K-theoretic Catalan functions

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UVA Algebra Seminar

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Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda \subseteq (n^m)}$ in variety $X = \mathrm{Gr}(m, n)$.

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$
\begin{array}{|c|ccc}
\cline{1-1}
5 & \multicolumn{1}{c}{} & & \\
\cline{1-2}
3 & \multicolumn{1}{c|}{4} & & \\
\cline{1-2}
2 & \multicolumn{1}{c|}{3} & & \\
\hline
1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\
\hline
\end{array}
\qquad\qquad
\underbrace{
\begin{array}{|c|ccc}
\cline{1-1}
8 & \multicolumn{1}{c}{} & & \\
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7 & \multicolumn{1}{c|}{9} & & \\
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1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\
\hline
\end{array}
}_{\text{standard} \,=\, \text{no repeated letters}}
$$

Example

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$$
\begin{array}{|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-2} 3 & 4 \\
\cline{1-2} 2 & 3 \\
\hline 1 & 2 & 2 & 5 \\
\hline
\end{array}
\qquad\qquad
\underbrace{
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\hline
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$$

Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$
\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline\end{array},\;
\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 1 \\ \hline\end{array},\;
\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 2 \\ \hline\end{array},\;
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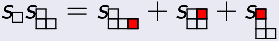
$$

$$
s_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3
$$

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$



Schur functions s_λ (cont.)

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$$s_{\ydiagram{1}} s_{\ydiagram{1,2}} = s_{\ydiagram{1,3}} + s_{\ydiagram{2,2}} + s_{\ydiagram{1,1,2}}$$

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^{\nu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^{\nu}$.

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$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$ have no tableaux description.

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Quantum cohomology of flag variety	Quantum Schuberts
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(Co)homology of affine Grassmannian	(dual) k-Schur functions
K-theory of Grassmannian	Grothendieck polynomials
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And many more!

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where $s_\lambda^{(k)}$ is a k-Schur symmetric function and $\mathrm{Gr}_{SL_{k+1}}$ is the "affine Grassmannian."

Upshot

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Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

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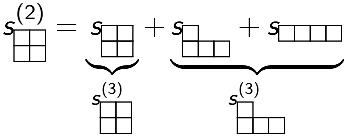

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  - Branching with t important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

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- K-theoretic Catalan functions

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Key: Catalan functions = large class of symmetric functions.

Ingredients for Catalan functions

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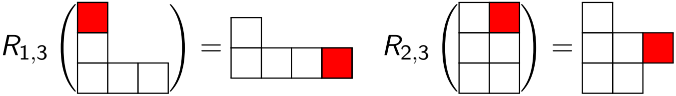
- Root ideals

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams



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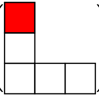
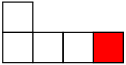
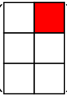
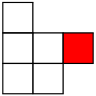
$R_{1,3}$ () $=$  $R_{2,3}$ () $=$ 

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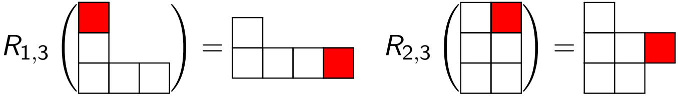

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$$s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$$

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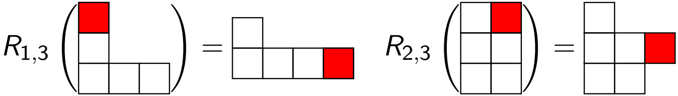

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$$\begin{aligned} s_{22} &= (1 - R_{12}) h_{22} = h_{22} - h_{31} \\ s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211} \\ &= h_{211} - h_{301} - h_{220} - h_{310} + h_{310} + \underbrace{h_{32\,\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\,\text{-}1}}_{=0} \end{aligned}$$

some terms cancel

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Simplifies formulas. E.g., for $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$ (note $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$),

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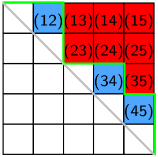
$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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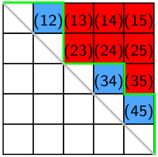

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For Ψ and $\gamma \in \mathbb{Z}^\ell$

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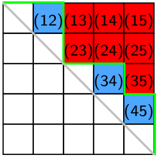
A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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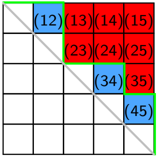

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Catalan functions

Intuition

Catalan functions interpolate between h_λ and s_λ.

