NILPOTENT, SOLVABLE, AND SEMISIMPLE LIE ALGEBRAS

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1. Introduction

Just as in groups, Lie algebras can have additional structure which make them "nice." Due to the resemblence these structures have to their group theoretic counter-parts, we talk about solvable, nilpotent, and simple Lie algebras. The goal of this monograph is to primarily understand the most important theorems surrounding these structures, namely Engel's theorem, Lie's theorem, and Cartan's theorem. **This monograph will borrow content freely from [Hum72] and makes no claim to originality of theorems, their statements, or their proofs.** The reader should think of this monograph as companion notes to [Hum72, Sections 1–5]. Unless otherwise stated, we are always working over algebraically closed ground field F of characteristic 0, $\mathfrak{g}$ is an arbitrary Lie algebra, and $\mathfrak{gl}(V) = \mathrm{End}(V)$ with the Lie bracket multiplication.

2. Basic Definitions

2.1. **Definition.** Consider a Lie algebra $\mathfrak{g}$. We say $\mathfrak{g}$ is *linear* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

While rather boring since the Lie bracket does not provide much additional structure, linear Lie algebras are important to keep in mind as easy examples. Linear Lie algebras play a similar role in Lie theory to abelian groups in group theory.

2.2. **Theorem.** *Any* 1*-dimensional Lie algebra is linear.*

Proof. By definition, a Lie algebra $\mathfrak{g}$ must have $[g, g] = 0$ for all $g \in \mathfrak{g}$. Now, let $\mathfrak{g}$ have basis element $\{x\}$. Then, $[x, x] = 0 \implies [\mathfrak{g}, \mathfrak{g}] = 0$. □

2.3. **Definition.** Consider a Lie algebra $\mathfrak{g}$. We say $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ has no proper non-trivial ideals.

Note that simple algebras are, in some sense, opposite to linear Lie algebras, since simple non-linear Lie algebras have the following property.

2.4. **Proposition.** *Let $\mathfrak{g}$ be a simple non-linear Lie algebra. Then,* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

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Proof. Since $[\mathfrak{g},\mathfrak{g}] \trianglelefteq \mathfrak{g}$ and $[\mathfrak{g},\mathfrak{g}] \neq 0$ because $\mathfrak{g}$ is non-linear, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ by the simplicity of $\mathfrak{g}$. $\square$

2.5. **Example.** Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the Lie algebra of 2 by 2 complex traceless matrices. Such a Lie algebra is simple and non-linear. This follows from the fact that the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

has the relations

$$[h,e] = 2e, [h,f] = -2f, [e,f] = h$$

So, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Now, suppose I is some nonempty ideal of $\mathfrak{sl}_2(\mathbb{C})$. Then, it contains some element $ae + bf + ch$ for $a, b, c \in \mathbb{C}$. Then,

$$[h,[h,ae+bf+ch]] = [h,a[h,e]+b[h,f]] = [h,2ae-2bf] = 4ae+4bf \implies c = 0 \text{ or } h \in I$$

since $4(ae+bf+ch)-(4ae+4bf) \in I$. However, if $h \in I$, then $[e,h] = -2e \in I$ and $[f,h] = 2f \in I$ and so $I = \mathfrak{g}$. So, it must be that $c = 0$. Now, consider since $ae - bf \in I$ by above,

$$[e, ae - bf] = -bh \in I \implies b = 0$$

However, if $ae \in I$, then $[f, ae] = -ah \in I$ and so we are done. It must be that $I = \mathfrak{sl}_2(\mathbb{C})$.

Once more theory is developed, there are easier and more elegant ways to show $\mathfrak{sl}_2(\mathbb{C})$ is simple.

2.6. **Definition.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$, and $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$. We call $\{\mathfrak{g}^{(i)}\}_{i\geq 0}$ the *derived series* of $\mathfrak{g}$.

2.7. **Definition.** Let $\mathfrak{g}$ be a Lie algebra. We say $\mathfrak{g}$ is *solvable* if there exists an $n \in \mathbb{N}$ such that $\mathfrak{g}^{(n)} = 0$.

Solvable Lie algebras have many of the same properties as solvable groups in regards to their behavior with ideals, homomorphisms, and short exact sequences. See [Hum72, p 11] for more information.

2.8. **Example.** The canonical example of a solvable Lie algebra is the Lie algebra of all upper triangular matrices over a field F. Let us denote such a Lie algebra as $\mathfrak{b}_n(F)$. Let $\mathfrak{g} = \mathfrak{b}_n(F)$. Then, we first seek to compute $[\mathfrak{g},\mathfrak{g}]$. Let $x, y \in \mathfrak{g}$. Then, $x = d_x + n_x$, where d_x is the diagonal part of x and n_x is the nilpotent part of x. We then check:

$$\begin{aligned}
[x,y] &= xy - yx \\
&= (d_x + n_x)(d_y + n_y) - (d_y + n_y)(d_x + n_x) \\
&= d_x d_y + n_x d_y + d_x n_y + n_x n_y - d_y d_x - n_y d_x - d_y n_x - n_y n_x \\
&= n_x d_y + d_x n_y + n_x n_y - n_y d_x - d_y n_x - n_y n_x && \text{since } d_x d_y \text{ commute.}
\end{aligned}$$

However, this final result must be *strictly* upper-triangular. Such matrices actually form another Lie algebra, denoted $\mathfrak{n}_n(F)$. Thus, $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}_n(F)$.

