# $\begin{array}{c} {\bf Quantum~Groups} \\ {\bf Notes~from~a~class~taught~by~Weiqiang} \\ {\bf Wang~in~Fall~2019} \end{array}$

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#### 1. q-Numbers

Let q be an indeterminate. Then, we will work in any of the following rings

$$\mathbb{Z}[q] \subseteq \mathbb{Z}[q, q^{-1}] \subseteq \mathbb{Q}(q) \subseteq \mathbb{C}(q)$$

1.1. DEFINITION. For an indeterminate q and  $n \in \mathbb{Z}$ , we define

(a) 
$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n}$$
  
(b)  $[0]_q! := 1$  and  $[n]_q! := [n]_q[n-1]_q \dots [1]_q$  for  $n \in \mathbb{Z}_{\geq 0}$ 

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(c) If  $m \in \mathbb{Z}, n \geq 0$ , then

$$\left[\begin{array}{c} m \\ n \end{array}\right]_q = \frac{[m]_q[m-1]_q\cdots[m-n+1]_q}{[n]_q!} \stackrel{\text{If}}{=} \frac{m\geq 0}{[n]_q![m-n]_q!}$$

1.2. Remark. When the q is clear, we will drop the q from the notation and say  $[n] := [n]_q$ , etc.

1.3. Example. We compute some examples of q-numbers.

(a) 
$$[0] = 0$$

(b) 
$$[1] = 1$$

(c) 
$$[2] = q + q^{-1}$$

1.4. Proposition. We have the following simple identities on q-numbers.

(a) 
$$[-n] = -[n]$$
 for any  $n \in \mathbb{Z}$ .

(b) 
$$\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} m \\ m \end{bmatrix}_q$$
 for all  $m \in \mathbb{Z}$ .

(c) 
$$\begin{bmatrix} m \\ n \end{bmatrix}_q^q = 0 \text{ for } 0 \le m < n.$$

1.5. Proposition. We have the identity

$$\left[\begin{array}{c} m+1 \\ n \end{array}\right]_q = q^{-n} \left[\begin{array}{c} m \\ n \end{array}\right]_q + q^{m-n+1} \left[\begin{array}{c} m \\ n-1 \end{array}\right]_q$$

and also that both  $[n]_q$  and  $\begin{bmatrix} m \\ n \end{bmatrix}_q$  are elements of  $\mathbb{Z}[q,q^{-1}]$ 

Proof. We compute directly that

$$\begin{split} q^{-n} \left[ \begin{array}{c} m \\ n \end{array} \right]_q + q^{m-n+1} \left[ \begin{array}{c} m \\ n-1 \end{array} \right]_q &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[m][m-1] \cdots [m-n+2]}{[n-1]_q!} \\ &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[n][m][m-1] \cdots [m-n+2]}{[n][n-1]_q!} \\ &= (q^{-n}[m-n+1] + q^{m-n+1}[n]) \frac{[m][m-1] \cdots [m-n+2]}{[n]_q!} \\ &= \left( \frac{q^{m-2n+1} - q^{-m-1}}{q-q^{-1}} + \frac{q^{m+1} - q^{m-2n+1}}{q-q^{-1}} \right) \frac{[m] \cdots [m-n+2]}{[n]_q!} \end{split}$$

$$= \left(\frac{q^{m+1} - q^{-(m+1)}}{q - q^{-1}}\right) \frac{[m] \cdots [m - n + 2]}{[n]_q!}$$

$$= \frac{[m+1][m] \cdots [(m+1) - n + 1]}{[n]_q!} = \begin{bmatrix} m+1 \\ n \end{bmatrix}_q$$

Now, observe that

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q}{q^n} \cdot \frac{q^{2n} - 1}{q^2 - 1} = \frac{1}{q^{n-1}} (q^{2n-2} + q^{2n-4} + \dots + q^2 + 1) \in \mathbb{Z}[q, q^{-1}]$$

This immediately gives that  $[n]_q! \in \mathbb{Z}[q,q^{-1}]$ . To show  $\begin{bmatrix} m \\ n \end{bmatrix}_q \in \mathbb{Z}[q,q^{-1}]$ , we proceed by induction on m. Namely,  $\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1 \in \mathbb{Z}[q,q^{-1}]$  for all  $m \in \mathbb{Z}$ . Then,

$$\begin{bmatrix} m+1 \\ n \end{bmatrix}_q = q^n \underbrace{\begin{bmatrix} m \\ n \end{bmatrix}_q}_{\in \mathbb{Z}[q,q^{-1}]} + q^{-m+n-1} \underbrace{\begin{bmatrix} m \\ n-1 \end{bmatrix}_q}_{\in \mathbb{Z}[q,q^{-1}]} \in \mathbb{Z}[q,q^{-1}]$$

1.6. THEOREM. q-Binomial Theorem For an indeterminate z and  $r \geq 0$ ,

$$\prod_{j=0}^{r-1} (1 + q^{2j}z) = \sum_{i=0}^{r-1} q^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q z^i$$

