A raising operator formula for Macdonald polynomials via LLT polynomials in the Schiffmann algebra

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Outline

1. **Background on symmetric functions and Macdonald polynomials**
2. A new formula for Macdonald polynomials
3. LLT polynomials in the elliptic Hall algebra

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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- E.g. for $n=3$,

$$\begin{aligned} e_1 &= x_1+x_2+x_3 = h_1 \\ e_2 &= x_1x_2+x_1x_3+x_2x_3 \quad h_2 = x_1^2+x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3+x_1^2x_2+x_1^2x_3+x_1x_2^2+\cdots \end{aligned}$$

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- Λ is a $\mathbb{Q}(q,t)$-algebra.

Symmetric functions and Schur functions

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$,

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Then, for $\rho = (n-1, n-2, \ldots, 1, 0)$,

$$s_\gamma = \begin{cases} \mathrm{sgn}(\gamma+\rho) s_{\mathrm{sort}(\gamma+\rho)-\rho} & \text{if } \gamma+\rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

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Example: $s_{201} = 0, s_{2\text{-}11} = -s_{200}$.

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

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Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_+^{S_3})$.

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Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

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Answer: Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X;q)$.

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- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

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	(12)	(13)	(14)	(15)
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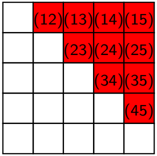
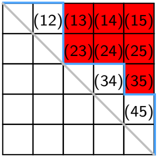


A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.



Ψ = Roots above Dyck path

Weyl symmetrization

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma}\colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$\boldsymbol{z}^\gamma \mapsto s_\gamma(X)$$

where $\boldsymbol{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

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$$H(\Phi; \gamma) = \boldsymbol{\sigma} \left(\frac{\boldsymbol{z}^\gamma}{\prod_{(i,j)\in\Psi}(1 - tz_i/z_j)} \right)$$

Denominator factors are understood as geometric series $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2(z_i/z_j)^2 + \cdots$

Catalan functions

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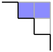
Example:

$\Psi =$  $\gamma = (1, 1, 1)$

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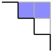
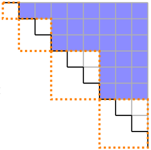
$\Psi =$  $\gamma = (1,1,1)$

$$\begin{aligned} H(\Psi;\gamma) &= \boldsymbol{\sigma}\left((1 + t\frac{z_1}{z_2} + t^2\frac{z_1^2}{z_2^2} + \cdots)(1 + t\frac{z_1}{z_3} + t^2\frac{z_1^2}{z_3^2} + \cdots)z_1z_2z_3\right) \\ &= s_{111} + t(s_{201} + s_{210}) + t^2(s_{3\text{-}10} + s_{300} + s_{31\text{-}1}) + \cdots \\ &= s_{111} + ts_{210} \end{aligned}$$

A Catalan function for modified Hall-Littlewoods

B_μ = set of roots above block diagonal matrix with block sizes $\mu_{\ell(\mu)}, \ldots, \mu_1$

$B_{3321} =$ 

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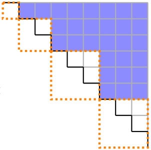
$B_{3321} =$ 

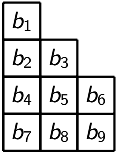
Theorem (Weyman, Shimozono-Weyman)

$$\tilde{H}_\mu(X; 0, t) = \omega \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu}(1 - t\boldsymbol{z}^\alpha)}\right),$$

where $\boldsymbol{z}^\alpha = z_i/z_j$.

$\omega(s_\lambda) = s_{\lambda'}$ for λ' the transpose partition of λ.

