

# $K$ -theoretic Catalan functions

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CAGE

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- Schubert calculus
- Catalan functions: a new approach to old problems
- $K$ -theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

# Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .

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## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

## Next Step: Flag Variety

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### Open Problem

Structure constants  $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$  are combinatorially unknown.

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(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
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And many more!



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$$\Phi: QH^*(Fl_{k+1}) \rightarrow H_*(Gr_{SL_{k+1}})_{loc}$$

$$\mathfrak{S}_w^Q \mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}$$

where  $s_\lambda^{(k)}$  is a  $k$ -Schur function.

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## Upshot

Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).

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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\lambda} + s_{\lambda}$$

The diagram shows the branching rule for  $k$ -Schur functions. The left side is  $s_{\lambda}^{(2)}$ , represented by a 2x2 square. The right side is the sum of three terms:  $s_{\lambda}^{(3)}$  (a 3x2 rectangle),  $s_{\lambda}^{(3)}$  (a 3x1 vertical rectangle), and  $s_{\lambda}^{(3)}$  (a 3x1 vertical rectangle). Brackets indicate that the second and third terms are grouped together as  $s_{\lambda}^{(3)}$ .

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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}$$

$\underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}^{(3)}} \quad \underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}^{(3)}}$

- Has geometric interpretation.

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The diagram shows the branching of the 2-partition  $(2,2)$  into 3-partitions. On the left,  $s_{\lambda}^{(2)}$  is represented by a 2x2 grid. On the right, it is equal to the sum of three 3-partitions:  $s_{\lambda}^{(3)}$  (2x2 grid),  $s_{\mu}^{(3)}$  (3x1 grid), and  $s_{\nu}^{(3)}$  (1x4 grid). Brackets below the 3-partitions group them into  $s_{(2,2)}^{(3)}$  and  $s_{(3,1)}^{(3)}$ .

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- No combinatorial interpretation of branching coefficients.

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- Definition with  $t$  important for Macdonald polynomials.

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- Has geometric interpretation.
- No combinatorial interpretation of branching coefficients.
- Definition with  $t$  important for Macdonald polynomials.
- Many definitions. A new one makes proofs easier!

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- $K$ -theoretic Catalan functions

# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

$$R_{1,3} \left( \begin{array}{|c|c|c|} \hline \text{red} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \text{red} \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|} \hline \text{red} \\ \hline \\ \hline \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \text{red} \\ \hline & \\ \hline \end{array}$$

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$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \underbrace{h_{310}}_{=0} + \underbrace{h_{310}}_{=0} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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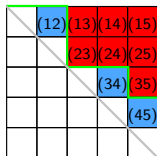
For  $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$ ,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi =$  Roots above Dyck path  
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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

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- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_{\gamma}$

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots

# Catalan functions

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$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2



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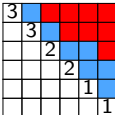
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$$\Delta^5(4, 4, 3, 3, 2, 2) =$$


Pieri:

$$s_1^\perp s_{\lambda+1}^{(k+1)} = \sum_\mu a_{\lambda+1}^{\ell, \mu} s_\mu^{(k+1)}$$

# Key ingredient of branching proof

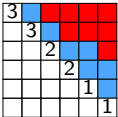
Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

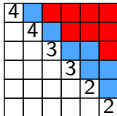
## Shift Invariance (Blasiak et al., 2019)

For partition  $\lambda$  of length  $\ell$  with  $\lambda_1 \leq k$ ,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$

Proof:  $k - \lambda_i = (k + 1) - (\lambda_i + 1)$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$


$$\Delta^5(4, 4, 3, 3, 2, 2) =$$


Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

- Schubert calculus
- Catalan functions: a new approach to old problems
- ***K*-theoretic Catalan functions**

# Dual Grothendieck polynomials

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- Dual to Grothendieck polynomials  $G_\lambda$ : Schubert representatives for  $K^*(Gr(m, n))$



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2-bounded partitions  $\leftrightarrow$  3-cores

The diagram illustrates the Pieri rule for  $K$ -Schur functions. It shows the product of a 1-strip ( $g_1$ ) and a 2-bounded partition ( $g_{211}^{(2)}$ ) resulting in the difference of two 2-bounded partitions ( $g_{2111}^{(2)} - 2g_{211}^{(2)}$ ). The partitions are represented as 5x5 grids of colored dots (red, blue, black) with some cells shaded gray to indicate the addition or subtraction of strips.

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- Conjecture:  $g_{\lambda}^{(k)}$  have positive branching into  $g_{\mu}^{(k+1)}$  (Lam et al., 2010; Morse, 2011).

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## Problem

No direct formula for  $g_{\lambda}^{(k)}$

# An Extra Ingredient: Lowering Operators

Lowering Operators  $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$

$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332}$$



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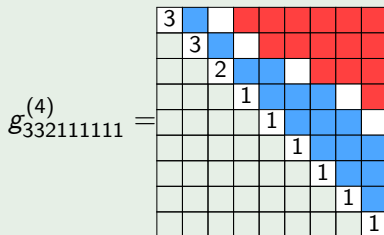
For  $K$ -homology of affine Grassmannian,  $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$  since this family satisfies the Pieri rule.

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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

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## Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

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- 3 Combinatorially describe  $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$ .

## Thank you!

- Anderson, David, Linda Chen, and Hsian-Hua Tseng. 2017. *On the quantum  $K$ -ring of the flag manifold*, preprint. arXiv: 1711.08414.
- Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and  $k$ -Schur Positivity*, J. Amer. Math. Soc. **32**, no. 4, 921–963.
- Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for  $k$ -Schur functions*, Ph.D. thesis.
- Ikedo, Takeshi, Shinsuke Iwao, and Toshiaki Maeno. 2018. *Peterson Isomorphism in  $K$ -theory and Relativistic Toda Lattice*, preprint. arXiv: 1703.08664.
- Lam, Thomas. 2008. *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. **21**, no. 1, 259–281.
- Lam, Thomas, Luc Lapointe, Jennifer Morse, and Mark Shimozono. 2010. *Affine insertion and Pieri rules for the affine Grassmannian*, Mem. Amer. Math. Soc. **208**, no. 977.
- Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010.  *$K$ -theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.
- Lapointe, Luc, Alain Lascoux, and Jennifer Morse. 2003. *Tableau atoms and a new Macdonald positivity conjecture*, Duke Mathematical Journal **116**, no. 1, 103–146.
- Morse, Jennifer. 2011. *Combinatorics of the  $K$ -theory of affine Grassmannians*, Advances in Mathematics.
- Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.