Catalanimals:
a walk through the zoo of shuffle theorems

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University of Pennsylvania Mathematics Colloquium

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Fundamental Object: Symmetric polynomials

Symmetric
polynomials

Fundamental Object: Symmetric polynomials

Chromatic graph theory

Permutations

Symmetric polynomials

Young tableaux

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Intersection theory of Grassmannians

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Characteristic classes in cohomology

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Galois theory

Representation theory of GL_n and S_n

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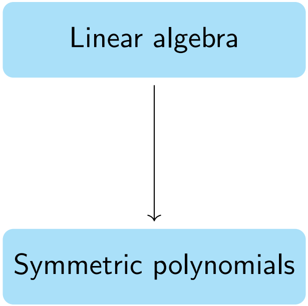

Some representation theory

Setup

Group $G \curvearrowright V \cong \mathbb{C}^n$.

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Over $\mathbb{C}$, $G \curvearrowright V \cong \bigoplus \text{Irreducibles}$
$\iff \mathrm{tr}(G \curvearrowright V) = \sum \mathrm{tr}(G \curvearrowright \text{Irreducible})$

The miraculous basis of Schur polynomials

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$\mu_i \in \mathbb{Z}_{\geqslant 0}$

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$$s_{(2,0)}(z_1, z_2) = z_1^2 + z_1 z_2 + z_2^2.$$

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Guiding principle

Positive integer sums of Schurs $\Longrightarrow$ representation-theoretically meaningful.

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Semistandard Young Tableaux SSYT(μ)

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$\downarrow$ Trace

Schur polynomial s_μ

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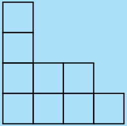
$\mu = (4, 3, 1, 1) \rightarrow$ 

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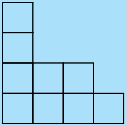
$\mu = (4, 3, 1, 1) \to$  $\to T =$

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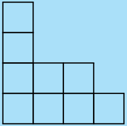
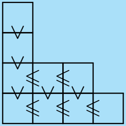
$\mu = (4, 3, 1, 1) \rightarrow$  $\rightarrow T =$ 

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$$s_\mu = \sum_{T \in \mathrm{SSYT}(\mu)} \prod_{z_i \in T} z_i$$

Sum of tableaux weights

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A combinatorial interpretation

$\dim V^\mu =$number of semistandard Young tableaux of μ.

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$$S_n \curvearrowright V_\mu \xrightarrow{\text{Frob}} \text{Schur polynomial } s_\mu$$

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\cline{1-1} z_5 \\
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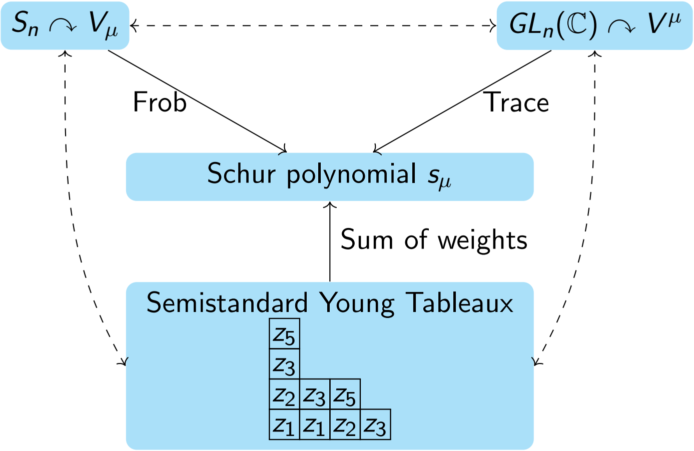

Upgrading with gradings

$$\begin{tikzcd} S_n \curvearrowright V \arrow[r, leftrightarrow] & \text{Symmetric polynomials} \arrow[r, leftrightarrow] & \text{Tableaux combinatorics} \end{tikzcd}$$

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Shuffle Theorem Idea

Bigraded S_n representation $\leftrightarrow$ Bigraded combinatorial formula

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Shuffle Theorem Idea

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Generalized Shuffle Theorem Idea (Blasiak, Haiman, Morse, Pun, S., 2023)

General family of symmetric polynomials with $\mathbb{Q}(q,t)$ coefficients $\leftrightarrow$ Bigraded combinatorial formula

S_n-coinvariant representation

Symmetric group $S_n \curvearrowright \mathbb{C}[x_1, \ldots, x_n]$

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$$\downarrow$$

$$S_n \curvearrowright \mathcal{R}_n = \mathbb{C}[x_1, \ldots, x_n]/I$$

$I =$ all positive degree symmetric polynomials

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Question

What are the irreducible (S_n-invariant) pieces of $\mathcal{R}_n$?

