K-theoretic Catalan functions

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April 26, 2021

Overview

- Schubert calculus: connecting geometry and combinatorics
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Polynomials informing Geometry

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 & h_1 &= x_1 + x_2 + x_3 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 & h_2 &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 & h_3 &= x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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- Bases indexed by integer partitions.

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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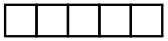
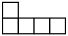
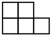
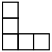
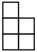
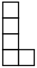
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

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- Schubert varieties $X_\lambda = \overline{\Omega_\lambda}$.

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Special basis of Schur polynomials $\{s_\lambda\}$ indexed by partitions such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Tableaux

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 2 & 5 \\ \hline \end{array} \qquad \qquad \begin{array}{|c|c|c|c|} \cline{1-1} 8 \\ \cline{1-2} 7 & 9 \\ \cline{1-2} 3 & 4 \\ \hline 1 & 2 & 5 & 6 \\ \hline \end{array}$$

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$$\mathrm{wt}(T) = (1,3,2,1,2) \quad (1,1,1,1,1,1,1,1,1)$$

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$$x^{\mathrm{wt}(T)} = x_1^1 x_2^3 x_3^2 x_4^1 x_5^2 \quad x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$$

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$\mathrm{SSYT}(\lambda) =$ all semistandard tableaux of shape λ.

$$\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Schur functions s_λ

Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 2 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$$

$$s_{\ydiagram{2,1}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2 x_1 x_2 x_3$$

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$$s_{\ydiagram{1,2}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2 x_1 x_2 x_3$$

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$s_\lambda(x)$ is homogeneous of degree $\lambda_1 + \cdots + \lambda_\ell$.

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\ydiagram{1}} s_{\ydiagram{1,2}} = s_{\ydiagram{1,3}} + s_{\ydiagram{2,2}} + s_{\ydiagram{1,1,2}}$$

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Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c^\nu_{\lambda\mu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$.

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Upshot

Let $\{f_\lambda\}$ be a basis of Λ such that

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2. $f_r f_\lambda$ satisfies the Pieri rule.

Then, $f_\lambda = s_\lambda$.

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Looking Ahead

This type of technique will be useful for establishing the equivalence of new formulas for other bases.

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When examining Schubert representatives in Λ, we ask

- Does it have a Pieri rule? $(s_r s_\lambda = \sum s_\nu)$
- Does it have a direct formula? $(s_\lambda = \sum_{T \in \mathsf{SSYT}(\lambda)} x^T)$

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(Co)homology of flag variety	Schubert polynomimals
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
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And many more!

Big Picture

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{(Co)homology of}\\ \text{Grassmannian/}\\ \text{Schur functions}\end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c}\text{Homology of Affine}\\ \text{Grassmannian/}\\ k\text{-Schur functions}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}K\text{-homology of}\\ \text{Grassmannian/}\\ \text{dual Grothendieck}\\ \text{polynomials}\end{array}} \arrow[r] & \boxed{\begin{array}{c}K\text{-homology of affine}\\ \text{Grassmannian/}\\ K\text{-}k\text{-Schur functions}\end{array}}
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- Branching with positive coefficients (Lam et al., 2010):

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$$s_{(2,2)}^{(2)} = \underbrace{s_{(2,2)}}_{s_{(2,2)}^{(3)}} + \underbrace{s_{(3,1)} + s_{(4)}}_{s_{(3,1)}^{(3)}}$$

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- Branching with positive coefficients (Lam et al., 2010):

$$s^{(2)}_{\ydiagram{2,2}} = \underbrace{s_{\ydiagram{2,2}}}_{s^{(3)}_{\ydiagram{2,2}}} + \underbrace{s_{\ydiagram{1,3}} + s_{\ydiagram{4}}}_{s^{(3)}_{\ydiagram{1,3}}}$$

  - (Lam et al., 2010) gives geometric interpretation,
  - but no combinatorial interpretation of branching coefficients.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
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- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Branching with positive coefficients (Lam et al., 2010):

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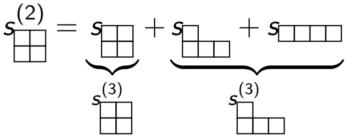

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- Many conjecturally equivalent definitions.

Overview

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- K-theoretic Catalan functions

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Key: $\{s_\lambda^{(k)}\}_\lambda \subseteq$ Catalan functions $=$ large class of symmetric functions.

