

# A Catalan formula for Macdonald polynomials

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FPSAC 2023

Based on arXiv:2307.06517

July 17, 2023

# Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$  denotes the set of positive roots for  $GL_n$ , where  $\alpha_{ij} = \epsilon_i - \epsilon_j$ .

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		(23)	(24)	(25)
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A root ideal  $\Psi \subseteq R_+$  is an upper order ideal of positive roots.

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$\Psi = \text{Roots above Dyck path}$

# Symmetric functions and Schur functions

- Let  $\Lambda(X)$  be the ring of symmetric functions in  $X = x_1, x_2, \dots$
- $h_d = h_d(X) = \sum_{i_1 \leq \dots \leq i_d} x_{i_1} \cdots x_{i_d}$  with  $h_0 = 1$  and  $h_d = 0$  for  $d < 0$ .
- For any  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$ ,

$$s_\gamma = s_\gamma(X) = \det(h_{\gamma_i + j - i}(X))_{1 \leq i, j \leq n}$$

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Then,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$  = weakly decreasing sequence obtained by sorting  $\beta$ ,
- $\operatorname{sgn}(\beta)$  = sign of the shortest permutation taking  $\beta$  to  $\operatorname{sort}(\beta)$ .

# Weyl symmetrization

Define the *Weyl symmetrization operator*  $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$  by linearly extending

$$\mathbf{z}^\gamma \mapsto s_\gamma(X)$$

where  $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$ .

# Modified Macdonald polynomials

The *modified Macdonald polynomials*  $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$  are Schur positive symmetric functions in  $X = x_1, x_2, \dots$  over  $\mathbb{Q}(q, t)$ .

They differ from the *integral form Macdonald polynomials* by

$$\tilde{H}_\mu(X; q, t) = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right].$$

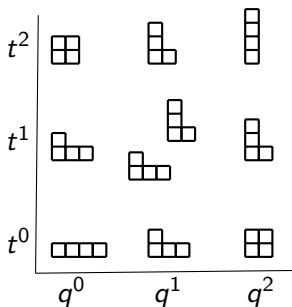
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$$\tilde{H}_{22} = s_4 + (q + t + qt)s_{31} + (q^2 + t^2)s_{22} + (qt + q^2t + qt^2)s_{211} + q^2t^2s_{1111}$$

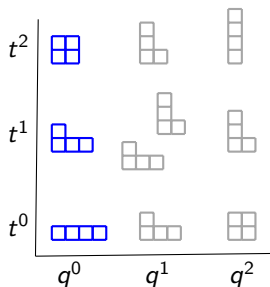




# Modified Hall-Littlewood polynomials

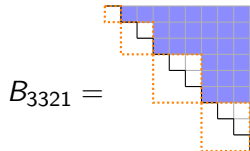
When  $q = 0$ , the modified Macdonald polynomials reduce to the *modified Hall-Littlewood polynomials*  $\tilde{H}_\mu(X; 0, t)$ .

$$\tilde{H}_{22}(X; 0, t) = s_4 + ts_{31} + t^2s_{22}$$



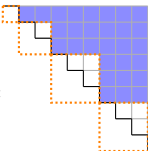
# A Catalan function for modified Hall-Littlewoods

$B_\mu$  = set of roots above block diagonal matrix with block sizes  $\mu_{\ell(\mu)}, \dots, \mu_1$



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$$B_{3321} =$$


Theorem (Weyman, Shimozono-Weyman)

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - t \mathbf{z}^\alpha)} \right),$$

where  $\mathbf{z}^\alpha = z_i / z_j$ .

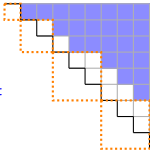
# Catalan functions for modified Hall-Littlewoods

$b_1$		
$b_2$	$b_3$	
$b_4$	$b_5$	$b_6$
$b_7$	$b_8$	$b_9$

row reading order

$$b_1 \prec b_2 \prec \cdots \prec b_n$$

$$R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \}.$$

$$R_{3321} =$$


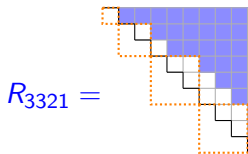
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$$\begin{aligned} \tilde{H}_\mu(X; 0, t) &= \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - t z^\alpha)} \right), \\ &= \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \end{aligned}$$

# A Catalan formula for $\tilde{H}_\mu(X; q, t)$

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$$\hat{R}_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j \}.$$

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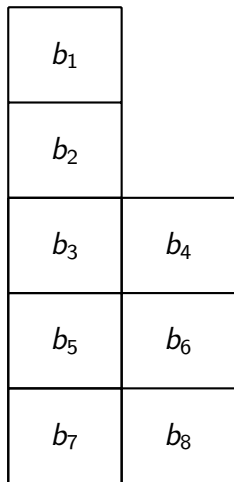
$$\hat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j\}.$$

