SCHUR Q-FUNCTIONS

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1. Introduction

These notes are a companion for [Mac79, III, Sections 2,4,8]. All results and proofs are from [Mac79], **usually verbatim or very close** with some extra detail added for my own sake.

Recall the Hall-Littlewood functions given by

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{V_\lambda(t)} \sum_{w \in \mathfrak{S}_n} w \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j} \right)$$

or alternatively

$$P_\lambda(x_1, \ldots, x_n; t) = \sum_{w \in \mathfrak{S}_n / \mathfrak{S}_n^\lambda} w \left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{\lambda_i > \lambda_j} \frac{x_i - tx_j}{x_i - x_j} \right)$$

One typically learns some remarkable specializations, namely

$$P_\lambda(x; 0) = s_\lambda(x) \text{ and } P(x; 1) = m_\lambda(x)$$

However, there exist other "variants" of the Hall-Littlewood functions, and we will explore one of these variants in this section.

1.1. **Definition.** For $r \geq 1$, we define

$$q_r(x; t) = (1-t) P_{(r)}(x; t)$$

and set $q_0(x; t) = 1$.

1.2. **Remark.** Notice that when $t = 0$, we have

$$q_r(x; 0) = P_{(r)}(x; 0) = s_{(r)}(x) = h_r(x)$$

1.3. **Proposition.** *In n variables,*

$$q_r(x_1, \ldots, x_n; t) = (1-t) \sum_{i=1}^{n} x_i^r \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}$$

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Proof. We observe that $\mathfrak{S}_n^{(r)}$ is all permutations that fix 1, giving us $\mathfrak{S}_n/\mathfrak{S}_n^{(r)} \cong \{(1,j)\mathfrak{S}_n^{(r)|1\leq j\leq n}\}$. Let us say $\tau_{1,j} = (1,j)$. From the definition,

$$q_r(x_1,\ldots,x_n;t) = (1-t)\sum_{w\in\mathfrak{S}_n/\mathfrak{S}_n^{(r)}} w\left(x_1^r\prod_{\lambda_i>\lambda_j}\frac{x_i-tx_j}{x_i-x_j}\right) = (1-t)\sum_{w\in\mathfrak{S}_n/\mathfrak{S}_n^{(r)}} w\left(x_1^r\prod_{j\neq 1}\frac{x_1-tx_j}{x_1-x_j}\right)$$

Next, we break up the sum based on where the permutation sends 1.

$$q_r(x_1,\ldots,x_n;t) = (1-t)\sum_{i=1}^n x_i^r\tau_{1,i}\left(\prod_{j\neq 1}\frac{x_1-tx_j}{x_1-x_j}\right)$$

which then yields the result. $\square$

1.4. **Example.** We compute using 2 variables

$$\begin{cases} q_1(x_1,x_2) = (1-t)\left(x_1\left(\frac{x_1-tx_2}{x_1-x_2}\right)+x_2\left(\frac{x_2-tx_1}{x_2-x_1}\right)\right) = (1-t)\frac{x_1^2-tx_1x_2-x_2^2+tx_1x_2}{x_1-x_2} \\ = (1-t)(x_1+x_2) = (1-t)m_1 \\ q_2(x_1,x_2) = (1-t)\frac{x_1^3-tx_1^2x_2-x_2^3+tx_1x_2^2}{x_1-x_2} = \frac{1-t}{x_1-x_2}(x_1^3-x_2^3+t(x_1x_2^2-x_1^2+x_2)) = \\ = (1-t)(x_1^2+x_1x_2+x_2^2-t(x_1x_2)) = (1-t)(m_2+(1-t)m_{11}) \end{cases}$$

Naturally, such computations are not much different than computing Hall-Littlewood polynomials.

1.5. **Proposition.** *The generating function for the q_r is given by*

$$\sum_{r=0}^{\infty} q_r(x;t)u^r = \prod_i \frac{1-x_itu}{1-x_iu} = \frac{H(u)}{H(tu)}$$

Proof. When using a finite number of variables,

$$\begin{aligned} \sum_{r=1}^{\infty} q_r(x;t)u^r &= \sum_{r=1}^{\infty}(1-t)\sum_{i=1}^n x_i^ru^r\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j} \\ &= (1-t)\sum_{i=1}^n\left(\sum_{r=1}^{\infty}x_i^ru^r\right)\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j} \\ &= (1-t)\sum_{i=1}^n\frac{x_iu}{1-x_iu}\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j} \end{aligned}$$

