

# Diagonal Harmonics and Shuffle Theorems

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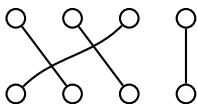
joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun  
ISU Algebra Seminar

27 October 2022

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# Symmetric Group

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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


The diagram illustrates the permutation  $\sigma$  as a mapping between two rows of four nodes. The top row represents the domain  $\{1, 2, 3, 4\}$  and the bottom row represents the codomain  $\{1, 2, 3, 4\}$ . Lines connect the nodes as follows: the first node in the top row connects to the second node in the bottom row; the second node in the top row connects to the third node in the bottom row; the third node in the top row connects to the first node in the bottom row; and the fourth node in the top row connects to the fourth node in the bottom row. This represents a 3-cycle (1 2 3) and a fixed point (4).

# Symmetric Group

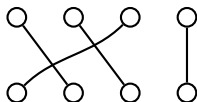
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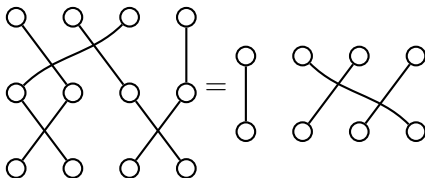
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# Polynomials

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- $\sigma \in S_n$  acts as  $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$



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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$  forms a vector space,  $\Lambda_{\mathbb{Q}}$ .

# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of  $\Lambda_{\mathbb{Q}}$ ?

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

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5  $\rightarrow$ 

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$4 + 1 \rightarrow$ 


$3 + 2 \rightarrow$ 


$3 + 1 + 1 \rightarrow$  

$2 + 2 + 1 \rightarrow$ 


$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$



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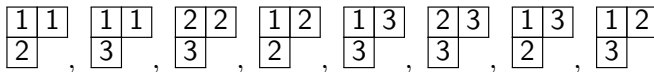
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For  $\lambda = (2, 1)$ ,

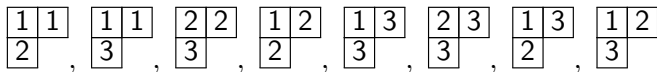


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Associate a polynomial to  $\text{SSYT}(\lambda)$ .

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2	

, 

1	1
3	

, 

2	2
3	

, 

1	2
2	

, 

1	3
3	

, 

2	3
3	

, 

1	3
2	

, 

1	2
3	

  
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)



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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for  $\Lambda_{\mathbb{Q}}$

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$



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①  $S_3$  action on  $M$  fixes vector subspaces!

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$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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Schur basis expansion counts multiplicity of irreducible  $S_n$  fixed subspaces!

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## Upshot

Via Frobenius characteristic map, questions about  $S_n$ -action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

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- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible  $S_n$ -fixed vector subspaces.

## Upshot

Via Frobenius characteristic map, questions about  $S_n$ -action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

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Break  $M$  up into smallest  $S_n$  fixed subspaces

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Solution: minimal  $S_n$ -fixed subspace of degree  $d \mapsto q^d s_\lambda$  (graded Frobenius)

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Answer: "Hall-Littlewood polynomial"  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .
- Does there exist a family of  $S_n$ -representations whose (bigraded) Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .

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## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

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- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ . (Still open!)

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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## Question

What symmetric function is the bigraded Frobenius characteristic of  $DH_n$ ?

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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Theorem (Haiman, 2002)

*The bigraded Frobenius characteristic of  $DH_n$  is given by  $\nabla e_n$ .*

# A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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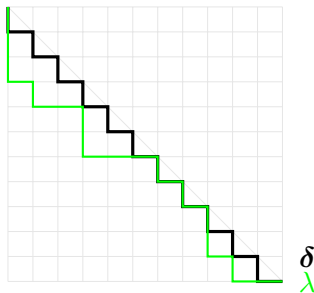
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- $\mathcal{G}_{\nu(\lambda)}(X; q)$  a symmetric LLT polynomial indexed by a tuple of offset rows.

