Diagonal Harmonics and Shuffle Theorems

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on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
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Capsule Research Talk

23 August 2021

Outline

- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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- E.g. for $n=3$,

$$\begin{aligned} e_1 &= x_1+x_2+x_3 = h_1 \\ e_2 &= x_1x_2+x_1x_3+x_2x_3 \quad h_2 = x_1^2+x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3+x_1^2x_2+x_1^2x_3+x_1x_2^2+\cdots \end{aligned}$$

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- Λ is a $\mathbb{Q}(q,t)$-algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \ldots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i<j}(1 - x_j/x_i)} \right)$$

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}[q,t]$) linear combinations in Schur polynomial basis are interesting.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

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- Algebraic LHS: ∇e_k doubly graded character of diagonal coinvariants for S_k ((Haiman, 2002) via Hilbert Scheme connection).

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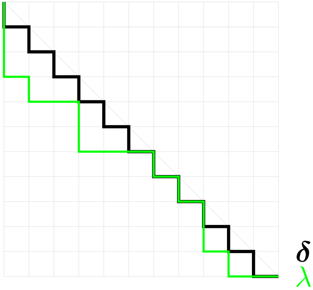
- $\mathrm{area}(\lambda)$ and $\mathrm{dinv}(\lambda)$ statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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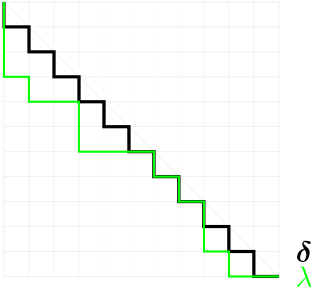

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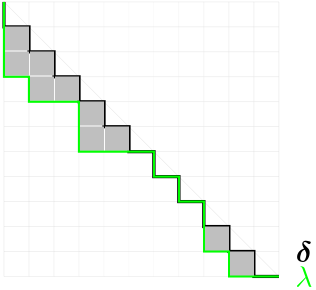

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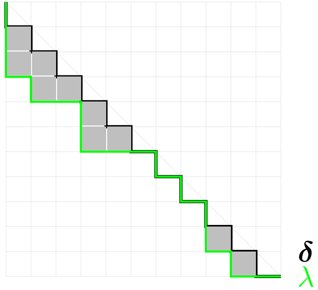

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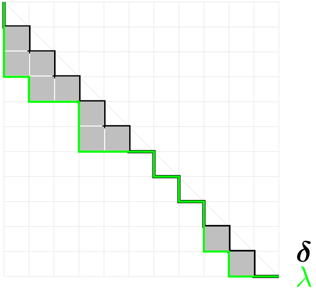

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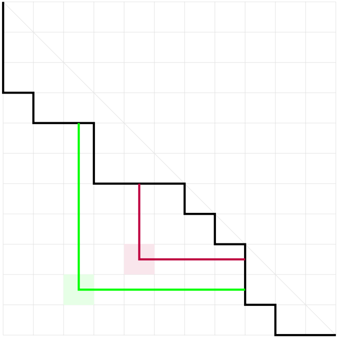
dinv

$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ.



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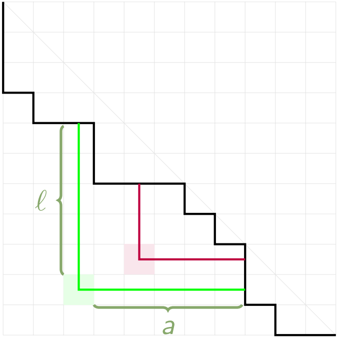

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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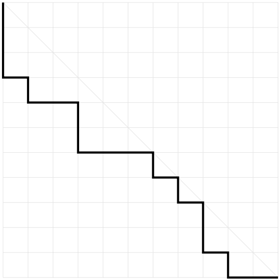

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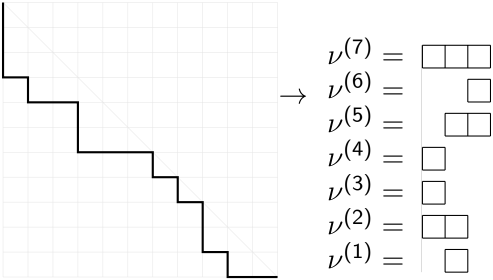

