Theory of Sheaves
Notes inspired by a class taught by Andrei Rapinchuk in Fall 2018

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1. Presheaves and Sheaves

1.1. Presheaves.

1.1. Definition. Let X be topological space. A *presheaf of sets* F on X is given by the following data.

(a) For each open set $U \subseteq X$, $F(U)$ is a set.
(b) If $V \subseteq U$, there exists a map $\rho_V^U \colon F(U) \to F(V)$ such that
  (a) $\rho_U^U = id_{F(U)}$
  (b) If $W \subseteq V \subseteq U$, then $\rho_W^U = \rho_W^V \circ \rho_V^U$. In other words,

$$\begin{tikzcd} F(U) \arrow[rr, "\rho_W^U"] \arrow[dr, "\rho_V^U"'] & & F(W) \\ & F(V) \arrow[ur, "\rho_W^V"'] & \end{tikzcd}$$

1.2. Remark. (a) Sometimes $F(\varnothing) = \{e\}$, the singleton set, is included in the definition, but it is also a formal consequence of the above statements.
(b) A presheaf of sets can also be defined as a contravariant functor from $\mathbf{Top}(X) \to \mathbf{Set}$ where $\mathbf{Top}(X)$ is the category with objects being the open sets of X and the morphisms being

$$\mathrm{Hom}(V, U) = \begin{cases} \{\iota \colon V \hookrightarrow U\} & V \subseteq U \\ \varnothing & V \not\subseteq U \end{cases}$$

Informally, we can think of presheaves as a collection of functions. Historically, sheaves were thought in the context of some "étale space", $\mathcal{F}$, mapping to X and then considering the set of sections over $U \subseteq X$.

$$\begin{tikzcd} \mathcal{F} \arrow[r] & X \\ & U \arrow[ul, "\sigma \in F(U)"] \end{tikzcd}$$

This leads to the terminology of a "sheaf of sections." Thus, we often call elements of $F(u)$ "sections of F over $U \subseteq X$." Similarly, one can think of ρ_V^U as a restriction map.

1.3. Example. Throughout, let X be a topological space.

(a) Fix another topological space Y. Then, for $U \subseteq X$, let

$$F(U) = \{\phi \colon U \to Y \mid \phi \text{ is continuous}\} \text{ and } F(\varnothing) = \{e\}$$

Then, for $V \subseteq U$, we define $\rho_V^U \colon F(U) \to F(V)$ via

$$(\phi \colon U \to Y) \mapsto (\phi|_V \colon V \to Y)$$

In fact, if we take $Y = \mathbb{R}$ or $\mathbb{C}$, then we would get a presheaf of groups or even rings.

(b) If we take a set S and set

$$F(U) = \{\text{Constant functions } \phi\colon U \to S\}$$

we get the *constant presheaf*. We can modify this slightly by instead letting $F(U)$ be all locally constant functions from U.

(c) Given Y as another topological space, recall that, given a fixed map $\pi\colon Y \to X$, a section $\sigma\colon X \to Y$ is a continuous map such that $\sigma \circ \pi = id_X$. Then, we can define the *presheaf of sections* (with respect to π) by defining, for $U \subseteq X$,

$$F(U) = \{\phi\colon U \to Y \text{ continuous} \mid \pi \circ \phi = id_U\}.$$

(d) Given any presheaf F on X, we can restrict it to an open subset $W \subseteq X$ by taking only the $F(U)$ where $U \subseteq W$.

(e) Fix a point $p \in X$ and a set S. Then, we can define the *skyscraper presheaf* by saying, for open $U \subseteq X$,

$$F(U) = \begin{cases} S & \text{if } p \in U \\ \{e\} & \text{if } p \notin U \end{cases}$$

Then, if $p \in V \subseteq U$, $\rho_V^U = id_S$ and if $p \notin V$, then there is a unique map $\rho_V^U\colon F(U) \to \{e\}$.

Let K be an algebriacally closed field (eg $K = \mathbb{C}$).

1.4. Definition. Given an ideal $I \trianglelefteq K[x_1, \ldots, x_n]$, define the *vanishing set*

$$V(I) := \{(a_1, \ldots, a_n) \in K^n \mid f(a_1, \ldots, a_n) = 0 \forall f \in I\}$$

We call a set $S \subseteq K^n$ for which there exists an ideal $I \trianglelefteq K[x_1, \ldots, x_n]$ such that $S = V(I)$ an *algebraic set*.

Then, there is a topology called the *Zariski topology* on K^n for which sets of the form $V(I)$ form the family of all closed sets.

1.5. Definition. If $X = V(I)$ is an algebraic set, a function $f\colon X \to K$ is *regular* if there exists a polynomial $p(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ such that $f(x) = p(x)$ for all $x \in X$. (In other words, there is a polynomial p such that $p|_X = f$.)

All such functions form a ring, called the *coordinate ring*, denoted $K[X]$. There is a natural map

$$\begin{aligned} K[x_1, \ldots, x_n] &\twoheadrightarrow K[X] \\ p &\mapsto p|_X \end{aligned}$$

which gives us

1.6. Proposition. *The coordinate ring* $K[X] \cong K[x_1, \ldots, x_n]/I(X)$ *where* $I(X) = \{p \mid p|_X = 0\}$.

Recall, however, that $X = V(I)$ for some $I \trianglelefteq K[x_1, \ldots, x_n]$. One can check that $I \subseteq I(X) = I(V(I))$, but this containement is not necessarily an equality due to the following example.

1.7. Example. Let $I = (x_1^2)$. Then, $X = V(I) = \{(a_1, \ldots, a_n) \in K^n \mid a_1 = 0\}$. From this, we see that any $p \in x_1 K[x_1, \ldots, x_n]$ has the property that $p|_X = 0$. Thus, $I(X) = (x_1)$.

One may then ask how $I(X)$ is related to X. The answer is given by the following famous theorem.

1.8. Theorem (Hilbert's Nullstellensatz). *Let K be an algebraically closed field. Then, for $J \trianglelefteq K[x_1, \ldots, x_n]$, we have that*

$$I(V(J)) = r(J)$$

where $r(J) = \{s \in K[x_1, \ldots, x_n] \mid s^m \in J \text{ for some } m \in \mathbb{N}\}$ is the "radical" of J.

