Spin Representation Theory of Symmetric Groups and Related Combinatorics
Notes from a reading course in Fall 2018

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1. Introduction (presented by Jinkui Wan)

When discussing the representation theory of the symmetric group, one considers *linear representations* which are group homomoprhisms

$$\mathfrak{S}_n \to GL(V)$$

In 1911, Schur started considering projective representations

$$\mathfrak{S}_n \to PGL(V) = GL(V)/\mathbb{C}^*$$

leading to the projective representation theory of $\mathfrak{S}_n$. It turns out that this corresponds to the linear representation theory of an extension of $\mathfrak{S}_n$, denoted $\tilde{\mathfrak{S}}_n$ and referred to as the *double cover of the symmetric group*, fitting into the short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{\mathfrak{S}}_n \to \mathfrak{S}_n \to 1$$

where, if $\mathbb{Z}/2\mathbb{Z} = \{1, z\}$, then z is central in $\tilde{\mathfrak{S}}_n$, which gives us that $z = 1$ or $z = -1$.

When $z = 1$, we have the representation theory of $\mathfrak{S}_n$. When, $z = -1$, we have the representation theory of the *spin symmetric group algebra*

$$\mathbb{C}\mathfrak{S}_n^- = \mathbb{C}\mathfrak{S}_n/\langle z+1\rangle = \left\langle t_1, \ldots, t_n \;\middle|\; \begin{array}{l} t_i^2 = 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ t_i t_j = -t_j t_i \text{ when } |i-j| > 1 \end{array} \right\rangle$$

which is equipped with a $\mathbb{Z}/2\mathbb{Z}$-grading. So, when we discuss spin representations of $\mathfrak{S}_n$, we are discussing linear representations of $\mathbb{C}\mathfrak{S}_n^-$. Our program to establish these ideas is as follows.

Part I

(1) Basics of associative superalgebras
(2) Connection to Hecke-Clifford (or Sengeev) algebra, $\mathcal{H}_n$
(3) Split conjugacy classes in a finite supergroup
(4) Characteristic map
(5) Schur-Q functions
(6) Schur-Sergeev duality
(7) Seminormal form of irreducible representations

Part II

(1) Centers of $\mathbb{C}\mathfrak{S}_n^-$ (analog of Farahat-Higman theory for $\mathbb{C}\mathfrak{S}_n$)
(2) Coinvariant theory for $\mathbb{C}\mathfrak{S}_n^-$
(3) Spin Kostka polynomials
(4) Quantum deformation (in particular, Olshanki-Sergeev duality)

2. Generalities for Associative Superalgebras (presented by Jinkui Wan)

2.1. Definitions and Examples.

2.1. Definition. (a) A *vector superspace* (over $\mathbb{C}$) is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where elements of $V_{\bar{0}}$ are called *even* and elements of $V_{\bar{1}}$ are called *odd*. For $v \in V_i$, $i \in \mathbb{Z}/2\mathbb{Z}$, we say $|v| = i$.

(b) If V is a vector superspace with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$, we say the *graded dimension* of V is (m, n), denoted $\mathbf{dim} V = (m, n)$.

(c) A *superalgebra* is a $\mathbb{C}$-algebra A with a $\mathbb{Z}/2\mathbb{Z}$-grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$ such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}/2\mathbb{Z}$.

(d) A *superalgebra ideal* is a homogeneous ideal, that is, a subset $I \subseteq A$ such that $I = I_{\bar{0}} \oplus I_{\bar{1}} = (I \cap A_{\bar{0}}) \oplus (I \cap A_{\bar{1}})$ as vector spaces and $A_i I_j \subseteq I_{i+j}$ for all $i, j \in \mathbb{Z}/2\mathbb{Z}$.

(e) A superalgebra that has no non-trivial ideals is called *simple*.

(f) A *superalgebra homomorphism* $\theta \colon A \to B$ is an even algebra homomorphism, that is, an algebra homomorphism sending $A_i \to B_i$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.

(g) Given superalgebras A and B, the tensor product $A \otimes B$ is a superalgebra with multiplication

$$(a \otimes b)(a' \otimes b') = (-1)^{|a||b|} aa' \otimes bb'$$

for homogeneous elements and extended by linearity.

(h) A *commutative superalgebra* is one that is graded commutative, that is

$$yx = (-1)^{|x||y|} xy$$

Thus, the *supercommutator* of a superalgebra is given by

$$[x, y] = xy - (-1)^{|x||y|} yx$$

and the *supercenter* is given by

$$Z(A) = \{a \in A \mid [a, x] = 0 \text{ for all } x \in A\}$$

which is different than the center of an ungraded algebra.

(i) Given a superalgebra A, we let $|A|$ be the associative algebra where we forget the grading on A.

2.2. Example. (a) Let $V = V(m|n)$, the vector superspace with $\mathbf{dim} V = (m, n)$. Then, $\mathrm{End}_{\mathbb{C}}(V)$ is a superalgebra and is isomorphic to the matrix superalgebra

$$M(m|n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{c} a \text{ is an } m \times m \text{ matrix} \\ b \text{ is an } m \times n \text{ matrix} \\ c \text{ is an } n \times m \text{ matrix} \\ d \text{ is an } n \times n \text{ matrix} \end{array} \right\}$$

or, in other words, $M(m|n)$ consists of all $m|n$-block matrices and has $\mathbf{dim}M(m|n) = (m^2+n^2, 2mn)$. Furthermore, $M(m|n)$ is a simple superalgebra since $|M(m|n)|$ is simple as a $\mathbb{C}$-algebra.

(b) Let $V = V(n|n)$ and $p \in \mathrm{End}_{\mathbb{C}}(V)$ be an odd involution (that is, it sends $V_i \to V_{i+1}$ for $i \in \mathbb{Z}/2\mathbb{Z}$). Then, we define

$$\mathcal{Q}(V) := \{f \in \mathrm{End}_{\mathbb{C}}(V) \mid fp = (-1)^{|f|}pf\} = \mathcal{Q}(V)_{\bar{0}} \oplus \mathcal{Q}(V)_{\bar{1}}$$

$\mathcal{Q}(V)$ is also a superalgebra. Moreover, if we pick a basis $\{v_1, \dots, v_n\}$ of $V_{\bar{0}}$ and let $v_i' = p(v_i)$ for $1 \le i \le n$, we have that, with respect to the basis $\{v_1, \dots, v_n, v_1', \dots, v_n'\}$, $\mathcal{Q}(V)$ is isomorphic to

$$\mathcal{Q}(n) := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M(n|n) \right\}$$

and it is simple.

(c) The Clifford algebra $\mathcal{C}\ell_n$ is the superalgebra generated by the odd elements $c_1, \dots, c_n$ subject to the relations

$$\begin{cases} c_i^2 = 1 \\ c_i c_j = -c_j c_i & \forall 1 \le i \ne j \le n \end{cases}$$

2.3. Lemma. *There exist isomorphisms of superalgebras*

(a) $M(m|n) \otimes M(k|l) \cong M(mk+nl|mk+nl)$
(b) $M(m|n) \otimes \mathcal{Q}(k) \cong Q((m+n)k)$
(c) $\mathcal{Q}(m) \otimes \mathcal{Q}(n) \cong M(mn|mn)$

Proof. For part (a), we note that

$$\mathrm{End}_{\mathbb{C}}(V(m|n)) \otimes \mathrm{End}_{\mathbb{C}}(V(k|l)) \cong \mathrm{End}_{\mathbb{C}}(V(mk+ml|mk+nl))$$

under the isomorphism sending $f \otimes g$ to the endomorphism of $V(mk+ml|mk+nl)$ mapping $v \otimes w$ to $(-1)^{|g||v|} f(v) \otimes g(w)$.

For part (b), we have

$$\mathrm{End}(V(m|n)) \otimes \mathcal{Q}(V(k|k), p) \cong Q(V(m|n) \otimes V(k|k), id \otimes p)$$

For (c), one explicitly checks that $\mathcal{Q}(1) \otimes \mathcal{Q}(1) \cong M(1|1)$ and then inductively applies (a) and (b) above. □

2.4. Corollary. *Since $\mathcal{C}\ell_{m+n} \cong \mathcal{C}\ell_m \otimes \mathcal{C}\ell_n$ under the isomorphism sending generators $c_1, \dots, c_n$ to $c_1 \otimes 1, \dots, c_n \otimes 1$ and $c_{n+1}, \dots, c_{n+m}$ to $1 \otimes c_1, \dots, 1 \otimes c_m$, we have the corollaries*

(a) $\mathcal{C}\ell_1 \cong \mathcal{Q}(1)$ *under the isomorphism* $c_1 \mapsto p(v_1)$
(b) $\mathcal{C}\ell_2 \cong M(1|1)$ *since* $\mathcal{C}\ell_2 \cong \mathcal{C}\ell_1 \otimes \mathcal{C}\ell_1 \cong \mathcal{Q}(1) \otimes \mathcal{Q}(1) \cong M(1|1)$
(c) $\mathcal{C}\ell_{2k} \cong M(2^{k-1}|2^{k-1})$
(d) $\mathcal{C}\ell_{2k-1} \cong \mathcal{Q}(2^{k-1})$
(e) and thus, $\mathcal{C}\ell_n$ *is simple by parts (c) and (d).*

2.2. Classification of Simple Superalgebras.

2.5. Theorem. *There are two types of finite dimensional simple associative superalgebras over* $\mathbb{C}$*:*

(a) $M(m|n)$
(b) $\mathcal{Q}(n)$

2.3. Wedderburn Theorem and Schur's Lemma.

2.6. Definition. (a) A *(super)module* over a superalgebra A is a vector space $M = M_{\overline{0}} \oplus M_{\overline{1}}$ with a left action of A on M such that $A_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}/2\mathbb{Z}$.

