Representation Theory of Finite Groups
Notes (loosely) inspired by a class taught by Brian Parshall in Fall 2017

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1. Introduction

From one perspective, the representation theory of finite groups is merely a special case of the representation theory of associative algebras by considering the fact that a sufficiently nice finite group ring $\Bbbk[G]$ is semisimple, and thus the results from Artin-Wedderburn theory apply. However, the extra structure of the group provides a rich connection between $\Bbbk[G]$-modules and linear algebra.

A representation of a finite group is a way to induce an action of a finite group on a vector space. Sometimes, such a perspective allows mathematicians to see structure or symmetries in groups that may not have been readily apparent from a purely group theoretic point of view, much like modules can provide insight into the structure of rings. In fact, a reprsentation of a finite group is fundamentally the same as a module of a finite group algebra, but the representation theoretic perspective allows us to leverage tools from linear algebra to gain additional insights.

One of the most oft-cited examples of representation theory in action is Burnside's pq-theorem (see 11), but representations are also often studied in their own right. In general, representation theories seem to follow a general program along the following lines.

(a) To what degree do representations decompose into direct sums of subrepresentations? The best answer is when all representations are completely reducible, that is, a representation breaks up into a direct sum of the irreducible subrepresentations. This happens for finite group representations over $\mathbb{C}$.
(b) Can we determine the irreducible and indecomposable representations of a given object? For finite group representations over $\mathbb{C}$, the irreducible representations are completely determined by the group's "character table."
(c) Given a tensor product of representations, which irreducible and/or indecomposable representations does the product contain?

These are the questions this monograph will try to answer for representations of finite groups over $\mathbb{C}$. Over fields where the characteristic of the field divides the order of the group, different techniques are often used in an area of math known as "modular representation theory," and such a situation will not be discussed much here.

While the classic texts in representation theory continue to be cited and used often, and contain many details ommitted here, I am eternally grateful for [**EGH$^+$11**] and [**Smi10**] for being approachable, providing simplifying insights, and using more modern language to rephrase classical results. I am also grateful for [**Tel05**], out of which I first learned the representation theory of finite groups, even if I did not understand a lot of it at the time. Much of the writings here are shamelessly borrowed and synthesized from these sources.

2. Group Actions

The following exposition is a summary of [**Gro**]. The reader is probably familiar with the notion of a group action.

2.1. DEFINITION. A *group action* on a set X is a homomorphism $\phi\colon G \to \mathrm{Aut}(X)$ such that $\phi(e) = Id_X$.

If the reader is unfamiliar with group actions, most introductory texts on group theory will provide more than adequate treatment.

2.2. DEFINITION. A group action is called *faithful* if ϕ is injective. A group action is called *transitive* if $\phi(G)$ induces only one orbit on X.

When an action is not transitive, we have multiple orbits and we can consider the group action on each orbit separately. Thus, in a sense, we can "decompose" this group action (on the level of sets). Thus, let us focus on transitive actions as our basic building blocks of group actions.

2.3. THEOREM. *Let ϕ be a transitive group action of G on a set X. Then, consider $H = \mathrm{Stab}_G(x)$. Then, $G/H \cong X$ as sets with a (left) G-action.*

PROOF. Given $H = \mathrm{Stab}_G(x)$, consider the correspondence

$$gH \leftrightarrow g.x = \phi(g)x$$

Then, the action of G preserves this correspondence, namely

$$g'.gH = (g'g)H \leftrightarrow (g'g).x = \phi(g'g)x = \phi(g')\phi(g)x = g'.\phi(g)x = g'.(g.x)$$

$\square$

Thus, we have reduced the classification of transitive G-actions on X to the study of conjugacy classes of subgroups of G. Thus, the structure of a group that controls a group action on a set X is the subgroup structure of G. However, understanding the subgroup structure of a group is, in general, incredibly difficult. For instance, recall Cayley's theorem.

2.4. THEOREM (Cayley's Theorem). *Any finite group G with $|G| = n < \infty$ can be embedded into $\mathfrak{S}_n$ via the action of G on itself.*

Thus, to understand the subgroups of $\mathfrak{S}_n$, one must understand all finite groups G with order less than n. Thus, while group actions are incredibly useful and general, they lack structure that can be otherwise useful.

3. Group Algebras

3.1. DEFINITION. The *group ring* or *group algebra* of a group G over a ring R, denoted $R[G]$ is a free R-module over the set

$$\{e_g \mid g \in G\}$$

with addition being formal sums. Additionally, $R[G]$ has the structure of a ring where multiplication is given by $e_g \cdot e_h = e_{gh}$ for $g, h \in G$. Thus, $1_{R[G]} = e_{1_G}$

Typically, we will take R to be a field, usually $\mathbb{C}$. Furthermore, we will often replace e_g by g when it is understood that we are working in the group algebra. That is, for $\lambda \in R$, we may rewrite

$$\lambda e_g \to \lambda g$$

Since group algebras are R-algebras, we can ask many mathematical questions about their modules.

3.2. Theorem (Maschke's Theorem). *Let G be a finite group and $\Bbbk$ be a field such that $\operatorname{char} \Bbbk \nmid |G|$. Then, $\Bbbk[G]$ is completely reducible as a module over itself.*

Proof. Let V be a proper $\Bbbk[G]$-submodule of $\Bbbk[G]$. Consider the $\Bbbk$-linear map map $\pi \colon \Bbbk[G] \to B$ and let $\phi \colon \Bbbk[G] \to V$ be given by

$$\phi(x) = \frac{1}{|G|} \sum_{s \in G} s.\pi(s^{-1}.x)$$

Thus, ϕ is also a projection since, for $v \in V$,

$$\phi(v) = \frac{1}{|G|} \sum_{s \in G} s.\pi(s^{-1}.v) = \frac{1}{|G|} \sum_{s \in G} s.s^{-1}.v = v$$

and it is $\Bbbk$-linear since it is simply the (appropriately scaled) sum of $\Bbbk$ linear maps. In fact,

$$\begin{aligned} \phi(t.x) &= \frac{1}{|G|} \sum_{s \in G} s.\pi(s^{-1}.t.x) \\ &= \frac{1}{|G|} \sum_{s' \in G} (ts') \cdot \pi(s'^{-1}.x) \qquad (s'^{-1} = s^{-1}t \implies s = ts') \\ &= t.\phi(x) \end{aligned}$$

so, ϕ is $\Bbbk[G]$-linear. Thus, $\Bbbk[G] \cong V \oplus \ker \phi$, but V was an arbitrary proper submodule. Thus, $\Bbbk[G]$ is completely reducible. □

3.3. Remark. The converse of Maschke's theorem is also true.

Thus, since $\Bbbk[G]$ is finite-dimensional and thus also artinian, $\Bbbk[G]$ is a semisimple ring when $\operatorname{char} \Bbbk \nmid |G|$ and, using artin-wedderburn theory, has decomposition as an module over itself

$$\Bbbk[G] \cong \bigoplus_{i=1}^{k} M_{n_i}(D_i)$$

for $n_i \in \mathbb{N}$ and D_i a division ring, and these direct summands represent all the irreducible $\Bbbk[G]$-modules. However, we still wish to understand what these division rings and n_i's actually are (see [**See17**]). Thus, we will shift our perspective to looking at *group representations* instead of group algebra modules.

4. Representations

In this section, we seek to establish some of the basic ideas of representations of finite groups. Much of this exposition is borrowed from [**EGH$^+$11**] where the results are presented more generally for R-algebras. Also, many of these results can be rephrased using the language in [**See17**].

4.1. Definition. A *representation of a finite group G in a (complex) vector space V* is a homomorphism $\rho\colon G \to GL(V)$.

4.2. Remark. Throughout this monograph, we will often simply say "representation" to mean a representation of a finite group. When we refer to a vector space V, we will mean a vector space over $\mathbb{C}$.

4.3. Theorem. *There is a natural bijection between $\mathbb{C}[G]$-modules and complex representations of G.*

Proof. Given a representation of G given by $\rho\colon G \to GL(V)$, we get a $\mathbb{C}[G]$ action on $v \in V$ via

$$e_g.v = \rho(g)v$$

and extend via linearity. Conversely, given M a $\mathbb{C}[G]$-module, we can consider M as a vector space over $\mathbb{C}e_1$ and then view the action of each e_g as an invertible linear transformation on this vector space. $\square$

4.4. Definition. We say two representations $\rho, \rho'\colon G \to GL(V)$ are *similar* or *isomorphic* if there is a linear transformation $\tau\colon V \to V'$ such that

$$\tau \circ \rho(g) = \rho'(g) \circ \tau$$

for all $g \in G$.

4.5. Definition. We say that a subspace $W \subseteq V$ is *G-stable* if $\rho(g)W \subseteq W$ for all $g \in G$.

4.6. Proposition. *Let $\rho\colon G \to GL(V)$ be a linear representation of G in V and let W be a subspace of V such that $\rho(g)W = W$ for all $g \in G$. Then, there exists a complement W' of W in V such that $\rho(g)W' \subseteq W'$ for all $g \in G$*

Proof. Let W' be a vector space complement of W in V (not necessarily G-stable) and let $p\colon V \to W$ be the standard projection. Then, consider the map

$$p^0 := \frac{1}{|G|}\sum_{t\in G} \rho(t) \circ p \circ \rho(t)^{-1}$$

Such a map is a projection of V onto a subspace of W. In fact, $p^0|_W = Id_W$ since, for $w \in W$,

$$p \circ \rho(t)^{-1} w = \rho(t)^{-1} w \Longrightarrow \rho(t) \circ p \circ \rho(t)^{-1} w = w \Longrightarrow p^0 w = w$$

Thus, we seek to show $W^0 := \ker p^0$ is stable under the G action via ρ. Indeed, it is easy to check $\rho(s)p^0\rho(s)^{-1} = p^0$ and thus $p^0 \circ \rho(s)x = \rho(s) \circ p^0 x = 0$, that is, $\rho(s)x \in W^0$. Thus, $V = W \oplus W^0$ is a decomposition into G-stable subspaces. $\square$

4.7. Remark. Note the similarity between this proof and the proof of Maschke's theorem above (3.2). Indeed, these proofs are more or less equivalent and the proposition above is the same as Maschke's theorem for $\Bbbk = \mathbb{C}$.

4.8. Definition. Given representations $\rho_1 \colon G \to GL(V_1)$ and $\rho_2 \colon G \to GL(V_2)$, we define $\rho := \rho_1 \oplus \rho_2 \colon G \to GL(V_1 \oplus V_2)$ by the map

$$g \mapsto \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

That is, $\rho(g)|_{V_i} = \rho_i(g)$.

4.9. Definition. We say that a representation $\rho \colon G \to GL(V)$ is *irreducible* or *simple* if $V \neq 0$ and there is no subspace $W \subseteq V$ such that W is stable under the action of G via ρ. By the proposition above, this is equivalent to saying that ρ does not break into a direct sum of representations.

4.10. Proposition. *Every representation is a direct sum of irreducible representations.*

Proof. The proof follows from induction on $\dim V$ and application of the proposition above. $\square$

4.11. Definition. Let $\rho_1 \colon G \to GL(V_1)$ and $\rho_2 \colon G \to GL(V_2)$ be representations. Then, we define $\rho := \rho_1 \otimes \rho_2 \colon G \to GL(V_1 \otimes V_2)$ by, for $s \in G$,

$$\rho(s)(v_1 \otimes v_2) = \rho_1(s)v_1 \otimes \rho_2(s)v_2$$

4.12. Remark. The tensor product of two irreducible representations is not typically irreducible.

4.13. Theorem. *Let $\rho \colon G \to GL(V)$ be a representation. Then, given $\rho \otimes \rho \to GL(V \otimes V)$ decomposes as*

$$V \otimes V \cong \frac{V \otimes V}{(x \otimes y - y \otimes x)} \oplus \frac{V \otimes V}{(x \otimes y + y \otimes x)} = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$$

Proof. Consider the automorphism of $V \otimes V$ given by $\tau(e_i \otimes e_j) = e_j \otimes e_i$ for basis $\{e_1, \ldots, e_n\}$ of V. Then, $\tau(v \otimes w) = w \otimes v$ for any $v, w \in V$ and $\tau^2 = Id_{V \otimes V}$. Thus, as vector spaces,

$$V \otimes V \cong \ker(\tau - Id) \oplus \operatorname{im}(\tau - Id)$$

However, $\ker(\tau - Id) = \langle x \otimes y \mid x \otimes y - y \otimes x = 0 \rangle \cong (V \otimes V)/(x \otimes y - y \otimes x)$. Now, consider that

$$(\tau - Id)(x \otimes y) + \tau(\tau - Id)(x \otimes y) = (\tau - Id)(x \otimes y) + (Id - \tau)(x \otimes y) = 0$$

and so, for any $v \in \operatorname{im}(\tau - Id)$, we get $v + \tau(v) = 0$. Thus, $\operatorname{im}(\tau - Id) \subseteq (V \otimes V)/(x \otimes y + y \otimes x)$. However, by dimension counting, we note that

$$\dim \ker(\tau - Id) = \dim \operatorname{Sym}^2(V) = \frac{n(n+1)}{2} \implies$$
$$\dim \operatorname{im}(\tau - Id) = n - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} = \dim \operatorname{Alt}^2(V)$$

Thus, $\operatorname{im}(\tau - Id) = \operatorname{Alt}^2(V)$.

