GRASSMANNIANS AND SCHUBERT VARIETIES

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1. Introduction

2. Projective Varieties

The language of projective varieties is inherently useful for describing Grassmannians. Therefore, this section contains a terse reiteration of the relevant notions about projective varieties. For a more complete treatment, consult any introductory text on algebraic geometry.

2.1. **Definition.** Given $n \in \mathbb{N}$, we define *projective n-space* over a field K, denoted $\mathbb{P}^n_K$ or $\mathbb{P}^n$ when K is understood, to be the set of all 1-dimensional linear subpsaces of the vector space K^{n+1}.

2.2. **Example.** The easiest example of a projective variety is $\mathbb{P}^1$. If we take two copies of the affine line $\mathbb{A}^1$, denoted X_1, X_2, and "glue" the open sets $U, U' = \mathbb{A}^1 \setminus \{0\}$ together via the map $f(x) = \frac{1}{x}$, we get $\mathbb{P}^1$. In such a space, we have the identity "$\frac{1}{0} = \infty$" in the persective of X_1 and so we have a compactification of the affine line. When $K = \mathbb{C}$, this resulting space is the same as the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

2.3. **Remark.** One often describes projective space using "homogeneous coordinates", which are invariant under scalar multiple, arising from the description

$$\mathbb{P}^n = (K^{n+1} \setminus \{0\})/\sim$$

where $\sim$ is given by

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \iff \exists \lambda \in K^\times \text{ such that } x_i = \lambda y_i \ \forall i$$

We denote such an equivalence class by $[x_0 : \cdots : x_n] \in \mathbb{P}^n$.

A projective variety can also be described by a set of homogeneous polynomials $f_1, \ldots, f_r \in K[x_0, \ldots, x_n]$ since, if f_i is homogeneous of degree d, then

$$f_i(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n), \quad \lambda \in K$$

and so all scalar multiples of a solution to $f_i(x_0, \ldots, x_n) = 0$ are solutions.

2.4. **Definition.** Let $\pi \colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ send $(x_0, \ldots, x_n) \mapsto [x_0 : \cdots : x_n]$. Then,

Date: August 2018.

(a) We say an affine variety $X \subseteq \mathbb{A}^{n+1}$ is a *cone* if $0 \in X$ and $\lambda x \in X$ for all $\lambda \in K$ and $x \in X$.
(b) Given a cone $X \subseteq \mathbb{A}^{n+1}$, we say

$$\mathbb{P}(X) := \pi(X \setminus \{0\}) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid (x_0, \ldots, x_n) \in X\} \subseteq \mathbb{P}^n$$

is the *projectivization* of X.
(c) Given a projective variety $X \subseteq \mathbb{P}^n$, we say

$$C(X) := \{0\} \cup \pi^{-1}(X) = \{0\} \cup \{(x_0, \ldots, x_n) \mid [x_0 : \cdots : x_n] \in X\} \subseteq \mathbb{A}^{n+1}$$

is the *cone* over X.

2.5. **Remark.** From above, with some additional work, one can show there is a one-to-one correspondence

$$\begin{aligned} \{\text{cones in } \mathbb{A}^{n+1}\} &\leftrightarrow \{\text{projective varieties in } \mathbb{P}^n\} \\ X &\mapsto \mathbb{P}(X) \\ C(X) &\leftarrow X \end{aligned}$$

2.6. **Lemma.** *For non-empty projective variety $X \subseteq \mathbb{P}^n$, the dimension of the cone $C(X) \subseteq \mathbb{A}^{n+1}$ is $\dim X + 1$.*

