

# A Window into Symmetric Function Theory

George H. Seelinger

*ghs9ae@virginia.edu*

UVA Math Club  
Lightning Round

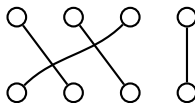
2 March 2021

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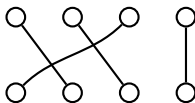
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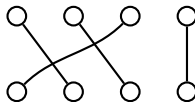
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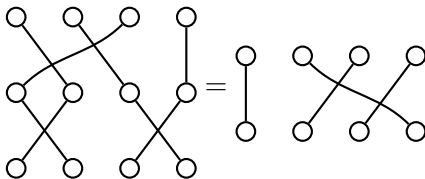
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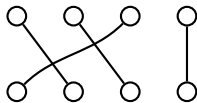
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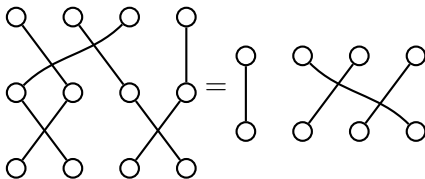


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- $S_n$  is a “group”

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- $\sigma \in S_n$  acts as  $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

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$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$  forms a vector space,  $\Lambda_{\mathbb{Q}}$ .

# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

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Basis of  $\Lambda_{\mathbb{Q}}$ ?

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .



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5  $\rightarrow$ 

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$4 + 1 \rightarrow$ 


$3 + 2 \rightarrow$ 


$3 + 1 + 1 \rightarrow$  

$2 + 2 + 1 \rightarrow$ 


$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

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- ② Many interesting connections to number theory (Ramanujan).
- ③ Generating function for  $p(n)$  = number of partitions of  $n$  is inverse of Euler  $\phi$  function.

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For  $\lambda = (2, 1)$ ,

1	1	1	1	1	1	1	1
2		3		3		3	

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Associate a polynomial to  $\text{SSYT}(\lambda)$ .

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1	1								
2									

, 

1	1								
3									

, 

2	2								
3									

, 

1	2								
2									

, 

1	3								
3									

, 

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda = \sum_{T \in \text{SSYT}} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for  $\Lambda_{\mathbb{Q}}$



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## Harmonic polynomials

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

# Harmonic polynomials

- ①  $S_3$  action on  $M$  fixes vector subspaces!

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Schur basis expansion counts multiplicity of irreducible  $S_n$  fixed subspaces!

## Upshot

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- 1 Schur functions  $\leftrightarrow S_n$ -invariant subspaces.

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- 2 Via Frobenius characteristic map, questions about  $S_n$ -action on vector spaces get translated to questions about Schur expansion coefficients in symmetric functions.

Interesting algebraic combinatorics questions

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- 1 Is a symmetric function Schur positive?



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- 2 What do the Schur expansion coefficients count?

# Getting more information

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Break  $M$  up into smallest  $S_n$  fixed subspaces

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Solution: minimal  $S_n$ -fixed subspace of degree  $d \mapsto q^d s_\lambda$  (graded Frobenius)

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- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .

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- Define  $\nabla$  by  $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$  for eigenvalue  $B_\mu(q, t) \in \mathbb{Q}[q, t]$ .

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Recover earlier story by taking  $t = 0$  and  $y_i = 1$  for all  $y_i$ 's.