

# Dens, nests, and Catalan animals: a walk through the zoo of shuffle theorems

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joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

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- $\sigma \in S_n$  acts as  $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$  forms a vector space,  $\Lambda_{\mathbb{Q}}$ .

# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$



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Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

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Basis of  $\Lambda_{\mathbb{Q}}$ ?

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\hline\hline\hline\hline\hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|}\hline\hline\hline\hline\hline\end{array}$$

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Collection is called  $\text{SSYT}(\lambda)$ .



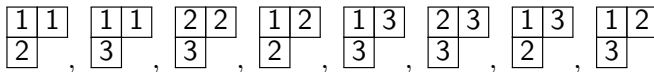
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For  $\lambda = (2, 1)$ ,

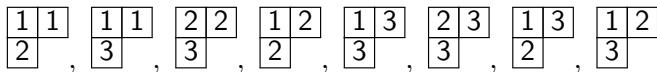


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 $(2,1,0)$   $(2,0,1)$   $(0,2,1)$   $(1,2,0)$   $(1,0,2)$   $(0,1,2)$   $(1,1,1)$   $(1,1,1)$

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for  $\Lambda_{\mathbb{Q}}$



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## Harmonic polynomials

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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- ①  $S_3$  action on  $M$  fixes vector subspaces!

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Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

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Solution: minimal  $S_n$ -fixed subspace of degree  $d \mapsto q^d s_\lambda$  (graded Frobenius)

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Answer: "Hall-Littlewood polynomial"  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .
- Does there exist a family of  $S_n$ -representations whose (bigraded) Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .



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## Theorem (Haiman, 2001)

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- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ . (Still open!)

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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## Question

What symmetric function is the bigraded Frobenius characteristic of  $DH_n$ ?

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Theorem (Haiman, 2002)

*The bigraded Frobenius characteristic of  $DH_n$  is given by  $\nabla e_n$ .*



# A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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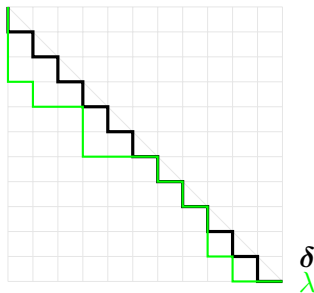
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# Dyck paths

## Dyck paths

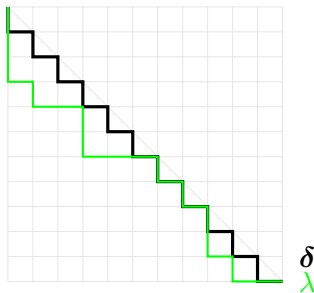
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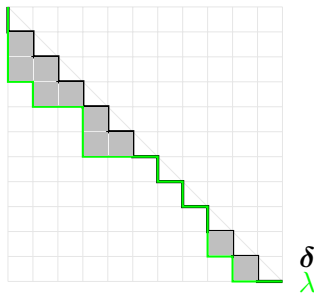


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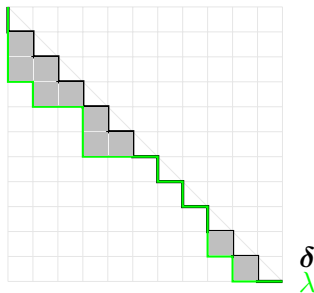
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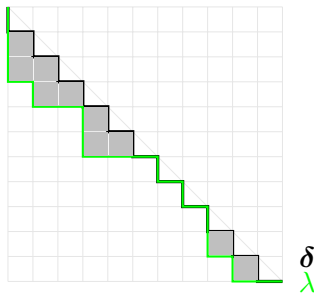


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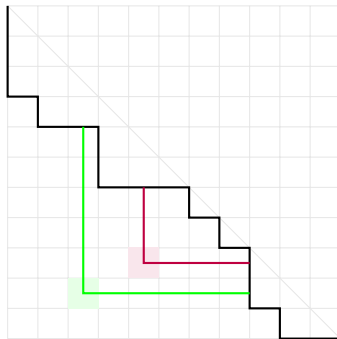
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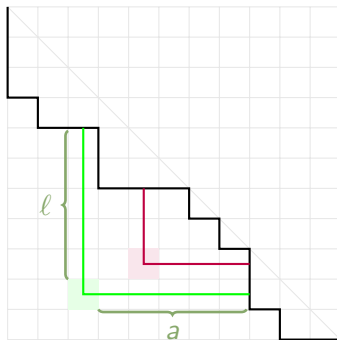