Let $e_{i,j}$ be the matrix with a 1 in the (i,j)th entry and 0 at all others. Then, $\mathfrak{b}_n(F)$ is spanned by all $e_{i,j}$ such that $j - i \geq 0$ and $\mathfrak{n}_n(F)$ is spanned by all $e_{i,j}$ such that $j - i \geq 1 = 2^0$. Furthermore,

$$[e_{i,j}, e_{k,\ell}] = \delta_{j,k} e_{i,\ell} - \delta_{\ell,i} e_{k,j}$$

Let us assume $j - i \geq 2^{k-1}$ and $\ell - k \geq 2^{k-1}$ for some $k \in \mathbb{N}$ and $\mathfrak{g}^{(k)}$ is spanned by such $e_{i,j}$. Then,

$$\begin{aligned}[e_{i,j}, e_{k,\ell}] &= \delta_{j,k} e_{i,\ell} - \delta_{\ell,i} e_{k,j} \\ &= \begin{cases} e_{i,\ell} & j = k, \ell \neq i \\ e_{k,j} & j \neq k, \ell = i \\ (e_{i,i} - e_{j,j}) & j = k, \ell = i \\ 0 & \text{else} \end{cases}\end{aligned}$$

However, if $j = k$ and $\ell = i$, then $i - j \geq 2^{k-1}$, which is a contradiction since $j - i \geq 2^{k-1}$. Thus, we actually have

$$[e_{i,j}, e_{k,\ell}] = \begin{cases} e_{i,\ell} & j = k, \ell \neq i \\ e_{k,j} & j \neq k, \ell = i \\ 0 & \text{else} \end{cases}$$

Moreover, if $j = k$, then $\ell - i \geq 2^{k-1} + k - i = 2^{k-1} + j - i \geq 2^{k-1} + 2^{k-1} = 2^k$, and similarly if $\ell = i$, then $j - k \geq 2^k$. Thus, $\mathfrak{g}^{(k+1)}$ has a basis consisting of all elements $e_{i,j}$ with $j - i \geq 2^k$.

Using this fact, we can see that $\mathfrak{g}^{\log_2 n + 1} = 0$ and thus $\mathfrak{g}$ is solvable.

2.9. **Example.** A specific and important Lie algebra in this family of Lie algebras is $\mathfrak{n}_3(F)$, which is isomorphic to *the Heisenberg algebra*, H, with basis $\{f, g, z\}$ and relation $[f, g] = z$, as well as the properties that $[H, H] \subseteq Z(H)$ and $z \in Z(H)$. The isomorphism between these two algebras is exhibited by

$$f \mapsto e_{1,2}, \;\; g \mapsto e_{2,3}, \;\; z \mapsto e_{1,3}$$

2.10. **Proposition.** *Let $\mathfrak{g}$ be an arbitrary Lie algebra. Then, $\mathfrak{g}$ has a unique maximal solvable ideal.*

Proof. Let $S \trianglelefteq \mathfrak{g}$ be a maximal solvable ideal. We know that, if $I, J \trianglelefteq \mathfrak{g}$ are both solvable, then $I + J$ is solvable. So, given another solvable ideal $I \trianglelefteq \mathfrak{g}$, it must be that $S + I$ is solvable, but since S is maximal, $S + I = S$. Thus, $I \subseteq S$. $\square$

2.11. **Definition.** If $\mathfrak{g}$ is a Lie algebra, we call its unique maximal solvable ideal the *radical of* $\mathfrak{g}$ denoted $\operatorname{Rad} \mathfrak{g}$.

2.12. **Definition.** If $\mathfrak{g}$ is a Lie algebra and $\operatorname{Rad}\mathfrak{g} = 0$, we call $\mathfrak{g}$ *semisimple.*

2.13. **Remark.** Since simple Lie algebras have no non-trivial ideals, their radical is 0 and thus any simple Lie algebra is also semisimple.

2.14. **Proposition.** *Any non-abelian solvable Lie algebra has a non-trivial abelian ideal.*

Proof. Consider that the derived series of $\mathfrak{g}$ has some minimal $n \in \mathbb{N}$ such that $\mathfrak{g}^{(n)} = 0$. Then, $\mathfrak{g}^{(n-1)} \neq 0$ but $[\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] = 0$, so $\mathfrak{g}^{(n-1)}$ is abelian and is an ideal of $\mathfrak{g}$ by repeated application of the fact that the Lie bracket of two ideals is an ideal of $\mathfrak{g}$. $\square$

2.15. **Proposition.** *Let* $\mathfrak{g}$ *be a Lie algebra. Then* $\mathfrak{g}/\operatorname{Rad}\mathfrak{g}$ *is semisimple.*

