on $K \times J$ space ($\alpha = 0.5, \beta = 0, \Delta = 1$ and $\omega_0 = 0$) for the frustrated Kuramoto model.

External drives

- We add an external periodic drive of the form $\vec{F} = (F\cos(\Omega t), F\sin(\Omega t))$ to Equation 1. By applying the OA ansatz, we obtained a set of non autonomous equations that describe the dynamic of the order parameter $\vec{p}$ for this system:

$$\dot{p} = -\Delta p + \frac{(1-p^2)}{2}\left[pK\cos\alpha - pJ\cos(2\phi + 2\Omega t + \beta) + F\cos\phi\right]$$

$$\dot{\phi} = (\omega_0 - \Omega) - \frac{(1+p^2)}{2}\left[K\sin\alpha - J\sin(2\phi + 2\Omega t + \beta) + \frac{F}{p}\sin(\phi)\right]$$

in which $\phi = \psi - \Omega t$.

- Due to the non-autonomous nature of the equations, analytical approaches are not feasible. Thus, we chose particular cases and investigated numerically.

Results

External drive on oscillatory states

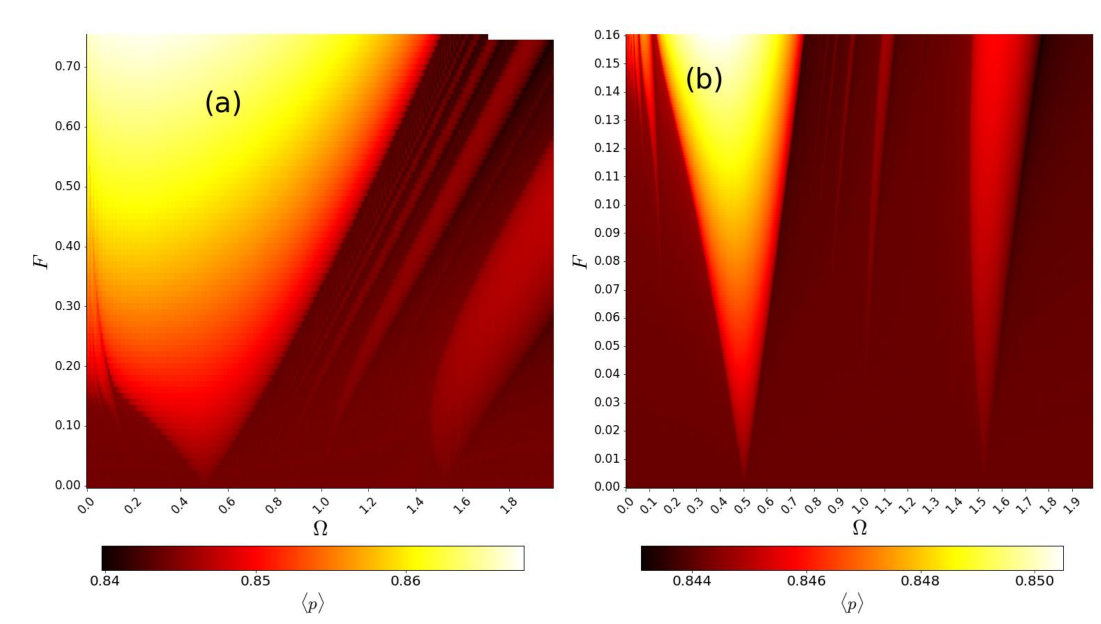

Fig. 2: Heatmaps in $F \times \Omega$ space showing time averages of the order parameter $\langle p\rangle$. ($K = 7; J = 1; \Delta = 1$ and $\omega_0 = 1$).