(b) A homomorphism between A-modules M and N is a linear map $f\colon M \to N$ such that

$$f(am) = (-1)^{|f||a|} a f(m) \text{ for all } a \in A, m \in M$$

and

$$\mathrm{Hom}_A(M, N) := \mathrm{Hom}_A(M, N)_{\overline{0}} \oplus \mathrm{Hom}_A(M, N)_{\overline{1}}$$

where $f \in \mathrm{Hom}_A(M, N)_{\overline{1}} \subseteq \mathrm{Hom}_{\mathbb{C}}(M, N)$ is such that $f(M_i) \subseteq N_{i+1}$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.

2.7. Definition. An A-module is said to be *simple* if it is nonzero and has no proper A-submodules. An A-module M is said to be *semisimple* if every A-submodule of M is a direct summand of M.

2.8. Theorem (Super Wedderburn Theorem). *The following are equivalent for a finite dimensional superalgebra A.*

(a) Every A-module is semisimple
(b) A is a finite direct sum of left simple superideals
(c) A is a direct product of a finite number of simple algebras

2.9. Definition. Thus, we say a superalgebra A is *semisimple* if it satisfies one of the three conditions.

2.10. Example. (a) $M(m|n) = I_1 \oplus I_2 \oplus \cdots \oplus I_m \oplus I_{m+1} \oplus \cdots \oplus I_{m+n}$ where $I_k = M(m|n)E_{k,k}$ for $1 \le k \le m+n$.

(b) $\mathcal{Q}(n) = J_1 \oplus \cdots \oplus J_n$ where

$$J_k = \mathcal{Q}(n)(E_{k,k} + E_{n+k,n+k})$$

(c) $\mathrm{Hom}_{M(m|n)}(I_k, I_k) \cong \mathbb{C}$ and $\mathrm{Hom}_{\mathcal{Q}(n)}(J_k, J_k) \cong \mathbb{C} \oplus \mathbb{C}p$. Importantly, the latter space is not 1-dimensional despite J_k being 1-dimensional!

Check this. Most likely depends on your choice of basis and involution.

2.11. Corollary. *A finite dimensional semisimple superalgebra A is isomorphic to*

$$A \cong \bigoplus_{i=1}^{m} M(r_i|s_i) \oplus \bigoplus_{j=1}^{n} \mathcal{Q}(n_j)$$

where $m = m(A)$ and $q = q(A)$ are invariants of A.

2.12. Definition. A simple A-module $V$is said to be of type M (resp. type Q) if it is annihilated by all but one summand of the form $M(r_i|s_i)$ (resp. $\mathcal{Q}(n_j)$).

2.13. Corollary. *(a) The number of non-isomorphic simple A-modules is given by $m(A) + q(A) = \dim(Z(|A|) \cap A_{\bar{0}})$.*

(b) The number of non-isomorphic simple A-modules of type Q is given by $q(A) = \dim(Z(|A|) \cap A_{\bar{1}})$.

2.14. Theorem (Schur's Lemma). *If M and L are simple A-modules, then*

$$\dim \operatorname{Hom}_A(M, L) = \begin{cases} 1 & \text{if } M \cong L \text{ of type } M \\ 2 & \text{if } M \cong L \text{ of type } Q \\ 0 & \text{otherwise} \end{cases}$$

2.15. Remark. (a) A simple A-module M is of type M if and only if $|M|$ is a simple $|A|$-module

(b) A simple A-module M is of type Q if and only if $|M|$ is a direct sum of two non-isomorphic simple $|A|$-modules.

3. Split Conjugacy Classses in a Finite Supergroup (presented by Jinkui Wan)

Throughout, let G be a finite group with index 2 subgroup $G_0 \leq G$.

3.1. Definition. (a) We say that the elements of G_0 are *even elements* and the elements of $G_1 := G \setminus G_0$ are *odd elements*,

(b) $\mathbb{C}G$ is a superalgebra, which we will denote $\mathbb{C}[G, G_0]$

3.2. Theorem (Super MAschke's Theorem). $\mathbb{C}[G, G_0]$ *is semisimple*

3.3. Proposition. *(a) If $g \in G_i$, $h \in G$, then $hgh^{-1} \in G_i$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.*

(b) The number of non-isomorphic simple $\mathbb{C}[G, G_0]$-modules is equal to the number of even conjugacy classes in G.

(c) The number of non-isomorphic simple $\mathbb{C}[G, G_0]$-module of type Q is equal to the number of odd conjugacy classes in G.

Proof. We note that, by the usual Artin-Wedderburn theorem, $\mathbb{C}G$ decomposes into a direct sum of simple matrix algebras, each of which has a 1-dimensional center and can be indexed by a conjugacy class of G via $c_i = \sum_{g \in \mathcal{C}_i} g$ where $\mathcal{C}_i$ is a conjugacy class of G. In fact, this shows in the classical theory that the number of conjugacy classes of G equal the number of irreducible representations.

Since conjugacy classes are either even or odd by (a), which is left as an exercise, (b) follows because $\dim(Z(\mathbb{C}G) \cap \mathbb{C}[G, G_0]_{\bar{0}})$ is equal to the number of non-isomorphic simple $\mathbb{C}[G, G_0]$-modules and (c) follow from the fact that $\dim(Z(\mathbb{C}G) \cap \mathbb{C}[G, G_0]_{\bar{1}})$ gives the number of those of type Q. $\square$

Now, consider the following situation. Let $\tilde{G}$ be a group such that there exists an index 2 subgroup $\tilde{G}_0 \leq \tilde{G}$ and there exists a short exact sequence

$$1 \to \{1, z\} \to \tilde{G} \xrightarrow{\theta} G \to 1$$

where $z^2 = 1$ and z is central in $\tilde{G}$. Then,

3.4. Proposition. *For C a conjugacy class of G, the preimage*

$$\theta^{-1}(C) = \{g, gz \mid g \in C\} \subseteq \tilde{G}$$

has that

(a) $\theta^{-1}(C)$ is a single conjugacy class in $\tilde{G}$ if g is conjugate to zg in $\tilde{G}$ or
(b) $\theta^{-1}(C)$ splits into two conjugacy classes in $\tilde{G}$ if there exists a $g \in C$ such that g is not conjugate to zg. In this case, we call C split.

3.5. Definition. We set

$$\mathbb{C}\tilde{G}^{-} := \mathbb{C}[G, G_0]/\langle z + 1\rangle$$

and call a $\mathbb{C}\tilde{G}^{-}$-module a *spin* $\mathbb{C}[\tilde{G}, \tilde{G}_0]$-module.

3.6. Proposition. *(a) We have the isomorphism of superalgebras*

$$\mathbb{C}[\tilde{G}, \tilde{G}_0] \cong \underbrace{\mathbb{C}[G, G_0]}_{(z=1)} \oplus \underbrace{\mathbb{C}\tilde{G}^{-}}_{(z=-1)}$$

(b) The number of non-isomorphic simple spin $\mathbb{C}[\tilde{G}, \tilde{G}_0]$-modules is equal to the number of even split conjugacy classes of G.
(c) The number of non-isomorphic simple spin $\mathbb{C}[\tilde{G}, \tilde{G}_0]$-modules of type Q is equal to the number of odd split conjugacy classes of G.

Proof. Part (a) follows from the semisimplicity of $\mathbb{C}[\tilde{G}, \tilde{G}_0]$.

Now, (a) tells us that

$$Z(|\mathbb{C}\tilde{G}^{-}|) = \{a \in Z(|\mathbb{C}[\tilde{G}, \tilde{G}_0]|) \mid za = -a\}$$

So let

$$\underbrace{D_1, zD_1, D_2, zD_2, \ldots, D_r, zD_r}_{\text{split}}, \underbrace{D_{r+1}, \ldots, D_{r+s}}_{\text{non-split}}$$

be the conjugacy classes of $\tilde{G}$ where r is the number of split conjugacy classes in G, $D_i \cap zD_i = \varnothing$ for $1 \leq i \leq r$ and $zD_j = D_j$ for $r + 1 \leq j \leq r + s$. Then,

$$Z(|\mathbb{C}\tilde{G}|) \cap \mathbb{C}\tilde{G}_{\overline{0}} = \{a \in Z(|\mathbb{C}\tilde{G}|) \mid a \text{ is even and } za = -a\}$$

has basis $d_{i_1} - zd_{i_1}, d_{i_2} - zd_{i_2}, \ldots, d_{i_k} - zd_{i_k}$ for d_i even and Check this part

$$Z(|\mathbb{C}\tilde{G}|) \cap \mathbb{C}\tilde{G}_{\overline{1}} = \{a \in Z(|\mathbb{C}\tilde{G}|) \mid a \text{ is odd and } za = -a\}$$

has basis $d_{j_1} - zd_{j_1}, d_{j_2} - zd_{j_2}, \ldots, d_{j_k} - zd_{j_\ell}$ for d_j odd. $\square$

3.7. Example. We have

$$1 \to \{1, z\} \to \tilde{\mathfrak{S}}_n \xrightarrow{\theta_n} \mathfrak{S}_n \to 1$$

where $z \in \tilde{\mathfrak{S}}_n$ is even and central, the subgroup of index 2 is $\tilde{A}_n$, and

$$\mathbb{C}\mathfrak{S}_n^- := \mathbb{C}\tilde{\mathfrak{S}}_n / \langle z + 1 \rangle$$

is the spin symmetric group algebra.

3.8. Definition. Throughout the remainder of these notes, we define $\theta_n \colon \tilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ to be the double covering may above.

4. A Morita Superequivalence (presented by Jinkui Wan)

Since $\mathfrak{S}_n$ acts on $\mathcal{C}\ell_n$ via $\sigma.c_i = c_{\sigma(i)}$, we can define the semidirect product $\mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$ with multiplication

$$(x, \sigma)(y, \tau) = (x\sigma(y), \sigma\tau)$$

4.1. Definition. We define the *Hecke-Clifford superalgebra* as

$$\mathcal{H} := \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$$

with the c_i having odd parity and the s_j having even parity.

