Diagonal Harmonics and Shuffle Theorems

George H. Seelinger

ghseeli@umich.edu

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Outline

- Symmetric polynomials and diagonal harmonics
- The Shuffle Theorem and its generalizations
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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Generators

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \text{ or } h_r = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

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- E.g. for $n = 3$,

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 = h_1 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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- Λ is a $\mathbb{Q}(q,t)$-algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \ldots, x_l) = \sum_{w \in S_l} w\left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i<j}(1 - x_j/x_i)}\right)$$

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

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Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_+^{S_3})$.

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Break M up into smallest S_n fixed subspaces

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Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

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Answer: Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X; q)$.

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- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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Corollary

$\tilde{H}_\lambda(X;q,t) = \tilde{K}_{\lambda\mu}(q,t)s_\mu$ *satisfies* $\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3 \tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

∇e_n

Frobenius characteristic of DH_3

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Operator ∇

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

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Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathsf{area}(\lambda)} q^{\mathsf{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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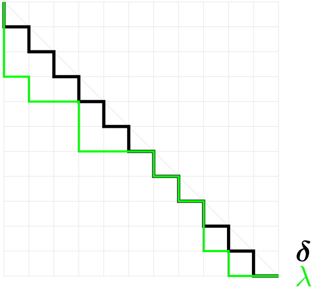
Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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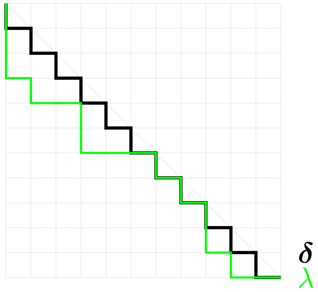

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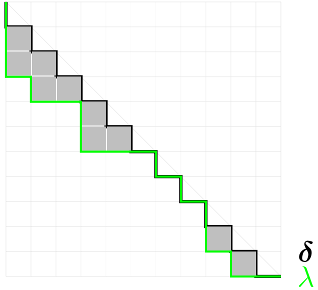

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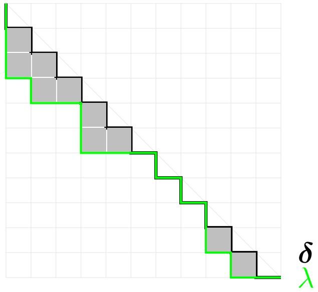

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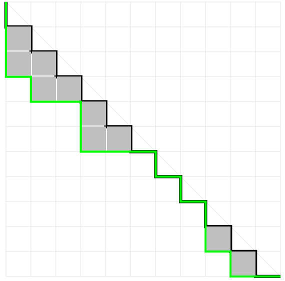

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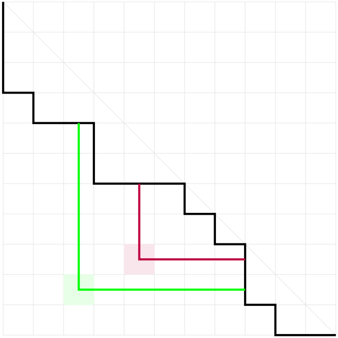
dinv

$\mathrm{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ.



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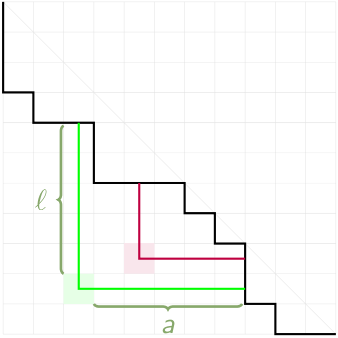

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$

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- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.
- $\mathcal{G}_\nu$ is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

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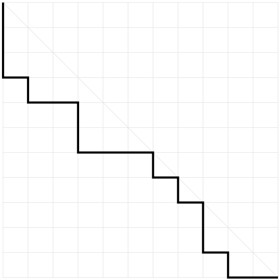

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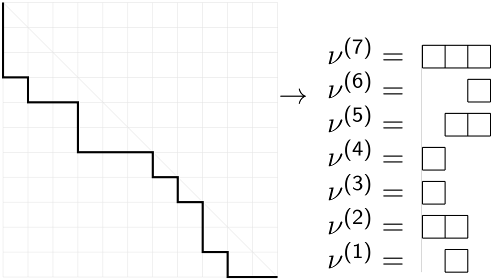

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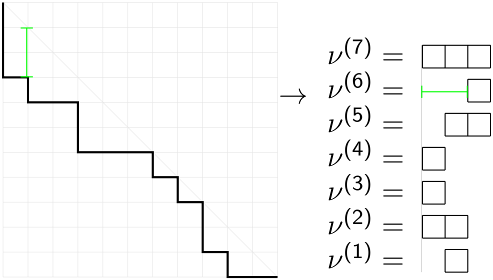

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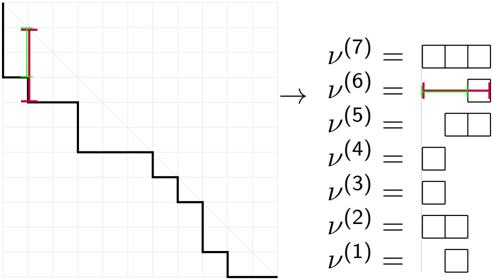

