

KAZHDAN-LUSZTIG BASIS FOR HECKE ALGEBRAS A CLASS PRESENTATION FOR QUANTUM GROUPS

GEORGE H. SEELINGER

1. INTRODUCTION

The Kazhdan-Lusztig basis was introduced in [KL79]. We will define the basis and give a proof of its existence and uniqueness, although we will mainly follow the proof in [Soe97]. Since their introduction, the so-called Kazhdan-Lusztig polynomials, which appear in the definition of the basis, have appeared in many other fields of mathematics. For a more detailed overview of connections, see [Bre03, p 5].

2. PRELIMINARIES

We work with the Hecke algebra, for which we will give two presentations.

2.1. Definition. [Hum90, Section 7.4] Let $A = \mathbb{Z}[q, q^{-1}]$. Then, the *Hecke algebra* \mathcal{H} associated to a Weyl group \mathcal{W} has a basis $\{T_w \mid w \in \mathcal{W}\}$ with relations

- (a) $T_x T_y = T_{xy}$ if $\ell(x) + \ell(y) = \ell(xy)$ and
- (b) $T_s^2 = (q - 1)T_s + qT_{id}$ for all simple reflections $s \in \mathcal{W}$.

2.2. Remark. We need not restrict \mathcal{W} to be a Weyl group. In full generality, we can replace \mathcal{W} with any Coxeter group.

For our purposes, it will also be convenient to work with the Hecke algebra over an enlarged ring. Let $v := q^{-\frac{1}{2}}$. Then, we have the following.

2.3. Proposition. *The Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ is given as the associative algebra with generators $\{H_s\}$ for $H_s = vT_s$ and relations*

- (a) $H_s^2 = 1 + (v^{-1} - v)H_s$ and
- (b) $H_s H_t \cdots H_s = H_t H_s \cdots H_t$ or $H_s H_t H_s \cdots H_t = H_t H_s H_t \cdots H_s$ if $st \cdots s = ts \cdots t$ or $sts \cdots t = tst \cdots s$, respectively, for simple reflections $s, t \in \mathcal{W}$

2.4. Proposition. *The Hecke algebra over $\mathbb{Z}[v, v^{-1}]$ has a basis given by $\{H_w \mid w \in \mathcal{W}\}$ where $H_w = v^{\ell(w)} T_w$. Furthermore, this basis has relation $H_x H_y = H_{xy}$ if $\ell(x) + \ell(y) = \ell(xy)$.*

Date: December 9, 2019.

2.5. Lemma. We have $H_s^{-1} = H_s + (v - v^{-1})$ and so all the H_x basis elements are units in \mathcal{H} .

Proof.

$$H_s^2 - (v^{-1} - v)H_s = 1 \implies H_s(H_s + (v - v^{-1})) = 1$$

□

2.6. Lemma. For simple reflection $s \in \mathcal{W}$, if $\ell(xs) < \ell(x)$, then

$$H_x H_s = H_{xs} + (v^{-1} - v)H_x.$$

Proof. We know that

$$H_x = H_{xs} H_s \implies H_x H_s = H_{xs} H_s^2 = H_{xs}(1 + (v^{-1} - v)H_s) = H_{xs} + (v^{-1} - v)H_x$$

□

3. THE KAZHDAN-LUSZTIG BASIS

3.1. Definition. Recall that $A = \mathbb{Z}[q, q^{-1}]$.

- (a) We define the \mathbb{Z} -linear map, called the *bar involution*, $-: A \rightarrow A$ given by sending $q \mapsto q^{-1}$
- (b) The Hecke algebra \mathcal{H} admits an extension of the bar involution, say $\iota: \mathcal{H} \rightarrow \mathcal{H}$, given by

$$\iota(T_w) := T_{w^{-1}}^{-1}$$

for any $w \in \mathcal{W}$. For convenience, we will overload notation and write

$$\overline{T_w} := \iota(T_w)$$

Note that $\iota(H_s) = v^{-1}T_s^{-1} = v^{-1}(v^2T_s - 1 + v^2) = H_s - v^{-1} + v = H_s^{-1}$ and, similarly, $\iota(H_w) = H_{w^{-1}}^{-1}$. Then, we have an ι -invariant of the form

$$C_s := q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_{id} = H_s - v^{-1}H_{id}$$

We can also introduce a similar ι -invariant of the form

$$C'_s := H_s + vH_{id}$$

This justifies why we introduced the H -basis in Proposition 2.4. In [Hum90, p 158], it is noted that it could be tempting to construct further ι -invariants by taking products of these C_s elements. However, if one has a word $sts = tst$ with $s, t \in \mathcal{W}$ both simple reflections and $\ell(sts) = 3 = \ell(tst)$, then one can check that $C_s C_t C_s \neq C_t C_s C_t$. However, if we compute (still assuming $\ell(sts) = 3$)

$$C_s C_t C_s - C_t = q^{-\frac{3}{2}}(T_{sts} - q(T_{st} - T_{ts}) + q^2(1 + q^{-1})(T_s + T_t) - q^3(1 + 2q^{-1})T_{id})$$

we get an ι -invariant expression where the s and t 's are interchangeable. Similarly, we can compute

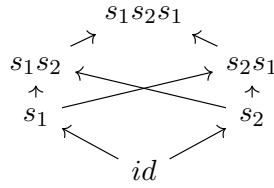
$$C'_s C'_t C'_s - C'_s = H_{sts} + v(H_{ts} + H_{st}) + v^2(H_s + H_t) + v^3H_{id}$$

since $vH_s^2 = H_s - v^2H_s + vH_{id}$ and so $vH_s^2 - C'_s = -v^2H_s$.