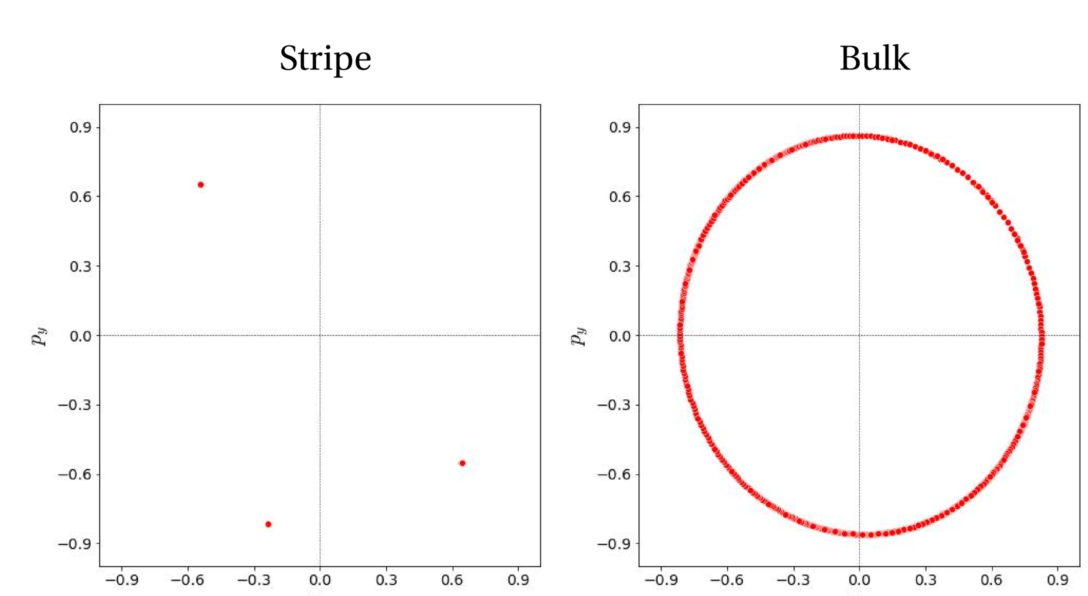

Fig. 3: Poincaré maps of typical trajectories of the order parameter, displaying periodic and aperiodic behavior.

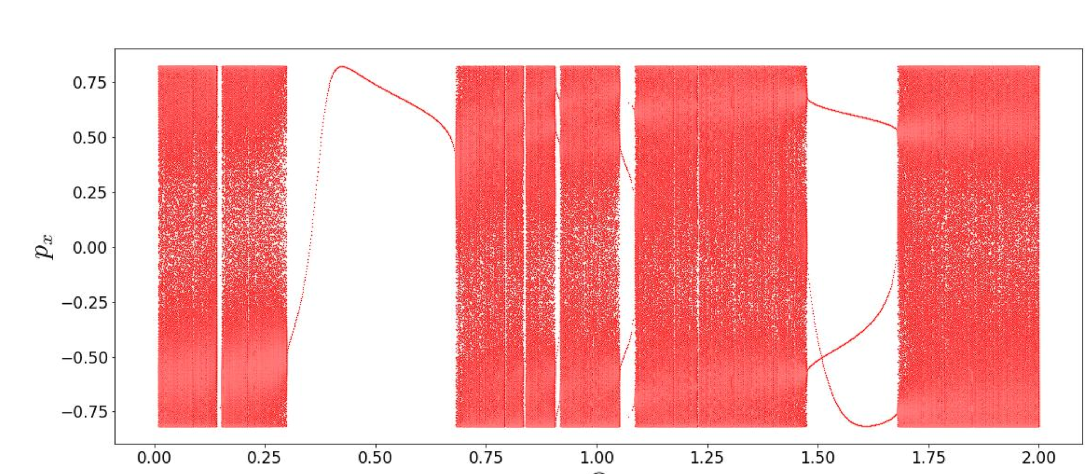

Fig. 4: Scatter of p_x extracted from the Poincaré maps as in Fig. 3, for different values of the external drive frequency Ω and fixed $F = 0.1$. It is possible to observe the periodic and aperiodic regions in detail.