Catalan functions for modified Hall-Littlewoods

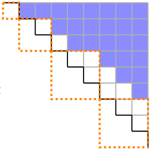


row reading order

$b_1 \prec b_2 \prec \cdots \prec b_n$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\}.$$

$R_{3321} =$



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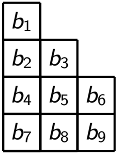

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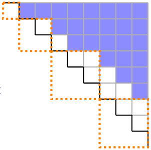

$$\tilde{H}_\mu(X; 0, t) = \omega \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu}(1 - t\boldsymbol{z}^\alpha)}\right),$$

$$= \omega \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu}(1 - t\boldsymbol{z}^\alpha)}\right)$$

A formula for $\tilde{H}_\mu(X; q, t)$

$$
\begin{array}{|c|c|}
\cline{1-1}
b_1 & \multicolumn{1}{c}{} \\
\cline{1-1}
b_2 & \multicolumn{1}{c}{} \\
\hline
b_3 & b_4 \\
\hline
b_5 & b_6 \\
\hline
b_7 & b_8 \\
\hline
\end{array}
$$

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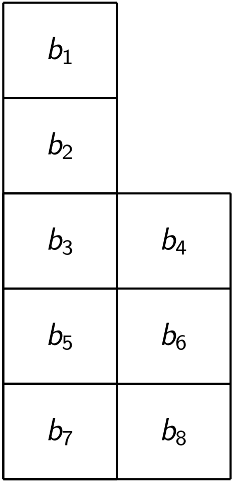
Theorem (Blasiak-Haiman-Morse-Pun-S.)

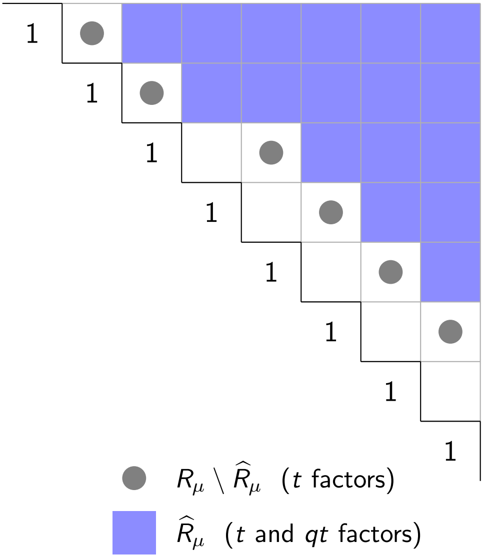
The modified Macdonald polynomial $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ is given by

$$\tilde{H}_\mu = \omega \boldsymbol{\sigma} \left(z_1 \cdots z_n \frac{\displaystyle\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i / z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

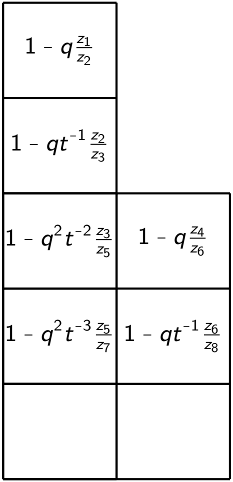
Example



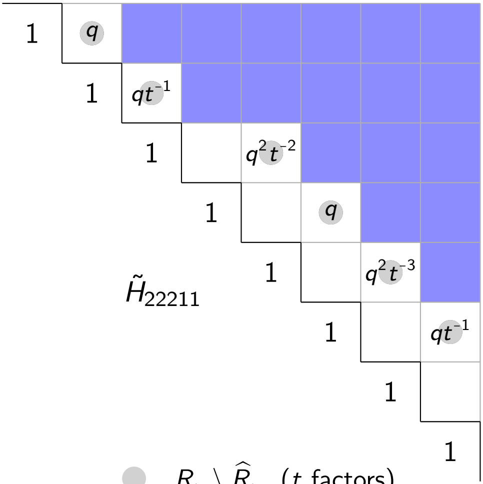
partition $\mu = 22211$



Example



numerator factors $1 - q^{\mathrm{arm}+1}t^{-\mathrm{leg}}z_i/z_j$



$R_\mu \setminus \widehat{R}_\mu$ (t factors)

$\widehat{R}_\mu$ (t and qt factors)