4.2. Lemma. *There exists a superalgebra isomoprhism*

$$\begin{aligned} \mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}\ell_n &\xrightarrow{\sim} \mathcal{H}_n \\ c_i &\mapsto c_i \\ t_j &\mapsto \frac{1}{\sqrt{-2}} s_j (c_j - c_{j+1}) \end{aligned}$$

Check this last map.

Recall that $\mathcal{C}\ell_n$ is a simple superalgebra and it has a unique simple module U_n. If n is even, U_n is of type M and if n is odd, U_n is of type Q. This leads us to define two functors

$$\begin{aligned} F_n := - \otimes U_n \colon \mathbb{C}\mathfrak{S}_n\textbf{-Mod} &\to \mathcal{H}_n\textbf{-Mod} \\ G_n := \mathrm{Hom}_{\mathcal{C}\ell_n}(U_n, -) \colon \mathcal{H}_n\textbf{-Mod} &\to \mathbb{C}\mathfrak{S}_n^-\textbf{-Mod} \end{aligned}$$

4.3. Lemma. [**Kle05**, Prop 13.2.2]

(a) If n is even, then $F_n \circ G_n \cong id$ and $G_n \circ F_n \cong id$.
(b) If n is odd, then $F_n \circ G_n \cong id \oplus \pi$ and $G_n \circ F_n \cong id \oplus \pi$ where $\pi(M)_i = M_{i+1}$ for all $i \in \mathbb{Z}/2\mathbb{Z}$.

Thus, because $(F_n \circ G_n)(M) = \mathrm{Hom}_{\mathcal{C}\ell_n}(U_n, M) \otimes U_n$, we have a (super)Morita equivalence between $\mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}\ell_n$ and $\mathcal{H}_n$.

5. A Double Cover $\tilde{B}_n$ (presented by Jinkui Wan)

Recall that $B_n = \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$. We define

5.1. Definition.

$$\Pi_n := \left\langle z, a_1, \dots, a_n \mid \begin{array}{c} z^2 = a_i^2 = 1,\ \forall 1 \le i \le n \\ a_i a_j = z a_j a_i,\ i \neq j \end{array} \right\rangle$$

Then, $\mathfrak{S}_n$ acts on Π_n via $\sigma(z) = z$ and $\sigma(a_i) = a_{\sigma(i)}$. This gives us the short exact sequence

$$\begin{array}{c} 1 \to \{1, z\} \to \Pi_n \rtimes \mathfrak{S}_n \to B_n \to 1 \\ \qquad\qquad\qquad a_i \to b_i \end{array}$$

and so we define

The flow here is not great.

5.2. Definition. Let $\tilde{B}_n$ be the supergroup on $\Pi_n \rtimes \mathfrak{S}_n$ with the a_i odd, z even, and $\sigma \in \mathfrak{S}_n$ even.

Since $\mathbb{C}\tilde{B}_n / \langle z + 1 \rangle = \mathcal{H}_n$, we wish to understand conjugacy classes in B_n. We will do so by example.

5.3. Example. Consider

$$x = ((+ + + - + + + - +-), (1234)(567)(89)) \in B_{10}$$

As an element of $\mathfrak{S}_{10}$, $(1234)(567)(89)$ has cycle type $(4, 3, 2, 1)$, but we wish to assign a parity to each of these cycles. To do so, we look at the $(+, -)$-array in the first coordinate and take the product of the entries corresponding to the cycle. So, (1234) gets cycle type $+ \times + \times + \times - = -$ since those are entries $1, 2, 3$, and 4 in the array. This gives the cycle type as a tuple of partitions $\rho = (\rho^+, \rho^-)$ and so $\rho(x) = ((3), (4, 2, 1))$. Similarly, if

$$y = ((+ - - - + - - - +-), (1386)(279)(45)) \in B_{10}$$

then the first cycle has parity $+ \times - \times - \times - = -$ since those are the $1, 3, 6$, and 8 entries of the array. One can check that y has the same cycle type as x.

5.4. Lemma. *Two elements of B_n are conjugate if and only if their cycle types are the same.*

5.5. Corollary. *The number of conjugacy classes in B_n is*

$$\#\{(\rho^+, \rho^-) \mid |\rho^+| + |\rho^-| = n\}$$

Now, the conjugacy class $\mathcal{C}_{\rho^+,\rho^-}$ is even if k is even for $\underbrace{b_{i_1} b_{i_2} \dots b_{i_k}}_{\in \mathbb{Z}_2^n} \sigma \in C_{\rho^+,\rho^-}$.

5.6. Theorem (Read). [**CW12**, Theorem 3.31]

(a) Even $\mathcal{C}_{\rho^+,\rho^-}$ splits if and only if $\rho^+ \in \mathcal{OP}_n$ and $\rho^- = \varnothing$
(b) Odd $\mathcal{C}_{\rho^+,\rho^-}$ splits if and only if $\rho^+ = \varnothing$ and $\rho^- \in \mathcal{SP}_n^-$

where $\mathcal{SP}_n^-$ is all partitions of n with strict parts and odd length.

5.7. Definition. For $\alpha \in \mathcal{OP}_n$, let $\mathcal{C}_\alpha^+$ be the split conjugacy class in $\tilde{B}_n$ satisfying

(a) $\mathcal{C}_\alpha^+ = \theta_n^{-1}(\mathcal{C}_{\alpha,\varnothing})$
(b) There exists $\sigma \in \mathcal{C}_\alpha^+$ such that $\sigma \in \mathfrak{S}_n$ with cycle type α.

6. A ring structure on R^- (presented by Jinkui Wan)

6.1. Definition. We give the following definition

(a) Let $R_n^- := [\mathcal{H}_n\text{-}\mathbf{Mod}]$, the Grothendieck group of $\mathcal{H}_n$-**Mod**.
(b) Let $R^- := \bigoplus_{n=0}^\infty R_n^-$ where $R_0^- = \mathbb{Z}$
(c) Let $R_\mathbb{Q}^- = \mathbb{Q} \otimes_\mathbb{Z} R^-$
(d) Let $\mathcal{H}_{m,n}$ be the subalgebra of $\mathcal{H}_{m+n}$ generated by $\mathcal{C}\ell_{m+n}$ and $S_m \times S_n$. Note that $\mathcal{H}_{m,n} \cong \mathcal{H}_m \otimes \mathcal{H}_n$ as a superalgebra.
(e) Given $M \in \mathcal{H}_m$-**Mod** and $N \in \mathcal{H}_n$-**Mod**, we define

$$[M] \cdot [N] := [\mathrm{Ind}_{\mathcal{H}_m \otimes \mathcal{H}_n}^{\mathcal{H}_{m+n}} M \otimes N]$$

6.2. Proposition. *R^- is commutative with respect to the above multiplication.*

6.3. Definition. Define a bilinear form via

$$\langle [M], [N] \rangle := \dim \mathrm{Hom}_{\mathcal{H}_n}(M, N)$$

for $M, N \in \mathcal{H}_n$

6.4. Lemma. *For $\phi \in R_n^-$ (viewed as a character of $\tilde{B}_n$), set $\phi_\alpha := \phi(x)$ for any $x \in \mathcal{C}_\alpha$. Then,*

(a) For $\phi \in R_m^-, \psi \in R_n^-$, and $\gamma \in \mathcal{OP}_{m+n}$,

$$(\phi \cdot \psi)_\gamma = \sum_{\substack{\alpha \in \mathcal{OP}_m, \beta \in \mathcal{OP}_n \\ \alpha \cup \beta = \gamma}} \frac{z_\gamma}{z_\alpha z_\beta} \phi_\alpha \psi_\beta$$

where z_α is the order of the centralizer of σ of cycle type α in $\mathfrak{S}_n$.

(b)

$$\langle \phi, \psi \rangle = \sum_{\alpha \in \mathcal{OP}} 2^{-\ell(\alpha)} z_\alpha^{-1} \phi_\alpha \psi_\alpha$$

6.5. Proposition. *The character value vanishes unless you are in an even split conjugacy class.*

7. The ring Γ (presented by Jinkui Wan)

7.1. Definition. Let $x = \{x_1, x_2, \ldots\}$.

(a) Define $q_r = q_r(x)$ via the generating function

$$Q(t) = \sum_{r \geq 0} q_r(x) t^r = \prod_{i \geq 1} \frac{1 + tx_i}{1 - tx_i}$$

(b) Let Γ be the $\mathbb{Z}$-subring of the ring of symmetric functions generated by q_r, $r \geq 0$.
(c) $\Gamma_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$

7.2. Proposition. *(a)* $\sum_{r+s=n}(-1)^r q_r q_s = 0$ *because* $Q(t)Q(-t) = 1$
(b) $q_n = \sum_{\alpha \in \mathcal{OP}_n} 2^{\ell(\alpha)} z_\alpha^{-1} p_\alpha$ *where* $p_\alpha = P_{\alpha_1} \cdots p_{\alpha_\ell}$ *because* $\ln Q(t) = \sum_{r \text{ odd}} \frac{2p_r(x)t^r}{r}$.

7.3. Theorem. *(a)* $\Gamma_{\mathbb{Q}}$ *is a polynomial algebra with polynomial generators* p_{2r-1} *for* $r \geq 1$.
(b) $\{p_\mu \mid \mu \in \mathcal{OP}\}$ *is a basis for* $\Gamma_{\mathbb{Q}}$

7.4. Definition. Let us define inner product on $\Gamma_{\mathbb{Q}}$ via

$$\langle p_\alpha, p_\beta \rangle := 2^{-\ell(\alpha)} z_\alpha \delta_{\alpha\beta}, \forall \alpha, \beta \in \mathcal{OP}$$

7.5. Definition. Define the *(spin) characteristic map* to be

$$\begin{aligned} \mathrm{ch}^- \colon R_{\mathbb{Q}}^- &\to \Gamma_{\mathbb{Q}} \\ \phi &\mapsto \sum_{\alpha \in \mathcal{OP}_n} z_\alpha^{-1} \phi_\alpha p_\alpha \end{aligned}$$

7.6. Proposition. *(a)* ch^- *is an algebra isomorphism.*
(b) ch^- *is an isometry (that is,* $\langle \phi, \psi \rangle = \langle \mathrm{ch}^-(\phi), \mathrm{ch}^-(\psi) \rangle$*).*

Now, we seek to construct the basic spin module.

