

**Spin Representation Theory of Symmetric
Groups and Related Combinatorics
Notes from a reading course in Fall 2018**

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1. Introduction (presented by Jinkui Wan)

When discussing the representation theory of the symmetric group, one considers *linear representations* which are group homomorphisms

$$\mathfrak{S}_n \rightarrow GL(V)$$

In 1911, Schur started considering projective representations

$$\mathfrak{S}_n \rightarrow PGL(V) = GL(V)/\mathbb{C}^*$$

leading to the projective representation theory of \mathfrak{S}_n . It turns out that this corresponds to the linear representation theory of an extension of \mathfrak{S}_n , denoted $\tilde{\mathfrak{S}}_n$ and referred to as the *double cover of the symmetric group*, fitting into the short exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n \rightarrow 1$$

where, if $\mathbb{Z}/2\mathbb{Z} = \{1, z\}$, then z is central in $\tilde{\mathfrak{S}}_n$, which gives us that $z = 1$ or $z = -1$.

When $z = 1$, we have the representation theory of \mathfrak{S}_n . When, $z = -1$, we have the representation theory of the *spin symmetric group algebra*

$$\mathbb{C}\mathfrak{S}_n^- = \mathbb{C}\mathfrak{S}_n / \langle z + 1 \rangle = \left\langle t_1, \dots, t_n \mid \begin{array}{l} t_i^2 = 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ t_i t_j = -t_j t_i \text{ when } |i - j| > 1 \end{array} \right\rangle$$

which is equipped with a $\mathbb{Z}/2\mathbb{Z}$ -grading. So, when we discuss spin representations of \mathfrak{S}_n , we are discussing linear representations of $\mathbb{C}\mathfrak{S}_n^-$. Our program to establish these ideas is as follows.

Part I

- (1) Basics of associative superalgebras
- (2) Connection to Hecke-Clifford (or Sergeev) algebra, \mathcal{H}_n
- (3) Split conjugacy classes in a finite supergroup
- (4) Characteristic map
- (5) Schur- Q functions
- (6) Schur-Sergeev duality
- (7) Seminormal form of irreducible representations

Part II

- (1) Centers of $\mathbb{C}\mathfrak{S}_n^-$ (analog of Farahat-Higman theory for $\mathbb{C}\mathfrak{S}_n$)
- (2) Coinvariant theory for $\mathbb{C}\mathfrak{S}_n^-$
- (3) Spin Kostka polynomials
- (4) Quantum deformation (in particular, Olshanki-Sergeev duality)

2. Generalities for Associative Superalgebras (presented by Jinkui Wan)

2.1. Definitions and Examples.

- 2.1. DEFINITION. (a) A *vector superspace* (over \mathbb{C}) is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where elements of $V_{\bar{0}}$ are called *even* and elements of $V_{\bar{1}}$ are called *odd*. For $v \in V_i$, $i \in \mathbb{Z}/2\mathbb{Z}$, we say $|v| = i$.
- (b) If V is a vector superspace with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$, we say the *graded dimension* of V is (m, n) , denoted $\mathbf{dim} V = (m, n)$.
- (c) A *superalgebra* is a \mathbb{C} -algebra A with a $\mathbb{Z}/2\mathbb{Z}$ -grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}/2\mathbb{Z}$.
- (d) A *superalgebra ideal* is a homogeneous ideal, that is, a subset $I \subseteq A$ such that $I = I_{\bar{0}} \oplus I_{\bar{1}} = (I \cap A_{\bar{0}}) \oplus (I \cap A_{\bar{1}})$ as vector spaces and $A_i I_j \subseteq I_{i+j}$ for all $i, j \in \mathbb{Z}/2\mathbb{Z}$.
- (e) A superalgebra that has no non-trivial ideals is called *simple*.
- (f) A *superalgebra homomorphism* $\theta: A \rightarrow B$ is an even algebra homomorphism, that is, an algebra homomorphism sending $A_i \rightarrow B_i$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.
- (g) Given superalgebras A and B , the tensor product $A \otimes B$ is a superalgebra with multiplication

$$(a \otimes b)(a' \otimes b') = (-1)^{|a||b|} aa' \otimes bb'$$

for homogeneous elements and extended by linearity.

- (h) A *commutative superalgebra* is one that is graded commutative, that is

$$yx = (-1)^{|x||y|}xy$$

Thus, the *supercommutator* of a superalgebra is given by

$$[x, y] = xy - (-1)^{|x||y|}yx$$

and the *supercenter* is given by

$$Z(A) = \{a \in A \mid [a, x] = 0 \text{ for all } x \in A\}$$

which is different than the center of an ungraded algebra.

- (i) Given a superalgebra A , we let $|A|$ be the associative algebra where we forget the grading on A .
- 2.2. EXAMPLE. (a) Let $V = V(m|n)$, the vector superspace with $\mathbf{dim} V = (m, n)$. Then, $\text{End}_{\mathbb{C}}(V)$ is a superalgebra and is isomorphic to the matrix superalgebra

$$M(m|n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{array}{l} a \text{ is an } m \times m \text{ matrix} \\ b \text{ is an } m \times n \text{ matrix} \\ c \text{ is an } n \times m \text{ matrix} \\ d \text{ is an } n \times n \text{ matrix} \end{array} \right\}$$

or, in other words, $M(m|n)$ consists of all $m|n$ -block matrices and has $\mathbf{dim} M(m|n) = (m^2 + n^2, 2mn)$. Furthermore, $M(m|n)$ is a simple superalgebra since $|M(m|n)|$ is simple as a \mathbb{C} -algebra.

- (b) Let $V = V(n|n)$ and $p \in \text{End}_{\mathbb{C}}(V)$ be an odd involution (that is, it sends $V_i \rightarrow V_{i+1}$ for $i \in \mathbb{Z}/2\mathbb{Z}$). Then, we define

$$\mathcal{Q}(V) := \{f \in \text{End}_{\mathbb{C}}(V) \mid fp = (-1)^{|f|}pf\} = \mathcal{Q}(V)_{\bar{0}} \oplus \mathcal{Q}(V)_{\bar{1}}$$

$\mathcal{Q}(V)$ is also a superalgebra. Moreover, if we pick a basis $\{v_1, \dots, v_n\}$ of $V_{\bar{0}}$ and let $v'_i = p(v_i)$ for $1 \leq i \leq n$, we have that, with respect to the basis $\{v_1, \dots, v_n, v'_1, \dots, v'_n\}$, $\mathcal{Q}(V)$ is isomorphic to

$$\mathcal{Q}(n) := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M(n|n) \right\}$$

and it is simple.

- (c) The Clifford algebra \mathcal{Cl}_n is the superalgebra generated by the odd elements c_1, \dots, c_n subject to the relations

$$\begin{cases} c_i^2 = 1 \\ c_i c_j = -c_j c_i \quad \forall 1 \leq i \neq j \leq n \end{cases}$$

2.3. LEMMA. *There exist isomorphisms of superalgebras*

- (a) $M(m|n) \otimes M(k|l) \cong M(mk + nl|mk + nl)$
- (b) $M(m|n) \otimes \mathcal{Q}(k) \cong \mathcal{Q}((m+n)k)$
- (c) $\mathcal{Q}(m) \otimes \mathcal{Q}(n) \cong M(mn|mn)$

PROOF. For part (a), we note that

$$\text{End}_{\mathbb{C}}(V(m|n)) \otimes \text{End}_{\mathbb{C}}(V(k|l)) \cong \text{End}_{\mathbb{C}}(V(mk + ml|mk + nl))$$

under the isomorphism sending $f \otimes g$ to the endomorphism of $V(mk + ml|mk + nl)$ mapping $v \otimes w$ to $(-1)^{|g||v|}f(v) \otimes g(w)$.

For part (b), we have

$$\text{End}(V(m|n)) \otimes \mathcal{Q}(V(k|k), p) \cong \mathcal{Q}(V(m|n) \otimes V(k|k), id \otimes p)$$

For (c), one explicitly checks that $\mathcal{Q}(1) \otimes \mathcal{Q}(1) \cong M(1|1)$ and then inductively applies (a) and (b) above. \square

2.4. COROLLARY. *Since $\mathcal{Cl}_{m+n} \cong \mathcal{Cl}_m \otimes \mathcal{Cl}_n$ under the isomorphism sending generators c_1, \dots, c_n to $c_1 \otimes 1, \dots, c_n \otimes 1$ and c_{n+1}, \dots, c_{n+m} to $1 \otimes c_1, \dots, 1 \otimes c_m$, we have the corollaries*

- (a) $\mathcal{Cl}_1 \cong \mathcal{Q}(1)$ under the isomorphism $c_1 \mapsto p(v_1)$
- (b) $\mathcal{Cl}_2 \cong M(1|1)$ since $\mathcal{Cl}_2 \cong \mathcal{Cl}_1 \otimes \mathcal{Cl}_1 \cong \mathcal{Q}(1) \otimes \mathcal{Q}(1) \cong M(1|1)$
- (c) $\mathcal{Cl}_{2^k} \cong M(2^{k-1}|2^{k-1})$
- (d) $\mathcal{Cl}_{2^{k-1}} \cong \mathcal{Q}(2^{k-1})$
- (e) and thus, \mathcal{Cl}_n is simple by parts (c) and (d).

2.2. Classification of Simple Superalgebras.

2.5. THEOREM. *There are two types of finite dimensional simple associative superalgebras over \mathbb{C} :*

- (a) $M(m|n)$
- (b) $\mathcal{Q}(n)$

2.3. Wedderburn Theorem and Schur's Lemma.

- 2.6. DEFINITION. (a) A (super)module over a superalgebra A is a vector space $M = M_{\bar{0}} \oplus M_{\bar{1}}$ with a left action of A on M such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}/2\mathbb{Z}$.
- (b) A homomorphism between A -modules M and N is a linear map $f: M \rightarrow N$ such that

$$f(am) = (-1)^{|f||a|} a f(m) \text{ for all } a \in A, m \in M$$

and

$$\text{Hom}_A(M, N) := \text{Hom}_A(M, N)_{\bar{0}} \oplus \text{Hom}_A(M, N)_{\bar{1}}$$

where $f \in \text{Hom}_A(M, N)_{\bar{1}} \subseteq \text{Hom}_{\mathbb{C}}(M, N)$ is such that $f(M_i) \subseteq N_{i+1}$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.

