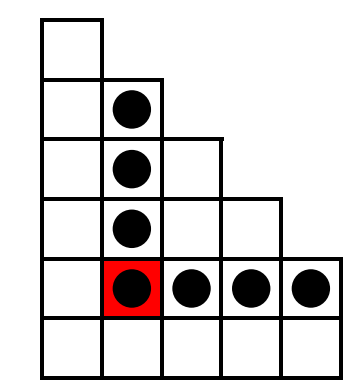


Overview

- Schur functions, s_λ , and Grothendieck polynomials, G_λ , give representatives for cohomology and K -theory of the Grassmannian.
- Pieri rules determine the structure constants of these rings.
- Representatives are known for (co)homology of affine Grassmannian.
- Aim: develop similar picture for affine K -theory.

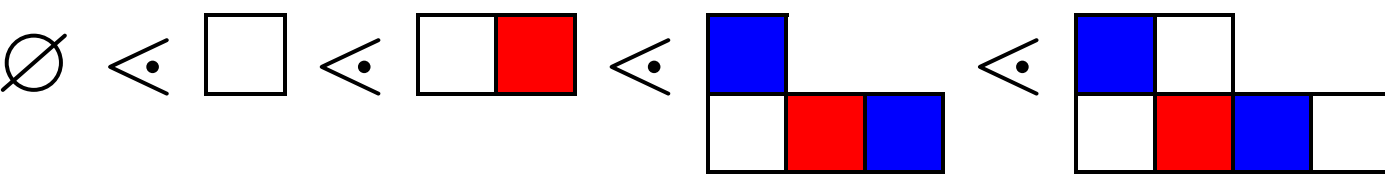
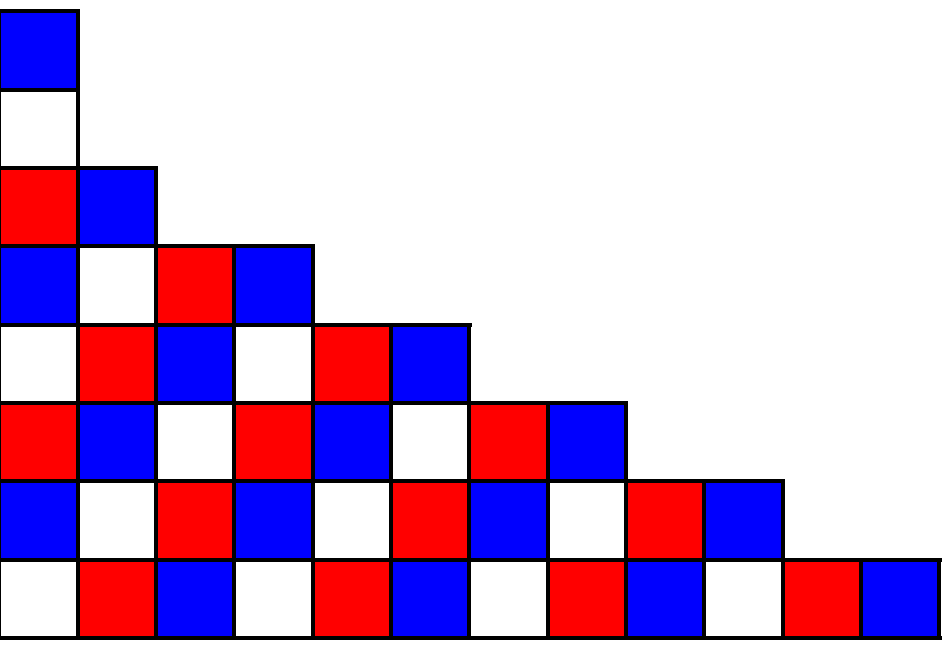
Affine Combinatorics

$w \in \tilde{S}_n \leftrightarrow n$ -cores
 n -core=partition with no cell of hook-length n
red cell has hook-length 7