There are other equivalent formulations of Hilbert's Nullstellensatz, such as

Check that this is actually equivalent.

1.9. Proposition. *Given proper ideal $I \subsetneqq K[x_1, \ldots, x_n]$, we have that $V(I) \neq \varnothing$.*

Furthermore, $K[X]$ allows us to recover X in a functorial way, since we have

1.10. Corollary. *For K an algebraically closed field, we have that*

(a) all maximal ideals $\mathfrak{m} \trianglelefteq K[x_1, \ldots, x_n]$ are of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ and
(b) $V(\mathfrak{m}) = \{(a_1, \ldots, a_n)\}$.

Thus, we can recover all the points of X by applying $V(\cdot)$ to the maximal ideals of X.

1.11. Definition. A topological space X is called *irreducible* if $X \neq X_1 \cup X_2$ for X_1, X_2 proper closed sets of X.

1.12. Remark. Note that this notion is relatively uninteresting for T_2 topological spaces since any non-trivial T_2 space is reducible. As such, this notion is rarely used outside algebraic geometry.

1.13. Proposition. *A space $X \subseteq K^n$ is irreducible if and only if $I(X)$ is a prime ideal if and only if $K[X]$ is an integral domain.*

Using Hilbert's Basis Theorem, we also get

1.14. Proposition. *Every algebraic set is a finite union of irreducible algebraic sets.*

1.15. Proposition. *If $f \in K[x_1, \ldots, x_n]$ is irreducible, then $X = V((f))$ is irreducible.*

Proof. By Hilbert's Nullstellensatz, we have that

$$I(V((f))) = r((f))$$

and so, $g \in r((f)) \Longrightarrow f \mid g^m$ and because f is irreducible, this gives that $f \mid g \Longrightarrow g \in (f)$. Thus, it must be that (f) is a prime ideal and so $V((f))$ is irreducible. □

1.16. Definition. If $X \subseteq K^n$ is irreducible, then $K[X]$ is an integral domains and so we define $K(X)$ to be the fraction field of $K[X]$, also referred to as the *field of rational functions*.

While it is nice that we have this definition, we have a fundamental problem because $f \in K(X)$ does not have a *canonical* presentation as $f = \frac{g}{h}$, and so we can run into problems if one choice of h is zero at a point x. Thus, we must do a little work to address this problem.

1.17. Definition. Given $f \in K(X)$ and $x \in X$, we say f is *defined at* x if there is $g_x, h_x \in K[X]$ such that $f = \frac{g_x}{h_x}$ and $h_x \neq 0$. Furthermore, we define

$$\mathrm{Dom}(f) := \{x \in X \text{ where } f \text{ is defined}\}$$

However, it is not clear that f is well-defined on the domain. Indeed, we check

Proof. Let $f = \frac{g_1}{h_1} = \frac{g_2}{h_2}$ such that $h_1(x), h_2(x) \neq 0$. Then, this gives

$$g_1 h_2 = g_2 h_1 \Longrightarrow g_1(x) h_2(x) = g_2(x) h_1(x) \Longrightarrow \frac{g_1(x)}{h_1(x)} = \frac{g_2(x)}{h_2(x)}$$

□

1.18. Proposition. *The domain of a rational function f is a non-empty Zariski open set.*

This follows immediately by considering that a Zariski closed set is of the form $V(I)$ and so an open set is of the form $K^n \setminus V(I) = \bigcup_{p \in I} D(p)$ where

1.19. Definition.

$$D(p) := \{x \in K^n \mid p(x) \neq 0\}$$

is called the *principal (distinguished) open set defined by p*.

1.20. Proposition. *(a)* $D(p_1) \cup D(p_2) = D(p_1 p_2)$
(b) The $D(p)$'s form a base for the Zariski topology.

1.21. Definition. Given that the map $K[x_1, \dots, x_n] \to K[X]$ sends $p \mapsto q$, we define

$$D_X(q) := \{x \in X \mid q(x) \neq 0\} = X \cap D(p)$$

Thus, we have that

$$\mathrm{Dom}(f) = \bigcup_{f=\frac{g}{h}} D_X(h)$$

1.22. Definition. Given a space $X \subseteq K^n$, we define the *structure presheaf*, by, for every open set $U \subseteq X$,

$$\mathcal{O}_X(U) = \{f \in K(X) \mid f \text{ is defined at every point of } U\}$$

and, for $V \subseteq U$,

$$\rho_V^U \colon \mathcal{O}_X(U) \hookrightarrow \mathcal{O}_X(V) \text{ is the identity embedding}$$

1.23. Proposition. *$\mathcal{O}_X(U)$ is a K-subalgebra of $K(X)$.*

One may ask, what is $\mathcal{O}_X(X)$. To answer this, we first define a general notion.

1.24. Definition. Given a presheaf F on X, $F(X)$ is called the *global sections*, and sometimes is denoted $\Gamma(F)$.

Thus, the global sections $\mathcal{O}_X(X)$ are the rational functions defined everywhere on X. From this, we can immediately conclude that $K[X] \subseteq \mathcal{O}_X(X)$, but in fact, we have

1.25. Proposition.

$$\mathcal{O}_X(X) = K[X]$$

Proof. Given $f \in \mathcal{O}_X(X)$, there is a presentation of f forever $x \in X$ as $f = \frac{g_x}{h_x}$ such that $h_x(x) \neq 0$. Now, let

$$J = (h_x \mid x \in X) \trianglelefteq K[X]$$

Then, J has no zeros in X and so, by Hilbert's Nullstellensatz, $J = K[X]$. Thus, there exists $\phi_i \in K[X]$ such that

$$\phi_1 h_{x_1} + \cdots + \phi_r h_{x_r} = 1$$

Thus,

$$f = f \cdot 1 = f(\phi_1 h_{x_1} + \cdots + \phi_r h_{x_r}) = \phi_1 g_{x_1} + \cdots + \phi_r g_{x_r}$$

which is a polynomial. □

Thus, with this idea, we can generalize our definition of a regular function.

1.26. Definition. The regular functions on an open set $U \subseteq X$ are given by $\mathcal{O}_X(U)$.

1.27. Remark. A generalization of this idea is the notion of the prime spectrum of an arbitrary commutative ring A, which allows us to put a Zariski topology and a presheaf on this space. This will be expanded on later.