2.7. DEFINITION. An A -module is said to be *simple* if it is nonzero and has no proper A -submodules. An A -module M is said to be *semisimple* if every A -submodule of M is a direct summand of M .

2.8. THEOREM (Super Wedderburn Theorem). *The following are equivalent for a finite dimensional superalgebra A .*

- (a) Every A -module is semisimple
- (b) A is a finite direct sum of left simple superideals
- (c) A is a direct product of a finite number of simple algebras

2.9. DEFINITION. Thus, we say a superalgebra A is *semisimple* if it satisfies one of the three conditions.

- 2.10. EXAMPLE. (a) $M(m|n) = I_1 \oplus I_2 \oplus \cdots \oplus I_m \oplus I_{m+1} \oplus \cdots \oplus I_{m+n}$ where $I_k = M(m|n)E_{k,k}$ for $1 \leq k \leq m+n$.
- (b) $\mathcal{Q}(n) = J_1 \oplus \cdots \oplus J_n$ where

$$J_k = \mathcal{Q}(n)(E_{k,k} + E_{n+k,n+k})$$

- (c) $\text{Hom}_{M(m|n)}(I_k, I_k) \cong \mathbb{C}$ and $\text{Hom}_{\mathcal{Q}(n)}(J_k, J_k) \cong \mathbb{C} \oplus \mathbb{C}p$. Importantly, the latter space is not 1-dimensional despite J_k being 1-dimensional!

2.11. COROLLARY. *A finite dimensional semisimple superalgebra A is isomorphic to*

$$A \cong \bigoplus_{i=1}^m M(r_i|s_i) \oplus \bigoplus_{j=1}^n \mathcal{Q}(n_j)$$

Check this. Most likely depends on your choice of basis and involution.

where $m = m(A)$ and $q = q(A)$ are invariants of A .

2.12. DEFINITION. A simple A -module V is said to be of type M (resp. type Q) if it is annihilated by all but one summand of the form $M(r_i|s_i)$ (resp. $Q(n_j)$).

2.13. COROLLARY. (a) The number of non-isomorphic simple A -modules is given by $m(A) + q(A) = \dim(Z(|A|) \cap A_{\bar{0}})$.

(b) The number of non-isomorphic simple A -modules of type Q is given by $q(A) = \dim(Z(|A|) \cap A_{\bar{1}})$.

2.14. THEOREM (Schur's Lemma). If M and L are simple A -modules, then

$$\dim \operatorname{Hom}_A(M, L) = \begin{cases} 1 & \text{if } M \cong L \text{ of type } M \\ 2 & \text{if } M \cong L \text{ of type } Q \\ 0 & \text{otherwise} \end{cases}$$

2.15. REMARK. (a) A simple A -module M is of type M if and only if $|M|$ is a simple $|A|$ -module

(b) A simple A -module M is of type Q if and only if $|M|$ is a direct sum of two non-isomorphic simple $|A|$ -modules.

3. Split Conjugacy Classes in a Finite Supergroup (presented by Jinkui Wan)

Throughout, let G be a finite group with index 2 subgroup $G_0 \leq G$.

3.1. DEFINITION. (a) We say that the elements of G_0 are *even elements* and the elements of $G_1 := G \setminus G_0$ are *odd elements*,

(b) $\mathbb{C}G$ is a superalgebra, which we will denote $\mathbb{C}[G, G_0]$

3.2. THEOREM (Super MASchke's Theorem). $\mathbb{C}[G, G_0]$ is semisimple

3.3. PROPOSITION. (a) If $g \in G_i$, $h \in G$, then $hgh^{-1} \in G_i$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.

(b) The number of non-isomorphic simple $\mathbb{C}[G, G_0]$ -modules is equal to the number of even conjugacy classes in G .

(c) The number of non-isomorphic simple $\mathbb{C}[G, G_0]$ -module of type Q is equal to the number of odd conjugacy classes in G .

PROOF. We note that, by the usual Artin-Wedderburn theorem, $\mathbb{C}G$ decomposes into a direct sum of simple matrix algebras, each of which has a 1-dimensional center and can be indexed by a conjugacy class of G via $c_i = \sum_{g \in \mathcal{C}_i} g$ where \mathcal{C}_i is a conjugacy class of G . In fact, this shows in the classical theory that the number of conjugacy classes of G equal the number of irreducible representations.

Since conjugacy classes are either even or odd by (a), which is left as an exercise, (b) follows because $\dim(Z(\mathbb{C}G) \cap \mathbb{C}[G, G_0]_{\bar{0}})$ is equal to the number of non-isomorphic simple $\mathbb{C}[G, G_0]$ -modules and (c) follow from the fact that $\dim(Z(\mathbb{C}G) \cap \mathbb{C}[G, G_0]_{\bar{1}})$ gives the number of those of type Q . \square

Now, consider the following situation. Let \tilde{G} be a group such that there exists an index 2 subgroup $\tilde{G}_0 \leq \tilde{G}$ and there exists a short exact sequence

$$1 \rightarrow \{1, z\} \rightarrow \tilde{G} \xrightarrow{\theta} G \rightarrow 1$$

where $z^2 = 1$ and z is central in \tilde{G} . Then,

3.4. PROPOSITION. *For C a conjugacy class of G , the preimage*

$$\theta^{-1}(C) = \{g, gz \mid g \in C\} \subseteq \tilde{G}$$

has that

- (a) $\theta^{-1}(C)$ is a single conjugacy class in \tilde{G} if g is conjugate to zg in \tilde{G}
or
- (b) $\theta^{-1}(C)$ splits into two conjugacy classes in \tilde{G} if there exists a $g \in C$ such that g is not conjugate to zg . In this case, we call C split.

3.5. DEFINITION. We set

$$\mathbb{C}\tilde{G}^- := \mathbb{C}[G, G_0] / \langle z + 1 \rangle$$

and call a $\mathbb{C}\tilde{G}^-$ -module a *spin* $\mathbb{C}[\tilde{G}, \tilde{G}_0]$ -module.

3.6. PROPOSITION. (a) *We have the isomorphism of superalgebras*

$$\mathbb{C}[\tilde{G}, \tilde{G}_0] \cong \underbrace{\mathbb{C}[G, G_0]}_{(z=1)} \oplus \underbrace{\mathbb{C}\tilde{G}^-}_{(z=-1)}$$

- (b) *The number of non-isomorphic simple spin $\mathbb{C}[\tilde{G}, \tilde{G}_0]$ -modules is equal to the number of even split conjugacy classes of G .*
- (c) *The number of non-isomorphic simple spin $\mathbb{C}[\tilde{G}, \tilde{G}_0]$ -modules of type Q is equal to the number of odd split conjugacy classes of G .*

PROOF. Part (a) follows from the semisimplicity of $\mathbb{C}[\tilde{G}, \tilde{G}_0]$.

Now, (a) tells us that

$$Z(|\mathbb{C}\tilde{G}^-|) = \{a \in Z(|\mathbb{C}[\tilde{G}, \tilde{G}_0]|) \mid za = -a\}$$

So let

$$\underbrace{D_1, zD_1, D_2, zD_2, \dots, D_r, zD_r}_{\text{split}}, \underbrace{D_{r+1}, \dots, D_{r+s}}_{\text{non-split}}$$

be the conjugacy classes of \tilde{G} where r is the number of split conjugacy classes in G , $D_i \cap zD_i = \emptyset$ for $1 \leq i \leq r$ and $zD_j = D_j$ for $r+1 \leq j \leq r+s$. Then,

$$Z(|\mathbb{C}\tilde{G}|) \cap \mathbb{C}\tilde{G}_0 = \{a \in Z(|\mathbb{C}\tilde{G}|) \mid a \text{ is even and } za = -a\}$$

has basis $d_{i_1} - zd_{i_1}, d_{i_2} - zd_{i_2}, \dots, d_{i_k} - zd_{i_k}$ for d_{i_i} even and

$$Z(|\mathbb{C}\tilde{G}|) \cap \mathbb{C}\tilde{G}_1 = \{a \in Z(|\mathbb{C}\tilde{G}|) \mid a \text{ is odd and } za = -a\}$$

has basis $d_{j_1} - zd_{j_1}, d_{j_2} - zd_{j_2}, \dots, d_{j_\ell} - zd_{j_\ell}$ for d_{j_j} odd. \square

Check this part

3.7. EXAMPLE. We have

$$1 \rightarrow \{1, z\} \rightarrow \tilde{\mathfrak{S}}_n \xrightarrow{\theta_n} \mathfrak{S}_n \rightarrow 1$$

where $z \in \tilde{\mathfrak{S}}_n$ is even and central, the subgroup of index 2 is \tilde{A}_n , and

$$\mathbb{C}\mathfrak{S}_n^- := \mathbb{C}\tilde{\mathfrak{S}}_n / \langle z + 1 \rangle$$

is the spin symmetric group algebra.

3.8. DEFINITION. Throughout the remainder of these notes, we define $\theta_n: \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n$ to be the double covering map above.

4. A Morita Superequivalence (presented by Jinkui Wan)

Since \mathfrak{S}_n acts on $\mathcal{C}\ell_n$ via $\sigma.c_i = c_{\sigma(i)}$, we can define the semidirect product $\mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$ with multiplication

$$(x, \sigma)(y, \tau) = (x\sigma(y), \sigma\tau)$$

4.1. DEFINITION. We define the *Hecke-Clifford superalgebra* as

$$\mathcal{H} := \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$$

with the c_i having odd parity and the s_j having even parity.