# Dyck paths

## Dyck paths

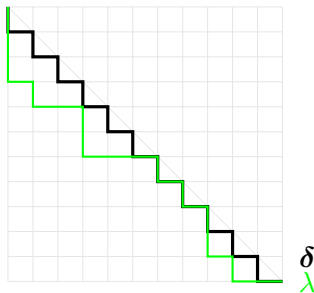
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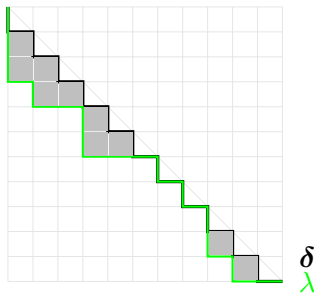


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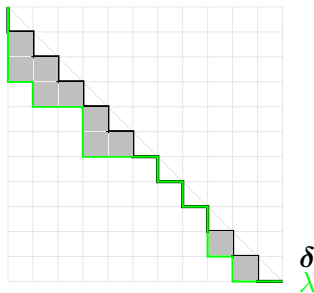


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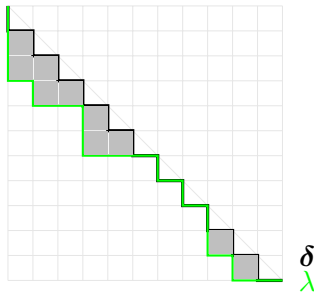


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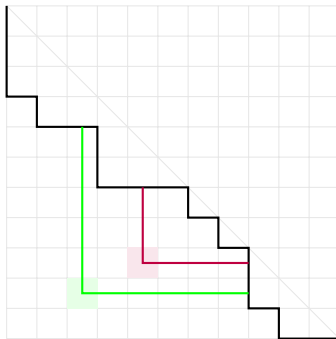
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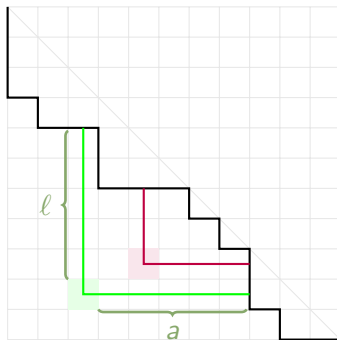
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# dinv

$\text{dinv}(\lambda) = \#$  of balanced hooks in diagram below  $\lambda$ .



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Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$



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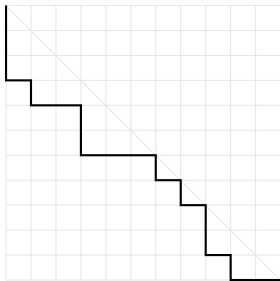
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- $\mathcal{G}_\nu$  is Schur-positive for any tuple of skew shapes  $\nu$  [Grojnowski-Haiman, 2007].

# LLT Polynomials

$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
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# LLT Polynomials

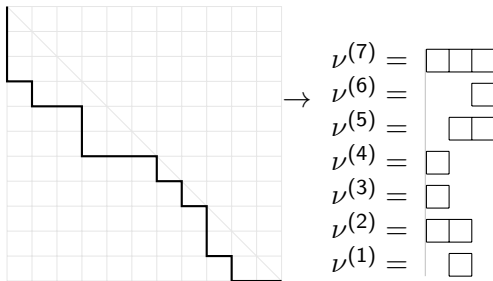
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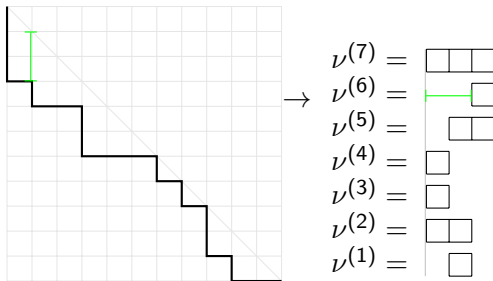
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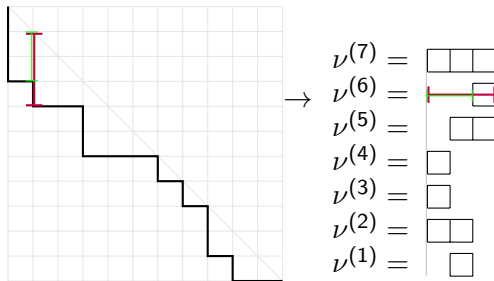
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$$\mathcal{G}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

## Example $\nabla e_3$

$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$








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




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




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- Entire quantity is  $q, t$ -symmetric
- Coefficient of  $s_{1,1,1}$  in sum is a “ $(q, t)$ -Catalan number”  
 $(q^3 + q^2t + qt + qt^2 + t^3)$ .

