

**TWO ELEMENTARY EXAMPLES OF EXTREME
CHARACTERS OF $U(\infty)$
INTEGRABLE PROBABILITY READING SEMINAR**

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1. INTRODUCTION

First, we recall some definitions.

1.1. Definition. An $N \times N$ matrix U is *unitary* if $UU^* = I_N$ where U^* is the conjugate transpose of U . Then, $U(N)$ is the compact Lie group of all $N \times N$ unitary matrices. Since $U(N-1) \hookrightarrow U(N)$ via a canonical embedding, we also define

$$U(\infty) := \bigcup_{N=1}^{\infty} U(N)$$

that is, $U(\infty)$ are all infinite $\mathbb{N} \times \mathbb{N}$ unitary matrices that differ from the identity matrix only in a fixed number of positions.

1.2. Definition. A *normalized character* of $U(N)$ is a function $\chi: U(N) \rightarrow \mathbb{C}$ such that

- (a) $\chi(e) = 1$ (normalized),
- (b) $\chi(ab) = \chi(ba)$ (constant on conjugacy classes),
- (c) $(\sum c_i \chi(a_i)) (\sum c_j \chi(a_j))^* = \sum c_i \bar{c}_j \chi(a_i a_j^{-1}) \geq 0$ (nonnegative definite),
- (d) χ is continuous.

Normalized characters form a convex set since $t\chi_1 + (1-t)\chi_2$ meets all the axioms of a normalized character for all $t \in [0, 1]$. Then, we can discuss the following notion.

1.3. Definition. An *extreme character* $\chi: U(N) \rightarrow \mathbb{C}$ is a normalized character such that $\chi \neq t\chi_1 + (1-t)\chi_2$ for any $t \in (0, 1)$ for normalized characters $\chi_1, \chi_2 \neq \chi$.

1.4. Definition. The N -dimensional torus is

$$\mathbb{T}^N := \{(x_1, \dots, x_N) \in \mathbb{C}^N \mid |x_i| = 1\}$$

and lies in $U(N)$ as diagonal matrices. The *finitary torus* is $\mathbb{T}_{fin}^\infty := \bigcup_{N=1}^{\infty} \mathbb{T}^N$.

Recall one of our main goals is to understand the following theorem.

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1.5. Theorem (Edrei-Voiculescu). *Extreme characters of $U(\infty)$ are functions $\chi: T_{fin}^\infty \rightarrow \mathbb{C}$ depending on countably many parameters*

$$\begin{cases} \alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \dots \geq 0); \\ \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \dots \geq 0); \\ \gamma^\pm \geq 0 \end{cases}$$

such that

$$\sum_i \alpha_i^+ + \sum_i \alpha_i^- + \sum_i \beta_i^+ + \sum_i \beta_i^- < \infty, \quad \beta_1^+ + \beta_1^- \leq 1$$

Furthermore, these functions have the form

$$\chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \dots) = \prod_{j=1}^{\infty} \Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_j)$$

where $\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}: \mathbb{T} \rightarrow \mathbb{C}$ is the continuous function

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) := e^{\gamma^+(x-1) + \gamma^-(x^{-1}-1)} \prod_{i=1}^{\infty} \left(\frac{1 + \beta_i^+(x-1)}{1 - \alpha_i^+(x-1)} \cdot \frac{1 + \beta_i^-(x^{-1}-1)}{1 - \alpha_i^-(x^{-1}-1)} \right).$$

1.6. Goal. In this presentation, we will outline two very special examples of this parameterization, namely when

(a) $\beta^+ = (\beta, 0, 0, \dots), \beta^- = \alpha^\pm = (0, 0, \dots), \gamma^\pm = 0$ for $\beta \in [0, 1]$ so that

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = 1 + \beta(x-1) \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \dots) = \prod_{j=1}^{\infty} (1 + \beta(x_j - 1))$$

(b) $\alpha^+ = (\alpha, 0, 0, \dots), \beta^\pm = \alpha^+ = (0, 0, \dots), \gamma^\pm = 0$ for $\alpha \in [0, 1]$ so that

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = \frac{1}{1 - \alpha(x-1)} \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \dots) = \prod_{j=1}^{\infty} \frac{1}{1 - \alpha(x_j - 1)}$$

2. SYMMETRIC FUNCTIONS

In the last lecture, we introduced the following.