4.2. LEMMA. *There exists a superalgebra isomorphism*

$$\begin{aligned} \mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}\ell_n &\xrightarrow{\sim} \mathcal{H}_n \\ c_i &\mapsto c_i \\ t_j &\mapsto \frac{1}{\sqrt{-2}} s_j (c_j - c_{j+1}) \end{aligned}$$

Check this last map.

Recall that $\mathcal{C}\ell_n$ is a simple superalgebra and it has a unique simple module U_n . If n is even, U_n is of type M and if n is odd, U_n is of type Q. This leads us to define two functors

$$\begin{aligned} F_n &:= - \otimes U_n: \mathbb{C}\mathfrak{S}_n\text{-Mod} \rightarrow \mathcal{H}_n\text{-Mod} \\ G_n &:= \text{Hom}_{\mathcal{C}\ell_n}(U_n, -): \mathcal{H}_n\text{-Mod} \rightarrow \mathbb{C}\mathfrak{S}_n^-\text{-Mod} \end{aligned}$$

4.3. LEMMA. [Kle05, Prop 13.2.2]

- (a) If n is even, then $F_n \circ G_n \cong \text{id}$ and $G_n \circ F_n \cong \text{id}$.
- (b) If n is odd, then $F_n \circ G_n \cong \text{id} \oplus \pi$ and $G_n \circ F_n \cong \text{id} \oplus \pi$ where $\pi(M)_i = M_{i+1}$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.

Thus, because $(F_n \circ G_n)(M) = \text{Hom}_{\mathcal{C}\ell_n}(U_n, M) \otimes U_n$, we have a (super)Morita equivalence between $\mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}\ell_n$ and \mathcal{H}_n .

5. A Double Cover \tilde{B}_n (presented by Jinkui Wan)

Recall that $B_n = \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$. We define

5.1. DEFINITION.

$$\Pi_n := \left\langle z, a_1, \dots, a_n \mid \begin{array}{l} z^2 = a_i^2 = 1, \forall 1 \leq i \leq n \\ a_i a_j = z a_j a_i, i \neq j \end{array} \right\rangle$$

Then, \mathfrak{S}_n acts on Π_n via $\sigma(z) = z$ and $\sigma(a_i) = a_{\sigma(i)}$. This gives us the short exact sequence

$$1 \rightarrow \{1, z\} \rightarrow \Pi_n \rtimes \mathfrak{S}_n \rightarrow B_n \rightarrow 1$$

$$a_i \rightarrow b_i$$

and so we define

The flow here is not great.

5.2. DEFINITION. Let \tilde{B}_n be the supergroup on $\Pi_n \rtimes \mathfrak{S}_n$ with the a_i odd, z even, and $\sigma \in \mathfrak{S}_n$ even.

Since $\mathbb{C}\tilde{B}_n / \langle z + 1 \rangle = \mathcal{H}_n$, we wish to understand conjugacy classes in B_n . We will do so by example.

5.3. EXAMPLE. Consider

$$x = ((++++-+-), (1234)(567)(89)) \in B_{10}$$

As an element of \mathfrak{S}_{10} , $(1234)(567)(89)$ has cycle type $(4, 3, 2, 1)$, but we wish to assign a parity to each of these cycles. To do so, we look at the $(+, -)$ -array in the first coordinate and take the product of the entries corresponding to the cycle. So, (1234) gets cycle type $+\times+\times+\times-=-$ since those are entries 1, 2, 3, and 4 in the array. This gives the cycle type as a tuple of partitions $\rho = (\rho^+, \rho^-)$ and so $\rho(x) = ((3), (4, 2, 1))$. Similarly, if

$$y = ((+----+----+-), (1386)(279)(45)) \in B_{10}$$

then the first cycle has parity $+\times-\times-\times-=-$ since those are the 1, 3, 6, and 8 entries of the array. One can check that y has the same cycle type as x .

5.4. LEMMA. *Two elements of B_n are conjugate if and only if their cycle types are the same.*

5.5. COROLLARY. *The number of conjugacy classes in B_n is*

$$\#\{(\rho^+, \rho^-) \mid |\rho^+| + |\rho^-| = n\}$$

Now, the conjugacy class $\mathcal{C}_{\rho^+, \rho^-}$ is even if k is even for $\underbrace{b_{i_1} b_{i_2} \dots b_{i_k}}_{\in \mathbb{Z}_2^n} \sigma \in$

$\mathcal{C}_{\rho^+, \rho^-}$.

5.6. THEOREM (Read). [CW12, Theorem 3.31]

- (a) Even $\mathcal{C}_{\rho^+, \rho^-}$ splits if and only if $\rho^+ \in \mathcal{OP}_n$ and $\rho^- = \emptyset$
- (b) Odd $\mathcal{C}_{\rho^+, \rho^-}$ splits if and only if $\rho^+ = \emptyset$ and $\rho^- \in \mathcal{SP}_n^-$

where \mathcal{SP}_n^- is all partitions of n with strict parts and odd length.

5.7. DEFINITION. For $\alpha \in \mathcal{OP}_n$, let \mathcal{C}_α^+ be the split conjugacy class in \tilde{B}_n satisfying

- (a) $\mathcal{C}_\alpha^+ = \theta_n^{-1}(\mathcal{C}_{\alpha, \emptyset})$
- (b) There exists $\sigma \in \mathcal{C}_\alpha^+$ such that $\sigma \in \mathfrak{S}_n$ with cycle type α .

6. A ring structure on R^- (presented by Jinkui Wan)

6.1. DEFINITION. We give the following definition

- (a) Let $R_n^- := [\mathcal{H}_n\text{-}\mathbf{Mod}]$, the Grothendieck group of $\mathcal{H}_n\text{-}\mathbf{Mod}$.
- (b) Let $R^- := \bigoplus_{n=0}^\infty R_n^-$ where $R_0^- = \mathbb{Z}$
- (c) Let $R_{\mathbb{Q}}^- = \mathbb{Q} \otimes_{\mathbb{Z}} R^-$
- (d) Let $\mathcal{H}_{m,n}$ be the subalgebra of \mathcal{H}_{m+n} generated by $\mathcal{C}\ell_{m+n}$ and $S_m \times S_n$. Note that $\mathcal{H}_{m,n} \cong \mathcal{H}_m \otimes \mathcal{H}_n$ as a superalgebra.
- (e) Given $M \in \mathcal{H}_m\text{-}\mathbf{Mod}$ and $N \in \mathcal{H}_n\text{-}\mathbf{Mod}$, we define

$$[M] \cdot [N] := [\mathrm{Ind}_{\mathcal{H}_m \otimes \mathcal{H}_n}^{\mathcal{H}_{m+n}} M \otimes N]$$

6.2. PROPOSITION. R^- is commutative with respect to the above multiplication.

6.3. DEFINITION. Define a bilinear form via

$$\langle [M], [N] \rangle := \dim \mathrm{Hom}_{\mathcal{H}_n}(M, N)$$

for $M, N \in \mathcal{H}_n$

6.4. LEMMA. For $\phi \in R_n^-$ (viewed as a character of \tilde{B}_n), set $\phi_\alpha := \phi(x)$ for any $x \in \mathcal{C}_\alpha$. Then,

- (a) For $\phi \in R_m^-, \psi \in R_n^-$, and $\gamma \in \mathcal{OP}_{m+n}$,

$$(\phi \cdot \psi)_\gamma = \sum_{\substack{\alpha \in \mathcal{OP}_m, \beta \in \mathcal{OP}_n \\ \alpha \cup \beta = \gamma}} \frac{z_\gamma}{z_\alpha z_\beta} \phi_\alpha \psi_\beta$$

where z_α is the order of the centralizer of σ of cycle type α in \mathfrak{S}_n .

- (b)

$$\langle \phi, \psi \rangle = \sum_{\alpha \in \mathcal{OP}} 2^{-\ell(\alpha)} z_\alpha^{-1} \phi_\alpha \psi_\alpha$$

6.5. PROPOSITION. The character value vanishes unless you are in an even split conjugacy class.

7. The ring Γ (presented by Jinkui Wan)

7.1. DEFINITION. Let $x = \{x_1, x_2, \dots\}$.

- (a) Define $q_r = q_r(x)$ via the generating function

$$Q(t) = \sum_{r \geq 0} q_r(x) t^r = \prod_{i \geq 1} \frac{1 + tx_i}{1 - tx_i}$$

(b) Let Γ be the \mathbb{Z} -subring of the ring of symmetric functions generated by q_r , $r \geq 0$.

(c) $\Gamma_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$

7.2. PROPOSITION. (a) $\sum_{r+s=n} (-1)^r q_r q_s = 0$ because $Q(t)Q(-t) = 1$

(b) $q_n = \sum_{\alpha \in \mathcal{OP}_n} 2^{\ell(\alpha)} z_{\alpha}^{-1} p_{\alpha}$ where $p_{\alpha} = P_{\alpha_1} \cdots p_{\alpha_{\ell}}$ because $\ln Q(t) = \sum_{r \text{ odd}} \frac{2p_r(x)t^r}{r}$.

7.3. THEOREM. (a) $\Gamma_{\mathbb{Q}}$ is a polynomial algebra with polynomial generators p_{2r-1} for $r \geq 1$.

(b) $\{p_{\mu} \mid \mu \in \mathcal{OP}\}$ is a basis for $\Gamma_{\mathbb{Q}}$

7.4. DEFINITION. Let us define inner product on $\Gamma_{\mathbb{Q}}$ via

$$\langle p_{\alpha}, p_{\beta} \rangle := 2^{-\ell(\alpha)} z_{\alpha} \delta_{\alpha\beta}, \forall \alpha, \beta \in \mathcal{OP}$$

7.5. DEFINITION. Define the (spin) characteristic map to be

$$\begin{aligned} \text{ch}^{-} : R_{\mathbb{Q}}^{-} &\rightarrow \Gamma_{\mathbb{Q}} \\ \phi &\mapsto \sum_{\alpha \in \mathcal{OP}_n} z_{\alpha}^{-1} \phi_{\alpha} p_{\alpha} \end{aligned}$$

7.6. PROPOSITION. (a) ch^{-} is an algebra isomorphism.

(b) ch^{-} is an isometry (that is, $\langle \phi, \psi \rangle = \langle \text{ch}^{-}(\phi), \text{ch}^{-}(\psi) \rangle$).

