K-THEORETIC CATALAN FUNCTIONS

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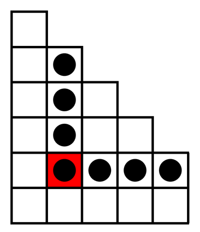
Overview

- Schur functions, s_λ, and Grothendieck polynomials, G_λ, give representatives for cohomology and K-theory of the Grassmannian.
- Pieri rules determine the structure constants of these rings.
- Representatives are known for (co)homology of affine Grassmannian.
- Aim: develop similar picture for affine K-theory.

Affine Combinatorics

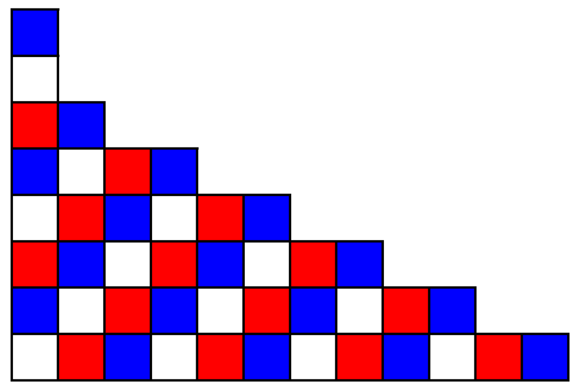
$w \in \tilde{S}_n \leftrightarrow n$-cores

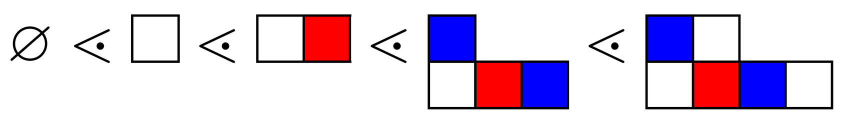
n-core=partition with no cell of hook-length n
red cell has hook-length 7



Weak Order

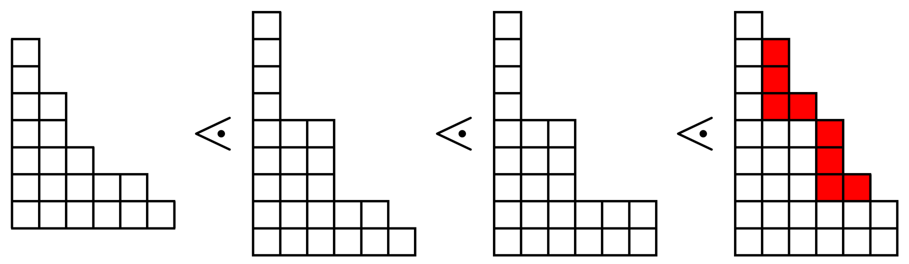
- Covers differ by boxes of same color.



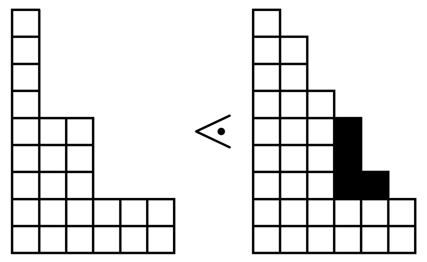


Strong (Bruhat) Order

- Ordered by containment of shapes.
- Covers differ by a ribbon + its copies.



- **Marked Cover:** Strong cover with selection of one ribbon

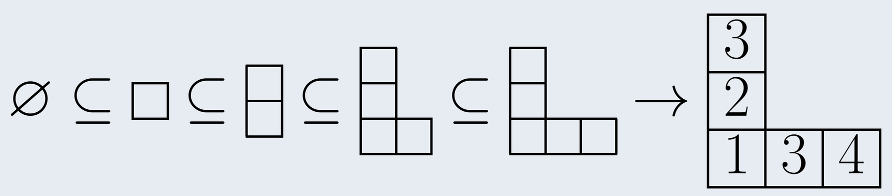


Dual k-Schur Functions

Generating functions of weak tableaux.

$$F_\lambda^{(k)} := \sum x^{\mathrm{weight}(T)}$$

Weak Tableaux:
Maximal chains in the weak order.

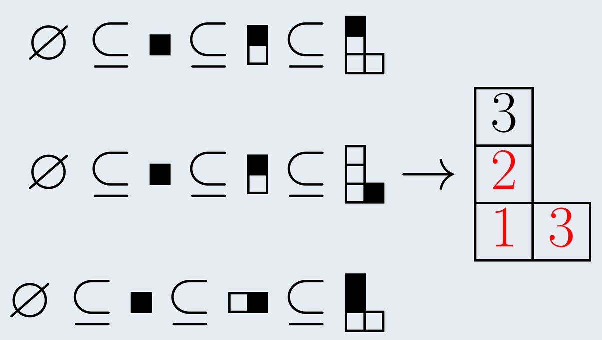


k-Schur Functions

Generating functons of strong tabelaux.