However, we must also check that these spaces are stable under G. However, for $\rho' = \rho \otimes \rho$,

$$\begin{aligned} \rho'(g) \circ \tau \circ \rho'(g^{-1})(v \otimes w) &= \rho'(g) \circ \tau(\rho^{-1}(g)v \otimes \rho^{-1}(g)w) \\ &= \rho'(g)(\rho^{-1}(g)w \otimes \rho^{-1}(g)v) \\ &= w \otimes v \\ &= \tau(v \otimes w) \end{aligned}$$

and so the image and kernel of $\tau - Id$ is stable under the action of G. □

We now also seek to prove some other useful tools in representation theory.

4.14. Theorem (Schur's Lemma). *Let V_1, V_2 be representations of G and let $\phi \colon V_1 \to V_2$ be a nontrivial homomorphism of representations. Then,*

(a) If V_1 is irreducible, then ϕ is injective.
(b) If V_2 is irreducible, then ϕ is surjective.

Proof. Exercise for the reader. □

4.15. Corollary (Schur's Lemma for algebraically closed fields). *Let V be a finite dimensional irreducible representation of a group G over an algebraically closed field $\Bbbk$, and $\phi \colon V \to V$ a commuting homomorphism (ie $\rho(g) \circ \phi = \phi \circ \rho(g)$). Then, $\phi = \lambda Id_V$ for some $\lambda \in \Bbbk$.*

4.16. Remark. (a) We sometimes call ϕ a "scalar operator" or say that ϕ "acts as a scalar" in this situation. In [**Ser97**], Serre calls such an operator a "homothey."
(b) This proposition is false over $\mathbb{R}$.

Proof. Let λ be an eigenvalue of ϕ, which exists since $\Bbbk$ is algebraically closed. Then, since ϕ commutes with $\rho(g)$, so does $\phi - \lambda Id \colon V \to V$. However, $\phi - \lambda Id$ is not an isomorphism since $\det(\phi - \lambda Id) = 0$. Thus, $\phi - \lambda Id = 0 \implies \phi = \lambda Id$. □

4.17. Corollary. *Let G be an abelian group. Then, every irreducible finite dimensional representation of G is 1-dimensional.*

Proof. Let V be an irreducible finite dimensional representation of G. Then, for $g, h \in G$, $v \in V$

$$\rho(g)\rho(h)v = \rho(gh)v = \rho(hg)v = \rho(h)\rho(g)v$$

Thus, $\rho(g)$ commutes with all $\rho(h)$ and so, by the corollary above, $\rho(g) = \lambda Id$ for some $\lambda \in \mathbb{C}$. Thus, every vector subspace of V is a subrepresentation, but V is irreducible. Thus, $\dim V = 1$. □

4.18. Example. Let $G = \mathfrak{S}_3$, the smallest order non-abelian group, and let V be a complex representation of G. Since $\mathfrak{S}_3$ is generated by $\sigma = (123)$ and $\tau = (12)$, we need only find subrepresentations of V that are stable under the actions of σ and τ. An important tool for this process will be to find eigenvectors. Let v be an eigenvector for σ with eigenvalue λ. Then, consider τv. Such a vector must be an eigenvector for σ as well since

$$\sigma\tau v = \tau\sigma^2 v = \tau\lambda^2 v = \lambda^2\tau v$$

Thus, the space $\operatorname{span}\{v, \tau v\}$ is stable under the actions of σ, τ and must be a subrepresentation of V.

Now, if we wish to find all irreducible representations of G, we know that the dimension must be less than or equal to 2. However, we can figure out more. We know that $\sigma^3 = Id$, and so it must be that $v = \sigma^3 v = \lambda^3 v$ and so $\lambda = 1, \zeta_3$, or ζ_3^2, where ζ_3 is a primitive 3rd root of unity.

- Let $\lambda \neq 1$. Then, $\lambda \neq \lambda^2$ and so v and τv have distinct eigenvalues and thus they are linearly independent. Thus, $\operatorname{span}\{v, \tau v\}$ defines an irreducible, 2-dimensional representation. This is called the *standard representation.* Explicitly, it is given by

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma \mapsto \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix}$$

- Let $\lambda = 1$ and $\tau v = -v$. Then, $\operatorname{span}\{v\}$ is an irreducible 1-dimensional representation, but it is distinct from the trivial representation. It is called the *alternating representation.*
- Let $\lambda = 1$ and $\tau v = v$. This is the trivial representation.

Thus, we have found 3 irreducible representations of $\mathfrak{S}_3$.

4.19. Definition. We say a representation $\rho\colon G \to GL(V)$ is *faithful* if $\ker\rho = \{1\}$.

4.20. Proposition. *Given a representation $\rho\colon G \to GL(V)$ and a normal subgroup $H \trianglelefteq G$ such that ρ acts trivially on H, (that is, $\rho(H) = \{Id_V\}$), ρ can descend to a representation of G/H, say $\overline{\rho}$, via*

$$\overline{\rho}(gH) = \rho(g)$$

PROOF. We must check that such an action is well defined. Let $g, g' \in gH$. Then, $g = hg'$ so

$$\overline{\rho}(gH) = \rho(g) = \rho(hg') = \rho(h)\rho(g') = \rho(g') = \overline{\rho}(g'H)$$

$\square$

4.21. PROPOSITION. *Given representation $\rho\colon G \to GL(V)$, we have that $\overline{\rho}\colon G/\ker\rho \to GL(V)$ is faithful.*

4.22. PROPOSITION. *Let $\rho\colon G/H \to GL(V)$ be a representation of G/H for $H \trianglelefteq G$. Then, we can lift ρ to a representation of G, say $\tilde{\rho}$, via*

$$\tilde{\rho}(g) = \rho(gH)$$

Furthermore, if ρ is a faitful representation of G/H, then $\ker\tilde{\rho} = H$.

PROOF. This construction is immediately well defined. If ρ is faithful, then we note that

$$\tilde{\rho}(g) = Id \Longleftrightarrow \rho(gH) = Id \Longleftrightarrow gH = H$$

$\square$

5. The Regular Representation

One of the most important representations is the regular representation. Later, we will see that every irreducible representation appears in the regular representation. Furthermore, the regular representation is a somewhat natural representation to define.

5.1. DEFINITION. Let G be a finite group. Then, the *regular representation* of G is a homomorphism $\rho_{reg}\colon G \to GL(V)$ where $V = \mathbb{C}^{|G|}$ given by

$$g \mapsto (g\colon V \to V, e_h \mapsto e_{gh})$$

Note that the regular representation is equivalent to viewing $\Bbbk[G]$ as a module over itself.

5.2. EXAMPLE. Consider $G = \mathfrak{S}_3$. Then, the regular representation of $\mathfrak{S}_3$ is given by

$$\sigma.e_{\sigma'} = e_{\sigma\sigma'}$$

Now, consider that the element

$$e_{(1)} + e_{(12)} + e_{(13)} + e_{(23)} + e_{(123)} + e_{(321)}$$

is a G-invariant subspace of $\mathbb{C}^6$ under the regular representation. This subrepresentation is isomorphic to the trivial represenation. Similarly, the space spanned by

$$e_{(1)} - e_{(12)} - e_{(13)} - e_{(23)} + e_{(123)} + e_{(321)}$$

is G-stable and, as a subrepresentation of the regular representation, is isomorphic to the alternating representation. Thus, we have

$$\rho_{reg} = \rho_{trivial} \oplus \rho_{alt} \oplus \rho'$$

where $\rho' \colon G \to GL(\mathbb{C}^6/(V_{trivial} \oplus V_{alt}))$. We know from our analysis above that ρ' must break up further, and we will see later that it breaks up as the direct sum of 2 standard representations.

6. Hom and dual representation

We take a quick detour to view some other methods for constructing new representations from old ones. These results will be useful later, so the reader may choose to skip them for now and come back later when they are needed.

6.1. PROPOSITION. *Let V, W be representations of a finite group G. Then, the action of G on the vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ given by, for $\phi \in \mathrm{Hom}_{\mathbb{C}}(V, W)$,*

$$\begin{aligned} g.\phi \colon V &\to W \\ v &\mapsto g.\phi(g^{-1}.v) \end{aligned}$$

makes $\mathrm{Hom}_{\mathbb{C}}(V, W)$ into a representation of G.

6.2. COROLLARY. *Let V be a representation of a finite group G. Then, $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ has a natural representation structure. Explicitly, for basis $f \in V^*$,*

$$\rho_{V^*}(g).f = f \circ \rho_V(g^{-1})$$

6.3. PROPOSITION. *Given finite-dimensional representations V, W of G, then there is a natural isomorphism*

$$V^* \otimes W \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V, W)$$

PROOF. Consider the $\mathbb{C}$-bilinear map

$$\begin{aligned} V^* \times W &\to \mathrm{Hom}_{\mathbb{C}}(V, W) \\ (f(\cdot), w) &\mapsto f(\cdot)w \end{aligned}$$

Then, by the universal property of tensor products, there is a unique homomorphism $\phi \colon V^* \otimes W \to \mathrm{Hom}_{\mathbb{C}}(V, W)$ such that $f(\cdot) \otimes w \mapsto f(\cdot)w$. Such a map is certainly injective since $\ker \phi = \{0\}$. It is surjective since $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is isomorphic to $\dim W \times \dim V$ matrices, so $\dim \mathrm{Hom}_{\mathbb{C}}(V, W) = \dim V \cdot \dim W = \dim V \otimes W = \dim V^* \otimes W$. $\square$

6.4. REMARK. The construction of the Hom representation more generally comes from the *Hopf algebra* structure on $\mathbb{C}[G]$.

7. Character Theory

Throughout this section, let G be a finite group and $F = \mathbb{C}$. We wish to extend the notion of the central character above to a general character $\chi_V \colon G \to \mathbb{C}$ that will still carry useful information about the group representation V.

7.1. PROPOSITION. *If V is a G-module, then every $g \in G$, viewed as a linear operator $g_V : V \to V$, is semisimple, and thus V has a basis of eigenvectors for g_V.*

PROOF. Since G is a finite group, g is of finite order, so $g_V^n = Id_V$. Thus, $g_V^n = (D+N)^n$ for diagonalizable D and nilpotent N (by Jordan Canonical Form since $\mathbb{C}$ is algebraically closed) such that $DN = ND$. Thus,

$$Id_V = g_V^n = (D+N)^n = D^n + nND^{n-1} + \binom{n}{2} N^2 D^{n-2} + \cdots + nN^{n-1}D + N^n$$

which thus tells us that $N = 0$ and $D^n = Id_V$ since Id_V has no nilpotent part. □

In general, it is quite annoying to compute the eigenvalues for each $g \in G$. However, we will see that it suffices to merely compute their sum using the trace.

7.2. DEFINITION. The *character* of group representation $\rho : G \to V$, denoted $\chi_\rho = \chi^\rho$, is the function $\chi_\rho : G \to \Bbbk$ defined by $\chi_\rho(g) = \operatorname{tr}(\rho(g))$.

7.3. PROPOSITION. *For $\Bbbk = \mathbb{C}$ and fixed group representation ρ,*

(a) $\chi(1) = \dim V$,
(b) $\chi(g^{-1}) = \overline{\chi(g)}$ *for all* $g \in G$,
(c) $\chi(tst^{-1}) = \chi(s)$ *for all* $s, t \in G$.

PROOF. All these properties follow from standard properties of trace. □

7.4. PROPOSITION. *Let V be a group representation and let χ_V be its character. Then,*

$$\overline{\chi_V(g)} = \chi_{V^*}(g)$$

PROOF. Consider that, by 6.2, $\chi_{V^*}(g) = \chi_V(g^{-1})$. However, since $\chi_V(g)$ is just the sum of eigenvalues of g (which must be roots of unity by the proof of 7.1), we have that

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi_V(g)}$$

□

7.5. DEFINITION. A function $f : G \to \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes. Note that $\operatorname{Class}(G)$ is the $\mathbb{C}$-algebra of all class functions on G and $\dim \operatorname{Class}(G) =$ the number of conjugacy classes in G.

