

A Catalan formula for Macdonald polynomials

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Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n , where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

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A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.

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$\Psi = \text{Roots above Dyck path}$

Symmetric functions and Schur functions

- Let $\Lambda(X)$ be the ring of symmetric functions in $X = x_1, x_2, \dots$
- $h_d = h_d(X) = \sum_{i_1 \leq \dots \leq i_d} x_{i_1} \cdots x_{i_d}$ with $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$,

$$s_\gamma = s_\gamma(X) = \det(h_{\gamma_i + j - i}(X))_{1 \leq i, j \leq n}$$

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Then,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β ,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\operatorname{sort}(\beta)$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$ by linearly extending

$$\mathbf{z}^\gamma \mapsto s_\gamma(X)$$

where $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

Modified Macdonald polynomials

The *modified Macdonald polynomials* $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ are Schur positive symmetric functions in $X = x_1, x_2, \dots$ over $\mathbb{Q}(q, t)$.

They differ from the *integral form Macdonald polynomials* by

$$\tilde{H}_\mu(X; q, t) = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right].$$

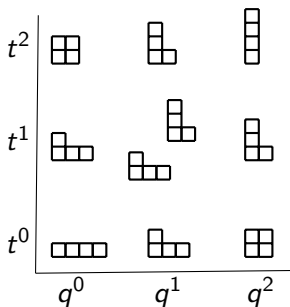
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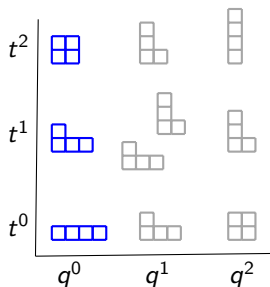
$$\tilde{H}_{22} = s_4 + (q + t + qt)s_{31} + (q^2 + t^2)s_{22} + (qt + q^2t + qt^2)s_{211} + q^2t^2s_{1111}$$



Modified Hall-Littlewood polynomials

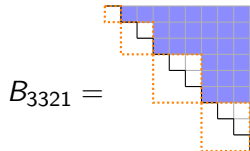
When $q = 0$, the modified Macdonald polynomials reduce to the *modified Hall-Littlewood polynomials* $\tilde{H}_\mu(X; 0, t)$.

$$\tilde{H}_{22}(X; 0, t) = s_4 + ts_{31} + t^2s_{22}$$



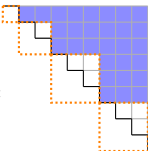
A Catalan function for modified Hall-Littlewoods

B_μ = set of roots above block diagonal matrix with block sizes $\mu_{\ell(\mu)}, \dots, \mu_1$



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$$B_{3321} =$$


Theorem (Weyman, Shimozono-Weyman)

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - t \mathbf{z}^\alpha)} \right),$$

where $\mathbf{z}^\alpha = z_i / z_j$.

Catalan functions for modified Hall-Littlewoods

b_1		
b_2	b_3	
b_4	b_5	b_6
b_7	b_8	b_9

row reading order

$$b_1 \prec b_2 \prec \cdots \prec b_n$$

$$R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \}.$$

$$R_{3321} =$$

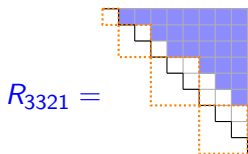
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A Catalan formula for $\tilde{H}_\mu(X; q, t)$

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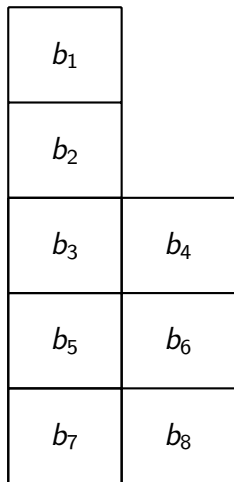
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Theorem (Blasiak-Haiman-Morse-Pun-S.)

