Schubert calculus and *K*-theoretic Catalan functions

George H. Seelinger (joint with J. Blasiak and J. Morse)

UVA Graduate Seminar

ghs9ae@virginia.edu

31 January 2020

Overview

1. An overview of Schubert calculus
2. Catalan functions: shedding new light on old problems
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

$\downarrow$

Cohomology

Schubert basis $\{\sigma_\lambda\}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

↓

Cohomology

Schubert basis $\{\sigma_\lambda\}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

↓

Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

$X = \text{Gr}_m(\mathbb{C}^{m+n}) = \{\text{all } m\text{-dimensional subspaces of } \mathbb{C}^{n+m}\}.$

Classical Example

$X = \mathrm{Gr}_m(\mathbb{C}^{m+n}) = \{\text{all } m\text{-dimensional subspaces of } \mathbb{C}^{n+m}\}$.

Decomposes into Schubert varieties indexed by partitions

$\lambda \subseteq (\underbrace{n, \ldots, n}_{m}) = (n^m)$.

Classical Example

$X = \mathsf{Gr}_m(\mathbb{C}^{m+n}) = \{\text{all } m\text{-dimensional subspaces of } \mathbb{C}^{n+m}\}$.

Decomposes into Schubert varieties indexed by partitions

$\lambda \subseteq (\underbrace{n, \ldots, n}_{m}) = (n^m)$.

Classical Example

$X = \mathrm{Gr}_m(\mathbb{C}^{m+n}) = \{$all m-dimensional subspaces of $\mathbb{C}^{n+m}\}$.
Decomposes into Schubert varieties indexed by partitions
$\lambda \subseteq (\underbrace{n, \ldots, n}_{m}) = (n^m)$.

- $H^*(\mathrm{Gr}_m(\mathbb{C}^{m+n})) = \bigoplus_{\lambda \subseteq (n^m)} \mathbb{Z}\sigma_\lambda$ as $\mathbb{Z}$-modules

Classical Example

$X = \mathsf{Gr}_m(\mathbb{C}^{m+n}) = \{\text{all } m\text{-dimensional subspaces of } \mathbb{C}^{n+m}\}$.

Decomposes into Schubert varieties indexed by partitions

$\lambda \subseteq (\underbrace{n, \ldots, n}_{m}) = (n^m)$.

- $H^*(\mathsf{Gr}_m(\mathbb{C}^{m+n})) = \bigoplus_{\lambda \subseteq (n^m)} \mathbb{Z}\sigma_\lambda$ as $\mathbb{Z}$-modules
- $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$

Classical Example

$X = \mathrm{Gr}_m(\mathbb{C}^{m+n}) = \{\text{all } m\text{-dimensional subspaces of } \mathbb{C}^{n+m}\}$.

Decomposes into Schubert varieties indexed by partitions

$\lambda \subseteq (\underbrace{n, \ldots, n}_{m}) = (n^m)$.

- $H^*(\mathrm{Gr}_m(\mathbb{C}^{m+n})) = \bigoplus_{\lambda \subseteq (n^m)} \mathbb{Z}\sigma_\lambda$ as $\mathbb{Z}$-modules
- $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$

$c_{\lambda\mu}^\nu =$ number of points in intersection of Schubert varieties.

Classical Example

$X = \mathsf{Gr}_m(\mathbb{C}^{m+n}) = \{$all m-dimensional subspaces of $\mathbb{C}^{n+m}\}$.
Decomposes into Schubert varieties indexed by partitions
$\lambda \subseteq (\underbrace{n, \ldots, n}_{m}) = (n^m)$.

- $H^*(\mathsf{Gr}_m(\mathbb{C}^{m+n})) = \bigoplus_{\lambda \subseteq (n^m)} \mathbb{Z}\sigma_\lambda$ as $\mathbb{Z}$-modules
- $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$

$c_{\lambda\mu}^\nu =$ number of points in intersection of Schubert varieties.
What are the structure constants $c_{\lambda\mu}^\nu$?

$\Lambda_m = \mathbb{C}[x_1, \ldots, x_m]^{S_m}$ is the ring of symmetric polynomials in m variables and has bases indexed by partitions.

$$\underbrace{12x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{symmetric}} \qquad \underbrace{5x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{not symmetric}}$$

Classical Example (cont.)

