

K -theoretic Catalan functions

George H. Seelinger (joint with J. Blasiak and J. Morse)

CAGE

ghs9ae@virginia.edu

6 February 2020

- Schubert calculus
- Catalan functions: a new approach to old problems
- K -theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of subvarieties in a variety X .

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda \subseteq (n^m)}$ in variety $X = \text{Gr}(m, n)$.

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$.

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$

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Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$ are combinatorially unknown.

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Theory	f_λ
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- P and Q functions
(Co)homology of affine Grassmannian	(dual) k -Schur functions
K -theory of Grassmannian	Grothendieck polynomials
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And many more!

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$$\Phi: QH^*(Fl_{k+1}) \rightarrow H_*(Gr_{SL_{k+1}})_{loc}$$

$$\mathfrak{S}_w^Q \mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}$$

where $s_\lambda^{(k)}$ is a k -Schur function.

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Upshot

Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

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- $s_{\lambda}^{(k)} = s_{\lambda}$ as $k \rightarrow \infty$.
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\lambda} + s_{\lambda}$$

The diagram shows the branching rule for k -Schur functions. It illustrates that $s_{\lambda}^{(2)}$ (a 2x2 square) can be expressed as the sum of three terms. The first term is $s_{\lambda}^{(3)}$ (a 3x2 rectangle). The second and third terms are $s_{\lambda}^{(3)}$ (a 3x1 vertical rectangle). Brackets indicate that the second and third terms are grouped together as $s_{\lambda}^{(3)}$.

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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}$$

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- Has geometric interpretation.

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- Has a tableaux formulation and Pieri rule: $s_1^r s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda}^{(3)} + s_{\mu}^{(3)} + s_{\nu}^{(3)}$$

The diagram shows the branching of the 2-partition $(2,2)$ into 3-partitions. On the left is the partition $s_{(2,2)}^{(2)}$ represented by a 2x2 grid. On the right is the sum of three partitions: $s_{(2,2)}^{(3)}$ (a 2x2 grid), $s_{(3,1)}^{(3)}$ (a 2x2 grid with an extra cell to the right of the bottom row), and $s_{(4)}^{(3)}$ (a single row of 4 cells). Brackets below the right side group these three terms under $s_{(2,2)}^{(3)}$ and $s_{(3,1)}^{(3)}$ respectively.

- Has geometric interpretation.
- No combinatorial interpretation of branching coefficients.

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The diagram shows the branching of the Schur function $s_{\lambda}^{(2)}$ into $s_{\lambda}^{(3)}$ and two $s_{\mu}^{(3)}$ terms. The first term $s_{\lambda}^{(2)}$ is a 2x2 square. The second term $s_{\lambda}^{(3)}$ is a 2x2 square. The third term $s_{\mu}^{(3)}$ is a 2x3 rectangle. The fourth term $s_{\nu}^{(3)}$ is a 2x3 rectangle. Brackets indicate the branching from $s_{\lambda}^{(2)}$ to $s_{\lambda}^{(3)}$ and $s_{\mu}^{(3)}$.

- Has geometric interpretation.
- No combinatorial interpretation of branching coefficients.
- Definition with t important for Macdonald polynomials.
- Many definitions. A new one makes proofs easier!

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- K -theoretic Catalan functions

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

$$R_{1,3} \left(\begin{array}{|c|c|c|} \hline \text{red} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \text{red} \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left(\begin{array}{|c|} \hline \text{red} \\ \hline \\ \hline \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \text{red} \\ \hline & \\ \hline \end{array}$$

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- Extend action to a symmetric function f_λ by $R_{i,j}(f_\lambda) = f_{\lambda + \epsilon_i - \epsilon_j}$.

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- For $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$, we have the *Jacobi-Trudi identity*

$$s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$$

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$$s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$$

$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \underbrace{h_{310}}_{=0} + \underbrace{h_{310}}_{=0} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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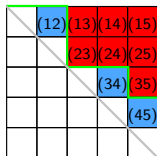
For $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



$\Psi =$ Roots above Dyck path
 $\Delta_{\ell}^{+} \setminus \Psi =$ Non-roots below

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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^{\ell}$

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- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_{\gamma}$

k -Schur root ideal for λ

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

\leftarrow row i has $4 - \lambda_i$ non-roots

Catalan functions

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k -Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$ for $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$.

Key ingredient of branching proof

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For partition λ of length ℓ with $\lambda_1 \leq k$,

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3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

Key ingredient of branching proof

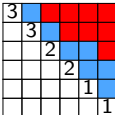
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Pieri:

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Key ingredient of branching proof

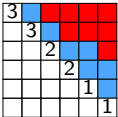
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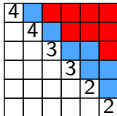
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$$\Delta^5(4, 4, 3, 3, 2, 2) =$$


Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

- Schubert calculus
- Catalan functions: a new approach to old problems
- **K -theoretic Catalan functions**

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- Dual to Grothendieck polynomials G_λ : Schubert representatives for $K^*(Gr(m, n))$

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2-bounded partitions \leftrightarrow 3-cores

The diagram illustrates the Pieri rule for K - k -Schur functions. It shows the product of a 1-strip (g_1) and a 2-bounded partition ($g_{211}^{(2)}$) resulting in the difference of two 2-bounded partitions ($g_{2111}^{(2)} - 2g_{211}^{(2)}$). The partitions are represented as 5x5 grids of colored dots (red, blue, black) with some cells shaded gray to indicate the addition or subtraction of strips.

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Problem

No direct formula for $g_{\lambda}^{(k)}$

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left(\begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

K -theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

Affine K -Theory Representatives with Raising Operators

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Example

non-roots of Ψ , roots of \mathcal{L}

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ = (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332} \end{aligned}$$

Answer (Blasiak-Morse-S., 2020)

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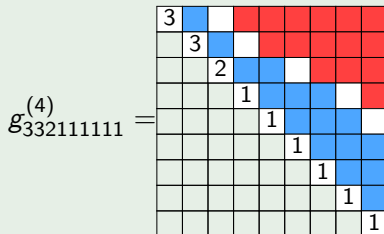
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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

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Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

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- 1 Combinatorially describe dual Pieri rule:

$$G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}, \quad 1 \leq r \leq k.$$

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Thank you!

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