

K -theoretic Catalan functions

George H. Seelinger

ghs9ae@virginia.edu

University of Virginia

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- Schubert calculus: connecting geometry and combinatorics
- Catalan functions: a new approach to old problems
- K -theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of subvarieties in a variety X .

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Algebra of Symmetric Functions

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$$e_1 = x_1 + x_2 + x_3 \quad h_1 = x_1 + x_2 + x_3$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.
- Bases indexed by integer partitions.

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Partitions

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

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- Schubert varieties $X_\lambda = \overline{\Omega_\lambda}$.

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Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda \subseteq (n^m)}$ in variety $X = \text{Gr}(m, n)$.

Classical Schubert Calculus

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ indexed by partitions such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$.

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$T =$$

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
3	4		
1	2	5	6

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$$\text{wt}(T) = (1, 3, 2, 1, 2) \quad (1, 1, 1, 1, 1, 1, 1, 1, 1)$$

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$$x^{\text{wt}(T)} = x_1^1 x_2^3 x_3^2 x_4^1 x_5^2 \quad x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$$

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$\text{SSYT}(\lambda) =$ all semistandard tableaux of shape λ .

$$\begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|} \hline 2 \\ \hline 1 \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array}$$

Schur functions s_λ

Schur function s_λ is a “weight generating function” of semistandard tableaux:

2		3		3		2		3		3		2		3	
1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

$$s_{\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda(x) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}$$

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$$s_\lambda(x) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}$$

$s_\lambda(x)$ is homogeneous of degree $\lambda_1 + \cdots + \lambda_\ell$.

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}$$

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Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$, subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu$.

Upshot

Let $\{f_\lambda\}$ be a basis of Λ such that

- ① $f_r = s_r$ and
- ② $f_r f_\lambda$ satisfies the Pieri rule.

Then, $f_\lambda = s_\lambda$.

Schur functions s_λ (cont.)

Upshot

Let $\{f_\lambda\}$ be a basis of Λ such that

- 1 $f_r = s_r$ and
- 2 $f_r f_\lambda$ satisfies the Pieri rule.

Then, $f_\lambda = s_\lambda$.

Looking Ahead

This type of technique will be useful for establishing the equivalence of new formulas for other bases.

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- Does it have a Pieri rule? ($s_r s_\lambda = \sum s_\nu$)
- Does it have a direct formula? ($s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T$)

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Schubert Calculus Variations

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Theory	f_λ
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- P and Q functions
(Co)homology of affine Grassmannian	(dual) k -Schur functions
K -theory of Grassmannian	Grothendieck polynomials
K -homology of affine Grassmannian	K - k -Schur functions

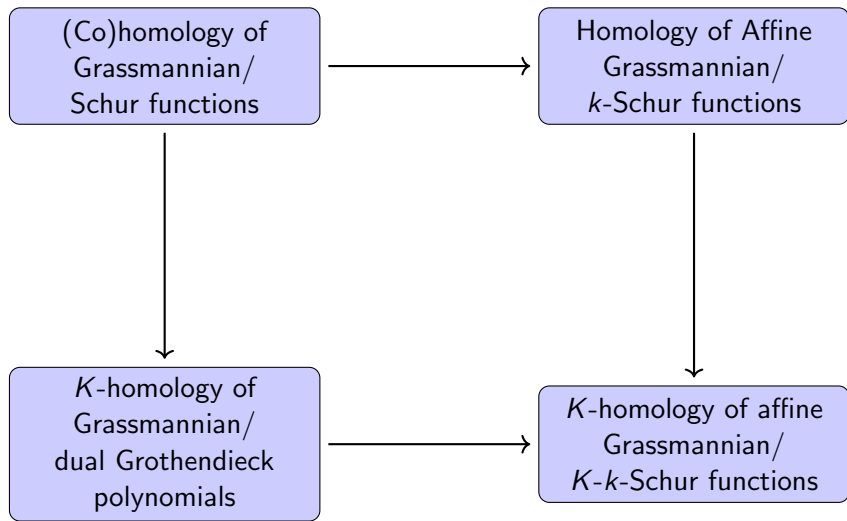
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And many more!

Big Picture



k -Schur functions

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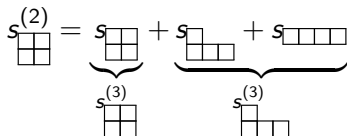
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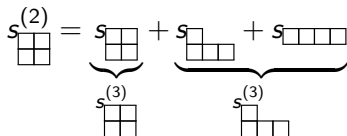
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$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\lambda} + s_{\lambda}$$



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$\underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}^{(3)}} \quad \underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}^{(3)}}$

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- $s_{\lambda}^{(k)} = s_{\lambda}$ as $k \rightarrow \infty$.
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda}^{(3)} + s_{\lambda}^{(3)}$$

The diagram shows the branching of the 2-partition $s_{(2)}^{(2)}$ into 3-partitions. On the left is a 2x2 square representing $s_{(2)}^{(2)}$. On the right is the sum of two 3-partitions: $s_{(2)}^{(3)}$ (a 2x2 square) and $s_{(1,1)}^{(3)}$ (a 1x1 square with a 1x2 rectangle attached to its right). Brackets below the right side group the terms as $s_{(2)}^{(3)} + s_{(1,1)}^{(3)}$.

