Diagonal Harmonics and Shuffle Theorems

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joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

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Outline

1. Symmetric functions, S_n-representations, and Frobenius characteristic
2. Diagonal harmonics and shuffle conjectures
3. Stable series approach
4. Application: extended Delta conjecture

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1. Symmetric functions, S_n-representations, and Frobenius characteristic
2. Diagonal harmonics and shuffle conjectures
3. Stable series approach
4. Application: extended Delta conjecture

Based off of slides from

- Mark Haiman: "A Shuffle Theorem for Paths Under Any Line" `https://www.math.uwaterloo.ca/~opecheni/2020-06-12-AlCoVE.pdf`
- Jennifer Morse: "Hey Series, Tell Me About the Extended Delta Conjecture" (ICERM, March 22, 2021)

Multivariate Polynomials

- $f \in \mathbb{Q}[x_1, \ldots, x_n]$ multivariate polynomial

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

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$$\begin{aligned} e_1 = x_1 + x_2 + x_3 = & h_1 \\ e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad & h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 = x_1x_2x_3 \quad & h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 = h_1 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

- $\{f \in \mathbb{Q}[x_1, \ldots, x_n] \mid \sigma.f = f \; \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the $e_1, e_2, \ldots$, or in the $h_1, h_2, \ldots$

$$3h_2 h_1^2 - h_2^2 + 6h_3 h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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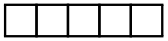
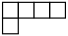
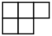
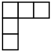
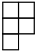
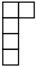
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

$1 + 1 + 1 + 1 + 1 \to$ 

Tableaux

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Collection is called SSYT(λ).

For $\lambda = (2, 1)$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur functions

Associate a polynomial to SSYT(λ).

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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Definition

For λ a partition

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Harmonic polynomials

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Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M =& \operatorname{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ =& \operatorname{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ & x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

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$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\ydiagram{1,1,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

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Schur basis expansion counts multiplicity of irreducible S_n-representations!

Schur positivity

Upshot

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1. Schur functions $\leftrightarrow$ irreducible S_n-representations.

Schur positivity

Upshot

1. Schur functions $\leftrightarrow$ irreducible S_n-representations.
2. Via Frobenius characteristic map, questions about S_n-action on vector spaces get translated to questions about Schur expansion coefficients in symmetric functions.

Getting more information

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Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\substack{\ydiagram{2,1} \\ \text{deg}=2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\substack{\ydiagram{2,1} \\ \text{deg}=1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

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Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Getting more information

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An example of bi-degree

Capturing even more information...

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- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of Δ_μ

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$$\Delta_{\yng(2,1)} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$$

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An example of bi-degree

Capturing even more information...

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$$\tilde{H}_\mu = qts_{\ydiagram{1,1,1}} + ts_{\ydiagram{2,1}} + qs_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Outline

1. Symmetric functions, S_n-representations, and Frobenius characteristic
2. **Diagonal harmonics and shuffle conjectures**
3. Stable series approach
4. Application: extended Delta conjecture

Diagonal harmonics

- $DH_n = \left\{ f \in \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}$.

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- E.g., Frobenius characteristic for DH_3:

$$(q^3 + q^2 t + qt^2 + t^3 + qt) s_{\ydiagram{1,1,1}} + (q^2 + qt + t^2 + q + t) s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

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Question

What symmetric function gives the Frobenius characteristic of DH_n?

Diagonal Harmonics

Frobenius characteristic of DH_3:

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$$\frac{t^3\tilde{H}_{111}}{-qt^2+t^3+q^2-qt}+\frac{(-q^2t-qt^2-qt)\tilde{H}_{21}}{-q^2t^2+q^3+t^3-qt}+\frac{-q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Definition

Define $\nabla:\Lambda\to\Lambda$ via

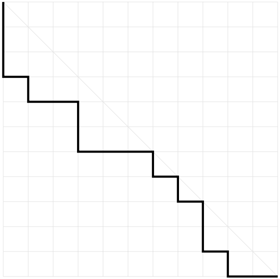
$$\nabla(\tilde{H}_\mu)=q^{n(\mu)}t^{n(\mu')}\tilde{H}_\mu$$

Nice, but not combinatorial...

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, n)$ to $(n, 0)$.



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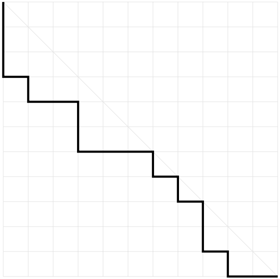

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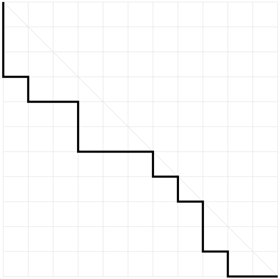

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- E.g., above $\text{area}(\lambda) = 10$.

