Flag Varieties
Notes from a reading course in Spring 2019

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1. Introduction (presented by Weiqiang Wang)

These notes will start with a quick introduction to vector bundles and Chern classes following [**BT82**, Sections 20–21,23]. Then, they will shift to a discussion of equivariant cohomology, loosely following [**Ful07**]. Note that citations may be incomplete because the author is basing these notes off of lectures presented in a reading course organized by Weiqiang Wang.

2. A Quick Introduction to Vector Bundles (not presented)

This section will list some facts about vector bundles without proof. For a more comprehensive treatment, see references such as [**BT82**] and [**MS74**]. The reader may skip this section and refer back to it as necessary.

2.1. Definition. A rank n *(complex) vector bundle* ξ, consists of

(a) a topological space E called *the total space*,
(b) a topological space B called *the base space*,
(c) and a continuous projection map $\rho\colon E \to B$ such that
  (i) for all $b \in B$, the preimage $\rho^{-1}(b)$, which is called the *fiber over* b is a vector space, and
  (ii) (local triviality) for all $b \in B$, there exists a neighborhood $U \subseteq B$ with $b \in U$ and a homeomorphism $h\colon U \times \mathbb{C}^n \to \rho^{-1}(U)$ (called a *local trivialization*) such that, for all $x \in U$, $h|_{\{x\}\times\mathbb{C}^n}\colon \mathbb{C}^n \to \rho^{-1}(x)$ is a vector space isomorphism.

$$\begin{tikzcd} U \times \mathbb{C}^n \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & \rho^{-1}(U) \arrow[dl, "\rho"] \\ & U & \end{tikzcd}$$

2.2. Remark. (a) One can just as easily define a *real vector bundle* by simply replacing $\mathbb{C}$ with $\mathbb{R}$ everywhere in the definition.
(b) One can define a vector bundle where the $\rho^{-1}(b) \cong \mathbb{C}^n$ and $\rho^{-1}(b') = \mathbb{C}^m$ for $n \neq m$ if b and b' are in different connected components of B. However, for simplicity, we will usually keep the rank of every fiber the same.
(c) If $b, b' \in B$ are in the same connected component, then the local triviality condition forces that $\rho^{-1}(b) \cong \mathbb{C}^n \cong \rho^{-1}(b')$.

2.3. Example. (a) Perhaps the most familiar example of a (smooth) real vector bundle is that of a "tangent bundle" of a manifold. That is, given an n-dimensional smooth manifold M, we have a $2n$-dimensional manifold $TM := \{(x, v) \in M \times \mathbb{R}^n \mid v \in T_xM\}$ where T_xM is the tangent space of x and map $\pi_1\colon TM \to M$ given by $\pi_1(x, v) = x$. Then, we have tangent bundle $\tau(M)$ given by

$$\begin{tikzcd} TM \arrow[d, "\pi_1"] \\ M \end{tikzcd}$$

with fiber $\pi_1^{-1}(x) = T_xM \cong \mathbb{R}^n$. Similarly, the "normal bundle" $\nu(M)$ is a vector bundle.

(b) Given a space X, we have the *trivial bundle* ϵ_n given by

$$\begin{tikzcd} X \times \mathbb{C}^n \arrow[d, "\pi_1"] \\ X \end{tikzcd}$$

with $\pi_1^{-1}(x) = \mathbb{C}^n$. If X is a single point, then the trivial bundle is essentially just the vector space $\mathbb{C}^n$. In this way, we see that vector bundles serve as a generalization of vector spaces. A particular example of a real trivial bundle would be

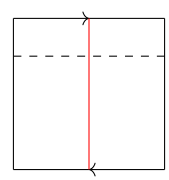
$$\begin{tikzcd} S^1 \times \mathbb{R} \arrow[d, "\pi_1"] \\ S^1 \end{tikzcd}$$

where $S^1 \times \mathbb{R}$ is a cylinder.

(c) Given a Möbius strip M, we have a real vector bundle

$$\begin{tikzcd} M \arrow[d, "\rho"] \\ S^1 \end{tikzcd}$$

where $\rho\colon M \to S^1$ sends a point in the Möbius strip to the circle embedded along the "middle". For instance, every point on the dashed line is sent by ρ to the point where the dashed line intersects the red line in the picture below.