Then, using the Heaviside cover-up method of partial sum decomposition, we have

$$\prod_{i=1}^n\frac{z-tx_i}{z-x_i} = 1+\frac{\prod_{i=1}^n(z-tx_i)-\prod_{i=1}^n(z-x_i)}{\prod_{i=1}^n z-x_i} = 1+\sum_{i=1}^n\frac{A_i}{z-x_i}$$

where

$$A_i = \mathrm{Res}_{z=x_i}\left(\frac{\prod_{j=1}^n (z-tx_j) - \prod_{j=1}^n (z-x_j)}{\prod_{j=1}^n z-x_j}\right)$$
$$= \frac{\prod_{j=1}^n (x_i-tx_j) - \prod_{j=1}^n (x_i-x_j)}{\prod_{j\neq i} x_i-x_j}$$
$$= (x_i-tx_i)\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j}$$

or, in other words,

$$\prod_{i=1}^n \frac{z-tx_i}{z-x_i} = 1 + \sum_{i=1}^n (1-t)\frac{x_i}{z-x_i}\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j}$$

Now, taking $z=u^{-1}$, we get

$$\prod_{i=1}^n \frac{1-tux_i}{1-ux_i} = 1 + \sum_{i=1}^n (1-t)\frac{ux_i}{1-ux_i}\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j}$$

And thus,

$$\sum_{r=0}^\infty q_r u^r = 1 + (1-t)\sum_{i=1}^n \frac{x_i u}{1-x_i u}\prod_{j\neq i}\frac{x_i-tx_j}{x_i-x_j} = \prod_{i=1}^n \frac{1-x_i tu}{1-x_i u}$$

□

1.6. **Definition.** Given a partition λ, we define

$$Q_\lambda(x;t) := b_\lambda(t)P_\lambda(x;t)$$

where

$$b_\lambda(t) := \prod_{i\geq 1}\phi_{m_i(\lambda)}(t)$$

with $m_i(\lambda)$ being the number of times i occurs as a part of λ and

$$\phi_r(t) = (1-t)(1-t^2)\cdots(1-t^r)$$

These Q_λ's are also called Hall-Littlewood functions.

1.7. **Proposition.** *Since* $b_{(r)}(t) = (1-t)$

$$Q_{(r)}(x;t) = (1-t)P_{(r)}(x;t) = q_r(x;t)$$

One can also prove that

$$Q_\lambda = \prod_{i<j}\frac{1-R_{ij}}{1-tR_{ij}}q_\lambda$$

which also gives that the transition matrix from $\{Q_\lambda\}_\lambda$ to $\{q_\mu\}_\mu$ is lower unitriangular. Thus, since Hall-Littlewood functions form a $\mathbb{Q}(t)$-basis of $\Lambda[t]$, we get

1.8. **Proposition.** *The set* $\{q_\lambda\}_\lambda$ *forms a* $\mathbb{Q}(t)$*-basis of* $\Lambda[t]$.

2. When $t = -1$

Above, we have mainly focused on the very basics of this "Q" class of Hall-Littlewood functions. However, in 1911, Schur published a paper on projective representations that realized the so-called "Schur Q-functions" as irreducible spin characters of $\mathfrak{S}_n$. In this section, we will define

$$q_r = q_r(x; t = -1), P_\lambda = P_\lambda(x; -1), Q_\lambda = Q_\lambda(x; -1)$$

We call such Q_λ's *Schur Q-function.* We will now lay some groundwork for $q_r(x; -1)$.

2.1. **Example.** Now we have

$$q_1(x; -1) = 2m_1 \qquad q_2(x; -1) = 2m_2 + 4m_{11}$$

2.2. **Corollary.** *Given $Q(t) = \sum_{r \geq 0} q_r t^r$, then,*

$$Q(t) = \prod_i \frac{1 + tx_i}{1 - tx_i} = E(t)H(t)$$

and thus, since $H(t)E(-t) = 1$, we get

$$Q(t)Q(-t) = E(t)H(t)E(-t)H(-t) = 1$$

Proof. The first part comes from specializing $t = -1$ in the formula $Q(u)$ in the section above (and then replacing u with t since t is now available as a variable). The second part follows from the exposition. $\square$

2.3. **Proposition.** *For $n = 2m$, we have the formula*

$$q_{2m} = \sum_{r=1}^{m-1} (-1)^{r-1} q_r q_{2m-r} + \frac{1}{2}(-1)^{m-1} q_m^2$$

and thus $q_{2m} \in \mathbb{Q}[q_1, q_2, \ldots, q_{2m-1}]$.

Proof. Since $Q(t)Q(-t) = 1$, we have

$$\sum_{r+s=n} (-1)^r q_r q_s = 0$$

and so, setting $n = 2m$, we get

$$0 = \sum_{r=0}^{2m} (-1)^r q_r q_{2m-r} = (-1)^m q_m^2 + 2\sum_{r=0}^{m-1} (-1)^r q_r q_{2m-r} \Longrightarrow 0 = q_{2m} + \frac{1}{2}(-1)^m q_m^2 + \sum_{r=1}^{m-1} (-1)^r q_r q_{2m-r}$$

and so the result is obtained by isolating q_{2m}. $\square$

2.4. **Corollary.** $q_{2m} \in \mathbb{Q}[q_1, q_3, q_5, \ldots, q_{2m-1}]$.

