K-theoretic Catalan functions

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Overview

- Schubert calculus: connecting geometry and combinatorics
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c^{\nu}_{\lambda\mu} f_\nu$

Polynomials informing Geometry

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 & h_1 &= x_1 + x_2 + x_3 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 & h_2 &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 & h_3 &= x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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- Bases indexed by integer partitions.

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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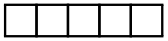
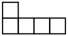
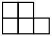
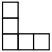
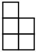
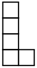
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

$1 + 1 + 1 + 1 + 1 \to$ 

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- Schubert varieties $X_\lambda = \overline{\Omega_\lambda}$.

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Special basis of Schur polynomials $\{s_\lambda\}$ indexed by partitions such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Tableaux

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$T = \begin{array}{|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-2} 3 & 4 \\
\cline{1-2} 2 & 3 \\
\hline 1 & 2 & 2 & 5 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|}
\cline{1-1} 8 \\
\cline{1-2} 7 & 9 \\
\cline{1-2} 3 & 4 \\
\hline 1 & 2 & 5 & 6 \\
\hline
\end{array}$$

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$$\mathrm{wt}(T) = (1,3,2,1,2) \quad (1,1,1,1,1,1,1,1,1)$$

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$$\mathrm{wt}(T) = (1,3,2,1,2) \quad (1,1,1,1,1,1,1,1,1)$$

$$x^{\mathrm{wt}(T)} = x_1^1 x_2^3 x_3^2 x_4^1 x_5^2 \quad x_1x_2x_3x_4x_5x_6x_7x_8x_9$$

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$$T = \begin{ytableau} 5 \\ 3 & 4 \\ 2 & 3 \\ 1 & 2 & 2 & 5 \end{ytableau} \qquad \begin{ytableau} 8 \\ 7 & 9 \\ 3 & 4 \\ 1 & 2 & 5 & 6 \end{ytableau}$$

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$\mathrm{SSYT}(\lambda) =$ all semistandard tableaux of shape λ.

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 2 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$$

Schur functions s_λ

Schur function s_λ is a “weight generating function” of semistandard tableaux:

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$$s_{\ydiagram{1,2}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$\begin{array}{|c|}\hline 2\\ \hline \end{array}\hspace{-4pt}\begin{array}{l}\\ \end{array}
\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array},\
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$$s_{\tiny\begin{array}{|c|c|}\cline{1-1} \, & \multicolumn{1}{c}{} \\ \hline \, & \, \\ \hline \end{array}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2 x_1 x_2 x_3$$

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$s_\lambda(x)$ is homogeneous of degree $\lambda_1 + \cdots + \lambda_\ell$.

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{(1)} s_{(2,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$$

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Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c^\nu_{\lambda\mu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$.

Schur functions s_λ (cont.)

Upshot

Let $\{f_\lambda\}$ be a basis of Λ such that

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2. $f_r f_\lambda$ satisfies the Pieri rule.

Then, $f_\lambda = s_\lambda$.

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Upshot

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Looking Ahead

This type of technique will be useful for establishing the equivalence of new formulas for other bases.

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When examining Schubert representatives in Λ, we ask

- Does it have a Pieri rule? $(s_r s_\lambda = \sum s_\nu)$
- Does it have a direct formula? $(s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T)$

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
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And many more!

Big Picture

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{(Co)homology of}\\ \text{Grassmannian/}\\ \text{Schur functions}\end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c}\text{Homology of Affine}\\ \text{Grassmannian/}\\ k\text{-Schur functions}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}K\text{-homology of}\\ \text{Grassmannian/}\\ \text{dual Grothendieck}\\ \text{polynomials}\end{array}} \arrow[r] & \boxed{\begin{array}{c}K\text{-homology of affine}\\ \text{Grassmannian/}\\ K\text{-}k\text{-Schur functions}\end{array}}
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- Branching with positive coefficients (Lam et al., 2010):

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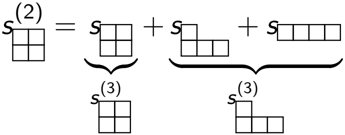