7.7. Proposition. $\mathcal{H}_n = \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$ *acts on* $\mathcal{C}\ell_n = \mathrm{span}\{c_I \mid I \subseteq \{1, 2, \ldots, n\}\}$ *via*

$$\begin{cases} c_i.(c_{i_1} \cdots c_{i_k}) = c_i c_{i_1} \cdots c_{i_k} \\ \sigma.(c_{i_1} \cdots c_{i_k}) = c_{\sigma(i_1)} \cdots c_{\sigma(i_k)} \end{cases}$$

where $c_i \in \mathcal{C}\ell_n$, $\sigma \in \mathfrak{S}_n$, *and the action is extended by linearity.*

7.8. Proposition. *Let* $\sigma = \sigma_1 \cdots \sigma_\ell$ *be a cycle decomposition of* σ. *Then*

$$\sigma c_I = \begin{cases} \pm c_I & \text{if } I \text{ is a union of some supports of } \sigma_1, \ldots, \sigma_\ell \\ \pm c_J (J \neq I) & \text{otherwise} \end{cases}$$

7.9. Example. Let $\sigma = (134)(25) \in \mathfrak{S}_5$. Then,

$$\sigma c_3 c_5 = c_4 c_2$$

but

$$\sigma c_1 c_3 c_4 = c_3 c_4 c_1 = c_1 c_3 c_4$$

Thus, the character of this action, say ξ^n, satisfies

$$\xi^n(\alpha) = 2^{\ell(\alpha)}, \alpha \in \mathcal{OP}_n$$

and thus $\mathrm{ch}^-(\xi^n) = \sum_{\alpha \in \mathcal{OP}_n} z_\alpha^{-1} 2^{\ell(\alpha)} p_\alpha = q_n$.

7.10. Definition. For $\lambda \in \mathcal{SP}$, we define ξ^λ via the recursive formulas

$$\xi^{(\lambda_1,\lambda_2)} = \xi^{\lambda_1}\xi^{\lambda_2} + 2\sum_{i=1}^{\lambda_2}(-1)^i\xi^{\lambda_1+i}\xi^{\lambda_2-i}$$

$$\xi^\lambda = \begin{cases} \sum_{j=2}^k(-1)^j\xi^{(\lambda_1,\lambda_j)}\xi^{(\lambda_2,\dots,\hat{\lambda}_j,\dots,\lambda_k)} & k=\ell(\lambda) \text{ is even} \\ \sum_{j=1}^k(-1)^{j-1}\xi^{\lambda_j}\xi^{(\lambda_1,\dots,\hat{\lambda}_j,\dots,\lambda_k)} & k=\ell(\lambda) \text{ is odd} \end{cases}$$

7.11. Theorem. *(a)* $\mathrm{ch}^-(\xi^\lambda) = Q_\lambda$*, the Schur-Q function (to be defined in the next lecture).*

(b) $\left\{\zeta^\lambda := 2^{-\frac{\ell(\lambda)-\delta(\lambda)}{2}}\xi^\lambda \mid \lambda \in \mathcal{SP}_n\right\}$*, where* $\delta(\lambda) = \chi\{\ell(\lambda) \text{ is odd}\}$*, is a complete list of simple characters.*

(c) ζ^λ *is of type M if* $\ell(\lambda)$ *is even and of type Q if* $\ell(\lambda)$ *is odd.*

(d) The degree of ζ^λ *is*

$$2^{n-\frac{\ell(\lambda)-\delta(\lambda)}{2}}\frac{n!}{\lambda_1!\cdots\lambda_\ell!}\left(\prod_{i<j}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}\right)$$

8. Schur-Q functions and related combinatorics (presented by George H. Seelinger)

9. Center of Symmetric Group Algebras and Spin Symmetric Group Algebras (presented by Jinkui Wan)

9.1. Farahat-Higman's Construction for $\mathfrak{S}_n$. Given a permutation σ, we note that its cycle type is not stable under inclusion from $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$.

9.1. Example. Let $\sigma = (134)(2576) \in \mathfrak{S}_8$. Then, σ has cycle type

$$(4,3,1) = \ydiagram{4,3,1}$$

but, when included into $\mathfrak{S}_n9$, σ has cycle type

$$(4,3,1,1) = \ydiagram{4,3,1,1}$$

9.2. Definition. Given a cycle $\sigma \in \mathfrak{S}_n$, its *modified cycle type*, λ, is given by removing the first column from its cycle type.

9.3. Example. The modified cycle type of $\sigma = (134)(2576) \in \mathfrak{S}_8$ is $\lambda = (3,2)$. Note that this is stable with respect to $\mathfrak{S}_8 \hookrightarrow \mathfrak{S}_9$.

9.4. Proposition. *(a) If σ is of modified type λ, then $|\lambda|$ is the minimal length for σ as a product of (not necessarily simple) transpositions.*

(b) If

$$\begin{cases} \sigma \text{ is of modified type } \lambda \\ \tau \text{ is of modified type } \mu \\ \sigma\tau \text{ is of modified type } \nu \end{cases}, \text{ then } |\nu| \leq |\lambda| + |\mu|$$

9.5. Example. $(134) = (13)(34)$ and has modified type (2).

9.6. Definition. (a) Let $\mathcal{C}_\lambda(n)$ be the conjugacy class of $\mathfrak{S}_n$ of modified type λ. Note $\mathcal{C}_\lambda(n) = \varnothing$ if $n < |\lambda| + \ell(\lambda)$.

(b) Let

$$C_\lambda(n) := \begin{cases} \text{Class sum of } \mathcal{C}_\lambda(n) & \text{if } n \geq |\lambda| + \ell(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

9.7. Example. $\mathcal{C}_0(n) = \{id\}$ and $\mathcal{C}_{(1)}(n)$ contains all transpositions of $\mathfrak{S}_n$. Thus, $C_{(1)}(n) = \sum_{1 \leq i < j \leq n} (ij)$.

9.8. Proposition. *$\{C_\lambda(n) \mid |\lambda| + \ell(\lambda) \leq n\}$ is a basis for $Z(\mathbb{Z}\mathfrak{S}_n)$.*

9.9. Definition. Write

$$C_\lambda(n) C_\mu(n) = \sum A^\nu_{\lambda\mu}(n) C_\nu(n)$$

9.10. Example.

$$C_{(1)}(n) C_{(1)}(n) = 3C_{(2)}(n) + 2C_{(1,1)}(n) + \frac{1}{2} n(n-1) C_0(n)$$

since $C_{(1)}(n)^2 = \sum (ij)(kl)$ for all transpositions $(ij), (kl)$ in $\mathfrak{S}_n$.

9.11. Theorem (Farahat-Higman). *Let λ, μ, ν be partitions. Then,*

(a) There is a unique polynomial $f^\nu_{\lambda\mu}(x) \in \mathcal{Q}[x]$ such that $a^\nu_{\lambda\mu}(n) = f^\nu_{\lambda\mu}(n)$ for all $n \geq |\nu| + \ell(\nu)$.

(b) $f^\nu_{\lambda\mu}(x) = 0$ unless $|\nu| \leq |\lambda| + |\mu|$

(c) If $|\nu| = |\lambda| + |\mu|$, then $f^\nu_{\lambda\mu}(x)$ is a constant. In other words, $a^\nu_{\lambda\mu}(n)$ is independent of n.

Proof Idea. Let $\Gamma = \{(\sigma, \tau) \mid \sigma \in \mathcal{C}_\lambda(n), \tau \in \mathcal{C}_\mu(n), \sigma\tau \in \mathcal{C}_\nu(n)\}$. Then, once computes

$$a^\nu_{\lambda\mu}(n) = \frac{\#\Gamma}{\#\mathcal{C}_\lambda(n)}$$

If we let $\mathfrak{S}_n$ act on Γ by conjugation, that is $\gamma.(\sigma, \tau) = (\gamma\sigma\gamma^{-1}, \gamma\tau\gamma^{-1})$, then we get that

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$$

Without loss of generality, take $(\sigma_1, \tau_1) \in \Gamma_1$. Then,

$$\#\Gamma_1 = \frac{n!}{\#\text{ of centralizers of } (\sigma_1, \tau_1) \text{ in } \mathfrak{S}_n}$$

Suppose $\gamma.(\sigma_1, \tau_1) = (\sigma_1, \tau_1)$. Then,

$$\gamma\sigma_1\gamma^{-1} = \sigma_1 \text{ and } \gamma\tau_1\gamma^{-1} = \tau_1$$

Thus,

$$\gamma \in \mathfrak{S}_{\mathrm{Supp}(\sigma_1,\tau_1)} \times \mathfrak{S}_{\{1,\ldots,n\}\setminus\mathrm{Supp}(\sigma_1,\tau_1)}$$

where $\mathrm{Supp}(\sigma_1, \tau_1) = \{j \in \{1, \ldots, n\} \mid \sigma_1(j) \neq j \text{ or } \tau_1(j) \neq j\}$. So,

$\#$ centralizers of (σ_1, τ_1) in $\mathfrak{S}_n = \#$ centralizers of (σ_1, τ_1) in $\mathfrak{S}_{\mathrm{Supp}(\sigma_1,\tau_1)} \times (n - \#\,\mathrm{Supp}(\sigma_1, \tau_1))!$

Thus, using our formula above for $a^{\nu}_{\lambda\mu}$, we arrive at

Finish this formula

$$a^{\nu}_{\lambda\mu}(n) = \frac{\sum_i \#\Gamma_i}{\#\mathcal{C}_\lambda(n)}$$

$\square$

9.12. Definition. (a) Let $\mathbb{B}$ be the ring of polynomials $f(x) \in \mathcal{Q}[x]$ such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

(b) Let $\mathcal{K}$ be the $\mathbb{B}$-algebra with basis $\{c_\lambda \mid \lambda \in \mathcal{P}\}$ such that

$$c_\lambda c_\mu := \sum_{\nu \in \mathcal{P}} f^{\nu}_{\lambda\mu}(x) c_\nu$$

We call this ring the *Farahat-Higman ring*.