Proof. This follows by induction on m using the proposition above. $\square$

2.5. **Corollary.** *Given a partition $\lambda \vdash n$, then either λ is strict or $q_\lambda = q_{\lambda_1} \ldots q_{\lambda_\ell}$ is a $\mathbb{Z}$-linear combination of the q_μ such that q_μ is strict and $\mu \trianglerighteq \lambda$.*

2.6. **Definition.** We define

$$\Gamma := \mathbb{Z}[q_1, q_2, q_3, \ldots] \subseteq \Lambda$$

and $\Gamma^n := \Gamma \cap \Lambda^n$. We also denote

$$\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q} = \mathbb{Q}[q_1, q_2, q_3, \ldots]$$

2.7. **Lemma.**

$$\frac{Q'(t)}{Q(t)} = \frac{E'(t)H(t) + E(t)H'(t)}{E(t)H(t)} = \frac{E'(t)}{E(t)} + \frac{H'(t)}{H(t)} = P(t) + P(-t)$$

and thus

$$\frac{Q'(t)}{Q(t)} = 2\sum_{r\geq 0} p_{2r+1}t^{2r}$$

2.8. **Proposition.**

$$rq_r = 2(p_1q_{r-1} + p_3q_{r-3} + \cdots)$$

Proof. Rearranging the results of our lemma above, we get

$$Q'(t) = Q(t)\left(2\sum_{r\geq 0} p_{2r+1}t^{2r}\right)$$

and so, looking at the t^{r-1} coefficient on both sides, we get

$$rq_r = \sum_{2s+u=r-1} p_{2s+1}q_u = \sum_{s=0}^{r-1} p_{2s+1}q_{r-1-2s}$$

where we take $q_u = 0$ if $u < 0$. □

2.9. **Corollary.** *Thus,*

$$\Gamma_{\mathbb{Q}} = \mathbb{Q}[p_r \mid r \textit{ is odd}] = \mathbb{Q}[q_r \mid r \textit{ is odd}]$$

and the q_r for r odd are algebraically independent over $\mathbb{Q}$

Proof. The formula above allows us to express odd power p_r's in terms of q's and vice versa. Since the odd p_r's are algebraically independent over $\mathbb{Q}$, we get the result. □

2.10. **Lemma.** *The number of odd partitions on n is equal to the number of strict partitions on n.*

Proof. Consider the generation function

$$\sum_{\lambda \text{ is odd}} t^{|\lambda|}$$

which would have a t^n term for every odd partition of n. Since every odd partition is some combination of odd terms, we can rewrite this sum as

$$\left(1 + t^{\mathrm{wt}(1)} + t^{\mathrm{wt}(1,1)} + \cdots\right)\left(1 + t^{\mathrm{wt}(3)} + t^{\mathrm{wt}(3,3)} + \cdots\right)\cdots = \left(1 + t + t^2 + \cdots\right)\left(1 + t^3 + t^6 + \cdots\right)\cdots = \prod_{r=1}^{\infty}\frac{1}{1 - t^{2r-1}}$$

However, using some clever algebra, we rewrite our fraction to have every $1-t^{2r}$ term in the numerator and all $1-t^r$ terms in the denominator (so after cancellation, only $1-t^{2r-1}$ terms remain in the denominator)

$$\prod_{r\geq 1}\frac{1}{1-t^{2r-1}} = \prod_{r\geq 1}\frac{1-t^{2r}}{1-t^r} = \prod_{r\geq 1}(1+t^r)$$

Finally, we observe by multiplying out the terms that

$$\prod_{r\geq 1}(1+t^r) = \sum_{\lambda \text{ distinct parts}} t^{|\lambda|}$$

$\square$

2.11. **Remark.** The proof above is originally due to Euler. In fact, one can give an explicit bijection using a method by Sylvester relying on the fact that every number can be expressed uniquely as a power of 2 multiplied by an odd number.

2.12. **Proposition.** *(a) The q_λ for λ odd form a $\mathbb{Q}$-basis of $\Gamma_{\mathbb{Q}}$.*
(b) The q_λ for λ strict form a $\mathbb{Z}$-basis of Γ.

