K-theoretic Catalan functions

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Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$
\begin{array}{|c|c|c|c|}
\cline{1-1}
5 \\
\cline{1-2}
3 & 4 \\
\cline{1-2}
2 & 3 \\
\hline
1 & 2 & 2 & 5 \\
\hline
\end{array}
\qquad\qquad
\underbrace{\begin{array}{|c|c|c|c|}
\cline{1-1}
8 \\
\cline{1-2}
7 & 9 \\
\cline{1-2}
3 & 4 \\
\hline
1 & 2 & 5 & 6 \\
\hline
\end{array}}_{\text{standard} \,=\, \text{no repeated letters}}
$$

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$$\begin{array}{|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 2 & 5 \\ \hline \end{array} \qquad \underbrace{\begin{array}{|c|c|c|c|}\cline{1-1} 8 \\ \cline{1-2} 7 & 9 \\ \cline{1-2} 3 & 4 \\ \hline 1 & 2 & 5 & 6 \\ \hline \end{array}}_{\text{standard } = \text{ no repeated letters}}$$

Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$\begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline \end{array}$$

$$s_{\ydiagram{2,1}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{(2,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$$

Schur functions s_λ (cont.)

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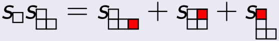

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^{\nu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^{\nu}$.

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$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$ have no tableaux description.

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
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And many more!

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where $s_\lambda^{(k)}$ is a k-Schur symmetric function and $\mathrm{Gr}_{SL_{k+1}}$ is the "affine Grassmannian."

Upshot

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Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

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- Many conjecturally equivalent definitions.

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Key: Catalan functions = large class of symmetric functions.

Ingredients for Catalan functions

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- Root ideals

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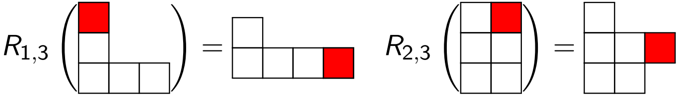

Raising Operators on Symmetric Functions

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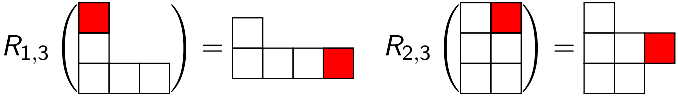

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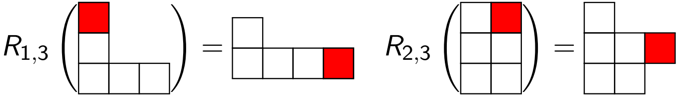

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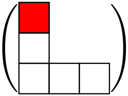
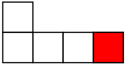
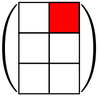
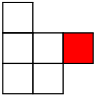
$R_{1,3}$  $=$  $R_{2,3}$  $=$ 

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$$\begin{aligned}
s_{22} &= (1 - R_{12}) h_{22} = h_{22} - h_{31} \\
s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211} \\
&= h_{211} - h_{301} - h_{220} \color{red}{- h_{310} + h_{310}} + \underbrace{h_{32\,\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\,\text{-}1}}_{=0}
\end{aligned}$$

some terms cancel

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Simplifies formulas. E.g., for $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

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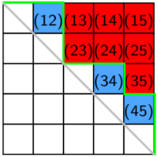
$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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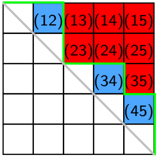

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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) h_\gamma(x)$$

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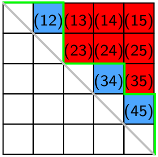

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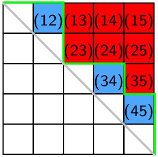
- $\Psi = \varnothing \implies H(\varnothing;\gamma) = s_\gamma$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



$\Psi =$ Roots above Dyck path

$\Delta_\ell^+ \backslash \Psi =$ Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+\backslash\Psi} (1 - R_{ij}) h_\gamma(x)$$

- $\Psi = \varnothing \implies H(\varnothing;\gamma) = s_\gamma$
- $\Psi =$ all roots $\implies H(\Psi;\gamma) = h_\gamma$

Catalan functions

Intuition

Catalan functions interpolate between h_λ and s_λ.

