Dens, nests, and Catalanimals: a walk through the zoo of shuffle theorems

George H. Seelinger

ghseeli@umich.edu

joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

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Polynomials

- $f \in \mathbb{Q}[x_1, \ldots, x_n]$ multivariate polynomial

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$

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$$

- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \;\forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the $e_1, e_2, \ldots$, or in the $h_1, h_2, \ldots$

$$3h_2 h_1^2 - h_2^2 + 6h_3 h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of* n is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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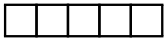
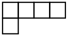
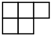
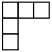
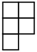
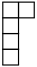
$5 \to$ 

$4+1 \to$ 

$3+2 \to$ 

$3+1+1 \to$ 

$2+2+1 \to$ 

$2+1+1+1 \to$ 

$1+1+1+1+1 \to$ 

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For $\lambda = (2, 1)$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur functions

Associate a polynomial to $\mathrm{SSYT}(\lambda)$.

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& \begin{ytableau} 1 & 1 \\ 2 \end{ytableau}, & \begin{ytableau} 1 & 1 \\ 3 \end{ytableau}, & \begin{ytableau} 2 & 2 \\ 3 \end{ytableau}, & \begin{ytableau} 1 & 2 \\ 2 \end{ytableau}, & \begin{ytableau} 1 & 3 \\ 3 \end{ytableau}, & \begin{ytableau} 2 & 3 \\ 3 \end{ytableau}, & \begin{ytableau} 1 & 3 \\ 2 \end{ytableau}, & \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \\
\text{Weight:} & (2,1,0) & (2,0,1) & (0,2,1) & (1,2,0) & (1,0,2) & (0,1,2) & (1,1,1) & (1,1,1)
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Weight: (2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- s_λ is a symmetric function
- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

Harmonic polynomials

1. S_3 action on M fixes vector subspaces!

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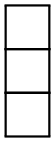


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Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

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Via Frobenius characteristic map, questions about S_n-representations get translated to questions about Schur expansion coefficients in symmetric functions.

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Via Frobenius characteristic map, questions about S_n-representations get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

Getting more information

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Solution: minimal S_n-fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\substack{\ydiagram{2,1}\\ \deg=2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\substack{\ydiagram{2,1}\\ \deg=1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Solution: minimal S_n-fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Answer: "Hall-Littlewood polynomial" $H_{\ydiagram{1,1,1}}(X;q)$.

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- Does there exist a family of S_n-representations whose (bigraded) Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$. (Still open!)

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

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$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Operator ∇

$$\nabla\tilde{H}_\lambda(X;q,t) = q^{n(\lambda)}t^{n(\lambda')}\tilde{H}_\lambda(X;q,t)$$

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathsf{area}(\lambda)} q^{\mathsf{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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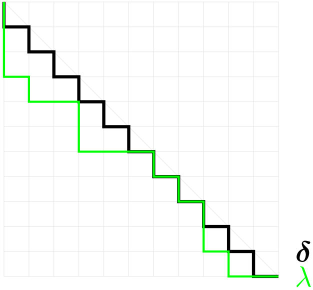
Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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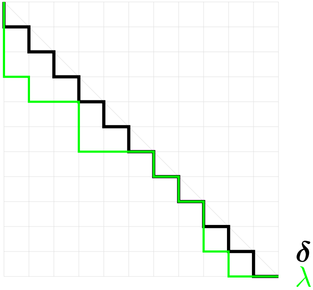

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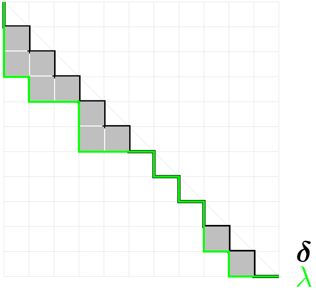

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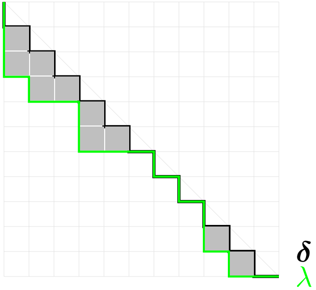

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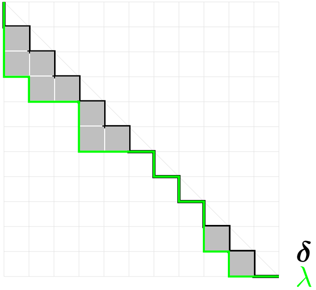

