

K -theoretic Catalan functions

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Junior Mathematician Research Archive

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- ① Schubert calculus
- ② Catalan functions
- ③ K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of subvarieties in a variety X .

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

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Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda \subseteq (n^m)}$ in variety $X = \text{Gr}(m, n)$.

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$ for combinatorially understood Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$.

- Complete homogeneous symmetric function: for $r \in \mathbb{Z}$,
$$h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}.$$

Schur polynomials and raising operators

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- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

$$R_{1,3} \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \blacksquare & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \blacksquare \\ \hline \square & & & \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \blacksquare \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \\ \hline \square & \blacksquare \\ \hline \end{array}$$

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- Schur function $s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$ (Jacobi-Trudi)

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(Co)homology of Grassmannian	Schur functions
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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- P and Q functions
(Co)homology of affine Grassmannian	(dual) k -Schur functions
K -theory of Grassmannian	Grothendieck polynomials
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K -theory and K -homology of the affine Grassmannian

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K -theory and K -homology of the affine Grassmannian

Simultaneously generalizes K -theory of Grassmannian and (co)homology of affine Grassmannian.

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- ① K -theory classes of Grassmannian (not affine!) represented by “Grothendieck polynomials.” We are interested in their dual:

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K -Theory of Affine Grassmannian

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- ② Homology classes of affine Grassmannian represented by k -Schur functions ($t = 1$).
- ③ (Lam et al., 2010) leave open the question: what is a direct formulation of the K -homology representatives of the affine Grassmannian (K - k -Schur functions)?

Remember?

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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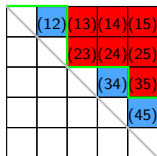
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

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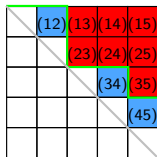
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- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_{\gamma}$

k -Schur root ideal for λ

For $k \in \mathbb{Z}_{\geq 0}$ and $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell) \in \mathbb{Z}^\ell$,

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3						
	3					
		2				
			2			
				1		
					1	

\leftarrow row i has $4 - \lambda_i$ non-roots

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k -Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

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Remark

(Blasiak et al., 2019) show results for k -Schur functions with parameter t , but $t = 1$ specialization is necessary for Schubert calculus.

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- Extra ingredient: lowering operators $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$

$$L_3 \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \quad L_1 \left(\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \color{red}\square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

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Affine K -Theory Representatives with Raising Operators

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Example

non-roots of Ψ in blue, roots of \mathcal{L} marked with •

	(12)		•	•
			•	•
		(34)		
			(45)	

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332} \end{aligned}$$

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For K -homology of affine Grassmannian,
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Example

$$g_{332111}^{(4)} =$$

3				•	•	•
	3				•	•
			2			
				1		
					1	
						1

$$\Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$$

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Property and Further Work

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Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ “branching coefficients” are alternating by degree, i.e. the $b_{\lambda\mu}^{(k)}$ in

$$g_\lambda^{(k)} = \sum_{\mu} b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

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- 3 Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$.

Thank you!

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