$\Lambda_m = \mathbb{C}[x_1, \ldots, x_m]^{S_m}$ is the ring of symmetric polynomials in m variables and has bases indexed by partitions.

$$\underbrace{12x_1^2 + 12x_2^2 - 7x_1 x_2}_{\text{symmetric}} \qquad \underbrace{5x_1^2 + 12x_2^2 - 7x_1 x_2}_{\text{not symmetric}}$$

There exists a basis of Λ_m denoted $\{s_\lambda\}_\lambda$ and a surjection of rings such that

$$\Lambda_m \to H^*(\mathrm{Gr}(m, n))$$

$$s_\lambda \mapsto \begin{cases} \sigma_\lambda & \lambda \subseteq (n^m) \\ 0 & \text{otherwise.} \end{cases}$$

Cohomology structure: $\sigma_\lambda \leftrightarrow s_\lambda$ when $\lambda \subseteq (n^m)$.

$$s_\lambda s_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu s_\nu + \sum_{\nu \nsubseteq (n^m)} c_{\lambda\mu}^\nu s_\nu \leftrightarrow \sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu \sigma_\nu$$

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$
\begin{array}{|c|}
\hline
5 \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
3 & 4 \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
2 & 3 \\
\hline
\end{array}
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 2 & 5 \\
\hline
\end{array}
\qquad
\begin{array}{|c|}
\hline
8 \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
7 & 9 \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
3 & 4 \\
\hline
\end{array}
\underbrace{
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 5 & 6 \\
\hline
\end{array}
}
$$

standard = no repeated letters

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$\begin{array}{l}
\boxed{5} \\
\boxed{3}\boxed{4} \\
\boxed{2}\boxed{3} \\
\boxed{1}\boxed{2}\boxed{2}\boxed{5}
\end{array}
\qquad\qquad
\begin{array}{l}
\boxed{8} \\
\boxed{7}\boxed{9} \\
\boxed{3}\boxed{4} \\
\underbrace{\boxed{1}\boxed{2}\boxed{5}\boxed{6}}
\end{array}$$

standard = no repeated letters

Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$\begin{array}{l}\boxed{2}\\\boxed{1}\boxed{1}\end{array},\;
\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{1}\end{array},\;
\begin{array}{l}\boxed{3}\\\boxed{2}\boxed{2}\end{array},\;
\begin{array}{l}\boxed{2}\\\boxed{1}\boxed{2}\end{array},\;
\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{3}\end{array},\;
\begin{array}{l}\boxed{3}\\\boxed{2}\boxed{3}\end{array},\;
\begin{array}{l}\boxed{2}\\\boxed{1}\boxed{3}\end{array},\;
\begin{array}{l}\boxed{3}\\\boxed{1}\boxed{2}\end{array}$$

$$s_{\square}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_\square s_{\square\square} = s_{\square\square\blacksquare} + s_{\square\blacksquare} + s_{\blacksquare}$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_\square s_{\square\square} = s_{\square\square\blacksquare} + s_{\square\blacksquare} + s_\blacksquare$$

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_\square s_\boxplus = s_{\boxplus\blacksquare} + s_{\boxplus\blacksquare} + s_{\blacksquare}$$

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1,\dots,\mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1,\dots,\mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu$.

Classical Schubert Calculus

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of Schubert varieties in variety $X = \mathrm{Gr}(m, n)$.

Classical Schubert Calculus

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of Schubert varieties in variety $X = \text{Gr}(m, n)$.

$\downarrow$

Cohomology

Schubert basis $\{\sigma_\lambda\}_{\lambda \subseteq (n^m)}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu}\sigma_\nu$

Classical Schubert Calculus

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of Schubert varieties in variety $X = \text{Gr}(m, n)$.

$\downarrow$

Cohomology

Schubert basis $\{\sigma_\lambda\}_{\lambda \subseteq (n^m)}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

$\downarrow$

Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by $w \in S_n$.

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by $w \in S_n$.
- $H^*(Fl_n(\mathbb{C}))$ supported by Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$.

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by $w \in S_n$.
- $H^*(Fl_n(\mathbb{C}))$ supported by Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$.
- Structure constants $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$ are combinatorially unknown.

Schubert Calculus Variations

There are many variations on classical Schubert calculus of the
Grassmannian (Type A).

Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	f_λ
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomimals
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
K-theory of Grassmannian	Grothendieck polynomials
K-homology of affine Grassmannian	K-k-Schur functions

Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	f_λ
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomimals
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
K-theory of Grassmannian	Grothendieck polynomials
K-homology of affine Grassmannian	K-k-Schur functions

And many more!

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ ($q \to 0$).

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ ($q \to 0$).
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$.

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ ($q \to 0$).
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$.
- Peterson isomorphism

$$\Psi \colon QH^*(Fl_{k+1}) \to H_*(Gr_{SL_{k+1}})_{loc}$$

$$\mathfrak{S}_w^Q \mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}$$

where $s_\lambda^{(k)}$ is a k-Schur function.