## Theorem (Blasiak-Haiman-Morse-Pun-S.)

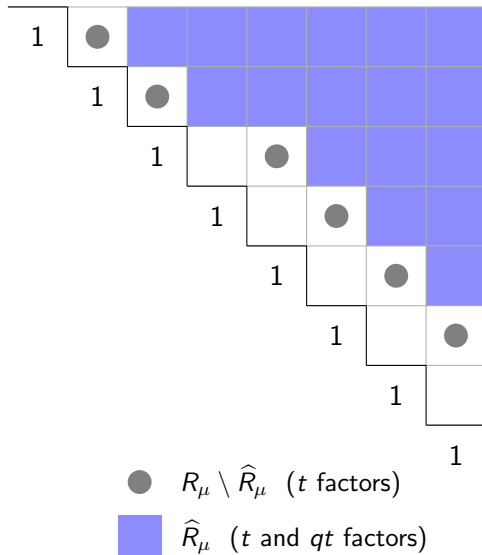
The modified Macdonald polynomial  $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$  is given by

$$\tilde{H}_\mu = \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right).$$

# Example



partition  $\mu = 22211$

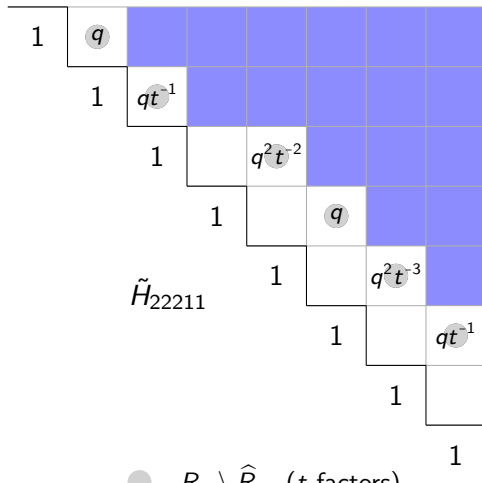




# Example

$1 - q^{\frac{z_1}{z_2}}$	
$1 - qt^{-1} \frac{z_2}{z_3}$	
$1 - q^2 t^{-2} \frac{z_3}{z_5}$	$1 - q^{\frac{z_4}{z_6}}$
$1 - q^2 t^{-3} \frac{z_5}{z_7}$	$1 - qt^{-1} \frac{z_6}{z_8}$

numerator factors  $1 - q^{\text{arm}+1} t^{-\text{leg}} z_i/z_j$



●  $R_\mu \setminus \hat{R}_\mu$  ( $t$  factors)

■  $\hat{R}_\mu$  ( $t$  and  $qt$  factors)

$q = t = 1$  specialization

$$\begin{aligned}
 & \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=t=1} \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \hat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\
 & = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\
 & = \omega h_1^n \\
 & = e_1^n
 \end{aligned}$$

$q = 0$  specialization

$$\begin{aligned}
 & \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=0} \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & = \tilde{H}_\mu(X; 0, t)
 \end{aligned}$$

# Proof of formula for $\tilde{H}_\mu$

## Definition

$\nabla$  is the linear operator on symmetric functions satisfying  $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$ , where  $n(\mu) = \sum_i (i-1)\mu_i$ .

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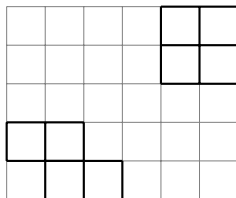
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- Start with the Haglund-Haiman-Loehr formula for  $\tilde{H}_\mu$  as a sum of LLT polynomials  $\mathcal{G}_\nu(X; q)$ .
- Apply  $\omega \nabla$  to both sides.
- Use Catalanimal formula for  $\omega \nabla \mathcal{G}_\nu(X; q)$  and collect terms.

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes.

$$\nu = \left( \begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$





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-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
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Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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$$\nu = \left( \begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes.