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Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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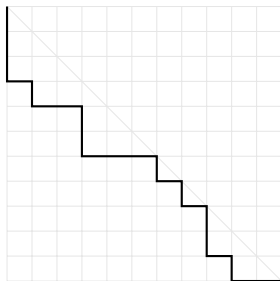
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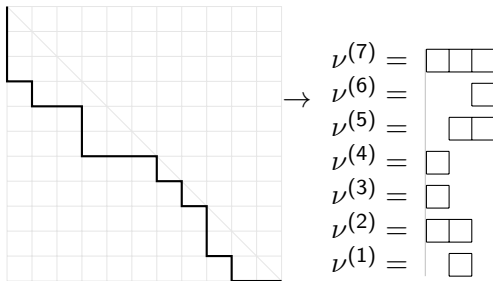
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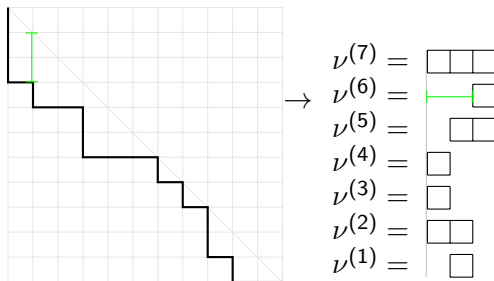
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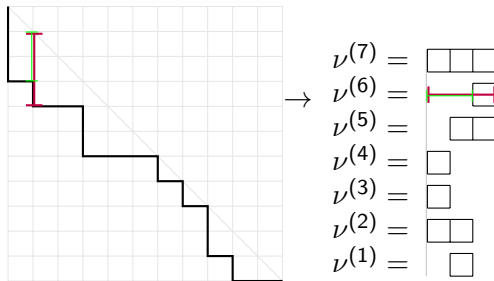
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$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
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$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

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$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

## Example $\nabla e_3$






$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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




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- Entire quantity is  $q, t$ -symmetric
- Coefficient of  $s_{1,1,1}$  in sum is a “ $(q, t)$ -Catalan number”  
 $(q^3 + q^2t + qt + qt^2 + t^3)$ .

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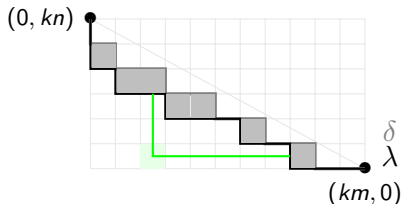
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- $\mathcal{E}$  contains subalgebra  $\Lambda(X^{m,n}) \cong \Lambda$  for each coprime pair  $(m, n) \in \mathbb{Z}^2$ .
- In general,  $\mathcal{E}$ -action can be a pain to compute in a nice way, but sometimes it is nice!



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Fix  $l \in \mathbb{Z}_{>0}$ . Let  $R_+ = \{(i, j) \mid 1 \leq i < j \leq l\}$  .

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- Can also be thought of as an infinite series of virtual  $GL_l$ -characters.
- We can take “polynomial part” (restrict to only polynomial  $GL_l$ -characters) to get a symmetric function.

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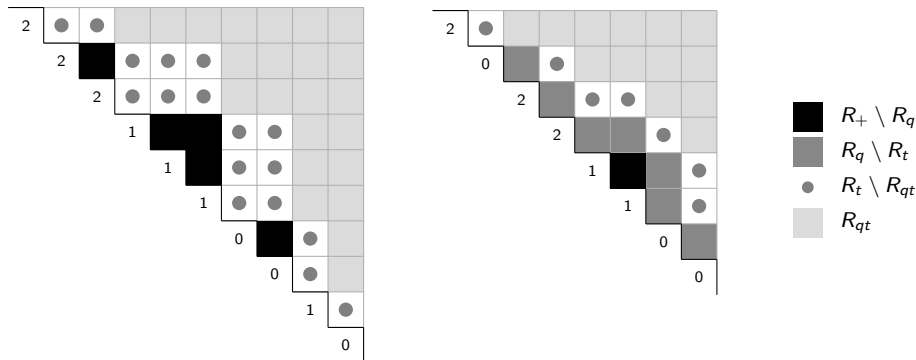
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- In this case, we set  $\operatorname{cub}(H) = f$ .
- The cuddly conditions allow a nice coproduct formula for  $f[X + Y]$  in terms of cubs of “restrictions” of  $H$ .