2.1. Definition. Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$, the *Schur polynomial* is given by

$$s_\lambda(x_1, \dots, x_N) = \frac{\det(x_j^{\lambda_i + N - i})_{i,j=1}^N}{\det(x_j^{N-i})_{i,j=1}^N}$$

Also, if λ has $\lambda_N \geq 0$, we can use “Littlewood’s Combinatorial Description” of Schur functions

2.2. Proposition. *Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$,*

$$s_\lambda(x_1, \dots, x_N) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}$$

where $x^{\text{wt}(T)} = \prod_{j=1}^{\sum \lambda_i} x_j^{\# \text{ of } j\text{'s in } T}$.

2.3. Example.

$$s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$$

| | |
|---|---|
| 1 | 1 |
| 2 | |

| | |
|---|---|
| 1 | 2 |
| 2 | |

We also proved that

2.4. Theorem. *The irreducible representations of $U(N)$ are in one-to-one correspondence with $\{\lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \dots \geq \lambda_N\}$ where the character of representation T_λ of $U(N)$ corresponding to λ has character given by*

$$\text{Tr} \left(T_\lambda \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix} \right) = s_\lambda(x_1, \dots, x_N)$$

We will work with two special cases of the Schur polynomials.

2.5. Definition. Let $e_m(x_1, \dots, x_N) := s_{(1^m)}(x_1, \dots, x_N)$ be the *elementary symmetric polynomials*.

2.6. Example. Using the semistandard Young tableaux formula for Schur functions (Littlewood's combinatorial description), we compute

(a)

$$e_2(x_1, x_2) = x_1 x_2$$

| |
|---|
| 1 |
| 2 |

(b)

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

| |
|---|
| 1 |
| 2 |

| |
|---|
| 1 |
| 3 |

| |
|---|
| 2 |
| 3 |

(c)

$$e_3(x_1, x_2, x_3) = x_1 x_2 x_3$$

| |
|---|
| 1 |
| 2 |
| 3 |

2.7. Remark. $e_N(x_1, \dots, x_N)$ encodes character of the “determinant representation” of $U(N)$, that is

$$T(U)v = (\det U)v = x_1 x_2 \cdots x_N v$$

since the determinant is just the product of the eigenvalues. More generally, $e_m(x_1, \dots, x_N)$ encodes the representation induced by the $U(N)$ -action on $\bigwedge^m \mathbb{C}^N$:

$$U \cdot (v_1 \wedge \cdots \wedge v_m) = (Uv_1 \wedge \cdots \wedge Uv_m)$$

Importantly, we also compute, generalizing our example above

2.8. Proposition. *For $0 < m \leq n$,*

$$e_m(x_1, x_2, \dots, x_n) = \sum_{T \in \text{SSYT}((1^m)) \text{ filled with elements of } \{1, \dots, n\}} x^{\text{wt}(T)} = \sum_{I \subseteq \{1, \dots, n\}, |I|=m} x^I$$

where $x^I := \prod_{i \in I} x_i$ and consequently,

$$e_m(\underbrace{1, \dots, 1}_n) = \binom{n}{m}$$

Proof. To see this, we simply observe that a single column semistandard tableau with m rows filled with letters $\{1, \dots, n\}$ is a choice of m distinct elements of $\{1, \dots, n\}$ since columns must be strictly increasing. \square

2.9. Definition. Let $h_m(x_1, \dots, x_N) := s_{(m)}(x_1, \dots, x_N)$ be the *complete homogeneous symmetric polynomials*.

2.10. Example. Using again our tableaux formula for Schur functions, we compute

(a)

$$h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$$

| | | | | | | | |
|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|
| $\boxed{1}$ | $\boxed{1}$ | $+$ | $\boxed{1}$ | $\boxed{2}$ | $+$ | $\boxed{2}$ | $\boxed{2}$ |
|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|

(b)

$$h_2(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

| | | | | | | | | | | | | | | | | |
|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|
| $\boxed{1}$ | $\boxed{1}$ | $+$ | $\boxed{1}$ | $\boxed{2}$ | $+$ | $\boxed{1}$ | $\boxed{3}$ | $+$ | $\boxed{2}$ | $\boxed{2}$ | $+$ | $\boxed{2}$ | $\boxed{3}$ | $+$ | $\boxed{3}$ | $\boxed{3}$ |
|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|-----|-------------|-------------|