Shuffle Conjecture

Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov, 2005)

$$\nabla e_n = \sum_{\lambda \in \mathbf{DP}_n} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(x; q^{-1}) \, .$$

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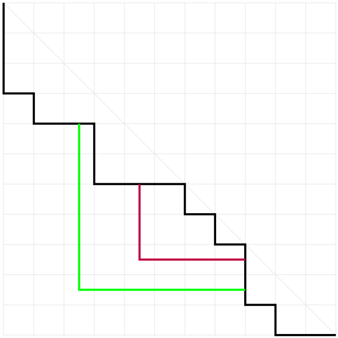
- $\mathrm{dinv}(\lambda) =$ number of balanced hooks.

dinv



Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell + 1}{a}.$$

LLT Polynomials

$$\mathcal{G}_\nu(x;q^{-1}) = \sum_{T \in \mathsf{SSYT}(\nu)} q^{-i(T)} x^T$$

for $i(T)$ the number of attacking inversions:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 3 & 5 \\
\hline
2 & 4 & 4 & 7 & 8 & 9 & 9 \\
\hline
 & & & 1 & 1 & 6 & 7 & 7 & 7 \\
\hline
\end{array}$$

- $\mathcal{G}_\nu$ is symmetric and Schur positive.

Shuffle Theorem

Representation Theory: Diagonal Harmonics

$$DH_n = \left\{ f \in \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \sum_{1 \le j \le n} \partial^a_{x_j} \partial^b_{y_j} f(x, y) = 0 \right\}$$

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Frobenius characteristic ∇e_n.

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Combinatorics: Shuffle Theorem (Carlsson-Mellit, 2018)

$$\nabla e_n = \sum_{\lambda \in \mathbf{DP}_n} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(x; q).$$

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Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

- For every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

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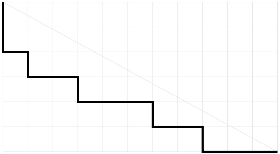
Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

$$e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all (kn, km)-Dyck paths.

Rational Path Combinatorics



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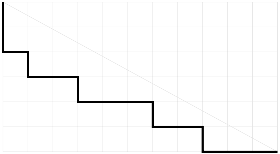

- area(λ) as before; number of boxes between λ and highest path δ.

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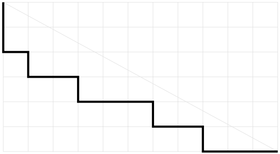
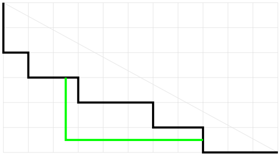


- area(λ) as before; number of boxes between λ and highest path δ.
- $\mathrm{dinv}_p(\lambda)$ = number of p-balanced hooks:



$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \qquad p = \frac{n}{m} - \epsilon$$

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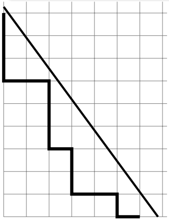

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Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \ldots, x_l) = H_{q,t} \left(\frac{x_1^{b_1} \cdots x_l^{b_l}}{\prod_{i=1}^{l-1}(1 - qtx_i/x_{i+1})} \right)_{\text{pol}}$$

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- $\chi_\lambda \leftrightarrow s_\lambda$ when $\lambda_l \geq 0$.
- Under polynomial truncation, $\mathcal{L}^{\sigma}_{\beta/\alpha} \to q^{\mathrm{dinv}_p(\lambda)}\mathcal{G}_{\nu(\lambda)}$

Cauchy Identity

- (Twisted) non-symmetric Hall-Littlewood polynomials $E^{\sigma}_{\lambda}(x;q)$ defined via Demazure-Lusztig operators.

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- $\mathcal{L}_{\beta/\alpha} = H_q(w_0(F_\beta^{\sigma^{-1}}(x;q)\overline{E_\alpha^{\sigma^{-1}}(x;q)}))$

What have we learned?

Shuffle Theorem for any path

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$$H_q \left(x^{\mathbf{b}} \frac{\prod_{i+1<j}(1 - qtx_i/x_j)}{\prod_{i<j}(1 - tx_i/x_j)} \right)$$
$$= \sum_{a_1,\ldots,a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0,a_{l-1},\ldots,a_1))/(a_{l-1},\ldots,a_1,0)}(x;q)$$

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Cauchy identity

$$\frac{\prod_{i<j}(1 - q\,t\,x_i\,y_j)}{\prod_{i\leq j}(1 - t\,x_i\,y_j)} = \sum_{\mathbf{a}\geq 0} t^{|\mathbf{a}|}\, E^{\sigma}_{\mathbf{a}}(x_1,\dots,x_l;q^{-1})\, F^{\sigma}_{\mathbf{a}}(y_1,\dots,y_l;q),$$

Outline

1. Symmetric functions, S_n-representations, and Frobenius characteristic
2. Diagonal harmonics and shuffle conjectures
3. Stable series approach
4. **Application: extended Delta conjecture**

Another family of symmetric function operators

Changing the eigenvalues of Macdonald polynomials:

$$\Delta_f H_\mu = f[B_\mu] H_\mu \qquad \Delta'_f H_\mu = f[B_\mu - 1] H_\mu$$

for any $f \in \Lambda$ and $B_\mu = \sum_{(i,j)\in\mu} q^{i-1} t^{j-1}$. (Note $\Delta'_{e_{n-1}} e_n = \nabla e_n$).