1.28. Definition. Given a fixed arbitrary topological space X, and 2 presheaves F and G on X, then a morphism between them, denoted $\phi\colon F \to G$ consists of appropriate morphisms ϕ_U for all open $U \subseteq X$ and $V \subseteq U$ such that the following diagram commutes

$$\begin{tikzcd} F(U) \arrow[r, "\phi_V"] \arrow[d, "\rho_V^U(F)"] & G(U) \arrow[d, "\rho_V^U(G)"] \\ F(V) \arrow[r, "\phi_V"] & G(V) \end{tikzcd}$$

1.29. Remark. If we think of a presheaf as a contravariant functor, then a morphism of presheaves is a natural transformation.

Now, suppose F, G are presheaves of some category with kernels, such as abelian groups. Then, our morphism diagrams can be extended

$$\begin{tikzcd} \ker\phi_U \arrow[r] & F(U) \arrow[r, "\phi_V"] \arrow[d, "\rho_V^U(F)"] & G(U) \arrow[d, "\rho_V^U(G)"] \\ \ker\phi_V \arrow[r] & F(V) \arrow[r, "\phi_V"] & G(V) \end{tikzcd}$$

Then, we have

1.30. Proposition. $\rho_V^U(F)(\ker\phi_U) \subseteq \ker\phi_V$

which allows us to define $K(U) = \ker\phi_U \subseteq F(U)$, giving us that $\{K(U), \rho_V^U(F)\}$ is a presheaf, and similarly for $I(U) = \operatorname{im}\phi_U \subseteq G(U)$.

1.31. Proposition. *(a)* ***Presheaves***(X) *forms a category.*

(b) We can send a presheaf $F \to F(X)$, the global sections of F over X, giving us a function. For example, if F is presheaf of abelian groups, then we have a functor

$$\mathbf{Presheaves}(X) \to \mathbf{AbGps}$$

called the functor of global sections.

Eventually, this will allow us to define the notion of a short exact sequence for (pre)sheaves. However, the functor of global sections turns out to be left exact but not right exact, so we will be able to look at right derived functors to measure the failure of this exactness.

1.2. Sheaves. We often want to glue together local information. For instance, if we had 2 functions $f_1\colon U_1 \to \mathbb{R}$ and $f_2\colon U_2 \to \mathbb{R}$, and wanted to build a function on $U = U_1 \cup U_2$, we would need that

$$f_1|_{U_1\cap U_2} = f_2|_{U_1\cap U_2}$$

1.32. Definition. Let X be a topological space. For all open sets $U \subseteq X$, take an open cover (not necessarily finite) $U = \bigcup_{\alpha\in I} U_\alpha$. A presheaf F on X is a *sheaf* if

(S1) For $x, t \in F(U)$ and $\rho^U_{U_\alpha}(s) = \rho^U_{U_\alpha}(t)$, then $s = t$ and

(S2) Given $s_\alpha \in F(U_\alpha)$ for $\alpha \in I$ such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$, there exists an $s \in F(U)$ such that $\rho^U_{U_\alpha}(s) = s_\alpha$ for all $\alpha \in I$.

1.33. Remark. A presheaf satisfying only condition (a) is called a *separated presheaf* or a *monopresheaf.*

Let us see how some of our presheaves meet this definition (and others do not).

1.34. Example. (1) Fix another topologyical space Y and define $F(U) = \{\text{all continuous maps } f\colon U \to Y\}$ and the restriction maps to be the usual restriction of functions. Then, if $U = \bigcup_{\alpha \in I} U_\alpha$,

(a) For $f, g\colon U \to Y$, we have

$$\rho^U_{U_\alpha}(f) = \rho^U_{U_\alpha}(g) \Longleftrightarrow f(x) = g(x)\ \forall x \in U_\alpha \Longleftrightarrow f(x) = g(x)\ \forall x \in U$$

(b) Given $f_\alpha \in F(U_\alpha)$, we get

$$\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(f_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(f_\beta) \Longrightarrow f_\alpha(x) = f_\beta(x)\ \forall x \in U_\alpha \cap U_\beta$$

Then, given $x \in U$, we have that there is at least one $\alpha \in I$ such that $x \in U_\alpha$ and, by above, we can define $f(x) = f_\alpha(x)$. This is continuous since, for open set $V \subseteq Y$, we get $f^{-1}(V) = \bigcup f_\alpha^{-1}(V)$, which is open.

(2) Consider the presheaf of bounded functions on $X = \mathbb{R}$, that is $F(U) = \{\text{bounded function } f\colon U \to \mathbb{R}\}$. We may cover $X = \mathbb{R}$ by overlapping intervals (eg $U_n = (n - \frac{1}{2}, n + \frac{4}{3})$). Then, on each U_n, take $f_n \in F(U_n)$ to be $f_n(x) = x$ for $x \in U_n$. Each of these f_n's is bounded on the domain, but if you glue them all together, you do not get a bounded function on $\mathbb{R}$.

(3) More generally, consider the constant presheaf on X to set S with at least 2 elements where $F(U) = S$ when $U \neq \varnothing$ and $F(\varnothing) = \{e\}$, the 1 element set. Then, take non-empty open sets U_1, U_2 such that $U_1 \cap U_2 = \varnothing$. If we take 2 distinct elements of S, say s_1, s_2 and let $U = U_1 \cup U_2$, we find that we cannot satisfy the second sheaf condition as follows. If there is an $s \in F(U) = S$ such that the condition holds, it would get mapped to s_1 under the restriction map to U_1 and to s_2 under the restriction map to U_2. However, this cannot happen because the restriction maps in these instances are just the identity maps.

(4) The outlook for the locally constant presheaf is better, since we no longer run into these gluing problems.

(5) Consider the skyscaper presheaf. We verify

(a) Let $s, t \in F(U)$. If $p \notin U$, then $s = t$ since $F(U) = \{e\}$. If $p \in U$, there exists an $\alpha \in I$ with $p \in U_\alpha$ and $\rho^U_{U_\alpha} = id_S$. Thus,

$$\rho^U_{U_\alpha}(s) = \rho^U_{U_\alpha}(t) \Longrightarrow s = t$$

(b) Let $s_\alpha \in F(U_\alpha)$ such that $\rho^{U_\alpha}_{U_\alpha\cap U_\beta(s_\alpha)=\rho^{U_\beta}_{U_\alpha\cap U_\beta}}(s_\beta)$. If $p \notin U$, then $F(U) = \{e\}$. If $p \in U$, then $p \in U_\alpha$ so $s_\alpha = s \in F(U_\alpha)$. The rest follows from the definition of the restriction maps.