Proof. Let

$$\varnothing \neq X_0 \subsetneqq \cdots \subsetneqq X_n \subseteq X$$

be a chain of irreducible closed subsets in X. Then,

$$\{0\} \subsetneqq C(X_0) \subsetneqq \cdots \subsetneqq C(X_n) \subseteq C(X)$$

is a chain of irreducible closed subsets in $C(X)$. So, we have that

$$\dim C(X) \geq \dim X + 1$$

Similarly, if we show

Do this explicitly

$$\operatorname{codim}(C(X)) \geq \operatorname{codim}(X) = n - \dim X$$

If we assume X is irreducible (if not, take an irreducible decomposition), then

$$\dim(C(X)) + \operatorname{codim}(C(X)) = \dim(\mathbb{A}^{n+1}) = n + 1$$

and so

$$\dim(C(X)) = n + 1 - \operatorname{codim}(C(X)) \leq n + 1 - n + \dim(X) = \dim(X) + 1$$

$\square$

3. Grassmannians

3.1. **Defining and Understanding Grassmannians.**

3.1. **Definition.** A *Grassmannian*, denoted $G(m,n)$, is the set of linear subpsaces of dimension m (and therefore of codimension n) in K^{m+n}.

In this way, one can represent the Grassmannian $G(m,n)$ as equivalence classes of full rank $m \times (m+n)$ matrices whose row space encodes a linear subspace in $G(m,n)$, where the equivalence class is given by row operations, or more formally

$$A \sim B \iff A = EB,\ E \in GL_m(K)$$

Therefore, we can always pick represenetatives of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & x_{11} & \cdots & x_{1n} \\ 0 & 1 & \cdots & 0 & x_{21} & \cdots & x_{2n} \\ \vdots & & \ddots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

3.2. **Proposition.** *The m-dimensional linear subspaces of K^{m+n} are in natural one-to-one correspondence with $(m-1)$-dimensional linear subspaces of $\mathbb{P}^{m+n-1}$.*

Proof. Given such a subspace of K^{m+n}, an m-dimensional linear subspace W (which is defined by a set of homogeneous polynomials and thus a cone) can be sent to $\mathbb{P}(W)$, which has dimension $m-1$ in $\mathbb{P}^{m+n-1}$. $\square$

Therefore, we may also think of $G(m,n)$ as the set of such projective linear subspaces.

3.2. **The Plücker Embedding.**

We wish to give a concrete realization of $G(m,n)$ as a subset of projective space via the "Plücker Embedding."

3.3. **Lemma.** *If $W \in G(m,n)$, then $\bigwedge^m W$ is a line (a cone of dimension 1) in $\bigwedge^m K^{m+n}$.*

Proof. Given $W \cong K^m = \operatorname{span}\{e_1, \ldots, e_n\}$, we have that $\bigwedge^m K^m \cong K$ since it only has a single basis vector, namely $e_1 \wedge \cdots \wedge e_n$. $\square$

3.4. **Lemma.** *Let $v_1, \ldots, v_k \in K^n$ and $w_1, \ldots, w_k \in K^n$ both be linearly independent. Then, $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are linearly dependent in $\bigwedge^k K^n$ if and only if $\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{w_1, \ldots, w_k\}$.*

Proof. This is an exercise using only multi-linear algebra skills. $\square$

3.5. **Theorem.** *The map*

$$\phi\colon G(m,n) \to \mathbb{P}\Big(\bigwedge^m K^{m+n}\Big)$$
$$W = \operatorname{span}\{w_1, \ldots, w_m\} \mapsto [w_1 \wedge \cdots \wedge w_m]$$

is an embedding, called the Plücker Embedding*.*

Proof. The map is well-defined by the $\Longleftarrow$ direction of the lemma above and is injective by the $\Longrightarrow$ direction of the lemma above. $\square$

However, the description given by the proposition is not very concrete. Using multi-linear algebra, we can remedy this problem.