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- What is $\mathrm{Frob}_q(\mathcal{R}_n)$?

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Foundational unsolved question

What are the combinatorics for this expression?

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$$\mathrm{Frob}_{q,t}(D\mathcal{R}_3) = (q^3+q^2t+qt^2+t^3+qt)s_{111}+(q^2+qt+t^2+q+t)s_{21}+s_3$$

The fluttering of a combinatorialist's heart

What do the following objects have in common?

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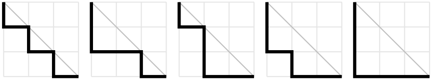

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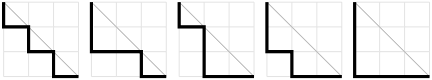
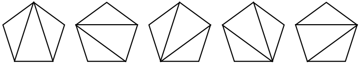


- Triangulations of an $(n+2)$-gon



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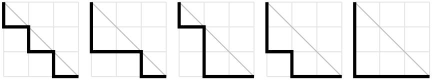

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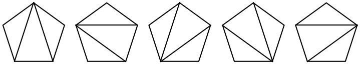

- The number of $2 \times n$ rectangle standard Young tableaux

4	5	6
1	2	3

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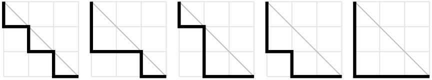

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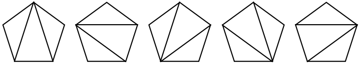

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- Number of terms in coefficient of $s_{(1,\dots,1)}$ in $\mathrm{Frob}_{q,t}(D\mathcal{R}_n)$?

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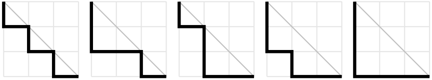

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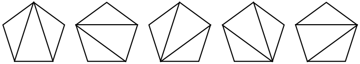

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Catalan numbers!

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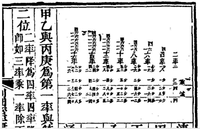
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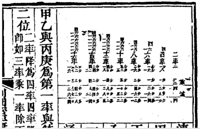
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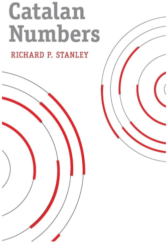

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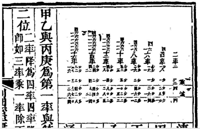
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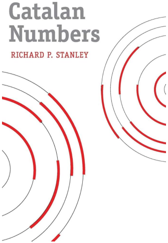

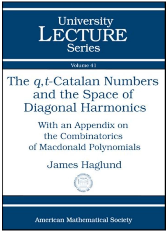

q, t Catalan "numbers"

Write $\mathrm{Frob}_{q,t}(D\mathcal{R}_n) = C_n(q,t) s_{(1,\ldots,1)} + \cdots$

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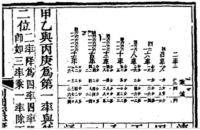
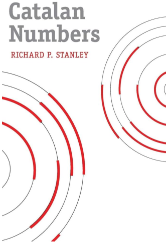
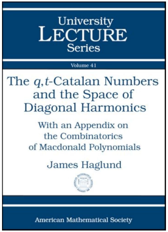
$1, 1, 2, 5, 14, 42, 132, \ldots$







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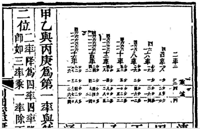

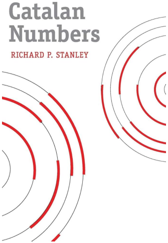

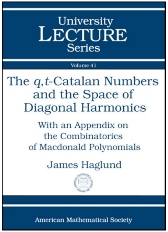

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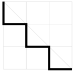

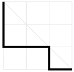

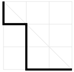

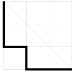

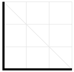

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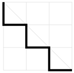
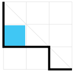
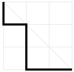
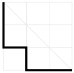
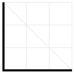




t^{1}







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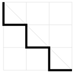
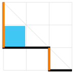
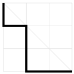
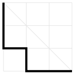
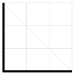
	
	$t^{1} e_{2} e_{1}$
	
	
	

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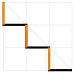
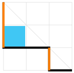
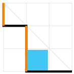
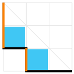
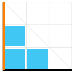
	$e_1 e_1 e_1$
	$t^1 e_2 e_1$
	$t^1 e_2 e_1$
	$t^2 e_2 e_1$
	$t^3 e_3$