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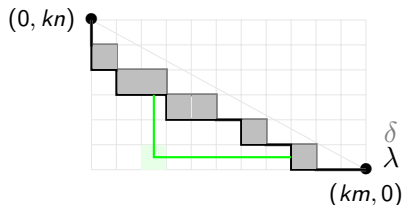
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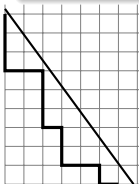
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$\text{area}(\lambda)$  as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks } \frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

# Proof Overview (algebraic side)

- $\psi: \mathcal{E}^+ \cong S$
- $\mathcal{E}^+$  is the “positive half” of  $\mathcal{E}$
- $S$  is an algebra of symmetric Laurent series in  $\mathbb{Q}(q, t)(z_1^{\pm 1}, \dots, z_l^{\pm 1})^{S_l}$  satisfying extra conditions and equipped with a “shuffle product”.

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## Key relationship

For  $\xi \in \mathcal{E}^+$ ,

$$\omega(\xi \cdot 1) = \text{pol}_X(\psi(\xi))$$

for automorphism  $\omega: \Lambda \rightarrow \Lambda$  and  $\text{pol}_X: S \rightarrow \Lambda$  a “polynomial truncation” operation.

# Proof Overview (combinatorial side)

- For  $\xi = D_{\mathbf{b}}$ , we get

$$\text{pol}_X \mathbf{H}_q \left( \frac{z^{\mathbf{b}} \prod_{i < j+1} (1 - qtz_i/z_j)}{\prod_{i < j} (1 - tz_i/z_j)} \right) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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Need an “infinite series” version of LLT polynomials!

# Cauchy identity

- For a fixed  $\sigma \in S_I$ , there exists a basis of  $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_I^{\pm 1}]$  called “non-symmetric Hall-Littlewood polynomials”, denoted  $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \dots, z_I; q)$  for  $\lambda \in \mathbb{Z}^I$ .

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$$\frac{\prod_{i < j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)} = \sum_{\mathbf{a} \geq 0} t^{|\mathbf{a}|} E_{\mathbf{a}}^\sigma(x_1, \dots, x_I; q^{-1}) F_{\mathbf{a}}^\sigma(y_1, \dots, y_I; q),$$

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- For a fixed  $\sigma \in S_I$ , there exists a basis of  $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_I^{\pm 1}]$  called “non-symmetric Hall-Littlewood polynomials”, denoted  $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \dots, z_I; q)$  for  $\lambda \in \mathbb{Z}^I$ .
- Under an inner-product coming from representation theory of affine Hecke algebras, there is a dual basis  $F_\lambda^\sigma = E_{-\lambda}^{\sigma w_0}(z_1^{-1}, \dots, z_I^{-1}; q^{-1}) = \overline{E_{-\lambda}^{\sigma w_0}}$

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- (Grojnowski-Haiman 2007) defines a (symmetric) “series LLT” polynomial  $\mathcal{L}_{\beta/\alpha}^\sigma(x_1, \dots, x_I; q) = H_q(w_0(F_\beta^{\sigma^{-1}} \overline{E_\alpha^{\sigma^{-1}}}))$

## Stable Shuffle Theorem (BHMPs 21a)

For  $\mathbf{b} \in \mathbb{Z}^I$  corresponding to highest path under a line of slope  $-r/s$ ,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{I-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_I, \dots, b_1) + (0, a_{I-1}, \dots, a_1)) / (a_{I-1}, \dots, a_1, 0)}^{\sigma}(x_1, \dots, x_I; q)$$

