

Diagonal Harmonics and Shuffle Theorems

George H. Seelinger

ghseeli@umich.edu

joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
ISU Algebra Seminar

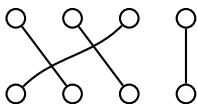
27 October 2022

Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


Symmetric Group

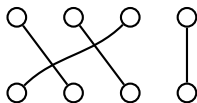
- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \begin{array}{c} \circ \quad \circ \quad \circ \quad \circ \\ \diagdown \quad \diagup \quad \diagdown \quad | \\ \circ \quad \circ \quad \circ \quad \circ \end{array}$$

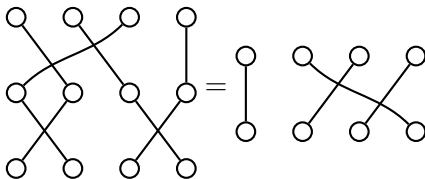
- Stacking = composition

Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \text{diagram}$$


- Stacking = composition



- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}[x_1, \dots, x_n]$ satisfying $\sigma.f = f$?

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}[x_1, \dots, x_n]$ satisfying $\sigma.f = f$?
- Symmetric polynomials ($n = 3$)

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}[x_1, \dots, x_n]$ satisfying $\sigma.f = f$?
- Symmetric polynomials ($n = 3$)

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric functions are polynomials in the e_1, e_2, \dots , or in the h_1, h_2, \dots

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric functions are polynomials in the e_1, e_2, \dots , or in the h_1, h_2, \dots

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

Basis of $\Lambda_{\mathbb{Q}}$?

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition* of n is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|}\hline\Box \\ \hline\end{array}$$

Definition

Filling of partition diagram of λ with numbers such that

Definition

Filling of partition diagram of λ with numbers such that

- 1 strictly increasing down columns

Definition

Filling of partition diagram of λ with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Definition

Filling of partition diagram of λ with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Collection is called $\text{SSYT}(\lambda)$.

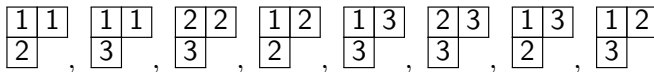
Definition

Filling of partition diagram of λ with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Collection is called $\text{SSYT}(\lambda)$.

For $\lambda = (2, 1)$,

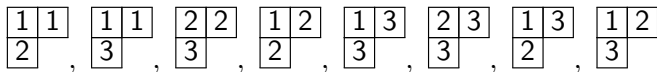


Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.



Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

Weight:

1	1
2	

,

1	1
3	

,

2	2
3	

,

1	2
2	

,

1	3
3	

,

2	3
3	

,

1	3
2	

,

1	2
3	

(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

Weight: $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

Weight: $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

Weight: $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

Weight: $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function
- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

Harmonic polynomials

① S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break M up into smallest S_n fixed subspaces

Harmonic polynomials

- 1 S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Harmonic polynomials

- 1 S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- 3 How many times does an S_n fixed subspace occur?

Harmonic polynomials

- 1 S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- 3 How many times does an S_n fixed subspace occur? Frobenius:

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ③ How many times does an S_n fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Harmonic polynomials

- 1 S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- 3 How many times does an S_n fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.

Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible S_n -fixed vector subspaces.

Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible S_n -fixed vector subspaces.

Upshot

Via Frobenius characteristic map, questions about S_n -action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible S_n -fixed vector subspaces.

Upshot

Via Frobenius characteristic map, questions about S_n -action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

Getting more information

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}_{\text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}_{\text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Answer: "Hall-Littlewood polynomial" $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n -representations whose (bigraded) Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu = \text{span of partial derivatives of}$
 $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu = \text{span of partial derivatives of}$
 $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\deg=(0,0)}$$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

Irreducible S_n -representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\deg=(0,0)}$$

Irreducible S_n -representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = qts \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + ts \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + qs \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + s \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via geometric connection to the Hilbert Scheme $\text{Hilb}^n(\mathbb{C}^2)$.