Now, we seek to construct the basic spin module.

7.7. PROPOSITION. $\mathcal{H}_n = \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$ acts on $\mathcal{C}\ell_n = \text{span}\{c_I \mid I \subseteq \{1, 2, \dots, n\}\}$ via

$$\begin{cases} c_i \cdot (c_{i_1} \cdots c_{i_k}) = c_i c_{i_1} \cdots c_{i_k} \\ \sigma \cdot (c_{i_1} \cdots c_{i_k}) = c_{\sigma(i_1)} \cdots c_{\sigma(i_k)} \end{cases}$$

where $c_i \in \mathcal{C}\ell_n$, $\sigma \in \mathfrak{S}_n$, and the action is extended by linearity.

7.8. PROPOSITION. Let $\sigma = \sigma_1 \cdots \sigma_{\ell}$ be a cycle decomposition of σ . Then

$$\sigma c_I = \begin{cases} \pm c_I & \text{if } I \text{ is a union of some supports of } \sigma_1, \dots, \sigma_{\ell} \\ \pm c_J (J \neq I) & \text{otherwise} \end{cases}$$

7.9. EXAMPLE. Let $\sigma = (134)(25) \in \mathfrak{S}_5$. Then,

$$\sigma c_3 c_5 = c_4 c_2$$

but

$$\sigma c_1 c_3 c_4 = c_3 c_4 c_1 = c_1 c_3 c_4$$

Thus, the character of this action, say ξ^n , satisfies

$$\xi^n(\alpha) = 2^{\ell(\alpha)}, \alpha \in \mathcal{OP}_n$$

and thus $\text{ch}^{-}(\xi^n) = \sum_{\alpha \in \mathcal{OP}_n} z_{\alpha}^{-1} 2^{\ell(\alpha)} p_{\alpha} = q_n$.

7.10. DEFINITION. For $\lambda \in \mathcal{SP}$, we define ξ^λ via the recursive formulas

$$\xi^{(\lambda_1, \lambda_2)} = \xi^{\lambda_1} \xi^{\lambda_2} + 2 \sum_{i=1}^{\lambda_2} (-1)^i \xi^{\lambda_1+i} \xi^{\lambda_2-i}$$

$$\xi^\lambda = \begin{cases} \sum_{j=2}^k (-1)^j \xi^{(\lambda_1, \lambda_j)} \xi^{(\lambda_2, \dots, \hat{\lambda}_j, \dots, \lambda_k)} & k = \ell(\lambda) \text{ is even} \\ \sum_{j=1}^k (-1)^{j-1} \xi^{\lambda_j} \xi^{(\lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_k)} & k = \ell(\lambda) \text{ is odd} \end{cases}$$

7.11. THEOREM. (a) $\text{ch}^-(\xi^\lambda) = Q_\lambda$, the Schur- Q function (to be defined in the next lecture).

(b) $\left\{ \zeta^\lambda := 2^{-\frac{\ell(\lambda)-\delta(\lambda)}{2}} \xi^\lambda \mid \lambda \in \mathcal{SP}_n \right\}$, where $\delta(\lambda) = \chi\{\ell(\lambda) \text{ is odd}\}$, is a complete list of simple characters.

(c) ζ^λ is of type M if $\ell(\lambda)$ is even and of type Q if $\ell(\lambda)$ is odd.

(d) The degree of ζ^λ is

$$2^{n-\frac{\ell(\lambda)-\delta(\lambda)}{2}} \frac{n!}{\lambda_1! \cdots \lambda_\ell!} \left(\prod_{i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right)$$

8. Schur- Q functions and related combinatorics (presented by George H. Seelinger)

9. Center of Symmetric Group Algebras and Spin Symmetric Group Algebras (presented by Jinkui Wan)

9.1. Farahat-Higman's Construction for \mathfrak{S}_n . Given a permutation σ , we note that its cycle type is not stable under inclusion from $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$.

9.1. EXAMPLE. Let $\sigma = (134)(2576) \in \mathfrak{S}_8$. Then, σ has cycle type

$$(4, 3, 1) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}$$

but, when included into \mathfrak{S}_9 , σ has cycle type

$$(4, 3, 1, 1) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array}$$

9.2. DEFINITION. Given a cycle $\sigma \in \mathfrak{S}_n$, its *modified cycle type*, λ , is given by removing the first column from its cycle type.

9.3. EXAMPLE. The modified cycle type of $\sigma = (134)(2576) \in \mathfrak{S}_8$ is $\lambda = (3, 2)$. Note that this is stable with respect to $\mathfrak{S}_8 \hookrightarrow \mathfrak{S}_9$.

9.4. PROPOSITION. (a) If σ is of modified type λ , then $|\lambda|$ is the minimal length for σ as a product of (not necessarily simple) transpositions.

(b) If

$$\begin{cases} \sigma \text{ is of modified type } \lambda \\ \tau \text{ is of modified type } \mu \\ \sigma\tau \text{ is of modified type } \nu \end{cases}, \text{ then } |\nu| \leq |\lambda| + |\mu|$$

9.5. EXAMPLE. $(134) = (13)(34)$ and has modified type (2).

9.6. DEFINITION. (a) Let $\mathcal{C}_\lambda(n)$ be the conjugacy class of \mathfrak{S}_n of modified type λ . Note $\mathcal{C}_\lambda(n) = \emptyset$ if $n < |\lambda| + \ell(\lambda)$.

(b) Let

$$C_\lambda(n) := \begin{cases} \text{Class sum of } \mathcal{C}_\lambda(n) & \text{if } n \geq |\lambda| + \ell(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

9.7. EXAMPLE. $\mathcal{C}_0(n) = \{id\}$ and $\mathcal{C}_{(1)}(n)$ contains all transpositions of \mathfrak{S}_n . Thus, $C_{(1)}(n) = \sum_{1 \leq i < j \leq n} (ij)$.

9.8. PROPOSITION. $\{C_\lambda(n) \mid |\lambda| + \ell(\lambda) \leq n\}$ is a basis for $Z(\mathbb{Z}\mathfrak{S}_n)$.

9.9. DEFINITION. Write

$$C_\lambda(n)C_\mu(n) = \sum A_{\lambda\mu}^\nu(n)C_\nu(n)$$

9.10. EXAMPLE.

$$C_{(1)}(n)C_{(1)}(n) = 3C_{(2)}(n) + 2C_{(1,1)}(n) + \frac{1}{2}n(n-1)C_0(n)$$

since $C_{(1)}(n)^2 = \sum (ij)(kl)$ for all transpositions $(ij), (kl)$ in \mathfrak{S}_n .

9.11. THEOREM (Farahat-Higman). Let λ, μ, ν be partitions. Then,

- (a) There is a unique polynomial $f_{\lambda\mu}^\nu(x) \in \mathbb{Q}[x]$ such that $a_{\lambda\mu}^\nu(n) = f_{\lambda\mu}^\nu(n)$ for all $n \geq |\nu| + \ell(\nu)$.
- (b) $f_{\lambda\mu}^\nu(x) = 0$ unless $|\nu| \leq |\lambda| + |\mu|$
- (c) If $|\nu| = |\lambda| + |\mu|$, then $f_{\lambda\mu}^\nu(x)$ is a constant. In other words, $a_{\lambda\mu}^\nu(n)$ is independent of n .

PROOF IDEA. Let $\Gamma = \{(\sigma, \tau) \mid \sigma \in \mathcal{C}_\lambda(n), \tau \in \mathcal{C}_\mu(n), \sigma\tau \in \mathcal{C}_\nu(n)\}$. Then, once computes

$$a_{\lambda\mu}^\nu(n) = \frac{\#\Gamma}{\#\mathcal{C}_\lambda(n)}$$

If we let \mathfrak{S}_n act on Γ by conjugation, that is $\gamma \cdot (\sigma, \tau) = (\gamma\sigma\gamma^{-1}, \gamma\tau\gamma^{-1})$, then we get that

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$$

Without loss of generality, take $(\sigma_1, \tau_1) \in \Gamma_1$. Then,

$$\#\Gamma_1 = \frac{n!}{\# \text{ of centralizers of } (\sigma_1, \tau_1) \text{ in } \mathfrak{S}_n}$$

Suppose $\gamma \cdot (\sigma_1, \tau_1) = (\sigma_1, \tau_1)$. Then,

$$\gamma \sigma_1 \gamma^{-1} = \sigma_1 \text{ and } \gamma \tau_1 \gamma^{-1} = \tau_1$$

Thus,

$$\gamma \in \mathfrak{S}_{\text{Supp}(\sigma_1, \tau_1)} \times \mathfrak{S}_{\{1, \dots, n\} \setminus \text{Supp}(\sigma_1, \tau_1)}$$

where $\text{Supp}(\sigma_1, \tau_1) = \{j \in \{1, \dots, n\} \mid \sigma_1(j) \neq j \text{ or } \tau_1(j) \neq j\}$. So,

$\# \text{ centralizers of } (\sigma_1, \tau_1) \text{ in } \mathfrak{S}_n = \# \text{ centralizers of } (\sigma_1, \tau_1) \text{ in } \mathfrak{S}_{\text{Supp}(\sigma_1, \tau_1)} \times (n - \# \text{Supp}(\sigma_1, \tau_1))!$

Thus, using our formula above for $a_{\lambda\mu}^\nu$, we arrive at

Finish this formula

$$a_{\lambda\mu}^\nu(n) = \frac{\sum_i \#\Gamma_i}{\#\mathcal{C}_\lambda(n)}$$

□

9.12. DEFINITION. (a) Let \mathbb{B} be the ring of polynomials $f(x) \in \mathcal{Q}[x]$ such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

(b) Let \mathcal{K} be the \mathbb{B} -algebra with basis $\{c_\lambda \mid \lambda \in \mathcal{P}\}$ such that

$$c_\lambda c_\mu := \sum_{\nu \in \mathcal{P}} f_{\lambda\mu}^\nu(x) c_\nu$$

We call this ring the *Farahat-Higman ring*.