2.11. Proposition. *For $0 < m \leq n$,*

$$h_m(x_1, x_2, \dots, x_n) = \sum_{T \in \text{SSYT}((m)) \text{ filled with elements of } \{1, \dots, n\}} x^{\text{wt}(T)} = \sum_{I \text{ multiset of } \{1, \dots, n\}, |I|=m} x^I$$

where $x^I := \prod_{i \in I} x_i$ and consequently,

$$h_m(\underbrace{1, \dots, 1}_n)$$

= Number of ways to choose a multiset of size m from n things

$$= \binom{n+m-1}{m} = \binom{n+m-1}{n-1}$$

2.12. Remark. The combinatorics of the identity above follow by considering a “stars and bars” approach, namely, both expressions are in bijection with the number of ways to place $n - 1$ bars among m stars, allowing bars to be consecutive with each other.

$$\{1, 1, 1, 2, 4, 5\} \rightarrow \star \star \star | \star || \star | \star$$

2.13. Definition. Let

$$\binom{n}{m} := \binom{n+m-1}{m}$$

be the number of ways to choose a multiset of size m from n things.

3. TWO EXAMPLES OF $U(\infty)$ CHARACTERS

Now, we wish to take a sequence of $U(N)$ characters to get a character of $U(\infty)$.

3.1. Definition. We say that a sequence of central functions f_N (i.e. f_N only depends on the eigenvalues of the input) on $U(N)$ converge to a central function f on $U(\infty)$ if, for every fixed K , we have

$$f_N(x_1, \dots, x_K, 1, 1, \dots, 1) \rightarrow f(x_1, \dots, x_K, 1, 1, \dots)$$

uniformly on the K -torus \mathbb{T}^K of diagonal matrices.

3.2. Proposition. Let $L: \mathbb{N} \rightarrow \mathbb{N}$ be a sequence such that $L(N)/N \rightarrow \beta \in [0, 1]$ as $N \rightarrow \infty$. Then,

$$\frac{e_{L(N)}(x_1, \dots, x_N)}{e_{L(N)}(1, \dots, 1)} \rightarrow \prod_{i=1}^{\infty} (1 + \beta(x_i - 1)), \quad (x_1, x_2, \dots) \in \mathbb{T}_{fin}^{\infty}$$

Proof. Fix $K \leq N$. Then,

$$\begin{aligned} e_{L(N)}(x_1, \dots, x_K, 1, \dots, 1) &= \sum_{T \in \text{SSYT}((1^{L(N)})) \text{ labelled with } \{1, \dots, N\}} x^{\text{wt}(T|_{\leq K})} \\ &= \sum_{\substack{\text{binary } K \text{ sequences } \epsilon \\ \text{such that } \sum_i \epsilon_i = L(N)}} \#\{\text{sequences with sum } L(N) \text{ that start with } (\epsilon_1, \dots, \epsilon_K)\} x^{(\epsilon_1, \dots, \epsilon_K)} \\ &= \sum_{\substack{\text{binary } K \text{ sequences } \epsilon \\ \text{such that } \sum_i \epsilon_i = L(N)}} \binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \end{aligned}$$

where the last equality comes from considering how to fill tableaux of the form

$$\begin{aligned}
& \left\{ \begin{array}{l} \text{Fill } \sum_{i=1}^K \epsilon_i \text{ boxes with } \{1, \dots, K\} \\ \text{Fill } L(N) - \sum_{i=1}^K \epsilon_i \text{ boxes with } \{K+1, \dots, N\} \end{array} \right\} \\
& \Rightarrow \frac{e_{L(N)}(x_1, \dots, x_K, 1, \dots, 1)}{e_{L(N)}(1, \dots, 1)} = \sum_{\text{binary } K \text{ sequences } \epsilon} \left(\binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} / \binom{N}{L(N)} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\end{aligned}$$

where

$$\begin{aligned}
\binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} / \binom{N}{L(N)} &= \frac{(N-K)!}{N!} \times \frac{(L(N))!}{(L(N) - \sum_{i=1}^K \epsilon_i)!} \times \frac{(N-L(N))!}{(N-L(N) - (K - \sum_{i=1}^K \epsilon_i))!} \\
&\xrightarrow{N \rightarrow \infty} \beta^{\sum_{i=1}^K \epsilon_i} (1-\beta)^{\sum_{i=1}^K \epsilon_i} \text{ since } L(N)/N \rightarrow \beta
\end{aligned}$$

Thus, taking the limit as $N \rightarrow \infty$ on our ratio, we get

$$\sum_{\text{binary } K \text{ sequences } \epsilon} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \beta^{\sum_{i=1}^K \epsilon_i} (1-\beta)^{K - \sum_{i=1}^K \epsilon_i} = \prod_{i=1}^K ((1-\beta) + \beta x_i)$$

and so, taking $K \rightarrow \infty$ completes the proof. \square

3.3. Remark. An astute reader may notice that $(1-\beta)^{K - \sum_{i=1}^K \epsilon_i} \beta^{\sum_{i=1}^K \epsilon_i}$ represents the probability of $\sum_{i=1}^K \epsilon_i$ successes in K trials where each attempt has probability of success β . One can use “de Finetti’s theorem” in order to derive the proposition directly from this observation. See [Pet12]§4.1.10 for this approach.

