

Diagonal Harmonics and Shuffle Theorems

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on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
arXiv:2102.07931

Capsule Research Talk

23 August 2021

- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

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- Let $\Lambda = \mathbb{Q}(q, t)[e_1, e_2, \dots] = \mathbb{Q}(q, t)[h_1, h_2, \dots]$. Call these “symmetric functions.”

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- Let $\Lambda = \mathbb{Q}(q, t)[e_1, e_2, \dots] = \mathbb{Q}(q, t)[h_1, h_2, \dots]$. Call these “symmetric functions.”
- Λ is a $\mathbb{Q}(q, t)$ -algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

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- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \dots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \dots \geq \mu_l \geq 0$.

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Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

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- Algebraic LHS: ∇e_k doubly graded character of diagonal coinvariants for S_k ((Haiman, 2002) via Hilbert Scheme connection).

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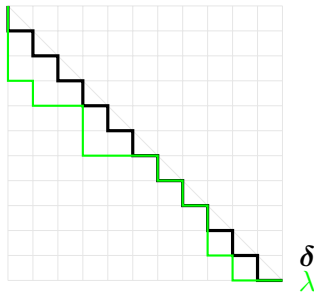
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- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.
- $G_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

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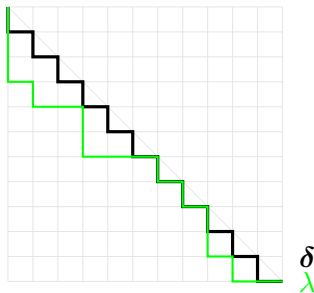
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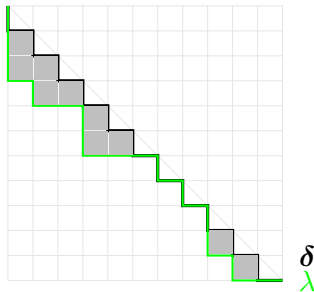


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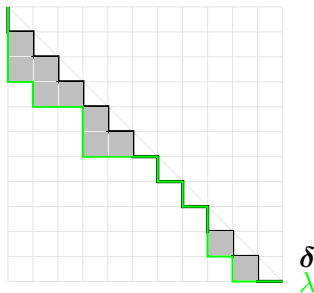


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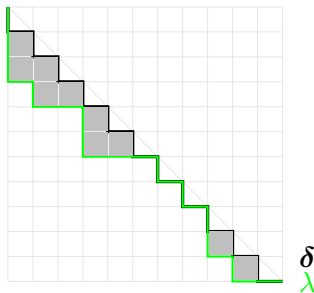


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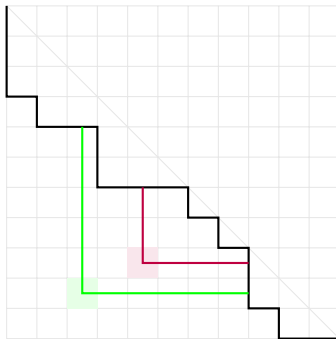
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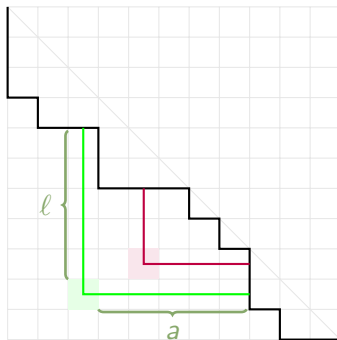
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dinv

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Balanced hook is given by a cell below λ satisfying

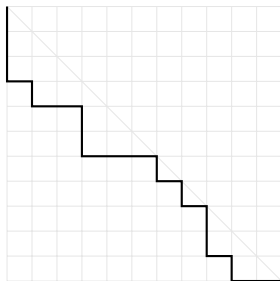
$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.