# Cuddly Catalananimals with cub $e_k$

- $H(R_+, R_+, [R_+, R_+], (1^k))$  is  $(1, 1)$ -cuddly with cub  $e_k$ .

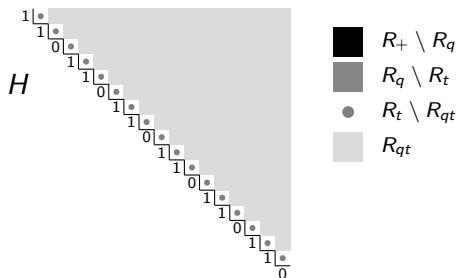
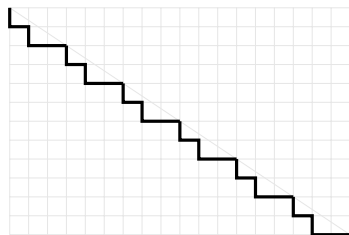
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$\delta = (1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0)$  and  $e_6[-MX^{3,2}] \cdot 1 = \omega \text{pol}_X H$

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


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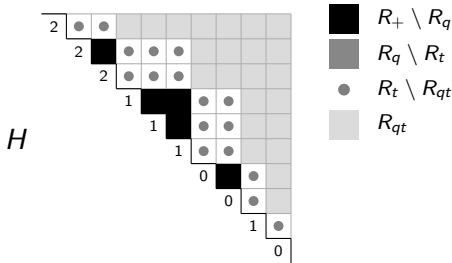
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## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

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- Conjectured by Loehr-Warrington (2008) when  $n = 1$  with different combinatorics (but bijectively related).

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- $h = m(\text{largest hook length in } \mu) = m(\mu_1 + \ell(\mu) - 1).$



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$$\begin{aligned} s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \end{aligned}$$

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- $p(\mu)$  = number of boxes with positive content.



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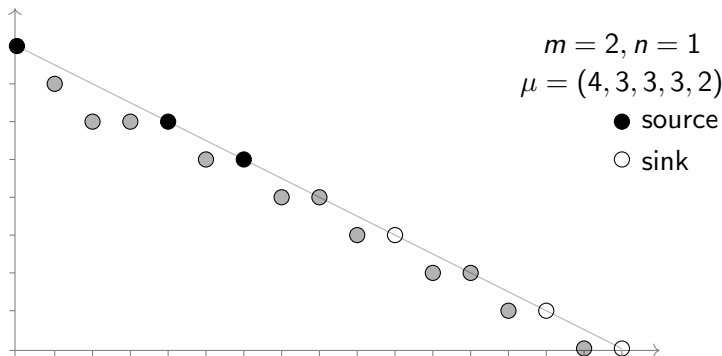
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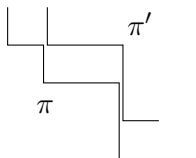
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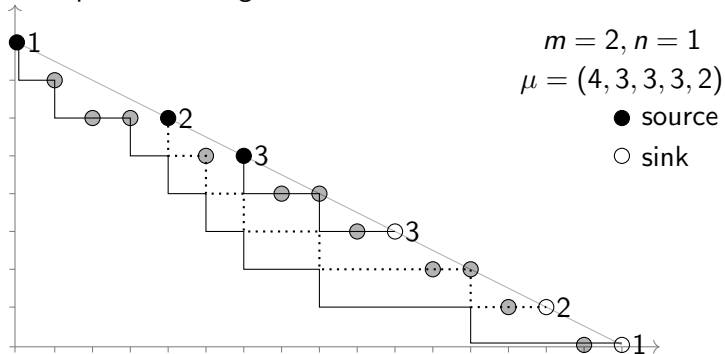
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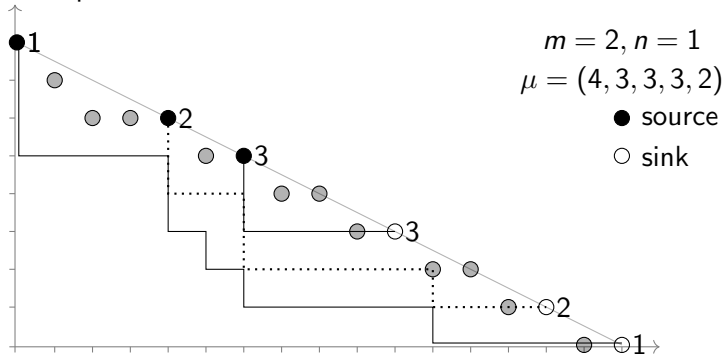
# Dens and nests

