

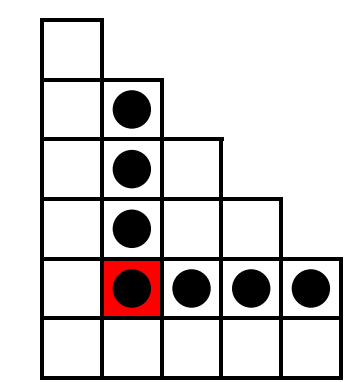
Overview

- Schur functions,  $s_\lambda$ , and Grothendieck polynomials,  $G_\lambda$ , give representatives for cohomology and  $K$ -theory of the Grassmannian.
- Pieri rules determine the structure constants of these rings.
- Representatives are known for (co)homology of affine Grassmannian.
- Aim: develop similar picture for affine  $K$ -theory.

Affine Combinatorics

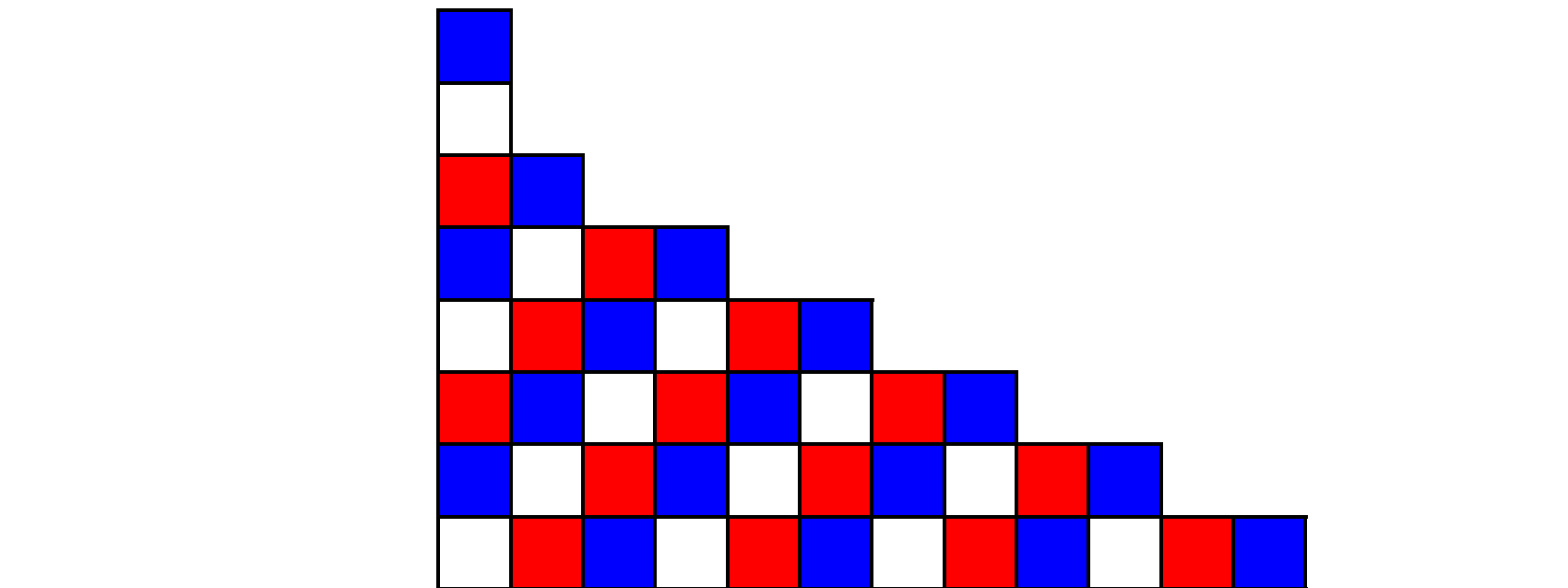
$w \in \tilde{S}_n \leftrightarrow n$ -cores

$n$ -core=partition with no cell of hook-length  $n$   
red cell has hook-length 7