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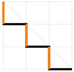
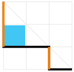
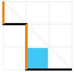
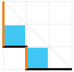
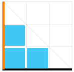
	$e_1 e_1 e_1$
	$t^1 e_2 e_1$
	$t^1 e_2 e_1$
	$t^2 e_2 e_1$
	$t^3 e_3$

$\implies \mathrm{Frob}_{q,t}(D\mathcal{R}_n)|_{q=1} = e_1^3 + (2t + t^2) e_2 e_1 + t^3 e_3$

The Shuffle Theorem

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Special Case (Garsia, Haglund, 2001+2002)

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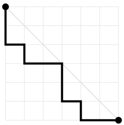

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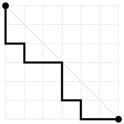

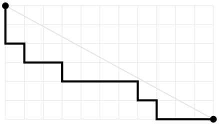

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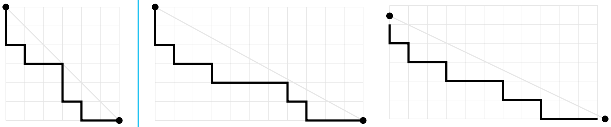

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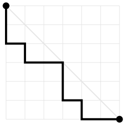

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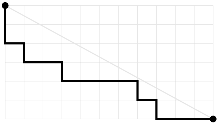

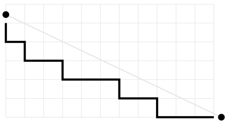

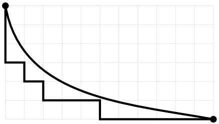

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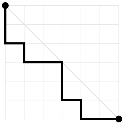

Shuffle Theorem

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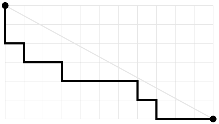

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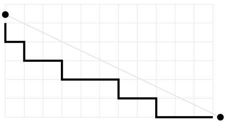

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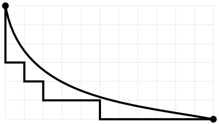

Generalizing with the Elliptic Hall Algebra

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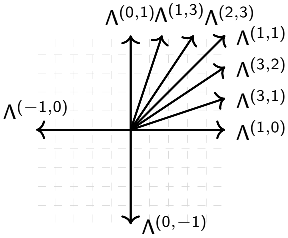

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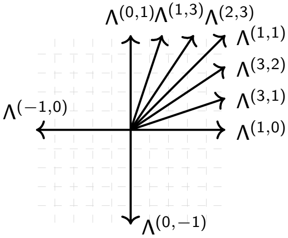

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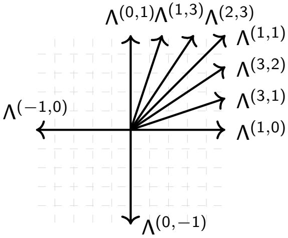

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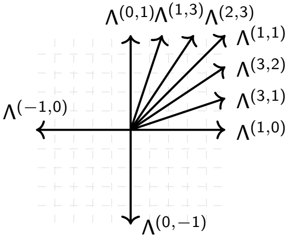

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Question

What can we say about $e_k^{(m,n)} \in \Lambda^{(m,n)}$ acting on $1 \in \Lambda$?

Rational Shuffle Theorem

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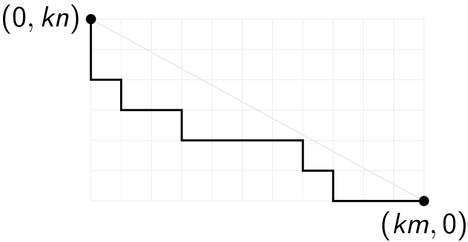

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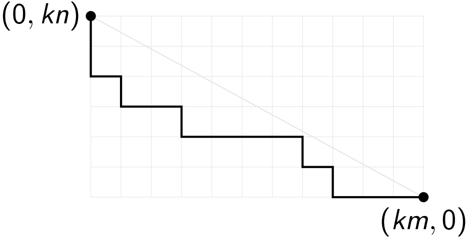

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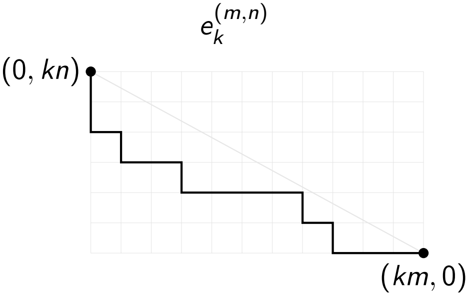