Proof. The first part follows immediately from the previous proposition. By 2.5, we have that the q_λ for λ strict span Γ^n (and thus also $\Gamma^n_{\mathbb{Q}}$). Furthermore, since, by the lemma above, the number of strict partitions is equal to the number of odd partitions, they must form a $\mathbb{Q}$ basis of $\Gamma^n_{\mathbb{Q}}$. Therefore, they are linearly independent over $\mathbb{Q}$ and thus also $\mathbb{Z}$, giving the second part. $\square$

2.13. **Proposition.** *Given a partition λ, we have*

$$Q_\lambda(x;-1) = \begin{cases} 2^{\ell(\lambda)}P_\lambda & \text{if } \lambda \text{ is strict} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Recall from 1.6 that

$$Q_\lambda(x;t) = b_\lambda(t)P_\lambda(x;t)$$

where $b_\lambda(t) = \prod_{i\geq 1}\phi_{m_i(\lambda)}(t)$ and $\phi_r(t) = (1-t)(1-t^2)\cdots(1-t^r)$. Then, if $r=1$, $\phi_r(-1)=2$, but for $r>1$, $\phi_r(-1)=0$. Thus, if λ does not have distinct parts, there is an i such that $\phi_{m_i(\lambda)}(-1) = 0 \Longrightarrow b_\lambda(-1)=0$. When λ has distinct parts, then $b_\lambda(-1) = 2^{\ell(\lambda)}$. Thus, we get our result. $\square$

2.14. **Definition.** We call $Q_\lambda(x;-1)$ the *Schur-Q* function indexed by λ.

2.15. **Proposition.** *The $\{Q_\lambda\}$ with λ strict form a $\mathbb{Z}$ basis of Γ.*

Proof. We have from the previous section that

$$Q_\lambda(x;-1) = \prod_{i<j}\frac{1-R_{ij}}{1+R_{ij}}q_\lambda$$

and thus $\{Q_\lambda\}$ is unitriangularly related to the $\{q_\lambda\}$. This proves the claim. $\square$

2.1. The Pfaffian Definition of the Schur-Q function (Optional).

Originally, Schur defined the Schur-Q functions using the "Pfaffian" of a matrix (indirectly named after a German mathematician Johann Friedrich Pfaff). From a combinatorial standpoint, this section is optional, but can paint a nice picture and give more intuition for working with Schur-Q functions.

2.16. **Proposition.** *A $2n \times 2n$ real skew-symmetric matrix A with eigenvalues $i\lambda_1, -i\lambda_1, i\lambda_2, -i\lambda_2, \ldots, i\lambda_r, -i\lambda_r$ with $i = \sqrt{-1}$ and $\lambda_j \in \mathbb{R}$ can be written in the form $A = QSQ^t$ where Q is an orthogonal matrix and*

$$S = \begin{pmatrix} 0 & \lambda_1 & & & & & & & & \\ -\lambda_1 & 0 & & & & & & & & \\ & & 0 & \lambda_2 & & & & & & \\ & & -\lambda_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & \lambda_r & & & \\ & & & & & -\lambda_r & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & 0 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 0 \end{pmatrix}$$

2.17. **Corollary.** *From the above characterization of a $2n \times 2n$ skew symmetric matrix A, we have that* $\det A$ *is a perfect square.*

2.18. **Definition.** The *Pfaffian* of a $2n \times 2n$ skew-symmetric matrix $A = (a_{ij})_{1 \le i,j \le 2n}$ is given by

$$\mathrm{Pf}(A) = \sum_{\substack{w \in \mathfrak{S}_{2n} \\ w(2r-1) < w(2r), 1 \le r \le n \\ w(2r-1) < w(2r+1), 1 \le r \le n-1}} \mathrm{sgn}(w) a_{w(1),w(2)} \cdots a_{w(2n-1)w(2n)}$$

2.19. **Lemma.**

$$\mathrm{Pf}(A) = \frac{1}{2^n n!} \sum_{w \in \mathfrak{S}_{2n}} \mathrm{sgn}(w) a_{w(1),w(2)} \cdots a_{w(2n-1)w(2n)}$$

2.20. **Proposition.** *Given a $2n \times 2n$ skew-symmetric matrix A,*

$$(\mathrm{Pf}(A))^2 = \det A$$

2.21. **Lemma.** *Given m an even positive integer, denote the Pfaffian of the $m \times m$ matrix*

$$P(t_1, \ldots, t_m) = \mathrm{Pf}\left(\frac{t_i - t_j}{t_i + t_j}\right)_{1 \le i,j \le m}$$

Then, from the definition of the Pfaffian, we get

$$P(t_1, \ldots, t_m) = \sum_{i=2}^{m} (-1)^i P(t_1, t_i) P(t_2, \ldots, \hat{t_i}, \ldots, t_m)$$

2.22. **Lemma.** *From the formula in the proof above, we have for $r > s \ge 0$,*

$$Q_{(r,s)} = q_r q_s + 2 \sum_{i=1}^{s} (-1)^i q_{r+i} q_{s-i}$$

Proof.

$$\begin{aligned} Q_{(r,s)}(x; -1) &= \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}} q_{(r,s)} \\ &= \prod_{i<j} (1 - R_{ij})(1 - R_{ij} + R_{ij}^2 - R_{ij}^3 + \cdots) q_{(r,s)} \\ &= \prod_{i<j} (1 - 2R_{ij} + 2R_{ij}^2 + \cdots) q_{(r,s)} \\ &= q_r q_s + 2 \sum_{i=1}^{s} (-1)^i q_{r+i} q_{s-i} \end{aligned}$$

