

Diagonal Harmonics and Shuffle Theorems

George H. Seelinger

ghseeli@umich.edu

joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
ISU Algebra Seminar

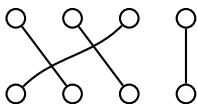
27 October 2022

Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

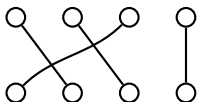
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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


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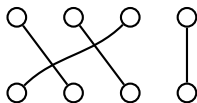
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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \text{Diagram}$$


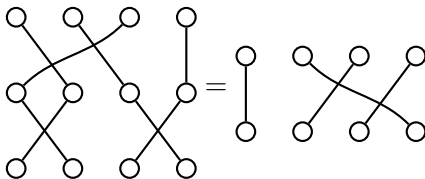
- Stacking = composition

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Polynomials

- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

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$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the e_1, e_2, \dots , or in the h_1, h_2, \dots

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Partitions

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5 \rightarrow

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$4 + 1 \rightarrow$

$3 + 2 \rightarrow$

$3 + 1 + 1 \rightarrow$ 

$2 + 2 + 1 \rightarrow$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

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Collection is called $\text{SSYT}(\lambda)$.

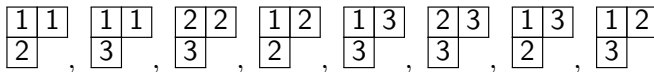
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For $\lambda = (2, 1)$,

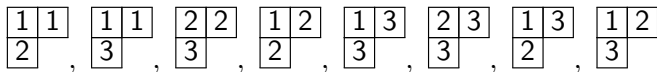


Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

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Weight:

1	1
2	

,

1	1
3	

,

2	2
3	

,

1	2
2	

,

1	3
3	

,

2	3
3	

,

1	3
2	

,

1	2
3	

(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

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Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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① S_3 action on M fixes vector subspaces!

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- 2 Break M up into smallest S_n fixed subspaces

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$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

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Via Frobenius characteristic map, questions about S_n -action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

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Via Frobenius characteristic map, questions about S_n -action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

Getting more information

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Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

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Answer: "Hall-Littlewood polynomial" $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

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- Does there exist a family of S_n -representations whose (bigraded) Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$. (Still open!)

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n ?

Frobenius characteristic of DH_3

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n .

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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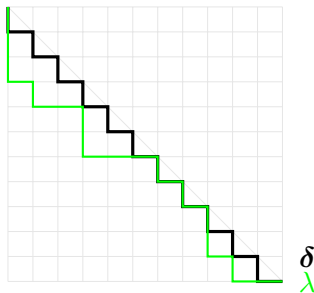
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- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

Dyck paths

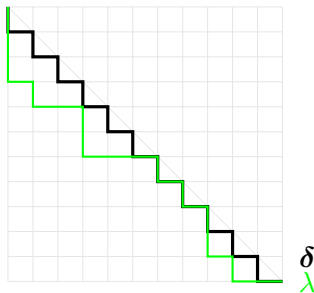
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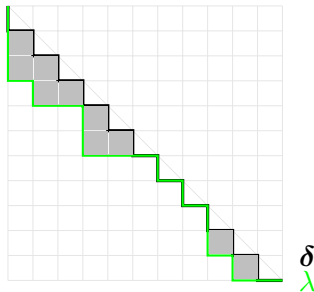


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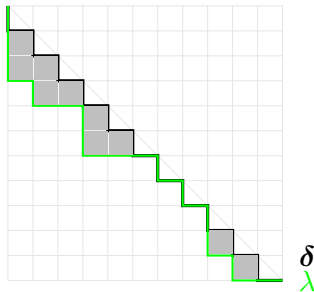


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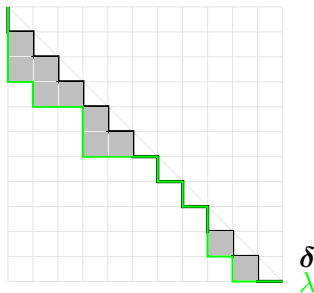