(6) Now, let us consider the structure presheaf $\mathcal{O}_X$ for $X \subseteq K^n$ and X irreducible. We will show it is a sheaf. Let open set $U \subseteq X$ have open cover $U = \bigcup_{\alpha\in I} U_\alpha$

(a) Take $f, g \in \mathcal{O}_X(U)$. Then, we have that $\rho^U_{U_\alpha}(f) = f \in K(X)$ and thus, if $\rho^U_{U_\alpha}(f) = \rho^U_{U_\alpha}(g)$, we immediately get that $f = g$ in $K(X)$.

(b) Given $f_\alpha \in F(U_\alpha)$, we similarly get that

$$\rho^{U_\alpha}_{U_\alpha\cap U_\beta}(f_\alpha) = \rho^{U_\beta}_{U_\alpha\cap U_\beta}(f_\beta) \Longrightarrow f_\alpha = \rho^{U_\alpha}_{U_\alpha\cap U_\beta}(f_\alpha) = \rho^{U_\beta}_{U_\alpha\cap U_\beta}(f_\beta) = g_\beta$$

and so, we may take $f = f_\alpha$.

1.35. Proposition. *$\mathcal{O}_X(D_X(p)) = S^{-1}K[X]$, the localization of $K[X]$ with $S = \{1, p, p^2, \dots\}$.*

Proof. The containment $\mathcal{O}_X(D_X(p)) \supseteq S^{-1}K[X]$ is immediate since every element of $S^{-1}K[X]$ is a rational function defined where $p \neq 0$. Let $f \in \mathcal{O}_X(D_X(p))$. For all $x \in D_X(p)$, we have a presentation of f as $f = \frac{g_x}{h_x}$ where $h_x(x) \neq 0$. Now, let

$$I = (h_x \mid x \in D_X(p)) \trianglelefteq K[X]$$

and consider that $V(I) \cap D_X(p) = \varnothing$. In fact, $V(I) \subseteq X \setminus D_X(p) = V_X(p)$, that is, p vanishes at all points where I vanishes.

By Hilbert's Nullstellensatz, some power of p, say p^m, is in I. Thus,

$$\begin{aligned} p^m &= a_1 h_{x_1} + \cdots + a_r h_{x_r} && \text{by Hilbert's Basis Theorem} \\ \Longrightarrow f p^m &= a_1 g_{x_1} + \cdots + a_r g_{x_r} \in K[X] \\ \Longrightarrow f &= \frac{a_1 g_{x_1} + \cdots + a_r g_{x_r}}{p^m} \in S^{-1}K[X] \end{aligned}$$

□

1.36. Example. Let $X = \mathbb{C}^2$ and $U = \mathbb{C}^2 \setminus \{(0,0)\}$. Then, we have $I = (x, y)$ and, since $\{(0,0)\} = \{x = 0\} \cap \{y = 0\}$, we get

$$U = D((x)) \cup D((y)) \Longrightarrow \mathcal{O}_X(U) = \mathcal{O}_X(U_1) \cap \mathcal{O}_X(U_2)$$

where

$$\begin{cases} \mathcal{O}_X(U_1) = S_1^{-1}K[x,y] & S_1 = \{1, x, x^2, \dots\} \\ \mathcal{O}_X(U_2) = S_2^{-1}K[x,y] & S_2 = \{1, y, y^2, \dots\} \end{cases}$$

However, this forces

$$O_X(U) = S_1^{-1}K[x,y] \cap S_2^{-1}K[x,y] = K[x,y]$$

Now, we observe that $\mathbf{Sh}(X)$ is a full subcategory of $\mathbf{PrSh}(X)$, so we have an embedding $\mathbf{Sh}(X) \hookrightarrow \mathbf{PrSh}(X)$. However, many things to construct new sheaves from old ones, like cokernel and tensor, land in $\mathbf{PrSh}(X)$. So, we need a way to go back.

1.37. Theorem. *For any presheaf F on a topological space X, there exists a sheaf F^+ on X together with a morphism $\theta\colon F \to F^+$ so that, for every sheaf G on X and a morphism $\phi\colon F \to G$, there exists a unique morphism $\psi\colon F^+ \to G$ such that the following diagram commutes*

$$\begin{tikzcd} F \arrow[rr, "\phi"] \arrow[dr, "\theta"'] & & G \\ & F^+ \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

F^+ is called the sheaf associated with presheaf F. The functor sending F to F^+ is called sheafification.

Proof. Postponed for now. $\square$

Using this universal property, we can take a map of presheaves and lift it to a map of their associated sheaves

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "\theta"] \arrow[dr, dashed, "\psi"] & F_1 \arrow[d, "\theta_1"] \\ F^+ \arrow[r, dotted, "\phi^+"] & F_1^+ \end{tikzcd}$$

This gives us

1.38. Proposition. $\mathrm{Hom}_{\mathbf{Sh}}(F^+, G) \cong \mathrm{Hom}_{\mathbf{PrSh}}(F, G)$. *In fact, such an identification is natural.*

1.39. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and let $S\colon \mathcal{A} \to \mathcal{B}$ and $T\colon \mathcal{B} \to \mathcal{A}$ be functors. We say S and T are *adjoint* if, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there is a natural bijection

$$\mathrm{Hom}_{\mathcal{B}}(SA, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, TB)$$

denoted $S \dashv T$.

So, in our setup with

$$\mathbf{PrSh}(X) \xrightarrow{S} \mathbf{Sh}(X),\ \mathbf{Sh}(X) \xrightarrow{T} \mathbf{PrSh}(X)$$

since $TG = G$, we get that S and T are adjoint.