9.13. PROPOSITION. *We have the following facts*

(a) \mathcal{K} is commutative and associative.

(b) \mathcal{K} is filtered via $\deg(c_\lambda) = |\lambda|$ for all $\lambda \in \mathcal{P}$.

9.14. REMARK. \mathcal{K} is not graded because $\sum_{\nu \in \mathcal{P}} f_{\lambda\mu}^\nu(x) c_\nu$ is not homogeneous. However, if we say $\mathcal{K}_r = \text{span}\{c_\lambda \mid |\lambda| \leq r\}$, then $\mathcal{K}_r \mathcal{K}_s \subseteq \mathcal{K}_{r+s}$, making \mathcal{K} filtered.

9.15. DEFINITION. Let $\text{gr } \mathcal{K}$ be the associated graded algebra, that is, $\text{gr } \mathcal{K}$ is defined by $(\text{gr } \mathcal{K})_r = \mathcal{K}_r / \mathcal{K}_{r-1}$ and then $\text{gr } \mathcal{K} = \bigoplus_{r \geq 0} (\text{gr } \mathcal{K})_r$.

9.16. LEMMA. *We have the following facts.*

(a) If $|\lambda| + s = m$, then

$$a_{\lambda, (s)}^{(m)} = \begin{cases} \frac{(m+1)s!}{\prod_{i \geq 0} m_i(\lambda)!} & \text{if } \ell(\lambda) \leq s+1 \\ 0 & \text{otherwise} \end{cases}$$

where $m_0(\lambda) = r+1 - \ell(\lambda)$.

(b) If $|\lambda| + s = |\nu|$, then

$$a_{\lambda, (s)}^\nu = \sum_{\substack{(i, \mu) \in \mathbb{N} \times \mathcal{P} \\ 1 \leq i \leq \ell(\lambda) \\ \mu \cup \nu = \lambda \cup (\nu_i)}} a_{\mu, (s)}^{(\nu_i)}$$

9.17. PROPOSITION. Let λ be a partition and let $m_i(\lambda)$ be the number of i 's in λ such that $\lambda = (i^{m_i(\lambda)})_{i \geq 1}$. Then, the top degree of $c_\lambda c_{(s)}$ is given by

$$(c_\lambda c_{(s)})^* = \sum_{|\nu|=|\lambda|+s} a_{\lambda,(s)}^\nu c_\nu = \sum_{\mu \subseteq \lambda, \ell(\mu) \leq s+1} \frac{(m_{s+|\mu|}(\lambda) + 1)(s + |\mu| + 1)s!}{(s + 1 - \ell(\mu))! \prod_{i \geq 1} m_i(\mu)!} c_{\lambda \cup (s+|\mu|) - \mu}$$

9.18. REMARK. $\mu = (i^{m_i(\mu)}) \subseteq \lambda \iff m_i(\mu) \leq m_i(\lambda) \ \forall i \geq i$.

9.19. EXAMPLE. To illustrate the formula $\lambda \cup (s + |\mu|) - \mu$, consider cycles $\sigma = (134)(2567)(8)$ and $\tau = (28)$. Then, $\lambda = (3, 2)$ is the modified cycle type of σ and $s = 1$ gives the modified cycle type of τ . Then,

$$\sigma\tau = (134)(28567)$$

which has modified cycle type $(4, 2) = (3, 2) \cup (1 + |(3)|) - (3)$.

9.20. COROLLARY.

$$c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_\ell} = \sum_{\mu \supseteq \lambda} d_{\lambda\mu} c_\mu$$

and $d_{\lambda\lambda} > 0$ in $\text{gr } \mathcal{K}$. Thus, c_1, c_2, \dots are algebraically independent elements of $\text{span}_{\mathbb{Z}}\{c_\lambda\}$.

9.21. PROPOSITION. $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gr } \mathcal{K}$ is a polynomial algebra generated by c_1, c_2, \dots (Note that $\mathbb{Z} \hookrightarrow \mathbb{B}$ as constants.)

9.22. REMARK. (a) There exists a ring isomorphism $\Lambda \xrightarrow{\sim} \text{gr } \mathcal{K}$ sending duals of h_λ^* (images of h_λ under a certain automorphism), called g_λ , to c_λ . See [Mac79, p 132–3].
(b) $\text{gr } Z(\mathbb{Z}\mathfrak{S}_n) \cong H^*(\text{Hilb}^n(\mathbb{C}^2); \mathbb{Z})$, the cohomology ring of the Hilbert Scheme of points on \mathbb{C}^2 , as a \mathbb{Z} -algebra.

9.23. THEOREM. The homomorphism given by

$$\begin{aligned} \Pi_n : \mathcal{K} &\rightarrow Z(\mathbb{Z}\mathfrak{S}_n) \\ \sum f_\lambda(x) c_\lambda &\mapsto \sum f_\lambda(n) c_\lambda(n) \end{aligned}$$

is a surjective homomorphism.

9.24. PROPOSITION. \mathcal{K} is generated by $K_m := \sum_{|\lambda|=m} c_\lambda$ for $m \geq 0$.

This tells us that

$$\Pi_m(\mathcal{K}) = \sum_{|\lambda|=m} c_\lambda(n) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \# \text{ of cycles in } \sigma = n-m}} \sigma$$

and so $Z(\mathbb{Z}\mathfrak{S}_n)$ is generated by $\Pi_n(\mathcal{K}_0), \Pi_n(\mathcal{K}_1), \dots, \Pi_n(\mathcal{K}_{n-1})$.

10. Double cover of $\tilde{\mathfrak{S}}_n$ and even split conjugacy classes

Recall the short exact sequence

$$1 \rightarrow \{1, z\} \rightarrow \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n \rightarrow 1$$

where

$$\tilde{\mathfrak{S}}_n = \left\langle z, t_1, t_2, \dots, t_{n-1} \mid \begin{cases} z \text{ is central} \\ z^2 = 1, t_i^2 = z \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ t_i t_j = z t_j t_i \quad |i - j| > 1 \end{cases} \right\rangle$$

10.1. DEFINITION. Define the element

$$x_i := t_i t_{i+1} \cdots t_{n-1} t_n t_{n-1} \cdots t_{i+1} t_i$$

which gets mapped to the transposition $(i, n) \in \mathfrak{S}_n$ under the map θ_n . Then, we let

$$[i_1 i_2 \cdots i_m] := \begin{cases} z & \text{if } m = 1 \\ x_{i_1} x_{i_m} x_{i_{m-1}} \cdots x_{i_2} x_{i_1} & \text{if } m \geq 2 \end{cases}$$

10.2. PROPOSITION. *Every element of $\tilde{\mathfrak{S}}_n$ is of the form*

$$z^q \underbrace{[i_1 i_2 \cdots i_m][j_1 j_2 \cdots j_k] \cdots}_{\text{disjoint}}$$

where $q = 0, 1$.

10.3. LEMMA. *For λ such that $|\lambda| + \ell(\lambda) \leq n$, $\theta_n^{-1}(\mathcal{C}_\lambda(n))$ splits if and only if*

- (a) λ has only even parts or
- (b) $\lambda \in \mathcal{SP}$, $|\lambda|$ odd, $|\lambda| + \ell(\lambda) = n$ or $n - 1$.

10.4. PROPOSITION. $\sigma \in \mathfrak{S}_n$ of modified type λ is even if and only if $|\lambda|$ is even.

10.5. DEFINITION. Let $\mathcal{D}_\lambda(n)$ be the even split conjugacy class in $\tilde{\mathfrak{S}}_n$ containing $[1, 2, \dots, \lambda_1 + 1][\lambda_1 + 2, \dots, \lambda_1 + \lambda_2 + 2] \dots$

- 10.6. PROPOSITION. (a) $\theta^{-1}(\mathcal{C}_\lambda(n)) = \mathcal{D}_\lambda(n) \cup z\mathcal{D}_\lambda(n)$.
- (b) $\{d_\lambda(n) \mid \lambda \in \mathcal{EP}, |\lambda| + \ell(\lambda) \leq n\}$ is a basis for the even center of $\mathbb{Z}\tilde{\mathfrak{S}}_n^- = \mathbb{Z}\tilde{\mathfrak{S}}_n / \langle z + 1 \rangle$.

10.7. DEFINITION. Define $b_{\lambda\mu}^\nu(n)$ by

$$d_\lambda(n) d_\mu(n) = \sum_{\nu \in \mathcal{EP}} b_{\lambda\mu}^\nu(n) c_\nu(n)$$

10.8. EXAMPLE.

$$d_{(4)}(8) d_{(2)}(8) = 13d_{(4)}(8) - 35d_{(2)}(8) - 18d_{(2,2)}(8) - 7d_{(6)}(8) + 2d_{(4,2)}(8) \in Z(\mathbb{Z}\tilde{\mathfrak{S}}_8^-)$$

10.9. THEOREM (Tysse-Wang). *Let $\lambda, \mu, \nu \in \mathcal{EP}$.*

- (a) There exists a unique $g_{\lambda\mu}^\nu(x) \in \mathcal{Q}[x]$ such that $b_{\lambda\mu}^\nu(n) = g_{\lambda\mu}^\nu(n)$ for all $n \geq |\nu| + \ell(\nu)$.
- (b) $g_{\lambda\mu}^\nu(x) = 0$ unless $|\nu| \leq |\lambda| + |\mu|$
- (c) If $|\nu| = |\lambda| + |\mu|$, then $g_{\lambda\mu}^\nu(x)$ is a constnat.