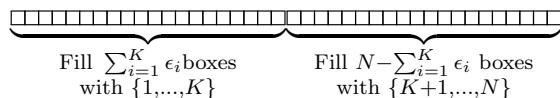
3.4. Proposition. Let $L: \mathbb{N} \rightarrow \mathbb{N}$ be a sequence such that $L(N)/N \rightarrow \alpha \in [0, 1]$ as $N \rightarrow \infty$. Then,

$$\frac{h_{L(N)}(x_1, \dots, x_N)}{h_{L(N)}(1, \dots, 1)} \rightarrow \prod_{i=1}^{\infty} \frac{1}{1 - \alpha(x_i - 1)}, \quad (x_1, x_2, \dots) \in \mathbb{T}_{fin}^{\infty}$$

Proof. We proceed much as in the proposition above. For a fixed $K \leq N$, we have

$$\begin{aligned}
& h_{L(N)}(x_1, \dots, x_K, 1, \dots, 1) \\
&= \sum_{\epsilon \in \mathbb{N}_0^K} \#\{N \text{ sequences with sum } L(N) \text{ starting with } (\epsilon_1, \dots, \epsilon_K)\} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \\
&= \sum_{\epsilon \in \mathbb{N}_0^K} \left(\binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K}
\end{aligned}$$

where the last line comes from thinking about



and so

$$\begin{aligned} & \frac{h_{L(N)}(x_1, \dots, x_K, 1, \dots, 1)}{h_{L(N)}(1, \dots, 1)} \\ &= \sum_{\epsilon \in \mathbb{N}_0^K} \left[\binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} / \binom{N}{L(N)} \right] x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \end{aligned}$$

Consider that, for fixed $K \leq N$, we have

$$\begin{aligned} & \binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} / \binom{N}{L(N)} \\ &= \binom{N-K+L(N)-\sum_{i=1}^K \epsilon_i-1}{L(N)-\sum_{i=1}^K \epsilon_i} / \binom{N+L(N)-1}{L(N)} \\ &= \frac{(N+L(N)-K-\sum \epsilon_i-1)!}{(N+L(N)-1)!} \times \frac{(L(N))!}{(L(N)-\sum \epsilon_i)!} \times \frac{(N-1)!}{(N-K-1)!} \\ &\approx \frac{(L(N))^{\sum \epsilon_i} N^K}{(N+L(N))^{K+\sum \epsilon_i}} \\ &= \left(\frac{L(N)}{N} \right)^{\sum \epsilon_i} \left(\frac{1}{1 + \frac{L(N)}{N}} \right)^{K+\sum \epsilon_i} \\ &\xrightarrow{N \rightarrow \infty} \left(\frac{\alpha}{1+\alpha} \right)^{\sum \epsilon_i} \left(\frac{1}{1+\alpha} \right)^K \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{h_{L(N)}(x_1, \dots, x_K, 1, \dots, 1)}{h_{L(N)}(1, \dots, 1)} &= \sum_{\epsilon} \left(\frac{1}{1+\alpha} \right)^K \left(\frac{\alpha}{1+\alpha} \right)^{\sum \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \\ &= \prod_{i=1}^K \left(\frac{1}{1+\alpha} \right) \left(1 + \frac{\alpha}{1+\alpha} x_i + \left(\frac{\alpha}{1+\alpha} \right)^2 x_i^2 + \dots \right) \\ &= \prod_{i=1}^K \frac{1}{1+\alpha} \times \frac{1}{1 - \frac{\alpha}{1+\alpha} x_i} \\ &= \prod_{i=1}^K \frac{1}{1+\alpha - \alpha x_i} \end{aligned}$$

So, taking $K \rightarrow \infty$ completes the proof. \square

REFERENCES

- [Pet12] L. Petrov, *Representation Theory of Big Groups and Probability* (2012). Accessed as a draft from <https://lpetrov.cc/reading-2019/> on January 31, 2019.