2. Limits

In class, there were several lectures on categorical colimits and direct limits that are ommitted here since they are relatively straightforward

3. Some Basic Theorems

3.1. Definition. Given a presheaf F on a topological space X, for $x \in X$, define the *stalk at x* to be

$$F_x := \varinjlim_{U \ni x} F(U)$$

where U is a neighborhood of x. Elements of the stalk are called *germs*. Finally, for all such U, we define the map

$$\rho_x^U : F(U) \to F_x$$

and, given a homomorphism of presheaves $\phi\colon F \to G$, we say

$$\phi_x\colon F_x \to G_x$$

is the induced homomorphism.

While it is easy to say $\phi_x\colon F_x \to G_x$ is the induced homomorphism (and well-defined), it is worth thinking about how via this commutative diagram exhibiting the universal property of $\varinjlim F(U)$:

$$\begin{tikzcd}
F(U) \arrow[rr, "\phi_U"] \arrow[dr, "\rho_x^U"] \arrow[dd, "\rho_V^U"'] & & G(U) \arrow[dd] \arrow[dr] & \\
& F_x = \varinjlim F(U) \arrow[rr, dashed] & & G_x \\
F(V) \arrow[rr, "\phi_V"] \arrow[ur, "\rho_x^V"] & & G(V) \arrow[ur] &
\end{tikzcd}$$

From this, we list a few consequences of the colimit consturction in this setting.

3.2. Proposition. Double check these are true.

(a) *Given an $s \in F_x$, there exists some open neighborhood U of x and some $t \in F(U)$ such that $\rho_x^U(t) = s$.*

(b) *Given a homomorphism of presheaves $\phi\colon F \to G$, we have*

$$\rho_x^U(G) \circ \phi_U = \phi_x \circ \rho_x^U(F)$$

from the commutative diagram.

3.3. Lemma. *Let F be a sheaf. Given open $U \subseteq X$ and $s, t \in F(U)$, if $\rho_x^U(s) = \rho_x^U(t)$ for all $x \in U$, then $s = t$. In other words,*

$$F(U) \to \prod_{x \in U} F_x \text{ is injective.}$$

Proof. Let $\rho_x^U(s) = \rho_x^U(t)$. Then, there exists an open neighborhood U_x of x such that $\rho_{U_x}^U(s) = \rho_{U_x}^U(t)$. Now, $U = \bigcup_{x \in U} U_x \implies s = t$ by (S1). $\square$

3.4. Corollary. *Let F be a presheaf, G a sheaf, and*

$$\phi_1, \phi_2 \colon F \to G$$

If $\phi_{1x} = \phi_{2x}$ for all $x \in X$, then $\phi_1 = \phi_2$.

Proof. Consider that

$$\rho_x^U(G)(\phi_{1U}(s)) = \phi_{1x}(\rho_x^U(F)(s)) = \phi_{2x}(\rho_x^U(G)(s)) = \rho_x^U(G)(\phi_{2U}(s))$$

Then, by the lemma above, the corollary follows. $\square$

3.5. Definition. Given a sheaf F and $s \in F(X)$, we define

$$\operatorname{Supp} s := \{x \in X \mid \rho_x^X(s) \neq 0\}$$

3.6. Proposition. *$\operatorname{Supp} s$ is closed in X.*

Proof. Let $x \in X \setminus \operatorname{Supp} s$. Then,

$$\rho_x^X(s) = 0 \Longrightarrow \exists \text{ open } V \ni x \text{ such that } \rho_V^X(s) = 0$$

Thus, there is a $y \in V$ such that $\rho_y^X(s) = 0$, so $V \subseteq X \setminus \operatorname{Supp} s$. $\square$

3.7. Proposition. *Let $\phi \colon F \to G$ be a morphism of presheaves.*

(a) Assume that F is a sheaf. Then

$$\phi_x \colon F_x \to G_x \text{ are injective } \forall x \in X \Longleftrightarrow \phi_U \colon F(U) \to G(U) \text{ are injective } \forall U \subseteq X$$

(b) Assume both F and G are sheaves. Then,

$$\phi_x \colon F_x \to G_x \text{ are bijective } \Longleftrightarrow \phi_U \text{ are bijective.}$$

Proof. (a) For ($\Longrightarrow$), let $s, t \in F(U)$ with $\phi_U(s) = \phi_U(t)$. Then,

$$\rho_x^U(\phi_U(s)) = \rho_x^U(\phi_U(t)) \Longleftrightarrow \phi_x(\rho_x^U(s)) = \phi_x(\rho_x^U(t))$$

(where ρ_x^U has domain for the appropriate (pre)sheaf). Then, since ϕ_x is injective, we have that $\rho_x^U(s) = \rho_x^U(t)$ and so $s = t$ by our lemma.

For ($\Longleftarrow$), let $s_x, t_x \in F_x$ and $\phi_x(s_x) = \phi_x(t_x)$. Then, there exists open set $U \ni x$ and $s_U, t_U \in F(U)$ such that $s_x = \rho_x^U(s_u)$ and $t_x = \rho_x^U(t_u)$. However, from the definition of the stalk of G, there exists an open set V such that $x \in V \subseteq U$ with

$$\rho_V^U(\phi_U(s_U)) = \rho_V^U(\phi_U(t_U)) \Longleftrightarrow \phi_V(\rho_V^U(s_U)) = \phi_V(\rho_V^U(t_U))$$

and so, by the injectivity of ϕ_V, $\rho_V^U(s_U) = \rho_V^U(t_U)$. Thus, by our lemma, $s_U = t_U$.

(b) For both parts, we need only show surjectivity.

For ($\Longleftarrow$), given $s_x \in G_x$, there exists a neighborhood U_x and an $s \in G(U_x)$ such that $\rho_x^{U_x}(s) = s_x$. Since ϕ_{U_x} is surjective, then there is a $t \in F(U_x)$ such that $\phi_{U_x}(t) = s$. Thus,

$$\phi_x(\rho_x^{U_x}(t)) = \rho_x^{U_x}(\phi_{U_x}(t)) = \rho_x^{U_x}(s) = s_x$$

and so ϕ_x is surjective.