10.10. DEFINITION. Let the *spin Farahat-Higman algebra* \mathbb{F} be a \mathbb{B} -algebra with basis $\{d_\lambda \mid \lambda \in \mathcal{EP}\}$ and

$$d_\lambda d_\mu = \sum_{\nu \in \mathcal{EP}} g_{\lambda\mu}^\nu(x) d_\nu$$

which is filtered with respect to $\deg(d_\lambda) = |\lambda|$.

10.11. PROPOSITION. Let λ be a partition and rewrite $\lambda = (i_{i \geq 1}^{m_i(\lambda)})$. Let $s \geq 0$ be event. Then,

$$(d_\lambda d_{(s)})^* = \sum_{\mu} (-1)^{\ell(\mu)} \frac{(m_{s+|\mu|}(\lambda) + 1)(s + |\mu| + 1)s!}{(s + 1 - \ell(\mu))! \prod_{i \geq 1} m_i(\mu)!} d_{\lambda \cup (s+|\mu|) - \mu}$$

What is this summation over?

- 10.12. COROLLARY. (a) $\mathbb{Q} \otimes_{\mathbb{Z}[\frac{1}{2}]} \text{gr } \mathbb{F}$ is generated by d_2, d_4, \dots
 (b) There exists an injective homomorphism $\mathcal{Q} \otimes_{\mathbb{Z}[\frac{1}{2}]} \text{gr } \mathbb{F} \hookrightarrow \mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{K}$ via
 (for $\lambda \in \mathcal{EP}$) $d_\lambda \mapsto (-1)^{\ell(\lambda)c_\lambda}$.

10.1. Connections to odd Jucys-Murphy elements.

10.13. DEFINITION. Let us define

$$M_k := \sum_{i=1}^{k-1} [i, k] \in \mathbb{Z}\mathfrak{S}_n^-$$

10.14. PROPOSITION. We have

- (a) $M_k M_l = -M_l M_k$ for $k \neq l$
- (b)

$$M_k^2 = -(k-1) - \sum_{1 \leq i \neq j \leq k-1} [i, j, k] \in \mathbb{Z}\mathfrak{S}_n^-$$

10.15. DEFINITION. we define

$$e_{r,n} := \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} M_{i_1}^2 M_{i_2}^2 \dots M_{i_r}^2 \in Z(\mathbb{Z}\mathfrak{S}_n^-)$$

- 10.16. PROPOSITION. (a) $e_{r,n}$ has top degree $2r$
 (b) $e_{r,n} = \sum_{\lambda \in \mathcal{EP}, |\lambda| + \ell(\lambda) \leq n} A_\lambda(n) d_\lambda(n)$ for some $A_\lambda(n)$.
 (c) $A_\lambda(n)$ is the coefficient of $[1, 2, \dots, \lambda_1 + 1][\lambda_1 + 2, \dots, \lambda_1 + \lambda_2 + 2] \dots$ in $e_{r,n}$ and is independent of n .

10.17. DEFINITION. In light of the proposition above, we write $A_\lambda := A_\lambda(n)$ and define

$$e_r^* := \sum_{\lambda \in \mathcal{EP}, |\lambda| = 2r} A_\lambda d_\lambda \in \mathbb{F}$$

10.18. EXAMPLE. $e_1^* = -d_2$ and $e_2^* = d_{(2,2)} - 2d_4$.

10.19. PROPOSITION. $A_\lambda = (-1)^{\ell(\lambda) \prod_{i \geq 1} c_{\frac{\lambda_i}{2}}}$ where $c_0 = 1$ and $c_r = \frac{1}{r+1} \binom{2r}{r}$ are the Cartan numbers.

10.20. THEOREM. $\mathbb{B} \left[\frac{1}{2} \right] \otimes_{\mathbb{B}} \mathbb{F}$ is generated by $e_1^*, e_2^*, e_3^*, \dots$

10.21. COROLLARY. Via the surjective homomorphism

$$\begin{aligned} \mathbb{B} \left[\frac{1}{2} \right] \otimes_{\mathbb{B}} \mathbb{F} &\rightarrow Z(\mathbb{Z} \left[\frac{1}{2} \right] \mathfrak{S}_n^-) \\ \sum_{\lambda \in \mathcal{EP}} f_\lambda(x) d_\lambda &\mapsto \sum_{\lambda \in \mathcal{EP}} f_\lambda(n) d_\lambda(n) \end{aligned}$$

the even center of $\mathbb{Z} \left[\frac{1}{2} \right] \mathfrak{S}_n^-$ is generated by (the top degree of) $e_{r,n}$.

11. Schur-Sergeev duality for $\mathfrak{q}(n)$ (presented by Chris Chung)

12. Seminormal form construction for irreducible \mathcal{H}_n -modules (presented by Jinkui Wan)

A review for the symmetric group case was presented, but not written up here yet.

12.1. DEFINITION. We define the *Jucys-Murphy elements* in $\mathcal{H}_n = \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$ as

$$J_k := \sum_{1 \leq j < k \leq n} (1 + c_j c_k)(jk),$$

12.2. PROPOSITION. *The Jucys-Murphy elements have the following properties.*

- (a) $J_k J_l = J_l J_k$ for $l \leq k \neq l \leq n$.
- (b) $c_k J_k = -J_k c_k$ and $c_l J_k = J_k c_l$ for $k \neq l$.
- (c) $s_k J_k = J_{k+1} s_k - (1 + c_k c_{k+1})$ for $1 \leq k \leq n-1$ and $s_l J_k = J_k s_l$ for $k \neq l, l+1$.

12.3. DEFINITION. The *degenerate affine Hecke-Clifford algebra* is given by

$$\hat{\mathcal{H}}_n = \langle s_1, \dots, s_{n-1}, c_1, \dots, c_n, x_1, \dots, x_n \rangle$$

with additional relations

$$\begin{cases} x_k x_l = x_l x_k \\ s_k x_k = x_{k+1} s_k - (1 + c_k c_{k+1}) \\ s_i x_k = x_k s_i & k \neq i, i+1 \\ x_k c_k = -c_k x_k \\ x_k c_l = c_l x_k & l \neq k \end{cases}$$

12.4. PROPOSITION. *There exists a projection $\pi: \hat{\mathcal{H}}_n \rightarrow \mathcal{H}_n$ such that*

$$\pi(s_i) = s_i, \pi(c_k) = c_k, \pi(x_l) = J_l$$

In particular, $\pi(x_1) = J_1 = 0$, so $x_1 \in \ker \pi$.

In fact, $\ker \pi = \langle x_1 \rangle$ and so $\mathcal{H}_n\text{-}\mathbf{Mod}$ can be identified as the subcategory of $\hat{\mathcal{H}}_n\text{-}\mathbf{Mod}$ on which $x_1 = 0$.

12.5. THEOREM (PBW Theorem). *We have that*

$$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} c_1^{\beta_1} \cdots c_n^{\beta_n} w \mid \alpha_i \in \mathbb{Z}_+, \beta_i \in \{0, 1\}, 1 \leq i \leq n, w \in \mathfrak{S}_n\}$$

is a basis for $\hat{\mathcal{H}}_n$.

12.6. COROLLARY (Corollary of PBW Theorem). *The subalgebra of $\hat{\mathcal{H}}_n$ generated by $x_1, \dots, x_n, c_1, \dots, c_n$ is isomorphic to*

$$\mathcal{C}\ell_n \otimes \mathbb{C}[x_1, \dots, x_n] / \langle x_k c_k = -c_k x_k, x_k c_l = c_l x_k, l \neq k \rangle = \underbrace{P_1^c \otimes P_1^c \otimes \cdots \otimes P_1^c}_{n \text{ copies}}$$

where $P_1^c = \langle x_1, c_1 \rangle$.

12.7. PROPOSITION. *The \mathfrak{S}_n fixed points $\mathbb{C}[x_1^2, x_2^2, \dots, x_n^2]^{\mathfrak{S}_n} \subseteq \text{Center of } \hat{\mathcal{H}}_n$.*

12.8. PROPOSITION. (a) *The eigenvalues of x_1^2, \dots, x_n^2 are of the form $q(i) := i(i+1)$ for $i \in \mathbb{Z}_+$.*

(b) *If all the eigenvalues of x_j^2 on a finite dimensional $\hat{\mathcal{H}}_n$ -module M for a fixed j are of the form $q(i)$, then M is integral.*

The second part of the proposition follows from the intertwining elements.

12.9. DEFINITION. Let *intertwining element* $\Phi_k \in \hat{\mathcal{H}}_n$ be given by

$$\begin{aligned} \Phi_k &:= s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}) \\ &= (x_{k+1}^2 - x_k^2)s_k - (x_k + x_{k+1}) - c_k c_{k+1}(x_k - x_{k+1}) \end{aligned}$$

Note, the second equality follows from the fact that

$$s_k x_k^2 = x_{k+1} s_k x_k - (1 + c_k c_{k+1})x_k = x_{k+1}^2 s_k - (1 + c_k c_{k+1})(1 + x_k)$$

and, using conjugation by s_k ,

$$x_k^2 s_k = s_k x_{k+1}^2 - s_k(1 + c_k c_{k+1})(1 + x_k)s_k \implies -s_k x_{k+1}^2 = -x_k^2 s_k - s_k(1 + c_k c_{k+1})(1 + x_k)s_k$$

Therefore, we get

$$s_k(x_k^2 - x_{k+1}^2) = (x_{k+1}^2 - x_k^2)s_k - (1 + c_k c_{k+1})(1 + x_k) - s_k(1 + c_k c_{k+1})(1 + x_k)s_k$$

12.10. PROPOSITION. *We have the following useful relations for the intertwining elements.*

$$\begin{cases} \Phi_k \Phi_l = \Phi_l \Phi_k & |k - l| > 1 \\ \Phi_k \Phi_{k+1} \Phi_k = \Phi_{k+1} \Phi_k \Phi_{k+1} \\ \Phi_k^2 = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2 \\ \Phi_k x_k = x_{k+1} \Phi_k, \Phi_k x_{k+1} = x_k \Phi_k, \Phi_k x_l = x_l \Phi_k & l \neq k, k+1 \\ \Phi_k c_k = c_{k+1} \Phi_k, \Phi_k c_{k+1} = c_k \Phi_k, \Phi_k c_l = c_l \Phi_k & l \neq k, k+1 \end{cases}$$

12.11. PROPOSITION. *If v is some eigenvector of x_{j+1}^2 , that is, if $x_{j+1}^2 v = av$, then $x_j^2 \Phi_j v = a \Phi_j v$, so $\Phi_j v$ is an x_j^2 eigenvector with the same eigenvalue.*

PROOF. Since $x_j^2 \Phi_j = \Phi_j x_{j+1}^2$, we get that

$$x_j^2 \Phi_j v = \Phi_j x_{j+1}^2 v = \Phi_j av = a \Phi_j v$$

□

12.12. DEFINITION. A finite dimensional $\hat{\mathcal{H}}_n$ -module M is called *completely splittable (CS)* if x_1, x_2, \dots, x_n act semisimply, ie the actions of x_1, \dots, x_n can be diagonalized simultaneously.