Proof. Consider the short exact sequence

$$0 \to \operatorname{Rad}\mathfrak{g} \to \mathfrak{g} \to \mathfrak{g}/\operatorname{Rad}\mathfrak{g} \to 0$$

If $\mathfrak{g} \neq \operatorname{Rad}\mathfrak{g}$, then it must be that $\mathfrak{g}/\operatorname{Rad}\mathfrak{g}$ is not solvable, otherwise $\mathfrak{g}$ would be solvable and thus $\mathfrak{g} = \operatorname{Rad}\mathfrak{g}$. Now, consider an solvable ideal of $\mathfrak{g}/\operatorname{Rad}\mathfrak{g}$ must have the form $I/\operatorname{Rad}\mathfrak{g}$ for an ideal $I \trianglelefteq \mathfrak{g}$. Then we would have short exact sequence of ideals

$$0 \to I \cap \operatorname{Rad}\mathfrak{g} \to I \to I/\operatorname{Rad}\mathfrak{g} \to 0$$

and I would be solvable, so $I = \operatorname{Rad}\mathfrak{g}$ and $I/\operatorname{Rad}\mathfrak{g} = 0$. Thus, $\mathfrak{g}/\operatorname{Rad}\mathfrak{g}$ is semisimple. $\square$

2.16. **Definition.** Let $\mathfrak{g}$ be a Lie algebra. Then, the short exact sequence used in the proof above, namely

$$0 \to \operatorname{Rad}\mathfrak{g} \to \mathfrak{g} \to \mathfrak{g}/\operatorname{Rad}\mathfrak{g} \to 0$$

is the *Levi decomposition* of $\mathfrak{g}$. That is, $\mathfrak{g}$ is the extension of a semisimple Lie algebra by a solvable algebra.

2.17. **Definition.** Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^0 := \mathfrak{g}, \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{g}^i := [\mathfrak{g}, \mathfrak{g}^{i-1}]$. We call $\{\mathfrak{g}^i\}_{i \geq 0}$ the *lower central series of* $\mathfrak{g}$ or the *descending central series of* $\mathfrak{g}$.

2.18. **Definition.** Let $\mathfrak{g}$ be a Lie algebra. If there exists an $n \in \mathbb{N}$ such that $\mathfrak{g}^n = 0$, we say that $\mathfrak{g}$ is *nilpotent.*

Just like solvability, nilpotency behaves similarly to the group theoretic version with respect to homomorphisms, ideals, and short exact sequences. See [Hum72, p 12] for more details.

2.19. **Example.** Consider $\mathfrak{g} = \mathfrak{sl}_2(F)$ where $\operatorname{char} F = 2$. Such a Lie algebra is nilpotent since

$$[h, e] = 2e = 0, \ [h, f] = -2f = 0, \ \ [e, f] = h$$

and so $\mathfrak{g}^2 = \langle h \rangle$ and thus $\mathfrak{g}^3 = 0$.

2.20. **Definition.** Let $g \in \mathfrak{g}$, a Lie algebra. Then, if ad_g is a nilpotent endomorphism, that is, there is an n such that $(\mathrm{ad}_g)^n = 0$, we say that g is *ad-nilpotent.*

2.21. **Proposition.** *Given a Lie algebra $\mathfrak{g}$, if $\mathfrak{g}$ is nilpotent, then all $g \in \mathfrak{g}$ are ad-nilpotent.*

Proof. Let $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}] = 0$. Now, for any $g \in \mathfrak{g}$, we know $[g, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$. So, for any $h \in \mathfrak{g}$, $(\mathrm{ad}_g)^i(h) \subseteq \mathfrak{g}^i$. Therefore, $(\mathrm{ad}_g)^n(h) \in \mathfrak{g}^n = 0 \implies (\mathrm{ad}\, g)^n = 0$. □

3. Engel's Theorem and Consequences

We have now laid out the appropriate groundwork to state and prove Engel's Theorem, which is a somewhat surprising partial converse to the proposition above.

3.1. **Theorem** (Engel's Theorem)**.** *Let $\mathfrak{g}$ be a Lie algebra such that, for all $g \in \mathfrak{g}$, g is ad-nilpotent. Then, $\mathfrak{g}$ is nilpotent.*

To prove Engel's theorem, we will make use of the following lemma, in the style of [Hum72, p 12]

3.2. **Lemma.** *Let $g \in \mathfrak{gl}(V)$ be a nilpotent endomorphism of V, then ad_g is nilpotent.*

Proof of Lemma. Consider that $\mathrm{ad}_g = \lambda_g + \rho_{-g}$ where $\lambda_g \colon \mathfrak{g} \to \mathfrak{g}$ is given by $x \mapsto gx$ and $\rho_{-g} \colon \mathfrak{g} \to \mathfrak{g}$ is given by $x \mapsto -xg$. Now, since g is a nilpotent element of $\mathrm{End}(V)$, it must be that λ_g, ρ_{-g} are nilpotent elements of $\mathrm{End}(\mathrm{End}(V))$. Using the binomial theorem, it is straightforward to prove that, in any ring, the sum of two nilpotent elements is again nilpotent, so ad_g must be nilpotent. □