7.6. PROPOSITION. *Given a representation of a finite group G and a representation ρ, $\chi_\rho : G \to \mathbb{C}$ is a class function.*

PROOF. This follows from part (c) of the previous proposition. □

7.7. Proposition. *Let $\rho_i \colon G \to GL(V_i)$, $i = 1, 2$ be representations of G with characters χ_{ρ_i}.*

(a) $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$
(b) $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \cdot \chi_{\rho_2}$

Proof. This follows from the definition of direct sum and tensor product of representations and the definition of the character. $\square$

Based on the identity (a) above, we see that the characters of irreducible representations will be of primary interest.

8. Orthogonality Relations of Irreducible Characters

Inspired by the proof of Maschke's theorem, we present the following proposition.

8.1. Proposition. *Let $h \colon V_1 \to V_2$ be a linear transformation and let $\rho^1 \colon G \to GL(V_1), \rho^2 \colon G \to GL(V_2)$ be irreducible representations of finite group G. Consider*

$$h^0 := \frac{1}{|G|} \sum_{t \in G} \rho^2(t^{-1}) h \rho^1(t)$$

h^0 is a homomorphism of representations. Furthermore

(a) If $\rho^1 \ncong \rho^2$, then $h^0 = 0$.
(b) If $\rho^1 \cong \rho^2$ (and thus $V_1 \cong V_2$), then $h^0 = \frac{1}{\dim V_1} \operatorname{tr}(h) Id_{V_1}$.

Proof. $\square$ Fill this in

8.2. Corollary. *Consider that $\rho(t) \in GL(V)$ and thus we can represent $\rho(t)$ as a $n \times n$ matrix, say $\rho^1(t) = [r_{i_1, j_1}(t)]$ and $\rho^2(t) = [r_{i_2, j_2}(t)]$. Then,*

(a) If $\rho^1 \cong \rho^2$,

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2, j_2}(t^{-1}) r_{j_1, i_1}(t) = \frac{1}{n} \delta_{j_1, j_2} \delta_{i_1, i_2}$$

(b) If $\rho^1 \ncong \rho^2$,

$$\frac{1}{|G|} \sum_{t \in G} r_{i_2, j_2}(t^{-1}) r_{j_1, i_1}(t) = 0$$

This provides us motivation for definiting a positive definite Hermitian inner product on the set of class functions on G.

8.3. Definition. For class functions $\phi, \psi \colon G \to \mathbb{C}$, we define inner product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g^{-1}) \psi(g)$$

and also

$$(\phi, \psi) = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}$$

8.4. Proposition. *Let* $\langle \cdot, \cdot \rangle$ *and* $(\cdot, \cdot)$ *be as above.*

(a) $\langle \cdot, \cdot \rangle$ *is a bilinear form.*
(b) $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$*, that is* $\langle \cdot, \cdot \rangle$ *is symmetric.*
(c) $(\phi, \psi) = \langle \phi, \overline{\psi \circ i} \rangle$ *where* $i \colon G \to G$ *takes* $i(g) = g^{-1}$.
(d) $(\cdot, \cdot)$ *is a positive definite Hermitian inner form.*

8.5. Proposition. *If* ρ *is a representation of* G *in* V *and if* $\psi(t) = \operatorname{tr} \rho(t)$*, then*

$$\langle \phi, \psi \rangle = (\phi, \psi)$$

Thus, it is with the form $(\cdot, \cdot)$ that we can arise at the following

8.6. Theorem. *For any representations* V, W,

$$(\chi_V, \chi_W) = \dim \operatorname{Hom}_G(W, V)$$

and, if V, W *are irreducible,*

$$(\chi_V, \chi_W) = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}$$

Proof. We first compute for arbitrary representations V, W,

$$\begin{aligned} (\chi_V, \chi_W) &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_{W^*}(g) && \text{by 7.4} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{V \otimes W^*}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{Hom}_{\mathbb{C}}(W,V)}(g) && \text{by 6.3} \\ &= \operatorname{tr}_{\operatorname{Hom}_{\mathbb{C}}(W,V)} \left(\frac{1}{|G|} \sum_{g \in G} g \right) \end{aligned}$$

However, $P := \frac{1}{|G|} \sum_{g \in G} g$ is a homomorphism of G representations (that is, $P.v$ is a representation of G) and the image of P is stable under any action of G since $g.Pv = Pv$. However, the only such irreducible representation is the trivial representation or the zero representation. Thus, for arbitrary representation U, $\operatorname{tr}_U(P)$ simply counts the number of times the trivial representation occurs as a subrepresentation of U. However, considering $\mathrm{Hom}_{\mathbb{C}}(W, V)$ as a G-representation, the G-invariant subrepresentation is exactly $\mathrm{Hom}_G(W, V)$. Thus,

$$(\chi_V, \chi_W) = \mathrm{tr}_{\mathrm{Hom}_{\mathbb{C}}(W,V)}\left(\frac{1}{|G|}\sum_{g\in G} g\right) = \dim \mathrm{Hom}_G(W, V)$$

Furthermore, if V, W are irreducible, Schur's lemma tells us that

$$\dim \mathrm{Hom}_G(W, V) = \begin{cases} 1 & V \cong W \\ 0 & \text{else} \end{cases}$$

$\square$

Thus, we have shown that the irreducible characters of a finite group are orthogonal to each other under $(\cdot, \cdot)$. In fact, we have the following theorem

8.7. Theorem. *The irreducible characters of G, $\{\chi_1, \ldots, \chi_k\}$ form an orthonormal basis for the set of class functions on G.*

8.8. Lemma. *Let $\rho\colon G \to GL(V)$ with $\dim V = n$ be an irreducible representation of G and let f be a class function on G. Then, given*

$$\rho_f\colon V \to V, \rho_f := \sum_{g\in G} f(g)\rho(g)$$

we have $\rho_f = \frac{1}{n}|G|(f, \overline{\chi_\rho})Id_V$.

Proof of Lemma. Let $h \in G$. Then,

$$\begin{aligned}
\rho_f\rho(h) &= \sum_{g\in G} f(g)\rho(g)\rho(h) \\
&= \sum_{g\in G} f(g)\rho(gh) \\
&= \sum_{g\in G} \rho(h)\rho(h^{-1})f(g)\rho(gh) \\
&= \rho(h)\sum_{g\in G} f(g)\rho(h^{-1}gh) \\
&= \rho(h)\sum_{g\in G} f(h^{-1}gh)\rho(h^{-1}gh) \\
&= \rho(h)\rho_f
\end{aligned}$$

Since V is irreducible, it must be that $\rho_f = \lambda 1_V$ by Schur's lemma. Thus, taking the trace of both sides, we get

$$n\lambda = \mathrm{tr}(\rho_f) = \sum_{g\in G} f(g)\chi_\rho(g)$$

and thus

$$\lambda = \frac{1}{n}|G|(f, \overline{\chi})$$

$\square$

Proof of Theorem. By the theorem above, we know that the irreducible characters form an orthonormal system. Thus, we need only show that they generate Class(G). It will suffice to show that if $f \in \text{Class}(G)$ has $(f, \overline{\chi_i}) = 0$ for all irreducible characters χ_i, then $f \equiv 0$. Assume $(f, \overline{\chi_i}) = 0$. Then, by the lemma above, $\rho_f = 0$ for any irreducible representation ρ. Thus, from the direct sum decomposition, we can conlude that $\rho_f = 0$ for any ρ. So, applying this to the regular representation, ρ_{reg}, and computing the image of a basis vector e_1, we get

$$0 = (\rho_{reg})_f e_1 = \sum_{g \in G} f(g)\rho_{reg}(g)e_1 = \sum_{g \in G} f(g)e_g \Longrightarrow f(g) = 0, \forall g \in G.$$

Thus, $f = 0$. $\square$

8.9. Corollary. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

8.10. Remark. We also know this from Wedderburn's theorem + Maschke's theorem.

Proof. Let $g \in G$ and $f_g \in \text{Class}(G)$ be defined by

$$f_g(h) := \begin{cases} 1 & \text{if } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases}$$

These f_g functions form a natural basis for Class(G) when one is chosen for each conjugacy class in G. Thus, $\dim \text{Class}(G)$ is the number of conjugacy classes of G. On the other hand, using the theorem above, $\dim \text{Class}(G)$ is equal to the number of irreducible representations of G. Thus, these two quantities are equal. $\square$

8.11. Theorem. *Let $g \in G$ and $C(g)$ be the number of elements in the conjugacy class of g. Then,*

(a) $\sum_{i=1}^{d} \overline{\chi_i}(g)\chi_i(g) = \frac{|G|}{C(g)}$
(b) $\sum_{i=1}^{d} \overline{\chi_i}(g)\chi_i(h) = 0$ *if g is not conjugate to h.*

8.12. Remark. In particular, $\sum_{i=1}^{d} \overline{\chi_i}(1)\chi_i(1) = |G|$.

Proof. Using $f_g \in \text{Class}(G)$ as above, write

$$f_g = \sum_{i=1}^{k} \lambda_i \chi_i, \lambda_i \in \mathbb{C}$$

Then, $\lambda_i = (f_g, \chi_i) = \frac{C(g)}{|G|}\overline{\chi_i}(g)$ where $C(g)$ is the number of elements in the conjugacy class of g. Therefore,

$$f_g(h) = \frac{C(g)}{|G|}\sum_{i=1}^{k}\overline{\chi_i(g)}\chi_i(h)$$

Thus, by definition of $f_g(h)$, the identities in the theorem are proved. $\square$

8.13. EXAMPLE. Let $G = \mathfrak{S}_3$. We know that $\mathfrak{S}_3$ has 3 conjugacy classes given by cycle type. Thus, $\mathfrak{S}_3$ has 3 distinct irreducible characters, denoted χ_1, χ_2, and θ for trivial, alternating, and standard representations. We then get the following "character table"

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
θ	2	0	-1

where the first 2 rows follow from just computing the trace of the trivial and alternating representations. The third row then follows from our orthogonality relations, since

$$\begin{aligned}
\theta(1) &= 2\\
0 = (\chi_1, \theta) &= \frac{1}{6}\sum_{g\in\mathfrak{S}_3}\overline{\theta(g)}\\
0 = (\chi_2, \theta) &= \frac{1}{6}\sum_{g\in\mathfrak{S}_3}\mathrm{sgn}(g)\overline{\theta(g)}\\
\implies 0 &= \frac{1}{3}\left(2 + \theta((123)) + \theta((321))\right)\\
\implies \theta((123)) &= -1
\end{aligned}$$

and

$$0 = \chi_1((1))\chi_1((12)) + \chi_2((1))\chi_2((12)) + \theta((1))\theta((12)) = 1 - 1 + 2\theta((12)) \implies \theta((12)) = 0$$

Finally, returning to the regular representation of $\mathfrak{S}_3$, we know that $\chi_{reg}(1) = 6$ and $\chi_{reg} = a\chi_1 + b\chi_2 + c\theta$. However, we also know that $\chi_{reg}(g) = 0$ for all non-identity $g \in \mathfrak{S}_3$ since no such elements fix any basis elements. In particular, $\chi_{reg}((123)) = 0$. Thus, the unique decomposition of χ_{reg} is

$$\chi_{reg} = \chi_1 + \chi_2 + 2\theta$$

From this, we will see that

$$\rho_{reg} = \rho_{trivial} \oplus \rho_{alt} \oplus \rho_{std} \oplus \rho_{std} \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$$

9. Results of Character Theory

Now, we wish to use character theory to tell us more about representations of a group.

9.1. Proposition. *Let $H \trianglelefteq G$ and let $\rho\colon G/H \to GL(V)$ be a representation. Then, ρ is irreducible if and only if its lifted representation to G, $\tilde{\rho}$, is irreducible. (See 4.22 for a reminder of how a lifted representation is constructed.)*

Proof. We first note that

$$\operatorname{tr}(\tilde{\rho}(g)) = \operatorname{tr}(\rho(gN))$$

by definition of $\tilde{\rho}$. Thus, if χ is the character of $\tilde{\rho}$ and χ' is the character of ρ, then $\chi(g) = \chi'(gH)$. Now, we compute

$$\begin{aligned}
(\chi', \chi') &= \frac{1}{|G/H|} \sum_{gH \in G/H} \chi'(gH)\overline{\chi'}(gH) \\
&= \frac{|H|}{|G|} \sum_{gH \in G/H} \chi(g)\overline{\chi(g)} \\
&= \frac{1}{|G|} \sum_{gH \in G/H} \sum_{h \in H} \chi(gh)\overline{\chi(gh)} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} \\
&= (\chi, \chi)
\end{aligned}$$

□

9.2. Example. Let $G = A_4$. Then, $|A_4| = 12$ and A_4 has 4 conjugacy classes given by

$$C_1 = \{(1)\}, C_2 = \{(123), \ldots\}, C_3 = \{(132), \ldots\}, C_4 = \{(12)(34), \ldots\}$$

and thus must have 4 irreducible characters, say $\chi_1, \chi_2, \chi_3, \chi_4$, and let χ_1 be the trivial representation.