Weak Order

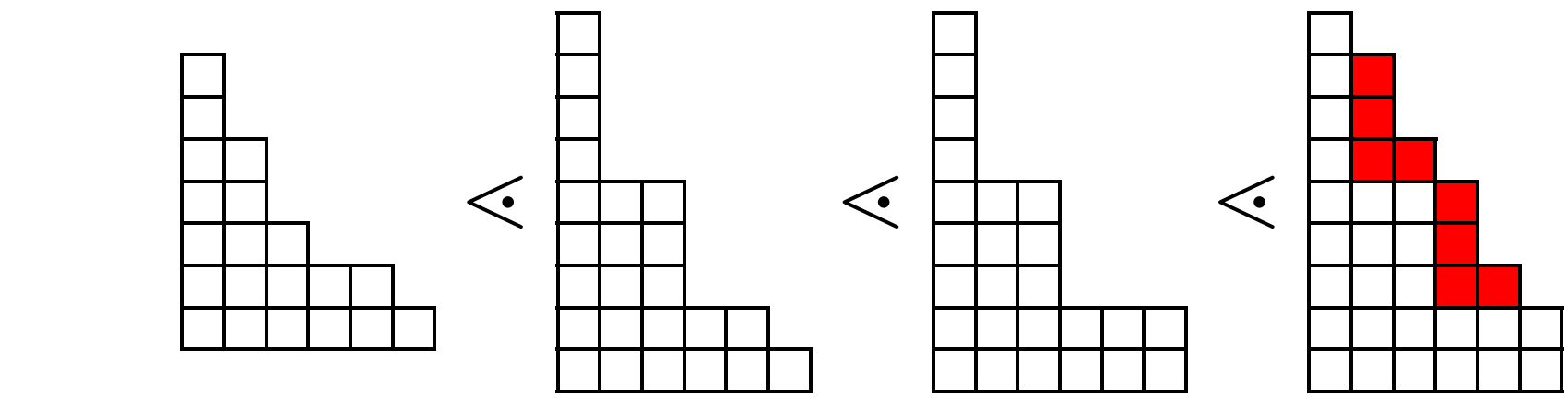
- Covers differ by boxes of same color.



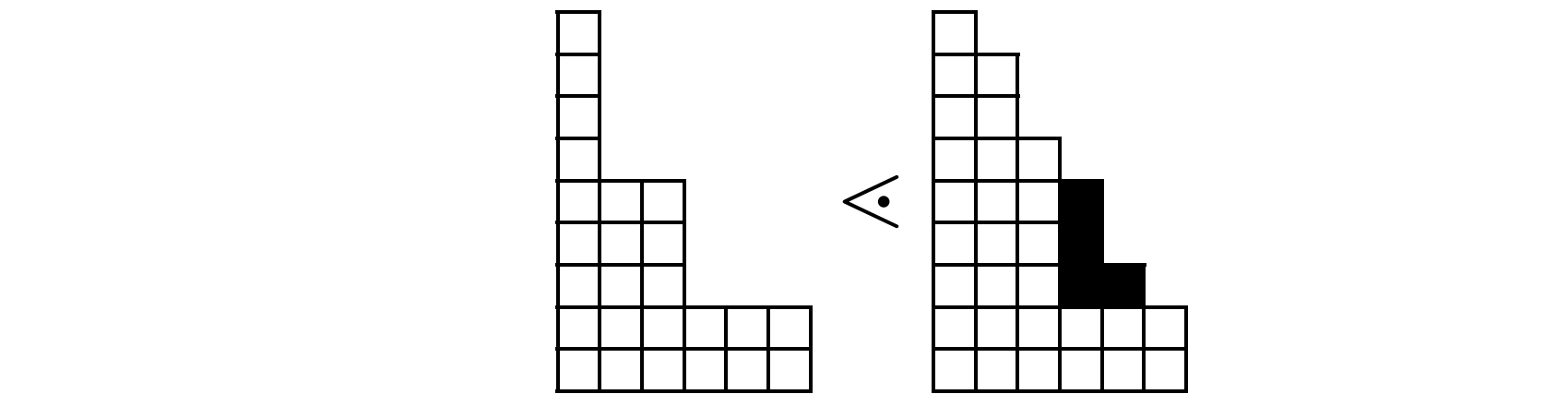
$\emptyset < \square < \begin{smallmatrix} \square & \square \end{smallmatrix} < \begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} < \begin{smallmatrix} \square & \square & \square \\ \square \end{smallmatrix}$

Strong (Bruhat) Order

- Ordered by containment of shapes.
- Covers differ by a ribbon + its copies.