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ ($q \to 0$).
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$.
- Peterson isomorphism

$$\Psi \colon QH^*(Fl_{k+1}) \to H_*(Gr_{SL_{k+1}})_{loc}$$

$$\mathfrak{S}_w^Q \mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}$$

where $s_\lambda^{(k)}$ is a k-Schur function.

Upshot

Computations for Schubert polynomials can be moved into symmetric functions.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.
- Schubert representatives for homology of affine Grassmannian, $\mathrm{Gr}_{SL_{k+1}}$.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.
- Schubert representatives for homology of affine Grassmannian, $\mathrm{Gr}_{SL_{k+1}}$.
- Has a tableaux formulation from which some properties were proven.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.
- Schubert representatives for homology of affine Grassmannian, $\mathrm{Gr}_{SL_{k+1}}$.
- Has a tableaux formulation from which some properties were proven.
- Branching

$$s_{\boxplus}^{(2)} = \underbrace{s_{\boxplus}}_{s_{\boxplus}^{(3)}} + \underbrace{s_{\boxplus} + s_{\boxplus}}_{s_{\boxplus}^{(3)}}$$

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.
- Schubert representatives for homology of affine Grassmannian, $\mathrm{Gr}_{SL_{k+1}}$.
- Has a tableaux formulation from which some properties were proven.
- Branching

$$s_{\boxplus}^{(2)} = \underbrace{s_{\boxplus}}_{s_{\boxplus}^{(3)}} + \underbrace{s_{\boxplus} + s_{\square\square\square}}_{s_{\boxplus}^{(3)}}$$

  - $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.
- Schubert representatives for homology of affine Grassmannian, $\mathrm{Gr}_{SL_{k+1}}$.
- Has a tableaux formulation from which some properties were proven.
- Branching

$$s_{\boxed{}}^{(2)} = \underbrace{s_{\boxed{}}}_{s_{\boxed{}}^{(3)}} + \underbrace{s_{\boxed{}} + s_{\boxed{}}}_{s_{\boxed{}}^{(3)}}$$

  - $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
  - Has geometric meaning for embedding of affine Grassmannians.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.
- Schubert representatives for homology of affine Grassmannian, $\mathrm{Gr}_{SL_{k+1}}$.
- Has a tableaux formulation from which some properties were proven.
- Branching

$$s_{\boxplus}^{(2)} = \underbrace{s_{\boxplus}}_{s_{\boxplus}^{(3)}} + \underbrace{s_{\text{⊟}} + s_{\text{▭}}}_{s_{\text{⊟}}^{(3)}}$$

- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Has geometric meaning for embedding of affine Grassmannians.
- Definition with t important for Macdonald polynomials.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$.
- Schubert representatives for homology of affine Grassmannian, $\mathrm{Gr}_{SL_{k+1}}$.
- Has a tableaux formulation from which some properties were proven.
- Branching

$$s_{\boxplus}^{(2)} = \underbrace{s_{\boxplus}}_{s_{\boxplus}^{(3)}} + \underbrace{s_{\boxplus} + s_{\boxplus\boxplus}}_{s_{\boxplus}^{(3)}}$$

  - $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
  - Has geometric meaning for embedding of affine Grassmannians.
  - Definition with t important for Macdonald polynomials.

- Many definitions. A new one makes proofs easier!

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

- Extend action to a symmetric function f_λ by $R_{i,j}(f_\lambda) = f_{\lambda + \epsilon_i - \epsilon_j}$.

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

$$R_{1,3}\left(\;\blacksquare\;\right) = \;\blacksquare\;\quad R_{2,3}\left(\;\blacksquare\;\right) = \;\blacksquare$$

- Extend action to a symmetric function f_λ by $R_{i,j}(f_\lambda) = f_{\lambda + \epsilon_i - \epsilon_j}$.
- For $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$, we have the *Jacobi-Trudi identity*

$$s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$$

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

$$R_{1,3}\left(\begin{array}{c}\blacksquare\\\square\square\square\end{array}\right) = \square\square\square\blacksquare \qquad R_{2,3}\left(\begin{array}{c}\blacksquare\\\square\end{array}\right) = \square\blacksquare$$

- Extend action to a symmetric function f_λ by $R_{i,j}(f_\lambda) = f_{\lambda+\epsilon_i-\epsilon_j}$.
- For $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$, we have the *Jacobi-Trudi identity*

$$s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$$

$$s_{22} = (1 - R_{12})h_{22} = h_{22} - h_{31}$$
$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13})h_{211}$$
$$= h_{211} - h_{301} - h_{220} \,{\color{red}-h_{310} + h_{310}} + h_{32\text{-}1} + h_{400} - h_{41\text{-}1}$$

some terms cancel

Raising Operators on Symmetric Functions

Advantage: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$.