Now, consider that C_4 is also a normal subgroup of A_4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so we can find irreducible representations of $A_4/C_4 \cong \mathbb{Z}/3\mathbb{Z}$, we could lift them to irreducible representations of A_4 by the proposition above. Since $\mathbb{Z}/3\mathbb{Z}$ is abelian, it immediately gives character table

	(1)	(123)	(132)
χ_1	1	1	1
χ_{ζ_3}	1	ζ_3	ζ_3^2
$\chi_{\zeta_3^2}$	1	ζ_3^2	ζ_3

Of course, the trivial representation with character χ_1 lifts to a trivial representation of A_4. However, the lifted representation $\tilde{\rho_{\zeta_3}}$ and $\tilde{\rho_{\zeta_3^2}}$ will be irreducible by the proposition above and thus give us irreducible characters of A_4. It is immediate that, up to reordering, $\chi_2 = \tilde{\rho_{\zeta_3}}$ and $\chi_3 = \tilde{\rho_{\zeta_3^2}}$, so we now fill in the rows of our character table to get

	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	ζ_3	ζ_3^2	1
χ_3	1	ζ_3^2	ζ_3	1
χ_4	a	b	c	d

However, we know that

$$1^2 + 1^2 + 1^2 + a^2 = 12 \Longrightarrow a = \pm 3$$

by 8.12 and $a = \chi_4(1) = \dim V$ for V the corresponding vector space of ρ_4. Thus $a > 0 \Longrightarrow a = 3$. Furthermore,

$$1^2 + 1^2 + 1^2 + ad = 0 \Longrightarrow ad = -3 \Longrightarrow d = -1$$

by 8.11. Now, we can use our orthogonality relations to determine χ_4. Using the orthongality of rows (8.6), we get

$$\begin{cases} (\chi_4, \chi_1) = 3 + 4b + 4c - 3 \cdot 1 = 0 \\ (\chi_4, \chi_2) = 3 + 4b\zeta_3^2 + 4c\zeta_3 - 3 \cdot 1 = 0 \\ (\chi_4, \chi_3) = 3 + 4b\zeta_3 + 4c\zeta_3^2 - 3 \cdot 1 = 0 \end{cases} \Longrightarrow \begin{cases} b = 0 \\ c = 0 \end{cases}$$

Thus, our completed character table is given by

	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	ζ_3	ζ_3^2	1
χ_3	1	ζ_3^2	ζ_3	1
χ_4	3	0	0	-1

Perhaps the most important reason to study character theory is the following theorem.

9.3. Corollary. *The multiplicity of an irreducible representation W in a representation V is (χ_W, χ_V).*

Proof. Let $V \cong W_1^{m_1} \oplus \cdots \oplus W_k^{m_k}$ be a decomposition of V into irreducibles. Then, we have

$$\chi_V = \chi_{W_1^{m_1} \oplus \cdots \oplus W_k^{m_k}} = m_1 \chi_{W_1} + \cdots + m_k \chi_{W_k}.$$

Moreover, since the irreducible characters form an *orthonormal basis* of class functions, we can apply $(\cdot, \chi_{W_i})$ to our equality to get

$$(\chi_V, \chi_{W_i}) = m_i$$

□

9.4. Corollary. *Two finite dimensional complex representations of a finite group G, say V, W, are isomorphic if and only if they have the same character. In other words*

$$V \cong W \iff \chi_V = \chi_W$$

Proof. The forward direction follows by definition of character. The backwards direction follows from the fact that the irreducible characters form a basis of Class(G). Namely, $\chi_V = \chi_W$ implies that χ_V and χ_W have the same unique decomposition into irreducible characters, and thus from the corollary above, V, W have the same multiplicity of each irreducible representation. $\square$

9.5. Corollary. *A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

Proof. The forward direction was proved earlier in 8.6. For the backwards direction, if $\chi_V = m_1\chi_{V_1} + \cdots + m_k\chi_{V_k}$ for irreducibles $V_1, \ldots, V_k$, then using the fact that the χ_{V_i}'s form an orthonormal basis, we get

$$1 = (\chi_V, \chi_V) = m_1^2 + \cdots + m_k^2$$

Thus, since each $m_i^2 \geq 0$ and are in $\mathbb{Z}$, it must be that exactly one $m_i = 1$ and all others are 0. Thus, V must be irreducible by the corollary above. $\square$

10. The Regular Representation Revisited

10.1. Corollary. *The multiplicity of any irreducible representation V in the regular representation equals its dimension.*

Proof. Recall that the dimension of the regular representation is $|G|$. Consider

$$(\chi_V, \chi_{reg}) = \frac{1}{|G|}\chi_V(1)\chi_{reg}(1) = \chi_V(1) = \dim V$$

$\square$

10.2. Corollary. *Let $\{V_1, \ldots, V_k\}$ be the set of all irreducible representations of finite group G. Then,*

$$|G| = \sum_{i=1}^{k}(\dim V_i)^2$$

Proof. From above, we have $\chi_{reg} = \sum_{i=1}^{k} \dim V_i \cdot \chi_{V_i}$, and so we simply take (χ_{reg}, χ_{reg}) to get

$$(\chi_{reg}, \chi_{reg}) = \sum_{1\leq i\leq k, 1\leq j\leq k} \dim V_i \dim V_j (\chi_{V_i}, \chi_{V_j}) = \sum_{i=1}^{k}(\dim V_i)^2$$

using the orthogonality of the irreducible characters. However, since

$$\chi_{reg}(g) = \begin{cases} |G| & g = 1 \\ 0 & \text{otherwise} \end{cases}$$

we also have $(\chi_{reg}, \chi_{reg}) = \frac{|G|^2}{|G|} = |G|$ by definition of the inner product $(\cdot, \cdot)$. □

10.3. REMARK. This result should not come as a surprise to a reader familiar with Artin-Wedderburn theory. However, the question of what the analogous decomposition of the regular representation looks like still remains.

10.4. LEMMA. *If ρ_1 and ρ_2 are irreducible representations of G_1 and G_2 respectively, then $\rho_1 \otimes \rho_2$ is an irreducible representation of $G_1 \times G_2$*

PROOF. We have, from above, that for characters χ_1 of ρ_1 and χ_2 of ρ_2, that

$$(\chi_1, \chi_1) = 1 = (\chi_2, \chi_2)$$

However, $\chi := \chi_{\rho_1 \otimes \rho_2}$ has

$$\begin{aligned}
(\chi, \chi) &= \frac{1}{|G_1||G_2|} \sum_{g \in G_1, g' \in G_2} \chi_1(g)\chi_2(g')\overline{\chi_1(g)\chi_2(g')} \\
&= \frac{1}{|G_1|} \left(\sum_{g \in G_1} \chi_1(g)\overline{\chi_1(g)} \right) \frac{1}{|G_2|} \left(\sum_{g' \in G_2} \chi_2(g')\overline{\chi_2(g')} \right) \\
&= (\chi_1, \chi_1) \cdot (\chi_2, \chi_2) \\
&= 1
\end{aligned}$$

Thus, it must be that $\rho_1 \otimes \rho_2$ is irreducible. □

10.5. REMARK. The converse, that each irreducible $G_1 \times G_2$-representation is isomorphic to a tensor product of G_i-representations is also true. The argument is to show that each class function of $G_1 \times G_2$ which is orthogonal to characters of the form $\chi_1 \cdot \chi_2$ is zero.

10.6. PROPOSITION. *One can realize $\mathbb{C}[G]$ as a $G \times G$-module via the action*

$$(h, k).g = hgk^{-1}$$

extended linearly.

PROOF. Indeed,

$$(h', k').(h, k).g = (h', k').hgk^{-1} = h'hgk^{-1}k'^{-1} = (h'h, k'k).g$$

□

10.7. Theorem. *The regular representation decomposes as a direct sum of $G \times G$ representations. More specifically,*

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{k} V_i \otimes V_i^*$$

for irreducible G-representations $\{V_1, \ldots, V_k\}$.

10.8. Remark. Thus, in a sense, we have arrived at the Artin-Wedderburn theorem in different language.

11. Some Applications of Representation Theory: Frobenius Divisibility and Burnside

We now seek to discuss some useful applications of representation theory that would be much more difficult to prove using only group theory. First, we will have to introduce the terminology of algebraic numbers.

11.1. Definition. We say $z \in R$, a commutative ring, is an *algebraic integer* or *integral over* $\mathbb{Z}$ if z is a root of a monic polynomial with coefficients in $\mathbb{Z}$.

11.2. Remark. With this definition, "integer" is a somewhat misleading word. For instance, if $R = \mathbb{C}$, then $i \in \mathbb{C}$ is an algebraic integer since it is a solution to the polynomial $x^2 + 1$.

11.3. Lemma. *The set of all algebraic integers is a ring.*

Proof. Let α be an eigenvalue of some $A \in M_n(R)$ with eigenvector v and β be an eigenvalue of some $B \in M_m(R)$ with eigenvector w. Then, $\alpha \pm \beta$ is an eigenvalue of

$$A \otimes Id_m \pm Id_n \otimes B$$

and $\alpha\beta$ is an eigenvalue of $A \otimes B$. In both cases, the appropriate eigenvector is $v \otimes w$. Thus, the lemma is proven. $\square$

11.4. Proposition. *Let z be an element of a commutative ring R. The following properties are equivalent.*

(a) *z is integral over $\mathbb{Z}$.*
(b) *The subring $\mathbb{Z}[z] \subseteq R$ generated by z is finitely generated as a $\mathbb{Z}$-module.*
(c) *There exists a finitely generated sub-module of R considered as a $\mathbb{Z}$-module containing $\mathbb{Z}[z]$*

11.5. Proposition. *Let χ be the character of a representation ρ of a finite group G. Then, $\chi(g)$ is an algebraic integer for each $g \in G$.*

Proof. $\chi(g) = \operatorname{tr}(\rho(g))$, so it is a sum of eigenvalues of $\rho(g)$. However, all the eigenvalues are roots of unity, which are clearly algebraic integers. Thus, since the algebraic integers form a ring, this sum is an algebraic integer. $\square$

11.6. THEOREM. *Given a complex irreducible representation V of G, $\dim V \mid |G|$.*

PROOF. Let $C_1, \ldots, C_k$ be the conjugacy classes of G and set

$$\lambda_i := \chi_V(C_i)\frac{|C_i|}{\dim V}$$

where $\chi_V(C_i) = \chi_V(g)$ for some $g \in C_i$ (recall that χ_V is a class function). Then, all of these λ_i are algebraic integers. To see this, consider

$$P_i = \sum_{h \in C_i} h$$

which is central in $\mathbb{Z}[G]$ by a quick computation. Thus, by Schur's lemma, $P_i = \mu_i Id_V$ since V is irreducible. Furthermore, $\mathbb{Z}[G]$ is finitely generated, so μ is an algebraic integer. Thus, we compute

$$\operatorname{tr}(P_i) = |C_i|\chi_V(C_i) = \mu_i \dim V \Longrightarrow \mu_i = \frac{|C_i|\chi_V(C_i)}{\dim V} = \lambda_i$$

Now, to finish our proof, we must show that

$$\sum_i \lambda_i \overline{\chi_V(C_i)}$$

is an algebraic integer. However, we already know that the algebraic integers form a ring and, by above, each λ_i is an algebraic integer. Moreover, $\chi_V(C_i)$ is an algebraic integer by 11.5. Thus, the sum is an algebraic integer by the ring structure on the algebraic integers.