□

2.23. **Proposition.** *Given λ a strict partition written in the form $\lambda_1 > \lambda_2 > \cdots > \lambda_{2n} \ge 0$, the $2n \times 2n$ matrix*

$$M_\lambda := (Q_{(\lambda_i, \lambda_j)})_{i,j}$$

is skew-symmetric. Then,

$$Q_\lambda(x; -1) = \mathrm{Pf}(M_\lambda) = \sum_{\substack{w \in \mathfrak{S}_{2n} \\ w(2r-1) < w(2r), 1 \le r \le n \\ w(2r-1) < w(2r+1), 1 \le r \le n-1}} \mathrm{sgn}(w) Q_{\lambda_{w(1)}, \lambda_{w(2)}} \cdots Q_{\lambda_{w(2n-1)} \lambda_{w(2n)}}$$

2.24. **Corollary.** *We have the following recursive relations for Q_λ*

$$Q_\lambda = \sum_{j=2}^{m} (-1)^j Q_{(\lambda_1, \lambda_j)} Q_{(\lambda_2, \ldots, \hat{\lambda_j}, \ldots, \lambda_\ell)} \qquad \textit{for } \ell \textit{ even}$$

$$Q_\lambda = \sum_{j=1}^{m} (-1)^{j-1} Q_{(\lambda_j)} Q_{(\lambda_1, \ldots, \hat{\lambda_j}, \ldots, \lambda_\ell)} \qquad \textit{for } \ell \textit{ odd}$$

Note that this description of Schur-Q functions is somewhat difficult to work with by hand, but is natural with respect to projective characters of the Symmetric group.

3. Orthogonality

When working with the Hall inner product on Λ, one typically encounters the Cauchy kernel

$$\Omega(x,y) = \prod_{i,j\geq 1} (1 - x_i y_i)^{-1}$$

A generalization of this formula can be given by

$$\Omega(x,y;t) = \prod_{i,j\geq 1} \frac{1 - t x_i y_j}{1 - x_i y_j}$$

so that, when $t = 0$, the usual Cauchy kernel is recovered. We will then use the t-generalized Cauchy kernel to show various orthogonality relations with a t-generalization of the Hall inner product.

3.1. **Theorem.** *We have*

(a) $\Omega(x,y;t) = \sum_\lambda z_\lambda(t)^{-1} p_\lambda(x) p_\lambda(y)$ *where* $z_\lambda(t) = z_\lambda \prod_{i\geq 1}(1 - t^{\lambda_i})^{-1}$
(b) $\Omega(x,y;t) = \sum_\lambda q_\lambda(x;t) m_\lambda(y) = \sum_\lambda m_\lambda(x) q_\lambda(y;t)$
(c) $\Omega(x,y;t) = \sum_\lambda P_\lambda(x;t) Q_\lambda(y;t)$

Proof. This is the subject of [Mac79, pp 222-224]. In brief,

(a) The first identity follows from generating function-ology. The key observation is that

$$\log \prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j} = \sum_{m=1}^{\infty} \frac{1 - t^m}{m} p_m(x) p_m(y)$$

and then exponentiating both sides.

(b) Since

$$Q(y_j) = \sum_{r=0}^{\infty} q_r(x;t) y_j^r = \prod_i \frac{1 - t x_i y_j}{1 - x_i y_j}$$

we get

$$\prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j} = \prod_j Q(y_j) = \prod_j \sum_{r_j=0}^{\infty} q_{r_j}(x;t) y_j^{r_j} = \sum_\lambda q_\lambda(x;t) m_\lambda(y)$$

and similarly for x's and y's interchanged.

(c) If we take the linear transformations

$$(A_{\lambda,\mu})(Q_\lambda)_\lambda = (q_\mu)_\mu \quad (B_{\lambda,\mu})(Q_\lambda)_\lambda = (m_\mu)_\mu \quad (C_{\lambda,\mu})(m_\lambda)_\lambda = (q_\mu)_\mu$$

We have that A is lower unitriangular by **??** and B is upper triangular because it is a product of upper triangular matrices ($m \to s \to P \to Q$). Thus, $D = B^t A$ is lower triangular. However, $D = B^t C B$ and since C is symmetric, D must also be symmetric. Thus, D must be a diagonal matrix with diagonal entries equal to those of B since A is unitriangular. Thus, $D = \mathrm{diag}(b_\lambda(t)^{-1})$. This gives us

$$\begin{aligned}\sum_\lambda q_\lambda(x;t)m_\lambda(y) &= \sum_{\lambda,\mu,\nu} A_{\lambda,\mu}B_{\lambda,\nu}Q_\mu(x;t)Q_\nu(y;t)\\ &= \sum_\mu b_\mu(t)^{-1}Q_\mu(x;t)Q_\mu(x;t)\\ &= \sum_\mu P_\mu(x;t)Q_\mu(y;t)\end{aligned}$$