- Marked Cover:** Strong cover with selection of one ribbon

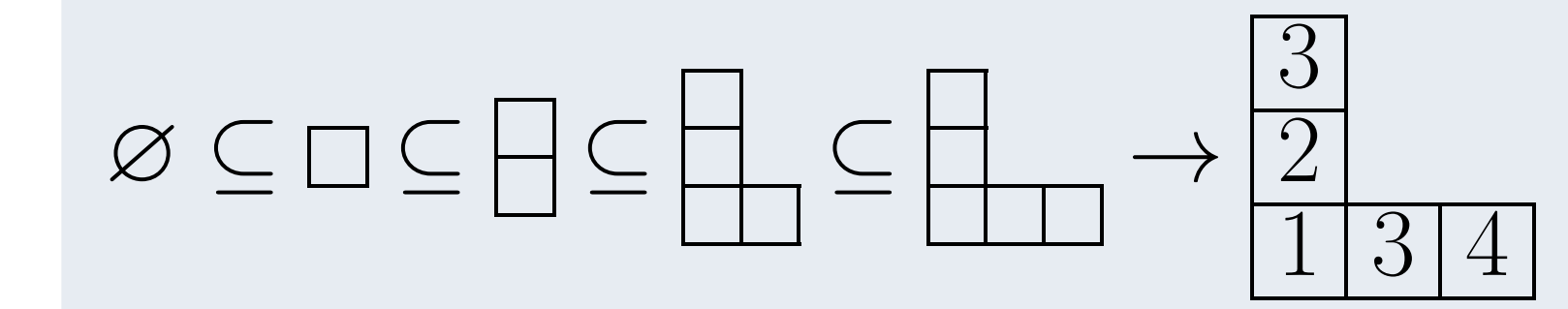


Dual  $k$ -Schur Functions

Generating functions of weak tableaux.

$$F_\lambda^{(k)} := \sum x^{\text{weight}(T)}$$

Weak Tableaux: Maximal chains in the weak order.



Pieri Rule:

$$e_r F_\lambda^{(k)} = \sum_{\mu=\lambda+ \text{ strong marked vertical strip of size } r} F_\mu^{(k)} \iff e_r^\perp s_\mu^{(k)} = \sum_{\lambda=\mu- \text{ strong marked vertical strip of size } r} s_\lambda^{(k)}$$

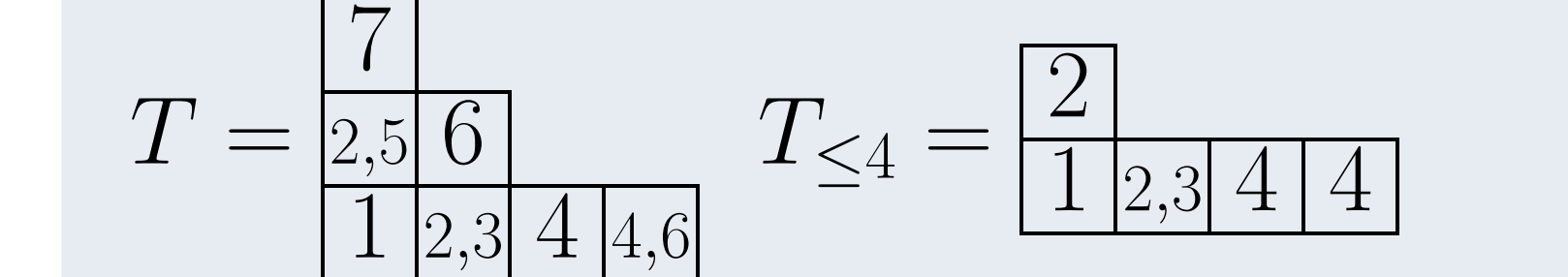
where a **strong vertical strip** is a chain of marked covers with markings proceeding north to south.