Now, we note, by definition of λ_i, so we get the following

$$\begin{aligned}
\sum_i \lambda_i \overline{\chi_V(C_i)} &= \frac{1}{\dim V}\sum_i |C_i|\chi_V(C_i)\overline{\chi_V(C_i)} \\
&= \frac{1}{\dim V}\sum_{g \in G} \chi_V(g)\overline{\chi_V(g)} && \text{since characters are class functions} \\
&= \frac{1}{\dim V}|G|(\chi_V, \chi_V) && \text{by definition of } (\cdot,\cdot) \\
&= \frac{|G|}{\dim V} && \text{since } V \text{ is irreducible, so } (\chi_V, \chi_V) = 1
\end{aligned}$$

Thus, it must be that $\frac{|G|}{\dim V}$ is an algebraic integer. However, $\frac{|G|}{\dim V} \in \mathbb{Q}$, so by Gauss's lemma, $\frac{|G|}{\dim V} \in \mathbb{Z}$. $\square$

While this result is useful in and of itself, we have something better, namely

11.7. THEOREM (Frobenius Divisibility). *Given an irreducible representation V of G,*

$$\dim V \mid \frac{|G|}{|Z(G)|}.$$

PROOF. For $g \in Z(G)$, we note that $\rho(g)$ commutes with $\rho(h)$ for all $h \in G$, so $\rho(g) = \lambda_g Id_V$ for some $\lambda \in \mathbb{C}$. Consider the map from $Z(G)$ to $\mathbb{C}$ given by $g \mapsto \lambda_g$. Such a map is, in fact, a homomorphism from $Z(G)$ to $\mathbb{C}^\times$. Now, for some $m \geq 0$, consider

$$\rho^{\otimes m} \colon G^{\times m} \to GL(V^{\otimes m})$$

By 10.4, this is an irreducible representation of $G^{\times m}$. So, given $(g_1, \ldots, g_m) \in Z(G) \times Z(G) \times \cdots \times Z(G) \leq Z(G^{\times m})$, it must be that $\rho(g_1, \ldots, g_m) = \lambda Id_{V^{\otimes m}}$. Now, consider $H = \{(g_1, \ldots, g_m) \in (Z(G))^{\times m} \mid g_1 \ldots g_m = 1_G\} \leq G$. Then, for $(h_1, \ldots, h_m) \in H$,

$$\begin{aligned}
(h_1, \ldots, h_m).(v_1 \otimes \cdots \otimes v_m) &= \rho(h_1)v_1 \otimes \cdots \otimes \rho(h_m)v_m \\
&= \lambda_{h_1} v_1 \otimes \cdots \otimes \lambda_{h_m} v_m \\
&= (\lambda_{h_1} \cdots \lambda_{h_m}) v_1 \otimes \cdots \otimes v_m \\
&= (h_1 \cdots h_m) v_1 \otimes \cdots \otimes v_m \\
&= v_1 \otimes \cdots \otimes v_m
\end{aligned}$$

so H acts trivially on $V^{\otimes m}$. Thus, consider that $\rho^{\otimes m}$ induces an irreducible representation on H $G^{\times m}/H$. By the theorem above, $(\dim V)^n \mid |G^{\times m}/H| = \frac{|G|^m}{|Z(G)|^{m-1}}$, where $|H| = |Z(G)|^{m-1}$ since $g_1 \cdots g_m = 1$ constrains a degree of freedom. Thus, $\left(\frac{|G|}{|Z(G)| \dim V}\right)^m \in |Z(G)|^{-1}\mathbb{Z}$ for all m and thus $\frac{|G|}{|Z(G)| \dim V}$ is an integer. $\square$

Another famous result is Burnside's theorem, which can be proved using representation theory.

11.8. THEOREM. *Any group G of order $p^a q^b$ for primes p, q and $a, b \geq 0$ is solvable.*

To prove Burnside's theorem, we will use the following results.

11.9. THEOREM. *Let V be an irreducible representation of finite group G and let C be a conjugacy class of G such that $\gcd(|C|, \dim V) = 1$. Then, for any $g \in C$, either $\chi_V(g) = 0$ or g acts as a scalar on V.*

11.10. LEMMA. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are roots of unity such that $\frac{1}{n}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ is an algebraic integer, then either $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ or $\lambda_1 + \cdots + \lambda_n = 0$.*

11.11. THEOREM. *Let G be a finite group and C be a conjugacy class in G with $|C| = p^k$ for p prime and $k \in \mathbb{Z}_+$. Then, G is not simple.*

11.12. LEMMA. *There exists a non-trivial irreducible representation of G, say V, with $p \nmid \dim V$ such that $\chi_V(g) \neq 0$.*

State and prove Burnside's theorem.

12. Induced Representations

Given a representation of a group G, there is a natural way to restrict the representation to a subgroup $H \leq G$.

12.1. Definition. Let $\rho\colon G \to GL(V)$ be a representation of G and $H \leq G$. Then, we define the *restricted representation* $\operatorname{Res}_H^G V$ to be

$$\rho_{\operatorname{Res}_H^G V} = \rho|_H$$

However, we may ask if we can go the other way. That is, given a representation V of a subgroup $H \leq G$, can we construct a representation of G? We present the following

12.2. Definition. Let $\rho\colon H \to GL(V)$ be a representation of $H \leq G$. Then, we define the *induced representation* $\operatorname{Ind}_H^G V$ to be

$$\operatorname{Ind}_H^G V = \{f\colon G \to V \mid f(hx) = \rho(h)f(x), \forall x \in G, h \in H\}$$
$$(g.f)(x) = f(xg), \forall g \in G$$

Less abstractly, this says that, $\operatorname{Ind}_H^G V = \bigoplus_{g \in G/H} g.V$ where each $g.V \cong V$ and the action is given by, where $gg_i = g_{\sigma(i)}h_i$ for $\{g_i\}$ a full set of coset representatives,

$$\rho_{\operatorname{Ind}_H^G V}(g) \cdot \left(\sum_{i=1}^{k} g_i v_i\right) = \sum_{i=1}^{k} g_{\sigma(i)}\rho(h_i)v_i$$

12.3. Proposition. *Given a representation $\rho\colon H \to GL(V)$ and $H \leq G$, we have that*

$$\operatorname{Ind}_H^G V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

as $\mathbb{C}[G]$-modules with G-action

$$g'.(e_g \otimes v) = e_{g'g} \otimes v = e_{g''} \otimes \rho(h)v$$

where $g'g = g''h$ for $g'' \in G, h \in H$.

Proof. This follows from the fact that $\mathbb{C}[G] \cong \bigoplus_{g \in G/H} g.\mathbb{C}[H]$, so

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \cong \left(\bigoplus_{g \in G/H} g.\mathbb{C}[H]\right) \otimes_{\mathbb{C}[H]} V \cong \bigoplus g.(\mathbb{C}[H] \otimes_{\mathbb{C}[H]} V) \cong \bigoplus_{g \in G/H} g.V$$

$\square$

12.4. Remark. Under this isomorphism, we get $\operatorname{Ind}_H^G V = \bigoplus_{g \in G/H} g.V$ where $V \cong g.V$ via $v \mapsto e_g \otimes v$

Check that all these actions are actually correct

12.5. Example. (a) Let $H = \{1\} \leq G$ and let W be the trivial representation. Then, since $W = \mathbb{C}$

$$\operatorname{Ind}_H^G W \cong \mathbb{C}[G] \otimes_{\mathbb{C}} W \cong \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}[G]$$

is the regular representation.

(b) Let $H \leq G$ and W be the trivial representation. Then,

$$\operatorname{Ind}_H^G W \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C} \cong \mathbb{C}[G/H]$$

since $g \otimes \lambda = g'h \otimes \lambda = g' \otimes h.\lambda = g' \otimes \lambda$ for $g' \in G, h \in H$.

12.6. Proposition. *Let V, V_1, V_2 be representations of $H \leq G$. Then,*

(a) $\operatorname{Ind}_H^G(V_1 \oplus V_2) \cong \operatorname{Ind}_H^G V_1 \oplus \operatorname{Ind}_H^G V_2$.
(b) $\dim \operatorname{Ind}_H^G V = |G/H| \dim V$.
(c) For $H \leq K \leq G$, $\operatorname{Ind}_K^G \operatorname{Ind}_H^K V \cong \operatorname{Ind}_H^G V$.

Proof. (b) has already been shown in the example above. For (a),

$$\operatorname{Ind}_H^G(V_1 \oplus V_2) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (V_1 \oplus V_2) = (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V_1) \oplus (\mathbb{C}[H] \otimes_{\mathbb{C}[H]} V_2) \cong \operatorname{Ind}_H^G V_1 \oplus \operatorname{Ind}_H^G V_2$$

and for (c),

$$\operatorname{Ind}_K^G \operatorname{Ind}_H^K V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} V) \cong C[G] \otimes_{\mathbb{C}[H]} V \cong \operatorname{Ind}_H^G V$$

□

12.7. Definition. Let $\phi \in \operatorname{Class}(H)$ for $H \leq G$. Then, we define the *induced class function* on G by

$$\operatorname{Ind}_H^G(\phi)(g) := \sum_{s \in G/H, s^{-1}gs \in H} \tilde{\phi}(s^{-1}gs) = \frac{1}{|H|} \sum_{t \in G} \tilde{\phi}(t^{-1}gt)$$

where

$$\tilde{\phi}(g) = \begin{cases} \phi(g) & g \in H \\ 0 & \text{else} \end{cases}$$

12.8. Example. Let $G = \mathfrak{S}_3$ and consider $H = \mathbb{Z}/3\mathbb{Z} \cong \{(1), (123), (132)\} \trianglelefteq \mathfrak{S}_3$. Then, since $\mathbb{Z}/3\mathbb{Z}$ is abelian, it has character table

	(1)	(123)	(132)
χ_1	1	1	1
χ_ζ	1	ζ	ζ^2
χ_{ζ^2}	1	ζ^2	ζ

where $\zeta = \zeta_3$, a third root of unity, since $g^3 = 1$ for all $g \in \mathbb{Z}/3\mathbb{Z}$. We also have, from earlier, the irreducible characters of $\mathfrak{S}_3$ as χ_1, χ_2, θ (see 8.13). Then, using the fact that H is normal and $G/H = \{H, (12)H\}$, we see that

$$\begin{aligned} \operatorname{Ind}_H^G(\chi)(h) &= \chi(h) + \chi((12)h(12)) && \forall h \in H \\ &= \chi(h) + \overline{\chi(h)} && \text{since } (12)(123)(12) = (132) \\ \operatorname{Ind}_H^G(\chi)((12)) &= \tilde{\chi}((12)) + \tilde{\chi}((12)(12)(12)) \\ &= 0 \end{aligned}$$

And thus

$$\begin{aligned} \operatorname{Ind}_H^G(\chi_1) &= \chi_1 + \chi_2 \\ \operatorname{Ind}_H^G(\chi_\zeta) &= \theta \end{aligned}$$

$$\operatorname{Ind}_H^G(\chi_{\zeta^2}) = \theta$$

since $\zeta + \zeta^2 = -1$.

12.9. Proposition. *Given* $\phi \in \operatorname{Class}(H)$ *for* $H \leq G$, *we get* $\operatorname{Ind}_H^G(\phi) \in \operatorname{Class}(G)$.

Proof. Let $g, k \in G$, then

$$\operatorname{Ind}_H^G(\phi)(k^{-1}gk) = \frac{1}{|H|}\sum_{t\in G}\tilde{\phi}(t^{-1}k^{-1}gkt) = \frac{1}{|H|}\sum_{t'\in G}\phi(t'^{-1}gt') = \operatorname{Ind}_H^G(\phi)(g)$$

$\square$

12.10. Theorem (The Mackey formula). *The character of the induced representation* $\operatorname{Ind}_H^G V$ *is* $\operatorname{Ind}_H^G(\chi_V)$.

Proof. We seek to directly compute the trace of the action of $s \in G$ on $\operatorname{Ind}_H^G V \cong \bigoplus_{g\in G/H} g.V$. Consider that $sgH = gH \iff g^{-1}sg = t \in H$. We only consider such cosets since all others will not contribute to the trace. On $g.V$, we get

$$s.(e_g \otimes v) = e_{sg} \otimes v = e_{gt} \otimes v = e_g \otimes \rho(t)w$$

so the action of s on $g.V$ corresponds to the action of $t = g^{-1}sg$ on V. Thus, the block corresponding to $g.V$ contributes $\chi_V(g^{-1}sg)$ to the trace of $\rho_{\operatorname{Ind}_H^G V}(s)$. Thus, summing over all such g, we get

$$\chi_{\operatorname{Ind}_H^G \rho}(s) = \sum_{g\in G/H, g^{-1}sg\in H}\chi_V(g^{-1}sg) = \sum_{g\in G/H}\tilde{\chi_V}(g^{-1}sg) = \operatorname{Ind}_H^G(\chi_V)(s)$$

$\square$

12.11. Example. Thus, from our example above, this actually tells us the decomposition of the induced representations of $\mathfrak{S}_3$ from $\mathbb{Z}/3\mathbb{Z}$.