9.13. Proposition. *We have the following facts*

(a) $\mathcal{K}$ *is commutative and associative.*

(b) $\mathcal{K}$ *is filtered via* $\deg(c_\lambda) = |\lambda|$ *for all* $\lambda \in \mathcal{P}$.

9.14. Remark. $\mathcal{K}$ is not graded because $\sum_{\nu\in\mathcal{P}} f^{\nu}_{\lambda\mu}(x)c_\nu$ is not homogeneous. However, if we say $\mathcal{K}_r = \mathrm{span}\{c_\lambda \mid |\lambda| \leq r\}$, then $\mathcal{K}_r\mathcal{K}_s \subseteq \mathcal{K}_{r+s}$, making $\mathcal{K}$ filtered.

9.15. Definition. Let $\mathrm{gr}\,\mathcal{K}$ be the associated graded algebra, that is, $\mathrm{gr}\,\mathcal{K}$ is defined by $(\mathrm{gr}\,\mathcal{K})_r = \mathcal{K}_r/\mathcal{K}_{r-1}$ and then $\mathrm{gr}\,\mathcal{K} = \bigoplus_{r\geq 0}(\mathrm{gr}\,\mathcal{K})_r$.

9.16. Lemma. *We have the following facts.*

(a) If $|\lambda| + s = m$, *then*

$$a^{(m)}_{\lambda,(s)} = \begin{cases} \frac{(m+1)s!}{\prod_{i\geq 0} m_i(\lambda)!} & \text{if } \ell(\lambda) \leq s+1 \\ 0 & \text{otherwise} \end{cases}$$

where $m_0(\lambda) = r + 1 - \ell(\lambda)$.

(b) If $|\lambda| + s = |\nu|$, *then*

$$a^{\nu}_{\lambda,(s)} = \sum_{\substack{(i,\mu)\in\mathbb{N}\times\mathcal{P} \\ 1\leq i\leq\ell(\lambda) \\ \mu\cup\nu=\lambda\cup(\nu_i)}} a^{(\nu_i)}_{\mu,(s)}$$

9.17. PROPOSITION. *Let λ be a partition and let $m_i(\lambda)$ be the number of i's in λ such that $\lambda = (i^{m_i(\lambda)})_{i\geq 1}$. Then, the top degree of $c_\lambda c_{(s)}$ is given by*

$$(c_\lambda c_{(s)})^* = \sum_{|\nu|=|\lambda|+s} a^{\nu}_{\lambda,(s)} c_\nu = \sum_{\mu\subseteq\lambda, \ell(\mu)\leq s+1} \frac{(m_{s+|\mu|}(\lambda)+1)(s+|\mu|+1)s!}{(s+1-\ell(\mu))!\prod_{i\geq 1} m_i(\mu)!} c_{\lambda\cup(s+|\mu|)-\mu}$$

9.18. REMARK. $\mu = (i^{m_i(\mu)}) \subseteq \lambda \iff m_i(\mu) \leq m_i(\lambda)\ \forall i \geq i$.

9.19. EXAMPLE. To illustrate the formula $\lambda \cup (s+|\mu|) - \mu$, consider cycles $\sigma = (134)(2567)(8)$ and $\tau = (28)$. Then, $\lambda = (3,2)$ is the modified cycle type of σ and $s = 1$ gives the modified cycle type of τ. Then,

$$\sigma\tau = (134)(28567)$$

which has modified cycle type $(4,2) = (3,2) \cup (1+|(3)|) - (3)$.

9.20. COROLLARY.

$$c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_\ell} = \sum_{\mu \trianglerighteq \lambda} d_{\lambda\mu} c_\mu$$

and $d_{\lambda\lambda} > 0$ in $\operatorname{gr}\mathcal{K}$. Thus, $c_1, c_2, \ldots$ are algebraically independent elements of $\operatorname{span}_{\mathbb{Z}}\{c_\lambda\}$.

9.21. PROPOSITION. *$\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{gr}\mathcal{K}$ is a polynomial algebra generated by $c_1, c_2, \ldots$ (Note that $\mathbb{Z} \hookrightarrow \mathbb{B}$ as constants.)*

9.22. REMARK.
(a) There exists a ring isomorphism $\Lambda \xrightarrow{\sim} \operatorname{gr}\mathcal{K}$ sending duals of h^*_λ (images of h_λ under a certain automorphism), called g_λ, to c_λ. See [**Mac79**, p 132–3].
(b) $\operatorname{gr} Z(\mathbb{Z}\mathfrak{S}_n) \cong H^*(\mathrm{Hilb}^n(\mathbb{C}^2);\mathbb{Z})$, the cohomology ring of the Hilbert Scheme of points on $\mathbb{C}^2$, as a $\mathbb{Z}$-algebra.

9.23. THEOREM. *The homomorphism given by*

$$\Pi_n \colon \mathcal{K} \to Z(\mathbb{Z}\mathfrak{S}_n)$$
$$\sum f_\lambda(x) c_\lambda \mapsto \sum f_\lambda(n) c_\lambda(n)$$

is a surjective homomorphism.

9.24. PROPOSITION. *$\mathcal{K}$ is generated by $K_m := \sum_{|\lambda|=m} c_\lambda$ for $m \geq 0$.*

This tells us that

$$\Pi_m(\mathcal{K}) = \sum_{|\lambda|=m} c_\lambda(n) = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \#\text{ of cycles in } \sigma = n-m}} \sigma$$

and so $Z(\mathbb{Z}\mathfrak{S}_n)$ is generated by $\Pi_n(\mathcal{K}_0), \Pi_n(\mathcal{K}_1), \ldots, \Pi_n(\mathcal{K}_{n-1})$.

10. Double cover of $\tilde{\mathfrak{S}}_n$ and even split conjugacy classes

Recall the short exact sequence

$$1 \to \{1, z\} \to \tilde{\mathfrak{S}}_n \to \mathfrak{S}_n \to 1$$

where

$$\tilde{\mathfrak{S}}_n = \left\langle z, t_1, t_2, \ldots, t_{n-1} \mid \begin{cases} z \text{ is central} \\ z^2 = 1, t_i^2 = z \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ t_i t_j = z t_j t_i & |i-j| > 1 \end{cases} \right\rangle$$

10.1. Definition. Define the element

$$x_i := t_i t_{i+1} \cdots t_{n-1} t_n t_{n-1} \cdots t_{i+1} t_i$$

which gets mapped to the transposition $(i, n) \in \mathfrak{S}_n$ under the map θ_n. Then, we let

$$[i_1 i_2 \cdots i_m] := \begin{cases} z & \text{if } m = 1 \\ x_{i_1} x_{i_m} x_{i_{m-1}} \cdots x_{i_2} x_{i_1} & \text{if } m \geq 2 \end{cases}$$

10.2. Proposition. *Every element of $\tilde{\mathfrak{S}}_n$ is of the form*

$$z^q \underbrace{[i_1 i_2 \cdots i_m][j_1 j_2 \cdots j_k] \cdots}_{\textit{disjoint}}$$

where $q = 0, 1$.

10.3. Lemma. *For λ such that $|\lambda| + \ell(\lambda) \leq n$, $\theta_n^{-1}(\mathcal{C}_\lambda(n))$ splits if and only if*

(a) λ has only even parts or
(b) $\lambda \in \mathcal{SP}$, $|\lambda|$ odd, $|\lambda| + \ell(\lambda) = n$ or $n-1$.

10.4. Proposition. *$\sigma \in \mathfrak{S}_n$ of modified type λ is even if and only if $|\lambda|$ is even.*

10.5. Definition. Let $\mathcal{D}_\lambda(n)$ be the even split conjugacy class in $\tilde{\mathfrak{S}}_n$ containing $[1, 2, \ldots, \lambda_1 + 1][\lambda_1 + 2, \ldots, \lambda_1 + \lambda_2 + 2] \ldots$.

10.6. Proposition. *(a) $\theta^{-1}(\mathcal{C}_\lambda(n)) = \mathcal{D}_\lambda(n) \cup z\mathcal{D}_\lambda(n)$.*
(b) $\{d_\lambda(n) \mid \lambda \in \mathcal{EP}, |\lambda| + \ell(\lambda) \leq n\}$ is a basis for the even center of $\mathbb{Z}\mathfrak{S}_n^- = \mathbb{Z}\tilde{\mathfrak{S}}_n / \langle z + 1 \rangle$.