Ingredients for Catalan functions

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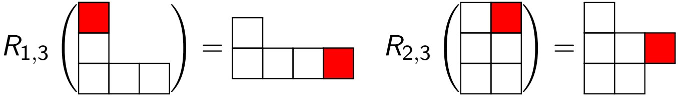

Raising Operators on Symmetric Functions

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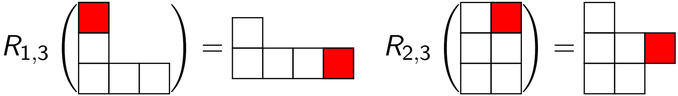

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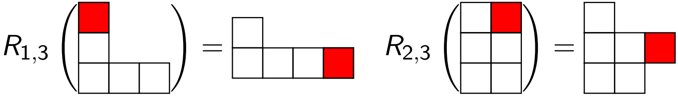

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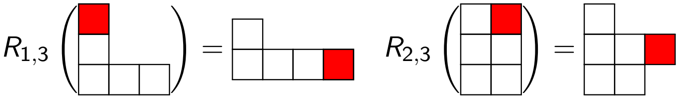

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$$\begin{aligned} s_{22} &= (1-R_{12})h_{22} = h_{22} - h_{31} \\ s_{211} &= (1-R_{12})(1-R_{23})(1-R_{13})h_{211} \\ &= h_{211} - h_{301} - h_{220} \color{red}{-h_{310} + h_{310}} + \underbrace{h_{32\,\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\,\text{-}1}}_{=0} \end{aligned}$$

some terms cancel

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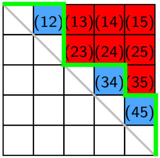
Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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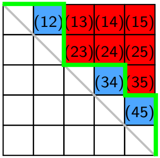

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For Ψ and $\gamma \in \mathbb{Z}^\ell$

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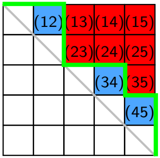

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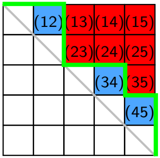

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Catalan functions

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Catalan functions interpolate between h_λ and s_λ.

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi;\lambda)$ is Schur positive!
Precisely, $H(\Psi;\lambda) = \sum_\nu c^\nu_{\Psi,\lambda} s_\nu$ satisfies $c^\nu_{\Psi,\lambda} \in \mathbb{Z}_{\geq 0}$.

Catalan functions

k-Schur root ideal for λ

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i,j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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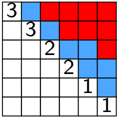
$\Delta^4(3,3,2,2,1,1) =$ 

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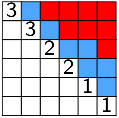
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

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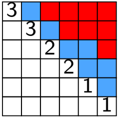
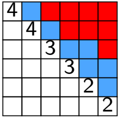
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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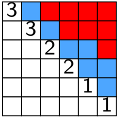
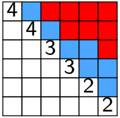
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Pieri:

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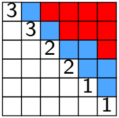
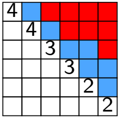
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

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Big Picture

$$
\begin{tikzcd}
\begin{array}{c}\text{(Co)homology of}\\ \text{Grassmannian}\\ s_\lambda = H(\varnothing;\lambda)\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{Homology of Affine}\\ \text{Grassmannian}\\ s_\lambda^{(k)} = H(\Delta^k(\lambda);\lambda)\end{array} \arrow[d] \\
\begin{array}{c}K\text{-homology of}\\ \text{Grassmannian/}\\ \text{dual Grothendieck}\\ \text{polynomials}\end{array} \arrow[r] & \begin{array}{c}K\text{-homology of affine}\\ \text{Grassmannian/}\\ K\text{-}k\text{-Schur functions}\end{array}
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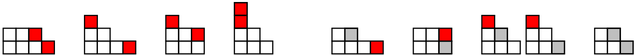

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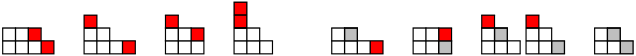

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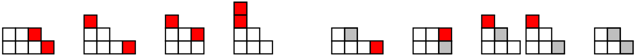

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- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m,n))$

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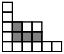
- A $(k+1)$-core is a partition with no cell of hook length $k+1$. 