Example of the “highest nest”  $\pi^0$



# Dens and nests

Example of another nest.

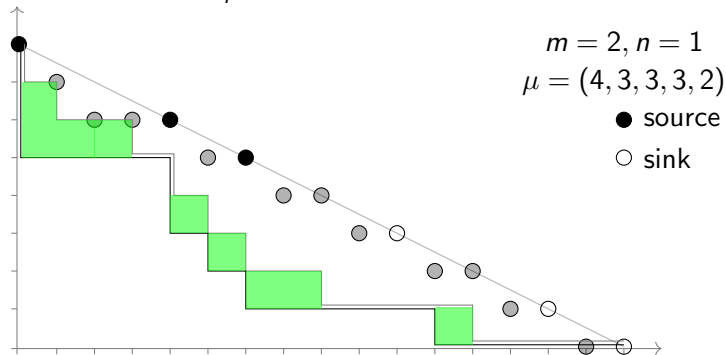




$\text{area}(\pi) = \sum_{i=1}^r \text{area}(\pi_i)$  where  $\text{area}(\pi_i) =$  number of lattice squares between  $\pi_i$  and  $\pi_i^0$ .

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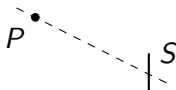
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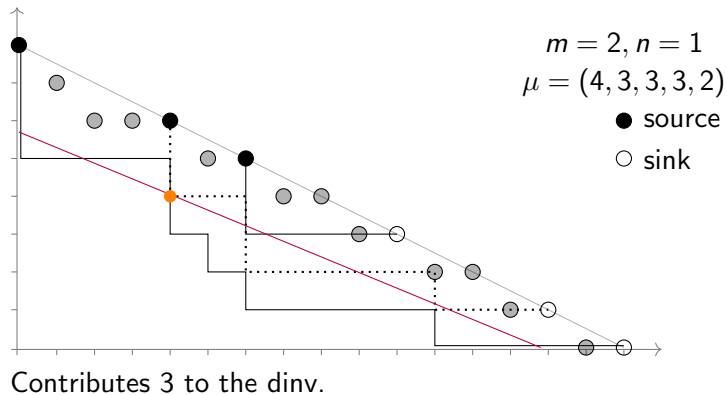