10.7. Definition. Define $b_{\lambda\mu}^\nu(n)$ by

$$d_\lambda(n) d_\mu(n) = \sum_{\nu \in \mathcal{EP}} b_{\lambda\mu}^\nu(n) c_\nu(n)$$

10.8. Example.

$$d_{(4)}(8) d_{(2)}(8) = 13 d_{(4)}(8) - 35 d_{(2)}(8) - 18 d_{(2,2)}(8) - 7 d_{(6)}(8) + 2 d_{(4,2)}(8) \in Z(\mathbb{Z}\mathfrak{S}_8^-)$$

10.9. Theorem (Tysse-Wang). *Let $\lambda, \mu, \nu \in \mathcal{EP}$.*

(a) *There exists a unique* $g^{\nu}_{\lambda\mu}(x) \in \mathcal{Q}[x]$ *such that* $b^{\nu}_{\lambda\mu}(n) = g^{\nu}_{\lambda\mu}(n)$ *for all* $n \geq |\nu| + \ell(\nu)$.
(b) $g^{\nu}_{\lambda\mu}(x) = 0$ *unless* $|\nu| \leq |\lambda| + |\mu|$
(c) *If* $|\nu| = |\lambda| + |\mu|$, *then* $g^{\nu}_{\lambda\mu}(x)$ *is a constnat.*

10.10. Definition. Let the *spin Farahat-Higman algebra* $\mathbb{F}$ be a $\mathbb{B}$-algebra with basis $\{d_\lambda \mid \lambda \in \mathcal{EP}\}$ and

$$d_\lambda d_\mu = \sum_{\nu \in \mathcal{EP}} g^{\nu}_{\lambda\mu}(x) d_\nu$$

which is filtered with respect to $\deg(d_\lambda) = |\lambda|$.

10.11. Proposition. *Let* λ *be a partition and rewrite* $\lambda = (i^{m_i(\lambda)}_{i\geq 1})$. *Let* $s \geq 0$ *be event. Then,*

$$(d_\lambda d_{(s)})^* = \sum_{\mu} (-1)^{\ell(\mu)} \frac{(m_{s+|\mu|}(\lambda)+1)(s+|\mu|+1)s!}{(s+1-\ell(\mu))! \prod_{i\geq 1} m_i(\mu)!} d_{\lambda \cup (s+|\mu|)-\mu}$$

What is this summation over?

10.12. Corollary. (a) $\mathbb{Q} \otimes_{\mathbb{Z}[\frac{1}{2}]} \operatorname{gr} \mathbb{F}$ *is generated by* $d_2, d_4, \ldots$
(b) *There exists an injective homomorphism* $\mathcal{Q} \otimes_{\mathbb{Z}[\frac{1}{2}]} \operatorname{gr} \mathbb{F} \hookrightarrow \mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{K}$ *via (for* $\lambda \in \mathcal{EP}$*)* $d_\lambda \mapsto (-1)^{\ell(\lambda)} c_\lambda$.

10.1. Connections to odd Jucys-Murphy elements.

10.13. Definition. Let us define

$$M_k := \sum_{i=1}^{k-1} [i,k] \in \mathbb{Z}\mathfrak{S}^-_n$$

10.14. Proposition. *We have*

(a) $M_k M_l = -M_l M_k$ *for* $k \neq l$
(b)
$$M_k^2 = -(k-1) - \sum_{1 \leq i \neq j \leq k-1} [i,j,k] \in \mathbb{Z}\mathfrak{S}^-_n$$

10.15. Definition. we define

$$e_{r,n} := \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} M_{i_1}^2 M_{i_2}^2 \cdots M_{i_r}^2 \in Z(\mathbb{Z}\mathfrak{S}^-_n)$$

10.16. Proposition. (a) $e_{r,n}$ *has top degree* $2r$
(b) $e_{r,n} = \sum_{\lambda \in \mathcal{EP}, |\lambda|+\ell(\lambda) \leq n} A_\lambda(n) d_\lambda(n)$ *for some* $A_\lambda(n)$.
(c) $A_\lambda(n)$ *is the coefficient of* $[1,2,\ldots,\lambda_1+1][\lambda_1+2,\ldots,\lambda_1+\lambda_2+2]\cdots$ *in* $e_{r,n}$ *and is independent of* n.

10.17. Definition. In light of the proposition above, we write $A_\lambda := A_\lambda(n)$ and define

$$e^*_r := \sum_{\lambda \in \mathcal{EP}, |\lambda| = 2r} A_\lambda d_\lambda \in \mathbb{F}$$

10.18. Example. $e_1^* = -d_2$ and $e_2^* = d_{(2,2)} - 2d_4$.

10.19. Proposition. *$A_\lambda = (-1)^{\ell(\lambda)\prod_{i\geq 1} c_{\frac{\lambda_i}{2}}}$ where $c_0 = 1$ and $c_r = \frac{1}{r+1}\binom{2r}{r}$ are the Cartan numbers.*

10.20. Theorem. *$\mathbb{B}\left[\frac{1}{2}\right] \otimes_{\mathbb{B}} \mathbb{F}$ is generated by $e_1^*, e_2^*, e_3^*, \ldots$*

10.21. Corollary. *Via the surjective homomorphism*

$$\mathbb{B}\left[\frac{1}{2}\right] \otimes_{\mathbb{B}} \mathbb{F} \to Z(\mathbb{Z}\left[\frac{1}{2}\right]\mathfrak{S}_n^-)$$
$$\sum_{\lambda\in\mathcal{EP}} f_\lambda(x) d_\lambda \mapsto \sum_{\lambda\in\mathcal{EP}} f_\lambda(n) d_\lambda(n)$$

the even center of $\mathbb{Z}\left[\frac{1}{2}\right]\mathfrak{S}_n^-$ is generated by (the top degree of) $e_{r,n}$.

11. Schur-Sergeev duality for $\mathfrak{q}(n)$ (presented by Chris Chung)

12. Seminormal form construction for irreducible $\mathcal{H}_n$-modules (presented by Jinkui Wan)

A review for the symmetric group case was presented, but not written up here yet.

12.1. Definition. We define the *Jucys-Murphy elements* in $\mathcal{H}_n = \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$ as

$$J_k := \sum_{1\leq j<k\leq n} (1 + c_j c_k)(jk),$$

12.2. Proposition. *The Jucys-Murphy elements have the following properties.*

(a) *$J_k J_l = J_l J_k$ for $l \leq k \neq l \leq n$.*
(b) *$c_k J_k = -J_k c_k$ and $c_l J_k = J_k c_l$ for $k \neq l$.*
(c) *$s_k J_k = J_{k+1} s_k - (1 + c_k c_{k+1})$ for $1 \leq k \leq n-1$ and $s_l J_k = J_k s_l$ for $k \neq l, l+1$.*

12.3. Definition. The *degenerate affine Hecke-Clifford algebra* is given by

$$\hat{\mathcal{H}}_n = \langle s_1, \ldots, s_{n-1}, c_1, \ldots, c_n, x_1, \ldots, x_n \rangle$$

with additional relations

$$\begin{cases} x_k x_l = x_l x_k & \\ s_k x_k = x_{k+1} s_k - (1 + c_k c_{k+1}) & \\ s_i x_k = x_k s_i & k \neq i, i+1 \\ x_k c_k = -c_k x_k & \\ x_k c_l = c_l x_k & l \neq k \end{cases}$$

12.4. Proposition. *There exists a projection* $\pi\colon \hat{\mathcal{H}}_n \twoheadrightarrow \mathcal{H}_n$ *such that*

$$\pi(s_i) = s_i, \pi(c_k) = c_k, \pi(x_l) = J_l$$

In particular, $\pi(x_1) = J_1 = 0$*, so* $x_1 \in \ker \pi$.

In fact, $\ker \pi = \langle x_1 \rangle$ and so $\mathcal{H}_n$**-Mod** can be identified as the subcategory of $\hat{\mathcal{H}}_n$**-Mod** on which $x_1 = 0$.

12.5. Theorem (PBW Theorem). *We have that*

$$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} c_1^{\beta_1} \cdots c_n^{\beta_n} w \mid \alpha_i \in \mathbb{Z}_+, \beta_i \in \{0,1\}, 1 \leq i \leq n, w \in \mathfrak{S}_n\}$$

is a basis for $\hat{\mathcal{H}}_n$.

12.6. Corollary (Corollary of PBW Theorem). *The subalgebra of* $\hat{\mathcal{H}}_n$ *generated by* $x_1, \ldots, x_n, c_1, \ldots, c_n$ *is isomorphic to*

$$\mathcal{C}\ell_n \otimes \mathbb{C}[x_1, \ldots, x_n]/\langle x_k c_k = -c_k x_k, x_k c_l = c_l x_k, l \neq k \rangle = \underbrace{P_1^c \otimes P_1^c \otimes \cdots \otimes P_1^c}_{n \text{ copies}}$$

where $P_1^c = \langle x_1, c_1 \rangle$.

12.7. Proposition. *The* $\mathfrak{S}_n$ *fixed points* $\mathbb{C}[x_1^2, x_2^2, \ldots, x_n^2]^{\mathfrak{S}_n} \subseteq$ *Center of* $\hat{\mathcal{H}}_n$.

12.8. Proposition. *(a) The eigenvalues of* $x_1^2, \ldots, x_n^2$ *are of the form* $q(i) := i(i+1)$ *for* $i \in \mathbb{Z}_+$.

(b) If all the eigenvalues of x_j^2 *on a finite dimensional* $\hat{\mathcal{H}}_n$*-module* M *for a fixed* j *are of the form* $q(i)$*, then* M *is integral.*

The second part of the proposition follows from the intertwining elements.