Affine Grothendieck Polynomials

Generating functions of affine SVTs.

$$G_\lambda^{(k)} := \sum (-1)^{|\lambda|+|\text{weight}(T)|} x^{\text{weight}(T)}$$

Affine Set-Valued Tableaux: Each  $T_{\leq x}$  is a  $k+1$ -core.



- $G_\lambda^{(k)} = F_\lambda^{(k)} + \text{higher order terms}$
- $G_\lambda^{(k)} = G_\lambda$  for large  $k$ .

Open Problem

Describe the  $G_\lambda^{(k)}$  Pieri rule.  $\iff$  Describe the  $g_\lambda^{(k)}$  dual Pieri rule.

Branching

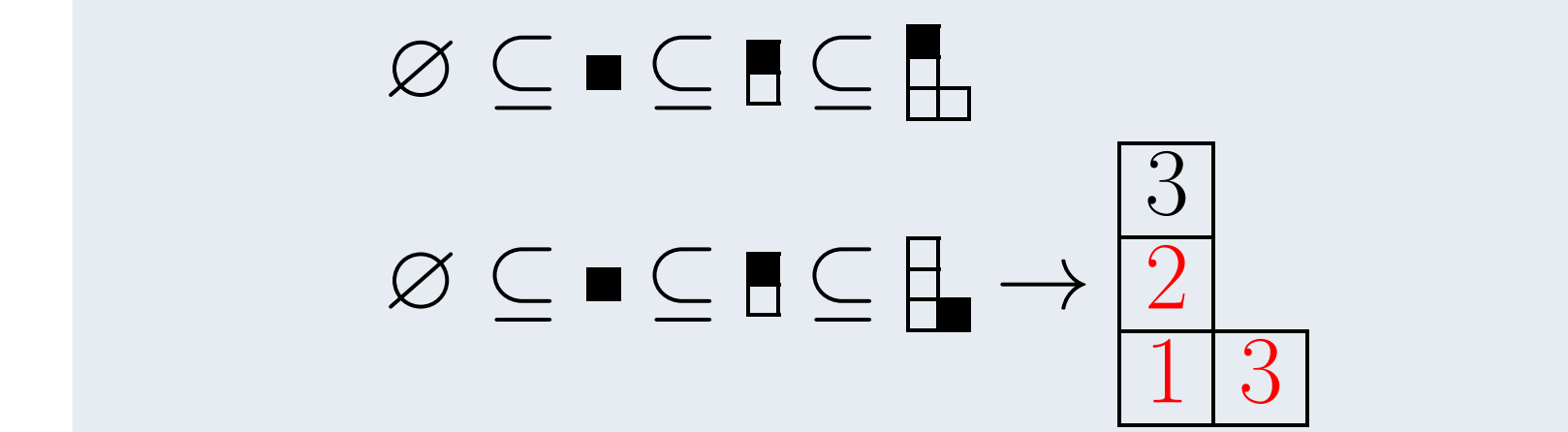
- $k$ -Schur functions are  $k+1$ -Schur positive:  $s_\lambda^{(k)} = \sum a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$  with  $a_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$ .
- Iteration gives Schur positivity of  $k$ -Schur functions.
- Conjecture:**  $g_\lambda^{(k)}$  is Schur positive.
- Conjecture:**  $g_\lambda^{(k)} = \sum_\mu (-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$  for  $b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$ .

$k$ -Schur Functions

Generating functons of strong tabelaux.

$$s_\lambda^{(k)} := \sum x^{\text{weight}(T)}$$

Strong Tableaux: Maximal strong order chains of marked covers



Dual Affine Grothendieck Polynomials

$g_\lambda^{(k)}$  is dual basis to  $G_\lambda^{(k)}$ .

- $g_\lambda^{(k)} = s_\lambda^{(k)} + \text{lower order terms}$
- $g_\lambda^{(k)} = g_\lambda$ , dual to  $G_\lambda$ , for large  $k$ .