13. Frobenius Reciprocity

We seek to find relationships between Ind and Res since Res is much easier to compute than Ind. By far the most important one is Frobenius reciprocity.

13.1. Theorem (Frobenius Reciprocity). *Let* $H \leq G$, *a finite group. Then, for* $\phi \in \operatorname{Class}(G)$ *and* $\psi \in \operatorname{Class}(H)$, *we have*

$$(\operatorname{Ind}_H^G \psi, \phi) = (\psi, \operatorname{Res}_H^G \phi)$$

which is to say, Ind_H^G *and* Res_H^G *are Hermetian adjoint.*

13.2. Lemma. *Given* M *an* R*-module and* N *an* S*-module and ring homomorphism* $f\colon R \to S$ *for* R, S *(unital) rings, then*

$$\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_S(S \otimes_R M, N)$$

where N is considered as an R module via $r.n = f(r)n$ (that is, the restriction of scalars).

Proof of Lemma. Given $u \in \mathrm{Hom}_R(M, N)$, let $F\colon \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(S \otimes_R M, N)$ be defined by sending u to the composition

$$\begin{aligned} S \otimes_R M &\overset{id_S \otimes u}{\to} S \otimes_R N \to N \\ & \qquad\qquad s \otimes n \mapsto sn \end{aligned}$$

Then, $F(u)$ is an S-module homomorphism and so F is well-defined and a homomorphism of abelian groups. We now construct and inverse to F. Let $G\colon \mathrm{Hom}_S(S \otimes_R M, N) \to \mathrm{Hom}_R(M, N)$ be defined by sending $v \in \mathrm{Hom}_S(S \otimes_R M, N)$ to the composition

$$\begin{aligned} &M \to R \otimes_R M \overset{f \otimes id_M}{\to} S \otimes_R M \overset{v}{\to} N \\ &m \mapsto 1 \otimes m \end{aligned}$$

One can check that F, G are inverse to each other. $\square$

13.3. Remark. One can also check that this isomorphism depends only on the ring homomorphism f. Thus, such an isomorphism is functorial and so the extensions of scalars, $S \otimes -$, is left adjoint to the restriction of scalars functor.

Proof of Frobenius Reciprocity. Frobenius reciprocity ends up being a special case of the lemma. That is,

$$\begin{aligned} (\mathrm{Ind}_H^G \chi_V, \chi_W) &= \dim \mathrm{Hom}_G(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, W) && \text{by 8.6} \\ &= \dim \mathrm{Hom}_H(V, W) && \text{by lemma} \\ &= (\chi_V, \mathrm{Res}_H^G \chi_W) \end{aligned}$$

$\square$

Frobenius reciprocity is useful for computing various representations of groups from representations of their subgroups.

13.4. Example. Consider $\mathbb{Z}/2\mathbb{Z} \cong \langle (12) \rangle \leq \mathfrak{S}_3$. Then, $\mathbb{Z}/2\mathbb{Z}$ has two irreducible 1-dimensional characters, let us say χ_1 and χ_{-1}. Then, $(\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_1, \chi_V) = (\chi_1, \mathrm{Res}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_V)$ for any irreducible representation V of $\mathfrak{S}_3$. Thus,

$$\begin{aligned} (\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_1, \chi_{trivial}) &= (\chi_1, \mathrm{Res}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{trivial}) = (\chi_1, \chi_1) = 1 \\ (\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_1, \chi_{alt}) &= (\chi_1, \mathrm{Res}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{alt}) = (\chi_1, \chi_{-1}) = 0 \\ (\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_1, \chi_{std}) &= (\chi_1, \mathrm{Res}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{std}) = (\chi_1, \chi_1 + \chi_{-1}) = 1 \\ (\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{-1}, \chi_{trivial}) &= (\chi_{-1}, \mathrm{Res}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{trivial}) = (\chi_{-1}, \chi_1) = 0 \\ (\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{-1}, \chi_{alt}) &= (\chi_{-1}, \mathrm{Res}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{alt}) = (\chi_{-1}, \chi_{-1}) = 1 \end{aligned}$$

$$(\operatorname{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{-1}, \chi_{std}) = (\chi_{-1}, \operatorname{Res}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{std}) = (\chi_{-1}, \chi_1 + \chi_{-1}) = 1$$

Thus, $\operatorname{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_1 = \chi_{trivial} + \chi_{std}$ and $\operatorname{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\mathfrak{S}_3} \chi_{-1} = \chi_{alt} + \chi_{std}$.

14. Mackey Theory

To gain more insight into induced representations, we wish to understand $\operatorname{Res}_K^G \operatorname{Ind}_H^G V$ for $H, K \leq G$ and V a representation of G. To describe our results, we will need the language of double cosets.

14.1. Definition. Let $H, K \leq G$, a group. The set of *double cosets* KnG/H is the set of $K \times H$-orbits on G where $(k, h).g = kgh$. Often, elements of such a set are denoted KgH for $g \in G$, a representative.

14.2. Remark. If $K = H \trianglelefteq G$, then $KnG/H = G/H$. Also note that there are other definitions of double cosets.

14.3. Definition. Following [**Ser97**], for $H, K \leq G$ and $s \in S$, a set of double-coset representatives, let

$$H_s := sHs^{-1} \cap K \leq K$$

and, given representation $\rho\colon H \to GL(V)$, let $\rho_s\colon H_s \to GL(V_s)$ given by

$$\rho_s(x) := \rho(s^{-1}xs)$$

be a representation of H_s. In other words, $V_s \cong s \otimes V$ with action of H_s given by

$$(shs^{-1})(s \otimes v) = s \otimes hv$$

14.4. Remark. Note that H_s is the stabilizer of sH under the action of K on G/H. Furthermore, if one embeds $H_s \hookrightarrow H$ via $x \mapsto s^{-1}xs$, then $\operatorname{im} H_s$ is the stabiliser of Ks under the action of H on HnG.

14.5. Proposition. *Given S, a set of double coset representatives for KnG/H, then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G V \cong \bigoplus_{s \in S} \operatorname{Ind}_{H_s}^K V_s$$

Proof. Let us consider $\mathbb{C}[G]$ as a $(\mathbb{C}[K], \mathbb{C}[H])$-bimodule. Then,

$$\mathbb{C}[G] \cong \bigoplus_{s \in S} \mathbb{C}[KsH]$$

Note, too, this viewpoint gives the isomorphism as $(\mathbb{C}[K], \mathbb{C}[H])$-bimodules

$$\mathbb{C}[KsH] \cong \mathbb{C}[K] \otimes_{\mathbb{C}[H_s]} (s \otimes \mathbb{C}[H])$$
$$ksh \mapsto k \otimes s \otimes h$$

Thus, using this decomposition, we get the following isomorphisms of $\mathbb{C}[K]$-modules

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G V \cong \operatorname{Res}_K^G \left(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V\right)$$

$$
\begin{aligned}
&\cong \bigoplus_{s\in S} \mathbb{C}[KsH] \otimes_{\mathbb{C}[H]} V \\
&\cong \bigoplus_{s\in S} \mathbb{C}[K] \otimes_{\mathbb{C}[H_s]} (s \otimes \mathbb{C}[H]) \otimes_{\mathbb{C}[H]} V \\
&\cong \bigoplus_{s\in S} \mathbb{C}[K] \otimes_{\mathbb{C}[H_s]} V_s \\
&\cong \bigoplus_{s\in S} \operatorname{Ind}_{H_s}^{K} V_s
\end{aligned}
$$

□

14.6. THEOREM (Mackey's Irreducibility Criterion). $\operatorname{Ind}_H^G V$ *is irreducible if and only if*

(a) V *is irreducible and*
(b) V_s *and* $\operatorname{Res}_{H_s}^H V$ *share no irreducibles for all* $s \in S$*, a set of double coset representatives of* KnG/H*.*

PROOF. We wish to show that $(\operatorname{Ind}_H^G V, \operatorname{Ind}_H^G V) = 1$. We compute

$$
\begin{aligned}
(\operatorname{Ind}_H^G V, \operatorname{Ind}_H^G V) &= (V, \operatorname{Res}_H^G \operatorname{Ind}_H^G V) && \text{by Frobenius Reciprocity} \\
&= (V, \bigoplus_{s\in HnG/H} \operatorname{Ind}_{H_s}^H(\rho_s)) && \text{by proposition above} \\
&= \sum_{s\in HnG/H} (\operatorname{Res}_{H_s}^H(\rho), \rho_s) && \text{by Frobenius reciprocity}
\end{aligned}
$$

Note that the last equality also makes use of the fact

$$\operatorname{Hom}(A, \bigoplus_{i\in I} B_i) = \bigoplus_{i\in I} \operatorname{Hom}(A, B_i) \text{ for finite index set } I$$

Now, when $s = 1$, $(\operatorname{Res}_H^H \rho, \rho_1) = (\rho, \rho) \geq 1$. Thus, for our sum to be exactly 1, it must be that

$$(\rho, \rho) = 1 \text{ and } (\operatorname{Res}_{H_s}^H \rho, \rho_s) = 0 \text{ for } s \neq 1$$

This will happen precisely when ρ is irreducible and $\operatorname{Res}_{H_s}^H \rho$ and ρ_s share no irreducibles. □

14.7. COROLLARY. *If* $H \trianglelefteq G$*, then* $\operatorname{Ind}_H^G V$ *is irreducible if and only if* V *is irreducible and* $V \ncong V_s$ *for all* $s \in G/H$

PROOF. If $H \trianglelefteq G$, then $H_s = H$ and $\operatorname{Res}_{H_s}^H \rho = \rho$, so it must be that ρ is irreducible and not isomorphic to any ρ_s, otherwise ρ_s and $\operatorname{Res}_{H_s}^H \rho = \rho$ will share an irreducible. □

14.8. Example. Consider $\langle r, s \mid r^4 = s^2 = 1, sr = r^{-1}s\rangle \cong D_8 \cong \mathbb{Z}_4 \rtimes \mathbb{Z}_2$. Then, we know $\langle r\rangle \cong \mathbb{Z}_4 \trianglelefteq D_8$ with character table

	1	r	r^2	r^3
χ_1	1	1	1	1
χ_2	1	i	-1	$-i$
χ_3	1	-1	1	-1
χ_4	1	$-i$	-1	i

If ρ^i is the irreducible representation of $\mathbb{Z}_4$ corresponding to χ_i, then we notice that $\operatorname{tr}\rho^3_s(r) = \operatorname{tr}\rho^3(srs) = \operatorname{tr}\rho^3(r^{-1}) = -1 = \operatorname{tr}\rho^3(r)$, so $\operatorname{Ind}_{\mathbb{Z}_4}^{D_8}\rho^3$ will *not* be irreducible. On the other hand, one can check $\operatorname{tr}\rho^2_s(r) = \operatorname{tr}\rho^2(srs) = \operatorname{tr}\rho^2(r^{-1}) = -i = \operatorname{tr}\rho^4(r)$, so ρ^2_s and ρ^2 are orthogonal, thus sharing no irreducibles. Thus, it must be that $\operatorname{Ind}_{\mathbb{Z}_4}^{D_8}\rho^2$ is irreducible. In fact, one can check $\operatorname{Ind}_{\mathbb{Z}_4}^{D_8}\rho^2 = \operatorname{Ind}_{\mathbb{Z}_4}^{D_8}\rho^4$ will give the only irreducible 2-dimensional representation of D_8.

15. Representations of Nilpotent Groups

15.1. Theorem. *Given an irreducible representation of a nilpotent group G, it is induced from a one-dimensional representation of some subgroup $H \leq G$.*

15.2. Remark. This is a specific case of Brauer's theorem (see 18.2). We will prove this theorem using the program given in [**Tel05**].

15.3. Lemma. *Given finite group G with $N \trianglelefteq G$ and irreducible representation V, then $V = \operatorname{Ind}_H^G W$ for some $N \leq H \leq G$ and $\operatorname{Res}_N^H W \cong U^{\oplus r}$ for U an irreducible representation of N, $r \in \mathbb{N}$.*

Proof. Let $\operatorname{Res}_N^G V = \bigoplus_i V_i$ for V_i irreducible N representations. If there is a single summand, then $H = G$ and we are done. Otherwise, given $\rho\colon N \to GL(V)$, note that $\rho_s(n) := \rho(s^{-1}ns)$ for $n \in N$, $s \in G$ will permute the summands of $\operatorname{Res}_N^G V$ since conjugation defines an automorphism of the normal subgroup N, and so must permute the irreducible characters of N. Thus, this conjugation action gives us an isomorphism of N-representations. Furthermore, this permutation action by G must be transitive since we assumed that V has no G-invariant subspaces and so each non-zero V_i has the same dimension.