The fiber is given by $\rho^{-1}(x) = (0, 1)$ which is homeomorphic to $\mathbb{R}$. Furthermore, this example is not equivalent to the trivial bundle $S^1 \times \mathbb{R} \to S^1$. Informally, this can be seen by noticing that one can pick a consistent choice of orientation on the trivial bundle but not on this bundle.

(d) Given a space $B = \mathbb{CP}^n$, the points $x \in \mathbb{CP}^n$ are lines. Then, we define the *tautological line bundle* γ_n^1 to be given by total space $E = \{(x, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \text{ lies on } x\}$ and map $\rho(x, v) = x$. Then, $\rho^{-1}(x) = \{x\} \times \mathbb{C}v \cong \mathbb{C}$.

(e) If we do the tautological line bundle construction over $\mathbb{RP}^1 \cong S^1$, we recover the Möbius strip example above. To see this in detail, see [**MS74**, pp 16–17]

2.4. Definition. A *line bundle* is a rank 1 vector bundle.

2.5. Proposition. *Given a rank n vector bundle ξ with base space B and total space E and $\rho\colon E \to B$, we can construct new vector bundles in a variety of ways.*

(a) *Given some map $f\colon B' \to B$, we can construct the* pullback bundle *$f^*(\xi)$ over B' by setting $f^*(E) = \{(x', v) \in B' \times E \mid f(x) = \rho(e)\}$ and $\rho'\colon f^*(E) \to B$ to be given by $\rho'(x', e) = x'$ so that $(\rho')^{-1}(x') = \rho^{-1}(f(x'))$. This is a specific version of a pullback square:*

$$\begin{tikzcd} f^*(E) \arrow[r, "\tilde{f}"] \arrow[d, "\rho'"] & E \arrow[d, "\rho"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

(b) *We can take the* Whitney sum *of two vector bundles over B induced by taking the pullback bundle of the diagonal inclusion $\iota\colon B \to B \times B$ that sends $b \mapsto (b, b)$ and this induces fibers $(\rho')^{-1}(b) \cong \rho_1^{-1}(b) \oplus \rho_2^{-1}(b)$.*

$$\begin{tikzcd} i^*(E_1 \times E_2) \arrow[r, "\tilde{\iota}"] \arrow[d, "\rho'"] & E_1 \times E_2 \arrow[d, "\rho_1 \times \rho_2"] \\ B \arrow[r, "\iota"] & B \times B \end{tikzcd}$$

We will denote this as $\xi_1 \oplus \xi_2$.

(c) *Any operation that makes sense to be done on a vector space can also be done on two vector bundles over B to get a vector bundle over B.*

Flush out with some explcit constructions.

(d) *Given a complex vector bundle ξ of rank n, one can get a rank $2n$ real vector bundle $\xi_{\mathbb{R}}$ by simply forgetting the complex structure.*

2.6. Definition. Given a vector bundle ξ, a *section* is a map $s\colon B \to E$ such that $\rho \circ s = Id_B$.

2.7. Example. (a) For every vector bundle, there exists a *zero section* $s_0\colon B \to E$ defined by $s_0(b) = (b, 0)$.

(b) In general, there are uncountably many sections whose images intersect the image of the zero section. These are called *vanishing sections* and are relatively uninteresting.

(c)

Add something not boring.

More generally, we wish to define a generalization of vector bundles that we will use sparringly.

2.8. Definition. A *fiber bundle* ξ with total space E, base space B, projection map $\rho\colon B \to E$, and fiber F is defined analogously to a complex vector bundle where every occurrence of $\mathbb{C}^n$ is replaced by F.