Open Problem

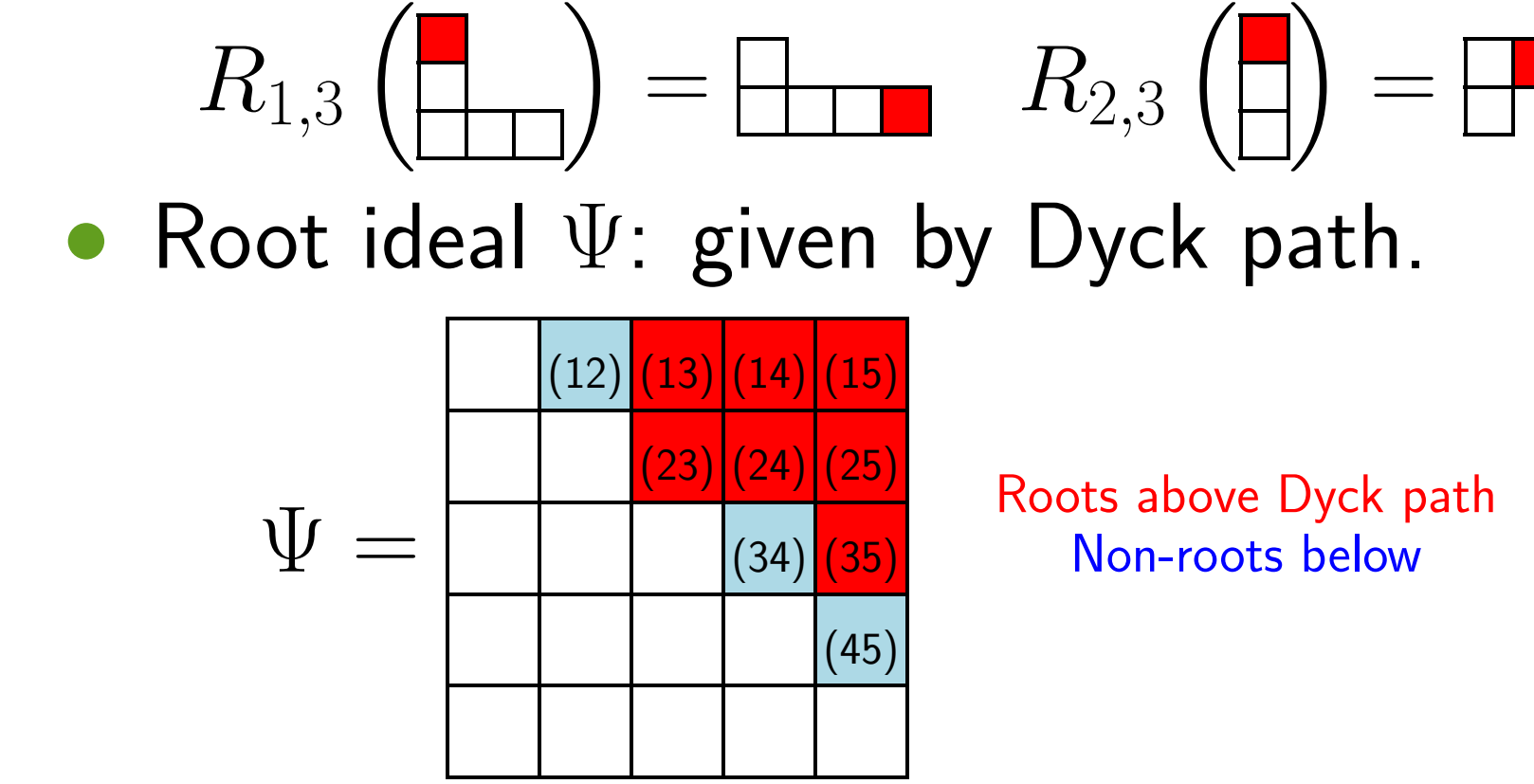
Find a direct definition of  $g_\lambda^{(k)}$ .

Catalan Functions

For  $\gamma \in \mathbb{Z}^\ell$ ,

$$H(\Psi; \gamma) := \prod_{(i,j) \notin \Psi} (1 - R_{ij}) h_\gamma$$

- Raising operators  $R_{i,j}(h_\lambda) = h_{\lambda+\epsilon_i-\epsilon_j}$
- Root ideal  $\Psi$ : given by Dyck path.