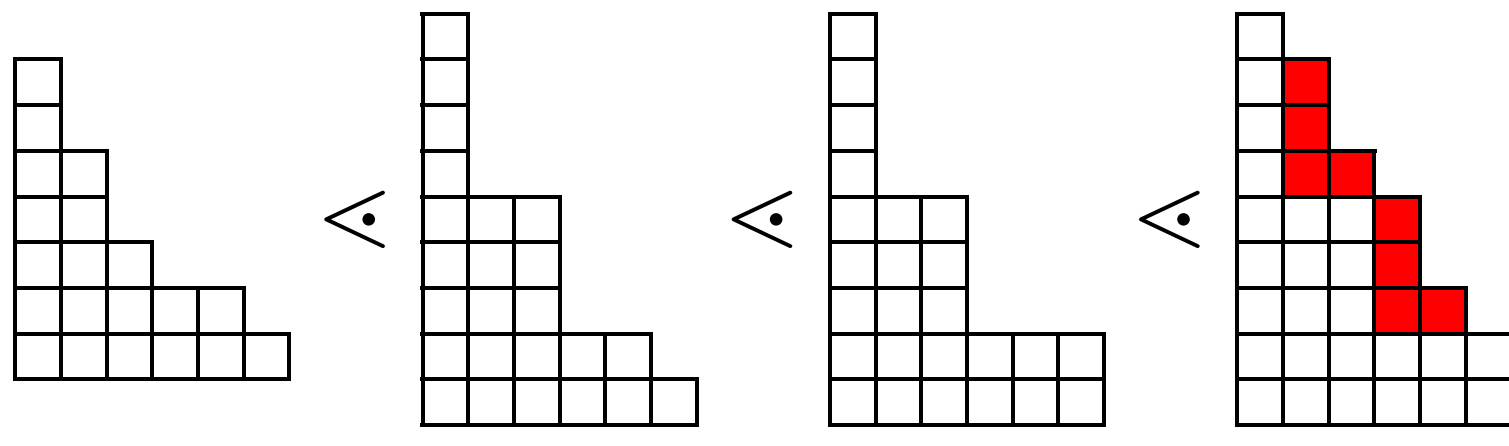
Weak Order

- Covers differ by boxes of same color.

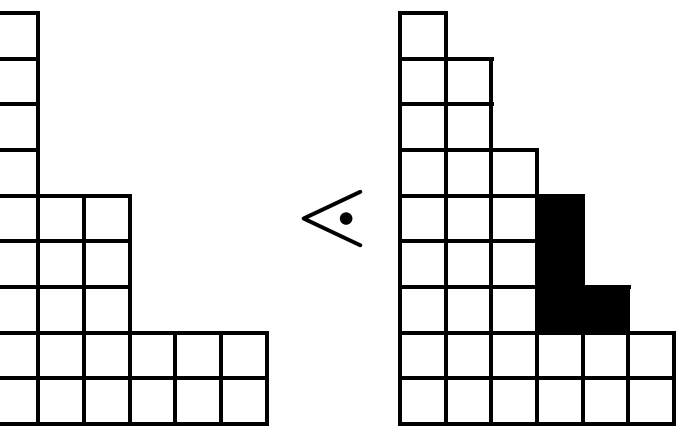


Strong (Bruhat) Order

- Ordered by containment of shapes.
- Covers differ by a ribbon + its copies.



- Marked Cover:** Strong cover with selection of one ribbon



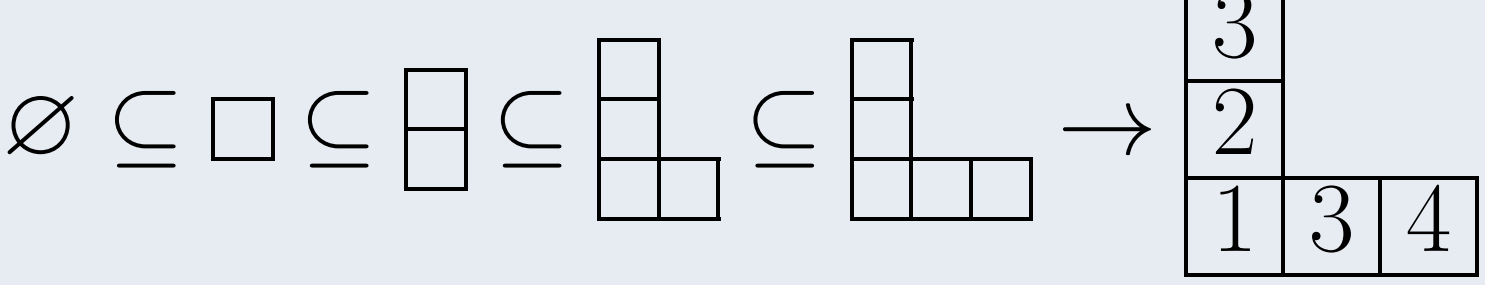
Dual k -Schur Functions

Generating functions of weak tableaux.