3.6. **Lemma.** *Given $0 \leq k \leq n$ and $v_1, \ldots, v_k \in K^n$ with $v_j = \sum_i a_{j,i} e_i$, we have*

$$v_1 \wedge \cdots \wedge v_k = \sum_{i_1, \ldots, i_k} a_{1,i_1} \cdots a_{k,i_k} \cdot e_{i_1} \wedge \cdots \wedge e_{i_k}$$

and, for stricly increasing index sequence $(j_1, \ldots, j_k)$, we get the coefficient of $e_{j_1} \wedge \cdots \wedge e_{j_k}$ in $v_1 \wedge \cdots \wedge v_k$ is given by

$$\sum \operatorname{sgn}(\sigma) a_{1,j_{\sigma(1)}} \cdots a_{k,j_{\sigma(k)}} = \det(a_{i,j})_{1 \leq i \leq k, j \in \{j_1, \ldots, j_k\}}$$

Thus, we have

3.7. **Theorem** (Plücker Coordinates)**.** *The homogeneous coordinates of $\phi(W)$ in $\mathbb{P}(\bigwedge^m K^{m+n})$ are the minors of order m of the matrix whose row space is W, say $(x_{i,j})_{1 \leq i \leq m, 1 \leq j \leq m+n}$, which we denote*

$$P_{i_1, \ldots, i_m} := \det(x_{p,i_q})_{1 \leq p,q \leq m}, \; i_1 < \cdots < i_m$$

3.8. **Example.** Consider $G(1, n-1)$ will have

$$\operatorname{span}\{a_1 e_1 + \cdots + a_n e_n\} \mapsto [a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}$$

Furthermore, since this map is surjective, we get $G(1, n-1) \cong \mathbb{P}^{n-1}$, as discussed above.

For a more complicated example, consider $G(2,1)$. If we take the space spanned by $e_1 + e_2, e_1 + e_3$, then we have

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mapsto [P_{1,2} : P_{1,3} : P_{2,3}] = \left[\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = [-1 : 1 : 1]$$

Thus, we have realized the Grassmannian in projective space. However, we wish to show it is a variety. To do that, we must show that it is a vanishing set of polynomials.

3.9. **Definition.**

(a) Let $J_{m,n}$ be the set of strictly increasing m-tuples consisting of integers between 1 and $m+n$.

(b) Let $K[P_J \mid J \in J_{m,n}]$ be the ring of polynomials in the Plücker coordinates.

(c) Let $I(G(m,n))$ be the ideal consisting of homogeneous polynomials vanishing identically on $G(m,n)$.

3.10. **Theorem.** *Let $i_1, \dots, i_m$ and $j_1, \dots, j_m$ be two sets of integers between 1 and $m+n$ and let l be an integer between 1 and m. Then, identically on $G(m,n)$, we have the relation*

$$\sum_{w \in \mathfrak{S}_{i_l,\dots,i_m,j_1,\dots,j_l}/\mathfrak{S}_{i_l,\dots,i_m} \times \mathfrak{S}_{j_1,\dots,j_l}} \operatorname{sgn}(w) P_{i_1,\dots,i_{l-1},w(i_l),\dots,w(i_m)} P_{w(j_1),\dots,w(j_l),j_{l+1},\dots,j_m} = 0$$

3.11. **Example.** [Man98, Exercise 3.1.5] If $m = n = 2$ and $l = 2$, then $\mathfrak{S}_{2,3,4}/\mathfrak{S}_2 \times \mathfrak{S}_{3,4} \cong \{id, (23), (243)\}$ and we have the single equation

$$P_{1,2}P_{3,4} - P_{1,3}P_{2,4} + P_{1,4}P_{2,3} = 0$$

One can use this relation to show that the Plücker embedding gives $G(2,2)$ as a quadratic hypersurface of $\mathbb{P}^5$.