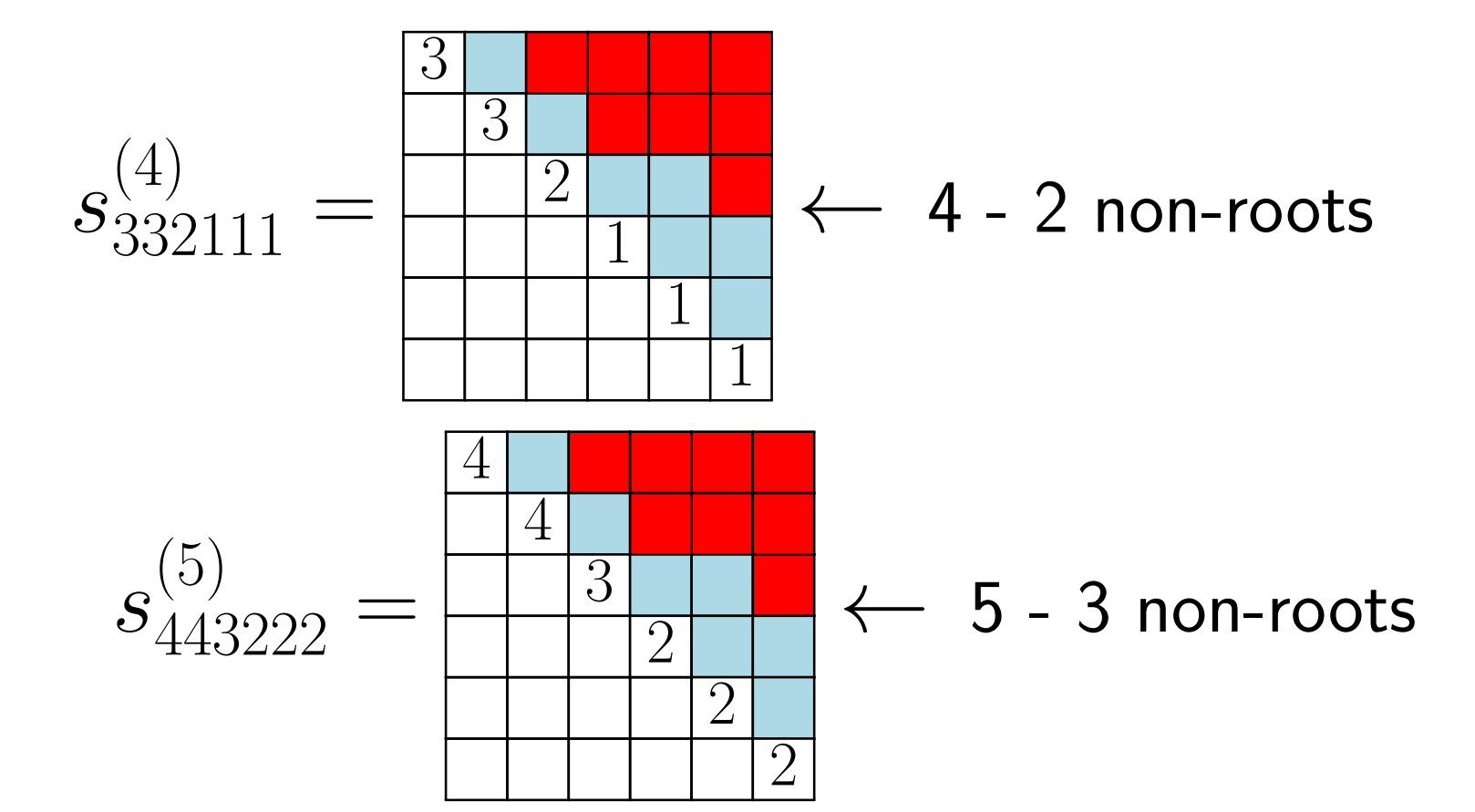
$H(\Psi; 54332)$

$$\begin{aligned} &= (1 - R_{12})(1 - R_{34})(1 - R_{45}) h_{54332} \\ &= h_{54332} - h_{45332} - h_{54422} - h_{54341} \\ &\quad + h_{45422} + h_{45341} + h_{54431} - h_{45431} \end{aligned}$$