$F_\lambda^{(k)} := \sum x^{\text{weight}(T)}$

Weak Tableaux:

Maximal chains in the weak order.



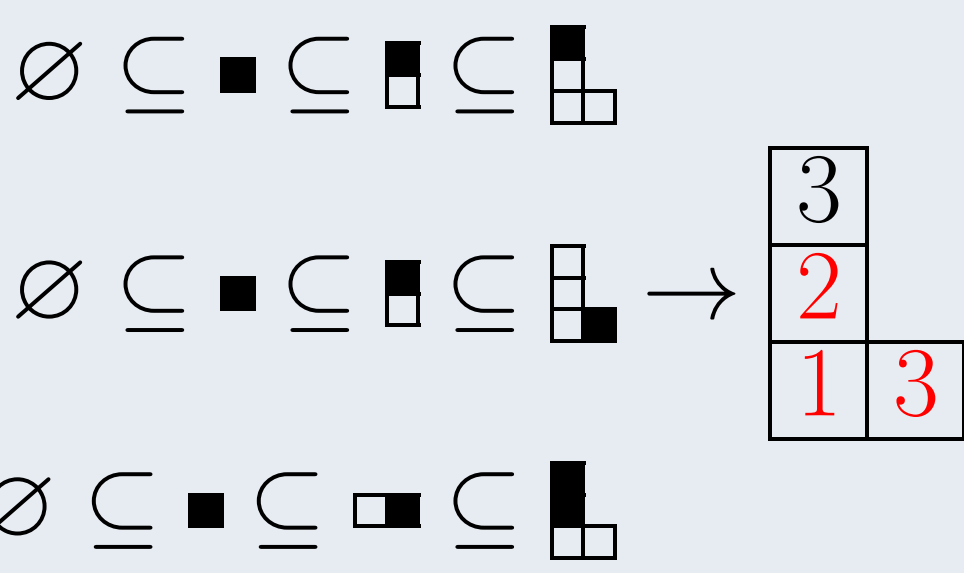
k -Schur Functions

Generating functons of strong tabelaux.

$s_\lambda^{(k)} := \sum x^{\text{weight}(T)}$

Strong Tableaux:

Maximal strong order chains of marked covers



Pieri Rule:

$e_r F_\lambda^{(k)} = \sum_{\mu=\lambda+ \text{ strong marked vertical strip of size } r} F_\mu^{(k)} \iff e_r^\perp s_\mu^{(k)} = \sum_{\lambda=\mu- \text{ strong marked vertical strip of size } r} s_\lambda^{(k)}$

where a **strong vertical strip** is a chain of marked covers with markings proceeding north to south.

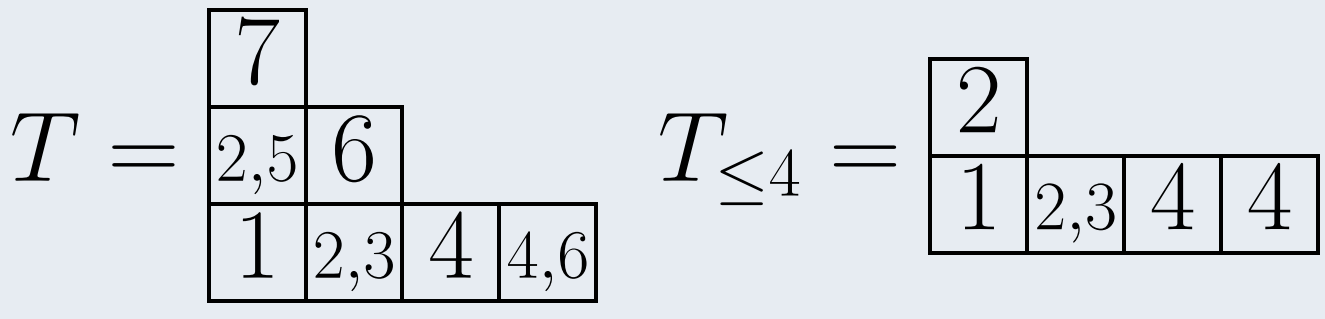
Affine Grothendieck Polynomials

Generating functions of affine SVTs.