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via geometric connection to the Hilbert Scheme $\text{Hilb}^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X; q, t) = \tilde{K}_{\lambda\mu}(q, t)s_\mu$ satisfies $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via geometric connection to the Hilbert Scheme $\text{Hilb}^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X; q, t) = \tilde{K}_{\lambda\mu}(q, t)s_\mu$ satisfies $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$. (Still open!)

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

Question

What symmetric function is the bigraded Frobenius characteristic of DH_n ?

Frobenius characteristic of DH_3

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Operator ∇

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Operator ∇

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n .

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k -by- k Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k -by- k Dyck paths.
- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

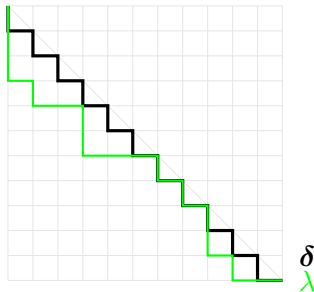
$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k -by- k Dyck paths.
- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

Dyck paths

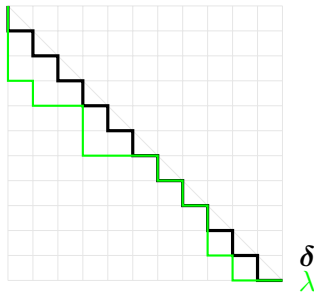
A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

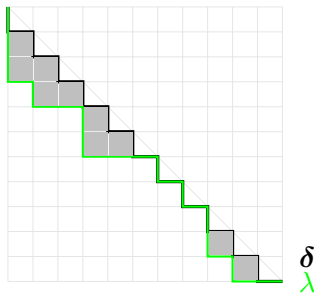


- $\text{area}(\lambda) = \text{number of squares above } \lambda \text{ but below the path } \delta \text{ of alternating S-E steps.}$

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

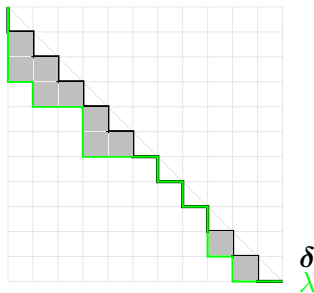


- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

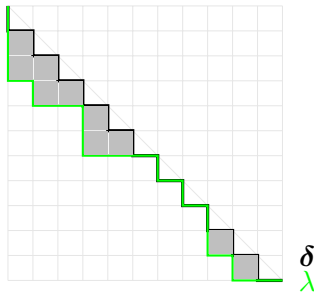


- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k .

Dyck paths

Dyck paths

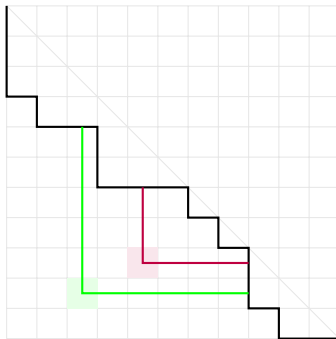
A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



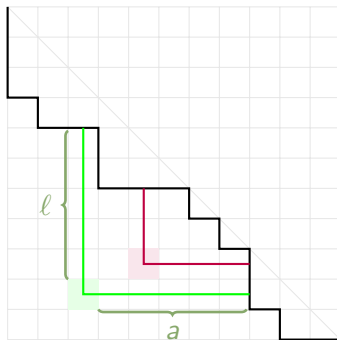
- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k . $(1, 2, 5, 14, 42, \dots)$

dinv

$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- \mathcal{G}_ν were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- \mathcal{G}_ν were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazhdan-Luzstig polynomials.

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

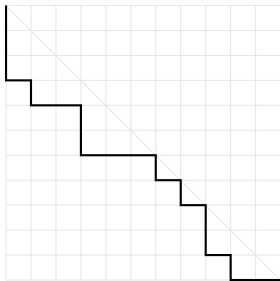
- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- \mathcal{G}_ν were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazhdan-Luzstig polynomials.
- \mathcal{G}_ν is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.