Raising Operators on Symmetric Functions

Advantage: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^{\ell}$. Amazingly:

$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 \end{cases}$$

For $\langle s_{1^r}^{\perp} s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^{\perp} s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

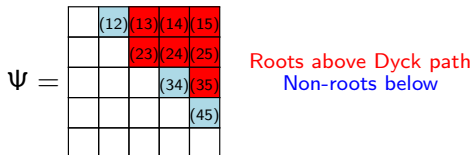A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$$\Psi = $$

Roots above Dyck path
Non-roots below

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$$\Psi =$$

Roots above Dyck path
Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) h_\gamma(x)$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$$\Psi = \begin{array}{|c|c|c|c|c|} \hline & (12) & (13) & (14) & (15) \\ \hline & & (23) & (24) & (25) \\ \hline & & & (34) & (35) \\ \hline & & & & (45) \\ \hline & & & & \\ \hline \end{array}$$

Roots above Dyck path
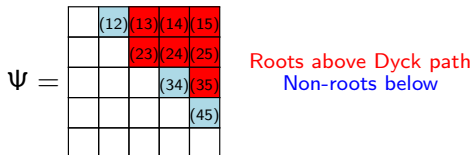Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) h_\gamma(x)$$

- $\Psi = \varnothing \implies H(\varnothing; \gamma) = s_\gamma$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$$\Psi = \begin{array}{|c|c|c|c|c|} \hline & (12) & (13) & (14) & (15) \\ \hline & & (23) & (24) & (25) \\ \hline & & & (34) & (35) \\ \hline & & & & (45) \\ \hline & & & & \\ \hline \end{array}$$

Roots above Dyck path
Non-roots below

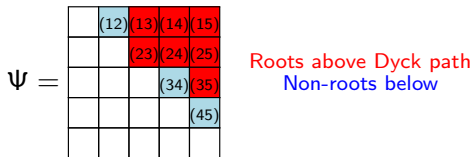## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) h_\gamma(x)$$

- $\Psi = \varnothing \implies H(\varnothing; \gamma) = s_\gamma$
- $\Psi = $ all roots $\implies H(\Psi; \gamma) = h_\gamma$

Catalan functions ($t = 1$)

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

$$\Delta^4(3,3,2,2,1,1) = \quad \boxed{} \quad \leftarrow \text{row } i \text{ has } 4 - \lambda_i \text{ non-roots}$$

Catalan functions ($t = 1$)

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

$$\Delta^4(3,3,2,2,1,1) = \begin{array}{|c|c|c|c|c|c|}\hline 3 & & & & & \\\hline & 3 & & & & \\\hline & & 2 & & & \\\hline & & & 2 & & \\\hline & & & & 1 & \\\hline & & & & & 1 \\\hline\end{array} \quad \leftarrow \text{row } i \text{ has } 4 - \lambda_i \text{ non-roots}$$

- For partition λ with $\lambda_1 \le k$, $s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)$.

Catalan functions

Key ingredient of branching proof:

Catalan functions

Key ingredient of branching proof:

Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$$

where $\langle s_{1^\ell}^{\perp} f, g \rangle = \langle f, s_{1^\ell} g \rangle$.

Catalan functions

Key ingredient of branching proof:

Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$$

where $\langle s_{1^\ell}^\perp f, g \rangle = \langle f, s_{1^\ell} g \rangle$.

$$\Delta^4(3,3,2,2,1,1) = \quad \Delta^5(4,4,3,3,2,2) =$$
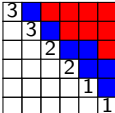
Catalan functions

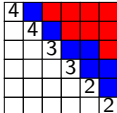Key ingredient of branching proof:

Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$$

where $\langle s_{1^\ell}^\perp f, g \rangle = \langle f, s_{1^\ell} g \rangle$.

$$\Delta^4(3,3,2,2,1,1) = \quad \Delta^5(4,4,3,3,2,2) =$$

Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
- Satisfies Pieri rule on "set-valued strips"

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
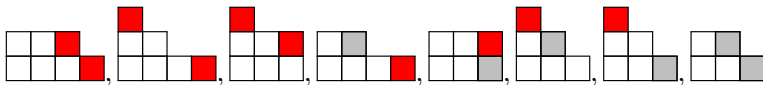- Satisfies Pieri rule on "set-valued strips"

$$g_{1^2} g_{3,2} = g_{43} + g_{421} + g_{331} - g_{42} - g_{33} - 2g_{321} + g_{31}$$

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
- Satisfies Pieri rule on "set-valued strips"

$$g_{1^2} g_{3,2} = g_{43} + g_{421} + g_{331} - g_{42} - g_{33} - 2g_{321} + g_{31}$$

- $g_\lambda = \prod_{i<j}(1 - R_{ij})k_\lambda$ for k_λ and inhomogeneous analogue of h_λ.