$G_\lambda^{(k)} := \sum (-1)^{|\lambda|+|\text{weight}(T)|} x^{\text{weight}(T)}$

Affine Set-Valued Tableaux:

Each $T_{\leq x}$ is a $k+1$ -core.



- $G_\lambda^{(k)} = F_\lambda^{(k)} + \text{higher order terms}$
- $G_\lambda^{(k)} = G_\lambda$ for large k .

Dual Affine Grothendieck Polynomials

$g_\lambda^{(k)}$ is dual basis to $G_\lambda^{(k)}$.

- $g_\lambda^{(k)} = s_\lambda^{(k)} + \text{lower order terms}$
- $g_\lambda^{(k)} = g_\lambda$, dual to G_λ , for large k .

Open Problem

Find a direct definition of $g_\lambda^{(k)}$.

Open Problem

Describe the $G_\lambda^{(k)}$ Pieri rule. \iff Describe the $g_\lambda^{(k)}$ dual Pieri rule.

Branching

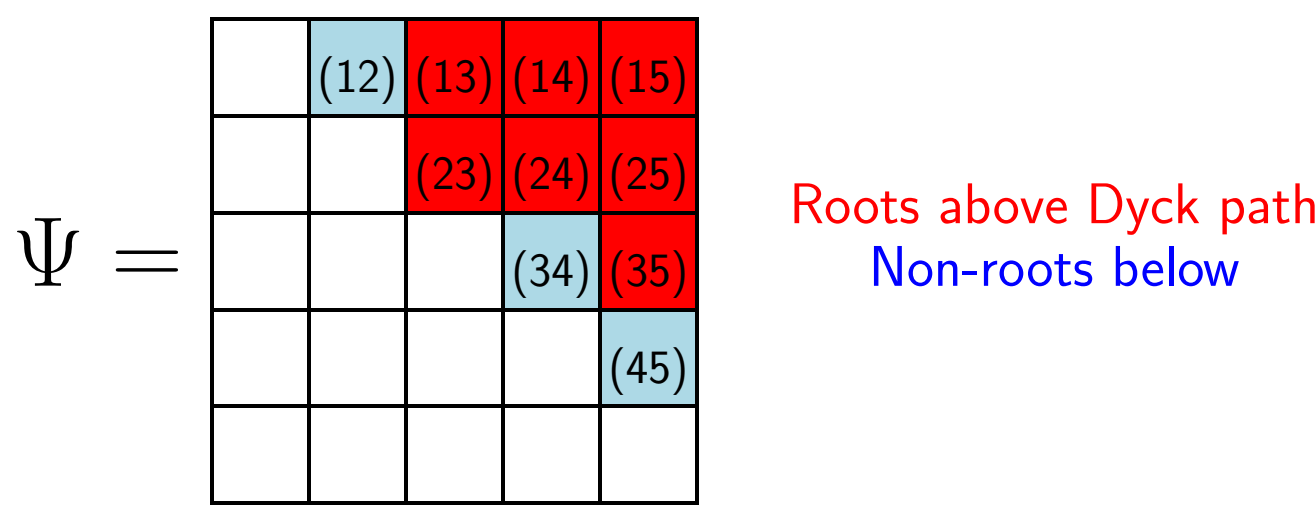
- k -Schur functions are $k+1$ -Schur positive: $s_\lambda^{(k)} = \sum a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$ with $a_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.
- Iteration gives Schur positivity of k -Schur functions.
- Conjecture:** $g_\lambda^{(k)}$ is Schur positive.
- Conjecture:** $g_\lambda^{(k)} = \sum_\mu (-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$ for $b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

Catalan Functions

For $\gamma \in \mathbb{Z}^\ell$,

$H(\Psi; \gamma) := \prod_{(i,j) \notin \Psi} (1 - R_{ij}) h_\gamma$

- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda+\epsilon_i-\epsilon_j}$
 $R_{1,3} \left(\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix} \right) = \begin{smallmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{smallmatrix}$ $R_{2,3} \left(\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix} \right) = \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$
- Root ideal Ψ : given by Dyck path.