12.13. PROPOSITION. *Every integral CS $\hat{\mathcal{H}}_n$ -module M can be decomposed as*

$$M = \bigoplus_{\mathbf{i} \in \mathbb{Z}_+^n} M_{\mathbf{i}}$$

where

$$M_{\mathbf{i}} = \{v \in M \mid x_k^2 v = q(i_k) v, 1 \leq k \leq n\}$$

is the common eigenspace of $x_1^2, x_2^2, \dots, x_n^2$ with eigenvalues $q(i_1), q(i_2), \dots, q(i_n)$. Furthermore, define

$$\text{wt}(M) := \{\mathbf{i} \in \mathbb{Z}_+^n \mid M_{\mathbf{i}} \neq 0\}$$

Our goal is to describe $\text{wt}(M)$ for integral irreducible CS $\hat{\mathcal{H}}_n$ -modules.

12.14. LEMMA. *If $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$ is in $\text{wt}(M)$ for some integral irreducible CS $\hat{\mathcal{H}}_n$ -module, then $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$.*

PROOF. Suppose $i_k = i_{k+1}$ so that $x_k^2 v = x_{k+1}^2 v$ for $v \in M_{\mathbf{i}}$. Since

$$x_k^4 s_k - 2q(i_k) x_k^2 s_k + q(i_k)^2 s_k = x_k^2 (s_k x_{k+1}^2 - s_k (1 + c_k c_{k+1}) (1 + x_k) s_k) - 2q(i_k) x_k^2 s_k + q(i_k)^2 s_k$$

Somehow we get that $s_k v \in M_{\mathbf{i}}$. Then, we get

figure this out

$$\implies x_k^2 s_k v = q(i_k) s_k v$$

$$\implies (x_k^2 - q(i_k)) s_k v = 0$$

$$\implies (s_k x_{k+1}^2 - x_k (1 - c_k c_{k+1}) - (1 - c_k c_{k+1}) x_{k+1}) v - q(i_k) s_k v = 0$$

However, $x_{k+1}^2 v = q(i_{k+1})v = q(i_k)v$ by assumption, so

$$\begin{aligned}
&\implies (x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_k)v = 0 \\
&\implies 2(x_k^2 + x_{k+1}^2)v = 0 \\
&\implies i_k = i_{k+1} = 0 \\
&\implies x_k^2 v = 0 = x_{k+1}^2 v \\
&\implies x_k v = 0 = x_{k+1} v \quad \text{since } M \text{ is CS} \\
&\implies v = 0 \quad \text{since}
\end{aligned}$$

□

Finish this proof.

12.15. LEMMA. Suppose $\mathbf{i} = (i_1, \dots, i_n) \in \text{wt}(M) \subseteq \mathbb{Z}_+^n$ for some integral, irreducible, CS $\hat{\mathcal{H}}_n$ -module. Fix $1 \leq k \leq n-1$.

- (a) If $i_k \neq i_{k+1} \pm 1$, then $\Phi_k z \neq 0$ for all $0 \neq z \in M_{\mathbf{i}}$.
- (b) If $i_k = i_{k+1} \pm 1$, then $\Phi_k = 0$ on $M_{\mathbf{i}}$.

PROOF. Since $\Phi_k^2 = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2$, then

$$\Phi_k^2 z = (2(q(i_k) + q(i_{k+1})) - q(i_k)^2 + 2q(i_k)q(i_{k+1}) - q(i_{k+1})^2)z = 0$$

if and only if

$$(q(i_k) - q(i_{k+1}))^2 = 2(q(i_k) + q(i_{k+1}))$$

Now, if we write $i_k = i_{k+1} + c$, then

$$\begin{cases} q(i_k) = i_{k+1}^2 + 2ci_{k+1} + c^2 + i_{k+1} + c \\ q(i_{k+1}) = i_{k+1}^2 + i_{k+1} \end{cases} \implies \begin{cases} q(i_k) - q(i_{k+1}) = 2ci_{k+1} + c^2 + c \\ q(i_k) + q(i_{k+1}) = 2i_{k+1}^2 + 2ci_{k+1} + c^2 + 2i_{k+1} + c \end{cases}$$

From here, one checks that $c = \pm 1$ certainly gives solutions independent of i_{k+1} . Thus, $i_k = i_{k+1} \pm 1 \implies \Phi_k^2 z = 0$. Furthermore, there are other formal solutions to these equations, namely $c = -2i_{k+1}$ and $c = -2(i_{k+1} + 1)$, but since $i_{k+1} \geq 0$, this would force $i_k < 0$ unless $i_k = i_{k+1} = 0$, which is not admissible by the previous lemma.

So, to prove the second part, since $\Phi_k^2 z = 0$, it must be that if $\Phi_k z \neq 0$ and so $\Phi_k z \in M_{s_k \mathbf{i}}$. Then, there exists a minimal sequence $\Phi_{j_1}, \dots, \Phi_{j_r}$ such that $\Phi_{j_1} \cdots \Phi_{j_r} \Phi_k z \in M_{\mathbf{i}}$ since M is irreducible. Then, if $\sigma = s_{j_1} \cdots s_{j_r} s_k \in \mathfrak{S}_n$, it must be that $\sigma \cdot \mathbf{i} = \mathbf{i}$. If one assumes $\sigma \neq 1$, this leads to a violation of Lemma 12.14 with some work. Then, using the exchange condition for Coxeter groups, one shows that $r = 1$ which gives $j_1 = k$, so $\Phi_k^2 z \neq 0$, contradicting what we showed above. □

Why is this true?

12.16. REMARK. Suppose V is an integral, irreducible $\hat{\mathcal{H}}_n$ -module. Let $\hat{\mathcal{H}}_{(n-r, 1^k)} = \langle s_1, \dots, s_{n-r-1}, c_1, \dots, c_n, x_1, \dots, x_r \rangle$. Then,

$$\begin{aligned}
V \text{ is CS} &\iff \forall \mathbf{i} \in \text{wt}(V), 1 \leq k \leq n-1, i_k \neq i_{k+1} \\
&\iff \text{Res}_{\hat{\mathcal{H}}_{(n-r, 1^r)}}^{\hat{\mathcal{H}}_n} V \text{ is semisimple } \forall 1 \leq r \leq n \\
&\iff \text{Res}_{\hat{\mathcal{H}}_{(1^k, n-k-r, 1^r)}}^{\hat{\mathcal{H}}_n} V \text{ is semisimple}
\end{aligned}$$

The second $\hat{\mathcal{H}}_{(1^k, n-k-r, 1^r)}$ is not defined.

12.17. COROLLARY. Suppose $\mathbf{i} \in \text{wt}(V)$ for some integral, irreducible, CS $\hat{\mathcal{H}}_n$ -module V . If $i_k = i_{k+2}$ for some $1 \leq k \leq n-2$, then $i_k = i_{k+2} = 0$ and $i_{k+1} = 1$.

PROOF. If $i_k \neq i_{k+1} \pm 1$, then $s_k \cdot \mathbf{i} \in \text{wt } V$ by the first part of 12.15, but $s_k \cdot \mathbf{i} = (\dots, i_{k+1}, i_k, i_{k+2}, \dots)$ and, by assumption, $i_k = i_{k+2}$ contradicting 12.14. So, it must be that $i_{k+1} = i_k \pm 1$ and so $\Phi_k = 0 = \Phi_{k+1}$ on $V_{\mathbf{i}}$ by the second part of 12.15. Thus, for $z \in V$,

$$\begin{cases} \Phi_k z = 0 & \implies (q(i_k) - q(i_{k+1}))s_k z = -((x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}))z \\ \Phi_{k+1} z = 0 & \implies (q(i_{k+1}) - q(i_{k+2}))s_{k+1} z = -((x_{k+1} + x_{k+2}) + c_{k+1} c_{k+2}(x_{k+1} - x_{k+2}))z \end{cases}$$

which gives us the s_k and s_{k+1} actions on $V_{\mathbf{i}}$. From here, we can use the braid relation $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ to arrive at the equality

$$((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z = 0$$

Now, $x_k z = \pm \sqrt{q(i_k)}z$. Furthermore, since $i_k = i_{k+2}$, we have that either $x_k z = x_{k+2} z$ or $x_k z = -x_{k+2} z$. Decompose

$$V_{\mathbf{i}} = W_1 \oplus W_2$$

where $W_1 := \{z \in V_{\mathbf{i}} \mid x_k z = x_{k+2} z\}$ and $W_2 := \{z \in V_{\mathbf{i}} \mid x_k z = -x_{k+2} z\}$. Now, we can break up our braid relation equality.

$$\begin{aligned} z \in W_1 \implies 0 &= ((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z \\ &= 2x_k(6x_{k+1}^2 + 2x_k^2)z \\ &= 2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k))z \\ z \in W_2 \implies 0 &= ((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z \\ &= c_k c_{k+2}2x_k(6x_{k+1}^2 + 2x_k^2)z \\ &= c_k c_{k+2}(2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k)))z \end{aligned}$$

Thus, we obtain that

$$2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k)) = 0$$

Moreover, we know that $i_{k+1} = i_k \pm 1$, so we get

$$\begin{cases} \sqrt{i_k(i_k+1)}(6(i_k-1)i_k + 2(i_k+1)i_k) = 0 & \text{if } i_{k+1} = i_k - 1 \\ \sqrt{i_k(i_k+1)}(6(i_k+1)(i_k+2) + 2(i_k+1)i_k) = 0 & \text{if } i_{k+1} = i_k + 1 \end{cases}$$

There are no solutions to the first equation that give i_k and i_{k+1} as non-negative integers and the only such solution for the second equation is $i_k = 0$ and $i_{k+1} = 1$. \square

In conclusion, we have the following.

