

Quantum Groups
Notes from a class taught by Weiqiang
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1. q -Numbers

Let q be an indeterminate. Then, we will work in any of the following rings

$$\mathbb{Z}[q] \subseteq \mathbb{Z}[q, q^{-1}] \subseteq \mathbb{Q}(q) \subseteq \mathbb{C}(q)$$

1.1. DEFINITION. For an indeterminate q and $n \in \mathbb{Z}$, we define

- (a) $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$
- (b) $[0]_q! := 1$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ for $n \in \mathbb{Z}_{\geq 0}$
- (c) If $m \in \mathbb{Z}, n \geq 0$, then

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q[m-1]_q \cdots [m-n+1]_q}{[n]_q!} \text{ if } m \geq 0 \quad \frac{[m]_q!}{[n]_q![m-n]_q!}$$

1.2. REMARK. When the q is clear, we will drop the q from the notation and say $[n] := [n]_q$, etc.

1.3. EXAMPLE. We compute some examples of q -numbers.

- (a) $[0] = 0$
- (b) $[1] = 1$
- (c) $[2] = q + q^{-1}$

1.4. PROPOSITION. *We have the following simple identities on q -numbers.*

- (a) $[-n] = -[n]$ for any $n \in \mathbb{Z}$.
- (b) $\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} m \\ m \end{bmatrix}_q$ for all $m \in \mathbb{Z}$.
- (c) $\begin{bmatrix} m \\ n \end{bmatrix}_q = 0$ for $0 \leq m < n$.

1.5. PROPOSITION. *We have the identity*

$$\begin{bmatrix} m+1 \\ n \end{bmatrix}_q = q^{-n} \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{m-n+1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$$

and also that both $[n]_q$ and $\begin{bmatrix} m \\ n \end{bmatrix}_q$ are elements of $\mathbb{Z}[q, q^{-1}]$

PROOF. We compute directly that

$$\begin{aligned} q^{-n} \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{m-n+1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[m][m-1] \cdots [m-n+2]}{[n-1]_q!} \\ &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[n][m][m-1] \cdots [m-n+2]}{[n][n-1]_q!} \\ &= (q^{-n}[m-n+1] + q^{m-n+1}[n]) \frac{[m][m-1] \cdots [m-n+2]}{[n]_q!} \\ &= \left(\frac{q^{m-2n+1} - q^{-m-1}}{q - q^{-1}} + \frac{q^{m+1} - q^{m-2n+1}}{q - q^{-1}} \right) \frac{[m] \cdots [m-n+2]}{[n]_q!} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{q^{m+1} - q^{-(m+1)}}{q - q^{-1}} \right) \frac{[m] \cdots [m - n + 2]}{[n]_q!} \\
&= \frac{[m+1][m] \cdots [(m+1) - n + 1]}{[n]_q!} = \left[\begin{matrix} m+1 \\ n \end{matrix} \right]_q
\end{aligned}$$

Now, observe that

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q}{q^n} \cdot \frac{q^{2n} - 1}{q^2 - 1} = \frac{1}{q^{n-1}} (q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1) \in \mathbb{Z}[q, q^{-1}]$$

This immediately gives that $[n]_q! \in \mathbb{Z}[q, q^{-1}]$. To show $\left[\begin{matrix} m \\ n \end{matrix} \right]_q \in \mathbb{Z}[q, q^{-1}]$,

we proceed by induction on m . Namely, $\left[\begin{matrix} m \\ 0 \end{matrix} \right]_q = 1 \in \mathbb{Z}[q, q^{-1}]$ for all $m \in \mathbb{Z}$. Then,

$$\left[\begin{matrix} m+1 \\ n \end{matrix} \right]_q = q^n \underbrace{\left[\begin{matrix} m \\ n \end{matrix} \right]_q}_{\in \mathbb{Z}[q, q^{-1}]} + q^{-m+n-1} \underbrace{\left[\begin{matrix} m \\ n-1 \end{matrix} \right]_q}_{\in \mathbb{Z}[q, q^{-1}]} \in \mathbb{Z}[q, q^{-1}]$$

□

1.6. THEOREM. *q-Binomial Theorem For an indeterminate z and $r \geq 0$,*

$$\prod_{j=0}^{r-1} (1 + q^{2j} z) = \sum_{i=0}^{r-1} q^{i(r-1)} \left[\begin{matrix} r \\ i \end{matrix} \right]_q z^i$$