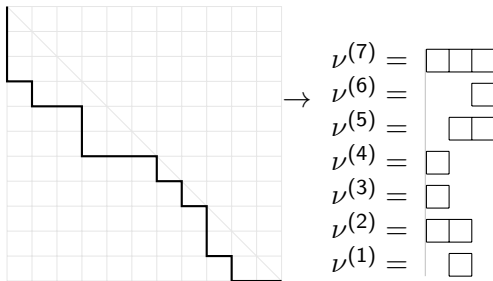
LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



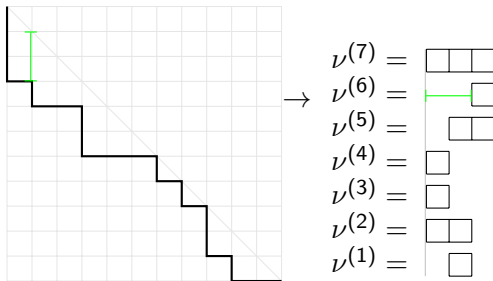
LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



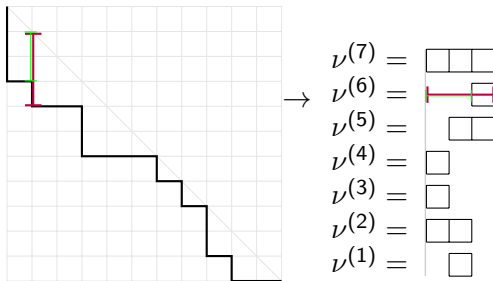
LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.



LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 3 & 5 \\ 2 & 4 & 4 & 7 & 8 & 9 & 9 \\ 1 & 1 & 6 & 7 & 7 & 7 \end{array}$$

LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

1	2	3	3	5
---	---	---	---	---

2	4	4	7	8	9	9
---	---	---	---	---	---	---

$$T = \begin{array}{cccccc} & 1 & 1 & 6 & 7 & 7 & 7 \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

1 2 3 3 5

2 4 4 7 8 9 9

$$T = \begin{array}{ccccccc} 1 & 1 & 6 & 7 & 7 & 7 \\ \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$



$$\mathcal{G}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

Example ∇e_3






$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

Example ∇e_3






$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$








Example ∇e_3

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

Example ∇e_3

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	q^2t	$qts_{2,1} + q^2ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2s_{2,1} + qt^2s_{1,1,1}$
	t^3	$t^3s_{1,1,1}$

Example ∇e_3

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	q^2t	$qts_{2,1} + q^2ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2s_{2,1} + qt^2s_{1,1,1}$
	t^3	$t^3s_{1,1,1}$

- Entire quantity is q, t -symmetric

Example ∇e_3

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	q^2t	$qts_{2,1} + q^2ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2s_{2,1} + qt^2s_{1,1,1}$
	t^3	$t^3s_{1,1,1}$

- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number”
 $(q^3 + q^2t + qt + qt^2 + t^3)$.

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

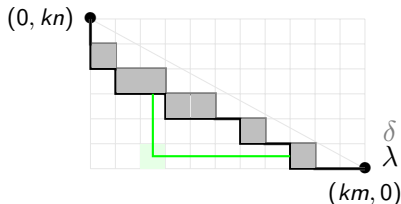
Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$



Schiffmann's Elliptic Hall Algebra \mathcal{E}

For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

Schiffmann's Elliptic Hall Algebra \mathcal{E}

For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

$$[A] \cdot [B] = \langle A, B \rangle \sum_{C \in \text{ob}(\mathcal{A})} \frac{P_{A,B}^C}{a_A a_B} [C]$$

where $P_{A,B}^C = \#\{0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0\}$ and $a_A = |\text{Aut}(A)|$, $a_B = |\text{Aut}(B)|$

Schiffmann's Elliptic Hall Algebra \mathcal{E}

For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

$$[A] \cdot [B] = \langle A, B \rangle \sum_{C \in \text{ob}(\mathcal{A})} \frac{P_{A,B}^C}{a_A a_B} [C]$$

where $P_{A,B}^C = \#\{0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0\}$ and $a_A = |\text{Aut}(A)|$, $a_B = |\text{Aut}(B)|$

- Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p

Schiffmann's Elliptic Hall Algebra \mathcal{E}

For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

$$[A] \cdot [B] = \langle A, B \rangle \sum_{C \in \text{ob}(\mathcal{A})} \frac{P_{A,B}^C}{a_A a_B} [C]$$

where $P_{A,B}^C = \#\{0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0\}$ and $a_A = |\text{Aut}(A)|$, $a_B = |\text{Aut}(B)|$

- Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p
- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$.