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi;\lambda)$ is Schur positive!

Catalan functions

k-Schur root ideal for λ

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i,j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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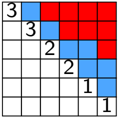
$\Delta^4(3,3,2,2,1,1) =$ 

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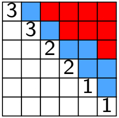
$\Delta^4(3,3,2,2,1,1) =$ 

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k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

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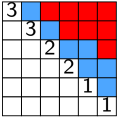
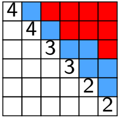
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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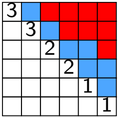
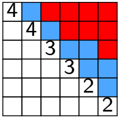
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Pieri:

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

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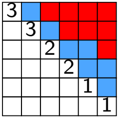
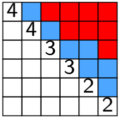
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

$$s_{\lambda}^{(k)} = s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- **K-theoretic Catalan functions**

Dual Grothendieck polynomials

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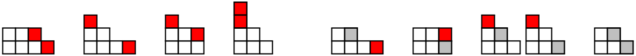

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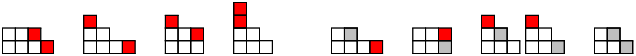

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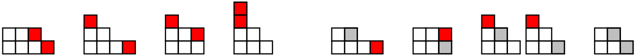

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- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m,n))$

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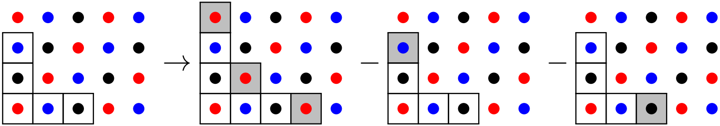
2-bounded partitions $\leftrightarrow$ 3-cores



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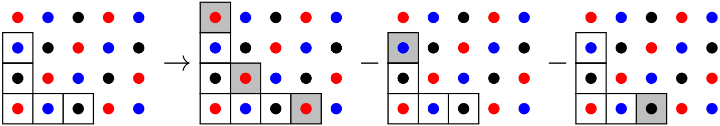

- Conjecture: $g^{(k)}_\lambda$ have positive branching into $g^{(k+1)}_\mu$ (Lam et al., 2010; Morse, 2011).

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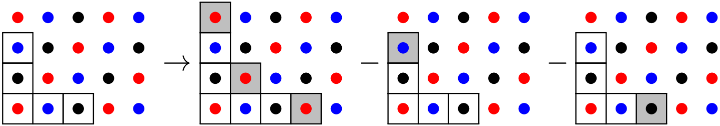

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Problem

No direct formula for $g_\lambda^{(k)}$

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Solution

Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

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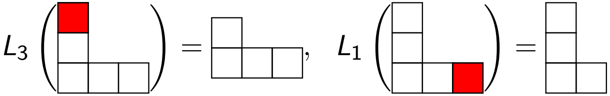
Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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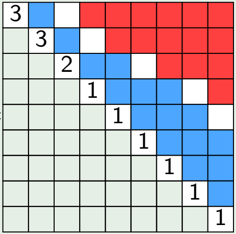
$g^{(4)}_{332111111} =$ 

$\Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$

Pieri Rule Illustrated (Recurrences)

A "graphical calculus."

$$g_1 g_{211}^{(2)}$$

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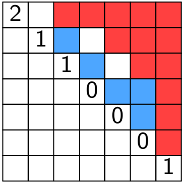

