

Diagonal Harmonics and Shuffle Theorems

George H. Seelinger

ghseeli@umich.edu

on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
arXiv:2102.07931

Capsule Research Talk

23 August 2021

- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

- Let $\Lambda = \mathbb{Q}(q, t)[e_1, e_2, \dots] = \mathbb{Q}(q, t)[h_1, h_2, \dots]$. Call these “symmetric functions.”

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

- Let $\Lambda = \mathbb{Q}(q, t)[e_1, e_2, \dots] = \mathbb{Q}(q, t)[h_1, h_2, \dots]$. Call these “symmetric functions.”
- Λ is a $\mathbb{Q}(q, t)$ -algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \dots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \dots \geq \mu_l \geq 0$.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \dots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \dots \geq \mu_l \geq 0$.
- Representation-theoretic and geometric significance.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \dots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \dots \geq \mu_l \geq 0$.
- Representation-theoretic and geometric significance.

Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- ∇ a symmetric function operator with (modified) Macdonald polynomials as eigenfunctions:

$$\nabla \tilde{H}_{\mu}(X; q, t) = t^{n(\mu)} q^{n(\mu')} \tilde{H}_{\mu}(X; q, t)$$

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- ∇ a symmetric function operator with (modified) Macdonald polynomials as eigenfunctions:

$$\nabla \tilde{H}_{\mu}(X; q, t) = t^{n(\mu)} q^{n(\mu')} \tilde{H}_{\mu}(X; q, t)$$

- Algebraic LHS: ∇e_k doubly graded character of diagonal coinvariants for S_k ((Haiman, 2002) via Hilbert Scheme connection).

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Combinatorial RHS: Combinatorics of Dyck paths.

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k -by- k Dyck paths.

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k -by- k Dyck paths.
- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.

Theorem (Carlsson-Mellit, 2018)

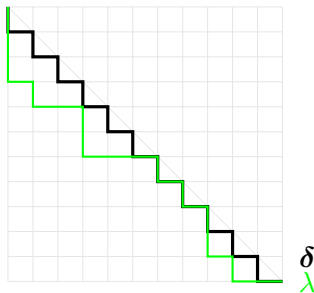
$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k -by- k Dyck paths.
- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.
- $G_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

Dyck paths

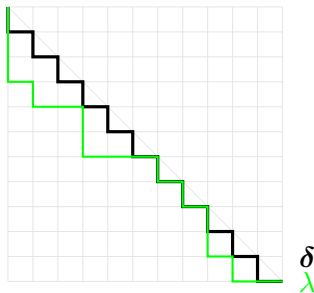
A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

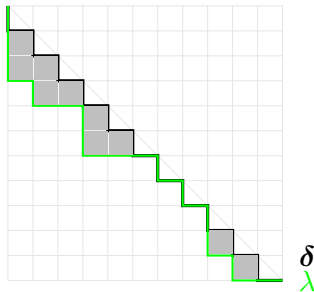


- $\text{area}(\lambda) = \text{number of squares above } \lambda \text{ but below the path } \delta \text{ of alternating S-E steps.}$

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

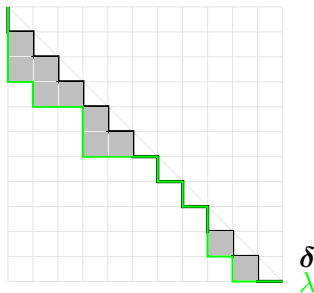


- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

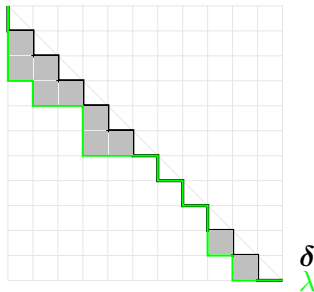


- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k .

