Catalanimals:
a walk through the zoo of shuffle theorems

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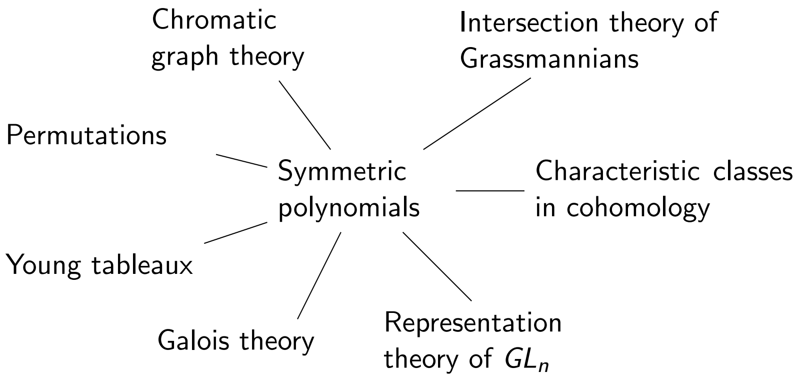
Introduction

1. Fundamental object: multivariate symmetric polynomials
2. Guiding conjecture: The shuffle conjecture
3. Recent breakthroughs

Fundamental Object: Symmetric polynomials



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Observation by Girard and Newton in 1600's

Consider relationships between symmetric polynomials in unspecified (sufficiently large) number of variables.

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- $e_2 = z_1z_2 + z_1z_3 + z_2z_3 + \cdots$

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$$
\begin{tikzcd}
\text{Linear algebra} \arrow[d] \\
\text{Symmetric polynomials}
\end{tikzcd}
$$

The miraculous basis of Schur polynomials

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$$GL_2(\mathbb{C}) \curvearrowright \mathsf{Sym}^2(\mathbb{C}^2)$$

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Integer partitions
$\mu = (\mu_1 \geqslant \cdots \geqslant \mu_n)$
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$\longleftrightarrow$

Irreducible (polynomial) $GL_n(\mathbb{C})$
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$s_{(2,0)}(z_1, z_2) = z_1^2 + z_1 z_2 + z_2^2.$

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$$\begin{tikzcd} \boxed{GL_n(\mathbb{C}) \curvearrowright V} \arrow[r, "\text{Trace}"] & \boxed{\text{Symmetric polynomial}} \arrow[l, bend left, "?"] \end{tikzcd}$$

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Guiding principle

Positive integer sums of Schurs $\implies$ representation-theoretically meaningful.

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$$GL_n(\mathbb{C}) \curvearrowright V^\mu$$

$$\Big\downarrow \text{Trace}$$

Symmetric "Schur polynomial" s_μ

$$s_\mu = \sum x^{\mathrm{wt}(T)}$$

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$\uparrow$ Sum of weights

Semistandard Young Tableaux

$$\mu = (4,3,1,1) \to \ydiagram{1,1,3,4} \to T = \begin{ytableau} 5 \\ 3 \\ 2 & 3 & 5 \\ 1 & 1 & 2 & 3 \end{ytableau}$$

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Consequence for irreducible V^μ

$\dim V^\mu =$number of semistandard Young tableaux of μ.

S_n-modules

Symmetric group $S_n \curvearrowright \mathbb{C}[x_1, \ldots, x_n]$

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$I =$ all positive degree symmetric polynomials

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Frobenius characteristic

S_n-representation $V \mapsto$ symmetric polynomial $\mathrm{Frob}(V)$.

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Integer partition of $n \leftrightarrow$ irreducible S_n representation V_μ

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$S_n \curvearrowright V_\mu$

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Semistandard Young Tableaux

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5 \\
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S_n-module of Harmonic polynomials

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Natural generalization: Diagonal Harmonics

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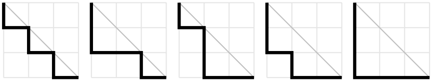

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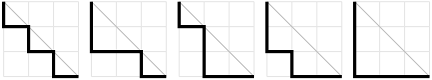

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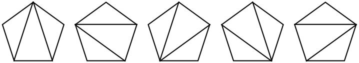

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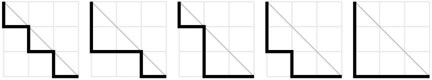

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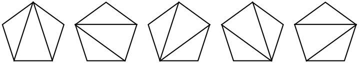

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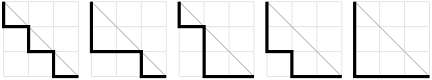

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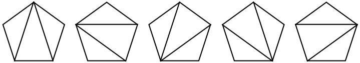