$q = t = 1$ specialization

$$\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n \frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)\\
\overset{q=t=1}{\to}&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\prod_{\alpha\in R_\mu\setminus\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in R_\mu}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega h_1^n\\
=&e_1^n
\end{aligned}$$

$q = 0$ specialization

$$\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\prod\limits_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod\limits_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)$$

$$\overset{q=0}{\rightarrow}\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_\mu}(1-t\boldsymbol{z}^\alpha)}\right)$$

$$=\tilde{H}_\mu(X;0,t)$$

Proof of formula for $\tilde{H}_\mu$

Definition

∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

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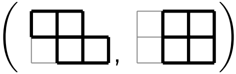
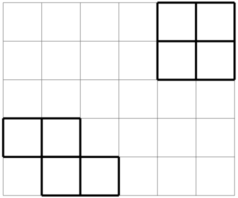
- Apply $\omega\nabla$ to both sides.
- Use Catalan-like ("Catalanimal") formula for $\omega\nabla\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ and collect terms.

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

$\boldsymbol{\nu} =$ 



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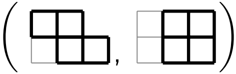
$\boldsymbol{\nu} =$ 

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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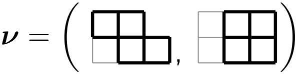

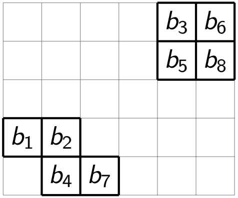

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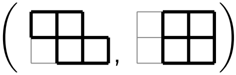
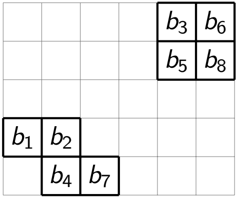
$\boldsymbol{\nu} =$ 



Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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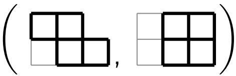
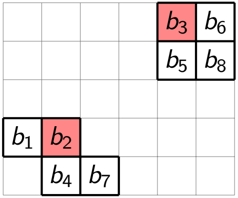
$\boldsymbol{\nu} =$ 



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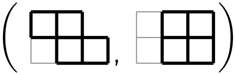
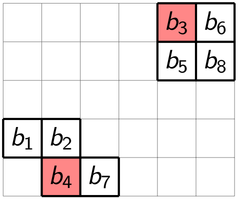
$\boldsymbol{\nu} =$ 



Attacking pairs: $(b_2, b_3), {\color{red}(b_3, b_4)}, (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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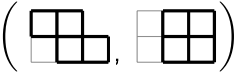
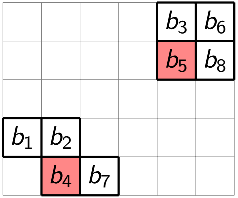
$\boldsymbol{\nu} =$ 



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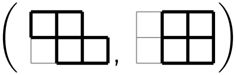
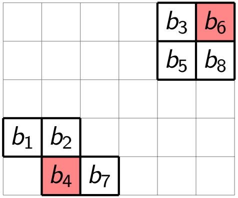
$\boldsymbol{\nu} =$ 



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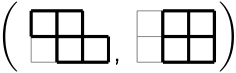
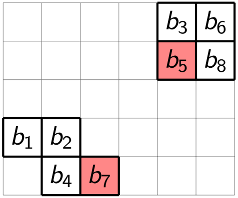
$\boldsymbol{\nu} =$ 



Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
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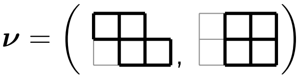

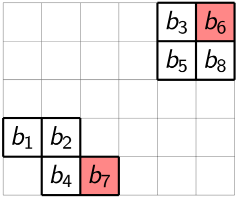

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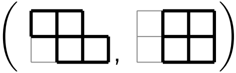
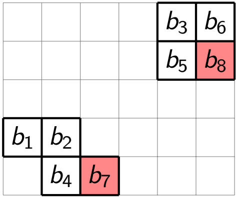
$\boldsymbol{\nu} =$ 



Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), \color{red}{(b_7, b_8)}$

LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T : \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$$T \quad = \quad \begin{array}{|c|c|c|c|c|c|}
\hline
 & & & & 5 & 6 \\
\hline
 & & & & 1 & 1 \\
\hline
 & & & & & \\
\hline
2 & 4 & & & & \\
\hline
 & 3 & 5 & & & \\
\hline
\end{array}$$

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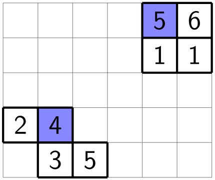

non-inversion

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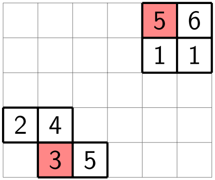

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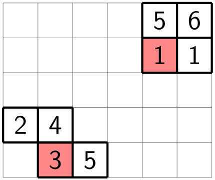

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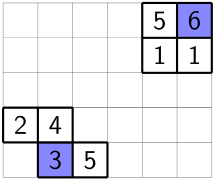

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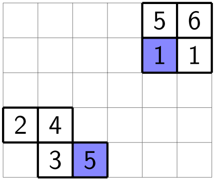

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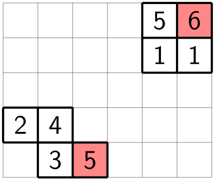

inversion

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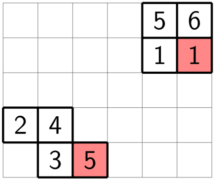
$T \quad =$ 

inversion

$\mathrm{inv}(T) = 4, \quad \boldsymbol{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$

Catalanimals

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha \in R_{qt}} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_q} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_t} \left(1 - t\boldsymbol{z}^\alpha\right)}\right).$$

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With $n = 3$,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1 \le i < j \le 3}(1 - qz_i/z_j)(1 - tz_i/z_j)}\right) \\ &= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2t + qt^2 + t^3)s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

LLT Catalanimals

For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal,
- $R_q \setminus R_t \; =$ the attacking pairs,
- $R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,
- λ: fill each diagonal D of $\boldsymbol{\nu}$ with
  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in reading order gives λ.

LLT Catalanimals

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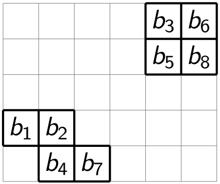
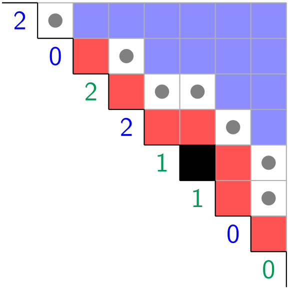


$\boldsymbol{\nu}$



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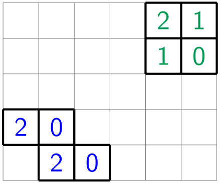
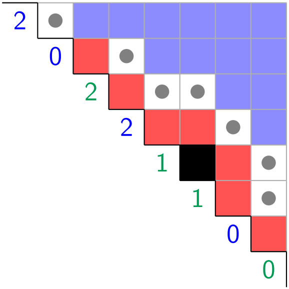


λ, as a filling of $\boldsymbol{\nu}$



LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let $\boldsymbol{\nu}$ be a tuple of skew shapes and let $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned} \nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) &= c_{\boldsymbol{\nu}}\, \omega H_{\boldsymbol{\nu}} \\ &= c_{\boldsymbol{\nu}}\, \omega \boldsymbol{\sigma} \left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} (1 - qt\, \boldsymbol{z}^{\alpha})}{\prod_{\alpha \in R_q} (1 - q\, \boldsymbol{z}^{\alpha}) \prod_{\alpha \in R_t} (1 - t\, \boldsymbol{z}^{\alpha})} \right) \end{aligned}$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

Haglund-Haiman-Loehr formula

Theorem (Haglund-Haiman-Loehr, 2005)