For ($\Longrightarrow$), let $t \in G(U)$ and let $x \in U$. Then, by the surjectivity of ϕ_x, there exists an $s_x \in F_x$ such that

$$\rho_x^U(t) = \phi_x(s_x)$$

Now, by definition of ϕ_x, there is some $s_{U_x} \in U_x \subseteq U$ such that $\phi_{U_x}(s_{U_x}) = \phi_{U_x}(t)$.

$$\begin{tikzcd}
 & F(U) \arrow[r, hook, "\phi_U"] \arrow[dl] \arrow[dd, "\rho_x^U"] & G(U) \arrow[dd, "\rho_x^U"] \arrow[dr] & \\
F(U_x) \arrow[dr, "\rho_x^{U_x}"'] & & & G(U_x) \arrow[dl] \\
 & F_x \arrow[r, hook, two heads, "\phi_x"] & G_x &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
s_{U_x} \arrow[dr, mapsto, "\rho_x^{U_x}"] & & t \arrow[d, mapsto, "\rho_x^U"] & \phi_{U_x}(s_{U_x}) \arrow[dl, mapsto] \\
 & s_x \arrow[r, mapsto, "\phi_x"] & \phi_x(s_x) = \rho_x^U(t) &
\end{tikzcd}$$

Note that $\phi_{U_x}(s_{U_x})$ gives the same element in G_x as t under the appropriate restriction maps. So, there exists a $V_x \subseteq U_x \subseteq U$ such that, for $s_{V_x} = \rho_{V_x}^{U_x}(s_{U_x})$,

$$\phi_{V_x}(s_{V_x}) = \rho_{V_x}^U(t)$$

Now, in order to glue this into a section in $F(U)$, we need to check the gluing condition. Namely, we want to show $\rho_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(s_{V_{x_1}}) = \rho_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(s_{V_{x_2}})$ for any $x_1, x_2 \in U$ (where $U = \bigcup_{x \in U} V_x$). Consider

$$\rho_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(\phi_{V_{x_1}}(s_{V_{x_1}})) = \rho_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(\rho_{V_{x_1}}^U(t)) = \rho_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(\rho_{V_{x_2}}^U(t)) = \rho_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(\phi_{V_{x_2}}(s_{V_{x_2}}))$$

However, we can then commute our homomorphism and restriction to get

$$\rho_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(\phi_{V_{x_1}}(s_{V_{x_1}})) = \rho_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(\phi_{V_{x_2}}(s_{V_{x_2}})) \Longrightarrow \phi_{V_{x_1} \cap V_{x_2}}(\rho_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(s_{v_{x_1}})) = \phi_{V_{x_1} \cap V_{x_2}}(\rho_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(s_{v_{x_2}}))$$

However, by part (a), $\phi_{V_{x_1} \cap V_{x_2}}$ is injective. Thus, we get

$$\rho_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(s_{v_{x_1}}) = \rho_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(s_{v_{x_2}})$$

Then, since F is a sheaf, there exists an $s \in F(U)$ such that $\rho_{V_x}^U(s) = s_{V_x}$. In conclusion, we observe that

$$\rho_{V_x}^U(t) = \phi_{V_x}(s_{V_x}) = \phi_{V_x}(\rho_{V_x}^U(s)) = \rho_{V_x}^U(\phi_U(s))$$

and so, because G is a sheaf, $\phi_U(s) = t$.

□

Insert stuff about direct limits and exactness

3.8. Definition. Given F, G, H all presheaves of abelian groups and $\phi\colon F \to G$ and $\psi\colon G \to H$ morphisms. Then, we say

$$F \xrightarrow{\phi} G \xrightarrow{\psi} H$$

is *exact in the category of presheaves* if

$$F(U) \stackrel{\phi_U}{\to} G(U) \stackrel{\psi_U}{\to} H(U)$$

is exact for every open set $U \subseteq X$.

3.9. Corollary. *If*

$$F \stackrel{\phi}{\to} G \stackrel{\psi}{\to} H$$

is exact in **PrSh**, *then, for every point $x \in X$,*

$$F_x \stackrel{\phi_x}{\to} G_x \stackrel{\psi_x}{\to} H_x$$

is also exact. In particular, if

$$0 \to F \stackrel{\phi}{\to} G \stackrel{\psi}{\to} H \to 0$$

is exact, then, for all $x \in X$,

$$0 \to F_x \stackrel{\phi_x}{\to} G_x \stackrel{\psi_x}{\to} H_x \to 0$$

3.10. Definition. A sequence of sheaves

$$F \stackrel{\phi}{\to} G \stackrel{\psi}{\to} H$$

is *exact in* **Sh** if

$$F_x \stackrel{\phi_x}{\to} G_x \stackrel{\psi_x}{\to} H_x$$

is exact for every $x \in X$.

3.11. Theorem. *Let*

$$0 \to F \to G \to H$$

be an exact sequence of sheaves. Then, for every open $U \subseteq X$,

$$0 \to F(U) \stackrel{\phi_U}{\to} G(U) \stackrel{\psi_U}{\to} H(U)$$

is exact.

Proof. The exactness at $F(U)$ follow from the fact that

$$0 \to F_x \to G_x \to H_x$$

is exact by definition of the exactness of sheaves and the above proposition 3.7.

For exactness of $G(U)$, first consider $s \in F(U)$. Then, for every $x \in U$, we get

$$\rho_x^U(\psi_U(\psi_U(s))) = \psi_x(\phi_x(\rho_x^U(s))) = 0$$

because $\phi_x \circ \phi_x = 0$ by the exactness assumption. Thus, $\psi_U \circ \phi_U = 0$ and so $\phi_U(F(U)) \subseteq \ker \psi_U$. To show the other inclusion, we require the following lemma.

3.12. Lemma. *Let $K(U) = \ker \psi_U$. Then, K is a sheaf and $K_x = \ker(G_x \stackrel{\psi_x}{\to} H_x)$.*

Proof of Lemma. Take $U = \bigcup_{\alpha \in I} U_\alpha$. Then, for $s \in K(U) \subseteq G(U)$, if $\rho^U_{U_\alpha}(s) = 0$ for all $\alpha \in I$, then $s = 0$ because G is a sheaf. Thus, $K(U)$ satisfies (S1).