- $H(\emptyset; \lambda) = s_\lambda$  (Jacobi-Trudi Identity)

$k$ -Schur Catalans

$s_\lambda^{(k)} = H(\Psi; \lambda)$  for particular  $\Psi$ , defined by  $\underbrace{\lambda_i + \# \text{non-roots in row } i}_{\text{bandwidth}} = k$ .



Why use Catalan  $k$ -Schurs?

Shift Invariance:

$$e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$$

Corollary

Dual Pieri rule  $\implies s_\lambda^{(k)}$  branching!

$$s_\lambda^{(k)} = e_\ell^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$$

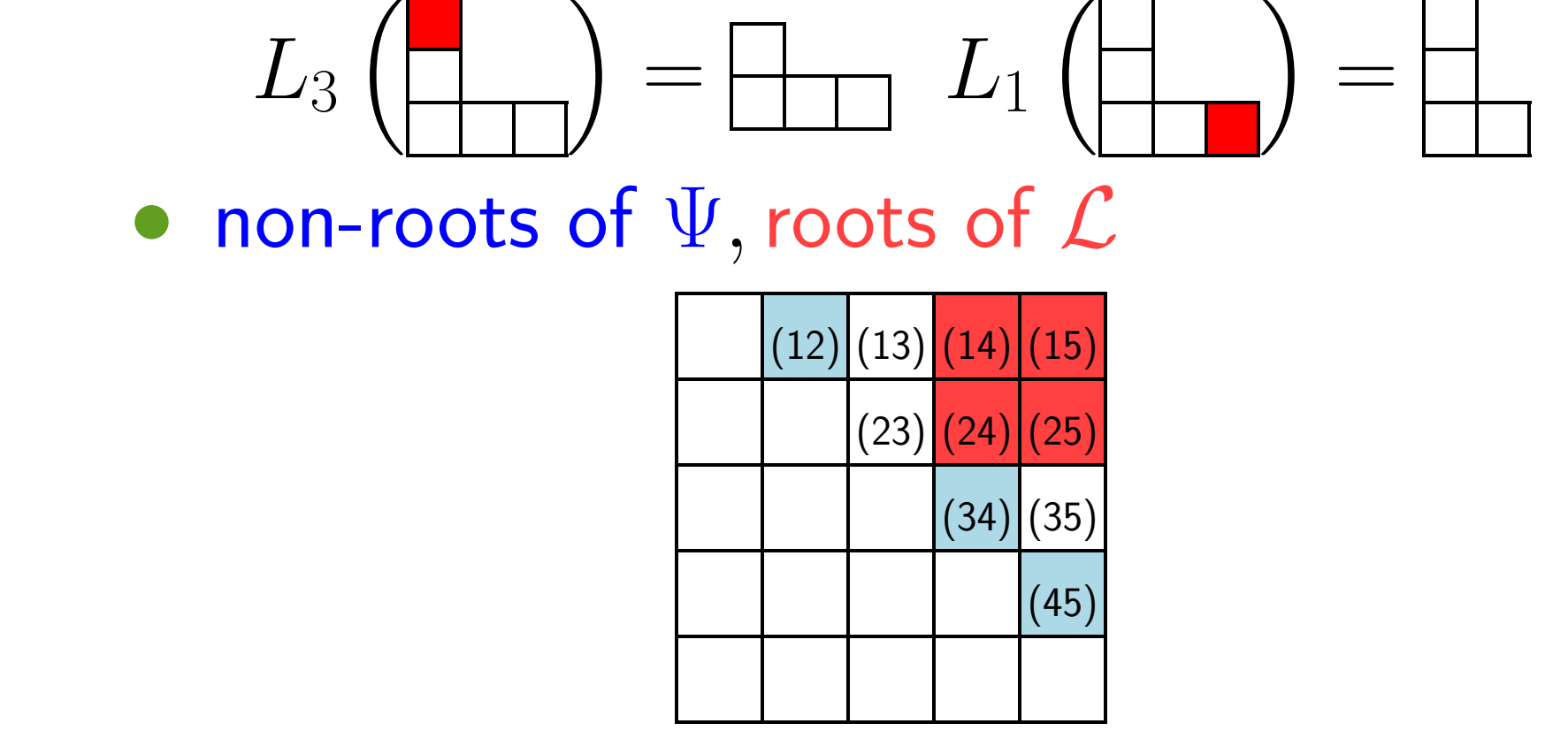
where  $a_{\lambda\mu}^{(k)}$  counts strong vertical strips.