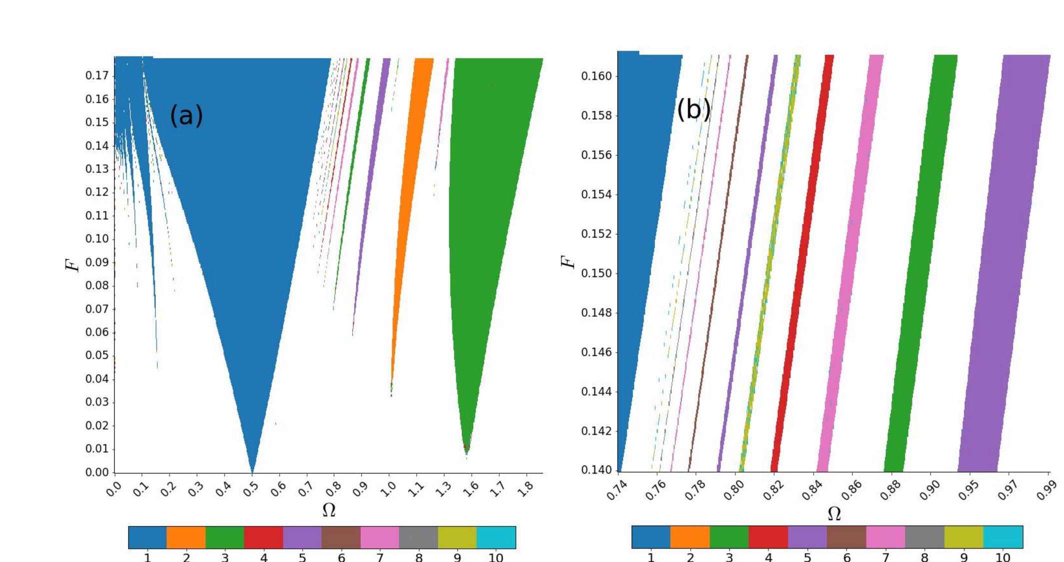

Fig. 5: (a) Visual representation of the period of $\vec{p}$ trajectories within the stripes found in the heatmaps of Fig. 2. (b) Zoom on (a) showing the stripes patterns.

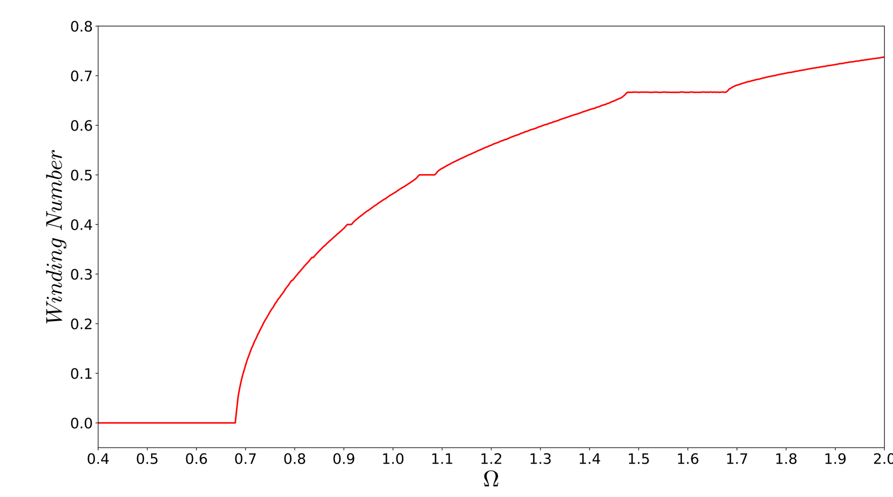

Fig. 6: Winding number in function of the external drive frequency.