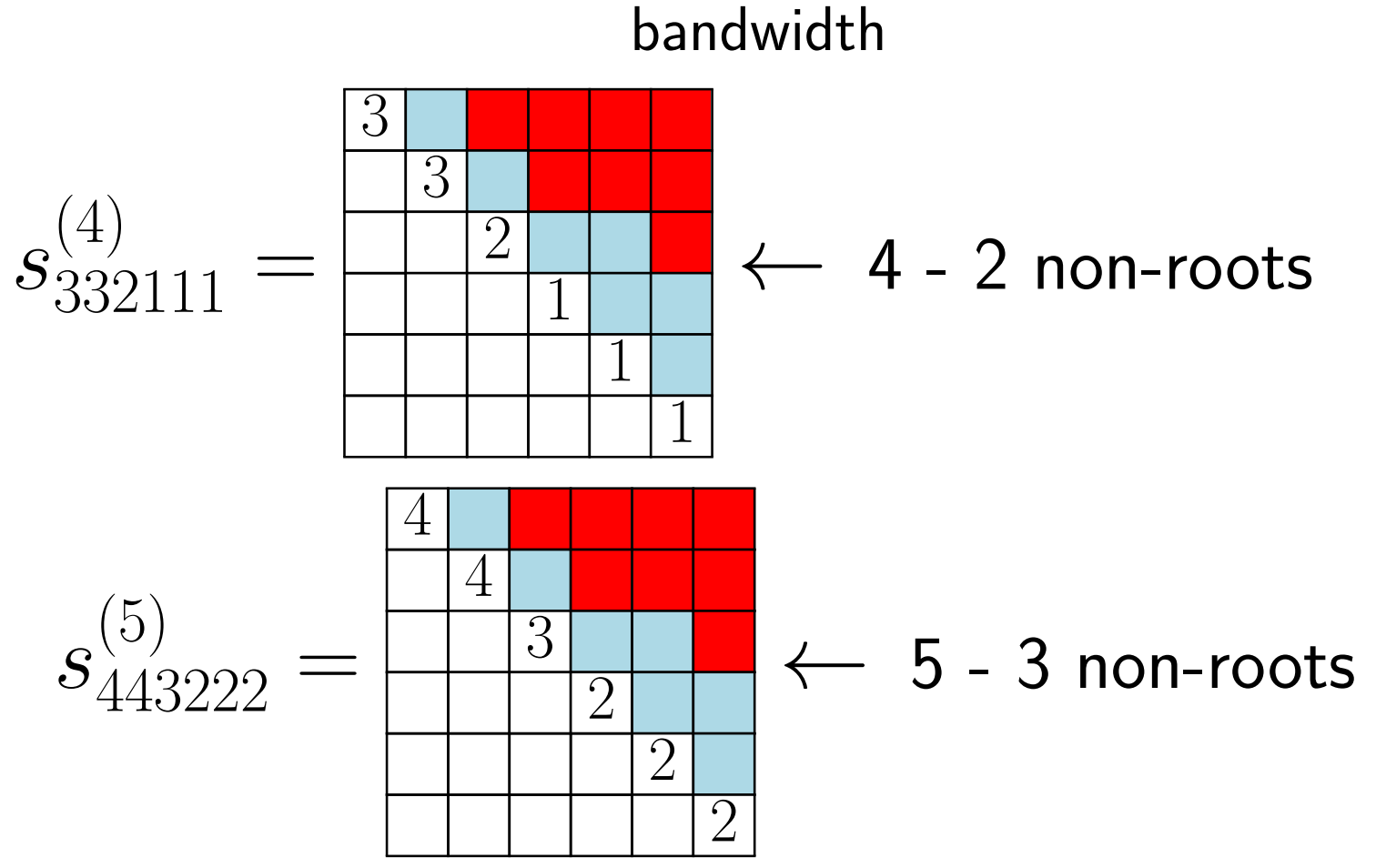


$H(\Psi; 54332)$
 $= (1 - R_{12})(1 - R_{34})(1 - R_{45}) h_{54332}$
 $= h_{54332} - h_{45332} - h_{54422} - h_{54341}$
 $+ h_{45422} + h_{45341} + h_{54431} - h_{45431}$

- $H(\emptyset; \lambda) = s_\lambda$ (Jacobi-Trudi Identity)

k -Schur Catalans

$s_\lambda^{(k)} = H(\Psi; \lambda)$ for particular Ψ , defined by $\lambda_i + \underbrace{\# \text{non-roots in row } i}_{\text{bandwidth}} = k$.