12.9. Definition. Let *intertwining element* $\Phi_k \in \hat{\mathcal{H}}_n$ be given by

$$\begin{aligned}\Phi_k &:= s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}) \\ &= (x_{k+1}^2 - x_k^2)s_k - (x_k + x_{k+1}) - c_k c_{k+1}(x_k - x_{k+1})\end{aligned}$$

Note, the second equality follows from the fact that

$$s_k x_k^2 = x_{k+1} s_k x_k - (1 + c_k c_{k+1}) x_k = x_{k+1}^2 s_k - (1 + c_k c_{k+1})(1 + x_k)$$

and, using conjugation by s_k,

$$x_k^2 s_k = s_k x_{k+1}^2 - s_k(1 + c_k c_{k+1})(1 + x_k)s_k \Longrightarrow -s_k x_{k+1}^2 = -x_k^2 s_k - s_k(1 + c_k c_{k+1})(1 + x_k)s_k$$

Therefore, we get

$$s_k(x_k^2 - x_{k+1}^2) = (x_{k+1}^2 - x_k^2)s_k - (1 + c_k c_{k+1})(1 + x_k) - s_k(1 + c_k c_{k+1})(1 + x_k)s_k$$

12.10. Proposition. *We have the following useful relations for the intertwining elements.*

$$\begin{cases} \Phi_k\Phi_l = \Phi_l\Phi_k & |k-l|>1 \\ \Phi_k\Phi_{k+1}\Phi_k = \Phi_{k+1}\Phi_k\Phi_{k+1} & \\ \Phi_k^2 = 2(x_k^2+x_{k+1}^2)-(x_k^2-x_{k+1}^2)^2 & \\ \Phi_k x_k = x_{k+1}\Phi_k, \Phi_k x_{k+1} = x_k\Phi_k, \Phi_k x_l = x_l\Phi_k & l\neq k,k+1 \\ \Phi_k c_k = c_{k+1}\Phi_k, \Phi_k c_{k+1} = c_k\Phi_k, \Phi_k c_l = c_l\Phi_k & l\neq k,k+1 \end{cases}$$

12.11. Proposition. *If v is some eigenvector of x_{j+1}^2, that is, if $x_{j+1}^2 v = av$, then $x_j^2\Phi_j v = a\Phi_j v$, so $\Phi_j v$ is an x_j^2 eigenvector with the same eigenvalue.*

Proof. Since $x_j^2\Phi_j = \Phi_j x_{j+1}^2$, we get that

$$x_j^2\Phi_j x = \Phi_j x_{j+1}^2 v = \Phi_j av = a\Phi_j x$$

$\square$

12.12. Definition. A finite dimensional $\hat{\mathcal{H}}_n$-module M is called *completely splittable (CS)* if $x_1, x_2, \ldots, x_n$ act semisimply, ie the actions of $x_1, \ldots, x_n$ can be diagonalized simultaneously.

12.13. Proposition. *Every integral CS $\hat{\mathcal{H}}_n$-module M can be decomposed as*

$$M = \bigoplus_{\mathbf{i}\in\mathbb{Z}_+^n} M_{\mathbf{i}}$$

where

$$M_{\mathbf{i}} = \{v \in M \mid x_k^2 v = q(i_k)v, 1\leq k\leq n\}$$

is the common eigenspace of $x_1^2, x_2^2, \ldots, x_n^2$ with eigenvalues $q(i_1), q(i_2), \ldots, q(i_k)$. Furthermore, define

$$\mathrm{wt}(M) := \{\mathbf{i}\in\mathbb{Z}_+^n \mid M_{\mathbf{i}}\neq 0\}$$

Our goal is to describe $\mathrm{wt}(M)$ for integral irreducible CS $\hat{\mathcal{H}}_n$-modules.

12.14. Lemma. *If $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}_+^n$ is in $\mathrm{wt}(M)$ for some integral irreducible CS $\hat{\mathcal{H}}_n$-module, then $i_k \neq i_{k+1}$ for all $1\leq k\leq n-1$.*

Proof. Suppose $i_k = i_{k+1}$ so that $x_k^2 v = x_{k+1}^2 v$ for $v\in M_{\mathbf{i}}$. Since

$$x_k^4 s_k - 2q(i_k)x_k^2 s_k + q(i_k)^2 s_k = x_k^2(s_k x_{k+1}^2 - s_k(1+c_kc_{k+1})(1+x_k)s_k) - 2q(i_k)x_k^2 s_k + q(i_k)^2 s_k$$

Somehow we get that $s_k v \in M_{\mathbf{i}}$. Then, we get figure this out

$$\implies x_k^2 s_k v = q(i_k)s_k v$$
$$\implies (x_k^2 - q(i_k))s_k v = 0$$
$$\implies (s_k x_{k+1}^2 - x_k(1-c_kc_{k+1}) - (1-c_kc_{k+1})x_{k+1})v - q(i_k)s_k v = 0$$

However, $x_{k+1}^2 v = q(i_{k+1})v = q(i_k)v$ by assumption, so

$$\begin{aligned}
&\implies (x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_k)v = 0 \\
&\implies 2(x_k^2 + x_{k+1}^2)v = 0 \\
&\implies i_k = i_{k+1} = 0 \\
&\implies x_k^2 v = 0 = x_{k+1}^2 v \\
&\implies x_k v = 0 = x_{k+1} v && \text{since } M \text{ is CS} \\
&\implies v = 0 && \text{since}
\end{aligned}$$

□

Finish this proof.

12.15. Lemma. *Suppose* $\mathbf{i} = (i_1, \ldots, i_n) \in \mathrm{wt}(M) \subseteq \mathbb{Z}_+^n$ *for some integral, irreducible, CS* $\hat{\mathcal{H}}_n$*-module. Fix* $1 \le k \le n-1$*.*

(a) If $i_k \neq i_{k+1} \pm 1$*, then* $\Phi_k z \neq 0$ *for all* $0 \neq z \in M_{\mathbf{i}}$*.*
(b) If $i_k = i_{k+1} \pm 1$*, then* $\Phi_k = 0$ *on* $M_{\mathbf{i}}$*.*

Proof. Since $\Phi_k^2 = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2$, then

$$\Phi_k^2 z = (2(q(i_k) + q(i_{k+1})) - q(i_k)^2 + 2q(i_k)q(i_{k+1}) - q(i_{k+1})^2)z = 0$$

if and only if

$$(q(i_k) - q(i_{k+1}))^2 = 2(q(i_k) + q(i_{k+1}))$$

Now, if we write $i_k = i_{k+1} + c$, then

$$\begin{cases} q(i_k) = i_{k+1}^2 + 2ci_{k+1} + c^2 + i_{k+1} + c \\ q(i_{k+1}) = i_{k+1}^2 + i_{k+1} \end{cases} \implies \begin{cases} q(i_k) - q(i_{k+1}) = 2ci_{k+1} + c^2 + c \\ q(i_k) + q(i_{k+1}) = 2i_{k+1}^2 + 2ci_{k+1} + c^2 + 2i_{k+1} + c \end{cases}$$

From here, one checks that $c = \pm 1$ certainly gives solutions independent of i_{k+1}. Thus, $i_k = i_{k+1} \pm 1 \implies \Phi_k^2 z = 0$. Furthermore, there are other formal solutions to these equations, namely $c = -2i_{k+1}$ and $c = -2(i_{k+1} + 1)$, but since $i_{k+1} \ge 0$, this would force $i_k < 0$ unless $i_k = i_{k+1} = 0$, which is not admissible by the previous lemma.

So, to prove the second part, since $\Phi_k^2 z = 0$, it must be that if $\Phi_k z \neq 0$ and so $\Phi_k z \in M_{s_k \mathbf{i}}$. Then, there exists a minimal sequence $\Phi_{j_1}, \ldots, \Phi_{j_r}$ such that $\Phi_{j_1} \cdots \Phi_{j_r} \Phi_k z \in M_{\mathbf{i}}$ since M is irreducible. Then, if $\sigma = s_{j_1} \cdots s_{j_r} s_k \in \mathfrak{S}_n$, it must be that $\sigma \cdot \mathbf{i} = \mathbf{i}$. If one assumes $\sigma \neq 1$, this leads to a violation of Lemma 12.14 with some work. Then, using the exchange condition for Coxeter groups, one shows that $r = 1$ which gives $j_1 = k$, so $\Phi_k^2 z \neq 0$, contradicting what we showed above. □

Why is this true?

12.16. Remark. Suppose V is an integral, irreducible $\hat{\mathcal{H}}_n$-module. Let $\hat{\mathcal{H}}_{(n-r,1^k)} = \langle s_1, \ldots, s_{n-r-1}, c_1, \ldots, c_n, x_1, \ldots, x_r \rangle$. Then,

$$\begin{aligned}
V \text{ is CS} &\iff \forall \mathbf{i} \in \mathrm{wt}(V), 1 \le k \le n-1, i_k \neq i_{k+1} \\
&\iff \mathrm{Res}^{\hat{\mathcal{H}}_n}_{\hat{\mathcal{H}}_{(n-r,1^r)}} V \text{ is semisimple } \forall 1 \le r \le n \\
&\iff \mathrm{Res}^{\hat{\mathcal{H}}_n}_{\hat{\mathcal{H}}_{(1^k, n-k-r, 1^r)}} V \text{ is semisimple}
\end{aligned}$$

The second $\hat{\mathcal{H}}_{(1^k,n-k-r,1^r)}$ is not defined.