See [**Hat01**, pp 376–377] for a more precise definition. One reason to study vector bundles is that they can give us insight into the cohomology

rings of the spaces involved. A general theorem which we will state without proof is

2.9. Theorem (Leray-Hirsch). [**Hat01**, Theorem 4D.1] *Let ξ be a fiber bundle with $\rho\colon E \to B$ such that, for some coefficient ring R,*

(a) $H^n(F;R)$ is a finitely generated free R-module for each n.
(b) There exist classes $e_j \in H^{k_j}(E;R)$ who restrictions $u^*(e_j)$ form a basis for $H^*(F;R)$ in each fiber F, where $i\colon F \to E$ is the inclusion.

Then, the map $\Phi\colon H^*(B;R) \otimes_R H^*(F;R) \to H^*(E;R)$ given by

$$\sum_{i,j} b_i \otimes i^*(e_j) \mapsto \sum_{i,j} \rho^*(b_i) \smile e_j$$

is an isomorphism of R-modules.

In particular, the Leray-Hirsch theorem tells us that $H^*(E;R)$ is a free $H^*(B;R)$-module with basis $\{e_j\}$ where we view $H^*(E;R)$ as a module over the ring $H^*(B;R)$ by defining the action

$$b.e = \rho^*(b) \smile e$$

for $b \in H^*(B;R)$ and $e \in H^*(E;R)$.

2.10. Definition. In the discussion that follows, for any vector space V, let $V_0 = V \setminus \{0\}$ and for any total space E, let $E_0 = E \setminus s_0(B)$ where s_0 is the zero section.

2.11. Definition. [**MS74**, p 96] An *orientation* of a real vector bundle ξ is a function which assigns an orientation to each fiber F of ξ such that for every point $b_0 \in B$, there is a neighborhood U containing b_0 with local trivialization $h\colon U \times \mathbb{C}^n \to \rho^{-1}(U)$ such that for each fiber $F = \rho^{-1}(b)$ for $b \in B$, the homomorphism $h_b\colon \mathbb{C}^n \to F$ given by $h_b(x) := h(b,x)$ is orientation preserving.

This definition tells us that, for an orientable rank n complex vector bundle ξ, for each fiber $F \cong \mathbb{R}^n$, we can pick a preferred generator $u_F \in H^n(F, F_0; \mathbb{Z}) = \mathbb{Z}$. Then, the local conditions in the definition of orientation imply that, for every $b_0 \in B$, there is a neighborhood U and a cohomology class $u \in H^n(\rho^{-1}(U), \rho^{-1}(U)_0; \mathbb{Z})$ so that for every fiber $F = \rho^{-1}(b)$ over U,

$$\iota^*(u) = u_F \in H^n(F, F_0; \mathbb{Z})$$

where $\iota\colon \rho^{-1}(b) \hookrightarrow \rho^{-1}(U)$.

2.12. Theorem. [**MS74**, Theorem 9.1] *Let ξ be an oriented real vector bundle of rank n with total space E. Then,*

(a) $H^i(E, E_0; \mathbb{Z}) = 0$ for $i < n$ and
(b) $H^n(E, E_0; \mathbb{Z})$ contains a unique cohomology class u, called the Thom class*, whose restriction*

$$\iota^*(u) = u_F \in H^n(F, F_0; \mathbb{Z})$$

for every fiber F of ξ where $\iota\colon F \hookrightarrow E$ is the standard inclusion.

(c) *The correspondence $y \mapsto y \smile u$ maps $H^k(E;\mathbb{Z})$ isomorphically to $H^{k+n}(E, E_0;\mathbb{Z})$ for every integer k.*

2.13. Definition. For an oriented real vector bundle ξ, the *Thom isomorphism*

$$\phi\colon H^k(B;\mathbb{Z}) \xrightarrow{\sim} H^k(E, E_0;\mathbb{Z})$$

is given by the formula

$$\phi(x) := \rho^* x \smile u$$

We will take for granted that this is an isomorphism of abelian groups. Now, the inclusion $(E, \varnothing) \hookrightarrow (E, E_0)$ induces a restriction $H^*(E, E_0;\mathbb{Z}) \to H^*(E;\mathbb{Z})$ which we will denote by $y \mapsto y|_E$. Then, for $u \in H^n(E, E_0;\mathbb{Z})$, we can take

$$u \mapsto u|_E \in H^n(E;\mathbb{Z}) \cong H^n(B;\mathbb{Z})$$

via the Thom isomorphism above. Thus, we define

2.14. Definition. The *Euler class* of an oriented rank n vector bundle ξ is the cohomology class

$$e(\xi) \in H^n(B;\mathbb{Z})$$

which corresponds to $u|_E$ under the Thom isomorphism $H^n(B;\mathbb{Z}) \cong H^n(E;\mathbb{Z})$.