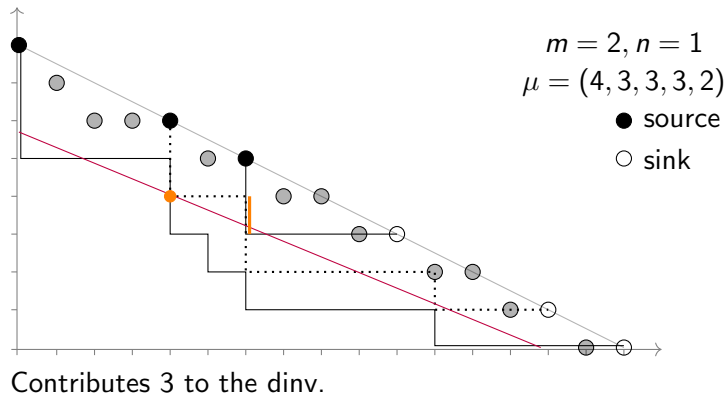
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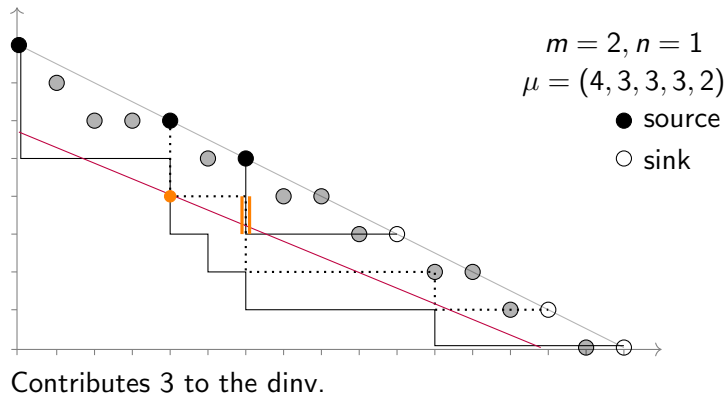
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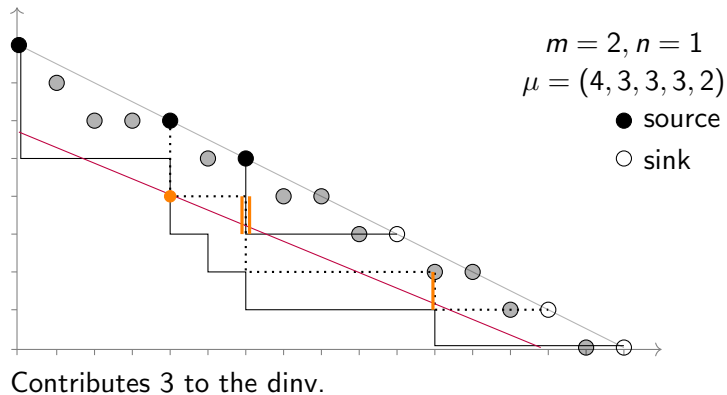












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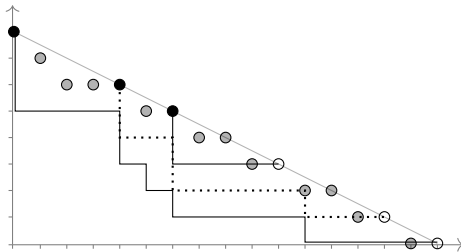
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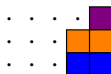
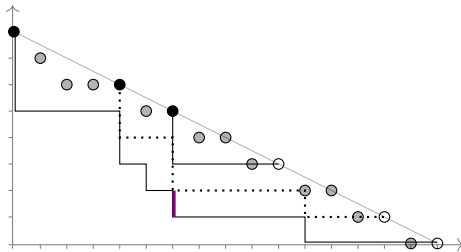
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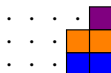
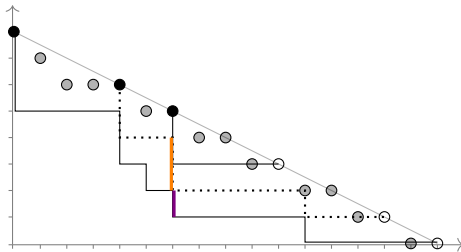
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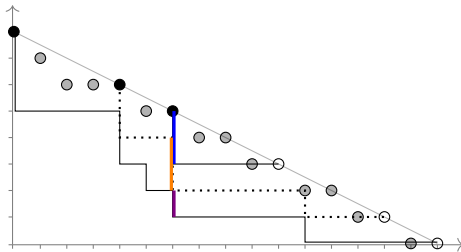
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- These results hold “stably.” In other words, a stronger result is proven before applying polynomial truncation.
- This allows us to simultaneously generalize the  $s_\lambda[-MX^{m,n}]$  formula and our “shuffle theorem for paths under any line” formula (BHMPs).

## Other exhibits for next time

- For each LLT polynomial  $\mathcal{G}_\nu$  and coprime  $(m, n)$  with  $m > 0$ , an  $m, n$ -cuddly Catalanimal with cub  $\mathcal{G}_\nu$  is given. (BHMPs)

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- Special cases include Schur functions and Hall-Littlewood polynomials.
- Unicorn Catalan animals (or Catalan functions) where  $R_t = R_{qt} = \emptyset$  also have a rich (older) results and combinatorics, but served as inspiration. (Chen-Haiman, Blasiak-Morse-Pun-Summers, Blasiak-Morse-Pun)



## Future work: exit through the gift shop

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- What connections do Catalan animals have with machinery used to prove other shuffle theorems, such as work by Carlsson-Mellit?

# Thank you for visiting!

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