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
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# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

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				1	1
2	4				
	3	5			

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inversion

$$\text{inv}(T) = 4, \quad \mathbf{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

The *Catalanimal* indexed by  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\lambda \in \mathbb{Z}^n$  is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qtz^\alpha)}{\prod_{\alpha \in R_q} (1 - qz^\alpha) \prod_{\alpha \in R_t} (1 - tz^\alpha)} \right).$$

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With  $n = 3$ ,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left( \frac{z^{111} (1 - qtz_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - qz_i/z_j) (1 - tz_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

# LLT Catalananimals

For a tuple of skew shapes  $\nu$ , the *LLT Catalananimal*  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  is determined by

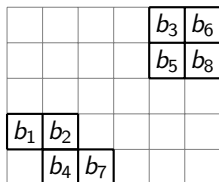
- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$ ,
- $R_+ \setminus R_q$  = pairs of boxes in the same diagonal,
- $R_q \setminus R_t$  = the attacking pairs,
- $R_t \setminus R_{qt}$  = pairs going between adjacent diagonals,
- $\lambda$ : fill each diagonal  $D$  of  $\nu$  with  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .  
Listing this filling in reading order gives  $\lambda$ .

# LLT Catalanimals

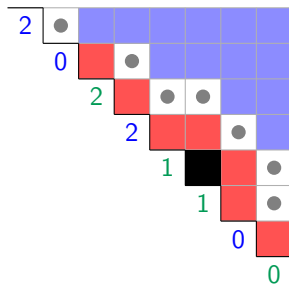
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- $R_{qt} =$  all other pairs,

$\lambda$ : fill each diagonal  $D$  of  $\nu$  with

$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end}).$



$\nu$

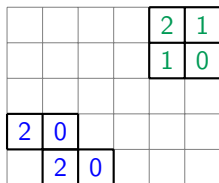


# LLT Catalanimals

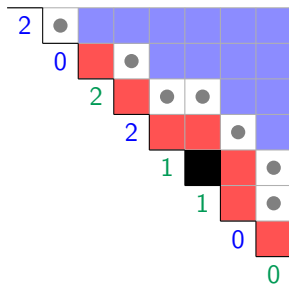
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$\lambda$ , as a filling of  $\nu$



## Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let  $\nu$  be a tuple of skew shapes and let  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega \operatorname{pol}_X(H_\nu) \\ &= c_\nu \omega \operatorname{pol}_X \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some  $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$ .



# Haglund-Haiman-Loehr formula

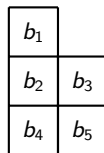
## Theorem (Haglund-Haiman-Loehr, 2005)

$$\tilde{H}_\mu(X; q, t) = \sum_D \left( \prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q),$$

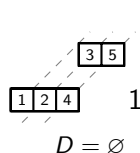
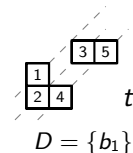
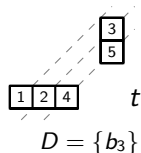
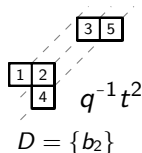
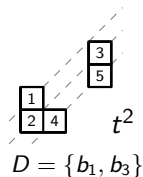
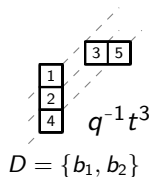
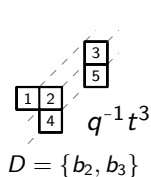
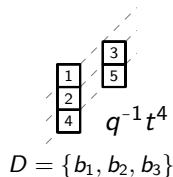
where

- the sum runs over all subsets  $D \subseteq \{(i, j) \in \mu \mid j > 1\}$ , and
- $\nu(\mu, D) = (\nu^{(1)}, \dots, \nu^{(k)})$  where  $k = \mu_1$  is the number of columns of  $\mu$ , and  $\nu^{(i)}$  is a ribbon of size  $\mu_i^*$ , i.e., box contents  $\{-1, -2, \dots, -\mu_i^*\}$ , and descent set  $\text{Des}(\nu^{(i)}) = \{-j \mid (i, j) \in D\}$ .

# Haglund-Haiman-Loehr formula example



$\mu$



## Putting it all together

- Take HHL formula  $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$  and apply  $\omega \nabla$ .

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- By construction, all the LLT Catalan animals  $H_{\nu(\mu,D)}$  appearing on the LHS will have the same root ideal data  $(R_q, R_t, R_{qt})$ .
- Collect terms to get  $\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$  factor.

$$\tilde{H}_\mu = \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

## A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

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$$\tilde{H}_{\mu}^{(s)} := \omega \sigma \left( (z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_{\mu} \setminus \hat{R}_{\mu}} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_{\mu}} (1 - qt z^{\alpha})}{\prod_{\alpha \in R_+} (1 - qz^{\alpha}) \prod_{\alpha \in R_{\mu}} (1 - tz^{\alpha})} \right)$$

## Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition  $\mu$  and positive integer  $s$ , the symmetric function  $\tilde{H}_{\mu}^{(s)}$  is Schur positive. That is, the coefficients in

$$\tilde{H}_{\mu}^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_{\nu}(X)$$

satisfy  $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$ .

# Thank you!

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