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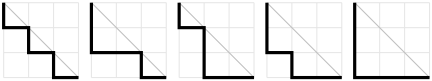

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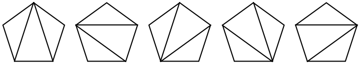

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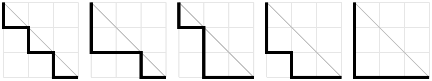

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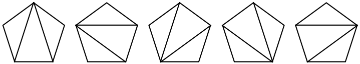

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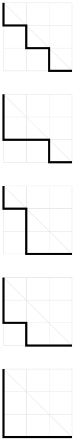

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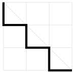
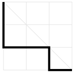
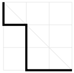
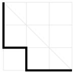
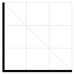
	e_1^3
	te_2e_1
	te_2e_1
	$t^2e_2e_1$
	t^3e_3

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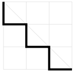
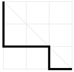
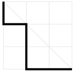
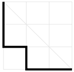
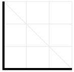
	e_1^3
	te_2e_1
	te_2e_1
	$t^2e_2e_1$
	t^3e_3

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q, t Catalan numbers

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Question

What can we say about $e_k^{(m,n)} \cdot 1$?

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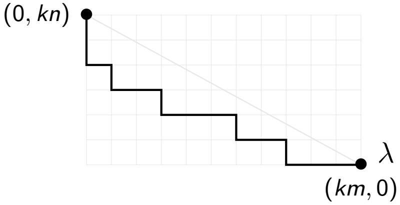

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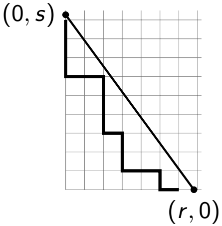

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- $\to$ Some S_n-module M such that $\mathrm{Frob}_{q,t}(M) = D^{(r,s)} \cdot 1$?

Some applications

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- $C^{(r,s)}(q,t) =$ subclass of "generalized q, t Catalan numbers" of (Gorsky, Hawkes, Schilling, Rainbolt, 2020).

Some applications

Generalized q, t Catalan numbers

- $D^{(r,s)} \cdot 1 = C^{(r,s)}(q,t)s_{(1,\ldots,1)} + \cdots$
- $C^{(r,s)}(q,t) =$ subclass of "generalized q, t Catalan numbers" of (Gorsky, Hawkes, Schilling, Rainbolt, 2020).
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- (Capran, González, Hogancamp, Mazin, 2024+) use Generalized Shuffle Theorem formula to give KR homology of associated "Coxeter knots."

A Mystery: Convex Curve Conjecture

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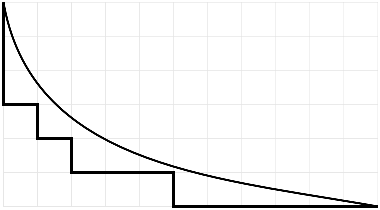

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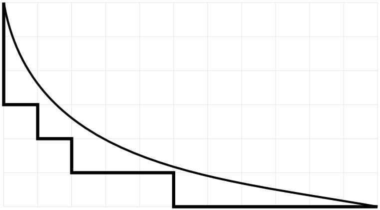

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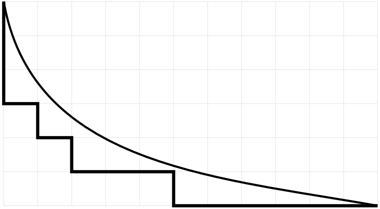

- S_n-module?
- $s_{(1,\ldots,1)}$ coefficient appears in context of "positroid varieties" (Galashin, Lam, 2024).

The Extended Delta Theorem

The Extended Delta Theorem (BHMPS, 2023)

For operators Δ_{h_l} and $\Delta'_{e_{k-1}}$ on symmetric polynomials,
$\Delta_{h_l}\Delta'_{e_{k-1}}e_n = \sum(q, t \text{ monomial})(\text{LLT symmetric polynomial})$

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Proof Technique

$\Delta_{h_l}\Delta'_{e_k}e_n = \sum_\alpha D_\alpha \cdot 1$

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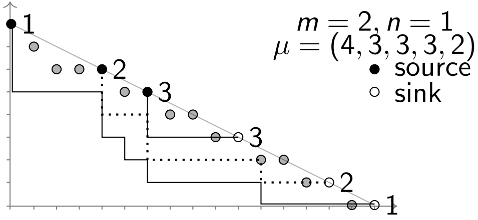

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Common theme: $\mathcal{E} \curvearrowright 1 = \text{Combinatorial sum}$

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Thank you!