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
- Satisfies Pieri rule on "set-valued strips"

$$g_{1^2} g_{3,2} = g_{43} + g_{421} + g_{331} - g_{42} - g_{33} - 2g_{321} + g_{31}$$

- $g_\lambda = \prod_{i<j}(1 - R_{ij})k_\lambda$ for k_λ and inhomogeneous analogue of h_λ.
- Dual to Grothendieck polynomials: Schubert representatives for $K^*(Gr(m, n))$

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)} +$ lower degree terms

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)} +$ lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)} +$ lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)} +$ lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

- Conjecture: $g_\lambda^{(k)}$ have branching into $g_\mu^{(k+1)}$.

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)} +$ lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

- Conjecture: $g_\lambda^{(k)}$ have branching into $g_\mu^{(k+1)}$.

Problem

No direct formula for $g_\lambda^{(k)}$

Lowering Operators $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$K(\Psi; \mathcal{L}; 54332)$

$= (1 - L_4)^2 (1 - L_5)^2$

$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$

Answer (Blasiak-Morse-S., 2020)

Answer (Blasiak-Morse-S., 2020)

For K-homology of affine Grassmannian,
$f_\lambda = g_\lambda^{(k)} = K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$ since this family satisfies the correct Pieri rule.

Affine K-Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

For K-homology of affine Grassmannian,
$f_\lambda = g_\lambda^{(k)} = K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$ since this family satisfies the correct Pieri rule.

Example

$$g_{332111}^{(4)} = $$

$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$

Theorem (Blasiak-Morse-S., 2020)

Branching Positivity

Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ are "shift invariant", i.e. for $\ell = \ell(\lambda)$

$$G_{1^\ell}^\perp g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Branching Positivity

Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ are "shift invariant", i.e. for $\ell = \ell(\lambda)$

$$G_{1^\ell}^\perp g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

K-theoretic Peterson isomorphism

$$\Phi \colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

K-theoretic Peterson isomorphism

$$\Phi \colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

K-theoretic Peterson isomorphism

$$\Phi \colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$ such that $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

K-theoretic Peterson isomorphism

$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$ such that $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

Theorem (Blasiak-Morse-S., 2020)

If $\lambda \subseteq (d^{k+1-d})$ for some $1 \leq d \leq k$, then $g_\lambda^{(k)} = g_\lambda \implies$ conjecture is true for w a Grassmannian permutation.

K-theoretic Peterson isomorphism

$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$ such that $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

Theorem (Blasiak-Morse-S., 2020)

If $\lambda \subseteq (d^{k+1-d})$ for some $1 \leq d \leq k$, then $g_\lambda^{(k)} = g_\lambda \implies$ conjecture is true for w a Grassmannian permutation.

Conjecture (Blasiak-Morse-S., 2020)

$$\tilde{g}_w = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule:
   $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}$, $1 \le r \le k$.

Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule:
   $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ??G_\lambda^{(k)}$, $1 \le r \le k$.

2. Combinatorially describe branching coefficients: $g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k+1)}$.

Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule:
   $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}$, $1 \le r \le k$.

2. Combinatorially describe branching coefficients: $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$.

3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$.

Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule:
   $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}$, $1 \le r \le k$.

2. Combinatorially describe branching coefficients: $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$.

3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$.

4. Describe the image of $\mathfrak{S}_w^Q$ under Peterson isomorphism for all $w \in S_{k+1}$.

References

Thank you!

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and k-Schur Positivity*, J. Amer. Math. Soc. **32**, no. 4, 921–963.

Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for k-Schur functions*, Ph.D. thesis.

Ikeda, Takeshi, Shinsuke Iwao, and Toshiaki Maeno. 2018. *Peterson Isomorphism in K-theory and Relativistic Toda Lattice*, preprint. arXiv: 1703.08664.

Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010. *K-theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.

Morse, Jennifer. 2011. *Combinatorics of the K-theory of affine Grassmannians*, Advances in Mathematics.

Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.