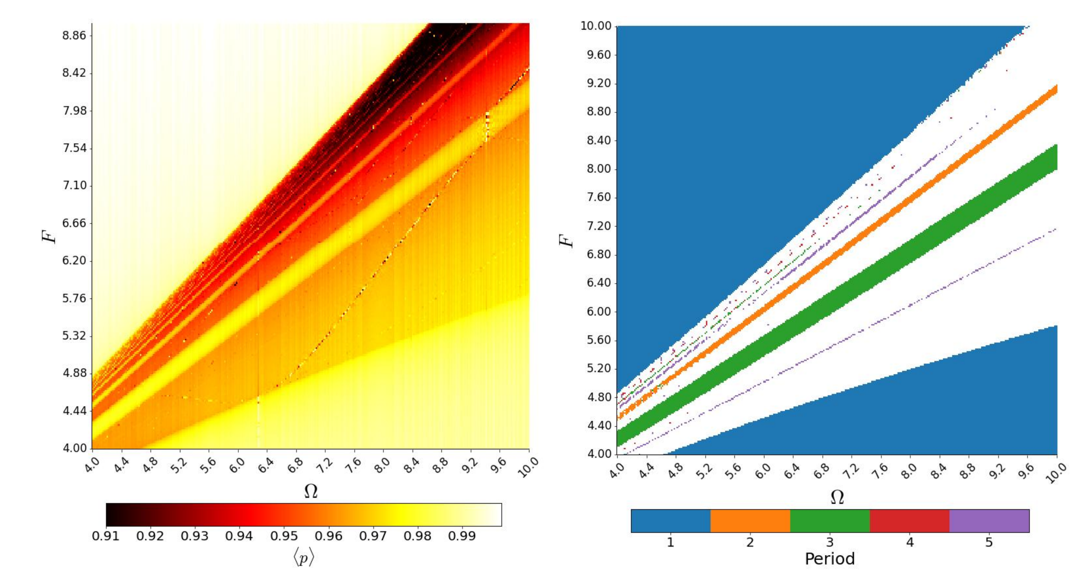

Fig. 7: **Left:** Heatmaps in $F \times \Omega$ space showing time averages of the order parameter $\langle p\rangle$ for a 10^5 nodes Érdos-Renyi network ($K = 7; J = 1; \Delta = 1$ and $\omega_0 = 1$). **Right:** Visual representation of the period of $\vec{p}$ trajectories within the stripes found in the heatmaps on the left.

Discussions and Perspectives

- We applied the OA ansatz to the frustrated Kuramoto with external drives to reduced the complexity of the problem to a set of non-autonomous equations that describe the dynamic of the order parameter instead of the individual oscillators.
- By selecting the particular case of oscillatory synchronization, we constructed heatmaps of the time-averaged order parameter $\langle p\rangle$, increasing both the external drive intesity F and frequency Ω, obtaining complex patterns, indicating novel synchronized states not found on the original Kuramoto model.
- By closely inspecting the dynamic for some parameters, we identified regions of periodic and aperiodic orbits resembling Arnold tongues, in which large regions are phase locked to different frequencies for some ratios. The calculation of the winding number for those cases yield curves that resembled the "Devil's staircases".
- We also obtained qualitatively similar results on simulations on Erdos-Renyi networks, backing up those findings.
- As a perspective, we intend to further investigate the effects of the external forces on other synchronized states of the frustrated Kuramoto model and also the robustness of these effects when the influence is partial.

References

[1] Y. Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators*, International Symposium on Mathematical Problems in Theoretical Physics (1975).

[2] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, *The Kuramoto model: A simple paradigm for synchronization phenomena*, Rev. Mod. Phys. 77, 137 (2005).

[3] G. L. Buzanello, A. E. D. Barioni, and M. A. M. de Aguiar,*Matrix coupling and generalized frustration in kuramoto oscillators*, Chaos: An Interdisciplinary J. Nonlinear Science 32,(2022).

[4] E. Ott and T. M. Antonsen, *Low dimensional behavior of large systems of globally coupled oscillators.*Chaos 18(3), (2008).

[5] Lauren M. Childs, Steven H. Strogatz, *Stability diagram for the forced Kuramoto model*, Chaos 18(4), (2008).

We acknowledge the financial support of FAPESP (grant 2023/03917-4), CNPq and CAPES.