Schiffmann's Elliptic Hall Algebra \mathcal{E}

For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

$$[A] \cdot [B] = \langle A, B \rangle \sum_{C \in \text{ob}(\mathcal{A})} \frac{P_{A,B}^C}{a_A a_B} [C]$$

where $P_{A,B}^C = \#\{0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0\}$ and $a_A = |\text{Aut}(A)|$, $a_B = |\text{Aut}(B)|$

- Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p
- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$.
- $e_k[-MX^{m,n}] \in \Lambda(X^{m,n})$.

Schiffmann's Elliptic Hall Algebra \mathcal{E}

For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

$$[A] \cdot [B] = \langle A, B \rangle \sum_{C \in \text{ob}(\mathcal{A})} \frac{P_{A,B}^C}{a_A a_B} [C]$$

where $P_{A,B}^C = \#\{0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0\}$ and $a_A = |\text{Aut}(A)|$, $a_B = |\text{Aut}(B)|$

- Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p
- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$.
- $e_k[-MX^{m,n}] \in \Lambda(X^{m,n})$.
- \mathcal{E} acts on Λ .

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_I) \in \mathbb{Z}^I$ to be the south step sequence of highest path δ under the line $y + px = s$.

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_I) \in \mathbb{Z}^I$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1, \dots, b_I)} \cdot 1$$

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1, \dots, b_l)} \cdot 1 = \sum_{\lambda} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,

Any Line

Negut Elements

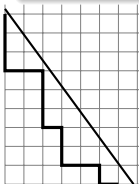
For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1, \dots, b_l)} \cdot 1 = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



$\text{area}(\lambda)$ as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks } \frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

Proof Overview (algebraic side)

- $\psi: \mathcal{E}^+ \cong S$
- \mathcal{E}^+ is the “positive half” of \mathcal{E}
- S is an algebra of symmetric Laurent series in $\mathbb{Q}(q, t)(z_1^{\pm 1}, \dots, z_l^{\pm 1})^{S_l}$ satisfying extra conditions and equipped with a “shuffle product”.

Proof Overview (algebraic side)

- $\psi: \mathcal{E}^+ \cong S$
- \mathcal{E}^+ is the “positive half” of \mathcal{E}
- S is an algebra of symmetric Laurent series in $\mathbb{Q}(q, t)(z_1^{\pm 1}, \dots, z_l^{\pm 1})^{S_l}$ satisfying extra conditions and equipped with a “shuffle product”.

Key relationship

For $\xi \in \mathcal{E}^+$,

$$\omega(\xi \cdot 1) = \text{pol}_X(\psi(\xi))$$

for automorphism $\omega: \Lambda \rightarrow \Lambda$ and $\text{pol}_X: S \rightarrow \Lambda$ a “polynomial truncation” operation.

Proof Overview (combinatorial side)

- For $\xi = D_{\mathbf{b}}$, we get

$$\text{pol}_X \mathbf{H}_q \left(\frac{z^{\mathbf{b}} \prod_{i < j+1} (1 - qtz_i/z_j)}{\prod_{i < j} (1 - t z_i/z_j)} \right) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

(H_q is a transformation that send this Laurent series to S)

Proof Overview (combinatorial side)

- For $\xi = D_{\mathbf{b}}$, we get

$$\text{pol}_X \mathbf{H}_q \left(\frac{z^{\mathbf{b}} \prod_{i < j+1} (1 - qtz_i/z_j)}{\prod_{i < j} (1 - tzi/z_j)} \right) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

(H_q is a transformation that send this Laurent series to S)

- Can we remove $\text{pol}_X H_q$ from left hand side and get something that goes to the righthand side?