12.18. THEOREM. Let $\mathcal{W}(n)$ be the set of weights of all integral irreducible CS $\hat{\mathcal{H}}_n$ -modules and let $\mathbf{i} \in \mathcal{W}(n)$. Then,

- (a) $i_k \neq i_{k+1}$ for all $1 \leq k \leq n-1$.
- (b) If $i_k = i_\ell = 0$ for some $1 \leq k < \ell \leq n$, then $1 \in \{i_{k+1}, \dots, i_{\ell-1}\}$.
- (c) If $i_k = i_\ell \geq 1$ for some $1 \leq k < \ell \leq n$, then $\{i_k - 1, i_k + 1\} \subseteq \{i_{k+1}, \dots, i_{\ell-1}\}$.

PROOF. The first part is just a restatement of 12.14.

For the next part, assume $1 \notin \{i_{k+1}, \dots, i_{\ell-1}\}$. Then, we can swap indices using ?? to get new weights until we obtain a weight of the form

$$(\dots, 0, 0, \dots)$$

which is not allowed by the previous part.

For the last part, let $u = i_k = i_\ell \geq 1$ be such that $k - \ell$ is minimal among such occurrences. If only one of $u + 1, u - 1$ appear in between i_k and i_ℓ , then it must appear twice because, otherwise, we could swap indices using 12.15 to get new weights until we are of the form

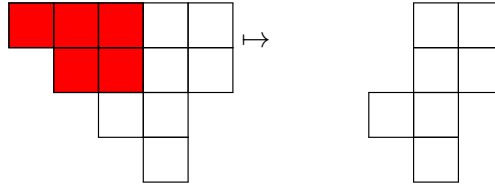
$$(\dots, u, u \pm 1, u, \dots)$$

which is not a weight by 12.17 since $u \geq 1$. Thus, we have violated the minimality of our choice of u . \square

Now, we wish to describe a bijection between $\mathcal{W}(n)$ and standard skew shifted tableaux of size n .

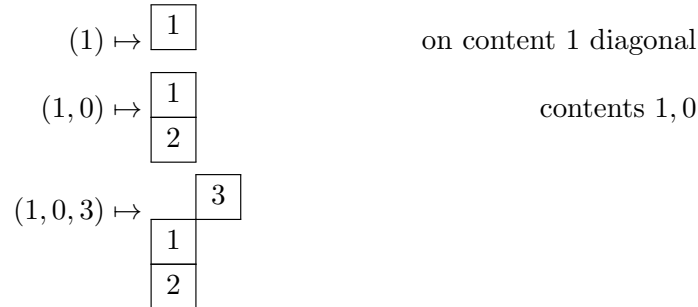
12.19. DEFINITION. Given strict partitions $\nu \subseteq \xi$, the *skew shifted Ferrers diagram* ξ/ν is given by removing the boxes of ν from ξ .

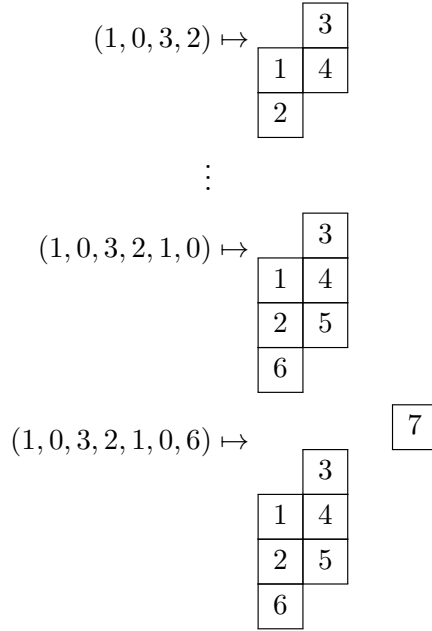
12.20. EXAMPLE. Consider $(3, 2) \subseteq (5, 4, 2, 1)$. Then,



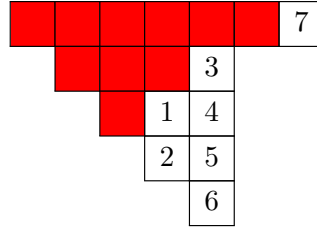
We now illustrate the bijection by example.

12.21. EXAMPLE. Let $\mathbf{i} = (1, 0, 3, 2, 1, 0, 6)$. Then, we construct our standard skew shifted tableau in steps by adding a box of content i_k labelled k on the k th step.





So, our final answer has outer shape $\xi = (7, 4, 3, 2, 1)$ and inner shape $\nu = (6, 3, 1)$:



12.22. DEFINITION. Given $\mathbf{i} \in \text{wt } V$, let $T(\mathbf{i})$ be the corresponding standard skew shifted tableau.

12.23. PROPOSITION. If $\mathbf{i}, \mathbf{j} \in \text{wt}(V)$ for some integral irreducible CS $\hat{\mathcal{H}}_n$ -module V , then $T(\mathbf{i})$ and $T(\mathbf{j})$ have the same shape.

PROOF. If $(i_1, \dots, i_k, i_{k+1}, \dots, i_n)$, then $(i_1, \dots, i_{k-1}, i_{k+1}, i_k, \dots, i_n)$ is a weight only if $i_k \neq i_{k+1} \pm 1$ by ??. However, under such a condition, it does not matter in which order we add the boxes corresponding to i_k and i_{k+1} . Thus, these two weights will yield the same shape. \square

What is the reference here?

12.24. DEFINITION. Given ξ/ν a skew-shifted Ferrer's diagram of size n , we define

$$\mathcal{F}(\xi/\nu) := \{\mathsf{T} \mid \mathsf{T} \text{ a standard Young tableau of shape } \xi/\nu\}$$

and

$$\hat{D}^{\xi/\nu} := \bigoplus_{\mathsf{T} \in \mathcal{F}(\xi/\nu)} \mathcal{C}\ell_n v_{\mathsf{T}}$$

as a vector space with actions

$$x_k v_{\mathbf{T}} = \sqrt{q(c(\mathbf{T}_k))} v_{\mathbf{T}}$$

where $c(\mathbf{T}_k)$ is the content of the box labelled by k in \mathbf{T} and where \mathcal{C}_{ℓ_n} acts by multiplication on the left.

12.25. PROPOSITION. *We have*

$$s_k v_{\mathbf{T}} = \left(\frac{1}{\sqrt{q(c(\mathbf{T}_{k+1}))} - \sqrt{q(c(\mathbf{T}_k))}} + \frac{1}{\sqrt{q(c(\mathbf{T}_{k+1}))} + \sqrt{q(c(\mathbf{T}_k))}} c_k c_{k+1} \right) v_{\mathbf{T}} + \sqrt{1 - \frac{2(q(c(\mathbf{T}_{k+1}))) + q(c(\mathbf{T}_k))}{(q(c(\mathbf{T}_{k+1})) - q(c(\mathbf{T}_k)))}}$$

PROOF. This fact follows formally from the fact that $\Phi_k v_{\mathbf{T}} = a v_{s_k \mathbf{T}}$ for some scalar a if $s_k \mathbf{T}$ is standard. \square

Actually do this proof.

12.26. COROLLARY. $\hat{D}^{\xi/\nu}$ is an integral CS $\hat{\mathcal{H}}_n$ -module.

PROOF. Mainly, one needs to check the Coxeter relations. \square

Fill in this proof.

13. Spin Kostka Polynomials

13.1. DEFINITION. Let $\nu \in \mathcal{SP}$ and μ be a partition. Then, the *spin Kostka polynomials* are the transition polynomials from the Hall-Littlewood P -functions to the Q -Schur functions. In other words,

$$Q_{\nu}(x) = \sum_{\mu} K_{\nu\mu}^{-}(t) P_{\mu}(x; t)$$

If we also let $b_{\nu\lambda}$ be such that

$$Q_{\nu}(x) = \sum_{\lambda} b_{\nu\lambda} s_{\lambda}(x)$$

then we get

13.2. PROPOSITION. *For $\nu \in \mathcal{SP}$ and μ a partition,*

$$K_{\nu\mu}^{-}(t) = \sum_{\lambda} b_{\nu\lambda} K_{\lambda\mu}(t)$$

where $K_{\lambda\mu}(t)$ are the Kostka-Foulkes polynomials.

PROOF. By definition of Kostka-Foulkes, we have

$$s_{\lambda} = \sum_{\mu} K_{\lambda\mu}(t) P_{\mu}(x; t)$$

and so

$$\sum_{\mu} K_{\nu\mu}(t) P_{\mu}(x; t) = Q_{\nu}(x) = \sum_{\lambda} b_{\nu\lambda} s_{\lambda} = \sum_{\lambda, \mu} b_{\nu\lambda} K_{\lambda\mu}(t) P_{\mu}(x; t)$$

\square

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