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive!

Catalan functions

k-Schur root ideal for λ

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i,j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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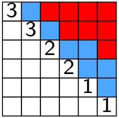
$\Delta^4(3,3,2,2,1,1) =$ 

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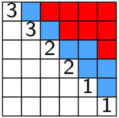
$\Delta^4(3,3,2,2,1,1) =$ 

$\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

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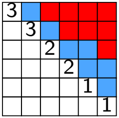
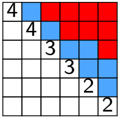
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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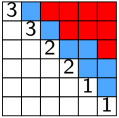
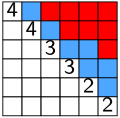
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Pieri:

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell,\mu} s_\mu^{(k+1)}$$

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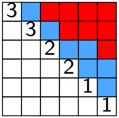
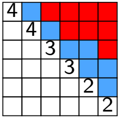
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

$$s_{\lambda}^{(k)} = s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- K-**theoretic Catalan functions**

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.

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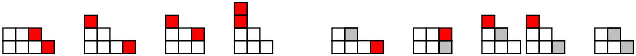

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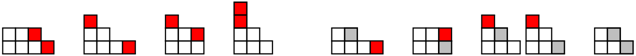

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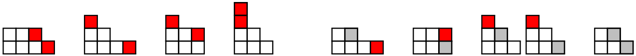

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- $g_\lambda = \prod_{i<j}(1 - R_{ij})k_\lambda$ for k_λ and inhomogeneous analogue of h_λ.
- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m,n))$

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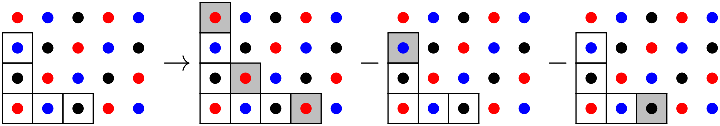
2-bounded partitions $\leftrightarrow$ 3-cores



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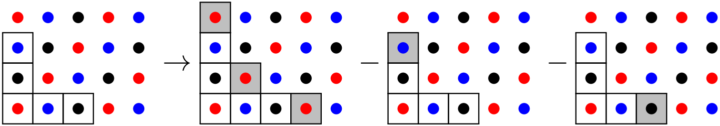

- Conjecture: $g_\lambda^{(k)}$ have positive branching into $g_\mu^{(k+1)}$ (Lam et al., 2010; Morse, 2011).

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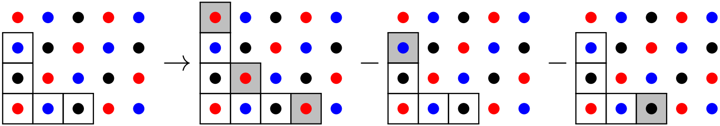

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Problem

No direct formula for $g_\lambda^{(k)}$

Solution

Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

K-k-Schur functions

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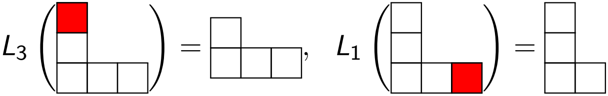
Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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Answer (Blasiak-Morse-S., 2020)

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For K-homology of affine Grassmannian, $g_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$ since this family satisfies the Pieri rule.

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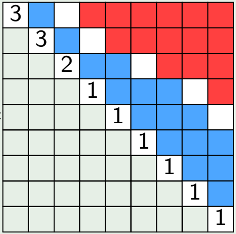
$g^{(4)}_{332111111} =$ 

$\Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$

Pieri Rule Illustrated (Recurrences)

A "graphical calculus."

$$g_1 g_{211}^{(2)}$$

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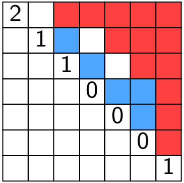

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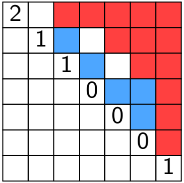
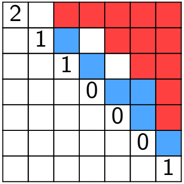
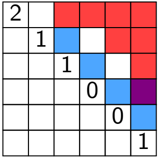
$g_1 g_{211}^{(2)}$ $=$ 