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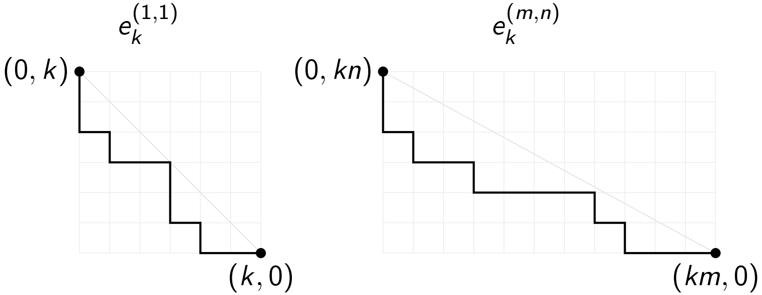

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$r, s \in \mathbb{R}_{>0} \to D^{(r,s)} \in \mathcal{E}$ such that $D^{(km,kn)} = e_k^{(m,n)}$.

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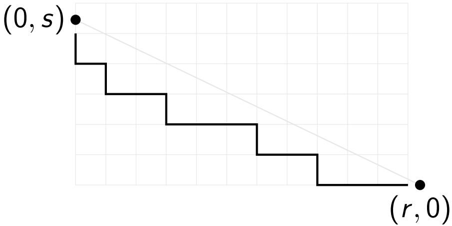

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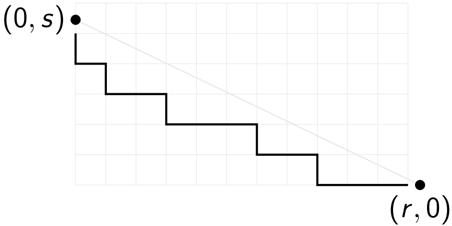

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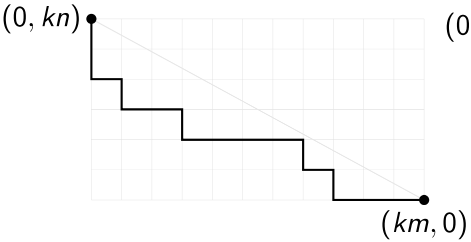

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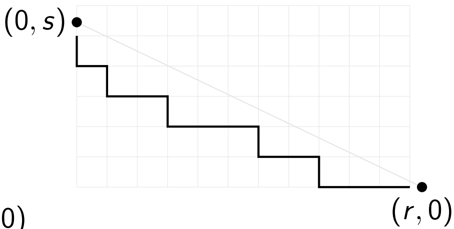

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Technique

Theory of “Catalanimals” bridging algebra and combinatorics.

Some applications

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Generalized q, t Catalan numbers

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(Capran, González, Hogancamp, Mazin, 2024+) use Generalized Shuffle Theorem formula to give KR homology of family of "Coxeter knots."

A Mystery: Convex Curve Conjecture

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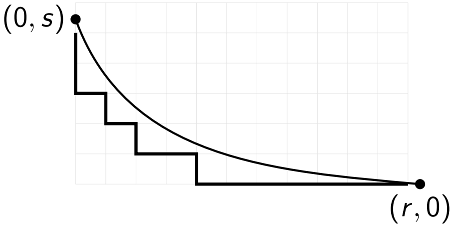
$D^{(C)} \in \mathcal{E}$ for convex curve C between $(0, s)$ and $(r, 0)$.



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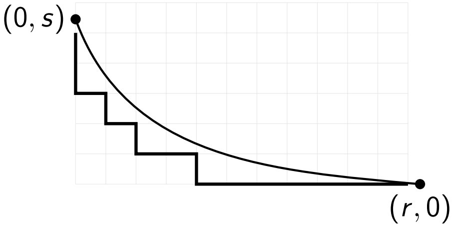

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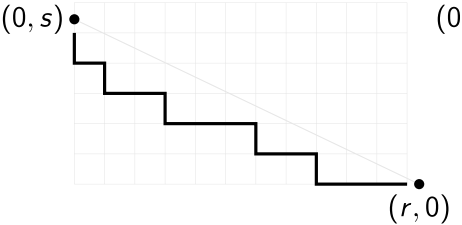