Under polynomial truncation,

$$\mathcal{L}_{\beta/\alpha}^{\sigma}(x_1, \dots, x_I; q) \rightarrow q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_I; q^{-1})$$

## Stable Shuffle Theorem (BHMPs 21a)

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$$\implies \omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_l; q^{-1}).$$

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$$\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} (1 + zt^{-r_i(\lambda)}).$$

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- $\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \sum_{\substack{s \in \mathbb{N}^{k+r}: |s|=n-k \\ 1 \in J \subseteq [k+r], |J|=k}} (D_{s+\epsilon_J} \cdot 1)$

## Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R) \in \operatorname{LNDP}_\mu} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

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Generalizing our methods further, we arrive at the following.

## Theorem (BHMP21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_{\pi} t^{a(\pi)} q^{\operatorname{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

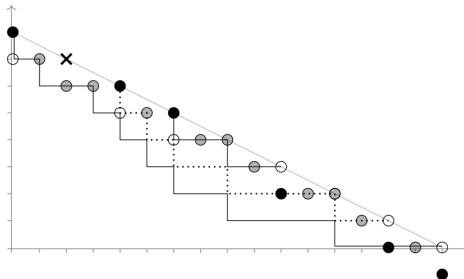
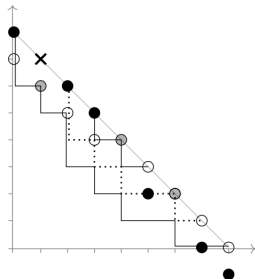
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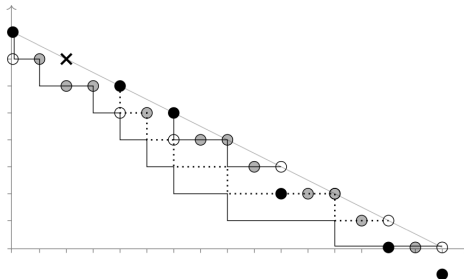
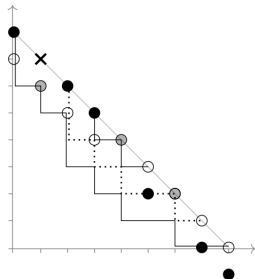


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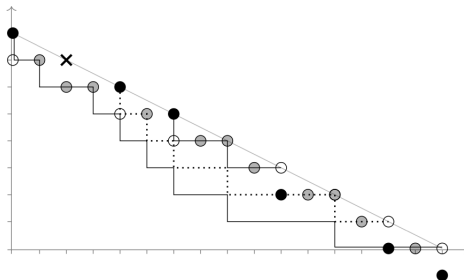
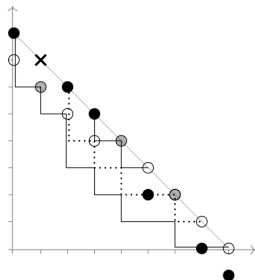
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For a sum over all “nests”  $\pi$  in a “den” corresponding to  $s_\mu$ .



- Implies the Loehr-Warrington Conjecture as a special case.
- Also proves  $\text{sgn}(\mu) \nabla s_\mu$  is Schur-positive.

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$D_{\mathbf{b}}$  defined for any  $\mathbf{b} \in \mathbb{Z}^I$ . When is  $D_{\mathbf{b}} \cdot 1$  nice?

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**Convex Curve Conjecture** (Blasiak-Haiman-Morse-Pun-S., 2021a)

For  $\mathbf{b} = (b_1, \dots, b_I)$  the south steps of highest path under a convex curve, the Schur expansion of  $D_{\mathbf{b}} \cdot 1$  has coefficients in  $\mathbb{N}[q, t]$ .

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- Experimental computation suggests this is “tight.”
- Coefficient of  $s_{1, \dots, 1}$  coincides with  $(q, t)$ -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

- What are the Schur expansion coefficients of  $D_{\mathbf{b}} \cdot 1$ ?



# Future Directions

- What are the Schur expansion coefficients of  $D_{\mathbf{b}} \cdot 1$ ?
- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)

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- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)
- What can we say about Macdonald polynomials?
- $S_I$ -representation theory interpretations?

# References

Thank you!

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