Now take $s_\alpha \in K(U_\alpha) \subseteq G(U_\alpha)$ such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$. Then, since G is a sheaf, there exists $s \in G(U)$ such that $\rho^U_{U_\alpha}(s) = s_\alpha$ for all α. From this, we get

$$\rho^U_{U_\alpha}(H)(\psi_U(s)) = \psi_{U_\alpha}(s_\alpha) = 0 \Longrightarrow \psi_U(s) = 0 \Longrightarrow s \in K(U)$$

thus giving us that K satisfies (S2). □

So now we wish to check if $\phi_U(F(U)) \subseteq K(U)$. From the lemma, we get that

$$\phi_x(F_x) = \ker \psi_x = K_x \Longrightarrow \phi_x \colon F_x \to K_x \text{ is an isomorphism for all } x$$

and so, by 3.7, ϕ is an isomorphism of the sheaves F and K. □

Finish this proof.

A question to ask is

$$\text{When does } 0 \to F \to G \to H \to 0 \text{ exact } \Longrightarrow 0 \to F(U) \to G(U) \to H(U) \to 0 \text{ exact?}$$

Given some surjection $G_x \twoheadrightarrow H_x$, some $t \in H(U)$, then for every $x \in U$, there is a neighborhood $U_x \subseteq U$ and $s \in G(U_x)$ such that $\psi_{U_x}(s) = \rho^U_{U_x}(t)$. In other words, we have some local lift. Can we glue all these local lifts together? Sometimes not.

3.13. Definition. Let F be a sheaf on X. G is called *flasque* (or *flabby*) if the restriction maps $\rho^U_V \colon F(U) \to F(V)$ are all surjective.

In particular, for a flasque sheaf, $\rho^X_V \colon F(X) \to F(V)$ is surjective.

3.14. Examples.
(a) The skyscraper sheaf is flasque.
(b) The sheaf of smooth functions on a connected differentiable manifold.

Check this

3.15. Theorem. *Let*

$$0 \to F \overset{\phi}{\to} G \overset{\psi}{\to} H \to 0$$

be an exact sequence of sheaves. If F is flasque, then

$$0 \to F(X) \overset{\phi_X}{\to} G(X) \overset{\psi_X}{\to} H(X) \to 0$$

is exact. If, in addition, G is flasque, then H is flasque

3.16. Remark. X can be replaced with any open set $U \subseteq X$ and the result still holds.

Proof. By the previous theorem (3.11), we need only show ψ_U is surjective. Let $t \in H(X)$. Then, for all $x \in X$, there exists an open neighborhood U with $x \in U, s \in G(U)$ such that $\psi_U(s) = \rho^X_U(t)$ by the surjectivity of the stalks. Now, consider all pairs (U, s) such that $U \subseteq X$ is open and s is a lift over U, ie $s \in F(U)$ and $\psi_U(s) = \rho^X_U(t)$.

Our first step is to construct a maximal pair with the partial order

$$(U_1, s_1) \leq (U_2, s_w) \iff U_1 \subseteq U_2 \text{ and } s_2|_{U_1} = s_1$$

Let $\{(U_\alpha, s_\alpha)\}_\alpha$ be a totally ordered subset. We wish to find an upper bound. Take $U := \bigcup U_\alpha$. Then, since we are in a totally ordered set, for $U_\alpha \subseteq U_\beta$, we have

$$s_\alpha = \rho^{U_\alpha}_{U_\alpha}(s_\alpha) = \rho^{U_\beta}_{U_\alpha}(s_\beta)$$

and so, by (S2), there exists an $s_U \in G(U)$ such that (U, s_U) is an upper bound. Thus, by Zorn's lemma, there is a maximal (U^*, s_{U^*}) for our set of pairs.

Assume there exists an $x \in X \setminus U^*$. Then, there is an open set V containing X and $s_V \in G(V)$ such that $\psi_V(s_V) = \rho^X_V(t)$. So, take

$$\psi_{U^*\cap V}(s_{U^*}|_{U^*\cap V} - s_V|_{U^*\cap V}) = \rho^X_{U^*\cap V}(t) - \rho^X_{U^*\cap V}(t) = 0$$

and thus $s_{U^*}|_{U^*\cap V} - s_V|_{U^*\cap V} \in \ker \psi_{U^*\cap V} = \phi_{U^*\cap V}(F(U^* \cap V))$ by Theorem 3.11. However, since F is flasque, we have that

$$F(V) \twoheadrightarrow F(U^* \cap V)$$

that is, the restriction map is surjective. Thus,

$$s_{U^*}|_{U^*\cap V} - s_V|_{U^*\cap V} = \phi_{U^*\cap V}(\rho^{U^*}_{U^*\cap V}(z))$$

for some $z \in F(V)$. Now, take $s'_V = s_V + \phi_V(z)$. Then, $s_{U^*}|_{U^*\cap V} = s'_V|_{U^*\cap V}$ by construction. Therefore, using (S2), there exists an element $s_{U^*\cup V} \in G(U^* \cup V)$ such that $s_{U^*\cup V}|_{U^*} = s_{U^*}$ and $s_{U^*\cup V}|_V = s'_V$. So, we have construced $s_{U^*\cup V}$ such that

$$(U^*, s_{U^*}) \prec (U^* \cup V, s_{U^*\cup V})$$

violating the maximality of (U^*, s_{U^*}). Thus, it must be that $X \setminus U^* = \varnothing$.

Our second step is to consider the commutative diagram with exact rows, $V \subseteq U$, and G flasque.

$$\begin{tikzcd}
0 \arrow[r] & F(U) \arrow[r] \arrow[d] & G(U) \arrow[r] \arrow[d, two heads] & H(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(V) \arrow[r] & G(V) \arrow[r] & H(V) \arrow[r] & 0
\end{tikzcd}$$

The surjectivity of $G(U) \to G(V)$ follows from G flasque, and so $H(U) \to H(V)$ must be surjective by a straightforward diagram chase. $\square$

Now, given an exact sequence of sheaves,

$$0 \to F \to G \to H \to 0$$

the sequence of global sections may not be exact. We provide an explicit example to illustrate what can go wrong.

