Schubert calculus and *K*-theoretic Catalan functions

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UVA Graduate Seminar

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1. An overview of Schubert calculus
2. Catalan functions: shedding new light on old problems
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

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Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Classical Example

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What are the structure constants $c^\nu_{\lambda\mu}$?

$\Lambda_m = \mathbb{C}[x_1, \ldots, x_m]^{S_m}$ is the ring of symmetric polynomials in m variables and has bases indexed by partitions.

$$\underbrace{12x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{symmetric}} \qquad \underbrace{5x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{not symmetric}}$$

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There exists a basis of Λ_m denoted $\{s_\lambda\}_\lambda$ and a surjection of rings such that

$$\Lambda_m \to H^*(\mathrm{Gr}(m, n))$$

$$s_\lambda \mapsto \begin{cases} \sigma_\lambda & \lambda \subseteq (n^m) \\ 0 & \text{otherwise.} \end{cases}$$

Cohomology structure: $\sigma_\lambda \leftrightarrow s_\lambda$ when $\lambda \subseteq (n^m)$.

$$s_\lambda s_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu s_\nu + \sum_{\nu \nsubseteq (n^m)} c_{\lambda\mu}^\nu s_\nu \leftrightarrow \sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu \sigma_\nu$$

Schur functions s_λ

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Semistandard tableaux: columns increasing and rows non-decreasing.

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Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$s_{\square}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_\square s_{\square\square} = s_{\square\square\blacksquare} + s_{\square\blacksquare} + s_\blacksquare$$

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$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} + $ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu$.

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Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

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- Structure constants $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$ are combinatorially unknown.

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(Co)homology of Grassmannian	Schur functions
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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
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And many more!

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$$\Psi \colon QH^*(Fl_{k+1}) \to H_*(Gr_{SL_{k+1}})_{loc}$$

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Upshot

Computations for Schubert polynomials can be moved into symmetric functions.

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.

k-Schur functions

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- Many definitions. A new one makes proofs easier!

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- For $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$, we have the *Jacobi-Trudi identity*

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$$s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$$

$$s_{22} = (1 - R_{12})h_{22} = h_{22} - h_{31}$$
$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13})h_{211}$$
$$= h_{211} - h_{301} - h_{220} \textcolor{red}{- h_{310} + h_{310}} + h_{32\text{-}1} + h_{400} - h_{41\text{-}1}$$

some terms cancel

Advantage: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$.

Raising Operators on Symmetric Functions

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$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 \end{cases}$$

For $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$$\Psi = $$

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Non-roots below

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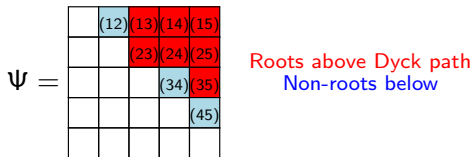
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

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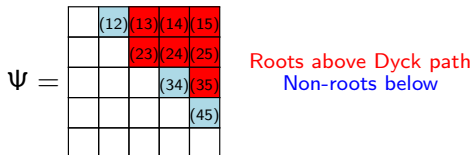Non-roots below

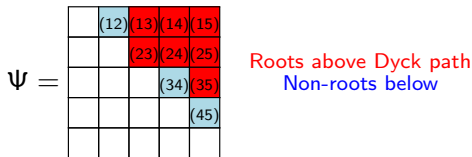## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

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- $\Psi = \varnothing \implies H(\varnothing; \gamma) = s_\gamma$
- $\Psi = $ all roots $\implies H(\Psi; \gamma) = h_\gamma$

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

Catalan functions ($t = 1$)

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- For partition λ with $\lambda_1 \leq k$, $s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)$.

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Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

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where $\langle s_{1^\ell}^\perp f, g \rangle = \langle f, s_{1^\ell} g \rangle$.

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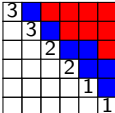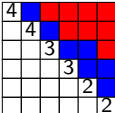
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where $\langle s_{1^\ell}^{\perp} f, g \rangle = \langle f, s_{1^\ell} g \rangle$.

$$\Delta^4(3,3,2,2,1,1) = \quad \Delta^5(4,4,3,3,2,2) =$$

Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell,\mu} s_\mu^{(k+1)}$$

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$$g_{1^2}g_{3,2} = g_{43} + g_{421} + g_{331} - g_{42} - g_{33} - 2g_{321} + g_{31}$$

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Problem

No direct formula for $g_\lambda^{(k)}$

Lowering Operators $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

$K(\Psi; \mathcal{L}; 54332)$

$= (1 - L_4)^2 (1 - L_5)^2$

$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$

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Example

$$g_{332111}^{(4)} = \qquad\qquad \Delta_6^+/\Delta^{(4)}(332111), \Delta^{(5)}(332111)$$

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satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

K-theoretic Peterson isomorphism

$$\Phi \colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

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For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothendieck polynomial",

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Future Directions

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4. Describe the image of $\mathfrak{S}_w^Q$ under Peterson isomorphism for all $w \in S_{k+1}$.

References

Thank you!

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and k-Schur Positivity*, J. Amer. Math. Soc. **32**, no. 4, 921–963.

Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for k-Schur functions*, Ph.D. thesis.

Ikeda, Takeshi, Shinsuke Iwao, and Toshiaki Maeno. 2018. *Peterson Isomorphism in K-theory and Relativistic Toda Lattice*, preprint. arXiv: 1703.08664.

Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010. *K-theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.

Morse, Jennifer. 2011. *Combinatorics of the K-theory of affine Grassmannians*, Advances in Mathematics.

Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.