Proof. □ Fill in proof

4. Schubert Cells and Schubert Varieties

4.1. **Definition.** [Man98, p 105] Fix a flag

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_i \subseteq \cdots \subseteq V_{n+m} = \mathbb{C}^{m+n}$$

Then, if λ is a partition contained in an $n \times m$ rectangle, that is $n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq 0$, we associate to it the *Schubert cell*

$$\Omega_\lambda := \{W \in G(m,n) \mid \dim(W \cap V_j) = i \text{ if } n + i - \lambda_i \leq j \leq n + i - \lambda_{i+1}\}$$

and the *Schubert variety*

$$X_\lambda = \{W \in G(m,n) \mid \dim(W \cap V_{n+i-\lambda_i}) \geq i, 1 \leq i \leq m\}$$

4.2. **Example.** (a) We have that $X_\varnothing = G(m,n)$ since

$$W \in G(m,n) \Longrightarrow \dim W = m \Longrightarrow \dim W + \dim V_{n+i} = m+n+i \Longrightarrow \dim(W \cap V_{n+i}) \geq i$$

(b) On the other extreme, $X_{(n^m)} = \{V_m\}$ since

$$\dim(W \cap V_{n+i-n}) = \dim(W \cap V_i) \geq i, \forall 1 \leq i \leq m \Longrightarrow V_i \subseteq W, \forall 1 \leq i \leq m \Longrightarrow W = V_m$$

(c) If $\lambda = (k)$, that is, λ has only one part, then

$$X_k = \{W \in G(m,n) \mid W \cap V_{n+1-k} \neq 0\}$$

since

$$\dim(W \cap V_{n+1-k}) \geq 1 \Longleftrightarrow W \cap V_{n+1-k} \neq 0$$

and we always have $\dim(W \cap V_{n+i}) \geq i$ for $2 \leq i \leq m$ as discussed in (a).

(d) If λ is a partition such that its diagram is the complement of a $q \times p$ rectangle inside an $n \times m$ rectangle, then

$$X_\lambda = \{W \in G(m,n) \mid V_{m-p} \subseteq W \subseteq V_{m+q}\} \cong G(p,q)$$

To see this, consider that λ is of the form

$$\lambda = (\underbrace{n, n, \dots, n}_{m-p}, \underbrace{n-q, \dots, n-q}_{p})$$

and so, as discussed in (b), we know that the first $m-p$ entries of n's will force $V_{m-p} \subseteq W$. Furthermore,

$$\dim(W \cap V_{n+i-(n-q)}) \geq i, \forall m-p+1 \leq i \leq m \iff \dim(W \cap V_{m+q}) \geq m \iff W \subseteq V_{m+q}$$

One important use of Schubert cells is that they form a cellular decomposition of the Grassmannian.

4.3. **Lemma.** *We have that*

$$G(m,n) = \bigsqcup_{\mu \text{ in an } n\times m \text{ rectangle}} \Omega_\mu$$

Proof. Given $W \in G(m,n)$, the sequence $\{\dim(W \cap V_i)\}_{1 \leq i \leq n+m}$ runs from 0 to m, increasing at most by 1 for each step. Thus, since there are m jumps in the sequence, say $j_1, \ldots, j_m$, we can define a partition μ via $\mu_i := n + i - j_i \iff j_i = n + i - \mu_i$ and, since $j_i \geq i$, get that μ is contained in an $n \times m$ rectangle. Thus, $W \in \Omega_\mu$ and, since we can recover all such partitions this way, we get the cellular decomposition result. □

4.4. **Proposition.** [Man98, Proposition 3.2.3] *For all partitions λ inside an $n \times m$ rectangle, we have*

- *(a) the Schubert variety X_λ is an algebraic subvariety of $G(m,n)$.*
- *(b) $X_\lambda = \bigsqcup_{\mu \supseteq \lambda} \Omega_\mu$*
- *(c) $\Omega_\lambda \cong \mathbb{C}^{mn-|\lambda|}$*
- *(d) $X_\lambda \supseteq X_\mu$ if and only if $\lambda \subseteq \mu$.*
- *(e) Ω_λ is a dense open set of X_λ contained in the set of nonsingular points.*

Proof. (a) follows from the fact that

$$\dim(W \cap V_i) \geq j \iff \operatorname{rank}(W \subseteq \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}/V_i) \leq m - j$$

which is equivalent to the vanishing of minors of order $m - j + 1$ of the matrix representing the map.