We will also use the following results to prove Engel's Theorem

3.3. **Theorem.** *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, with $V \neq 0$ finite-dimensional. If all $g \in \mathfrak{g}$ are nilpotent, then there exists $0 \neq v \in V$ such that $\mathfrak{g}.v = 0$.*

Proof. See [Hum72, p 13] □

Proof of Engel's Theorem. We proceed by induction on the dimension of $\mathfrak{g}$. If $\mathfrak{g}$ is 1-dimensional, we are done, since $\mathfrak{g}$ must be abliean. So, assume the result is true for all Lie algebras with dimension less than n and let $\mathfrak{g}$ have dimension n. Since $\mathfrak{g}$ consists of ad-nilpotent elements, then $\mathrm{ad}\, \mathfrak{g} \subseteq \mathfrak{gl}(\mathfrak{g})$ satisfies the above theorem. Thus, there is a $0 \neq x \in \mathfrak{g}$ such that $ad_g x \neq 0$ for all $g \in \mathfrak{g}$, that is to say$[\mathfrak{g}, x] = 0$. Therefore, $Z(\mathfrak{g}) \neq 0$ since $x \in Z(\mathfrak{g})$. However, $\mathfrak{g}/Z(\mathfrak{g})$ must also consist of ad-nilpotent elements and have dimension strictly less than $\mathfrak{g}$, since $Z(\mathfrak{g}) \neq 0$. Thus, $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, which implies that $\mathfrak{g}$ is nilpotent. □

These result have some interesting corollaries which give us insight into the structure of nilpotent Lie algebras.

Add some consequences.

3.4. **Corollary.** *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, with $V \neq 0$ finite dimensional, and all $g \in \mathfrak{g}$ are nilpotent. Then, there exists a flag (V_i) that is stable under $\mathfrak{g}$, with $g.V_i \subseteq V_{i-1}$ for all i. In other words, there is a basis of V relative to which the matrices of $\mathfrak{g}$ are all strictly upper triangular.*

Proof. By Engel's theorem, there is a $v \in V$ such that $\mathfrak{g}.v = 0$. So, proceeding by induction, $V/\operatorname{span}\{v\}$ has such a flag and we are done. □

Thus, we see that any nilpotent Lie algebra will "look like" the Lie algebra of strictly upper triangular matrices with the appropriate choice of basis.

3.5. **Corollary.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra. If $0 \neq I \trianglelefteq \mathfrak{g}$, then $I \cap Z(\mathfrak{g}) \neq 0$. In particular, $Z(\mathfrak{g}) \neq 0$ if $\mathfrak{g}$ is not trivial.*

Proof. This essentially follows from the proof of Engel's theorem given above. By 3.3, there is a $0 \neq x \in I$ such that $[g, x] = 0$ for all $g \in \mathfrak{g}$. Thus, $x \in Z(\mathfrak{g}) \Longrightarrow 0 \neq x \in I \cap Z(\mathfrak{g})$. □

4. Lie's Theorem and Consequences

Engel's theorem provided us with insight into the structure of nilpotent Lie algebras. Solvability is a slightly weaker condition, but Lie's theorem will still give us some useful information. For this section, we must assume that F is algebraically closed and $\operatorname{char} F = 0$. In essence, Lie's theorem is a corollary to the following theorem which most closely resembles 3.3, and its proof will follow the same rough structure.

4.1. **Theorem.** *Let $\mathfrak{g}$ be a solvable sub-algebra of $\mathfrak{gl}(V)$, where $0 \neq V$ is finite dimensional. Then, V contains a common eigenvector for all the endomorphisms in $\mathfrak{g}$.*

Proof. See [Hum72, pp 15–16] □

4.2. **Corollary** (Lie's Theorem)**.** *Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{gl}(V)$, V finite dimensional. Then, $\mathfrak{g}$ stabilizes some flag in $\mathfrak{g}$, that is to say that the matrices of $\mathfrak{g}$ are upper triangular relative to some suitable basis of V.*

4.3. **Remark.** Lie's theorem also holds in prime characteristic provided $\dim V < \operatorname{char} F$.

Proof. Let $\dim V = 1$. Then, $\mathfrak{g}$ trivially stabilizes the flag $0 \subseteq V$. Now, assume the theorem is true for all V with dimension less than n. Then, consider that, by the theorem above, there is a $v \in V$ such that v is an eigenvector for all $g \in \mathfrak{g}$, that is, $g.v = \lambda(g)v$ for some $\lambda\colon \mathfrak{g} \to F$. Then, consider that $\mathfrak{g}$ stabalizes some flag $V/\operatorname{span}\{v\}$ by the inductive hypothesis, say $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} = V/\operatorname{span}\{v\}$. Thus, $\mathfrak{g}$ stabalizes the flag

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} = V/\operatorname{span}\{v\} \subseteq V_n = V.$$

□

Using this result, we can get an interesting fact about the structure of solvable Lie algebras.