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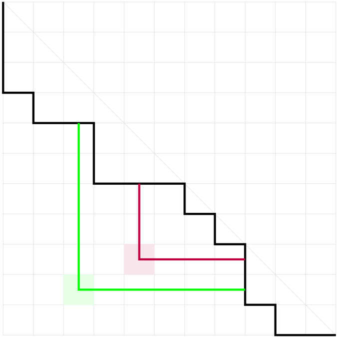

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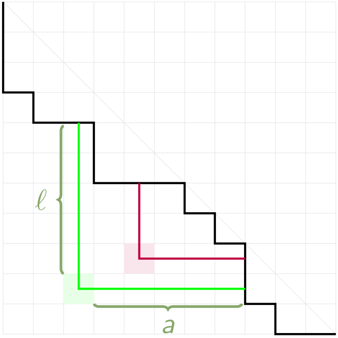

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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- $\mathcal{G}_\nu$ is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

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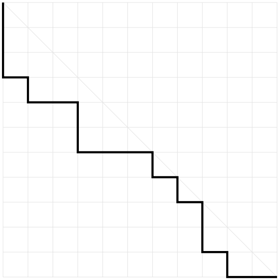

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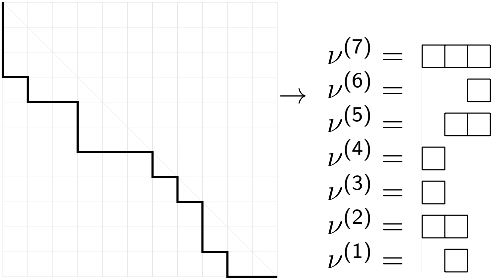

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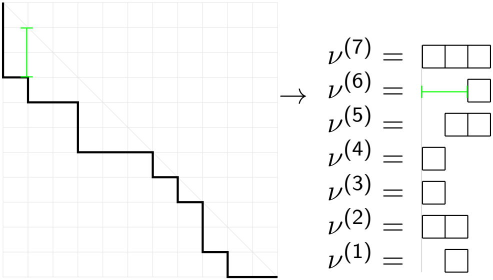

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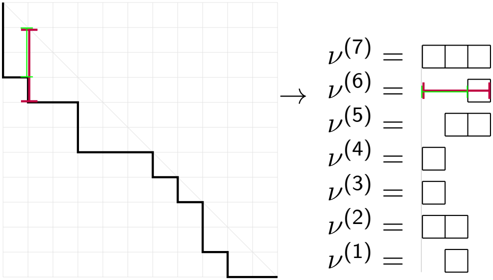

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-

$$\mathcal{G}_{\square\square,\square}(x_1,x_2;q) = x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + qx_1^2x_2 + qx_1x_2^2$$

$$\begin{array}{cccccc} \boxed{1}\boxed{1} & \boxed{1}\boxed{2} & \boxed{1}\boxed{2} & \boxed{2}\boxed{2} & \boxed{1}\boxed{\color{red}1} & \boxed{\color{blue}2}\boxed{2} \\ \quad\boxed{1} & \quad\boxed{1} & \quad\boxed{2} & \quad\boxed{2} & \quad\boxed{\color{red}2} & \quad\boxed{\color{blue}1} \end{array}$$

$$= s_3 + qs_{2,1}$$

Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)}t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)}t^{\mathsf{area}(\lambda)}\mathcal{G}_{\nu(\lambda)}(X;q^{-1})$

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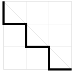		
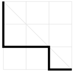		
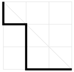		
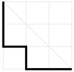		
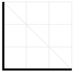		

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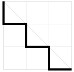
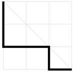
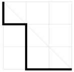
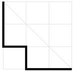
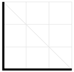
λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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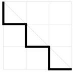
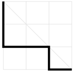
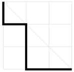
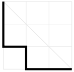
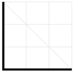
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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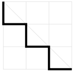
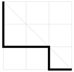
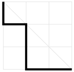
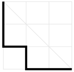
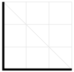
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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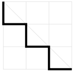
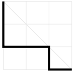
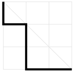
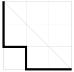
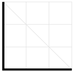
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + q t s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q,t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

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For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

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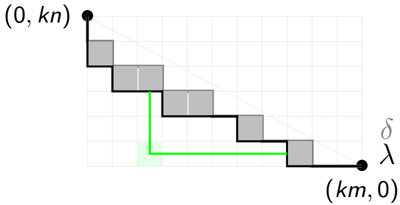

Welcome to the Zoo

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- $\mathcal{E}$ contains subalgebra $\Lambda(X^{m,n}) \cong \Lambda$ for each coprime pair $(m, n) \in \mathbb{Z}^2$.
- In general, $\mathcal{E}$-action can be a pain to compute in a nice way, but sometimes it is nice!