For (b), we first note that, as in the proof of the lemma above, given a $W \in X_\lambda$, we have a sequence of dimensions $\{\dim(W \cap V_j)\}_{1 \leq j \leq n+m}$, but moreover we also have that $\dim(W \cap V_{n-i+\lambda_j}) \geq i$ for $1 \leq i \leq m$ and so, the first i jumps in our sequence must occur before $n + i - \lambda_i$, which means $n + i - \lambda_i \geq j_i = n + i - \mu_i \implies \lambda_i \leq \mu_i$. Thus, this gives

$$X_\lambda = \bigsqcup_{\mu \supseteq \lambda} \Omega_\mu$$

For (c), we choose a basis of $\mathbb{C}^{m+n}$ that is compatible with our flag, that is $v_1, \ldots, v_{m+n}$ such that $V_i = \operatorname{span}\{v_1, \ldots, v_i\}$. Then, for $W \in \Omega_\lambda$, we have a unique basis consisting of vectors of the form

$$w_i = v_{n+i-\lambda_i} + \sum_{\substack{1 \leq j \leq n+i-\lambda_i \\ j \neq n+k-\lambda_k \\ k \leq i}} x_{ij} v_j$$

and thus the matrix defined by (x_{ij}) defines a linear transformation from Ω_λ to $\mathbb{C}^{mn-|\lambda|}$.

To finish the proof, one argues that $\Omega_\mu \subseteq \overline{\Omega_\lambda}$ by looking at elements of Ω_λ and considering the limiting values. Thus, we get $\Omega_\lambda \subseteq X_\lambda = \bigsqcup_{\mu \supseteq \lambda} \Omega_\mu \subseteq \overline{\Omega_\lambda}$, but since X_λ is closed, it must be that $X_\lambda = \overline{\Omega_\lambda}$. □

Now, we are particularly interested in the consequences of this result on the cohomology of the Grassmannian.

4.5. **Corollary.** *(a) Thus, the Poincaré duals of the cohomology classes of $\overline{\Omega_\lambda} = X_\lambda$, say $\sigma_\lambda := [X_\lambda]^*$, form a basis for the integral cohomology of $G(m, n)$. We call such classes* Schubert classes.

(b) For all partitions λ contained in an $n \times m$ rectangle, σ_λ is an element of $H^{2|\lambda|(G(m,n))}$

(c) In fact, we have the decomposition

$$H^*(G(m, n)) = \bigoplus_{\lambda \textit{ in an } n\times m \textit{ rectangle}} \mathbb{Z}\sigma_\lambda$$

Proof. For part (a), since each of the Schubert cells are isomorphic to $\mathbb{C}^{mn-|\lambda|}$, it must be that all the cells are concentrated in even (real) dimension, thus giving that there are no relations for for the fundamental classes in the cohomology groups, and so they form a basis.

For (b), this follows from the fact that Ω_λ has real dimension $2(mn-|\lambda|)$, and so when we take the Poincaré dual, we will end up in $H^{2|\lambda|}$.

(c) is an immediate consequence of (a). □

To analyze the cohomology further, we define

4.6. **Definition.** Given a flag

$$0 = V_0 \subseteq \cdots \subseteq C_{m+n} = \mathbb{C}^{m+n}$$

with compatible basis $v_1, \ldots, v_{m+n}$, we define the *dual flag* to be given by

$$V_i' = \text{span}\{v_{m+n-i+1}, \ldots, v_{m+n}\}$$

To the dual flag correspond Schubert cells and Schubert varieties, which we will denote by Ω_λ' and X_λ'.