□

3.2. **Definition.** Given a partition λ, we define

$$S_\lambda(x;t) := \det(q_{\lambda_i - i + j}(x;t))$$

3.3. **Proposition.** *We can also express*

$$S_\lambda(x;t) = \prod_{i<j}(1 - R_{ij})q_\lambda = \prod_{i<j}(1 - tR_{ij})Q_\lambda$$

Proof. The first equality follows using an identical argument for $s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$ from the Jacobi-Trudi identity. For the second equality, we have

$$\prod_{i<j}(1 - tR_{ij})Q_\lambda = \prod_{i<j}(1 - tR_{ij})\left(\prod_{i<j}\frac{1 - R_{ij}}{1 - tR_{ij}}q_\lambda\right) = \prod_{i<j}(1 - R_{ij})q_\lambda$$

□

3.4. **Lemma.** *For $r \geq 1$, we have*

$$q_r(x;t) = h_r[(1-t)p_1]$$

where $h_r[(1-t)p_1]$ is a plethystic substitution.

Proof. We write

$$h_r(x) = \sum_\lambda z_\lambda^{-1}p_\lambda(x)$$

$$\implies h_r[(1-t)p_1] = \sum_\lambda z_\lambda^{-1}p_\lambda[(1-t)p_1] = \sum_{\lambda\vdash r} z_\lambda^{-1}\prod_{i=1}^{\ell(\lambda)}(1 - t^{\lambda_i})p_{\lambda_i} = \sum_{\lambda\vdash r} z_\lambda^{-1}p_\lambda\prod_{i=1}^{\ell(\lambda)}(1 - t^{\lambda_i})$$

since $p_1(x_1^{\lambda_i}, x_2^{\lambda_i}, \ldots) = p_{\lambda_i}(x)$. Now, using some generating function-ology[1], we get

$$\sum_{r=0}^\infty h_r[(1-t)p_1]y^r = \sum_\lambda \frac{\prod_{j=1}^{\ell(\lambda)}(1 - t^{\lambda_j})}{z_\lambda}p_\lambda y^{|\lambda|}$$

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[1]The reader may find it helpful to review the proof that $\sum_{r=0}^\infty h_r y^r = \sum_\lambda \frac{p_\lambda}{z_\lambda}y^{|\lambda|}$. See [See18, 2.2] or [Mac79, p 25]

$$
\begin{aligned}
&= \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} \frac{((1-t^k)p_k y^k)^{m_k}}{m_k! k^{m_k}} \\
&= \prod_{k=1}^{\infty} \exp\left(\frac{(1-t^k)p_k y^k}{k}\right) \\
&= \exp\left(\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \frac{(1-t^k)(x_i y)^k}{k}\right) \\
&= \prod_i \exp\left(\sum_{k=1}^{\infty} \frac{(1-t^k)(x_i y)^k}{k}\right) \\
&= \prod_i \exp\left(\sum_{k=1}^{\infty} \frac{(x_i y)^k}{k} - \sum_{k=1}^{\infty} \frac{(t x_i y)^k}{k}\right) \\
&= \prod_i \exp\left(-\log(1-x_i y) + \log(1 - t x_i y)\right) \\
&= \prod_i \frac{\exp(\log(1-t x_i y))}{\exp(\log(1-x_i y))} \\
&= \prod_i \frac{1 - t x_i y}{1 - x_i y} \\
&= \sum_{r=0}^{\infty} q_r(x;t) y^r \qquad \text{by } 1.5
\end{aligned}
$$

Thus, we have proven the lemma. □

3.5. **Remark.** The plethystic substitution $f \mapsto f[(1-t)p_1]$ has explicit inverse given by $g \mapsto g[\frac{p_1}{1-t}]$, which can be seen by direct computation of $p_\lambda[\frac{p_1}{1-t}] = p_\lambda \prod_{i=1}^{\ell(\lambda)}(1-t^{\lambda_i})^{-1}$.

As some trivia, the plethystic substitutions have a representation theoretic meaning. If $f(x;t) = \sum_r t^r \operatorname{ch}(\chi_{A_r})$ for $A = \bigoplus_r A_r$ a graded $\mathfrak{S}_n$-module and $V = \mathbb{C}^n$ the defining representation, then

$$f[(1-t)p_1] = \sum_k (-1)^k t^k \left(\sum_r t^r \operatorname{ch}(\chi_{A \otimes \wedge^k V})\right)$$

and

$$f[\frac{p_1}{1-t}] = \sum_k t^k \left(\sum_r t^r \operatorname{ch}(\chi_{A \otimes \operatorname{Sym}^k V})\right)$$

which follows from proving the result for $f = h_n$ and doing some extra representation theoretic work.