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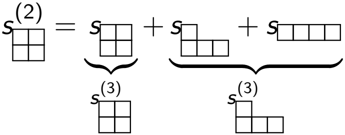

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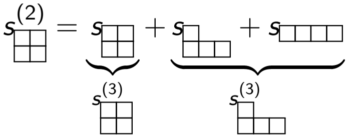
k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Branching with positive coefficients (Lam et al., 2010):



  - (Lam et al., 2010) gives geometric interpretation,
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  - Branching with t important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

Overview

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- K-theoretic Catalan functions

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Key: $\{s_\lambda^{(k)}\}_\lambda \subseteq$ Catalan functions $=$ large class of symmetric functions.

Ingredients for Catalan functions

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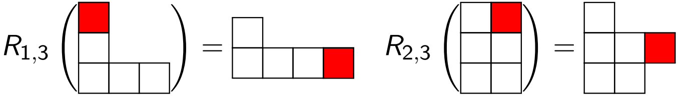

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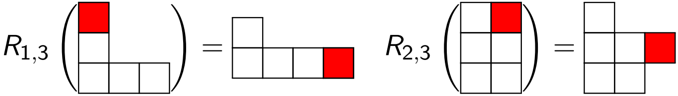

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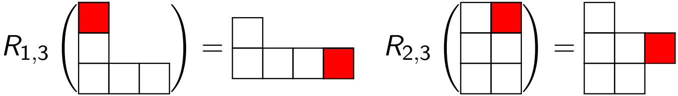

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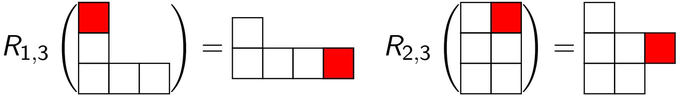

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$$\begin{aligned} s_{22} &= (1 - R_{12}) h_{22} = h_{22} - h_{31} \\ s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211} \\ &= h_{211} - h_{301} - h_{220} \color{red}{- h_{310} + h_{310}} + \underbrace{h_{32\,\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\,\text{-}1}}_{=0} \end{aligned}$$

some terms cancel

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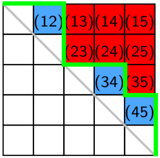
Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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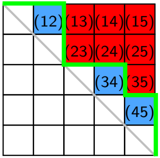

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For Ψ and $\gamma \in \mathbb{Z}^\ell$

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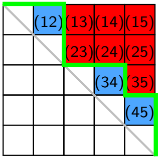

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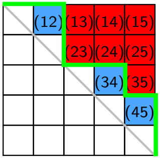

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Catalan functions

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive!
Precisely, $H(\Psi; \lambda) = \sum_\nu c^\nu_{\Psi,\lambda} s_\nu$ satisfies $c^\nu_{\Psi,\lambda} \in \mathbb{Z}_{\geq 0}$.

Catalan functions

k-Schur root ideal for λ

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i,j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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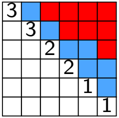
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

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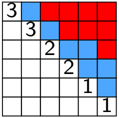

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k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

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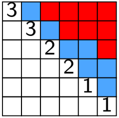
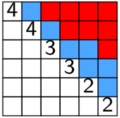
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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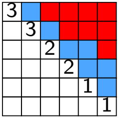
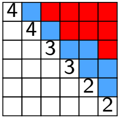
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Pieri:

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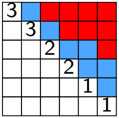
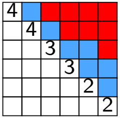
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

$$s_{\lambda}^{(k)} = s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

Big Picture

$$\begin{tikzcd}
\begin{array}{c}\text{(Co)homology of}\\ \text{Grassmannian}\\ s_\lambda = H(\varnothing;\lambda)\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{Homology of Affine}\\ \text{Grassmannian}\\ s_\lambda^{(k)} = H(\Delta^k(\lambda);\lambda)\end{array} \arrow[d] \\
\begin{array}{c}K\text{-homology of}\\ \text{Grassmannian/}\\ \text{dual Grothendieck}\\ \text{polynomials}\end{array} \arrow[r] & \begin{array}{c}K\text{-homology of affine}\\ \text{Grassmannian/}\\ K\text{-}k\text{-Schur functions}\end{array}
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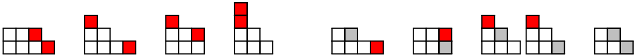