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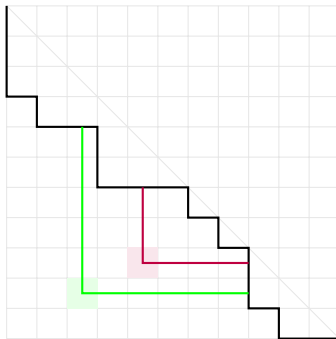
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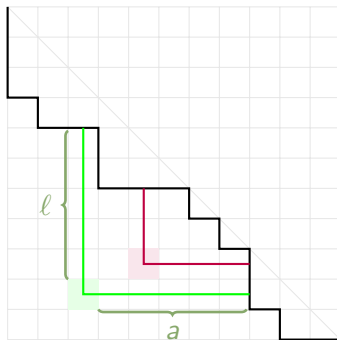
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dinv

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Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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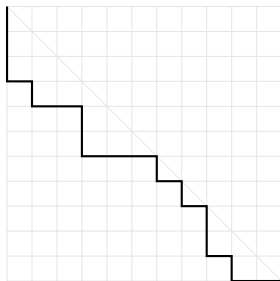
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- \mathcal{G}_ν is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

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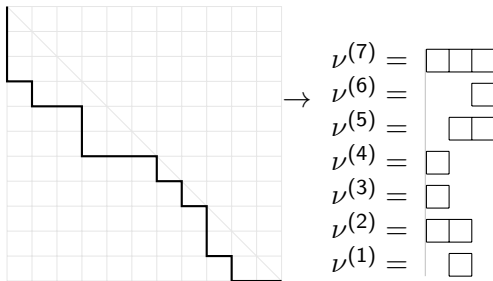
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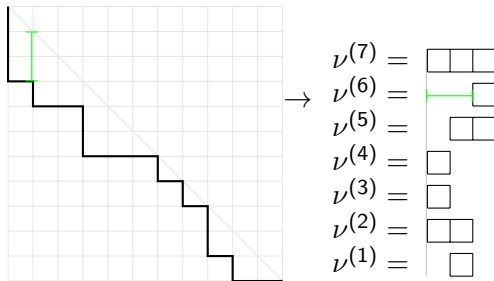
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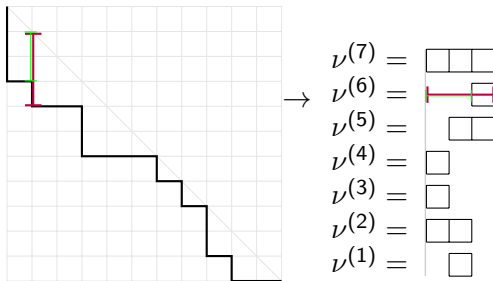
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for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

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$$T = \begin{array}{cccccc} 1 & 1 & 6 & 7 & 7 & 7 \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

1 2 3 3 5

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$$\mathcal{G}_{\begin{array}{cc} \square & \square \\ \square & \end{array}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{cc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

Example ∇e_3






$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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




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




Example ∇e_3

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
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	$q^2 t$	
	qt	
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- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number”
 $(q^3 + q^2t + qt + qt^2 + t^3)$.

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When a problem is too difficult, try generalizing!

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For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

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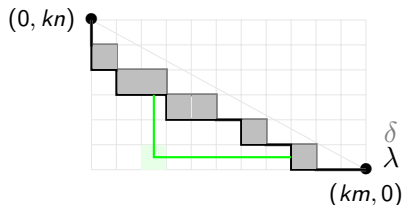
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For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

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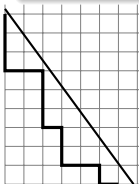
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$\text{area}(\lambda)$ as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks } \frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

Proof Overview (algebraic side)

- $\psi: \mathcal{E}^+ \cong S$
- \mathcal{E}^+ is the “positive half” of \mathcal{E}
- S is an algebra of symmetric Laurent series in $\mathbb{Q}(q, t)(z_1^{\pm 1}, \dots, z_l^{\pm 1})^{S_l}$ satisfying extra conditions and equipped with a “shuffle product”.

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Key relationship

For $\xi \in \mathcal{E}^+$,

$$\omega(\xi \cdot 1) = \text{pol}_X(\psi(\xi))$$

for automorphism $\omega: \Lambda \rightarrow \Lambda$ and $\text{pol}_X: S \rightarrow \Lambda$ a “polynomial truncation” operation.

Proof Overview (combinatorial side)

- For $\xi = D_{\mathbf{b}}$, we get

$$\text{pol}_X \mathbf{H}_q \left(\frac{z^{\mathbf{b}} \prod_{i < j+1} (1 - qtz_i/z_j)}{\prod_{i < j} (1 - t z_i/z_j)} \right) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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Need an “infinite series” version of LLT polynomials!