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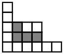

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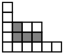

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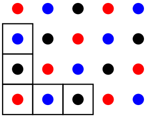

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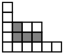

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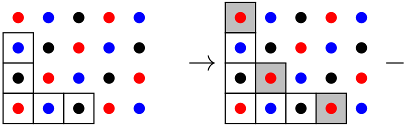

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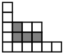

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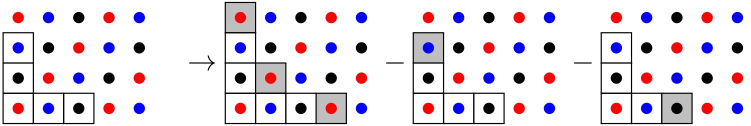

K-k-Schur functions

Conjecture (Lam et al., 2010; Morse, 2011)

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Problem

No direct formula for $g_\lambda^{(k)}$

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Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

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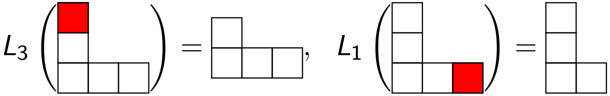
Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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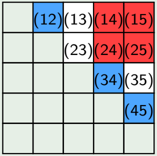
Example

non-roots of Ψ, roots of $\mathcal{L}$



$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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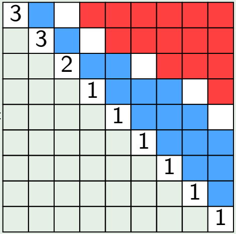
$g_{332111111}^{(4)} =$ 

$\Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$

Pieri Rule Illustrated (Recurrences)

A "graphical calculus."

$$g_1 g_{211}^{(2)}$$

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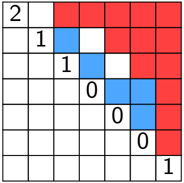

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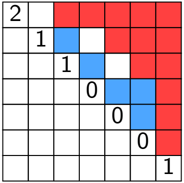

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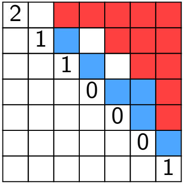

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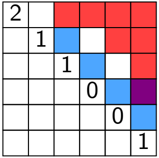

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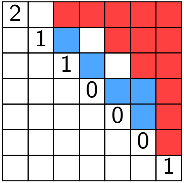
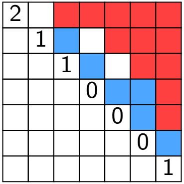
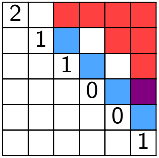
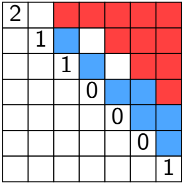
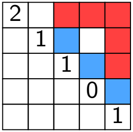
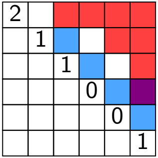
$g_1 g_{211}^{(2)} =$ 

$=$ 

$+$ 

$=$ 

$+$ 

$+$ 

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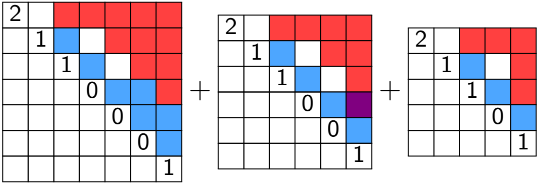

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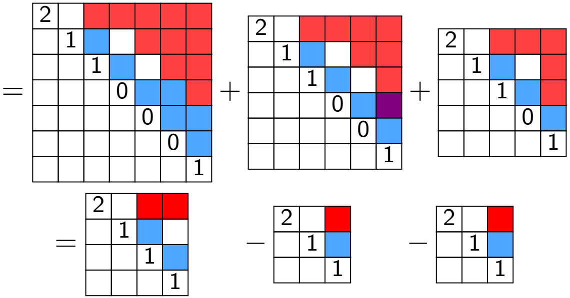

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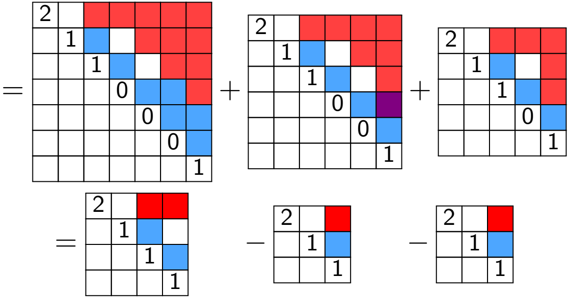