K-theoretic Catalan Functions

For  $\gamma \in \mathbb{Z}^\ell$ , root ideals  $\Psi, \mathcal{L}$

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \notin \Psi} (1 - R_{ij}) K h_\gamma$$

- Lowering operators  $L_j(K h_\lambda) = K h_{\lambda-\epsilon_j}$
- non-roots of  $\Psi$ , roots of  $\mathcal{L}$

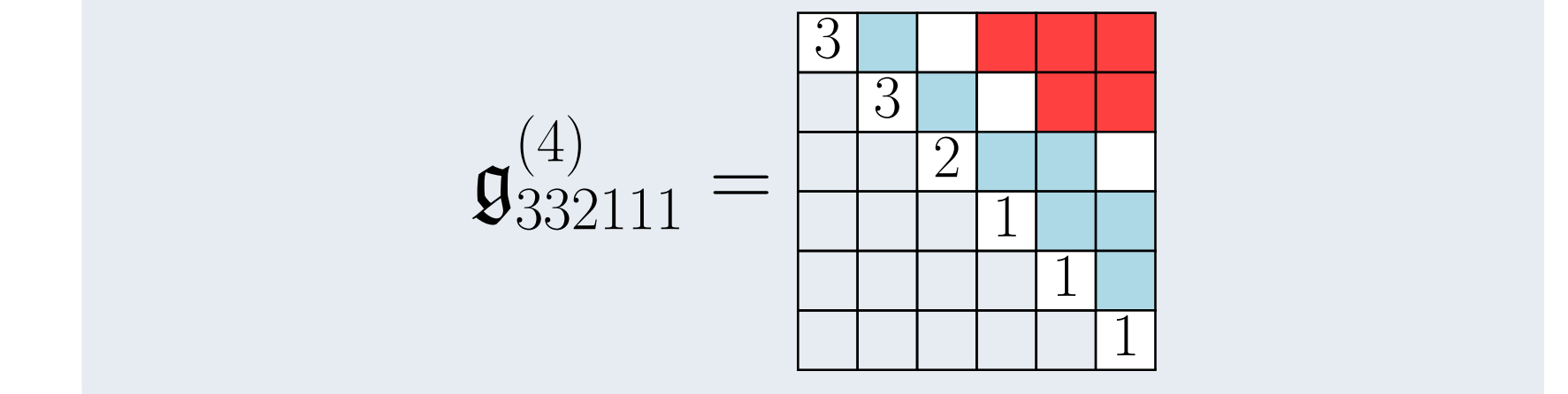


$K(\Psi; \mathcal{L}; 54332)$

$$\begin{aligned} &= (1 - L_4)^2 (1 - L_5)^2 \\ &\quad \cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) K h_{54332} \end{aligned}$$

- $K(\emptyset; \emptyset; \lambda) = g_\lambda$ .

$\mathfrak{g}_\lambda^{(k)} := K(\Psi; \mathcal{L}; \lambda)$  with  $\text{band}(\Psi) = k$ ,  $\text{band}(\mathcal{L}) = k+1$



Theorem: Shift Invariance

$$G_{1^\ell}^\perp \mathfrak{g}_{\lambda+1^\ell}^{(k+1)} = \mathfrak{g}_\lambda^{(k)}$$

Corollary

$\mathfrak{g}_\lambda^{(k)}$  branching follows from dual Pieri rule.

Conjecture

$$\mathfrak{g}_\lambda^{(k)} = g_\lambda^{(k)}$$