4.7. **Proposition.** *Two Schubert varieties associated to the same partition λ (thus differing only by their defining flags) have the same Schubert class σ_λ.*

Proof. The connected group $GL_{m+n}(\mathbb{C})$ acts transitively on the set of complete flags, so there is an element $g \in GL_{m+n}(\mathbb{C})$ such that g sends flag F to flag F'. Since $GL_{m+n}(\mathbb{C})$ is connected, there is a path $H\colon [0, 1] \to GL_{m+n}(\mathbb{C})$ such that $H(0) = g$ and $H(1) = I_{m+n}$. Furthermore, $GL_{m+n}(\mathbb{C})$ acts continuously on $G(m, n)$, so this path induces a homotopy sending $g\colon G(m, n) \to G(m, n)$ to the identity map on $G(m, n)$. Thus, since the two cells are homotopic, they are represented by the same (co)homology class. ∎

In particular, X_λ and X'_λ have the same Schubert class σ_λ.

4.8. **Lemma.** *Given partitions η, τ inside an $n \times m$ rectangle, we have that*

$$\Omega_\eta \cap \Omega'_\tau \neq \varnothing \Longrightarrow \eta \subseteq \hat{\tau}$$

Proof. See exposition in [Man98, p 108]. Essentially, this result follows from analyzing what an element of $\Omega_\eta \cap \Omega'_\nu$ must look like, and it forces the desired containment. ∎

4.9. **Lemma.** *Given irreducible algebraic subvarieties Y and Y' of a compact and connected complex variety X with codimensions c and c', respectively, we have*

$$Y \cap Y' = \bigcap_{i \in I} Z_i$$

for Z_i some irreducible subvarieties of X and, furthermore, if this intersection is "transverse" (which implies Z_i has codimension $c + c'$),

$$[Y] \smile [Y'] = \sum_{i \in I} [Z_i] \in H^{2c+2c'}(X)$$

Proof. This is a proof from algebraic geometry. See, for instance, [Ful97] appendix B. ∎

Of course, since cohomology forms a ring, we may ask about the ring structure on $H^*(G(m,n))$. We get

4.10. **Proposition.** [Man98, Proposition 3.2.7] *Given two partitions λ, μ contained in an $n \times m$ rectangle such that $|\lambda| + |\mu| = mn$, we get that*

$$\sigma_\lambda \smile \sigma_\mu = \delta_{\mu, \hat{\lambda}}$$

where $\hat{\lambda}$ is the complementary partition partition of λ in an $n \times m$ rectangle, that is, the partition such that $\lambda_i + \hat{\lambda}_{m-i} \leq n$.

Proof. By the first lemma above, since $|\lambda| + |\mu| = mn$, it must be that

$$\mu \neq \hat{\lambda} \Longrightarrow X_\lambda \cap X'_\mu = \varnothing \Longrightarrow \sigma_\lambda \smile \sigma_\mu = 0$$

since

$$\sigma_\lambda \smile \sigma_\mu = [X_\lambda \cap X_\mu]^*$$

by the second lemma above. Similarly, if $\mu = \hat{\lambda}$, then $\mu_i + \lambda_{m+1-i} = n$ for all i and $X_\lambda \cap X'_\mu \neq \varnothing$. So, take $W \in X_\lambda \cap X'_\mu$. This requires that

$$\begin{cases} \dim(W \cap \operatorname{span}\{v_1, \ldots, v_{n+i-\lambda_i}\}) \geq i \\ \dim(W \cap \operatorname{span}\{v_{m+n-(n+i-\mu_i)+1}, \ldots, v_{m+n}\}) \geq i \\ \Longleftrightarrow \dim(W \cap \operatorname{span}\{v_{n+(m+1-i)-\lambda_{m+1-i}}, \ldots, v_{m+n}\}) \geq i \end{cases}$$

The only W that can meet these conditions is $W^\lambda := \text{span}\{v_{n+1-\lambda_1}, \dots, v_{n+m-\lambda_m}\}$. Thus,

$$\sigma_\lambda \smile \sigma_\mu = [X_\lambda \cap X_\mu]^* = [\{W^\lambda\}]^* = 1$$

granting that X_λ and X_μ intersect transversely. $\square$

Apparently this is "obvious" because Ω_λ and Ω'_μ correspond to coordinate subspaces.