3.6. **Corollary.** *Let variables ξ_i be such that $h_r(\xi) = h_r[(1-t)p_1]$. Then,*

$$q_r(x;t) = h_r(\xi) \text{ and } S_\lambda(x;t) = s_\lambda(\xi)$$

Proof. It is a general fact that one can always introduce variables ξ_i such that $f(\xi) = f[g]$. Thus, the first part is immediate from the above lemma. Then, from above, we have

$$S_\lambda(x;t) = \det(q_{\lambda_i - i + j}(x;t)) = \det(h_{\lambda_i - i + j}(\xi)) = s_\lambda(\xi)$$

by the Jacobi-Trudi identity. □

3.7. **Proposition.** *We have*

$$\prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j} = \sum_\lambda S_\lambda(x;t) s_\lambda(y) = \sum_\lambda s_\lambda(x) S_\lambda(y;t)$$

Proof. Since the Cauchy kernel has equality

$$\Omega(x,y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

we compute in the spirit of the proof of the corollary above,

$$\prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j} = \prod_{i,j} \frac{1}{1 - \xi_i y_j} = \sum_\lambda s_\lambda(\xi) s_\lambda(y) = \sum_\lambda S_\lambda(x;t) s_\lambda(y)$$

and similarly for the other equality. □

3.8. **Definition.** We define a bilinear product on $\Lambda[t]$

$$\langle q_\lambda(x;t), m_\mu(x) \rangle_t := \delta_{\lambda\mu}$$

3.9. **Remark.** This is a t-generaliation of the Hall-inner product, but is not the same one used in [See18, Section 4.2].

3.10. **Lemma.** *Given $\{u_\lambda\}, \{v_\lambda\}$ as $\mathbb{Q}(t)$-bases of $\Lambda[t]$, the following are equivalent.*

(a) $\langle u_\lambda, v_\mu \rangle_t = \delta_{\lambda\mu}$ *for all* λ, μ
(b) $\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j}$

Proof. If we let

$$u_\lambda = \sum_\nu a_{\lambda\nu} q_\nu \quad v_\mu = \sum_\sigma b_{\mu\sigma} m_\sigma$$

then

$$\begin{aligned} \sum_\lambda u_\lambda(x) v_\lambda(y) &= \prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j} \\ \iff \sum_\lambda u_\lambda(x) v_\lambda(y) &= \sum_\mu q_\mu(x;t) m_\mu(y) \\ \iff \sum_\lambda a_{\lambda\nu} b_{\lambda\sigma} &= \delta_{\nu\sigma} \\ \iff \langle u_\lambda, v_\mu \rangle_t &= \sum_\rho a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu} \end{aligned}$$

□

3.11. **Proposition.** [Mac79, p 225] *Given the definition of* $\langle \cdot, \cdot \rangle_t$ *above, we have*

(a) $\langle P_\lambda(x;t), Q_\mu(x;t)\rangle_t = \delta_{\lambda,\mu}$
(b) $\langle S_\lambda(x;t), s_\mu(x)\rangle_t = \delta_{\lambda,\mu}$
(c) $\langle p_\lambda(x), p_\mu(x)\rangle_t = \delta_{\lambda,\mu} z_\lambda \prod_i (1-t^{\lambda_i})^{-1}$
(d) $\langle \cdot, \cdot \rangle_t$ *is symmetric.*

Proof. The first three identites follow by applying lemma 3.10 to 3.1. The symmetry follows from the self-duality of the power-sum basis. □

3.12. **Proposition.** *The* t*-generalized Hall inner product is given by*

$$\langle f, g\rangle_t = \langle f, g[(1-t)^{-1}p_1]\rangle$$

where the inner product on the right is the standard Hall-inner product.

Proof. Let $\phi, \psi \in \Lambda$ over $\mathbb{Q}$. Then, we may write

$$\phi = \sum_\lambda a_\lambda m_\lambda \quad \psi = \sum_\mu b_\mu h_\mu$$

for $a_\lambda, b_\mu \in \mathbb{Q}$. Then,

$$\langle \phi, \psi\rangle = \sum_\lambda \sum_\mu a_\lambda b_\mu \langle m_\lambda, b_\mu\rangle = \sum_\lambda a_\lambda b_\lambda = \langle \phi, \sum_\mu b_\mu q_\mu\rangle_t = \langle \phi, \psi[(1-t)p_1]\rangle_t$$

and thus the result follows by remark 3.5. □

3.13. **Remark.** The t-generalization of the Hall inner product in [See18] is given by $\langle p_\lambda, p_\mu\rangle_t = \langle p_\lambda, p_\mu[(1-t)p_1]\rangle$.