PROOF. This follows by induction. If $r = 0$, then we simply have $1 = 1$. Now, proceed by induction. Then,

$$\prod_{j=0}^r (1 + q^{2j} z) = (1 + q^{2r} z) \left(\sum_{i=0}^{r-1} q^{i(r-1)} \left[\begin{matrix} r \\ i \end{matrix} \right]_q z^i \right) = \sum_{i=0}^{r-1} q^{i(r-1)} \left[\begin{matrix} r \\ i \end{matrix} \right]_q z^i + \sum_{i=0}^{r-1} q^{i(r-1)+2r} \left[\begin{matrix} r \\ i \end{matrix} \right]_q z^{i+1}$$

Then, if we fix the z power for some $1 \leq k \leq r-1$, we get coefficient

$$\begin{aligned}
q^{k(r-1)} \left[\begin{matrix} r \\ k \end{matrix} \right]_q + q^{(k-1)(r-1)+2r} \left[\begin{matrix} r \\ k-1 \end{matrix} \right]_q &= q^{k(r-1)} \left[\begin{matrix} r \\ k \end{matrix} \right]_q + q^{k(r-1)+r+1} \left[\begin{matrix} r \\ k-1 \end{matrix} \right]_q \\
&= q^{kr} \left(q^{-k} \left[\begin{matrix} r \\ k \end{matrix} \right]_q + q^{-k+r+1} \left[\begin{matrix} r \\ k-1 \end{matrix} \right]_q \right) \\
&= q^{kr} \left[\begin{matrix} r+1 \\ k \end{matrix} \right]_q
\end{aligned}$$

where the last equality follows from 1.5. □

1.7. COROLLARY. *As consequences to 1.6, we get*

(a) For $r \geq 1$,

$$\sum_{i=0}^r (-1)^i q^{-i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q = 0$$

(b) Assume $xy = q^2 yx$. Then,

$$(x + y)^n = \sum_{t=0}^n q^{t(n-t)} \begin{bmatrix} n \\ t \end{bmatrix}_q y^t x^{n-t}$$

Sometimes in the literature, q -numbers are encoded slightly differently. We present the alternate definition here.

1.8. DEFINITION. $\{n\}_v := 1 + v + v^2 + \cdots + v^{n-1} = \frac{v^n - 1}{v - 1}$

Then, the two definitions are related as follows.

1.9. PROPOSITION. Setting $v = q^2$,

$$\{n\}_v = q^{n-1} [n]_q$$

2. The Quantum Group $\mathcal{U}_q(\mathfrak{sl}_2)$

Throughout this section, we will let $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$. Let \mathbb{k} be a field of characteristic 0 with $q \in \mathbb{k}$, $q \neq 0$, and q is not a root of 1.

2.1. DEFINITION. We define the *quantum group* $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$ to be the \mathbb{k} -algebra generated by elements E, F, K, K^{-1} with relations

- (a) $KK^{-1} = 1 = K^{-1}K$
- (b) $KE = q^2 EK$
- (c) $KF = q^{-1} FK$
- (d) $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$

2.2. DEFINITION. We define the *Drinfeld double* $\tilde{\mathcal{U}} = \langle E, F, K, K' \rangle$ to be the \mathbb{k} -algebra with relations

- (a) $K'E = q^{-2} EK'$
- (b) $K'F = q^2 EK'$
- (c) $EF - FE = \frac{K - K'}{q - q^{-1}}$

2.3. REMARK. Note that $\tilde{\mathcal{U}} / \langle KK' - 1 \rangle \cong \mathcal{U}$ and that KK' is central in $\tilde{\mathcal{U}}$.

2.4. DEFINITION. We define the following maps.

(a) The \mathbb{k} -linear involution ω acts on \mathcal{U} by

$$\omega(E) = F, \omega(F) = E, \omega(K) = K^{-1}$$

(b) The \mathbb{k} -linear anti-involution τ (ie $\tau(xy) = \tau(y)\tau(x)$) acts on \mathcal{U} by

$$\tau(E) = E, \tau(F) = F, \tau(K) = K^{-1}$$

2.5. DEFINITION. For making computations more compact, we define the short hand

- (a) $[K; n] = \frac{q^n K - q^{-n} K^{-1}}{q - q^{-1}}$
- (b) For $n \in \mathbb{Z}_{\geq 0}$, $E^{(n)} = \frac{E^n}{[n]_q!}$ and $F^{(n)} = \frac{F^n}{[n]_q!}$.

2.6. THEOREM (PBW Theorem). $\{F^s K^n E^r \mid s, r \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}$ forms a basis for \mathcal{U} .

SKETCH OF PROOF. (a) Use the commutation relations of \mathcal{U} to show that this is a spanning set; when commuting an E past an F , one only picks up lower degree correction terms.