4.11. **Definition.** Given the above, we say that the classes σ_λ and $\sigma_{\hat{\lambda}}$ are *dual* to each other.

4.12. **Corollary.** *Given $x \in H^*(G(m,n))$, we have*

$$x = \sum_{\lambda \text{ in an } n\times m \text{ rectangle}} (x \smile \sigma_{\hat{\lambda}})\sigma_\lambda$$

Proof. This follows immediately from the cup product structure outlined in the above proposition and the fact that the Schubert classes form a $\mathbb{Z}$-basis of the cohomology (4.5). $\square$

4.13. **Theorem** (Pieri Rule)**.** [Man98, 3.2.8] *If λ is a partition contained in an $n \times m$ rectangle and $1 \le k \le n$, then*

$$\sigma_\lambda \smile \sigma_k = \sum_{\substack{\nu \text{ contained in an } n\times m \text{ rectangle} \\ \nu=\lambda+\text{a horizontal } k\text{-strip}}} \sigma_\nu$$

Proof. By the previous corollary, we need only show that

$$(\sigma_\lambda \smile \sigma_k) \smile \sigma_{\hat{\nu}} = \begin{cases} 1 & \text{if } \nu \text{ is a summand of the sum above} \\ 0 & \text{otherwise} \end{cases}$$

For μ a partiton contained in an $n \times m$ rectangle and $|\lambda| + |\mu| = nm - k$, then

$$\begin{aligned} \mu = \text{ complement of } (\lambda + \text{a horizontal } k \text{ strip}) &\iff \hat{\lambda} = \mu + \text{a horizontal } k \text{ strip} \\ &\iff n - \lambda_m \ge \mu_1 \ge n - \lambda_{m-1} \ge \mu_2 \ge \dots \ge n - \lambda_1 \ge \mu_m \\ &\implies \sigma_\lambda \smile \sigma_\mu \smile \sigma_k = 1 \end{aligned}$$

and, if the inequalities are not satisfied, then the product is zero. If $\lambda_i + \mu_{n+1-i} > n$, then $\sigma_\lambda \smile \sigma_\mu = 0$, so we will assume that $\lambda_i + \mu_{n+1-i} \le n$. Now, we set

$$\begin{aligned} A_i &= \text{span}\{v_1, \dots, v_{n+i-\lambda_i}\} && = V_{n+i-\lambda_i} \\ B_i &= \text{span}\{v_{\mu_{m+1-i}+i}, \dots, v_{m+n}\} && = V'_{n+m+1-i-\mu_{m+1-i}} \\ C_i &= \text{span}\{v_{\mu_{m+1-i}+i}, \dots, v_{n+i-\lambda_i}\} && = A_i \cap B_i \end{aligned}$$

We note that $\dim C_i = n - \lambda_i - \mu_{m+1-i}$, so if $\mu_{m+1-i} \ge n - \lambda_i$, then $A_i \cap B_i = C_i = 0$. $\square$

Finish this proof; maybe find a different one.

The important fact about the Pieri rule is that it totally determines the structure constants of the algebra. Furthermore, the ring of symmetric function has its structure constants completely determined by the Pieri rule.

4.14. **Corollary.** *Let Λ_m be the symmetric functions in m variables. Then, the map*

$$\phi_{m,n} \colon \Lambda_m \to H^*(G(m,n))$$
$$s_\lambda \to \begin{cases} \sigma_\lambda & \lambda \text{ in an } n \times m \text{ rectangle} \\ 0 & \text{else} \end{cases}$$

is a surjective ring homomorphism.

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