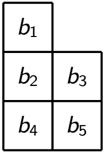
$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u \in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1} \right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X;q)\,,$$

where

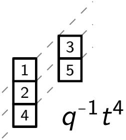
- *the sum runs over all subsets* $D \subseteq \{(i,j) \in \mu \mid j > 1\}$, *and*
- $\boldsymbol{\nu}(\mu, D) = (\nu^{(1)}, \ldots, \nu^{(k)})$ *where* $k = \mu_1$ *is the number of columns of* μ, *and* $\nu^{(i)}$ *is a ribbon of size* μ_i^*, *i.e., box contents* $\{-1, -2, \ldots, -\mu_i^*\}$, *and descent set* $Des(\nu^{(i)}) = \{-j \mid (i,j) \in D\}$.

Haglund-Haiman-Loehr formula example

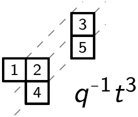
$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u\in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}\right) \mathcal{G}_{\nu(\mu,D)}(X;q)$$



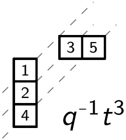
μ



$D = \{b_1, b_2, b_3\}$



$D = \{b_2, b_3\}$



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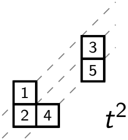

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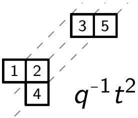

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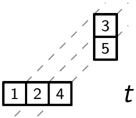

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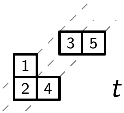
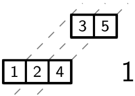


$D = \{b_1\}$



$D = \varnothing$

Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D}\mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.

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- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j)$ factor.

$$\tilde{H}_\mu = \omega\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

Outline

1. Background on symmetric functions and Macdonald polynomials
2. A new formula for Macdonald polynomials
3. **LLT polynomials in the elliptic Hall algebra**

Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra $\mathcal{E}$ of the Hall algebra of coherent sheaves on an elliptic curve over $\mathbb{F}_p$.

The *elliptic Hall algebra* $\mathcal{E}$ is generated by subalgebras $\Lambda(X^{a,b})$ isomorphic to the ring of symmetric functions Λ over $\mathbb{k} = \mathbb{Q}(q,t)$, one for each coprime pair $(a,b) \in \mathbb{Z}^2$, along with an additional central subalgebra.

Shuffle algebra

Define a linear map

$$\sigma_\Gamma \colon \bigoplus_n \mathbb{k}(z_1, \ldots, z_n) \to \bigoplus_n \mathbb{k}(z_1, \ldots, z_n)^{S_n}$$

whose graded components σ_Γ^n are given by

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$$\sigma_\Gamma^n(f) = \sum_{w \in S_n} w\Big(f(z_1, \dots, z_n) \prod_{1 \le i < j \le n} \Gamma(z_i, z_j)\Big),$$

$$\text{where } \Gamma(z_i, z_j) = \frac{1 - qtz_i/z_j}{(1 - z_j/z_i)(1 - qz_i/z_j)(1 - tz_i/z_j)}$$

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The *shuffle algebra* $\mathcal{S}_\Gamma$ is the image of $\bigoplus_n \Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ under the map σ_Γ, equipped with a variant of the concatenation product.

Shuffle algebra

Define a linear map

$$\sigma_\Gamma \colon \bigoplus_n \Bbbk(z_1, \ldots, z_n) \to \bigoplus_n \Bbbk(z_1, \ldots, z_n)^{S_n}$$

whose graded components σ_Γ^n are given by

$$\sigma_\Gamma^n \colon \Bbbk(z_1, \ldots, z_n) \to \Bbbk(z_1, \ldots, z_n)^{S_n}$$

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Nice fact (up to some modifications of definitions)

Some Catalanimals are elements in $\mathcal{S}_\Gamma$. ("Tame Catalanimals")

Shuffle to elliptic Hall isomorphism

- The *right half-plane subalgebra* $\mathcal{E}^+ \subseteq \mathcal{E}$ is generated by $\Lambda(X^{a,b})$ for $a > 0$.

