

# What is a Catalanimal?

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ICERM: What is...? Seminar

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# Resources

- My IPAC 2024 Lectures:  
<https://www2.math.upenn.edu/~jhaglund/IPAC/2024.html>
- Anna Pun's 2021 Lectures:  
<https://www2.math.upenn.edu/~jhaglund/IPAC/2021.html>

# Hall-Littlewood Examples

$$\begin{aligned} H_{(4,2)} &= \text{pol} \left( \sigma \left( \frac{x_1^4 x_2^2}{1 - tx_1/x_2} \right) \right) \\ &= \text{pol} \left( \sigma \left( x_1^4 x_2^2 + tx_1^5 x_2 + t^2 x_1^6 x_2^0 + t^3 x_1^7 x_2^{-1} + \dots \right) \right) \\ &= \text{pol} \left( \chi_{42} + t\chi_{51} + t^2\chi_{60} + t^3\chi_{7,-1} + \dots \right) \\ &= s_{42} + ts_{51} + t^2 s_6 . \end{aligned}$$

$$\begin{aligned} H_{(4,2,1)} &= \text{pol} \left( \sigma \left( \frac{x_1^4 x_2^2 x_3^1}{(1 - tx_1/x_2)(1 - tx_1/x_3)(1 - tx_2/x_3)} \right) \right) \\ &\dots \\ &= s_{421} + ts_{43} + ts_{511} + (t^2 + t)s_{52} + (t^3 + t^2)s_{61} + t^4 s_7 \end{aligned}$$

# Root Ideals

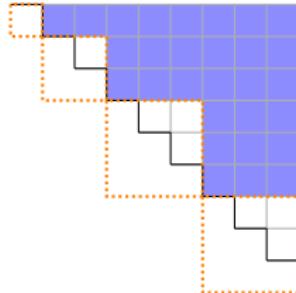
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	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

# Catalan Function Examples

- Parabolic Hall-Littlewood polynomials  $H_{\eta, \lambda} = H(R_\eta^+, \lambda)$

$$\eta = (1, 2, 3, 3) \qquad R_\eta^+ =$$



- $k$ -Schur functions  $s_\lambda^{(k)} = H(\Delta^k(\lambda), \lambda)$

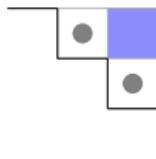
$$\Delta^4(432211111) =$$

# Catalanimal Example

Special case:  $R_+ = R_q = R_t \supseteq R_{qt}$

- $R_t \setminus R_{qt}$

- $R_{qt}$



$$H(R_+, R_+, \{\alpha_{13}\}, 111)$$

$$= \text{pol } \sigma \left( \frac{x_1 x_2 x_3 (1 - qtx_1/x_3)}{\prod_{1 \leq i < j \leq 3} (1 - qx_i/x_j)(1 - tx_i/x_j)} \right)$$

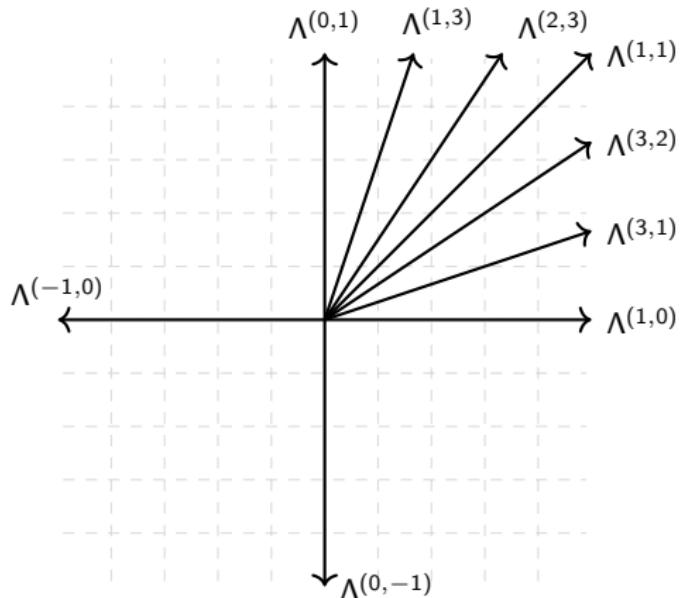
$$= \dots$$

$$= s_{111} + (q^2 + qt + t^2 + q + t)s_{21} + (q^3 + q^2t + qt^2 + t^3 + qt)s_3$$

$$= \omega \nabla e_3$$

# Elliptic Hall Algebra $\mathcal{E}$

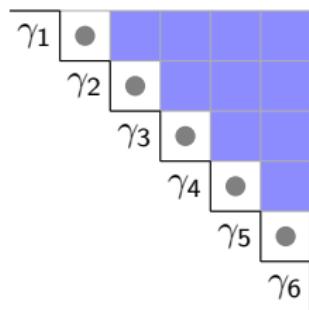
$$\Lambda^{(m,n)} = \Lambda(X^{m,n}) \cong \Lambda$$



# Negut Catalanimal

Special case:  $R_+ = R_q = R_t \supseteq R_{qt}$

- $R_t \setminus R_{qt}$
- $R_{qt}$

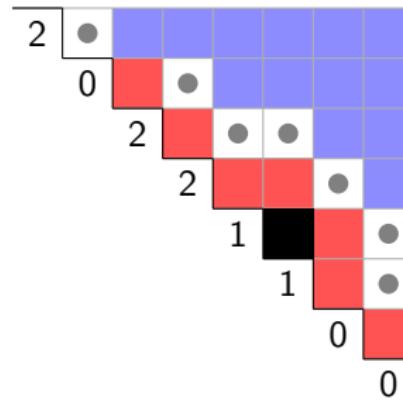
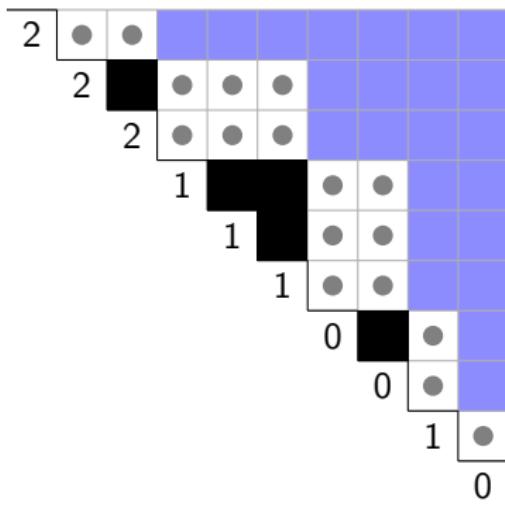


$$H(R_+, R_+, [R_+, R_+], \gamma) = \text{pol } \sigma \left( \frac{x^\gamma \prod_{i+1 < j} (1 - qtx_i/x_j)}{\prod_{i < j} (1 - qx_i/x_j)(q - tx_i/x_j)} \right)$$

# Some $(1, 1)$ -Cuddly Catalanimals

Special case:  $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$

- $R_+ \setminus R_q$
- $R_q \setminus R_t$
- $R_t \setminus R_{qt}$
- $R_{qt}$



# Catalanimals and shuffle theorems

$$\begin{array}{ccc} \mathcal{E}^+ & \xrightarrow{\sim} & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathcal{E}^+ \curvearrowright 1 \in \Lambda & = & \text{Tame} \quad \text{Catalanimal} \\ \parallel & & (\text{explicit description}) \\ & & \downarrow \text{Cauchy formula} \\ \text{Algebraic quantity} & = & \text{Combinatorial quantity} \end{array}$$

# Mystery