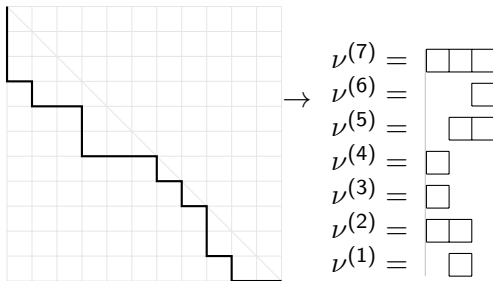
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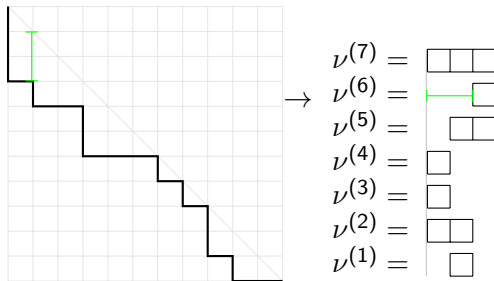
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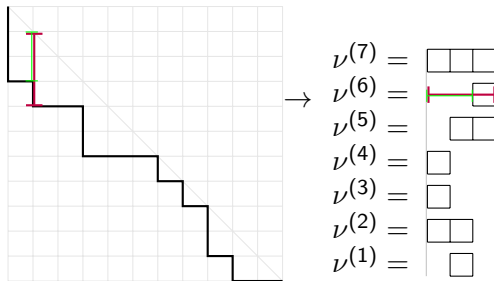
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for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

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---	---	---	---	---

2	4	4	7	8	9	9
---	---	---	---	---	---	---

$$T = \begin{array}{c} \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 5 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 4 & 4 & 7 & 8 & 9 & 9 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 6 & 7 & 7 & 7 \\ \hline \end{array} \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

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• \mathcal{G}_ν is symmetric and Schur positive.

Example ∇e_3






$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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	qt	
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- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number”
 $(q^3 + q^2t + qt + qt^2 + t^3)$.

- Symmetric polynomials and The Shuffle Theorem
- **Generalizations of The Shuffle Theorem**
- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra \mathcal{E}

- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

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where summation is over all (kn, km) -Dyck paths.

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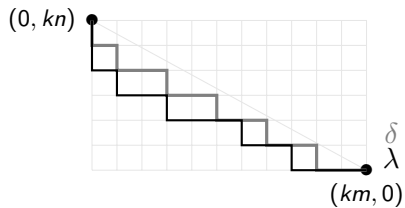
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$$e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

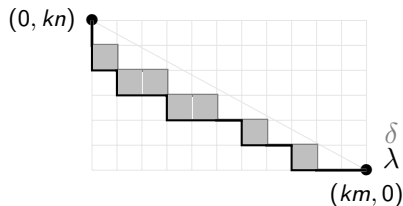
where summation is over all (kn, km) -Dyck paths.

- Coefficient of $s_{1,\dots,1}$ is “rational (q, t) -Catalan number”

Rational Path Combinatorics

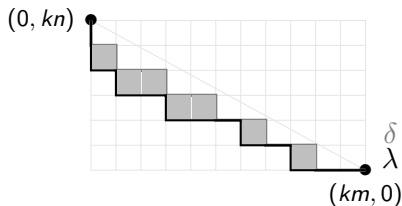


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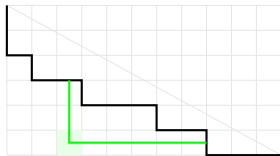


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$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \quad p = \frac{n}{m} - \epsilon$$

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Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

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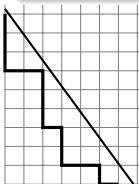
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- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- **Proof techniques and new progress**

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1 \leq j} (1 - qtx_i/x_{i+1})}{\prod_{i < j} ((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

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Proof Idea

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For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1)) / (a_{l-1}, \dots, a_1, 0)}^{\sigma}(x_1, \dots, x_l; q)$$

for infinite formal sum $\mathcal{L}_{\beta/\alpha}^{\sigma}$ a “series LLT.” (Grojnowski-Haiman, 2007).

Cauchy Identity

- (Twisted) non-symmetric Hall-Littlewood polynomials $E_{\lambda}^{\sigma}(x_1, \dots, x_I; q)$ defined via Demazure-Lusztig operators

$$T_i = qs_i + (1 - q) \frac{s_i - 1}{1 - x_{i+1}/x_i}$$

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Loehr-Warrington Conjecture

$$\nabla s_\mu = \text{sgn}(\mu) \sum_{(G,R) \in \text{LNDP}_\mu} t^{\text{area}(G,R)} q^{\text{dinv}(G,R)} x^R$$

Generalizations

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Thank you!

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