This illustrates the problem more generally we wish to solve. For every $w \in \mathcal{W}$, we want to associate an ι -invariant element, C_w , which is a linear combination of T_x for $x \leq w$, thus giving us a basis. In order to follow [Soe97], we will actually produce elements C'_w as a linear combination of H_x 's, but the idea remains the same. To do this, we first recall a partial ordering on the Weyl group.

3.2. Definition. For $u, v \in \mathcal{W}$, we say $u \leq v$ in the (strong) *Bruhat order* on \mathcal{W} if some substring of some reduced word for v is a reduced word for u .

3.3. Example. Let $\mathcal{W} = \mathfrak{S}_3 = \langle s_1, s_2 \rangle$. Then, the Bruhat order is given by the following poset.



3.4. Theorem. [Soe97, Theorem 2.1] *For each $w \in \mathcal{W}$, there exists a unique element $C'_w \in \mathcal{H}$ having the following properties:*

- (a) $\iota(C'_w) = C'_w$
- (b) $C'_w \in H_w + \sum_{x < w} v\mathbb{Z}[v]H_x$ where $x < w$ in the (strong) Bruhat order.

Then, one may wish to construct

3.5. Example. (a) From the above, we already see that if $s \in \mathcal{W}$ is a simple reflection, then it must be that

$$C'_s = H_s + vH_{id}$$

- (b) We can compute the basis for $\mathfrak{S}_3 = \langle s_1, s_2 \rangle$ by hand. We know that the simple reflections must be of the form.

$$\begin{aligned} C'_{s_1} &= H_{s_1} + vH_{id} \\ C'_{s_2} &= H_{s_2} + vH_{id} \end{aligned}$$

Then, to form ι -invariants of length 2, we check

$$C'_{s_1} C'_{s_2} = H_{s_1 s_2} + v(H_{s_1} + H_{s_2}) + v^2 H_{id}$$

is ι -invariant. If we apply ι to this, we get

$$\begin{aligned} \iota(C'_{s_1} C'_{s_2}) &= H_{s_1 s_2} + (v - v^{-1})(H_{s_1} + H_{s_2}) + (v - v^{-1})^2 + v^{-1}(H_{s_1} + H_{s_2} + 2(v - v^{-1})) + v^{-2} \\ &= H_{s_1 s_2} + v(H_{s_1} + H_{s_2}) + (v - v^{-1})^2 + 2(1 - v^{-2}) + v^{-2} \\ &= H_{s_1 s_2} + v(H_{s_1} + H_{s_2}) + v^2 \end{aligned}$$

So, by uniqueness, it must be $C'_{s_1 s_2} = C'_{s_1} C'_{s_2}$. A similar computation gives $C'_{s_2 s_1}$. For length 3, we already computed above that

$$C'_{s_1 s_2 s_1} = C'_{s_1} C'_{s_2} C'_{s_1} - C'_{s_1} = H_{s_1 s_2 s_1} + v(H_{s_1 s_2} + H_{s_2 s_1}) + v^2(H_{s_1} + H_{s_2}) + v^3$$

Proof of Theorem 3.4. We have already established the formula for C'_s for s a simple reflection. Now, we compute

$$H_x C'_s = \begin{cases} H_{xs} + v H_x & \text{if } xs > x; \\ H_{xs} + v^{-1} H_x & \text{if } xs < x \end{cases}$$

where the first case is immediate from the definition of the Hecke algebra and the second case is a straightforward application of Lemma 2.6. To show existence, we proceed by induction on the Bruhat order. Certainly, $C'_{id} = H_{id} = 1$. Now, let $x \in \mathcal{W}$ be given and suppose we know C'_y exists for all $y < x$. If $x \neq id$, we can find a simple reflection s such that $xs < x$ and by induction, we get

$$C'_{xs} C'_s = H_x + \sum_{y < x} h_y H_y$$

for some $h_y \in \mathbb{Z}[v]$. Then, we say

$$C'_x = C'_{xs} C'_s - \sum_{y < x} h_y(0) C'_y.$$

C'_x is ι -invariant because it is a sum of ι -invariant elements and it lies in $H_x + \sum_{y < x} v\mathbb{Z}[v] H_y$ since, if $C'_y = H_y + \sum_{z < y} h_{z,y} H_x$ for $h_{z,y} \in v\mathbb{Z}[v]$, then

$$C'_x = H_x + \sum_{y < x} \left((h_y - h_y(0)) H_y - \sum_{z < y} h_y(0) h_{z,y} H_z \right).$$

For uniqueness, we prove the following.