$$\begin{array}{ccc} z^{\mathbf{b}} \frac{\prod_{i < j+1} (1 - qtz_i/z_j)}{\prod_{i < j} (1 - tzi/z_j)} & = & ?? \\ \downarrow & & \downarrow \\ D_{\mathbf{b}} \cdot 1 & = & \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1}) \end{array}$$

Proof Overview (combinatorial side)

- For $\xi = D_{\mathbf{b}}$, we get

$$\text{pol}_X \mathbf{H}_q \left(\frac{z^{\mathbf{b}} \prod_{i < j+1} (1 - qtz_i/z_j)}{\prod_{i < j} (1 - tz_i/z_j)} \right) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

(H_q is a transformation that send this Laurent series to S)

- Can we remove $\text{pol}_X H_q$ from left hand side and get something that goes to the righthand side?

$$\begin{array}{ccc} z^{\mathbf{b}} \frac{\prod_{i < j+1} (1 - qtz_i/z_j)}{\prod_{i < j} (1 - tz_i/z_j)} & = & ?? \\ \downarrow & & \downarrow \\ D_{\mathbf{b}} \cdot 1 & = & \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1}) \end{array}$$

Need an “infinite series” version of LLT polynomials!

Cauchy identity

- For a fixed $\sigma \in S_I$, there exists a basis of $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_I^{\pm 1}]$ called “non-symmetric Hall-Littlewood polynomials”, denoted $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \dots, z_I; q)$ for $\lambda \in \mathbb{Z}^I$.

Cauchy identity

- For a fixed $\sigma \in S_I$, there exists a basis of $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_I^{\pm 1}]$ called “non-symmetric Hall-Littlewood polynomials”, denoted $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \dots, z_I; q)$ for $\lambda \in \mathbb{Z}^I$.
- Under an inner-product coming from representation theory of affine Hecke algebras, there is a dual basis $F_\lambda^\sigma = E_{-\lambda}^{\sigma w_0}(z_1^{-1}, \dots, z_I^{-1}; q^{-1}) = \overline{E_{-\lambda}^{\sigma w_0}}$

Cauchy identity

- For a fixed $\sigma \in S_I$, there exists a basis of $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_I^{\pm 1}]$ called “non-symmetric Hall-Littlewood polynomials”, denoted $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \dots, z_I; q)$ for $\lambda \in \mathbb{Z}^I$.
- Under an inner-product coming from representation theory of affine Hecke algebras, there is a dual basis $F_\lambda^\sigma = E_{-\lambda}^{\sigma w_0}(z_1^{-1}, \dots, z_I^{-1}; q^{-1}) = \overline{E_{-\lambda}^{\sigma w_0}}$

Cauchy identity

$$\frac{\prod_{i < j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)} = \sum_{\mathbf{a} \geq 0} t^{|\mathbf{a}|} E_{\mathbf{a}}^\sigma(x_1, \dots, x_I; q^{-1}) F_{\mathbf{a}}^\sigma(y_1, \dots, y_I; q),$$

Cauchy identity

- For a fixed $\sigma \in S_I$, there exists a basis of $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_I^{\pm 1}]$ called “non-symmetric Hall-Littlewood polynomials”, denoted $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \dots, z_I; q)$ for $\lambda \in \mathbb{Z}^I$.
- Under an inner-product coming from representation theory of affine Hecke algebras, there is a dual basis $F_\lambda^\sigma = E_{-\lambda}^{\sigma w_0}(z_1^{-1}, \dots, z_I^{-1}; q^{-1}) = \overline{E_{-\lambda}^{\sigma w_0}}$

Cauchy identity

$$\frac{\prod_{i < j} (1 - q t x_i y_j)}{\prod_{i \leq j} (1 - t x_i y_j)} = \sum_{\mathbf{a} \geq 0} t^{|\mathbf{a}|} E_{\mathbf{a}}^\sigma(x_1, \dots, x_I; q^{-1}) F_{\mathbf{a}}^\sigma(y_1, \dots, y_I; q),$$

- (Grojnowski-Haiman 2007) defines a (symmetric) “series LLT” polynomial $\mathcal{L}_{\beta/\alpha}^\sigma(x_1, \dots, x_I; q) = H_q(w_0(F_\beta^{\sigma^{-1}} \overline{E_\alpha^{\sigma^{-1}}}))$