$$s_\lambda^{(k)} := \sum x^{\mathrm{weight}(T)}$$

Strong Tableaux: Maximal strong order chains of marked covers



Pieri Rule:

$$e_r F_\lambda^{(k)} = \sum_{\substack{\mu = \lambda + \text{ strong marked vertical} \\ \text{strip of size } r}} F_\mu^{(k)} \iff e_r^\perp s_\mu^{(k)} = \sum_{\substack{\lambda = \mu - \text{ strong marked vertical} \\ \text{strip of size } r}} s_\lambda^{(k)}$$

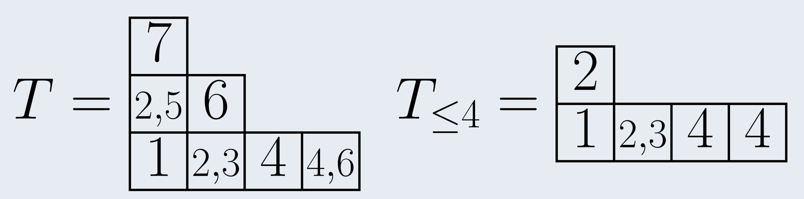
where a **strong vertical strip** is a chain of marked covers with markings proceeding north to south.

Affine Grothendieck Polynomials

Generating functions of affine SVTs.

$$G_\lambda^{(k)} := \sum (-1)^{|\lambda| + |\mathrm{weight}(T)|} x^{\mathrm{weight}(T)}$$

Affine Set-Valued Tableaux:
Each $T_{\leq x}$ is a $k+1$-core.



- $G_\lambda^{(k)} = F_\lambda^{(k)} +$ higher order terms
- $G_\lambda^{(k)} = G_\lambda$ for large k.

Dual Affine Grothendieck Polynomials

$g_\lambda^{(k)}$ is dual basis to $G_\lambda^{(k)}$.

- $g_\lambda^{(k)} = s_\lambda^{(k)} +$ lower order terms
- $g_\lambda^{(k)} = g_\lambda$, dual to G_λ, for large k.

Open Problem

Find a direct definition of $g_\lambda^{(k)}$.

Open Problem

Describe the $G_\lambda^{(k)}$ Pieri rule. $\iff$ Describe the $g_\lambda^{(k)}$ dual Pieri rule.

Branching

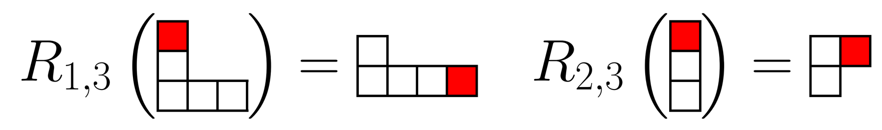
- k-Schur functions are $k+1$-Schur positive: $s_\lambda^{(k)} = \sum a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$ with $a_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.
- Iteration gives Schur positivity of k-Schur functions.
- **Conjecture:** $g_\lambda^{(k)}$ is Schur positive.
- **Conjecture:** $g_\lambda^{(k)} = \sum_\mu (-1)^{|\lambda| - |\mu|} b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$ for $b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

Catalan Functions

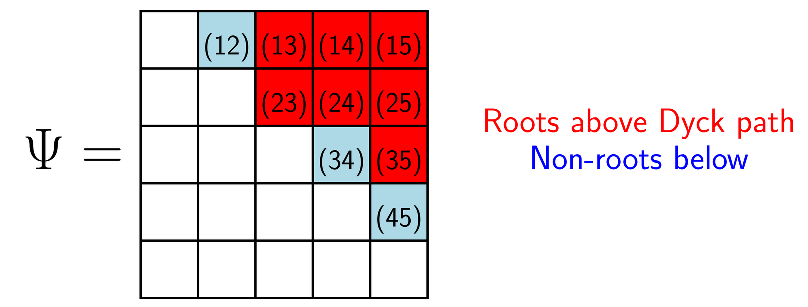
For $\gamma \in \mathbb{Z}^\ell$,

$$H(\Psi; \gamma) := \prod_{(i,j) \notin \Psi} (1 - R_{ij}) h_\gamma$$

- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$



- Root ideal Ψ: given by Dyck path.