4.4. **Corollary.** *Let* $\mathfrak{g}$ *be a solvable Lie algebra. Then, there exist a chain of ideals of* $\mathfrak{g}$*,*

$$0 = \mathfrak{g}_0 \subseteq \mathfrak{g}_1 \subseteq \cdots \subseteq \mathfrak{g}_n = \mathfrak{g}$$

such that $\dim \mathfrak{g}_i = i$*.*

Proof. Let $\mathfrak{g}$ be a solvable Lie algebra and $\phi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{g}$. Then, $\phi(\mathfrak{g})$ must be solvable and, by Lie's theorem, stabilize some flag in V. So, if ϕ is the adjoint representation, then $V = \mathfrak{g}$ and the subspaces of the flag are ideals of $\mathfrak{g}$, each with codimension 1 in the next. □

4.5. **Corollary.** *Let* $\mathfrak{g}$ *be a solvable Lie algebra. Then,* $\mathrm{ad}_{\mathfrak{g}}\, x$ *is nilpotent for any* $x \in [\mathfrak{g}, \mathfrak{g}]$*.*

Proof. $\mathfrak{g}$ has a flag of ideals as in the corollary above. Then, take a basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$ for which $(x_1, \ldots, x_i)$ spans $\mathfrak{g}_i$. Relative to this basis, the matrices of $\mathrm{ad}\, \mathfrak{g}$ must be upper triangular since $\mathfrak{g}.x_i \in \mathrm{span}\{x_1, \ldots, x_i\}$. Thus, ad_g is upper triangular for all $g \in \mathfrak{g}$. Thus, since the Lie algebra of upper triangular matrices has derived subalgebra the Lie algebra of all *strictly* upper triangular matrices, it must be that $[\mathrm{ad}\, \mathfrak{g}, \mathrm{ad}\, \mathfrak{g}] = \mathrm{ad}_{\mathfrak{g}}[\mathfrak{g}, \mathfrak{g}]$ contains only upper triangular matrices. Thus, since every strictly upper triangular matrix is nilpotent, every element of $[\mathfrak{g}, \mathfrak{g}]$ is ad-nilpotent. □

5. Representation Theory of Solvable and Nilpotent Lie Algebras

We can also consider the following:

5.1. **Definition.** A *1-dimensional representation* of a Lie algebra $\mathfrak{g}$ is a linear map $\rho\colon \mathfrak{g} \to \mathrm{End}(\mathbb{C}) \cong \mathbb{C}$ such that $\rho([x, y]) = [\rho(x), \rho(y)]$ for all $x, y \in \mathfrak{g}$.

Then, we have

5.2. **Lemma.** *A linear map* $\rho\colon \mathfrak{g} \to \mathbb{C}$ *is a 1-dimensional representation of* $\mathfrak{g}$ *if and only if* ρ *vanishes on* $[\mathfrak{g}, \mathfrak{g}]$*.*

Proof. This is a straightforward exercise using definitions since $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$ is 1-dimensional as a Lie algebra. □

5.3. **Corollary** (Alternative Lie's Theorem)**.** *Let* $\mathfrak{g}$ *be a solvable Lie algebra and* V *a finite dimensional irreducible* $\mathfrak{g}$*-module. Then* $\dim V = 1$*.*

Proof. Let V be an irreducible $\mathfrak{g}$-module. We know by Lie's theorem that $\mathfrak{g}$ stabalizes some flag $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$. However, V is irreducible, so the only $\mathfrak{g}$-submodule of V is itself and 0. Thus, the flag must be $0 \subseteq V$

and thus V is one-dimensional.

Conversely, one could prove Lie's theorem from this corollary since it implies that every $\mathfrak{g}$-module would contain a 1-dimensional subrepresentation. $\square$

This reinforces the picture that all solvable Lie algebra modules can be given a basis such that the action of $\mathfrak{g}$ is given by an upper-triangular matrix. However, we can say even more about the action of a nilpotent $\mathfrak{g}$, namely,

5.4. **Theorem.** *Let $\mathfrak{g}$ be a nilpotent Lie algebra and V be a $\mathfrak{g}$-module. Let $y \in \mathfrak{g}$ and $\rho(y)\colon V \to V$ be the map $v \mapsto yv$. Then, the generalized eigenspaces of the map $\rho(y)$ are submodules of V.*

Proof. The proof of this fact is rather ugly but relies on the fact that

$$(\rho(y) - (\alpha+\beta)1)^n xv = \sum_{i=0}^{n} \binom{n}{i} ((\operatorname{ad}_y - \beta 1)^i x)((\rho(y) - \alpha 1)^{n-1} v)$$

for $v \in V, x, y \in \mathfrak{g}, \alpha, \beta \in \mathbb{C}$. Then, one takes $\alpha = \lambda_i$ and $\beta = 0$. See [Car05, pp 17–18] for the full proof. $\square$

Essentially, this theorem is saying that, if $\rho(y)$ is put in Jordan-Canonical form, then the action $y \mapsto$ Jordan block of $\rho(y)$ forms a Lie algebra submodule.