Stable Shuffle Theorem (BHMPs 21a)

For $\mathbf{b} \in \mathbb{Z}^I$ corresponding to highest path under a line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{I-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_I, \dots, b_1) + (0, a_{I-1}, \dots, a_1)) / (a_{I-1}, \dots, a_1, 0)}^{\sigma}(x_1, \dots, x_I; q)$$

Under polynomial truncation,

$$\mathcal{L}_{\beta/\alpha}^{\sigma}(x_1, \dots, x_I; q) \rightarrow q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_I; q^{-1})$$

Stable Shuffle Theorem (BHMPs 21a)

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to highest path under a line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_l, \dots, b_1) + (0, a_{l-1}, \dots, a_1)) / (a_{l-1}, \dots, a_1, 0)}^{\sigma}(x_1, \dots, x_l; q)$$

Under polynomial truncation,

$$\mathcal{L}_{\beta/\alpha}^{\sigma}(x_1, \dots, x_l; q) \rightarrow q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_l; q^{-1})$$

$$\implies \omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1, \dots, x_l; q^{-1}).$$

Generalizations

Same paradigm works to show the following formula.

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b) \in \mu} q^{a-1} t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \hline \end{array} \rightarrow B_\mu = 1 + q + q^2 + t + qt$

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b) \in \mu} q^{a-1} t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \hline \end{array} \rightarrow B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu$

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b) \in \mu} q^{a-1} t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \hline \end{array} \rightarrow B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu$
- $\Delta'_f \tilde{H}_\mu = f[B_\mu - 1] \tilde{H}_\mu$

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b) \in \mu} q^{a-1} t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \hline \end{array} \rightarrow B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu$
- $\Delta'_f \tilde{H}_\mu = f[B_\mu - 1] \tilde{H}_\mu$

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

$$\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda) = r_{i-1}(\lambda) + 1} (1 + zt^{-r_i(\lambda)}).$$

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b) \in \mu} q^{a-1} t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \hline \end{array} \rightarrow B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu$
- $\Delta'_f \tilde{H}_\mu = f[B_\mu - 1] \tilde{H}_\mu$

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

$$\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda) = r_{i-1}(\lambda) + 1} (1 + zt^{-r_i(\lambda)}).$$

- $\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \sum_{\substack{s \in \mathbb{N}^{k+r}: |s|=n-k \\ 1 \in J \subseteq [k+r], |J|=k}} (D_{s+\epsilon_J} \cdot 1)$

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R) \in \operatorname{LNDP}_\mu} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R) \in \operatorname{LNDP}_\mu} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

Generalizing our methods further, we arrive at the following.

Theorem (BHMP21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_{\pi} t^{a(\pi)} q^{\operatorname{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

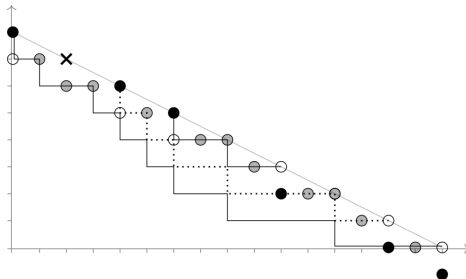
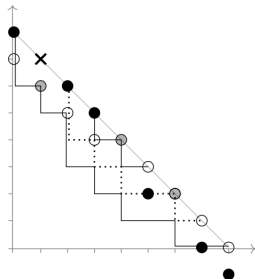
For a sum over all “nests” π in a “den” corresponding to s_μ .

Generalizations

Theorem (BHMP21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_{\pi} t^{a(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

For a sum over all “nests” π in a “den” corresponding to s_μ .

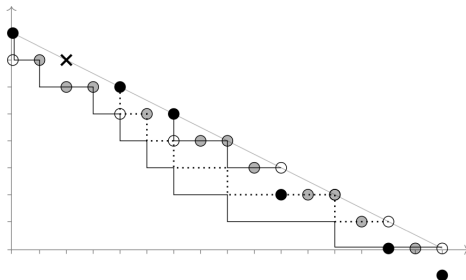
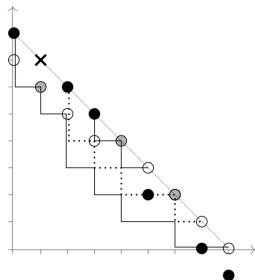


Generalizations

Theorem (BHMP21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_{\pi} t^{a(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

For a sum over all “nests” π in a “den” corresponding to s_μ .