3.14. **Corollary.** *The plethystic substitution of* $(1-t)p_1$ *is self-adjoint with the the standard Hall inner product,* $\langle \cdot, \cdot \rangle$. *In other words, for* $f, g \in \Lambda$,

$$\langle f, g[(1-t)p_1]\rangle = \langle f[(1-t)p_1], g\rangle$$

Proof. We observe

$$\langle f, g[(1-t)p_1]\rangle = \langle f, g\rangle_t = \langle g, f\rangle_t = \langle g, f[(1-t)p_1]\rangle = \langle f[(1-t)p_1], g\rangle$$

□

4. Bernstein and Jing Operators

First, recall from [Mac79, pp 75–76] that, given a symmetric function $f \in \Lambda$, there exists $f^\perp \in \mathrm{End}(\Lambda)$ given as an adjoint to f under the Hall inner product, that is

$$\langle f^\perp u, v\rangle = \langle u, fv\rangle$$

In particular, we may express

$$p_n^\perp = n\frac{\partial}{\partial p_n}$$

Then, from [Mac79, p 96], we may define the following.

4.1. **Definition.** The *Bernstein operators* on Λ are given by

$$B_n := \sum_{i\geq 0}(-1)^i h_{n+i}e_i^{\perp}$$

which are also encoded in the generating function

$$B(t)=\sum_{n\in\mathbb{Z}} B_n t^n = H(t)E^{\perp}(-t^{-1}) = \exp\left(\sum_{k\geq 1}\frac{t^k}{k}p_k\right)\exp\left(-\sum_{k\geq 1}\frac{t^{-k}}{k}p_k^{\perp}\right) = \exp\left(\sum_{k\geq 1}\frac{t^k}{k}p_k\right)\exp\left(-\sum_{k\geq 1}t^{-k}\frac{\partial}{\partial p_k}\right)$$

However, we can also rephrase the description using plethysm and the Cauchy kernel. Recall the Cauchy kernel

$$\Omega(x,y)=\prod_{i,j\leq 1}(1-x_iy_i)^{-1}$$

and let us set

$$\Omega(x) := \prod_{i\leq 1}(1-x_i)^{-1}$$

We may then we have

4.2. **Proposition.** *Given $f\in\Lambda$, we have $B(t)f(x) = f[p_1-\frac{1}{t}]\Omega(tx)$ and thus $B_m f$ is the coefficient of the mth term.*

Prove this.

4.3. **Theorem.** *Given a Schur function s_λ with $\lambda=(\lambda_1,\ldots,\lambda_\ell)$, if $m\geq\lambda_1$, then*

$$B_m s_\lambda = s_{m,\lambda} = s_{m,\lambda_1,\ldots,\lambda_\ell}$$

Proof. [Mac79] gives a proof using his definition on page 96. Using the proposition above, we observe first that

$$\Omega(tx)=\prod_i\frac{1}{1-tx_i}=\sum_i\frac{1}{1-tx_i}\prod_{j\neq i}\frac{1}{1-\frac{x_j}{x_i}}$$

via partial fraction expansion (similar to the proof of 1.5). Next, we observe that the coefficient of t^m in $f(t^{-1})(1-tx)^{-1}$ is the coefficient of t^0 in $t^{-m}f(t^{-1})(1-tx)^{-1}$, giving us $x^m f(t^{-1})$. Thus,

$$Bs_\lambda = s_\lambda[p_1-t^{-1}]\Omega(tx) = s_\lambda[p_1-t^{-1}]\sum_i\frac{1}{1-tx_i}\prod_{j\neq i}\frac{1}{1-\frac{x_j}{x_i}}$$

Thus, we get that

How did this happen?

$$B_m s_\lambda = \sum_i x_i^m\frac{s_\lambda[p_1-x_i]}{\prod_{j\neq i}(1-\frac{x_j}{x_i})}$$

However, $s_\lambda[p_1-x_i]$ is simply evaluating s_λ in all the other variables, removing x_i. □

4.4. **Definition.** The *Jing operators* $S^k(t)$ are defined by

$$S^k(t)f(x) := f[p_1+(k-1)t^{-1}]\Omega(tx)$$

where $S^k_n(t)f(x)$ is the coefficient of t^n in $S^k(t)f(x)$.

Of course, when $k = 0$, we recover the Bernstein operators. However, their general action on Schur functions is slightly more complicated. Namely, we have

4.5. **Lemma.** [Hai03, Lem 3.4.6] *If $n \geq \mu_1$ and $\lambda \trianglerighteq \mu$, then*

$$S_n^k s_\lambda \in \mathbb{Z}[t]\{s_\gamma \mid \gamma \trianglerighteq (m, \mu)\}$$

Furthermore, $s_{(m,\mu)}$ occurs with coefficient 1 in $S_n^k s_\mu$.

References

[Hai03] M. Haiman, *Combinatorics, Symmetric Functions, and Hilbert Schemes* (2003). `https://math.berkeley.edu/~mhaiman/ftp/cdm/cdm.pdf`.

[Mac79] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 1979. 2nd Edition, 1995.

[See18] G. H. Seelinger, *Algebraic Combinatorics*, 2018. [Online] `https://ghseeli.github.io/grad-school-writings/class-notes/algebraic-combinatorics.pdf`.