Dyck paths

Dyck paths

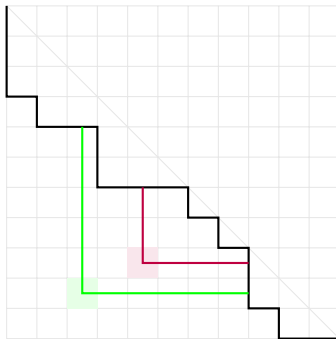
A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



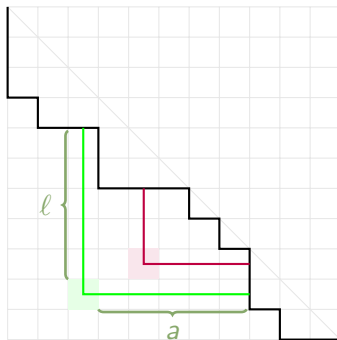
- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k . $(1, 2, 5, 14, 42, \dots)$

dinv

$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



Balanced hook is given by a cell below λ satisfying

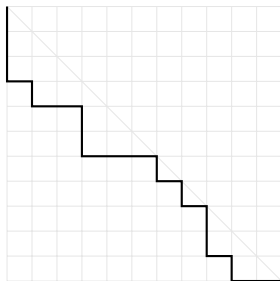
$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.

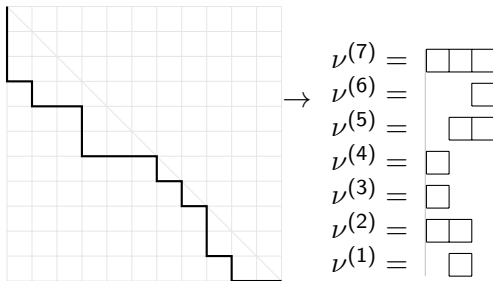
LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



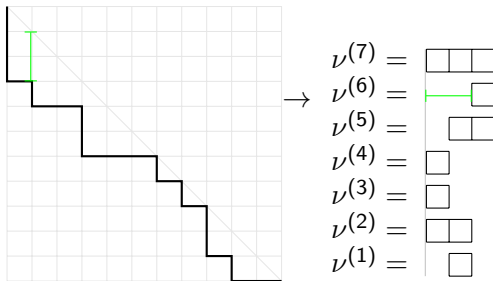
LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



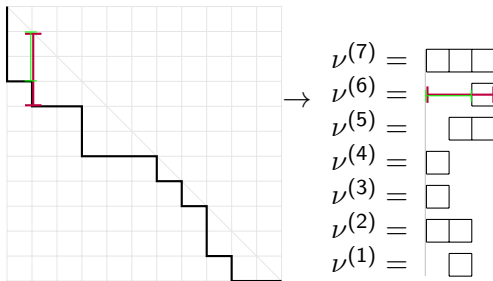
LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

1 2 3 3 5

2 4 4 7 8 9 9

$T =$ 1 1 6 7 7 7

LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 5 |
|---|---|---|---|---|

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 4 | 4 | 7 | 8 | 9 | 9 |
|---|---|---|---|---|---|---|

$$T = \begin{array}{c} \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 3 & 5 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & 4 & 4 & 7 & 8 & 9 & 9 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 6 & 7 & 7 & 7 \\ \hline \end{array} \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

1 2 3 3 5

2 4 4 7 8 9 9

$$T = \begin{array}{cccccc} 1 & 1 & 6 & 7 & 7 & 7 \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

$$\mathcal{G}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

1 2 3 3 5

2 4 4 7 8 9 9

$$T = \begin{array}{cccccc} 1 & 1 & 6 & 7 & 7 & 7 \\ 2 & 4 & 4 & 7 & 8 & 9 & 9 \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

•

$$\mathcal{G}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

• \mathcal{G}_ν is symmetric and Schur positive.

Example ∇e_3






$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

Example ∇e_3

$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$








Example ∇e_3

| λ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$ |
|---|---|---|
|  | q^3 | |
|  | $q^2 t$ | |
|  | qt | |
|  | qt^2 | |
|  | t^3 | |

Example ∇e_3

| λ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$ |
|---|---|---|
|  | q^3 | $s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$ |
|  | q^2t | $qts_{2,1} + q^2ts_{1,1,1}$ |
|  | qt | $ts_{2,1} + qts_{1,1,1}$ |
|  | qt^2 | $t^2s_{2,1} + qt^2s_{1,1,1}$ |
|  | t^3 | $t^3s_{1,1,1}$ |

Example ∇e_3

| λ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$ |
|---|---|---|
|  | q^3 | $s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$ |
|  | q^2t | $qts_{2,1} + q^2ts_{1,1,1}$ |
|  | qt | $ts_{2,1} + qts_{1,1,1}$ |
|  | qt^2 | $t^2s_{2,1} + qt^2s_{1,1,1}$ |
|  | t^3 | $t^3s_{1,1,1}$ |

- Entire quantity is q, t -symmetric

Example ∇e_3

| λ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$ | $q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$ |
|-----------|---|---|
| | q^3 | $s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$ |
| | q^2t | $qts_{2,1} + q^2ts_{1,1,1}$ |
| | qt | $ts_{2,1} + qts_{1,1,1}$ |
| | qt^2 | $t^2s_{2,1} + qt^2s_{1,1,1}$ |
| | t^3 | $t^3s_{1,1,1}$ |

- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number”
 $(q^3 + q^2t + qt + qt^2 + t^3)$.