2.15. Proposition. (a) [**MS74**, Property 9.2] *If $f\colon B \to B'$ is continuous and covered by an orientation preserving map $f\colon \xi \to \xi'$, then $e(\xi) = f^* e(\xi')$.*

(b) [**MS74**, Property 9.6] *The Euler class of a Whitney sum is given by $e(\xi \oplus \xi') = e(\xi) \smile e(\xi')$.*

(c) $e(\xi \otimes \xi') = e(\xi) + e(\xi')$.

(d) [**MS74**, Property 9.7] *If the oriented real vector bundle ξ possesses a nowhere zero section, then $e(\xi)$ must be zero*

3. Chern Classes (presented by Thomas Sale)

The material in this section will roughly correspond to [**BT82**, Sections 20–21]. Because of this, throughout this section all cohomology is de Rham cohomology and thus over $\mathbb{R}$. All spaces can be assumed to be smooth manifolds.

3.1. Remark. As one gets more accustomed to using vector bundles, typically one stops denoting the bundle by something like ξ and just denotes it by the total space E when the bundle structure is clear.

3.1. The first Chern class of a line bundle.

3.2. Definition. Given a complex line bundle ξ, we define the *first Chern class of* ξ to be $c_1(\xi) := e(\xi_{\mathbb{R}})$.

3.3. Proposition. *Given line bundles* ξ, ξ',

(a) $c_1(\xi \otimes \xi') = c_1(\xi) + c_1(\xi')$.
(b) $0 = c_1(\xi \otimes \xi^*) = c_1(\xi) + c_1(\xi^*) \Longrightarrow c_1(\xi^*) = -c_1(\xi)$.

Proof. The first part is immediate from 2.15(c) and the second follows from the fact that $\xi \otimes \xi^* = \mathrm{Hom}(\xi, \xi)$ always has a nonvanishing section sending $b \in B$ to the identity map in $\mathrm{Hom}(E, E)$ and so $e(L_{\mathbb{R}}) = 0$. $\square$

3.2. Projectivized vector bundles and Chern clases.

3.4. Definition. (a) Given a rank n complex vector bundle ξ, we define the *projectivized bundle* $\mathbb{P}(\xi)$ to be the bundle

$$\begin{tikzcd} \mathbb{P}(E) \arrow[d, "\pi"] \\ B \end{tikzcd}$$

where $\pi^{-1}(b) = \mathbb{P}(\rho^{-1}(b)) \cong \mathbb{P}(\mathbb{C}^n) \cong \mathbb{CP}^{n-1}$. Note that $\pi \circ \mathbb{P} = \rho$ and that a point $x \in \mathbb{P}(E)$ is a line in $\rho^{-1}(\pi(x))$.

$$\begin{tikzcd} & E \arrow[dl, dashed, "\mathbb{P}"'] \arrow[d, "\rho"] \\ \mathbb{P}(E) \arrow[r, "\pi"] & B \end{tikzcd}$$

(b) We define the pullback vector bundle $\pi^*(\xi)$ of the projectivization map $\pi\colon \mathbb{P}(E) \to M$ and the fiber $(\rho')^{-1}(x) = \rho^{-1}(\pi(x))$.

$$\begin{tikzcd} \pi^*(E) \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho'"] & E \arrow[d, "\rho"] \\ \mathbb{P}(E) \arrow[r, "\pi"] & B \end{tikzcd}$$

(c) The *universal sub-bundle* of $\pi^*(\xi)$ is defined by taking the total space

$$S = \{(x, v) \in \pi^*(E) \mid v \in x\}$$

and is a copy of the tautological line bundle over $\mathbb{P}(E)$ (see 2.3(d)) and so S is a rank 1 vector bundle.