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- We can take "polynomial part" (restrict to only polynomial GL_l-characters) to get a symmetric function.

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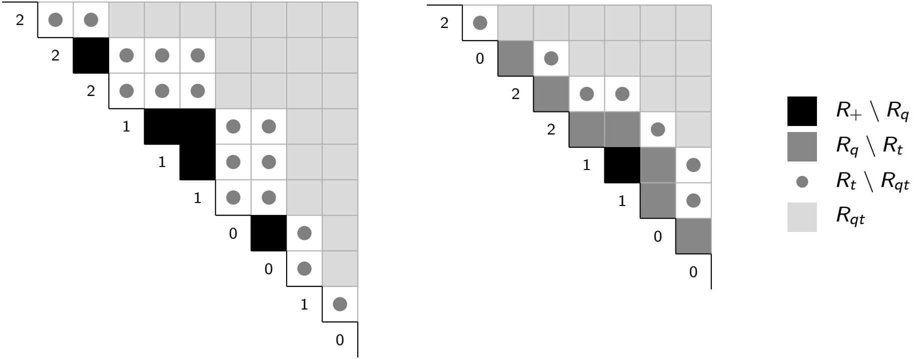

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- In this case, we set $\operatorname{cub}(H) = f$.
- The cuddly conditions allow a nice coproduct formula for $f[X+Y]$ in terms of cubs of "restrictions" of H.

Cuddly Catalanimals with cub e_k

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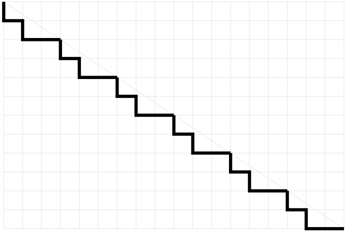

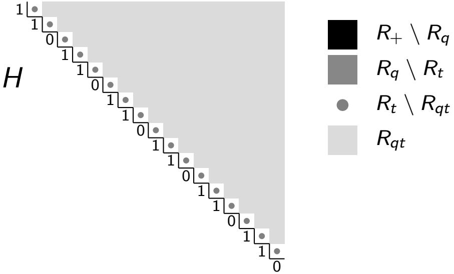

$\delta = (1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0)$ and
$e_6[-MX^{3,2}] \cdot 1 = \omega \, \mathrm{pol}_X \, H$

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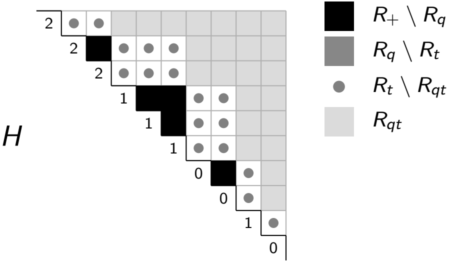

$$s_\mu[-MX^{1,1}] \cdot 1 = \nabla s_\mu = \omega \, \mathrm{pol}_X \, H \text{ (up to } q, t\text{-monomial)}$$

Dens and nests

Theorem (Blasiak-Haiman-Morse-Pun-S. (2021^+))

For every partition μ and coprime positive integers m, n, we have

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- Combinatorial RHS: Over all *nests* π in a *den* associated to μ and m, n.
- Conjectured by Loehr-Warrington (2008) when $n = 1$ with different combinatorics (but bijectively related).

Dens and nests

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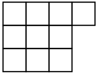
$\mu =$ 

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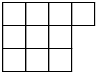
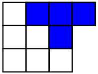
$\mu =$ 

- $p(\mu) =$ number of boxes with positive content. 

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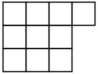
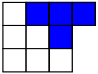
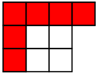
$\mu =$ 

- $p(\mu) =$ number of boxes with positive content. 
- $h = m(\text{largest hook length in } \mu) = m(\mu_1 + \ell(\mu) - 1)$. 

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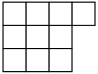
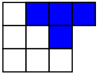
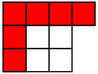
$\mu =$ 

- $p(\mu) =$ number of boxes with positive content. 
- $h = m(\text{largest hook length in } \mu) = m(\mu_1 + \ell(\mu) - 1)$. 