- Symmetric polynomials and The Shuffle Theorem
- **Generalizations of The Shuffle Theorem**
- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra \mathcal{E}

- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

Schiffmann's Elliptic Hall Algebra \mathcal{E}

- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)
- \mathcal{E} acts on Λ , e.g., for $M = (1 - q)(1 - t)$ and automorphism ω ,

$$e_k[-MX^{m,1}] \cdot 1 = \omega \nabla^m e_k$$

Schiffmann's Elliptic Hall Algebra \mathcal{E}

- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)
- \mathcal{E} acts on Λ , e.g., for $M = (1 - q)(1 - t)$ and automorphism ω ,

$$e_k[-MX^{m,1}] \cdot 1 = \omega \nabla^m e_k$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

$$e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all (kn, km) -Dyck paths.

Schiffmann's Elliptic Hall Algebra \mathcal{E}

- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)
- \mathcal{E} acts on Λ , e.g., for $M = (1 - q)(1 - t)$ and automorphism ω ,

$$e_k[-MX^{m,1}] \cdot 1 = \omega \nabla^m e_k$$

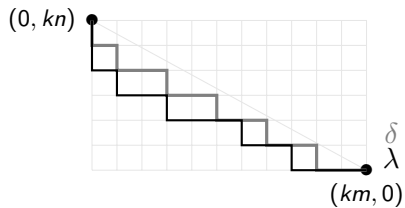
Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

$$e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

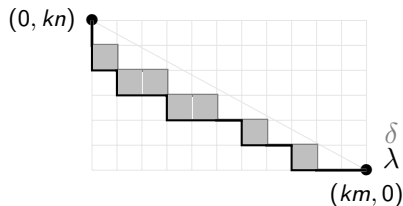
where summation is over all (kn, km) -Dyck paths.

- Coefficient of $s_{1,\dots,1}$ is “rational (q, t) -Catalan number”

Rational Path Combinatorics

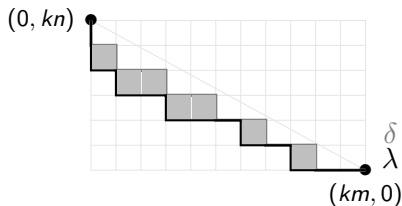


Rational Path Combinatorics

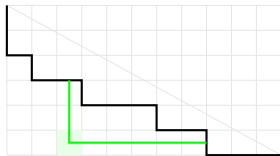


- $\text{area}(\lambda)$ as before; number of boxes between λ and highest path δ below $y + \frac{n}{m}x = kn$.

Rational Path Combinatorics



- $\text{area}(\lambda)$ as before; number of boxes between λ and highest path δ below $y + \frac{n}{m}x = kn$.
- $\text{dinv}_p(\lambda) = \text{number of } p\text{-balanced hooks:}$



$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \quad p = \frac{n}{m} - \epsilon$$

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_I) \in \mathbb{Z}^I$ to be the south step sequence of highest path δ under the line $y + px = s$.

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_I) \in \mathbb{Z}^I$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1, \dots, b_I)} \cdot 1$$

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_I) \in \mathbb{Z}^I$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1, \dots, b_I)} \cdot 1 = \sum_{\lambda} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,

Negut Elements

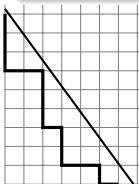
For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1, \dots, b_l)} \cdot 1 = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



$\text{area}(\lambda)$ as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks } \frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- **Proof techniques and new progress**

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1 \leq j} (1 - qtx_i/x_{i+1})}{\prod_{i < j} ((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1 \leq j} (1 - qtx_i/x_{i+1})}{\prod_{i < j} ((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

Let $\psi D_{\mathbf{b}}$ be RHS without applying pol. Easier to prove a “shuffle theorem-like” result on infinite series:

Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1 \leq j} (1 - qtx_i/x_{i+1})}{\prod_{i < j} ((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

Let $\psi D_{\mathbf{b}}$ be RHS without applying pol. Easier to prove a “shuffle theorem-like” result on infinite series:

Stable Shuffle Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1)) / (a_{l-1}, \dots, a_1, 0)}^{\sigma}(x_1, \dots, x_l; q)$$

for infinite formal sum $\mathcal{L}_{\beta/\alpha}^{\sigma}$ a “series LLT.” (Grojnowski-Haiman, 2007).