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- (Lam et al., 2010) gives geometric interpretation,
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- Branching with t important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- K -theoretic Catalan functions

Why a new definition of k -Schur?

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Answer

- 1 (Blasiak et al., 2019) gives a new definition of $s_{\lambda}^{(k)}$ and shows it is equivalent to many other previous definitions.

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- 2 From a new definition, (Blasiak et al., 2019) shows the branching coefficients $b_{\lambda\mu}$ in the expansion $s_{\lambda}^{(k)} = \sum_{\mu} b_{\lambda\mu} s_{\mu}^{(k+1)}$ have combinatorial interpretation!

Key:

Why a new definition of k -Schur?

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Key: $\{s_{\lambda}^{(k)}\}_{\lambda} \subseteq \text{Catalan functions} = \text{large class of symmetric functions.}$

Ingredients for Catalan functions

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Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

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$$s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$$

$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \text{red } h_{310} + \text{red } h_{310} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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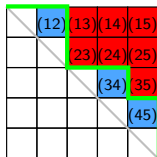
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Root Ideals

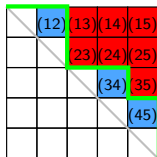
A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



Ψ = Roots above Dyck path
 $\Delta_{\ell}^{+} \setminus \Psi$ = Non-roots below

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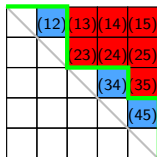
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

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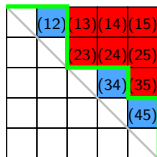
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Intuition

Catalan functions interpolate between h_λ and s_λ .

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive!
Precisely, $H(\Psi; \lambda) = \sum_\nu c_{\Psi, \lambda}^\nu s_\nu$ satisfies $c_{\Psi, \lambda}^\nu \in \mathbb{Z}_{\geq 0}$.

k -Schur root ideal for λ

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

\leftarrow row i has $4 - \lambda_i$ non-roots

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k -Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$ for $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$.

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3					
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			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

Key ingredient of branching proof

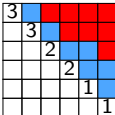
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Pieri:

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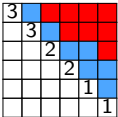
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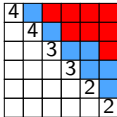
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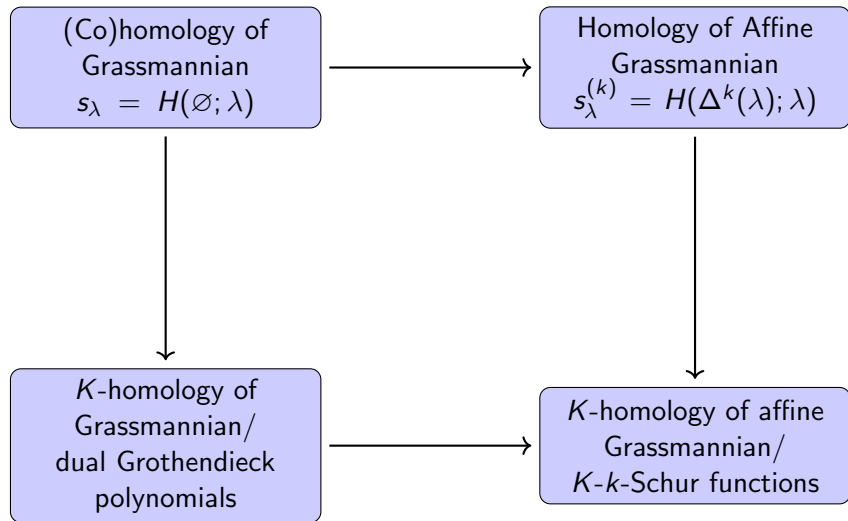
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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

Big Picture



- Schubert calculus
- Catalan functions: a new approach to old problems
- ***K*-theoretic Catalan functions**

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- Inhomogeneous basis: $g_\lambda = s_\lambda + \text{lower degree terms}$.

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$$g_{1^2}g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{32}$$



Add (addable) or mark (removable) in any combination of r boxes, but only once per row.