PROOF. This follows by induction. If r = 0, then we simply have 1 = 1. Now, proceed by induction. Then,

$$\prod_{j=0}^{r} (1+q^{2j}z) = (1+q^{2r}z) \left(\sum_{i=0}^{r-1} q^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q z^i \right) = \sum_{i=0}^{r-1} q^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q z^i + \sum_{i=0}^{r-1} q^{i(r-1)+2r} \begin{bmatrix} r \\ i \end{bmatrix}_q z^{i+1}$$

Then, if we fix the z power for some  $1 \le k \le r - 1$ , we get coefficient

$$\begin{split} q^{k(r-1)} \left[ \begin{array}{c} r \\ k \end{array} \right]_q + q^{(k-1)(r-1)+2r} \left[ \begin{array}{c} r \\ k-1 \end{array} \right]_q &= q^{k(r-1)} \left[ \begin{array}{c} r \\ k \end{array} \right]_q + q^{k(r-1)+r+1} \left[ \begin{array}{c} r \\ k-1 \end{array} \right]_q \\ &= q^{kr} \left( q^{-k} \left[ \begin{array}{c} r \\ k \end{array} \right]_q + q^{-k+r+1} \left[ \begin{array}{c} r \\ k-1 \end{array} \right]_q \right) \\ &= q^{kr} \left[ \begin{array}{c} r+1 \\ k \end{array} \right]_q \end{split}$$

where the last equality follows from 1.5.

1.7. COROLLARY. As consequences to 1.6, we get

(a) For  $r \geq 1$ ,

$$\sum_{i=0}^{r} (-1)^{i} q^{-i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_{q} = 0$$

(b) Assume  $xy = q^2yx$ . Then,

$$(x+y)^n = \sum_{t=0}^n q^{t(n-t)} \begin{bmatrix} n \\ t \end{bmatrix}_q y^t x^{n-t}$$

Sometimes in the literature, q-numbers are encoded slightly differently. We present the alternate definition here.

1.8. Definition. 
$$\{n\}_v := 1 + v + v^2 + \dots + v^{n-1} = \frac{v^n - 1}{v - 1}$$

Then, the two definitions are related as follows.

1.9. Proposition. Setting  $v = q^2$ ,

$${n}_v = q^{n-1}[n]_q$$

#### 2. The Quantum Group $\mathcal{U}_q(\mathfrak{sl}_2)$

Throughout this section, we will let  $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$ . Let  $\mathbb{k}$  be a field of characteristic 0 with  $q \in \mathbb{k}$ ,  $q \neq 0$ , and q is not a root of 1.

- 2.1. Definition. We define the quantum group  $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$  to be the  $\mathbb{k}$ -algebra generated by elements  $E, F, K, K^{-1}$  with relations
  - (a)  $KK^{-1} = 1 = K^{-1}K$
  - (b)  $KE = q^2 EK$
  - (c)  $KF = q^{-1}FK$
  - (d)  $EF \hat{F}E = \frac{K K^{-1}}{q q^{-1}}$
- 2.2. Definition. We define the Drinfield double  $\tilde{\mathcal{U}}=\langle E,F,K,K'\rangle$  to be the  $\Bbbk$ -algebra with relations
  - (a)  $K'E = q^{-2}EK'$
  - $(b) K'F = q^2 E K'$
  - (c)  $EF FE = \frac{K K'}{q q^{-1}}$
- 2.3. Remark. Note that  $\tilde{U}/\langle KK'-1\rangle\cong \mathcal{U}$  and that KK' is central in  $\tilde{U}$ .
  - 2.4. Definition. We define the following maps.
  - (a) The k-linear involution  $\omega$  acts on  $\mathcal{U}$  by

$$\omega(E) = F, \omega(F) = E, \omega(K) = K^{-1}$$

(b) The k-linear anti-involution  $\tau$  (ie  $\tau(xy) = \tau(y)\tau(x)$ ) acts on  $\mathcal{U}$  by

$$\tau(E) = E, \tau(F) = F, \tau(K) = K^{-1}$$

2.5. Definition. For making computations more compact, we define the short hand

(a) 
$$[K; n] = \frac{q^n K - q^{-n} K^{-1}}{q - q^{-1}}$$

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(b) For  $n \in \mathbb{Z}_{\geq 0}$ ,  $E^{(n)} = \frac{E^n}{[n]_q!}$  and  $F^{(n)} = \frac{F^n}{[n]_q!}$ .

2.6. Theorem (PBW Theorem).  $\{F^sK^nE^r \mid s, r \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}\ forms$ a basis for  $\mathcal{U}$ .

(a) Use the commutation relations of  $\mathcal{U}$  to Sketch of Proof. show that this is a spanning set; when commuting an E past an F, one only picks up lower degree correction terms.