- (b) Construct a “regular representation” $M = \mathbb{k}[\tilde{F}, \tilde{E}, \tilde{K}, \tilde{K}^{-1}]$ on which \mathcal{U} acts to show linear independence. This argument is more sophisticated, but since this is a faithful representation, you get that the map $\theta: \mathcal{U} \rightarrow \text{End}_{\mathbb{k}}(M)$ is injective and since $\theta(F^s K^n E^r)(1) = \tilde{F}^s \tilde{K}^n \tilde{E}^r$, which is known to be linearly independent, then the set $\{\theta(F^s K^n E^r)\}$ is linearly independent, thus giving us the desired linear independence by the injectivity of θ . See [Jan95, Theorem 1.5].

□

2.7. LEMMA (Useful Identities). (a) $[K; n]E = E[K; n + 2]$

(b) $[K; n]F = F[K; n - 2]$

(c) $EF^s = F^s E + [s]F^{s-1}[K; 1 - s]$ for $s \geq 0$

(d) $E^r F^s = \sum_{i=0}^{\min(r,s)} \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} s \\ i \end{bmatrix}_q [i]! F^{s-i} \left(\prod_{j=1}^i [K; i - (r + s) + j] \right) E^{r-i}$

d' $E^{(r)} F^{(s)} = \sum_{i=0}^{\min(r,s)} F^{(s-i)} \begin{bmatrix} K; 2i - (r + s) \\ i \end{bmatrix}_q E^{(r-i)}$ where $\begin{bmatrix} K; c \\ i \end{bmatrix}_q := \frac{[K; c][K; c-1] \cdots [K; c-i+1]}{[i]!}$.

Identity (d') gives one reason why divided powers are sometimes more convenient; writing identities with them can sometimes be simpler.

2.8. REMARK. $\mathcal{U}_q(\mathfrak{sl}_2)$ has no zero-divisors.

2.1. Finite-dimensional Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$.

2.9. EXAMPLE. Let $M = \mathbb{k}m_0 \oplus \mathbb{k}m_1$ with $Km_0 = qm_0$ and $Km_1 = q^{-1}m_1$ and E, F actions given by

$$0 \xleftarrow{F} m_1 \begin{array}{c} \xrightarrow{E} \\ \xleftarrow{F} \end{array} m_0 \xrightarrow{E} 0$$

2.10. LEMMA. Let M be a finite-dimensional \mathcal{U} -module. Then, there exists an $r > 0$ such that $E^r M = 0$ and $F^r M = 0$.

2.11. DEFINITION. For $M \in \mathcal{U}\text{-}\mathbf{mod}$, $\lambda \in \mathbb{k}^\times$, let $M_\lambda := \{m \in M \mid Km = \lambda m\}$ be called the λ -weight subspace of M .

2.12. LEMMA. (a) $EM_\lambda \subseteq M_{q^2\lambda}$ and $FM_\lambda \subseteq M_{q^{-2}\lambda}$.
(b) If $M_\lambda \neq 0$ and M is simple, then

$$M = \bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}$$

2.13. PROPOSITION. Let M be a finite-dimensional \mathcal{U} -module. Then,

$$M = \bigoplus_{a \in \mathbb{Z}} M_{+q^a} \oplus M_{-q^a}$$

PROOF. It is equivalent to show that the minimal polynomial of K on M is of the form $\prod_i (K - \lambda_i)$ with distinct $\lambda_i \in \pm q^{\mathbb{Z}}$. To do this, set

$$h_r := \prod_{j=1-r}^{r-1} [K; r-s+j], \quad r > 0, h_0 = 1$$

Now, for $s > 0$, if $F^s M = 0$, then $F^{s-r} h_r M = 0$ for all $0 \leq r \leq s$ because

$$\left(E^r F^s \prod_{j=1}^{r-1} [K; r-s+j] \right) M = \left(\sum_{i=0}^r a_i F^{s-i} h_i \prod_{j=0}^{r-i-1} [K; -s-j] E^{r-i} \right) M$$

for $a_i = \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} s \\ i \end{bmatrix}_q [i]!$ by 2.7(d) allows us to use induction. Then, we have

$$0 = h_s M = \prod_{j=1-s}^{s-1} \left[\underbrace{(q - q^{-1})^{-1} q^j K^{-1}}_{\text{Invertible scalar}} \quad \underbrace{(K^2 - q^{-2j})}_{\text{Minimal polynomial divides this}} \right] M$$

and thus we have distinct $\lambda_i \in \pm q^{\mathbb{Z}}$ □

Bibliography

[Jan95] J. C. Jantzen, *Lectures on Quantum Groups*, 1995.