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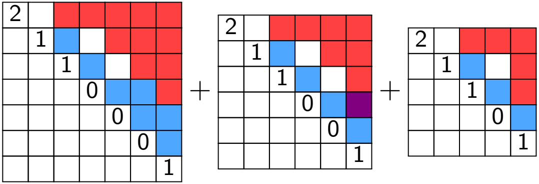

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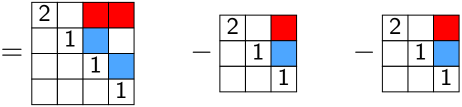

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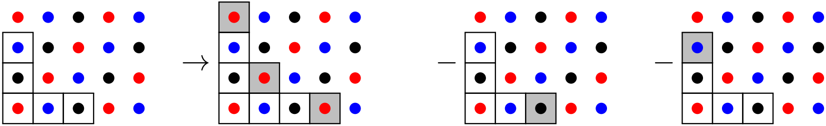
3-core perspective:



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Big Picture

$$\begin{tikzcd}
\begin{array}{c}\text{Schur functions}\\ H(\varnothing;\lambda)\end{array} \arrow[r] \arrow[d] & \begin{array}{c}k\text{-Schur functions}\\ H(\Delta^k(\lambda);\lambda)\end{array} \arrow[d] \\
\begin{array}{c}\text{dual Grothendieck}\\ \text{polynomials}\\ K(\varnothing;\varnothing;\lambda)\end{array} \arrow[r] & \begin{array}{c}K\text{-}k\text{-Schur functions}\\ {\color{red}K(\Delta^k(\lambda);\Delta^{k+1}(\lambda);\lambda)}\end{array}
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For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothtendieck polynomial",

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Theorem (S., 2021)

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- k-Rectangle Property fails for $g^{(k)}_{\lambda}$.

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What can be said about K-theoretic Catalan functions in general?

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Beyond K-theory

Raising operator techniques extend beyond Schubert calculus! Work by Blasiak-Haiman-Morse-Pun-S.:

- Shuffle theorems (Blasiak et al., 2021a; 2021b).
- Macdonald polynomials and LLT polynomials (Blasiak et al., 2021c).
- Much more work to be done!

References

Thank you!

Anderson, David, Linda Chen, and Hsian-Hua Tseng. 2017. *On the quantum K-ring of the flag manifold*, preprint. arXiv: 1711.08414.

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H Seelinger. 2021a. *A Shuffle Theorem for Paths Under Any Line*, arXiv e-prints, available at `arXiv:2102.07931`.

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021b. *A proof of the Extended Delta Conjecture*, arXiv e-prints, available at `arXiv:2102.08815`.

______. 2021c. *LLT polynomials in the Schiffmann algebra*. In preparation.

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and k-Schur Positivity*, J. Amer. Math. Soc. **32**, no. 4, 921–963.

Blasiak, Jonah, Jennifer Morse, and Anna Pun. 2020. *Demazure crystals and the Schur positivity of Catalan functions*, preprint. arXiv: 2007.04952.

Blasiak, Jonah, Jennifer Morse, and George H. Seelinger. 2020. *K-theoretic Catalan functions*, preprint. arXiv: 2010.01759.

Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for k-Schur functions*, Ph.D. thesis.

Fomin, Sergey, Sergei Gelfand, and Alexander Postnikov. 1997. *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10**, no. 3, 565–596, DOI 10.1090/S0894-0347-97-00237-3. MR1431829

Ikeda, Takeshi, Shinsuke Iwao, and Toshiaki Maeno. 2018. *Peterson Isomorphism in K-theory and Relativistic Toda Lattice*, preprint. arXiv: 1703.08664.

Lam, Thomas. 2008. *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. **21**, no. 1, 259–281.

Lam, Thomas, Luc Lapointe, Jennifer Morse, and Mark Shimozono. 2010. *Affine insertion and Pieri rules for the affine Grassmannian*, Mem. Amer. Math. Soc. **208**, no. 977.

Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010. *K-theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.

Lapointe, Luc, Alain Lascoux, and Jennifer Morse. 2003. *Tableau atoms and a new Macdonald positivity conjecture*, Duke Mathematical Journal **116**, no. 1, 103–146.

Morse, Jennifer. 2011. *Combinatorics of the K-theory of affine Grassmannians*, Advances in Mathematics.

Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.

Details

$$k_m^{(r)} = \sum_{i=0}^{m} \binom{r+i-1}{i} h_{m-i} = s_m(X+r)\,,$$

a specialization of "multiSchur functions." See, e.g., Lascoux-Naruse (2014).

$$k_\gamma = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_\ell}^{(\ell-1)}$$