Now choose a non-zero block V_k and let $H = \operatorname{Stab}_G(V_k)$. Then, it must be that $\dim V = \frac{|G|}{|H|}\dim V_k = \dim \operatorname{Ind}_H^G V_k$ by the orbit-stabilizer theorem. Furthermore, we get

$$\dim \operatorname{Hom}_G(\operatorname{Ind}_H^G V_k, V) = \dim \operatorname{Hom}_H(V_k, \operatorname{Res}_H^G V) \geq 1$$

Thus, since V is irreducible as a G-representation and $\dim V = \dim \operatorname{Ind}_H^G V_k$, it must be that

$$(\operatorname{Ind}_H^G V_k, V) = \dim \operatorname{Hom}_G(\operatorname{Ind}_H^G V_k, V) = 1$$

and so $V = \operatorname{Ind}_H^G V_k$. $\square$

15.4. COROLLARY. *If $N \trianglelefteq G$ is abelian, then either its action on V is a scalar or $V = \operatorname{Ind}_H^G W$ for some proper subgroup $H \subsetneqq G$.*

We will also use the following (standard) group theory fact.

15.5. LEMMA. *If G is nilpotent, then it contains a characteristic, abelian, non-central subgroup.*

PROOF OF THEOREM. Let $\rho\colon G \to GL(V)$ be an irreducible representation of a nilpotent group G. Then, $G/\ker\rho$ is nilpotent. If $G/\ker\rho$ is abelian, then $\overline{\rho}\colon G/\ker\rho \to GL(V)$ is one-dimensional and so $\rho\colon G \to GL(V)$ must be as well. Otherwise, take $A \trianglelefteq G/\ker\rho$ to be a characteristic, abelian, and non-central subgroup given by the lemma above. Then, for non-central element $a \in A$, $\overline{\rho}(a)$ cannot be a scalar.

Thus, from the corollary above, there is a proper subgroup $H \subsetneqq G$ with a representation $\rho'\colon H/\ker\rho \to GL(W)$ such that $\overline{\rho} = \operatorname{Ind}_{H/\ker\rho}^{G/\ker\rho} \rho' = \operatorname{Ind}_H^G \tilde{\rho}'$ where $\tilde{\rho}'\colon H \to GL(W)$ is a lift of ρ'. Since H is also nilpotent, we can repeat this argument until we find a 1-dimensional representation ρ'. $\square$

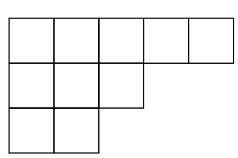
16. Example: Representations of the Symmetric Group

Elements of the symmetric group $\mathfrak{S}_n$ can be represented in 1-line notation as products of disjoint cycles, and to each element, we can assign a *cycle type* in the form of a partition of n.

16.1. EXAMPLE. Consider $(12345)(876)(9,10) \in \mathfrak{S}_{10}$. This cycle has cycle type $(5,3,2) \vdash 10$.

Partitions are useful combinatorial tools with many applications beyond what is described here. It is also useful to realize partitions of n as Young diagrams. Once again, we shall define the correspondence via an example.

16.2. EXAMPLE. We can represent a partition $(\lambda_1, \lambda_2, \ldots)$ as a *Ferrers diagram* with λ_i boxes on the ith row. So, consider $(5,3,2) \vdash 10$. The corresponding diagram would be



Furthermore, we can fill in the boxes of the diagram to get a *Young tableau*. For example

1	2	3	4	5
6	7	8		
9	10			

Finally, $\mathfrak{S}_n$ can act on a Young tableau by permuting the numbers. So, $(12345)(876)(9,10) \in \mathfrak{S}_{10}$ would act on the above diagram to yield

2	3	4	5	1
8	6	7		
10	9			

Now, using this combinatorial device, we can define the following.

16.3. DEFINITION. Let T be a tableau of shape $\lambda \vdash n$ for $n \in \mathbb{N}$. Then, we define the *row stabalizer subgroup* $R(\mathsf{T}) \leq \mathfrak{S}_n$ to be the subgroup of permutations such that every permutation in $R(\mathsf{T})$ preserves the elements in the rows of T. One analogously defines $C(\mathsf{T})$ to be the *column stabalizer subgroup*. Finally, one defines $\mathfrak{S}_\lambda := R(\mathsf{T}')$ where $\mathsf{T}' = \{(1, 2, \dots, \lambda_1), (\lambda_1 + 1, \dots, \lambda_1 + \lambda_2), \dots, (n - \lambda_\ell + 1, \dots, n)\}$.

16.4. EXAMPLE. In the example above, $(12345)(876)(9,10)$ is in the row stabalizer of the tableau. In fact, in that setup $R(\mathsf{T}) = \mathfrak{S}_{\{1,2,3,4,5\}} \times \mathfrak{S}_{\{6,7,8\}} \times \mathfrak{S}_{\{9,10\}} \cong \mathfrak{S}_5 \times \mathfrak{S}_3 \times \mathfrak{S}_2$.

Now, given $\mathfrak{S}_n$, we automatically know of 2 irreducible, 1-dimensional, complex representations, namely the trivial representation and the sign/alternating representation. Furthermore, we know that conjugacy classes of $\mathfrak{S}_n$ are encoded by cycle type, which are encoded by partitions of n. So, since character tables are square, there must be as many irreducible representations of $\mathfrak{S}_n$ as there are partitions of n. Thus, we have

$$\{\text{Partitions of } n\} \xleftrightarrow{\text{1-1}} \{\text{Irreducible representations of } \mathfrak{S}_n\}$$

Our task, then, is to figure out what these irreducible representations are. A systematic treatment will not be given here, but some of the tools in the previous chapters will be used to get an idea of some results about the representation theory of symmetric groups. For more, see [**Jam78**], [**Ful97**]. We will follow the program of [**Sag91**], strategically omitting details and adapting the content to the language in these notes.

16.5. DEFINITION. We say two λ-tableaux T_1 and T_2 are *row equivalent* if the corresponding rows of the two tableaux contain the same elements, that is, $R(\mathsf{T}_1) = R(\mathsf{T}_2)$. We then define a *tabloid of shape* λ or a λ-*tabloid* $\{\mathsf{T}\}$ to be

$$\{\mathsf{T}\} = \{\mathsf{T}' \mid \mathsf{T}' \sim \mathsf{T}, \text{shape}\,\mathsf{T} = \lambda\}$$

16.6. EXAMPLE. If

$$\mathsf{T} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

then,

$$\{\mathsf{T}\} = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

16.7. PROPOSITION. *If $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, then the number of λ-tableaux in any given λ-tabloid is*

$$\lambda_1!\lambda_2! \cdots \lambda_\ell! =: \lambda!$$

and thus the number of λ-tabloids is $n!/\lambda!$.

PROOF. This follows immediately since, if $\mathsf{T}_1 \sim \mathsf{T}_2$, then $g.\mathsf{T}_1 = \mathsf{T}_2$ for some $g \in R(\mathsf{T}_1) = R(\mathsf{T}_2)$, that is, $\mathsf{T}_1 \sim \mathsf{T}_2$ if and only if T_1 and T_2 are in the same $R(\mathsf{T}_1)$-orbit and $|R(\mathsf{T}_1)| = \lambda!$. $\square$

16.8. PROPOSITION. *Given T of shape λ, $R(\mathsf{T}) \leq S_n$ and trivial representation 1 of $R(\mathsf{T})$, we get that*

$$\mathrm{Ind}_{R(\mathsf{T})}^{S_n} \cong \mathbb{C}[S_n/R(\mathsf{T})] = \mathbb{C}[\pi_1 R(\mathsf{T}), \ldots, \pi_k R(\mathsf{T})]$$

and we can index each π_i by a λ-tabloid, say $\{\mathsf{T}_i\}$.

16.9. DEFINITION. Let $\lambda \vdash n$. We define

$$M^\lambda := \mathbb{C}[\{\mathsf{T}_1\}, \ldots, \{\mathsf{T}_k\}]$$

where $\{\mathsf{T}_1\}, \ldots, \{\mathsf{T}_k\}$ is a complete list of λ-tabloids as the *permutation module corresponding to λ.*

16.10. EXAMPLE. Let $\lambda = (n) = \square\square\cdots\square$. Then, there is only a single λ-tabloid consisting of all λ-tableaux. Thus,

$$M^{(n)} = \mathbb{C}\left[\begin{array}{c}\hline 1\ 2 \cdots n \\ \hline\end{array}\right] \cong \mathbb{C}$$

with a trivial action. Thus, $M^{(n)}$ is the trivial representation.

16.11. EXAMPLE. Let $\lambda = (1^n)$. Then, each λ-tabloids consists of a single λ-tableau and thus they are in bijection with $\mathfrak{S}_n$. Thus,

$$M^{(1^n)} = \mathbb{C}[\{\mathsf{T}_1\}, \ldots, \{T_{n!}\}] \cong \mathbb{C}\mathfrak{S}_n$$

is the regular representation of $\mathfrak{S}_n$.

16.12. EXAMPLE. Let $\lambda = (n-1, 1) = \begin{array}{l}\square\square\cdots\square \\ \square\end{array}$. Then, each λ-tabloid is uniquely determined by the number in the second row, so there are n λ-tabloids. Thus, we get

$$M^{(n-1,1)} = \mathbb{C}[\{\mathsf{T}_1\}, \ldots, \{\mathsf{T}_n\}]$$

with action $\sigma.\mathsf{T}_j = \mathsf{T}_{\sigma(j)}$ for $\sigma \in \mathfrak{S}_n$. Thus, we have the "standard" or "defining" representation of $\mathfrak{S}_n$, that is, the action of $\mathfrak{S}_n$ on $\{1, \ldots, n\}$.

16.13. Proposition. *If $\lambda \vdash n$, then M^λ is cyclicly generated by any given λ-tabloid. Furthermore,* $\dim M^\lambda = n!/\lambda!$*, that is, the number of λ-tabloids.*

Proof. This is a relatively straightforward application of the transitivity of the $\mathfrak{S}_n$-action on the set of λ-tabloids, which initially came from cosets of $\mathfrak{S}_n/R(\mathsf{T})$ for some T of shape λ. The dimension computation is immediate from the construction as well. $\square$

16.14. Theorem. *Let $\lambda \vdash n$ and let S_λ correspond to tabloid $\{\mathsf{T}^\lambda\}$. Then, $\mathbb{C}\mathfrak{S}_n\mathfrak{S}_\lambda \cong \mathbb{C}\mathfrak{S}_n\{\mathsf{T}^\lambda\}$ as $\mathfrak{S}_n$-modules.*

Proof. Define a map by sending $\pi_i S_\lambda \mapsto \{\pi_i \mathsf{T}^\lambda\}$ and extend linearly. $\square$

We can also define a partial order on tableau.

16.15. Definition. Given partitions $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots)$, we say that $\lambda \trianglelefteq \mu$ if

$$\begin{cases} \lambda_1 \leq \mu_1 \\ \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2 \\ \vdots \\ \lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k \\ \vdots \end{cases}$$

16.16. Example. Using the notation that $(2^2, 1^2) = (2, 2, 1, 1)$, the following partial order is induced on the partitions of 6.

$$\begin{tikzcd}
 & (6) \arrow[d] & \\
 & (5,1) \arrow[d] & \\
 & (4,2) \arrow[dl] \arrow[dr] & \\
(3,3) \arrow[dr] & & (4,1^2) \arrow[dl] \\
 & (3,2,1) \arrow[dl] \arrow[dr] & \\
(3,1^3) \arrow[dr] & & (2^3) \arrow[dl] \\
 & (2^2,1^2) \arrow[d] & \\
 & (2,1^4) \arrow[d] & \\
 & (1^6) &
\end{tikzcd}$$

16.17. Lemma (Dominance Lemma). [**Sag91**, Lem 2.2.4]. *Let T and S be tableaux of shape λ and μ, respectively. If, for each index i, the elements of row i of S are all in different columns in T, then $\lambda \trianglerighteq \mu$.*

16.18. Definition. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ abd $\mu = (\mu_1, \dots, \mu_m)$ be partitions of n. Then, $\lambda \leq \nu$ if, for some index i,

$$\lambda_j = \mu_j \text{ for } j < i \text{ and } \lambda_i < \mu_i$$

This order is called the *lexicographic order*.