12.17. Corollary. *Suppose* $\mathbf{i} \in \mathrm{wt}(V)$ *for some integral, irreducible, CS* $\hat{\mathcal{H}}_n$*-module* V*. If* $i_k = i_{k+2}$ *for some* $1 \le k \le n-2$*, then* $i_k = i_{k+2} = 0$ *and* $i_{k+1} = 1$*.*

Proof. If $i_k \neq i_{k+1} \pm 1$, then $s_k \cdot \mathbf{i} \in \mathrm{wt}\, V$ by the first part of 12.15, but $s_k \cdot \mathbf{i} = (\cdots, i_{k+1}, i_k, i_{k+2} \cdots)$ and, by assumption, $i_k = i_{k+2}$ contradicting 12.14. So, it must be that $i_{k+1} = i_k \pm 1$ and so $\Phi_k = 0 = \Phi_{k+1}$ on $V_{\mathbf{i}}$ by the second part of 12.15. Thus, for $z \in V$,

$$\begin{cases} \Phi_k z = 0 & \Longrightarrow (q(i_k) - q(i_{k+1}))s_k z = -((x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}))z \\ \Phi_{k+1} z = 0 & \Longrightarrow (q(i_{k+1}) - q(i_{k+2}))s_{k+1} z = -((x_{k+1} + x_{k+2}) + c_{k+1} c_{k+2}(x_{k+1} - x_{k+2}))z \end{cases}$$

which gives us the s_k and s_{k+1} actions on $V_{\mathbf{i}}$. From here, we can use the braid relation $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ to arrive at the equality

$$((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z = 0$$

Now, $x_k z = \pm\sqrt{q(i_k)}z$. Furthermore, since $i_k = i_{k+2}$, we have that either $x_k z = x_{k+2} z$ or $x_k z = -x_{k+2} z$. Decompose

$$V_{\mathbf{i}} = W_1 \oplus W_2$$

where $W_1 := \{z \in V_{\mathbf{i}} \mid x_k z = x_{k+2} z\}$ and $W_2 := \{z \in V_{\mathbf{i}} \mid x_k z = -x_{k+2} z\}$. Now, we can break up our braid relation equality.

$$\begin{aligned} z \in W_1 \Longrightarrow 0 &= ((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z \\ &= 2x_k(6x_{k+1}^2 + 2x_k^2)z \\ &= 2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k))z \\ z \in W_2 \Longrightarrow 0 &= ((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z \\ &= c_k c_{k+2} 2x_k(6x_{k+1}^2 + 2x_k^2)z \\ &= c_k c_{k+2}(2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k)))z \end{aligned}$$

Thus, we obtain that

$$2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k)) = 0$$

Moreover, we know that $i_{k+1} = i_k \pm 1$, so we get

$$\begin{cases} \sqrt{i_k(i_k+1)}(6(i_k-1)i_k + 2(i_k+1)i_k) = 0 & \text{if } i_{k+1} = i_k - 1 \\ \sqrt{i_k(i_k+1)}(6(i_k+1)(i_k+2) + 2(i_k+1)i_k) = 0 & \text{if } i_{k+1} = i_k + 1 \end{cases}$$

There are no solutions to the first equation that give i_k and i_{k+1} as non-negative integers and the only such solution for the second equation is $i_k = 0$ and $i_{k+1} = 1$. $\square$

In conclusion, we have the following.

12.18. Theorem. *Let* $\mathcal{W}(n)$ *be the set of weights of all integral irreducible CS* $\hat{\mathcal{H}}_n$*-modules and let* $\mathbf{i} \in \mathcal{W}(n)$*. Then,*

(a) $i_k \neq i_{k+1}$ *for all* $1 \leq k \leq n-1$.
(b) If $i_k = i_\ell = 0$ *for some* $1 \leq k < \ell \leq n$*, then* $1 \in \{i_{k+1}, \ldots, i_{\ell-1}\}$.
(c) If $i_k = i_\ell \geq 1$ *for some* $1 \leq k < \ell \leq n$*, then* $\{i_k - 1, i_k + 1\} \subseteq \{i_{k+1}, \ldots, i_{\ell-1}\}$.

PROOF. The first part is just a restatement of 12.14.

For the next part, assume $1 \notin \{i_{k+1}, \ldots, i_{\ell-1}\}$. Then, we can swap indices using **??** to get new weights until we obtain a weight of the form

$$(\ldots, 0, 0, \ldots)$$

which is not allowed by the previous part.

For the last part, let $u = i_k = i_\ell \geq 1$ be such that $k - \ell$ is minimal among such occurrences. If only one of $u+1, u-1$ appear in between i_k and i_ℓ, then it must appear twice because, otherwise, we could swap indices using 12.15 to get new weights until we are of the form

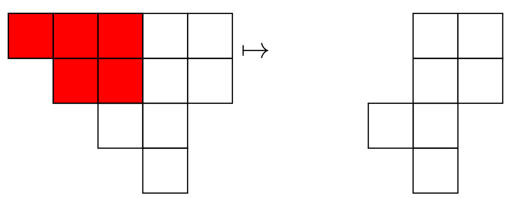
$$(\ldots, u, u \pm 1, u, \ldots)$$

which is not a weight by 12.17 since $u \geq 1$. Thus, we have violated the minimality of our choice of u. $\square$

Now, we wish to describe a bijection between $\mathcal{W}(n)$ and standard skew shifted tableaux of size n.

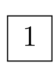
12.19. DEFINITION. Given strict partitions $\nu \subseteq \xi$, the *skew shifted Ferrers diagram* ξ/ν is given by removing the boxes of ν from ξ.

12.20. EXAMPLE. Consider $(3, 2) \subseteq (5, 4, 2, 1)$. Then,

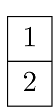


We now illustrate the bijection by example.

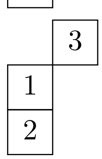
12.21. EXAMPLE. Let $\mathbf{i} = (1, 0, 3, 2, 1, 0, 6)$. Then, we construct our standard skew shifted tableau in steps by adding a box of content i_k labelled k on the kth step.

$(1) \mapsto$ 

on content 1 diagonal

$(1, 0) \mapsto$ 

contents $1, 0$

$(1, 0, 3) \mapsto$ 

$$(1,0,3,2) \mapsto \begin{array}{|c|c|} \hline & 3 \\ \hline 1 & 4 \\ \hline 2 & \\ \hline \end{array}$$

$$\vdots$$

$$(1,0,3,2,1,0) \mapsto \begin{array}{|c|c|} \hline & 3 \\ \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 6 & \\ \hline \end{array}$$

$$(1,0,3,2,1,0,6) \mapsto \begin{array}{|c|c|c|c|} \hline & & & 7 \\ \hline & 3 & & \\ \hline 1 & 4 & & \\ \hline 2 & 5 & & \\ \hline 6 & & & \\ \hline \end{array}$$

So, our final answer has outer shape $\xi = (7,4,3,2,1)$ and inner shape $\nu = (6,3,1)$:

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 7 \\ \hline & \blacksquare & \blacksquare & \blacksquare & 3 & & \\ \hline & & \blacksquare & 1 & 4 & & \\ \hline & & & 2 & 5 & & \\ \hline & & & & 6 & & \\ \hline \end{array}$$

12.22. Definition. Given $\mathbf{i} \in \operatorname{wt} V$, let $T(\mathbf{i})$ be the corresponding standard skew shifted tableau.

12.23. Proposition. *If* $\mathbf{i}, \mathbf{j} \in \operatorname{wt}(V)$ *for some integral irreducible CS* $\hat{\mathcal{H}}_n$*-module* V*, then* $T(\mathbf{i})$ *and* $T(\mathbf{j})$ *have the same shape.*

Proof. If $(i_1, \ldots, i_k, i_{k+1}, \ldots, i_n)$, then $(i_1, \ldots, i_{k-1}, i_{k+1}, i_k, \ldots, i_n)$ is a weight only if $i_k \neq i_{k+1} \pm 1$ by **??**. However, under such a condition, it does not matter in which order we add the boxes corresponding to i_k and i_{k+1}. Thus, these two weights will yield the same shape. $\square$

What is the reference here?

12.24. Definition. Given ξ/ν a skew-shifted Ferrer's diagram of size n, we define

$$\mathcal{F}(\xi/\nu) := \{\mathsf{T} \mid \mathsf{T} \text{ a standard Young tableau of shape } \xi/\nu\}$$

and

$$\hat{D}^{\xi/\nu} := \bigoplus_{\mathsf{T} \in \mathcal{F}(\xi/\nu)} \mathcal{C}\ell_n v_{\mathsf{T}}$$

as a vector space with actions

$$x_k v_{\mathsf{T}} = \sqrt{q(c(\mathsf{T}_k))}\, v_{\mathsf{T}}$$

where $c(\mathsf{T}_k)$ is the content of the box labelled by k in T and where $\mathcal{C}\ell_n$ acts by multiplication on the left.

12.25. Proposition. *We have*

$$s_k v_{\mathsf{T}} = \left(\frac{1}{\sqrt{q(c(\mathsf{T}_{k+1}))} - \sqrt{q(c(\mathsf{T}_k))}} + \frac{1}{\sqrt{q(c(\mathsf{T}_{k+1}))} + \sqrt{q(c(\mathsf{T}_k))}} c_k c_{k+1} \right) v_{\mathsf{T}} + \sqrt{1 - \frac{2(q(c(\mathsf{T}_{k+1}))) + q}{(q(c(\mathsf{T}_{k+1})) - q(c}}$$

Proof. This fact follows formally from the fact that $\Phi_k v_{\mathsf{T}} = a v_{s_k \mathsf{T}}$ for some scalar a if $s_k \mathsf{T}$ is standard. $\square$

Actually do this proof.

12.26. Corollary. *$\hat{D}^{\xi/\nu}$ is an integral CS $\hat{\mathcal{H}}_n$-module.*

Proof. Mainly, one needs to check the Coxeter relations. $\square$

Fill in this proof.

13. Spin Kostka Polynomials

13.1. Definition. Let $\nu \in \mathcal{SP}$ and μ be a partition. Then, the *spin Kostka polynomials* are the transition polynomials from the Hall-Littlewood P-functions to the Q-Schur functions. In other words,

$$Q_\nu(x) = \sum_\mu K^-_{\nu\mu}(t) P_\mu(x;t)$$

If we also let $b_{\nu\lambda}$ be such that

$$Q_\nu(x) = \sum_\lambda b_{\nu\lambda} s_\lambda(x)$$

then we get

13.2. Proposition. *For $\nu \in \mathcal{SP}$ and μ a partition,*

$$K^-_{\nu\mu}(t) = \sum_\lambda b_{\nu\lambda} K_{\lambda\mu}(t)$$

where $K_{\lambda\mu}(t)$ are the Kostka-Foulkes polynomials.

Proof. By definition of Kostka-Foulkes, we have

$$s_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu(x;t)$$

and so

$$\sum_\mu K_{\nu\mu}(t) P_\mu(x;t) = Q_\nu(x) = \sum_\lambda b_{\nu\lambda} s_\lambda = \sum_{\lambda,\mu} b_{\nu\lambda} K_{\lambda\mu}(t) P_\mu(x;t)$$

$\square$

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