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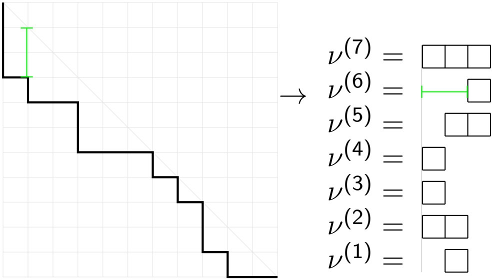

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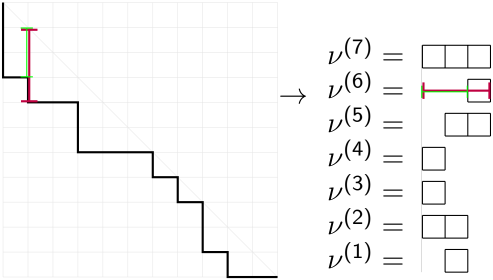

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for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

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- $\mathcal{G}_\nu$ is symmetric and Schur positive.

Example ∇e_3

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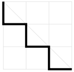		
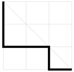		
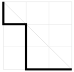		
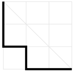		
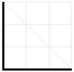		

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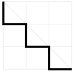
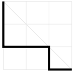
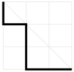
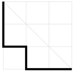
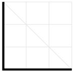
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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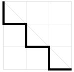
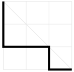
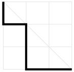
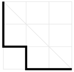
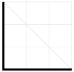
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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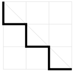
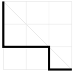
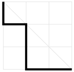
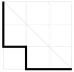
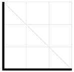
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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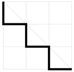
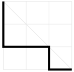
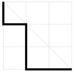
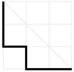
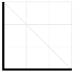
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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- Symmetric polynomials and The Shuffle Theorem
- **Generalizations of The Shuffle Theorem**
- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

- $\mathcal{E}$ contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

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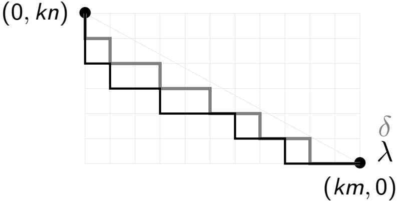

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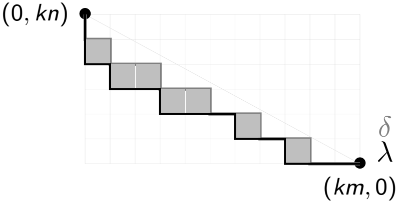

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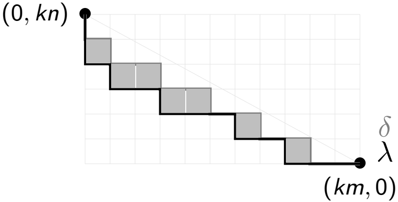

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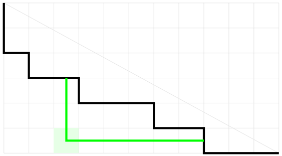

$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \qquad p = \frac{n}{m} - \epsilon$$

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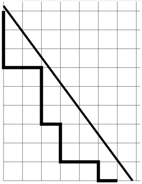

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Proof Idea

Key Relationship

$$\omega(D_{\mathrm{b}} \cdot 1)(x_1, \ldots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1<j}(1 - qtx_i/x_{i+1})}{\prod_{i<j}((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\mathrm{pol}}$$

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for infinite formal sum $\mathcal{L}^{\sigma}_{\beta/\alpha}$ a "series LLT." (Grojnowski-Haiman, 2007).

Cauchy Identity

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$$H_q(f) = \sum_{w\in S_l} w\left(f\prod_{i<j}((1 - x_j/x_i)(1 - qx_i/x_j))^{-1}\right)$$

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Loehr-Warrington Conjecture

$$\nabla s_\mu = \mathrm{sgn}(\mu) \sum_{(G,R)\in LNDP_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} x^R$$

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- Coefficient of $s_{1,\dots,1}$ coincides with (q, t)-polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

References

Thank you!

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