

# $K$ -theoretic Catalan functions

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Junior Mathematician Research Archive

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- ① Schubert calculus
- ② Catalan functions
- ③ K-theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

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Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

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Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

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Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .



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## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for combinatorially understood Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

- Complete homogeneous symmetric function: for  $r \in \mathbb{Z}$ ,  
$$h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}.$$

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- Raising operators  $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

$$R_{1,3} \left( \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \blacksquare & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \blacksquare \\ \hline \square & & & \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \blacksquare \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \\ \hline \square & \blacksquare \\ \hline \end{array}$$

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- Schur function  $s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$  (Jacobi-Trudi)

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(Co)homology of Grassmannian	Schur functions
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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
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Simultaneously generalizes  $K$ -theory of Grassmannian and (co)homology of affine Grassmannian.

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# $K$ -Theory of Affine Grassmannian

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- ② Homology classes of affine Grassmannian represented by  $k$ -Schur functions ( $t = 1$ ).
- ③ (Lam et al., 2010) leave open the question: what is a direct formulation of the  $K$ -homology representatives of the affine Grassmannian ( $K$ - $k$ -Schur functions)?

# Remember?

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).

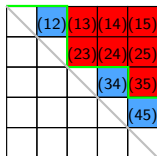


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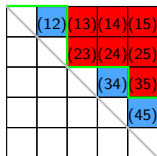
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

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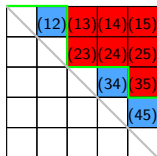
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- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_{\gamma}$

## $k$ -Schur root ideal for $\lambda$

For  $k \in \mathbb{Z}_{\geq 0}$  and  $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell) \in \mathbb{Z}^\ell$ ,

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

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$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

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## Remark

(Blasiak et al., 2019) show results for  $k$ -Schur functions with parameter  $t$ , but  $t = 1$  specialization is necessary for Schubert calculus.

# Lowering Operators

- Recall  $K$ -theory/homology of affine Grassmannian simultaneously generalizes:
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  - Homology of affine Grassmannian:  $s_\lambda^{(k)} = \prod_{(i,j) \in \Delta^+ \setminus \Delta^k(\lambda)} (1 - R_{ij}) h_\lambda$
- Extra ingredient: lowering operators  $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \color{red}\square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

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$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

for  $k_\gamma$  an inhomogeneous analogue of  $h_\gamma$ .

# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$  in blue, roots of  $\mathcal{L}$  marked with •

	(12)		•	•
			•	•
		(34)		
			(45)	

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332} \end{aligned}$$



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 $f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$  since this family has the correct structure constants.

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Example

$$g_{332111}^{(4)} = \begin{array}{|c|c|c|c|c|c|c|} \hline 3 & \text{blue} & & \bullet & \bullet & \bullet & \\ \hline & 3 & \text{blue} & & \bullet & \bullet & \\ \hline & & & 2 & \text{blue} & \text{blue} & \\ \hline & & & & 1 & \text{blue} & \text{blue} \\ \hline & & & & & 1 & \text{blue} \\ \hline & & & & & & 1 \\ \hline \end{array} \quad \Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$$

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## Theorem (Blasiak-Morse-S., 2020)

The  $g_{\lambda}^{(k)}$  “branching coefficients” are alternating by degree, i.e. the  $b_{\lambda\mu}^{(k)}$  in

$$g_{\lambda}^{(k)} = \sum_{\mu} b_{\lambda\mu}^{(k)} g_{\mu}^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$ .

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- 3 Combinatorially describe  $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$ .

Thank you!

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