Cauchy Identity

- (Twisted) non-symmetric Hall-Littlewood polynomials $E_{\lambda}^{\sigma}(x_1, \dots, x_I; q)$ defined via Demazure-Lusztig operators

$$T_i = qs_i + (1 - q) \frac{s_i - 1}{1 - x_{i+1}/x_i}$$

Cauchy Identity

- (Twisted) non-symmetric Hall-Littlewood polynomials $E_{\lambda}^{\sigma}(x_1, \dots, x_l; q)$ defined via Demazure-Lusztig operators

$$T_i = qs_i + (1 - q) \frac{s_i - 1}{1 - x_{i+1}/x_i}$$

- $F_{\lambda}^{\sigma} = \overline{E_{-\lambda}^{\sigma w_0}}$.

Cauchy Identity

- (Twisted) non-symmetric Hall-Littlewood polynomials $E_{\lambda}^{\sigma}(x_1, \dots, x_l; q)$ defined via Demazure-Lusztig operators

$$T_i = qs_i + (1 - q) \frac{s_i - 1}{1 - x_{i+1}/x_i}$$

- $F_{\lambda}^{\sigma} = \overline{E_{-\lambda}^{\sigma w_0}}$.

Cauchy identity

$$\frac{\prod_{i < j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)} = \sum_{\mathbf{a} \geq 0} t^{|\mathbf{a}|} E_{\mathbf{a}}^{\sigma}(x_1, \dots, x_l; q^{-1}) F_{\mathbf{a}}^{\sigma}(y_1, \dots, y_l; q),$$

Cauchy Identity

- (Twisted) non-symmetric Hall-Littlewood polynomials $E_{\lambda}^{\sigma}(x_1, \dots, x_l; q)$ defined via Demazure-Lusztig operators

$$T_i = qs_i + (1 - q) \frac{s_i - 1}{1 - x_{i+1}/x_i}$$

- $F_{\lambda}^{\sigma} = \overline{E_{-\lambda}^{\sigma w_0}}$.

Cauchy identity

$$\frac{\prod_{i < j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)} = \sum_{\mathbf{a} \geq 0} t^{|\mathbf{a}|} E_{\mathbf{a}}^{\sigma}(x_1, \dots, x_l; q^{-1}) F_{\mathbf{a}}^{\sigma}(y_1, \dots, y_l; q),$$

- $\mathcal{L}_{\beta/\alpha}^{\sigma} = H_q(w_0(F_{\beta}^{\sigma^{-1}}(x; q) \overline{E_{\alpha}^{\sigma^{-1}}(x; q)}))$ for

$$H_q(f) = \sum_{w \in S_l} w \left(f \prod_{i < j} ((1 - x_j/x_i)(1 - qx_i/x_j))^{-1} \right)$$

Proof Idea

$$\text{Note } \psi D_b = H_q \left(x^{\mathbf{b}} \frac{\prod_{i+1 \leq j} (1 - qtx_i/x_j)}{\prod_{i < j} (1 - tx_i/x_j)} \right)$$

Proof Idea

Note $\psi D_b = H_q \left(x^{\mathbf{b}} \frac{\prod_{i+1 \leq j} (1 - q t x_i / x_j)}{\prod_{i \leq j} (1 - t x_i / x_j)} \right)$ (looks related to $\frac{\prod_{i < j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)}$)

Proof Idea

Note $\psi D_b = H_q \left(x^{\mathbf{b}} \frac{\prod_{i+1 \leq j} (1 - q t x_i / x_j)}{\prod_{i \leq j} (1 - t x_i / x_j)} \right)$ (looks related to $\frac{\prod_{i \leq j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)}$)

Stable Shuffle Theorem

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1)) / (a_{l-1}, \dots, a_1, 0)}^{\sigma} (x_1, \dots, x_l; q)$$