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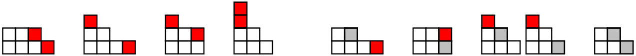

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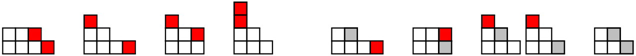

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- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m,n))$

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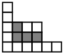
- A $(k+1)$-core is a partition with no cell of hook length $k+1$. 

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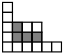
- A $(k+1)$-core is a partition with no cell of hook length $k+1$. 

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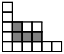
- A $(k+1)$-core is a partition with no cell of hook length $k+1$. 

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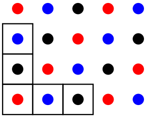

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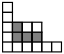

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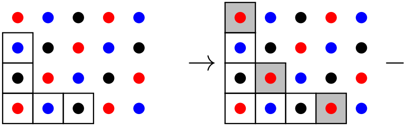

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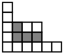
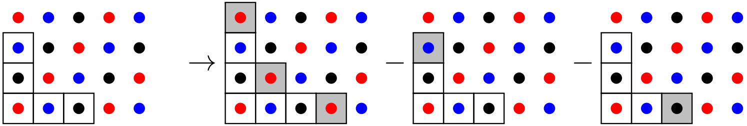
- A $(k+1)$-core is a partition with no cell of hook length $k+1$.



$$g_1 g_{211}^{(2)} \quad = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions $\leftrightarrow$ 3-cores



K-k-Schur functions

Conjecture (Lam et al., 2010; Morse, 2011)

$g_\lambda^{(k)}$ have positive branching into $g_\mu^{(k+1)}$.

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Problem

No direct formula for $g_\lambda^{(k)}$

K-k-Schur functions

Solution

Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

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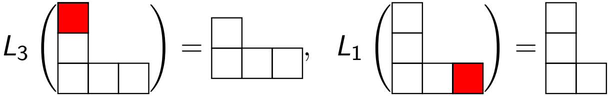
Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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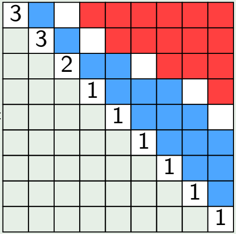
Example

$g^{(4)}_{332111111} =$



$\Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$

Pieri Rule Illustrated (Recurrences)

A "graphical calculus."

$$g_1 g_{211}^{(2)}$$

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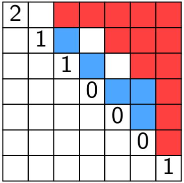

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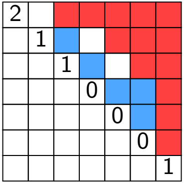

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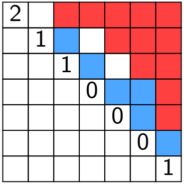

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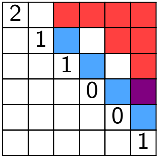

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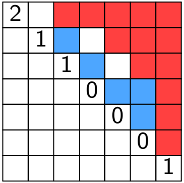

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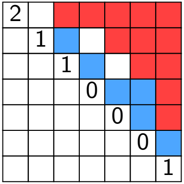

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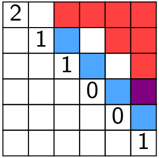

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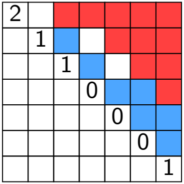

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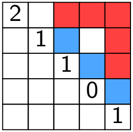

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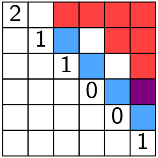