$=$  $+$ 

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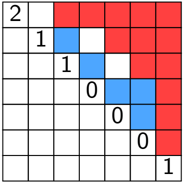
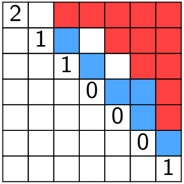
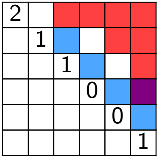
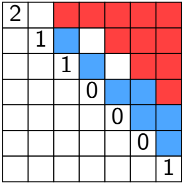
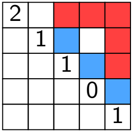
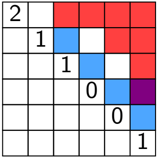
$g_1 g^{(2)}_{211}$ = 

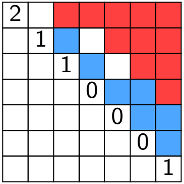
= 

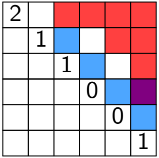
+ 

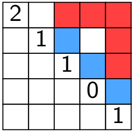
= 

+ 

+ 

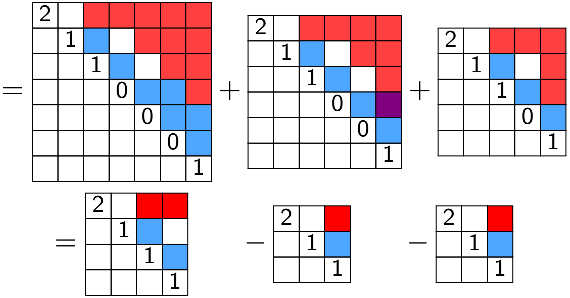
$g_1 g_{211}^{(2)} =$ 

$+$ 

$+$ 

Pieri Rule Illustrated (Straightening)

$g_1 g^{(2)}_{211}$



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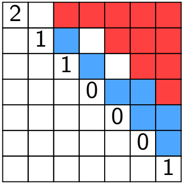

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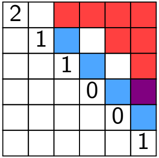

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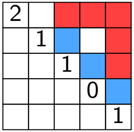

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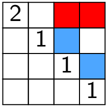

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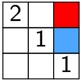

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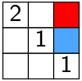

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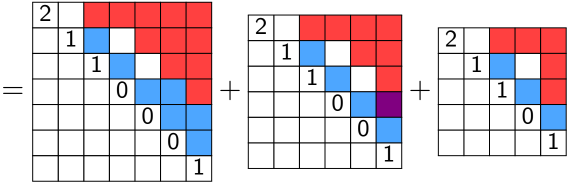
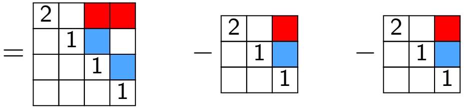


$$=$$



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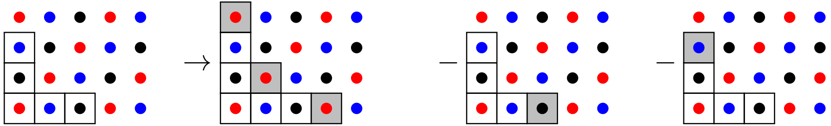
3-core perspective:



Branching Positivity

Theorem (Blasiak-Morse-S., 2020)

Branching Positivity

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The $g_\lambda^{(k)}$ *are "shift invariant", i.e. for* $\ell = \ell(\lambda)$

$$G_{1^\ell}^{\perp} g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Branching Positivity

Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ *are "shift invariant", i.e. for* $\ell = \ell(\lambda)$

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Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

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satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

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$$\tilde{g}_w = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

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- k-Rectangle Property fails for $g^{(k)}_\lambda$.

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- $K(\Psi; RC^a(\Psi); \lambda) = \sum_\mu b_\mu s_\mu$ satisfies $b_\mu \in \mathbb{Z}_{\geq 0}$.

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Thank you!

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