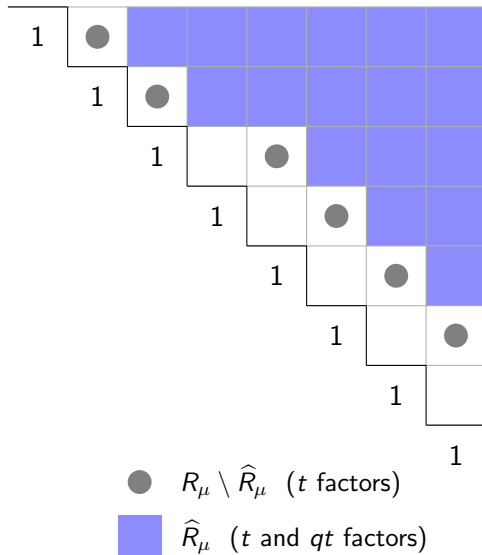
The modified Macdonald polynomial $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ is given by

$$\tilde{H}_\mu = \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right).$$

Example



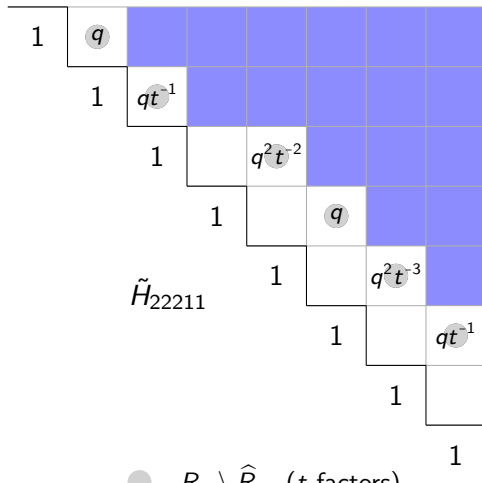
partition $\mu = 22211$



Example

$1 - q^{\frac{z_1}{z_2}}$	
$1 - qt^{-1} \frac{z_2}{z_3}$	
$1 - q^2 t^{-2} \frac{z_3}{z_5}$	$1 - q^{\frac{z_4}{z_6}}$
$1 - q^2 t^{-3} \frac{z_5}{z_7}$	$1 - qt^{-1} \frac{z_6}{z_8}$

numerator factors $1 - q^{\text{arm}+1} t^{-\text{leg}} z_i/z_j$



● $R_\mu \setminus \hat{R}_\mu$ (t factors)

■ \hat{R}_μ (t and qt factors)

$q = t = 1$ specialization

$$\begin{aligned}
 & \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=t=1} \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \hat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\
 & = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\
 & = \omega h_1^n \\
 & = e_1^n
 \end{aligned}$$

$q = 0$ specialization

$$\begin{aligned}
 & \omega\sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \widehat{R}_\mu} (1 - qt\mathbf{z}^\alpha)}{\prod_{\alpha \in R_+} (1 - q\mathbf{z}^\alpha) \prod_{\alpha \in R_\mu} (1 - t\mathbf{z}^\alpha)} \right) \\
 & \xrightarrow{q=0} \omega\sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t\mathbf{z}^\alpha)} \right) \\
 & = \tilde{H}_\mu(X; 0, t)
 \end{aligned}$$

Proof of formula for \tilde{H}_μ

Definition

∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

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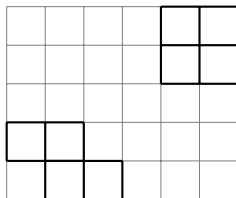
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- Apply $\omega \nabla$ to both sides.
- Use Catalanimal formula for $\omega \nabla \mathcal{G}_\nu(X; q)$ and collect terms.

LLT Polynomials

Let $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$ be a tuple of skew shapes.

$$\nu = \left(\begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$



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-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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				b_3	b_6
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Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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LLT Polynomials

- A *semistandard tableau* on ν is a map $T: \nu \rightarrow \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where $\text{inv}(T)$ is the number of attacking inversions in T and $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$.

$T =$

				5	6
				1	1
2	4				
	3	5			

LLT Polynomials

- A *semistandard tableau* on ν is a map $T: \nu \rightarrow \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where $\text{inv}(T)$ is the number of attacking inversions in T and $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$.

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$$\text{inv}(T) = 4, \quad \mathbf{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left(\frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qtz^\alpha)}{\prod_{\alpha \in R_q} (1 - qz^\alpha) \prod_{\alpha \in R_t} (1 - tz^\alpha)} \right).$$

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With $n = 3$,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left(\frac{z^{111} (1 - qt z_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - q z_i/z_j) (1 - t z_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

LLT Catalananimals

For a tuple of skew shapes ν , the *LLT Catalananimal* $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

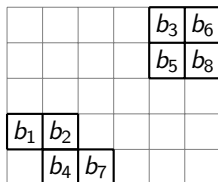
- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
- $R_q \setminus R_t$ = the attacking pairs,
- $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- λ : fill each diagonal D of ν with $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
Listing this filling in reading order gives λ .