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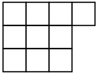
$\mu =$ 

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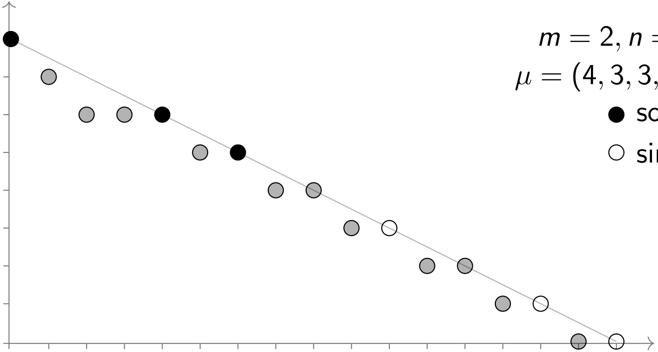

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○ sink

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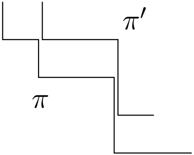
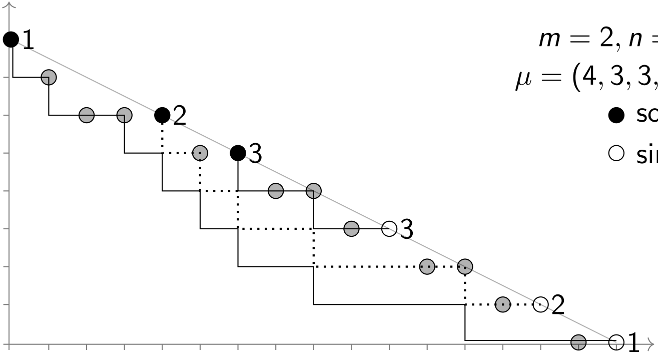


Dens and nests

Example of the "highest nest" π^0



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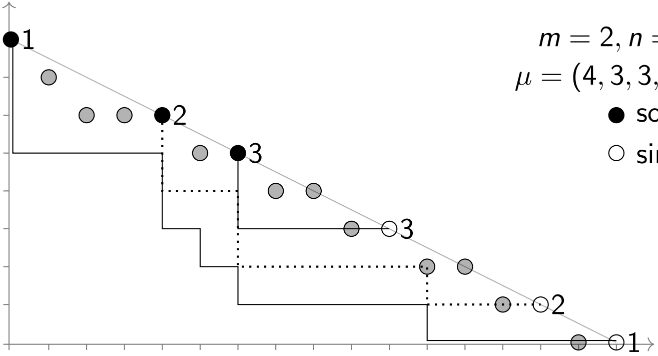
Dens and nests

Example of another nest.



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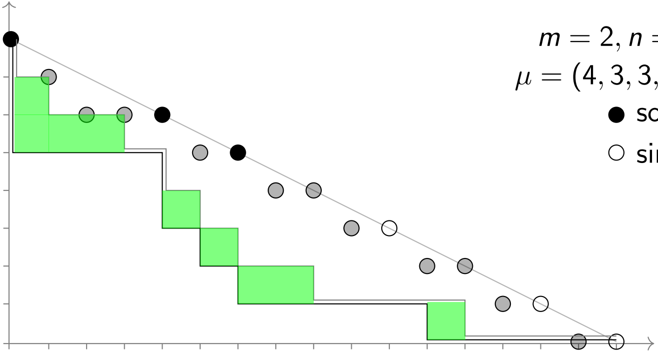

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● source

○ sink

$\text{area}(\pi_1) = 9$

Dens and nests

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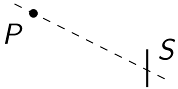
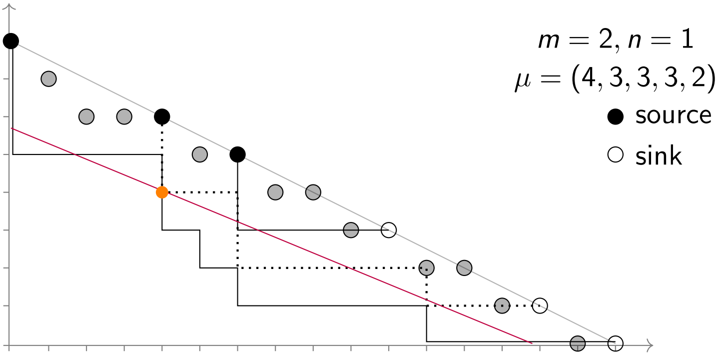
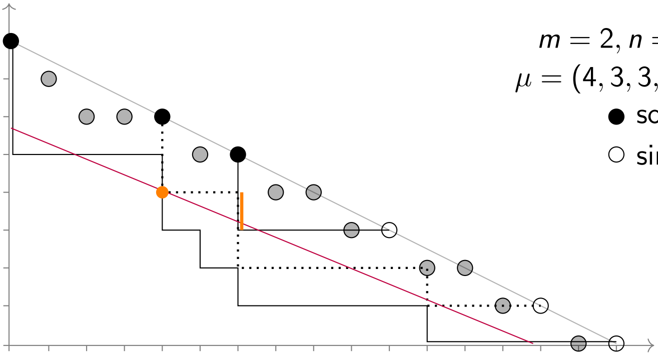


dinv



Contributes 3 to the dinv.

