# $K$-theoretic Catalan functions

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# Overview

1. Schubert calculus
2. Catalan functions
3. K-theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety $X$.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c^{\nu}_{\lambda\mu} f_\nu$

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## Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu$ for combinatorially understood Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$.

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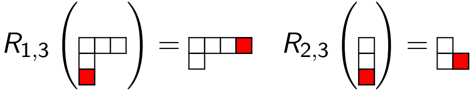

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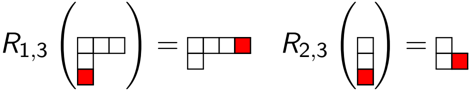


- Schur function $s_\lambda = \prod_{i<j}(1 - R_{ij}) h_\lambda$ (Jacobi-Trudi)

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| (Co)homology of Types BCD Grassmannian | Schur-$P$ and $Q$ functions |
| (Co)homology of affine Grassmannian | (dual) $k$-Schur functions |
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$K$-theory and $K$-homology of the affine Grassmannian

Simulatenously generalizes $K$-theory of Grassmannian and (co)homology of affine Grassmannian.

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1. $K$-theory classes of Grassmannian (not affine!) represented by "Grothendieck polynomials." We are interested in their dual:

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2. Homology classes of affine Grassmannian represented by $k$-Schur functions ($t = 1$).
3. (Lam et al., 2010) leave open the question: what is a direct formulation of the $K$-homology representatives of the affine Grassmannian ($K$-$k$-Schur functions)?

# Goal

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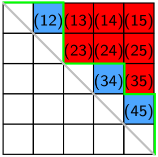
Identify $K$-$k$-Schur functions in explicit (simple) terms amenable to calculation and proofs.

# Root Ideals

A root ideal $\Psi$ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).

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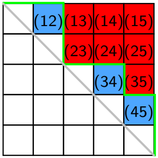


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## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For $\Psi$ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) h_\gamma(x)$$

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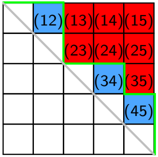


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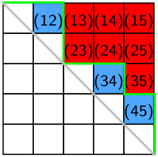


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- $\Psi =$ all roots $\implies H(\Psi;\gamma) = h_\gamma$

# Catalan functions

## $k$-Schur root ideal for $\lambda$

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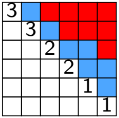
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row $i$ has $4 - \lambda_i$ non-roots

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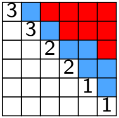
$\Delta^4(3,3,2,2,1,1) =$ 

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## $k$-Schur is a Catalan function (Blasiak et al., 2019).

For partition $\lambda$ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

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**Remark**

(Blasiak et al., 2019) show results for $k$-Schur functions with parameter $t$, but $t = 1$ specialization is necessary for Schubert calculus.

# Lowering Operators

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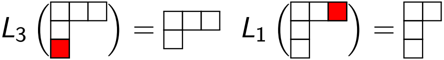
- Extra ingredient: lowering operators $L_j(h_\lambda) = h_{\lambda-\epsilon_j}$



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## Example

non-roots of $\Psi$ in blue, roots of $\mathcal{L}$ marked with $\bullet$

| | | | | |
|---|---|---|---|---|
| | (12) | | $\bullet$ | $\bullet$ |
| | | | $\bullet$ | $\bullet$ |
| | | | (34) | |
| | | | | (45) |
| | | | | |

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2$$
$$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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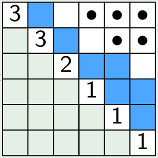
## Example

$g_{332111}^{(4)} =$ 

$\Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$

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**Theorem (Blasiak-Morse-S., 2022)**

*The* $g_\lambda^{(k)}$ *"branching coefficients" are alternating by degree, i.e. the* $b_{\lambda\mu}^{(k)}$ *in*

$$g_\lambda^{(k)} = \sum_\mu b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$$

*satisfy* $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

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*Under the Peterson Isomorphism, the "quantum Grothendieck polynomials" $\mathfrak{G}_w(z;Q)$ get sent to "closed $K$-$k$-Schur functions", $\mathfrak{g}_\lambda^{(k)} = K(\Delta^{(k)};\Delta^{(k)};\lambda)$ with suitable localization.*

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Proved using "Katalan function description."

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3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ??s_\mu^{(k)}$.
4. Answer same questions for "closed $K$-$k$-Schur's."

# Other results using Catalan function methods

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3. New formulas for Macdonald polynomials using raising operators (Blasiak-Haiman-Morse-Pun-S.)

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Thank you!


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