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\phantom{\boxed{0}\boxed{0}\boxed{0}}\boxed{1}\boxed{\color{blue}1}\boxed{6}\boxed{7}\boxed{7}\boxed{7}
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- $$\mathcal{G}_{\substack{\square\square\\ \square}}(x_1,x_2;q) = x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + qx_1^2x_2 + qx_1x_2^2$$

  $$\begin{array}{cccccc} \boxed{1}\boxed{1} & \boxed{1}\boxed{2} & \boxed{1}\boxed{2} & \boxed{2}\boxed{2} & \boxed{1}\boxed{\color{red}{1}} & \boxed{\color{blue}{2}}\boxed{2} \\ \quad\;\boxed{1} & \quad\;\boxed{1} & \quad\;\boxed{2} & \quad\;\boxed{2} & \quad\;\boxed{\color{red}{2}} & \quad\;\boxed{\color{blue}{1}} \end{array}$$

  $$= s_3 + qs_{2,1}$$

Example ∇e_3

λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

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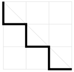		
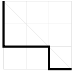		
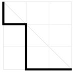		
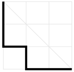		
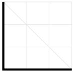		

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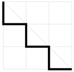
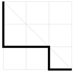
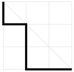
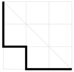
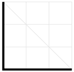
	q^3	
	q^2t	
	qt	
	qt^2	
	t^3	

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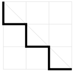
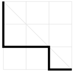
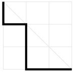
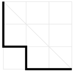
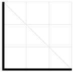
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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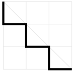
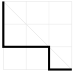
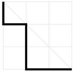
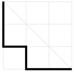
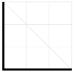
	q^3	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	q^2t	$qts_{2,1} + q^2ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2s_{2,1} + qt^2s_{1,1,1}$
	t^3	$t^3s_{1,1,1}$

- Entire quantity is q, t-symmetric

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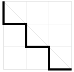
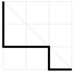
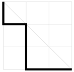
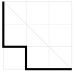
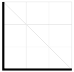
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q,t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Outline

- Symmetric polynomials and diagonal harmonics
- **The Shuffle Theorem and its generalizations**
- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

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- Burban and Schiffmann studied a subalgebra $\mathcal{E}$ of the Hall algebra of coherent sheaves on an elliptic curve over $\mathbb{F}_p$
- $\mathcal{E}$ contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

- $\mathcal{E}$ acts on Λ, e.g., for $M = (1-q)(1-t)$ and automorphism ω,

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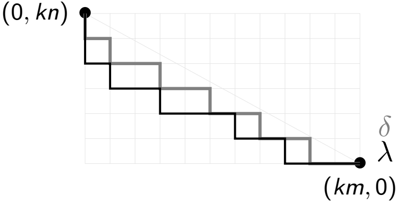
Rational Path Combinatorics



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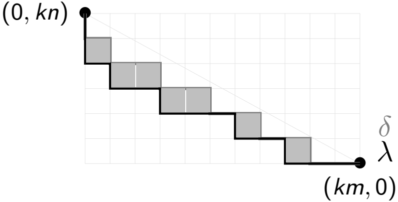

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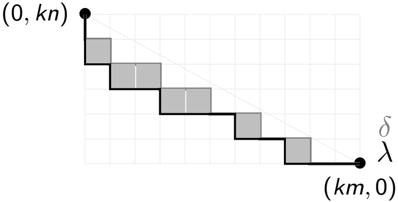
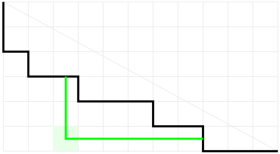


- $\text{area}(\lambda)$ as before; number of boxes between λ and highest path δ below $y + \frac{n}{m}x = kn$.
- $\text{dinv}_p(\lambda) =$ number of p-balanced hooks:



$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \qquad p = \frac{n}{m} - \epsilon$$

Any Line

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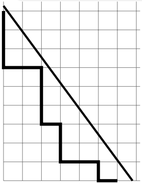

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- Generalizations of The Shuffle Theorem
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Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \ldots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1<j}(1 - qtx_i/x_{i+1})}{\prod_{i<j}((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

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- Need an "infinite series" version of LLT polynomials!

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$$\frac{\prod_{i<j}(1 - q\, t\, x_i\, y_j)}{\prod_{i\leq j}(1 - t\, x_i\, y_j)} = \sum_{\mathbf{a}\geq 0} t^{|\mathbf{a}|}\, E_{\mathbf{a}}^\sigma(x_1, \ldots, x_l; q^{-1})\, F_{\mathbf{a}}^\sigma(y_1, \ldots, y_l; q),$$

Proof Idea

Let $H_q(f) = \sigma\left(\frac{f}{\prod_{i<j}(1-qx_i/x_j)}\right)$.

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Stable Shuffle Theorem

For $\mathbf{b}\in\mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

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$$\Delta_{h_r}\Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda,P} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} \left(1 + zt^{-r_i(\lambda)}\right).$$

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- Coefficient of $s_{1,\ldots,1}$ coincides with (q, t)-polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

Future Directions

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \mathrm{sgn}(\mu) \sum_{(G,R) \in LNDP_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} x^R$$

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- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)
- S_l-representation theory interpretations?

References

Thank you!

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