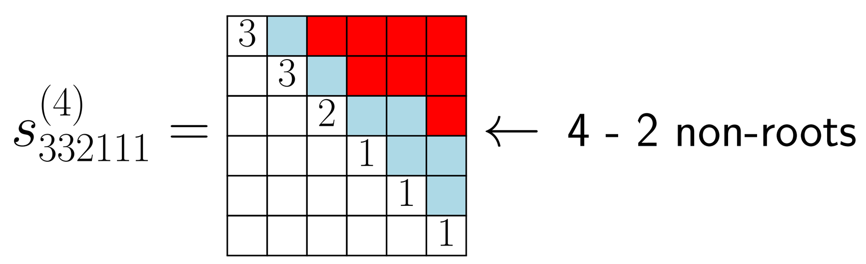


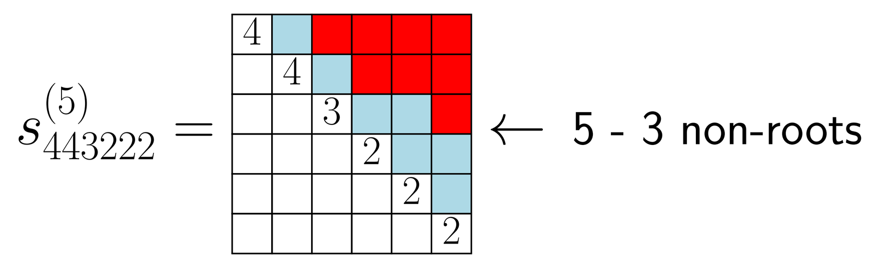
$$\begin{aligned} &H(\Psi; 54332) \\ &= (1 - R_{12})(1 - R_{34})(1 - R_{45}) h_{54332} \\ &= h_{54332} - h_{45332} - h_{54422} - h_{54341} \\ &+ h_{45422} + h_{45341} + h_{54431} - h_{45431} \end{aligned}$$

- $H(\varnothing; \lambda) = s_\lambda$ (Jacobi-Trudi Identity)

k-Schur Catalans

$s_\lambda^{(k)} = H(\Psi; \lambda)$ for particular Ψ, defined by $\underbrace{\lambda_i + \#\text{non-roots in row } i}_{\text{bandwidth}} = k$.





Why use Catalan k-Schurs?

Shift Invariance:

$$e_\ell^\perp s_{\lambda + 1^\ell}^{(k+1)} = s_\lambda^{(k)}$$

Corollary

Dual Pieri rule $\Longrightarrow s_\lambda^{(k)}$ branching!

$$s_\lambda^{(k)} = e_\ell^\perp s_{\lambda + 1^\ell}^{(k+1)} = \sum_\mu a_{\lambda\mu}^{(k)} s_\mu^{(k+1)}$$

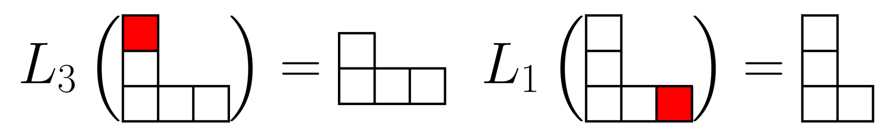
where $a_{\lambda\mu}^{(k)}$ counts strong vertical strips.

K-theoretic Catalan Functions

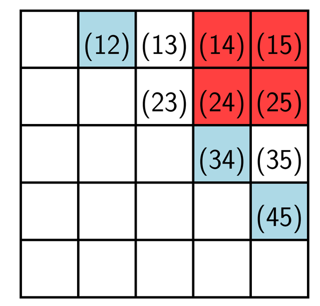
For $\gamma \in \mathbb{Z}^\ell$, root ideals $\Psi, \mathcal{L}$

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \notin \Psi} (1 - R_{ij}) K h_\gamma$$

- Lowering operators $L_j(K h_\lambda) = K h_{\lambda - \epsilon_j}$



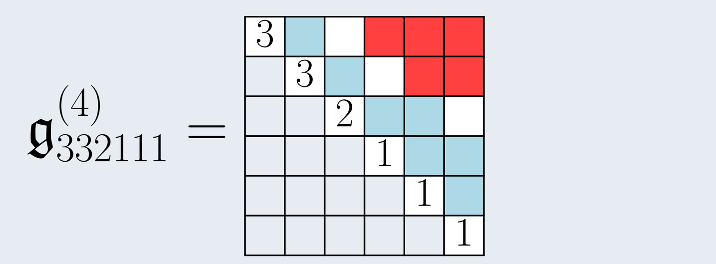
- non-roots of Ψ, roots of $\mathcal{L}$



$$\begin{aligned} &K(\Psi; \mathcal{L}; 54332) \\ &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) K h_{54332} \end{aligned}$$

- $K(\varnothing; \varnothing; \lambda) = g_\lambda$.

$\mathfrak{g}_\lambda^{(k)} := K(\Psi; \mathcal{L}; \lambda)$ with $\mathrm{band}(\Psi) = k$, $\mathrm{band}(\mathcal{L}) = k+1$



Theorem: Shift Invariance

$$G_{1^\ell}^\perp \mathfrak{g}_{\lambda + 1^\ell}^{(k+1)} = \mathfrak{g}_\lambda^{(k)}$$

Corollary

$\mathfrak{g}_\lambda^{(k)}$ branching follows from dual Pieri rule.

Conjecture

$$\mathfrak{g}_\lambda^{(k)} = g_\lambda^{(k)}$$