3.6. Lemma. *If $H \in \sum_y v\mathbb{Z}[v] H_y$ is ι -invariant, then $H = 0$.*

We have $H_z \in C'_z + \sum_{y < z} \mathbb{Z}[v, v^{-1}] C'_y$ for the C'_x elements described earlier in the proof by the unitriangularity condition. Now, if $H = \sum_y h_y H_y$ and we choose z maximal such that $h_z \neq 0$, then $\iota(H) = H$ implies that $\overline{h_z} = h_z$. However, this contradicts $h_z \in v\mathbb{Z}[v]$, so it must be that $H = 0$.

Thus, if there were two ι -invariant elements C'_w and D'_w satisfying the hypotheses of Theorem 3.4, then it must be that $C'_w - D'_w \in v\mathbb{Z}[v]$ is ι -invariant, but the lemma shows that $C'_w - D'_w = 0$. Thus, uniqueness is established. \square

3.7. Definition. For $x, y \in \mathcal{W}$, we define the *Kazhdan-Lusztig polynomials* $h_{y,x} \in \mathbb{Z}[v, v^{-1}]$ by the equality

$$C'_x = \sum_y h_{y,x} H_y$$

3.8. Remark. These polynomials are related to the Kazhdan-Lusztig polynomials in [KL79], denoted $P_{y,x}$, by

$$h_{y,x} = v^{\ell(x) - \ell(y)} P_{y,x}$$

3.9. Proposition. *Let \mathcal{W} be finite, $w_\circ \in \mathcal{W}$ be the longest element, and $r = \ell(w_\circ)$ its length. Then, we have $C'_{w_\circ} = \sum_{y \in \mathcal{W}} v^{r-\ell(y)} H_y$.*

3.1. Further Properties of Kazhdan-Lusztig Polynomials. Since their introduction, the Kazhdan-Lusztig polynomials have been an area of intense research. Now, much more is known than when they were first introduced.

3.10. Proposition. [KL80] *For any Weyl group \mathcal{W} and $x, y \in \mathcal{W}$, we have that the coefficients a_i occurring in*

$$P_{y,x}(q) = \sum_i a_i q^i$$

satisfy $a_i \in \mathbb{Z}_{\geq 0}$.

3.11. Remark. This has been proved by [EW14] for general Coxeter systems.

In [KL79], the following was conjectured. It was proven in [BB81] and [BK81].

3.12. Proposition. *Given a semisimple Lie algebra \mathfrak{g} with Weyl group \mathcal{W} , for each $w \in \mathcal{W}$, let M_w be the Verma module with highest weight $-w(\rho) - \rho$ and let L_w be its unique irreducible quotient. Then, we have the equivalent identities*

$$\begin{aligned} (a) \quad \text{ch } L_w &= \sum_{y \leq w} (-1)^{\ell(w)+\ell(y)} P_{y,w}(1) \text{ch } M_y \\ (b) \quad \text{ch } M_w &= \sum_{y \leq w} P_{w_\circ w, w_\circ y}(1) \text{ch } L_y \end{aligned}$$

where w_\circ is the longest element of \mathcal{W} .

Finally, there exists a geometric interpretation of the Kazhdan-Lusztig polynomials using perverse sheaves.

3.2. Historical Note. Kazhdan and Lusztig were originally interested in using the Kazhdan-Lusztig basis to construct representations of the Hecke algebra, but their significance has extended far beyond this goal. Our exposition here does not follow [KL79] and our definitions do not match those in [KL79], although it is straightforward to translate between [KL79] and these notes. The proof given for existence and uniqueness here is simpler; notably, this exposition does not include the R -polynomials. Such a proof can be found in [Hum90].

REFERENCES

- [BB81] A. Beilinson and J. Bernstein, *Localisation de \mathfrak{g} -modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18.
- [BK81] J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), no. 3, 387–410.
- [Bre03] F. Brenti, *Kazhdan-Lusztig polynomials: History, Problems, and Combinatorial Invariance*, Séminaire Lotharingien de Combinatoire **49** (2003).
- [EW14] B. Elias and G. Williamson, *The Hodge theory of Soergel bimodules* **180** (2014), 1089–136.

- [Hum90] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, 1990.
- [KL79] D. Kazhdan and G. Lusztig, *Representation of Coxeter Groups and Hecke Algebras*, *Inventiones math.* **53** (1979), 165–184.
- [KL80] ———, *Schubert varieties and Poincaré duality*, *Geometry of the Laplace operator* (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979) (1980), 185–203.
- [Soe97] W. Soergel, *Kazhdan-Lusztig polynomials and a combinatoric for Tilting modules*, *Representation Theory* **1** (1997), 83–114.
- [Wil03] G. Williamson, *Mind your P and Q -symbols: Why the Kazhdan-Lusztig basis of the Hecke algebra of type A is cellular*, 2003.