

# Raising operators in Schubert Calculus

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GarsiaFest 2019

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18 June 2019

# Overview of Schubert Calculus Combinatorics

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Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

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- $f_\lambda = s_\lambda$ , the Schur functions
- $s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$  (Jacobi-Trudi)
- Raising operators  $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

$$R_{1,3} \left( \begin{array}{|c|c|c|} \hline \text{red} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \text{red} \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|} \hline \text{red} \\ \hline \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \text{red} \\ \hline & \\ \hline \end{array}$$

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## Focus

$K$ -theory and  $K$ -homology of the affine Grassmannian

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## Focus

### $K$ -theory and $K$ -homology of the affine Grassmannian

- Simultaneously generalizes  $K$ -theory of Grassmannian and (co)homology of affine Grassmannian.

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- ①  $K$ -theory classes of Grassmannian (not affine!) represented by “Grothendieck polynomials.” We are interested in their dual:

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- ② Homology classes of affine Grassmannian represented by  $k$ -Schur functions ( $t = 1$ )

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- ③ (Lam et al., 2010) leave open the question: what is a direct formulation of the  $K$ -homology representatives of the affine Grassmannian ( $K$ - $k$ -Schur functions)?

# Remember?

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

# Lowering Operators and Root Ideals

- Lowering Operators  $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

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- Root ideal  $\Psi$ : given by Dyck path.

$$\Psi = \begin{array}{|c|c|c|c|c|} \hline & (12) & (13) & (14) & (15) \\ \hline & & (23) & (24) & (25) \\ \hline & & & (34) & (35) \\ \hline & & & & (45) \\ \hline & & & & \\ \hline \end{array} \quad \begin{array}{l} \text{Roots above Dyck path} \\ \text{Non-roots below} \end{array}$$

## Definition

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) Kh_\gamma$$

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# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) Kh_{54332} \end{aligned}$$

# Affine $K$ -Theory Representatives with Raising Operators

## Definition

The  $k$ -Schur root ideal,  $\Delta^{(k)}(\lambda)$  is the unique root ideal with  $\lambda_i + \#\text{non-roots in row } i = k$ .

## Example

$k = 4, \lambda = 332111$

$$\Delta^{(4)}(332111) =$$

3					
	3				
		2			
			1		
				1	
					1

← 4 - 2 non-roots



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For  $K$ -homology of affine Grassmannian,  
 $f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$  since this family satisfies the correct Pieri rule.

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Example

$$g_{332111}^{(4)} =$$

3						
	3					
		2				
			1			
				1		
					1	

$$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$$

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$$G_{1^{\ell}}^{\perp} g_{\lambda+1^{\ell}}^{(k+1)} = g_{\lambda}^{(k)}$$

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## Further work

For  $G_{\lambda}^{(k)}$  an affine Grothendieck polynomial (dual to  $g_{\lambda}^{(k)}$ ),

- 1 Dual Pieri rule:  $G_{1^r}^{\perp} g_{\lambda}^{(k)} = \sum_{\mu} ?? g_{\mu}^{(k)} \iff G_{1^r} G_{\mu}^{(k)} = \sum_{\lambda} ?? G_{\lambda}^{(k)},$   
 $1 \leq r \leq k$
- 2 Branching rule:  $g_{\lambda}^{(k)} = \sum_{\mu} ?? g_{\mu}^{(k+1)}$

Thank you!

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and  $k$ -Schur Positivity*, Journal of the AMS.

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