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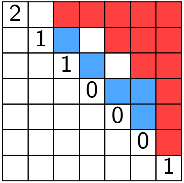
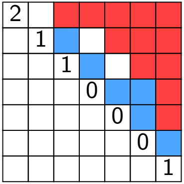
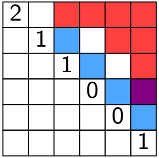
$g_1 g_{211}^{(2)}$ $=$ 

$=$ 

$+$ 

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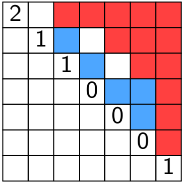

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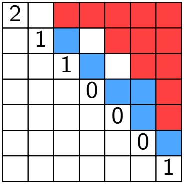

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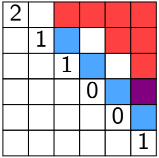

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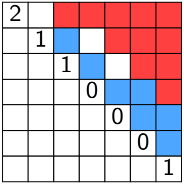

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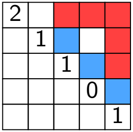

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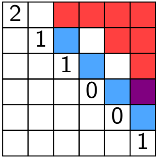

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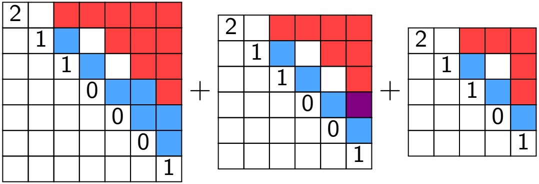

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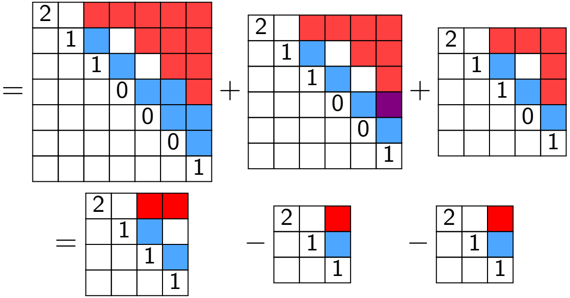

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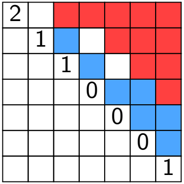
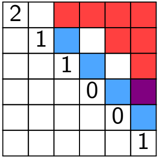
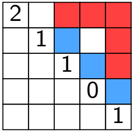
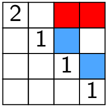
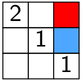
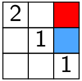
$g_1 g^{(2)}_{211}$ $=$ 

$+$ 

$+$ 

$=$ 

$-$ 

$-$ 

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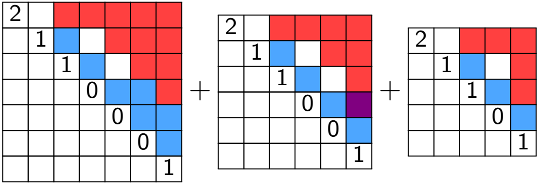

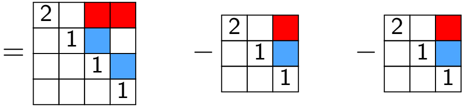

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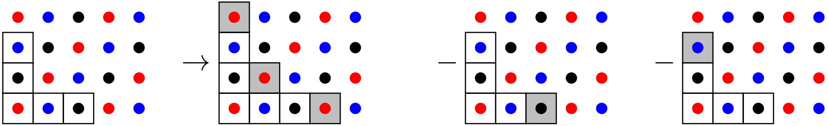
3-core perspective:



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- k-Rectangle Property fails for $g^{(k)}_{\lambda}$.

Positivity of Katalan functions

Recall (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive.

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Thank you!

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Details

$$k_m^{(r)} = \sum_{i=0}^{m} \binom{r+i-1}{i} h_{m-i} = s_m(X+r)\,,$$

a specialization of "multiSchur functions." See, e.g., Lascoux-Naruse (2014).

$$k_\gamma = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_\ell}^{(\ell-1)}$$