6. Cartan's Criterion and Consequences

While Lie's theorem gives us some idea of the structure of a solvable Lie algebra, we still only have the actual definition of a solvable Lie algebra to prove a Lie algebra is, in fact, solvable. Cartan's criterion provides us a way to test a Lie algebra $\mathfrak{g}$ for solvability.

6.1. **Theorem** (Cartan's Criterion)**.** *Let a Lie algebra $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, V finite-dimensional. Then, if $\operatorname{tr}(xy) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$, $\mathfrak{g}$ is solvable.*

To prove this, we will use a somewhat technical lemma from [Hum72, p 19].

6.2. **Lemma.** *Let $A \subseteq B$ be two subspaces of $\mathfrak{gl}(V)$ with $\dim V < \infty$. Set $M = \{x \in \mathfrak{gl}(V) \mid [x, B] \subseteq A\}$. If $x \in M$ satisfies $\operatorname{tr}(xy) = 0$ for all $y \in M$, then x is nilpotent.*

Proof. See [Hum72, p 19]. $\square$

6.3. **Lemma.** *Given $x, y, z \in \mathfrak{gl}_n(F)$, we get*

$$\operatorname{tr}([x, y]z) = \operatorname{tr}(x[y, z])$$

Proof of Cartan's Criterion. If $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent, we know that $\mathfrak{g}$ is solvable since $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}]^{(i)} \subseteq [\mathfrak{g}, \mathfrak{g}]^i$. To do this, we will show that all $x \in [\mathfrak{g}, \mathfrak{g}]$ are nilpotent endomorphisms and appeal to 3.2 followed by Engel's theorem to conclude $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Now, using the first lemma above, take $A = [\mathfrak{g}, \mathfrak{g}]$ and $B = \mathfrak{g}$. Then, $\mathfrak{g} \subseteq M = \{x \in \mathfrak{gl}(V) \mid [x, \mathfrak{g}] \subseteq \mathfrak{g}\}$ trivially. Now, take $[x, y] \in [\mathfrak{g}, \mathfrak{g}]$ and $z \in M$. The second lemma above tells us

$$\operatorname{tr}([x, y]z) = \operatorname{tr}(x[y, z]) = \operatorname{tr}([y, z]x) = 0$$

where the last equality follows from $[y, z] \in M$ and the given hypothesis on trace. Thus, we have satisfied the hypotheses for the first lemma above and get that any generator $[x, y] \in [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

6.4. **Corollary.** *Let $\mathfrak{g}$ be a Lie algebra. If $\operatorname{tr}(\operatorname{ad}_x, \operatorname{ad}_y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.*

Proof. We can apply Cartan's criterion to the adjoint representation of $\mathfrak{g}$ to get that $\operatorname{ad} \mathfrak{g}$ is solvable using the given trace condition. Then, since $\ker \operatorname{ad} = Z(\mathfrak{g})$, by definition of $Z(\mathfrak{g})$, and $Z(\mathfrak{g})$ is an abelian Lie algebra, $\ker \operatorname{ad}$ is also solvable. Thus, we have short exact sequence

$$0 \to Z(\mathfrak{g}) \to \mathfrak{g} \to \operatorname{ad} \mathfrak{g} \to 0$$

with $Z(\mathfrak{g}), \operatorname{ad} \mathfrak{g}$ solvable and so $\mathfrak{g}$ must be solvable. □

This corollary will be useful due to the way the Killing form is defined.

7. The Killing Form

7.1. **Definition.** Given a Lie algebra $\mathfrak{g}$, we define the *Killing form of* $\mathfrak{g}$, denoted $\kappa \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ to be defined by, for $x, y \in \mathfrak{g}$

$$\kappa(x, y) = \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y)$$

7.2. **Proposition.** *(a) The Killing form on a Lie algebra $\mathfrak{g}$ is a symmetric bilinear form on $\mathfrak{g}$.*

(b) The Killing for is associative, that is

$$\kappa([x, y], z) = \kappa(x, [y, z])$$

(c) The radical of the Killing form of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$.

(d) Given $x, y \in I \trianglelefteq \mathfrak{g}$, then

$$\kappa_{\mathfrak{g}}(x, y) = \kappa_I(x, y)$$

that is, the Killing form of $\mathfrak{g}$ restricted to I is the same as the Killing form on I.

(e) If $I \trianglelefteq \mathfrak{g}$, then

$$I^{\perp} := \{g \in \mathfrak{g} \mid \kappa(x, g) = 0, \forall x \in I\} \textit{ is an ideal of } \mathfrak{g}$$

Proof. For (a), observe that, for arbitrary matrices $A, B \in M_n(F)$ and $c \in F$, we have

$$\operatorname{tr}(A+B) = \sum_{i=1}^{n}(A+B)_{i,i} = A_{i,i}+B_{i,i} = \operatorname{tr}(A)+\operatorname{tr}(B) \text{ and } \operatorname{tr}(cA) = \sum_{i=1}^{n} cA_{i,i} = c\sum_{i=1}^{n} A_{i,i} = c\operatorname{tr}(A)$$

and also that

$$\operatorname{tr}(AB) = \sum_{i=1}^{n}(AB)_{i,i} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}B_{j,i} = \sum_{j=1}^{n}\sum_{i=1}^{n} B_{j,i}A_{i,j} = \sum_{j=1}^{n}(BA)_{j,j} = \operatorname{tr}(BA)$$

Thus, it must be that the Killing form is symmetric and bilinear.