Proof Idea

Note $\psi D_b = H_q \left(x^{\mathbf{b}} \frac{\prod_{i+1 \leq j} (1 - q t x_i / x_j)}{\prod_{i \leq j} (1 - t x_i / x_j)} \right)$ (looks related to $\frac{\prod_{i \leq j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)}$)

Stable Shuffle Theorem

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1)) / (a_{l-1}, \dots, a_1, 0)}^{\sigma} (x_1, \dots, x_l; q)$$

Under polynomial truncation,

$$\mathcal{L}_{\beta/\alpha}^{\sigma} (x_1, \dots, x_l; q) \rightarrow q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)} (x_1, \dots, x_l; q^{-1})$$

Proof Idea

Note $\psi D_b = H_q \left(x^{\mathbf{b}} \frac{\prod_{i+1 \leq j} (1 - q t x_i / x_j)}{\prod_{i \leq j} (1 - t x_i / x_j)} \right)$ (looks related to $\frac{\prod_{i < j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)}$)

Stable Shuffle Theorem

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1)) / (a_{l-1}, \dots, a_1, 0)}^{\sigma}(x_1, \dots, x_l; q)$$

Under polynomial truncation,

$$\mathcal{L}_{\beta/\alpha}^{\sigma}(x_1, \dots, x_l; q) \rightarrow q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_l; q^{-1})$$

$$\implies \omega(D_b \cdot 1)(x_1, \dots, x_l) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_l; q^{-1}).$$

Same paradigm works to show the following formulas.

Generalizations

Same paradigm works to show the following formulas.

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

For Δ_{h_I} , $\Delta'_{e_{k-1}}$ operators generalizing ∇ ,

$$\Delta_{h_I} \Delta'_{e_{k-1}} e_n = \langle z^k \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} (1 + zt^{-r_i(\lambda)}).$$

Generalizations

Same paradigm works to show the following formulas.

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

For Δ_{h_I} , $\Delta'_{e_{k-1}}$ operators generalizing ∇ ,

$$\Delta_{h_I} \Delta'_{e_{k-1}} e_n = \langle z^k \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} (1 + zt^{-r_i(\lambda)}).$$

Loehr-Warrington Conjecture

$$\nabla s_\mu = \text{sgn}(\mu) \sum_{(G,R) \in \text{LNDP}_\mu} t^{\text{area}(G,R)} q^{\text{dinv}(G,R)} x^R$$

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_I)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_I)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

- Experimental computation suggests this is “tight.”

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_I)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

- Experimental computation suggests this is “tight.”
- Coefficient of $s_{1, \dots, 1}$ coincides with (q, t) -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

References

Thank you!

- Bergeron, Francois, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. 2016. *Compositional (km, kn) -shuffle conjectures*, Int. Math. Res. Not. IMRN **14**, 4229–4270, DOI 10.1093/imrn/rnv272. MR3556418
- Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H Seelinger. 2021a. *A Shuffle Theorem for Paths Under Any Line*, arXiv e-prints, available at arXiv:2102.07931.
- Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021b. *A proof of the Extended Delta Conjecture*, arXiv e-prints, available at arXiv:2102.08815.
- Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373
- Carlsson, Erik and Anton Mellit. 2018. *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31**, no. 3, 661–697, DOI 10.1090/jams/893. MR3787405
- Galashin, Pavel and Thomas Lam. 2021. *Positroid Catalan numbers*, arXiv e-prints, arXiv:2104.05701, available at arXiv:2104.05701.
- Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091
- Gorsky, Eugene, Graham Hawkes, Anne Schilling, and Julianne Rainbolt. 2020. *Generalized q, t -Catalan numbers*, Algebr. Comb. **3**, no. 4, 855–886, DOI 10.5802/alco.120. MR4145982
- Grojnowski, Ian and Mark Haiman. 2007. *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, Unpublished manuscript.
- Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.
- Haglund, J., J. B. Remmel, and A. T. Wilson. 2018. *The delta conjecture*, Trans. Amer. Math. Soc. **370**, no. 6, 4029–4057, DOI 10.1090/tran/7096. MR3811519
- Haiman, Mark. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676
- Mellit, Anton. 2016. *Toric braids and (m, n) -parking functions*, arXiv e-prints, arXiv:1604.07456, available at arXiv:1604.07456.
- Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004