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Theorem (Schiffmann-Vasserot)

There is an algebra isomorphism $\psi \colon \mathcal{S}_\Gamma \to \mathcal{E}^+$.

Elliptic Hall algebra action

Schiffmann-Vasserot and Feigin-Tsymbaliuk constructed an action of $\mathcal{E}$ on Λ, where $f(X^{0,1})$ acts by multiplication by $f(X)$.

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Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let H be a Catalanimal such that $\psi(H) = f(X^{1,1})$. Then

$$\nabla f = \omega H\,.$$

Shuffle to elliptic Hall summary

$$
\begin{tikzcd}
 & \mathcal{E} \curvearrowright \Lambda \\
\bigoplus_{\substack{a>0 \\ b\in\mathbb{Z} \\ (a,b)=1}} \Lambda(X^{a,b}) \arrow[r, phantom, "\underset{\text{v.sp.}}{\cong}"] & \mathcal{E}^+ \arrow[u, hook] \\
\sigma_\Gamma \left(\bigoplus_n \Bbbk[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]\right) \arrow[r, phantom, "\underset{\text{v.sp.}}{\cong}"] & \mathcal{S}_\Gamma \ni H \arrow[u, "\psi", "\cong"']
\end{tikzcd}
$$

"tame" Catalanimal

Theorem (Blasiak-Haiman-Morse-Pun-S.)

$\psi(H) = f(X^{1,1}) \Longrightarrow f(X^{1,1}) \cdot 1 = \nabla f = \omega H.$

Proof of $\nabla \mathcal{G}_{\nu}$ formula

1. LLT Catalanimals $H_{\boldsymbol{\nu}}$ are tame.
2. LLT Catalanimals lie in $\psi^{-1}(\Lambda(X^{1,1}))$.
3. Describe coproduct Δ on $\mathcal{E}$ explicitly on tame Catalanimals and show $\Delta H_{\boldsymbol{\nu}}$ matches $\Delta \mathcal{G}_{\boldsymbol{\nu}}$.
4. Conclude $\psi(H_{\boldsymbol{\nu}}) = c_{\boldsymbol{\nu}}^{-1} \mathcal{G}_{\boldsymbol{\nu}}(X^{1,1}) \in \mathcal{E}$.
5. Apply previous theorem to conclude $\nabla \mathcal{G}_{\boldsymbol{\nu}} = c_{\boldsymbol{\nu}} \omega H_{\boldsymbol{\nu}}$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

A positivity conjecture

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$$\tilde{H}_\mu^{(s)} := \omega\boldsymbol{\sigma}\left((z_1\cdots z_n)^s \frac{\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_\nu K_{\nu,\mu}^{(s)}(q,t)\, s_\nu(X)$$

satisfy $K_{\nu,\mu}^{(s)}(q,t)\in\mathbb{N}[q,t]$.

Thank you!

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021. *LLT Polynomials in the Schiffmann Algebra*, arXiv e-prints, arXiv:2112.07063.

———. 2023. *A Raising Operator Formula for Macdonald Polynomials*, arXiv e-prints, arXiv:2307.06517.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Feigin, B. L. and Tsymbaliuk, A. I. 2011. *Equivariant K-theory of Hilbert Schemes via Shuffle Algebra*, Kyoto J. Math. **51**, no. 4, 831–854.

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

Haglund, J., M. Haiman, and N. Loehr. 2005. *A Combinatorial Formula for Macdonald Polynomials* **18**, no. 3, 735–761 (electronic).

Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919

Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sém. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754

Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004

Schiffmann, Olivier and Vasserot, Eric. 2013. *The Elliptic Hall Algebra and the K-theory of the Hilbert Scheme of A2*, Duke Mathematical Journal **162**, no. 2, 279–366, DOI 10.1215/00127094-1961849.