For (b), we note

$$\kappa([x,y],z) = \operatorname{tr}(\operatorname{ad}([x,y])\operatorname{ad}(z)) = \operatorname{tr}((xy-yx)z) = \operatorname{tr}(xyz) - \operatorname{tr}(yxz)$$

and also

$$\kappa(x,[y,z]) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}([y,z])) = \operatorname{tr}(x(yz-zy)) = \operatorname{tr}(xyz) - \operatorname{tr}(xzy).$$

Since $\operatorname{tr}(y(xz)) = \operatorname{tr}((xz)y)$, we are done.

For (c), recall that the radical of κ is given by

$$S := \{x \in \mathfrak{g} \mid \kappa(x,y) = 0, \forall y \in \mathfrak{g}\}$$

Then, since κ is bilinear, it is clear that, for $x, z \in S$, $x+z \in S$. Furthermore, for $g \in \mathfrak{g}$ and $x \in S$, we get, using the lemma above,

$$\kappa([g,x],y) = -\kappa(x,[g,y]) = 0 \Longrightarrow [g,x] \in S$$

Thus, S is an ideal.

For (d), pick a basis for I and extend it to a basis of $\mathfrak{g}$. Then, since $x, y \in I$, the linear maps are given by

$$\operatorname{ad}_x = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{ad}_y = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}$$

Thus, is matrix form

$$\operatorname{ad}_x\operatorname{ad}_y = \begin{pmatrix} A_1B_1 & A_2B_2 \\ 0 & 0 \end{pmatrix} \Longrightarrow \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}_x\operatorname{ad}_y) = \operatorname{tr} A_1B_1 = \operatorname{tr}_I(\operatorname{ad}_x\operatorname{ad}_y)$$

Thus, $\kappa_{\mathfrak{g}}(x,y) = \kappa_I(x,y)$.

For (e), if $g \in \mathfrak{g}$ and $x \in I^\perp$, we wish to show $[x,g] \in I^\perp$. Given $y \in I$, we check

$$\kappa([x,g],y) = \kappa(x,[g,y]) = 0 \text{ since } [g,y] \in I$$

Thus, $[x,g] \in I^\perp \Longrightarrow I^\perp \trianglelefteq \mathfrak{g}$. □

However, the most useful fact about the Killing form is the following. To prove it, we will use the following lemma.

7.3. **Theorem.** *Let $\mathfrak{g}$ be a Lie algebra. Then, $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.*

Proof. ($\Longrightarrow$) Let $\operatorname{Rad}\mathfrak{g} = 0$. Then, we wish to show the ideal given by the radical of κ, say S, is contained in $\operatorname{Rad}\mathfrak{g}$. Now, by Cartan's criterion, $\operatorname{ad}_{\mathfrak{g}} S$ must be solvable since $\operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y) = 0$ for all $x \in S, y \in \mathfrak{g}$ and therefore, since $\ker \operatorname{ad}$ is solvable, S is solvable. Therefore, $S \subseteq \operatorname{Rad}\mathfrak{g} = 0$, so $S = 0$.

($\Longleftarrow$) Let $S = 0$. We wish to show that every abelian ideal of $\mathfrak{g}$ is contained in S. Let $x \in I$, an abelian ideal of $\mathfrak{g}$, and $y \in \mathfrak{g}$. Then, $\operatorname{ad} x \operatorname{ad} y \colon \mathfrak{g} \to I$ and thus $(\operatorname{ad} x \operatorname{ad} y)^2 \colon \mathfrak{g} \to I \to [I, I] = 0$. Thus, $\operatorname{ad} x \operatorname{ad} y$ is a nilpotent linear transformation and so its only eigenvalues are 0. So,

$$0 = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = \kappa(x, y) \Longrightarrow I \subseteq S = 0.$$

Therefore, $\mathfrak{g}$ has no nontrivial abelian ideals. Furthermore, nontrivial $\operatorname{Rad}\mathfrak{g}$ always contains a nontrivial abelian ideal by 2.14. So, $\mathfrak{g}$ must therefore be semisimple. $\square$

7.4. **Remark.** Note that, in ($\Longrightarrow$) direction of the proof, we did not use $S = 0$ to show $S \subseteq \operatorname{Rad}\mathfrak{g}$, so this is always true.

7.5. **Example.** Let $\mathfrak{g} = \mathfrak{sl}_2$ with standard basis $\beta = \{e, h, f\}$. Then, we wish to compute the dual basis to β under the Killing form. Consider that $A = (\kappa(\beta_i \beta_j))_{i,j}$ has

$$A = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

and so its non-zero determinant tells us the form is nonzero and that the dual basis γ is given by

$$A^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{8} & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}$$

So, $\kappa(\beta_i, \gamma_j) = e_i^t A(A^{-1} e_j) = \delta_{ij} \Longrightarrow \gamma = \{\frac{1}{4} f, \frac{1}{8} h, \frac{1}{4} e\}$.