3.17. Example. Take $X = \mathbb{C} \setminus \{(0,0)\} \subseteq \mathbb{C}$ and consider $\mathcal{O}$ the sheaf of holomorphic functions. In otherwords, for $U \subseteq X, \mathcal{O}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions on U. Next, consider the induced derivative map

$$\mathcal{O} \to \mathcal{O}$$
$$f \mapsto f'$$

which has kernel given by the locally constant functions on X, say C_X. Then, we have exact sequence of sheaves

$$0 \to C_X \to \mathcal{O} \xrightarrow{\psi} \mathcal{O} \to 0$$

However, every holomorphic function has a local antiderivative. In other words, given $f \in \mathcal{O}(U)$, for all $x \in U$, there exists a $U_x \subseteq U$ with $x \in U_x$ and $g_x \in \mathcal{O}(U_x)$ such that $g'_x = \psi(g_x) = f|_{U_x}$. However, $\psi_X(\mathcal{O}(X))$ consists of holomorphic functions with a global antiderivative, which is not all of $\mathcal{O}(X)$. Thus, ψ_X is not surjective and so we only have exactness up to

$$0 \to C_x \to \mathcal{O}(X) \xrightarrow{\psi_X} \mathcal{O}(X)$$

To capture the failure of the exactness, we introduce the long exact sequence of cohomology groups (using the necessary homological algebra)

$$\mathcal{O}(X) \xrightarrow{\psi_X} \mathcal{O}(X) \to H^1(X, C_X) \to H^1(X, \mathcal{O}) \to \cdots$$

However, because X is a Riemann surface but not compact, $H^1(X, \mathcal{O}) = 0$ and thus

$$H^1(X, C_X) \cong \mathcal{O}(X)/\psi_X(\mathcal{O}(X)) \cong \mathbb{C}$$

because $\psi_X(\mathcal{O}(X))$ is the $\mathbb{C}$-algebra of all functions that integrate to zero on the unit circle. Note that this gives the same result as the singular cohomology of X.

3.18. Proposition. *$\phi\colon F \to G$ is an epimorphism in the category of presheaves if and only if $\phi_U\colon F(U) \to G(U)$ is surjective for every $U \subseteq X$.*

Proof. Let $C := \operatorname{coker} \phi$ be the presheaf cokernel of ϕ. Then, ϕ is an epimorphism if and only if C is the zero sheaf. Thus,

$$\phi \text{ is an epimorphism } \iff C = 0 \iff C(U) := \operatorname{coker} \phi_U = 0, \forall U \subseteq X \iff \phi_U \text{ is surjective } \forall U \subseteq X$$

$\square$

In the category of sheaves, however, the cokernel presheaf is not necessarily a sheaf, so we have to do a little more work. We will make use of the following lemma.

3.19. Proposition. *Let $\phi\colon F \to G$ be a morphism of sheaves of abelian groups. Then, ϕ is an epimorphism in the category of sheaves of abelian groups if and only if $\phi_x\colon F_x \to G_x$ is surjective for all $x \in X$*

Proof. Let $C(U) := G(U)/\phi_U(F(U)) = \operatorname{coker}\phi_U$. From above, this is a presheaf.

Assume $\phi_x \colon F_x \to G_x$ is surjective for all x. Then, consider the diagram

$$\begin{tikzcd} F \arrow[r, "\phi"] & G \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & H \end{tikzcd}$$

such that $g_1 \circ \phi = g_2 \circ \phi$. Then, on stalks, we have $(g_1)_x \circ \phi_x = (g_2)_x \circ \phi_x$ and ϕ_x is surjective by assumption. So, this means ϕ_x is a epimorphism in the category of abelian groups, and thus $(g_1)_x = (g_2)_x \Longrightarrow g_1 = g_2$ by 3.4.

Conversely, assume that ϕ is an epimorphism in $\mathbf{Sh}(X)$. We then define maps via the commutative diagram

$$\begin{tikzcd} F \arrow[r, "\phi"] & G \arrow[r, bend left, "\pi"] \arrow[r, bend right, "0"'] & C \end{tikzcd}$$

where $C = \operatorname{coker}\phi$ is the cokernel presheaf and π is the natural projection sending $\pi \colon F \to C \cong G/\phi(F)$. Now, $\pi \circ \phi = 0 \circ \phi = 0$, so $\pi = 0$ since ϕ is an epimorphism. However, this tells us that $C = 0$ and thus $0 = C_x = \operatorname{coker}(\phi_x \colon F_x \to G_x)$. Therefore, ϕ_x must be surjective for all x. $\square$

3.20. Proposition. *Given sheaves F, G, we have*

(a) $\phi \colon F \to G$ is a monomorphism $\Longleftrightarrow$ ϕ is injective, ie $\phi_U \colon F_U \to G_U$ is injective for all open $U \subseteq X$.
(b) K is the kernel in the categorical sense.

4. Étale space of a presheaf and sheafification

We recall some basic facts from topology.

4.1. Definition. A map $\pi \colon E \to X$ between topological spaces is a *local homeomorphism* if every $e \in E$ has a neighborhood O_e such that $\pi(O_e) = U_x$ where U_x is a neighborhood of $x = \pi(e)$.

In what follows, we will have the following setup. Given a projection $\pi \colon E \to X$,

- we will call X the "base space",
- we will call E the "total space",
- we will denote $E_x := \pi^{-1}(x)$ and call these the "stalks of π",
- we will denote $\Gamma(U, \pi) := \{\sigma \colon U \to E \mid U \subseteq X, \sigma$ continuous such that $\pi \circ \sigma = id_U\}$ and we will call this the "set of sections over U",
- and, for $V \subseteq U$, we will denote $\sigma|_V$ to ve the restriction of σ to V.

Our first goal is to show that Γ is a sheaf.

4.2. Proposition. *Let $\pi \colon E \to X$ be a local homeomorphism. Then,*

(a) π is an open map.

(b) $E = \bigsqcup_{x\in X} E_x$
(c) The induced topology on each stalk is discrete.
(d) If $s\colon U \to E$*is a section, then for every* $V \subseteq U$*,* $s|_V : V \to E$ *is a section.*
(e) If two sections coincide at a point, then they coincide at a neighborhood of that point.
(f) Sets of the form $s(U)$ *for open* $U \subseteq X$*,* $s \in \Gamma(U,\pi)$ *form a base of the topology on* E*.*

4.3. THEOREM. *Let* $\pi\colon E \to X$ *be a local homeomorphism. Then,* $F(U) = \Gamma(U,\pi)$ *is a sheaf on* X *and* F_x *can be identified with* $\pi^{-1}(x) = E_x$*.*

Bibliography