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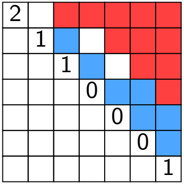

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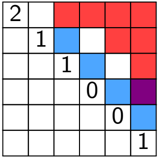

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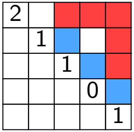

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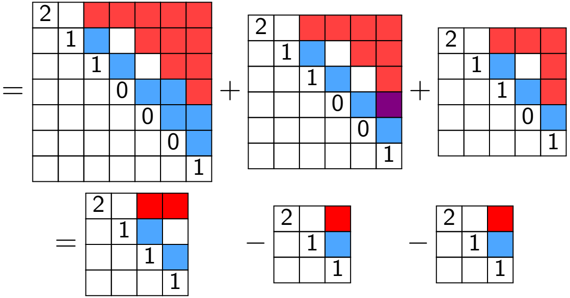

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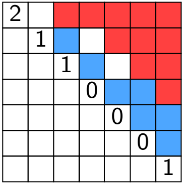

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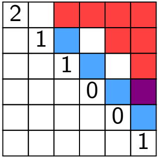

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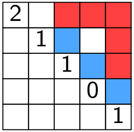

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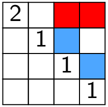

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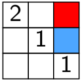

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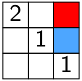

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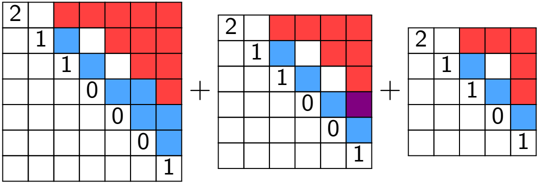

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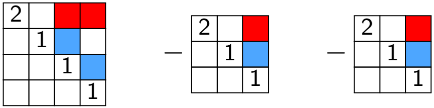

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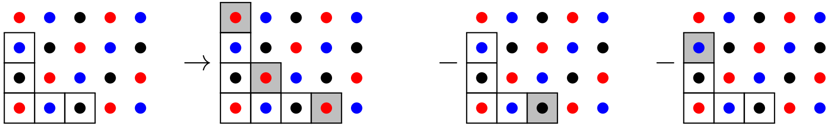
3-core perspective:



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Big Picture

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Schur functions}\\ H(\varnothing;\lambda)\end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c}k\text{-Schur functions}\\ H(\Delta^k(\lambda);\lambda)\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\text{dual Grothendieck}\\ \text{polynomials}\\ K(\varnothing;\varnothing;\lambda)\end{array}} \arrow[r] & \boxed{\begin{array}{c}K\text{-}k\text{-Schur functions}\\ {\color{red}K(\Delta^k(\lambda);\Delta^{k+1}(\lambda);\lambda)}\end{array}}
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$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

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For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothtendieck polynomial",

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What can be said about K-theoretic Catalan functions in general?

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- $K(\Psi; RC^a(\Psi); \lambda) = \sum_\mu b_\mu s_\mu$ satisfies $b_\mu \in \mathbb{Z}_{\geq 0}$.

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For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule:
   $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ??G_\lambda^{(k)}$, $1 \leq r \leq k$.
2. Combinatorially describe branching coefficients: $g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k+1)}$.
3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ??s_\mu^{(k)}$.

Beyond K-theory

Raising operator techniques extend beyond Schubert calculus! Work by Blasiak-Haiman-Morse-Pun-S.:

- Shuffle theorems (Blasiak et al., 2021a; 2021b).
- Macdonald polynomials and LLT polynomials (Blasiak et al., 2021c).
- Much more work to be done!

Thank you!

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Details

$$k_m^{(r)} = \sum_{i=0}^{m} \binom{r+i-1}{i} h_{m-i} = s_m(X+r)\,,$$

a specialization of "multiSchur functions." See, e.g., Lascoux-Naruse (2014).

$$k_\gamma = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_\ell}^{(\ell-1)}$$