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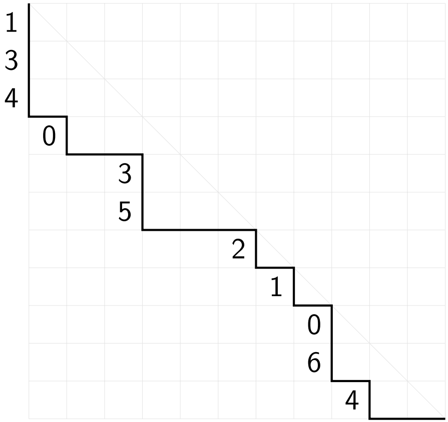
Extended Delta Conjecture (Haglund-Remmel-Wilson, 2018)

$$\Delta_{h_l} \Delta'_{e_{k-1}} e_n =$$

$$\langle z^{n-k} \rangle \sum_{\lambda \in \mathbf{DP}_{n+l}} \sum_{P \in LD_{n+l,l}(\lambda)} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^{\mathrm{wt}_+(P)} \prod_{r_i(\lambda) = r_{i-1}(\lambda)+1} \left(1 + z\, t^{-r_i(\lambda)}\right)$$

Delta Combinatorics



i	r_i
1	0
2	1
3	2
4	2
5	1
6	2
7	0
8	0
9	0
10	1
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- Label colums strictly decreasing south to north

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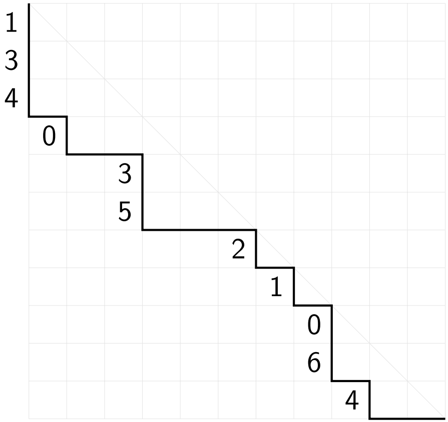

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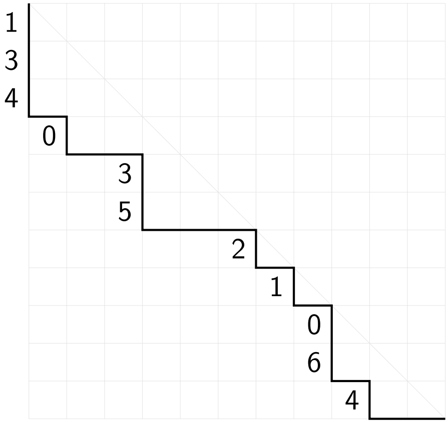

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- Label colums strictly decreasing south to north
- $\mathrm{wt}_+ = x_1^2 x_2 x_3^2 x_4^2 x_5 x_6$
- dinv $\leftrightarrow i(T)$ under suitable translation.

Application of previous program

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S.)

$$\big((h_l[B]e_{k-1}[B-1]e_n)\big)(x_1,\ldots,x_{k+l})$$

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$$
\begin{aligned}
&\big((h_l[B]e_{k-1}[B-1]e_n)\big)(x_1,\ldots,x_{k+l}) \\
&= \sum_{\substack{s\in\mathbb{N}^{k+r},|s|=n-k\\ 1\in J\subseteq[k+r],|J|=k}} \omega\left(D_{s+\varepsilon_J}\cdot 1\right) = \\
&H_{q,t}\left(\frac{(x_1\cdots x_{k+l})}{\prod_i(1-qtx_i/x_{if1})} h_{n-k}(x_1,\ldots,x_{k+l})\overline{e_l(x_2,\ldots,x_{k+l})}\right)_{\text{pol}} \\
&= \sum_{\substack{J\subseteq[k+l-1]\\ |J|=l}} \sum_{\substack{(0,\mathbf{a}),\tau\in\mathbb{N}^{k+l}\\ |\tau|=n-k}} t^{|\mathbf{a}|}q^{d(\mathbf{a},\tau,J)}(\mathcal{L}^{w_0}_{\beta/\alpha})_{\text{pol}}
\end{aligned}
$$

Stabilizing

Stable Extended Delta Theorem

$$H_q\left(\frac{\prod_{i+1<j}(1-qtx_i/x_j)}{\prod_{i<j}(1-tx_i/x_j)}(x_1\cdots x_{k+l})h_{n-k}(x_1,\dots,x_{k+l})\overline{e_l(x_2,\dots,x_{k+l})}\right)$$
$$=\sum_{\substack{J\subseteq[k+l-1]\\ |J|=l}}\ \sum_{\substack{(0,\mathbf{a}),\tau\in\mathbb{N}^{k+l}\\ |\tau|=n-k}} t^{|\mathbf{a}|}q^{d(\mathbf{a},\tau,J)}\mathcal{L}^{w_0}_{\beta/\alpha}$$

What next?

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3. Loehr-Warrington conjecture for ∇s_λ.

References

Thank you!

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