dinv



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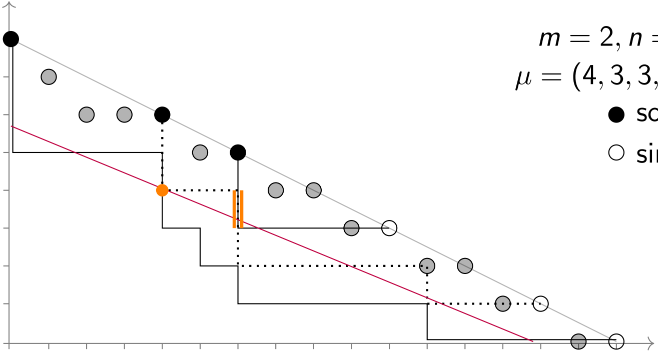

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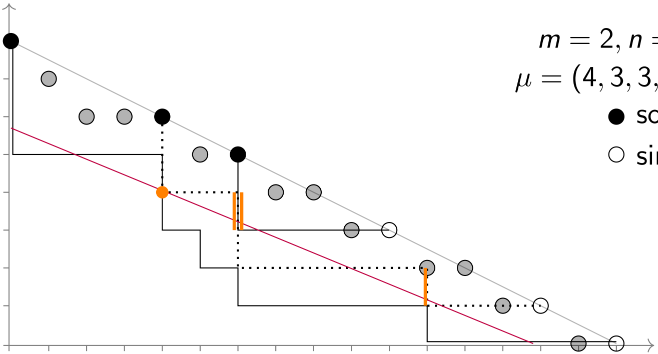

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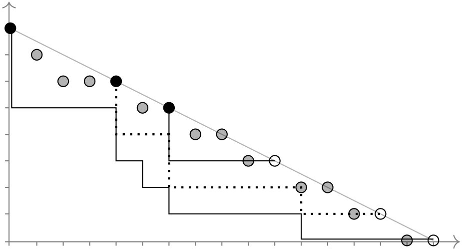

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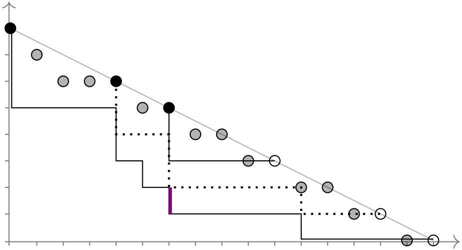

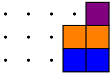

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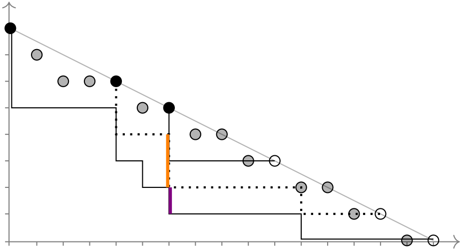

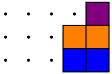

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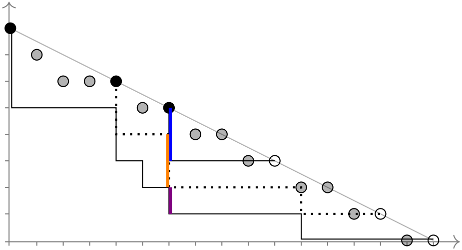

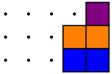

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- This allows us to simultaneously generalize the $s_\lambda[-MX^{m,n}]$ formula and our "shuffle theorem for paths under any line" formula (BHMPS).

Other exhibits for next time

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- For each LLT polynomial $\mathcal{G}_\nu$ and coprime (m, n) with $m > 0$, an m, n-cuddly Catalanimal with cub $\mathcal{G}_\nu$ is given. (BHMPS)
- Special cases include Schur functions and Hall-Littlewood polynomials.
- Unicorn Catalanimals (or Catalan functions) where $R_t = R_{qt} = \varnothing$ also have a rich (older) results and combinatorics, but served as inspiration. (Chen-Haiman, Blasiak-Morse-Pun-Summers, Blasiak-Morse-Pun)

Future work: exit through the gift shop

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- What connections do Catalanimals have with machinery used to prove other shuffle theorems, such as work by Carlsson-Mellit?

Thank you for visiting!

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