(b) Construct a "regular representation"  $M = \mathbb{k}[\tilde{F}, \tilde{E}, \tilde{K}, K^{-1}]$  on which  $\mathcal{U}$  acts to show linear independence. This argument is more sophisticated, but since this is a faithful representation, you get that the map  $\theta \colon \mathcal{U} \to \operatorname{End}_{\mathbb{K}}(M)$  is injective and since  $\theta(F^s K^n E^r)(1) =$  $\tilde{F}^s \tilde{K}^n \tilde{E}^r$ , which is known to be linearly independent, then the set  $\{\theta(F^sK^NE^r)\}\$  is linearly independent, thus giving us the desired linear independence by the injectivity of  $\theta$ . See [Jan95, Theorem [1.5].

2.7. Lemma (Useful Identities). (a) 
$$[K; n]E = E[K; n+2]$$

(b) 
$$[K; n]F = F[K; n-2]$$

(c) 
$$EF^s = F^sE + [s]F^{s-1}[K; 1-s]$$
 for  $s \ge 0$ 

(b) 
$$[K; n]F = F[K; n-2]$$
  
(c)  $EF^s = F^sE + [s]F^{s-1}[K; 1-s] \text{ for } s \ge 0$   
(d)  $E^rF^s = \sum_{i=0}^{\min(r,s)} \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} s \\ i \end{bmatrix}_q [i]!F^{s-i} \left(\prod_{j=1}^i [K; i-(r+s)+j]\right) E^{r-i}$ 

$$d' \ E^{(r)} F^{(s)} = \sum_{i=0}^{\min(r,s)} F^{(s-i)} \begin{bmatrix} K; 2i - (r+s) \\ i \end{bmatrix}_q E^{(r-i)} \ where \begin{bmatrix} K; c \\ i \end{bmatrix}_q := \frac{[K;c][K;c-1]\cdots[K;c-i+1]}{[i]!}.$$

Identity (d') gives one reason why divided powers are sometimes more convenient; writing identities with them can sometimes be simpler.

2.8. Remark.  $\mathcal{U}_q(\mathfrak{sl}_2)$  has no zero-divisors.

### 2.1. Finite-dimensional Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ .

2.9. Example. Let  $M = \mathbb{k} m_0 \oplus \mathbb{k} m_1$  with  $Km_0 = qm_0$  and  $Km_1 =$  $q^{-1}m_1$  and E, F actions given by

$$0 \stackrel{F}{\longleftarrow} m_1 \stackrel{E}{\longleftarrow} m_0 \stackrel{E}{\longrightarrow} 0$$

2.10. Lemma. Let M be a finite-dimensional U-module. Then, there exists an r > 0 such that  $E^r M = 0$  and  $F^r M = 0$ .

- 2.11. Definition. For  $M \in \mathcal{U}\text{-}\mathbf{mod}, \ \lambda \in \mathbb{k}^{\times}, \ \mathrm{let} \ M_{\lambda} := \{m \in M \mid$  $Km = \lambda m$ } be called the  $\lambda$ -weight subspace of M.
  - 2.12. Lemma. (a)  $EM_{\lambda} \subseteq M_{q^2\lambda}$  and  $FM_{\lambda} \subseteq M_{q^{-2}\lambda}$ . (b) If  $M_{\lambda} \neq 0$  and M is simple, then

$$M = \bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}$$

2.13. Proposition. Let M be a finite-dimensional U-module. Then,

$$M = \bigoplus_{a \in \mathbb{Z}} M_{+q^a} \oplus M_{-q^a}$$

PROOF. It is equivalent to show that the minimal polynomial of K on M is of the form  $\prod_i (K - \lambda_i)$  with distinct  $\lambda_i \in \pm q^{\mathbb{Z}}$ . To do this, set

$$h_r := \prod_{j=1-r}^{r-1} [K; r-s+j], \quad r > 0, h_0 = 1$$

Now, for s > 0, if  $F^s M = 0$ , then  $F^{s-r} h_r M = 0$  for all  $0 \le r \le s$  because

$$\left(E^r F^s \prod_{j=1}^{r-1} [K; r-s+j]\right) M = \left(\sum_{i=0}^r a_i F^{s-i} h_i \prod_{j=0}^{r-i-1} [K; -s-j] E^{r-i}\right) M$$

for  $a_i = \begin{bmatrix} r \\ i \end{bmatrix}_a \begin{bmatrix} s \\ i \end{bmatrix}_a [i]!$  by 2.7(d) allows us to use induction. Then, we have

$$0 = h_s M = \prod_{j=1-s}^{s-1} \left[ \underbrace{(q-q^{-1})^{-1} q^j K^{-1}}_{\text{Invertible scalar}} \underbrace{(K^2 - q^{-2j})}_{\text{Minimal polynomial divides this}} \right] M$$

and thus we have distinct  $\lambda_i \in \pm q^{\mathbb{Z}}$ 

## Bibliography

 $[\mathrm{Jan95}] \ \mathrm{J.\ C.\ Jantzen}, \ \mathit{Lectures\ on\ Quantum\ Groups}, \ 1995.$