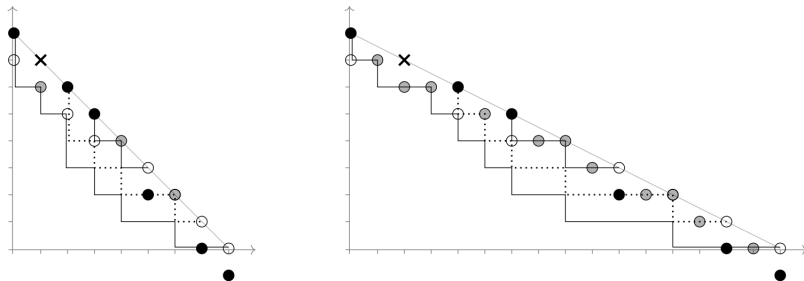
- Implies the Loehr-Warrington Conjecture as a special case.

Generalizations

Theorem (BHMP21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_{\pi} t^{a(\pi)} q^{\text{din}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

For a sum over all “nests” π in a “den” corresponding to s_μ .



- Implies the Loehr-Warrington Conjecture as a special case.
- Also proves $\text{sgn}(\mu) \nabla s_\mu$ is Schur-positive.

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_I)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_I)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

- Experimental computation suggests this is “tight.”

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?



Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_I)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

- Experimental computation suggests this is “tight.”
- Coefficient of $s_{1, \dots, 1}$ coincides with (q, t) -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?
- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?
- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)
- What can we say about Macdonald polynomials?

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?
- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)
- What can we say about Macdonald polynomials?
- S_I -representation theory interpretations?

References

Thank you!

- Bergeron, Francois, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. 2016. *Compositional (km, kn) -shuffle conjectures*, Int. Math. Res. Not. IMRN **14**, 4229–4270, DOI 10.1093/imrn/rnv272. MR3556418
- Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021a. *A Shuffle Theorem for Paths Under Any Line*, arXiv e-prints, available at arXiv:2102.07931.
- . 2021b. *A proof of the Extended Delta Conjecture*, arXiv e-prints, available at arXiv:2102.08815.
- . 2021c. *Dens, nests and the Loehr-Warrington conjecture*, arXiv e-prints, available at arXiv:2112.07070.
- Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373
- Carlsson, Erik and Anton Mellit. 2018. *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31**, no. 3, 661–697, DOI 10.1090/jams/893. MR3787405
- Galashin, Pavel and Thomas Lam. 2021. *Positroid Catalan numbers*, arXiv e-prints, arXiv:2104.05701, available at arXiv:2104.05701.
- Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091
- Gorsky, Eugene, Graham Hawkes, Anne Schilling, and Julianne Rainbolt. 2020. *Generalized q, t -Catalan numbers*, Algebr. Comb. **3**, no. 4, 855–886, DOI 10.5802/alco.120. MR4145982

- Grojnowski, Ian and Mark Haiman. 2007. *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, Unpublished manuscript.
- Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.
- Haglund, J., J. B. Remmel, and A. T. Wilson. 2018. *The delta conjecture*, Trans. Amer. Math. Soc. **370**, no. 6, 4029–4057, DOI 10.1090/tran/7096. MR3811519
- Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919
- . 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676
- Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sémin. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754
- Loehr, Nicholas A. and Gregory S. Warrington. 2008. *Nested quantum Dyck paths and $\nabla(s_\lambda)$* , Int. Math. Res. Not. IMRN **5**, Art. ID rnm 157, 29, DOI 10.1093/imrn/rnm157. MR2418288
- Mellit, Anton. 2016. *Toric braids and (m, n) -parking functions*, arXiv e-prints, arXiv:1604.07456, available at arXiv:1604.07456.
- Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004