LLT Catalanimals

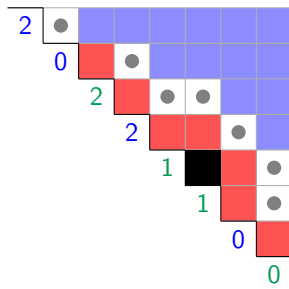
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal,
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- $R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,
- $R_{qt} =$ all other pairs,

λ : fill each diagonal D of ν with

$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end}).$



ν

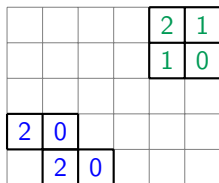


LLT Catalanimals

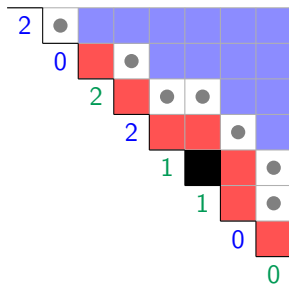
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λ , as a filling of ν



Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let ν be a tuple of skew shapes and let $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega \operatorname{pol}_X(H_\nu) \\ &= c_\nu \omega \operatorname{pol}_X \sigma \left(\frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

Haglund-Haiman-Loehr formula

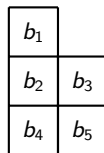
Theorem (Haglund-Haiman-Loehr, 2005)

$$\tilde{H}_\mu(X; q, t) = \sum_D \left(\prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q),$$

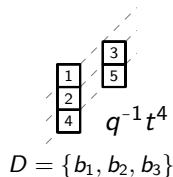
where

- the sum runs over all subsets $D \subseteq \{(i, j) \in \mu \mid j > 1\}$, and
- $\nu(\mu, D) = (\nu^{(1)}, \dots, \nu^{(k)})$ where $k = \mu_1$ is the number of columns of μ , and $\nu^{(i)}$ is a ribbon of size μ_i^* , i.e., box contents $\{-1, -2, \dots, -\mu_i^*\}$, and descent set $\text{Des}(\nu^{(i)}) = \{-j \mid (i, j) \in D\}$.

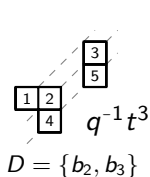
Haglund-Haiman-Loehr formula example



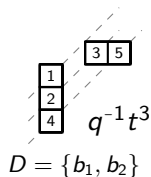
μ



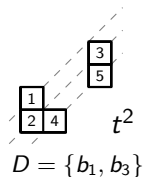
$D = \{b_1, b_2, b_3\}$



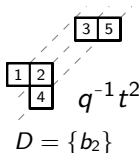
$D = \{b_2, b_3\}$



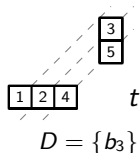
$D = \{b_1, b_2\}$



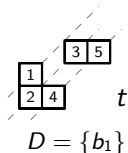
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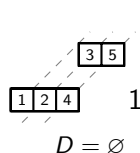
$D = \{b_2\}$



$D = \{b_3\}$



$D = \{b_1\}$



$D = \emptyset$

Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$ and apply $\omega \nabla$.

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- By construction, all the LLT Catalan animals $H_{\nu(\mu,D)}$ appearing on the LHS will have the same root ideal data (R_q, R_t, R_{qt}) .
- Collect terms to get $\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$ factor.

$$\tilde{H}_\mu = \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

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$$\tilde{H}_{\mu}^{(s)} := \omega \sigma \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_{\mu} \setminus \hat{R}_{\mu}} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_{\mu}} (1 - q t z^{\alpha})}{\prod_{\alpha \in R_+} (1 - q z^{\alpha}) \prod_{\alpha \in R_{\mu}} (1 - t z^{\alpha})} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s , the symmetric function $\tilde{H}_{\mu}^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_{\mu}^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_{\nu}(X)$$

satisfy $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$.

Thank you!

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