K-theoretic Catalan functions

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Junior Mathematician Research Archive

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Overview

1. Schubert calculus

2. Catalan functions

3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

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Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for combinatorially understood Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

- Complete homogeneous symmetric function: for $r \in \mathbb{Z}$,
  $h_r = \sum_{i_1 \le \cdots \le i_r} x_{i_1} \cdots x_{i_r}$.

Schur polynomials and raising operators

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- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

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- Schur function $s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$ (Jacobi-Trudi)

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K-theory and K-homology of the affine Grassmannian

Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

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2. Homology classes of affine Grassmannian represented by k-Schur functions ($t = 1$).

3. (Lam et al., 2010) leave open the question: what is a direct formulation of the K-homology representatives of the affine Grassmannian (K-k-Schur functions)?
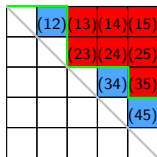
Remember?

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Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).

$\Psi =$ Roots above Dyck path
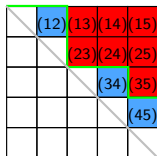$\Delta_\ell^+ \backslash \Psi =$ Non-roots below

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For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_\ell^+ \backslash \Psi} (1 - R_{ij}) h_\gamma(x)$$

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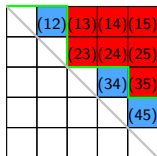$\Delta_\ell^+ \backslash \Psi = $ Non-roots below

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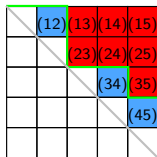## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

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- $\Psi = $ all roots $\implies H(\Psi; \gamma) = h_\gamma$

Catalan functions

k-Schur root ideal for λ

For $k \in \mathbb{Z}_{\geq 0}$ and $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell) \in \mathbb{Z}^\ell$,

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
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$$\Delta^4(3,3,2,2,1,1) = \begin{array}{} \text{(grid)} \end{array} \qquad \leftarrow \text{row } i \text{ has } 4 - \lambda_i \text{ non-roots}$$

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k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

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Remark

 (Blasiak et al., 2019) show results for k-Schur functions with parameter t, but $t = 1$ specialization is necessary for Schubert calculus.

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Lowering Operators

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  - K-theory of Grassmannian: $g_\lambda = \prod_{i<j}(1 - R_{ij})k_\lambda$ and
  - Homology of affine Grassmannian: $s_\lambda^{(k)} = \prod_{(i,j)\in\Delta^+\setminus\Delta^k(\lambda)}(1 - R_{ij})h_\lambda$
- Extra ingredient: lowering operators $L_j(h_\lambda) = h_{\lambda-\epsilon_j}$

$$L_3\left(\boxed{}\right) = \boxed{} \quad L_1\left(\boxed{}\right) = \boxed{}$$

Affine K-Theory Representatives with Raising Operators

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$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ in blue, roots of $\mathcal{L}$ marked with •

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2$$
$$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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$f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$ since this family has the correct structure constants.

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Example

$$g_{332111}^{(4)} =$$

$\Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$

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Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ "branching coefficients" are alternating by degree, i.e. the $b_{\lambda\mu}^{(k)}$ in

$$g_\lambda^{(k)} = \sum_\mu b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

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Future Directions

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1. Combinatorially describe dual "Pieri rule":
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3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$.

References

Thank you!

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