7.6. **Theorem.** *If the Killing form of $\mathfrak{g}$ is identically zero, then $\mathfrak{g}$ is solvable.*

Proof. In this case, $\mathfrak{g} = \operatorname{Rad}\kappa$ by definition, and by the remark above, $\operatorname{Rad}\kappa = S \subseteq \operatorname{Rad}\mathfrak{g}$, so $\mathfrak{g}$ is itself solvable. $\square$

Check this proof; seems too easy

Using the Killing form, we can then arrive at the following characterization of semisimple Lie algebras, mirroring the one for semisimple Artinian rings.

7.7. **Theorem.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is isomorphic to a direct sum of non-trivial simple Lie algebras.*

Proof. First, assume $\mathfrak{g}$ is semisimple. If $\mathfrak{g}$ is nontrivial and simple, we are done, so assume $\mathfrak{g}$ is not simple. Then, consider a minimal non-zero ideal $I \trianglelefteq \mathfrak{g}$. Now, $I^\perp$ is also an ideal and, since κ is non-degenerate, $I \cap I^\perp = \varnothing$. Thus,

$$\mathfrak{g} = I \oplus I^\perp$$

Thus, we must show that I is a *simple* Lie algebra. Take $J \trianglelefteq I$. Then, it must be

This seems too fast. Carter spends much more time to do this.

$$[J, \mathfrak{g}] \subseteq [J, I] + [J, I^\perp] \subseteq [J, I] + [I, I^\perp] \subseteq J$$

Thus, $J \trianglelefteq \mathfrak{g}$. However, I is minimal, so $J = I$ or $J = 0$, so I is simple.

Next, we wish to see that $I^\perp$ is semisimple by showing all its solvable ideals are trivial. If $J \trianglelefteq I^\perp$ is solvable, then

$$[J, \mathfrak{g}] \subseteq [J, I] + [J, I^\perp] \subseteq [J, I^\perp] \subseteq J$$

and so $J \trianglelefteq \mathfrak{g} \implies J = 0$ since $\mathfrak{g}$ is semisimple. Thus, $I^\perp$ is semisimple.

For the converse, suppose

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

where each $\mathfrak{g}_i$ is non-trivial and simple. We wish to show that the Killing form on $\mathfrak{g}$ is non-degenerate. Each $\mathfrak{g}_i$ has non-degenerate Killing form. Take $x \in \operatorname{Rad} \kappa \subseteq \mathfrak{g} \implies x = x_1 + \cdots + x_k$ with $x_i \in \mathfrak{g}_i$. For $y_i \in \mathfrak{g}_i$, we have

$$\langle x_i, y_i \rangle = \langle x, y_i \rangle = 0 \implies x_i = 0$$

Thus, $x = 0$ since the above is true for all i. Therefore, $\operatorname{Rad} \kappa = 0 \implies \mathfrak{g}$ is semisimple. □

Thus, in summary, we have shown,

7.8. **Theorem.** *Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent.*

(a) $\mathfrak{g}$ is semisimple, that is, $\operatorname{Rad} \mathfrak{g} = 0$.
(b) $\mathfrak{g}$ does not contain any non-trivial, abelian ideals.
(c) The Killing form is non-degenerate on $\mathfrak{g}$.
(d) $\mathfrak{g}$ is isomorphic to the direct sum of finitely many non-trivial simple Lie algebras.

Proof. $(a) \implies (b)$ is immediate since any abelian ideal is also a solvable ideal, but the maximal solvable ideal must be trivial. For $(b) \implies (a)$, let $\mathfrak{g}$ have no non-trivial, abelian ideals. Then, $\mathfrak{g}$ does not have any solvable ideals, since the last ideal of a derived series must be abelian.

$(a) \iff (c)$ is 7.3
$(a) \iff (d)$ is 7.7 □

8. Further Directions

In light of the above discussion, we see that semisimple Lie algebras break up into a direct sum of simple Lie algebras. Furthermore, in a strengthening of the Levi decomposition presented above (2.16), one gets that an arbitrary Lie algebra in characteristic 0 breaks up as $\mathfrak{g} = \mathfrak{l} \ltimes \operatorname{rad} \mathfrak{g}$ where $\mathfrak{l} \cong \mathfrak{g}/\operatorname{rad} \mathfrak{g}$ is a semisimple subalgebra of $\mathfrak{g}$ called a *Levi subalgebra*. The existence of Levi subalgebras was proved by Levi in 1905 and Malcev proved that any two Levi subalgebras are conjugate in 1942; the combination of these results is sometimes called the Levi-Malcen Theorem.

This could be expanded upon. [Mil13] has more details.

Thus, in characteristic 0, if we could classifly *all* simple Lie algebras, we would understand all the possible pieces an arbitrary Lie algebra could decompose into. An incredibly important tool for doing this is the representation theory of semisimple Lie algebras. In fact, using this representation, mathematicisna developed a beautiful and complete classification of simple Lie algebras.

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