16.19. Proposition. *(a) The lexicographic order is a total order.*
(b) If $\lambda, \mu \vdash n$ with $\lambda \trianglerighteq \mu$, then $\lambda \geq \mu$.

16.20. Definition. We define the *row symmetrizer* of a tableau T of shape λ to be

$$P(\mathsf{T}) := \sum_{\alpha \in R(\mathsf{T})} \alpha$$

and the *column symmetrizer* to be

$$Q(\mathsf{T}) := \sum_{\alpha \in C(\mathsf{T})} \operatorname{sgn}(\alpha)\alpha$$

16.21. Definition. Let T be a tableau of shape λ. Then, we define

$$e_{\mathsf{T}} = Q(\mathsf{T}) \cdot \{\mathsf{T}\} \in \mathbb{C}\mathfrak{S}_n / \sim$$

to be the *polytabloid* associated with T.

16.22. Remark. For the knowledgeable, e_{T} is the image of the *Young symmetrizer*

$$y(T) = \frac{Q(\mathsf{T})P(\mathsf{T})}{h_\lambda}$$

under the quotient map induced by $\sim$ where h_λ is the hook length formula. Is this actually true?

16.23. Definition. For any partition λ, the corresponding *Specht module*, S^λ, is the submodule of M^λ spanned by the polytabloids e_{T} where T is of shape λ.

Now, given the information we have above, we list some important results without proofs about the irreducible $\mathfrak{S}_n$ modules.

16.24. Proposition. *We have the following fundamental results.*

(a) The S^λ are cyclic modules generated by any given polytabloid.
(b) Let U be a submodule of M^λ. Then, $U \supseteq S^\lambda$ or $U \subseteq S^{\lambda\perp}$. Thus, over $\mathbb{C}$, the S^λ's are irreducible.
(c) The S^λ for $\lambda \vdash n$ form a complete list of irreducible $\mathfrak{S}_n$-modules over $\mathbb{C}$.

(d) *(Branching Rule) If $\lambda \vdash n$, then*

$$\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}(S^\lambda) \cong \bigoplus_{\lambda^-} S^{\lambda^-} \qquad \mathrm{Ind}^{\mathfrak{S}_{n+1}}_{\mathfrak{S}_n} \cong \bigoplus_{\lambda^+} S^{\lambda^+}$$

where λ^- is the set of all Ferrers diagrams with a removable corner of λ revmoved and λ^+ is the set of all Ferrers diagrams with an addable corner of λ added.

(e) *(Young's Rule) Over an algebraically closed field, the permutation modules decompose as*

$$M^\mu = \bigoplus_{\lambda} K_{\lambda,\mu} S^\lambda$$

where $K_{\lambda,\mu}$ is the number of "semistandard Young tableaux" of shape λ and content μ, referred to as the Kostka number.

16.25. Example. Let $\lambda = (n)$. Then, $e_{\square\square\cdots\square} = \begin{array}{c}\hline 1 \quad 2 \quad \cdots \quad n \\ \hline\end{array}$ and so the trivial representation is a subrepresentation of $S^{(n)}$. Therefore, $S^(n)$ is the trivial representation, which was the only possibility anyways since $M^{(n)}$ was also trivial.

16.26. Example. Let $\lambda = (1^n)$. Then, for any T of shape λ, e_T is the signed sum of all $n!$ permutations viewed as tabloids and $e_{\sigma\mathsf{T}} = \sigma e_\mathsf{T} = (\operatorname{sgn}\sigma)e_\mathsf{T}$. Thus, every polytabloid is a scalar multiple of e_T and thus $S^{(1^n)} = \mathbb{C}\{e_\mathsf{T}\}$ with action given by $\sigma e_\mathsf{T} = (\operatorname{sgn}\sigma)e_\mathsf{T}$, thus giving the sign representation.

16.27. Example. For $\lambda = (n-1,1)$, then we have polytabloids of the form

$$\{\mathsf{T}\} = \begin{array}{l}\hline i \ \cdots \ j \\ \hline k \\ \hline\end{array} \Longrightarrow Q(\mathsf{T}) = 1 - (i,k) \Longrightarrow e_\mathsf{T} = \begin{array}{l}\hline i \ \cdots \ j \\ \hline k \\ \hline\end{array} - \begin{array}{l}\hline k \ \cdots \ j \\ \hline i \\ \hline\end{array}$$

The span of all these e_T is given by

$$S^{(n-1,1)} = \{c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\mathbf{n} \mid c_1 + c_2 + \cdots + c_n = 0\} \text{ where } \mathbf{k} := \begin{array}{l}\hline i \ \cdots \ j \\ \hline k \\ \hline\end{array}$$

Thus, $\dim S^{(n-1,1)} = n-1$ and we can pick a basis $\{\mathbf{2}-\mathbf{1}, \mathbf{3}-\mathbf{1}, \ldots, \mathbf{n}-\mathbf{1}\}\}$ with character $\chi_{S^{(n-1,1)}}(\sigma)$ to be one less than the number of fixed points of $\sigma \in \mathfrak{S}_n$.

17. Artin's Theorem

Artin's theorem allows us to gain more insight into how induced characters or representations may be related to the irreducible representations of a finite group.

17.1. Definition. Let G be a finite group. A *virtual representation* of G is an integer linear combination of irreducible representations of G.

17.2. Proposition. *Let V be an virtual representation with character χ_V. If $(\chi_V, \chi_V) = 1$ and $\chi_V(1) > 0$, then χ_V is a character of an irreducible representation of G.*

Proof. Let $V = \sum n_i V_i$ for irreducible representations V_i and $n_i \in \mathbb{Z}$. Then, using the fact that the irreducible characters form an orthonomal basis, we get $1 = (\chi_V, \chi_V) = \sum n_i^2$. Thus, $n_i = \pm 1$ for one i and 0 otherwise. However, $\chi_V(1) > 0$ by assumption, so $n_i = 1$. $\square$

17.3. Theorem (Artin's theorem). *Let X be a conjugation-invariant family of subgroups of a finite group G. Then, the following are equivalent.*

(a) Any element of G belongs to a subgroup $H \in X$, that is, $G = \bigcup_{H \in X} H$.
(b) The character of any irreducible representation of G belongs to the $\mathbb{Q}$-span of characters of induced representations $\operatorname{Ind}_H^G V$ where $H \in X$ and V is an irreducible representation of H.

17.4. Remark. Part (b) is equivalent to the statement that $\operatorname{coker} \operatorname{Ind}$ is a finite group where Ind is defined by

$$\begin{aligned} \operatorname{Ind}\colon \bigoplus_{H \in X} R(H) &\to R(G) \\ (\theta_H)_{H \in X} &\mapsto \sum_{H \in X} \operatorname{Ind}_H^G \theta_H \end{aligned}$$

For $R(H)$ the ring of virtual characters of H and $R(G)$ similarly defined. In fact, this map can be represented by a matrix

$$\operatorname{Ind} = \left[(W_i, \operatorname{Ind}_H^G V_{H,j})\right]$$

where W_i ranges over all irreducible representations of G and $V_{H,j}$ ranges over all irreducible representations of H for al $H \in X$. Thus, the cokernel is a finite group if and only if the rows of this matrix are linearly independent.

Proof. For $(b) \implies (a)$, let $S = \bigcup_{H \in X} H$. Then, take $g \in G \setminus S$. Since S is conjugation invariant, no conjugate of g is in S. So, by the Mackey formula, $\chi_{\operatorname{Ind}_H^G V}(g) = 0$ for all $H \in X$ and V. Thus, by assumption, for any irreducible representation W of G is a $\mathbb{Q}$-span of characters of induced representations, so $\chi_W(g) = 0$. However, irreducible characters also span the class functions, so this would mean that any class function would vanish of g, a contradiction.

For $(a) \implies (b)$, let U be a virtual representation of G such that $(\chi_U, \chi_{\operatorname{Ind}_H^G V}) = 0$ for all H and V. Then, by Frobenius reciprocity, we get $(\chi_{\operatorname{Res}_H^G U}, \chi_V) = 0$. Thus, χ_U vanishes on H for any $H \in X$ and so $\chi_U \equiv 0$. $\square$ Expand on this

17.5. Corollary. *Any irreducible character of a finite group is a rational linear combination of induced characters from its cyclic subgroups.*

17.6. Example. Let $G = A_4$. Then, $|A_4| = 12$. Let $H = \langle 1, (123), (321) \rangle$, $K = \langle 1, (12)(34) \rangle$. Then, given characters of A_4, $\chi_1, \chi_\zeta, \chi_{\zeta^2}$, and θ, characters of H, $\chi_1, \chi_\zeta, \chi_{\zeta^2}$, and characters of K, χ_1, χ_{-1}, one can use Frobenius reciprocity to check

$$\begin{aligned}
\operatorname{Ind}_H^G \chi_1 &= \chi_1 + \theta \\
\operatorname{Ind}_H^G \chi_\zeta &= \chi_\zeta + \theta \\
\operatorname{Ind}_H^G \chi_{\zeta^2} &= \chi_{\zeta^2} + \theta \\
\operatorname{Ind}_K^G \chi_1 &= \chi_1 + \chi_\zeta + \chi_{\zeta^2} + \theta \\
\operatorname{Ind}_K^G \chi_{-1} &= 2\theta
\end{aligned}$$

and so

$$\begin{aligned}
\chi_1 &= \operatorname{Ind}_H^G \chi_1 - \frac{1}{2} \operatorname{Ind}_K^G \chi_{-1} \\
\chi_\zeta &= \operatorname{Ind}_H^G \chi_\zeta - \frac{1}{2} \operatorname{Ind}_K^G \chi_{-1} \\
\chi_{\zeta^2} &= \operatorname{Ind}_H^G \chi_{\zeta^2} - \frac{1}{2} \operatorname{Ind}_K^G \chi_{-1} \\
\theta &= \frac{1}{2} \operatorname{Ind}_K^G \chi_{-1}
\end{aligned}$$

Thus, $A_4 = \bigcup_{g \in A_4} gHg^{-1} \cup \bigcup_{g \in A_4} gKg^{-1}$.

18. Brauer's Theorem

While Artin's theorem gives us a way to write characters as a rational linear combination of characters induced from cyclic subgroups, the use of the rational numbers leaves something to be desired. We would rather have integral linear combinations. In fact, if we are willing to look a little more broadly than cyclic groups, we can arrive at such a decomposition.

18.1. Definition. Let p be a prime. A group G is called *p-elementary* if

$$G \cong Z \times P$$

where Z is a cyclic group of order prime to p, (that is $\gcd(p, |Z|) = 1$), and P is a p-group (i.e. $|P| = p^r$). G is *elementary* if it is p-elementary for some prime p.

18.2. Theorem (Brauer's Theorem). *Every complex character is a $\mathbb{Z}$-linear combination of linear characters of elementary subgroups P induced to G, that is*

$$\chi = \sum_P c_P \operatorname{Ind}_P^G \xi_p, \;\; c_P \in \mathbb{Z}$$

where ξ_P is a linear character of P.

However, the proof of Brauer's theorem is not so straight-forward. It requires use of the following theorem.

18.3. Theorem. *Let G be a finite group and let V_p be the subgroup of $R(G)$ generated by characters induced from those of p-elementary subgroups of G. Then,* $[R(G) : V_p] < \infty$ *and* $\gcd([R(G) : V_p], p) = 1$.

Actually prove Brauer's theorem.

Bibliography

[Ben95] D. J. Benson, *Representations and Cohomology I*, 1995.

[CR62] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, 1962.

[EGH+11] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina, *Introduction to Representation Theory* (2011). `http://math.mit.edu/~etingof/replect.pdf`.

[Ful97] W. Fulton, *Young Tableaux*, 1997.

[Gro] I. Gronjnowski, *Representation Theory*, The Princeton Companion to Mathematics, 419–431.

[Jam78] G. D. James, *The Representation Theory of the Symmetric Groups*, 1978.

[Sag91] B. E. Sagan, *The Symmetric Group: Representations, Combinatorical Algorithms, and Symmetric Functions*, 1991. 2nd Edition, 2001.

[See17] G. H. Seelinger, *Artin-Wedderburn Theory* (2017). See `https://ghseeli.github.io/grad-school-writings/class-notes/artin-wedderburn-notes.pdf`.

[Ser97] J.-P. Serre, *Linear Representations of Finite Groups*, 1997. Translated from the French by Leonard L. Scott.

[Smi10] K. Smith, *Groups and their Representations* (2010). `http://www.math.lsa.umich.edu/~kesmith/rep.pdf`.

[Tel05] C. Teleman, *Representation Theory* (2005). `https://math.berkeley.edu/~teleman/math/RepThry.pdf`.