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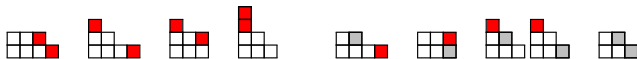
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- $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$ for k_λ and inhomogeneous analogue of h_λ .
- Dual to Grothendieck polynomials G_λ : Schubert representatives for $K^*(Gr(m, n))$

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$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)} \quad \text{2-bounded partitions} \leftrightarrow \text{3-cores}$$

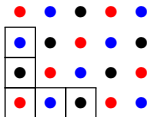
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Conjecture (Lam et al., 2010; Morse, 2011)

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Problem

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Find a formula for $g_{\lambda}^{(k)}$ analogous to raising operator formula for $s_{\lambda}^{(k)}$.

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Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left(\begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

K -theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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“ Ψ =raising ideal, \mathcal{L} =lowering ideal.”

Affine K -Theory Representatives with Raising Operators

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Example

non-roots of Ψ , roots of \mathcal{L}

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ = (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332} \end{aligned}$$

Answer (Blasiak-Morse-S., 2020)

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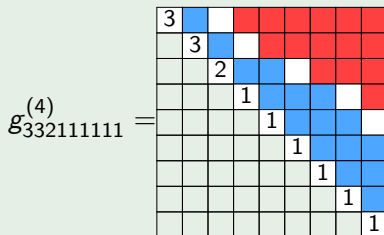
For K -homology of affine Grassmannian, $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$ since this family satisfies the Pieri rule.

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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

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$$=$$

2							
	1						
		1					
			0				
				0			
					0		
						1	

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$$=$$

2						
	1					
		1				
			0			
				0		
					0	
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$$=$$

2						
	1					
		1				
			0			
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					0	
						1

$$+$$

2						
	1					
		1				
			0			
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					0	
						1

Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$g_1 g_{211}^{(2)}$

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
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					0	
						1

$$+$$

2						
	1					
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					0	
						1

$$+$$

2						
	1					
		1				
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					0	
						1

Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
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						1

 $+$

2						
	1					
		1				
			0			
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						1

 $+$

2						
	1					
		1				
			0			
				0		
					0	
						1

Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
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					0	
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$$+$$

2						
	1					
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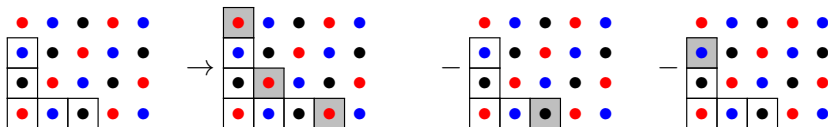
Pieri Rule Illustrated (Straightening)

$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
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3-core perspective:



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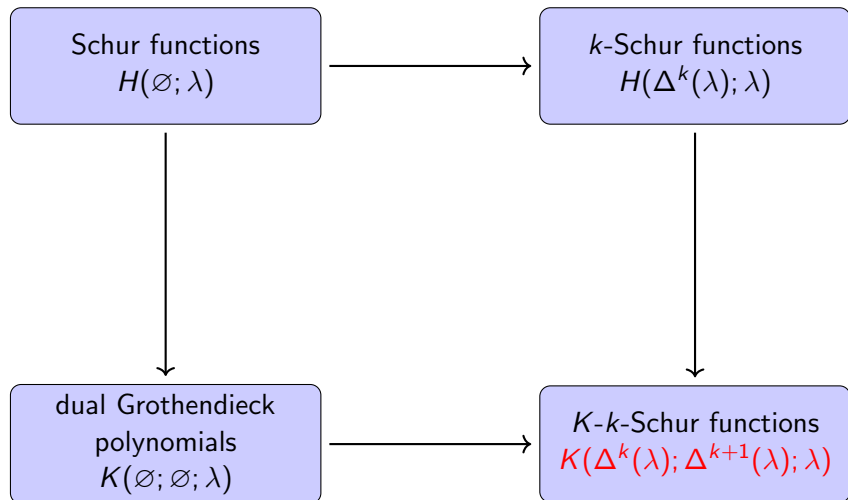
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Big Picture



K -theoretic Peterson isomorphism

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What can be said about K -theoretic Catalan functions in general?

Positivity of K -theoretic Catalan functions

Recall (Blasiak et al., 2020)

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- $K(\Psi; RC^a(\Psi); \lambda) = \sum_{\mu} b_{\mu} s_{\mu}$ satisfies $b_{\mu} \in \mathbb{Z}_{\geq 0}$.

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Raising operator techniques extend beyond Schubert calculus! Work by Blasiak-Haiman-Morse-Pun-S.:

- Shuffle theorems (Blasiak et al., 2021a; 2021b).
- Macdonald polynomials and LLT polynomials (Blasiak et al., 2021c).
- Much more work to be done!

Thank you!

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$$k_m^{(r)} = \sum_{i=0}^m \binom{r+i-1}{i} h_{m-i} = s_m(X+r),$$

a specialization of “multiSchur functions.” See, e.g., Lascoux-Naruse (2014).

$$k_{\gamma} = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_{\ell}}^{(\ell-1)}$$