Shimozono, Mark and Jerzy Weyman. 2000. *Graded Characters of Modules Supported in the Closure of a Nilpotent Conjugacy Class*, European Journal of Combinatorics **21**, no. 2, 257–288, DOI 10.1006/eujc.1999.0344.

Weyman, J. 1989. *The Equations of Conjugacy Classes of Nilpotent Matrices*, Inventiones mathematicae **98**, no. 2, 229–245, DOI 10.1007/BF01388851.

Catalanimals in the shuffle algebra

For $\lambda \in \mathbb{Z}^n$,

$$\begin{aligned}\sigma_\Gamma^n(\boldsymbol{z}^\lambda) &= \sum_{w \in S_n} w\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha \in R_+}\left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+}\left(\left(1 - \boldsymbol{z}^{-\alpha}\right)\left(1 - q\boldsymbol{z}^\alpha\right)\left(1 - t\boldsymbol{z}^\alpha\right)\right)}\right)\\ &= H(R_+, R_+, R_+, \lambda) \in \mathcal{S}_\Gamma.\end{aligned}$$

- Technicality: we have redefined $\boldsymbol{\sigma}(\boldsymbol{z}^\gamma) = \sum_{w \in S_n}\left(\frac{\boldsymbol{z}^\gamma}{\prod_{\alpha \in R_+}(1 - \boldsymbol{z}^{-\alpha})}\right) = \chi_\gamma$, the irreducible GL_n character.
- Let pol_X send $\chi_\lambda \mapsto s_\lambda$ if $\lambda_n \geq 0$, otherwise $\chi_\lambda \mapsto 0$.
- The $\boldsymbol{\sigma}$ from before is given by $\boldsymbol{\sigma}_{\mathrm{old}} = \mathrm{pol}_X\, \boldsymbol{\sigma}_{\mathrm{new}}$.

Catalanimals in the Shuffle algebra

$\sigma^n_\Gamma(f)$ can lie in $\mathcal{S}_\Gamma$ even when f is not a Laurent polynomial.

Theorem (Negut)

The following family of Catalanimals lie in the shuffle algebra:

$$\sigma^n_\Gamma\left(\frac{\boldsymbol{z}^\lambda}{\prod_{i=1}^{n-1}(1-qtz_i/z_{i+1})}\right) = H(R_+, R_+, R'_+, \lambda) \in \mathcal{S}_\Gamma,$$

where $\quad R'_+ = \{\alpha_{ij} \in R_+ \mid i+1 < j\}.$

The wheel condition

- A symmetric Laurent polynomial $g(\boldsymbol{z})$ satisfies the *wheel condition* if it vanishes whenever any three of the variables z_i, z_j, z_k are in the ratio $(z_i : z_j : z_k) = (1 : q : qt) = (1 : t : qt)$.
- Let $\mathcal{S}_{\check{\Gamma}} \cong \mathcal{S}_{\Gamma}$ for $\check{\Gamma}(z_i, z_j) = (1 - z_i/z_j)(1 - qz_j/z_i)(1 - tz_j/z_i)(1 - qtz_i/z_j)$.

Theorem (Negut)

A symmetric Laurent polynomial $g(z_1, \dots, z_n)$ belongs to $\mathcal{S}_{\check{\Gamma}}$ if and only if it satisfies the wheel condition and vanishes whenever $z_i = z_j$ for $i \neq j$.

The wheel condition and tame Catalanimals

A Catalanimal $H(R_q, R_t, R_{qt}, \lambda)$ is *tame* if

$$R_q + R_t \subseteq R_{qt},$$

where $R_q + R_t = \{\alpha + \beta \mid \alpha \in R_q, \beta \in R_t\}$.

■ $R_+ \setminus R_q$
■ $R_q \setminus R_t$
• $R_t \setminus R_{qt}$
■ R_{qt}

The Catalanimals $H(R_+, R_+, R'_+, \lambda)$ and the LLT Catalanimals are tame.

Using Negut's theorem, we show: Tame Catalanimals belong to the shuffle algebra $\mathcal{S}_\Gamma$.