Why use Catalan k -Schurs?

Shift Invariance:

$e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$

Corollary

Dual Pieri rule $\implies s_\lambda^{(k)}$ branching!

$s_\lambda^{(k)} = e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$

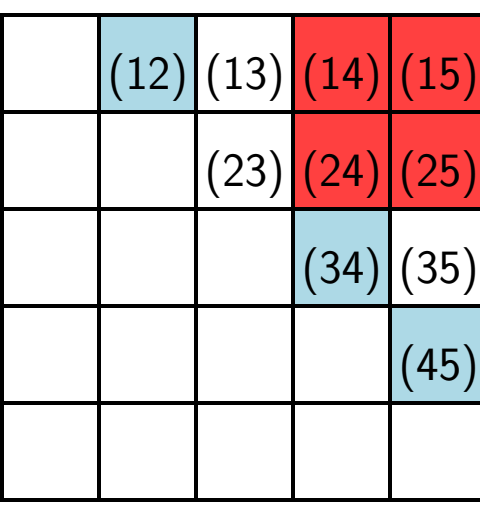
where $a_{\lambda\mu}^{(k)}$ counts strong vertical strips.

K-theoretic Catalan Functions

For $\gamma \in \mathbb{Z}^\ell$, root ideals Ψ, \mathcal{L}

$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \notin \Psi} (1 - R_{ij}) K h_\gamma$

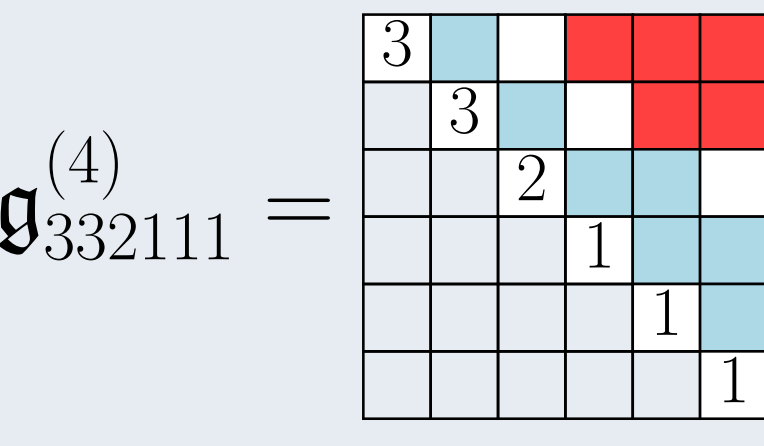
- Lowering operators
 $L_j(K h_\lambda) = K h_{\lambda-\epsilon_j}$
 $L_3 \left(\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix} \right) = \begin{smallmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{smallmatrix}$ $L_1 \left(\begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix} \right) = \begin{smallmatrix} \blacksquare \\ \blacksquare \end{smallmatrix}$
- non-roots of Ψ , roots of \mathcal{L}



$K(\Psi; \mathcal{L}; 54332)$
 $= (1 - L_4)^2 (1 - L_5)^2$
 $\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) K h_{54332}$

- $K(\emptyset; \emptyset; \lambda) = g_\lambda$.

$\mathfrak{g}_\lambda^{(k)} := K(\Psi; \mathcal{L}; \lambda)$ with $\text{band}(\Psi) = k$, $\text{band}(\mathcal{L}) = k+1$



Theorem: Shift Invariance

$G_{1^\ell}^\perp \mathfrak{g}_{\lambda+1^\ell}^{(k+1)} = \mathfrak{g}_\lambda^{(k)}$

Corollary

$\mathfrak{g}_\lambda^{(k)}$ branching follows from dual Pieri rule.

Conjecture

$\mathfrak{g}_\lambda^{(k)} = g_\lambda^{(k)}$