Cauchy identity

- For a fixed $\sigma \in S_I$, there exists a basis of $\mathbb{Q}(q)[z_1^{\pm 1}, \dots, z_I^{\pm 1}]$ called “non-symmetric Hall-Littlewood polynomials”, denoted $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \dots, z_I; q)$ for $\lambda \in \mathbb{Z}^I$.

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- (Grojnowski-Haiman 2007) defines a (symmetric) “series LLT” polynomial $\mathcal{L}_{\beta/\alpha}^\sigma(x_1, \dots, x_I; q) = H_q(w_0(F_\beta^{\sigma^{-1}} \overline{E_\alpha^{\sigma^{-1}}}))$

Stable Shuffle Theorem (BHMPs 21a)

For $\mathbf{b} \in \mathbb{Z}^I$ corresponding to highest path under a line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1, \dots, a_{I-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_I, \dots, b_1) + (0, a_{I-1}, \dots, a_1)) / (a_{I-1}, \dots, a_1, 0)}^{\sigma}(x_1, \dots, x_I; q)$$

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Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

$$\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} (1 + zt^{-r_i(\lambda)}).$$

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b) \in \mu} q^{a-1} t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \hline \end{array} \rightarrow B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu$
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- $\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \sum_{\substack{s \in \mathbb{N}^{k+r}: |s|=n-k \\ 1 \in J \subseteq [k+r], |J|=k}} (D_{s+\epsilon_J} \cdot 1)$

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R) \in \operatorname{LNDP}_\mu} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

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Generalizing our methods further, we arrive at the following.

Theorem (BHMP21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_{\pi} t^{a(\pi)} q^{\operatorname{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

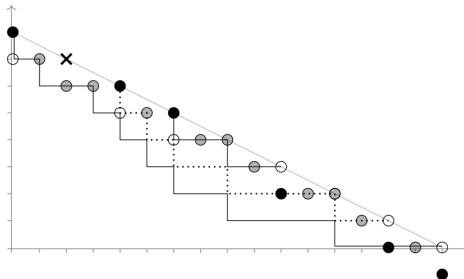
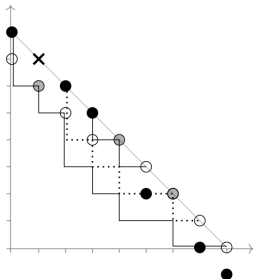
For a sum over all “nests” π in a “den” corresponding to s_μ .

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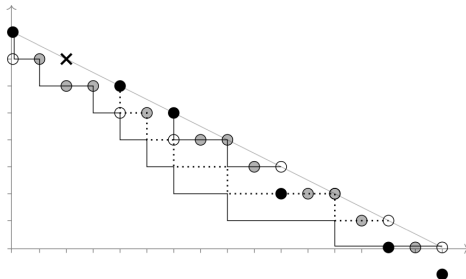
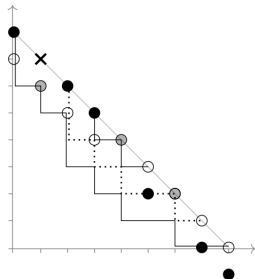


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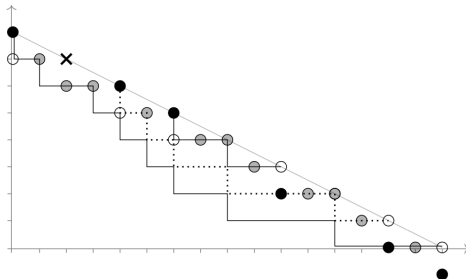
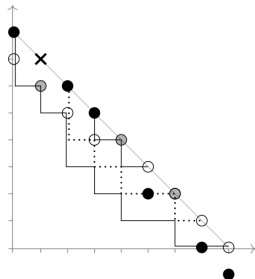
- Implies the Loehr-Warrington Conjecture as a special case.

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For a sum over all “nests” π in a “den” corresponding to s_μ .



- Implies the Loehr-Warrington Conjecture as a special case.
- Also proves $\text{sgn}(\mu) \nabla s_\mu$ is Schur-positive.

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^I$. When is $D_{\mathbf{b}} \cdot 1$ nice?

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Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

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- Experimental computation suggests this is “tight.”
- Coefficient of $s_{1, \dots, 1}$ coincides with (q, t) -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?

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- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)
- What can we say about Macdonald polynomials?
- S_I -representation theory interpretations?

References

Thank you!

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