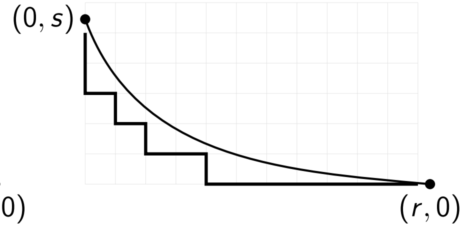

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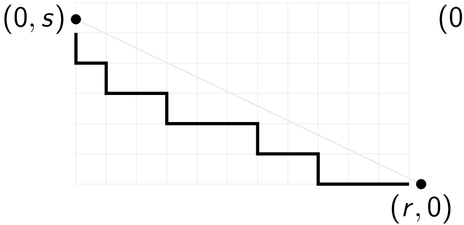

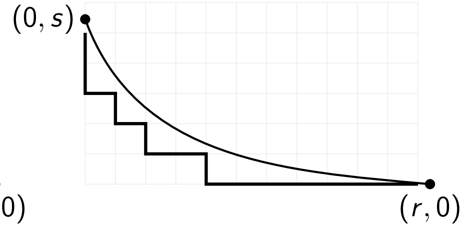

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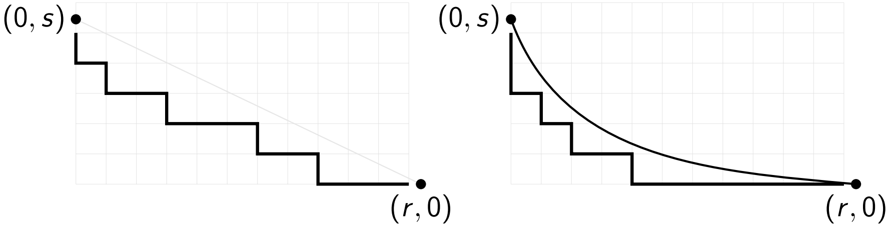

- S_n-module?
- $s_{(1,\ldots,1)}$ coefficient appears in context of "positroid varieties" (Galashin, Lam, 2024).

Overview

Common theme: $\mathcal{E} \curvearrowright \Lambda =$ Combinatorial sum

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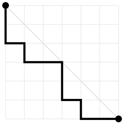

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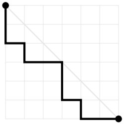

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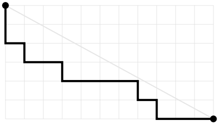

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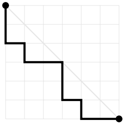

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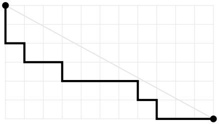

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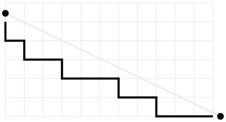

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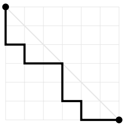

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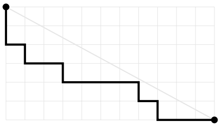

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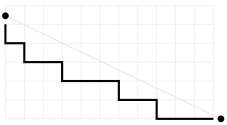

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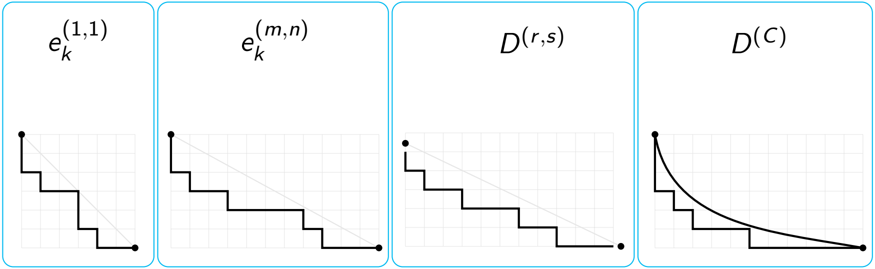

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Question

What about $s_\mu^{(m,n)} \in \Lambda^{(m,n)} \subseteq \mathcal{E}$?

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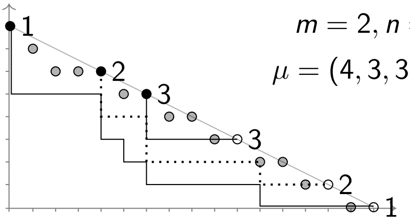

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$\mu = (4, 3, 3, 3, 2)$

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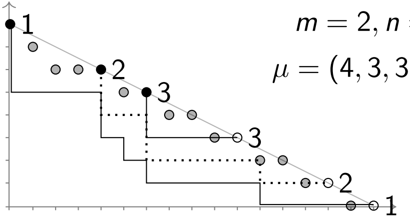

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Loehr-Warrington Conjecture (2008)

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Sum over "nested Dyck paths"

Catalanimals

Common theme: $\mathcal{E} \curvearrowright \Lambda =$ Combinatorial sum

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Thank you!