(d) The *universal quotient bundle* Q of $\pi^*(E)$ is defined by taking the short exact sequence

$$0 \to S \to \pi^* E \to Q \to 0$$

Then, Q is a rank $n-1$ vector bundle.

(e) If we restrict the unviversal sub-bundle S to be over a fiber $\pi^{-1}(b) \cong \mathbb{CP}^{n-1}$, then we have a bundle with total space $\tilde{S}$ which is a copy of the tautological line bundle over $\mathbb{CP}^{n-1}$.

Now, recall the fact that $H^*(\mathbb{P}(\pi^{-1}(b)); \mathbb{R}) = H^*(\mathbb{CP}^{n-1}; \mathbb{R}) \cong \mathbb{R}[a]/(a^n)$ as rings. If we set $x = c_1(\tilde{S}^*) = -c_1(\tilde{S}) \in H^2(\mathbb{P}(E))$ and $i\colon \mathbb{P}(\pi^{-1}(b)) \hookrightarrow \mathbb{P}(E)$, then $-i^*(x) = a \in H^2(\mathbb{CP}^{n-1}; \mathbb{R})$. Since this works for every fiber simultaneously, we can apply Explain why

3.5. Corollary (Leray-Hirsch (special case)). *The cohomology of $H^*(\mathbb{P}(E); \mathbb{R})$ is a free module over $H^*(B; \mathbb{R})$ with basis $\{1, x, \ldots, x^{n-1}\}$ with action $y.x^k = \pi^*(y) \smile x$.*

From this fact, we can write $x^n \in H^*(P(E); \mathbb{R})$ as a linear combination with coefficients in $H^*(B)$.

3.6. Definition. (a) The *Chern classes* of the bundle ξ with data $\rho\colon E \to B$ are elements $c_i(\xi) \in H^*(B; \mathbb{R})$ which are the unique coefficients such that

$$x^n + \pi^*(c_1(\xi))x^{n-1} + \cdots + \pi^*(c_n(\xi)) = 0 \in H^*(\mathbb{P}(E); \mathbb{R})$$

Thus, $c_i(\xi) \in H^{2i}(B; \mathbb{R})$.

(b) We define the *total Chern class* to be

$$c(\xi) := 1 + c_1(\xi) + \cdots + c_n(\xi) \in H^*(B; \mathbb{R})$$

3.7. Proposition (Main properties of Chern classes). *(a) Given a map $f\colon B' \to B$, Chern classes have naturality $c(f^*(\xi)) = f^*(c(\xi))$. As an immediate corollary, this menas that Chern classes are an invariant of vector bundles, that is, $\xi \cong \xi' \implies c(\xi) = c(\xi')$.*

(b) Given γ^1 to be the tautological complex line bundle over $\mathbb{P}^n$, then $H^*(\mathbb{P}^n; \mathbb{R}) \cong \mathbb{R}[c_1(\gamma^1)]/((c_1(\gamma^1)))^{n+1}$.

(c) $c(\xi \oplus \xi') = c(\xi) \smile c(\xi')$, called the Whitney product formula.

(d) If ξ has rank n, then $c_i(\xi) := 0$ for $i > n$.

(e) If ξ has a non-vanishing section, then $c_n(\xi) = 0$.

(f) The top Chern class of a complex vector bundle ξ is the Euler class of $\xi_{\mathbb{R}}$. In other words, $c_n(\xi) = e(\xi_{\mathbb{R}})$.

3.8. Corollary. *As a consequence of the above properties, the total Chern class of a line bundle λ is always of the form*

$$c(\lambda) = 1 + c_1(\lambda)$$

where $c_1(\lambda) = e(\lambda_{\mathbb{R}})$ by definition.

3.3. The Splitting Principle.

3.9. Definition. Let ξ with data $\tau\colon E \to M$ be a smooth complex vector bundle of rank n over a manifold M. Then, we define a *split manifold* of τ to be a space $F(E)$ with map $\sigma\colon F(E) \to M$ such that

(a) $\sigma^*\xi$ is a direct sum of line bundles and
(b) the induced map on cohomology $\sigma^*\colon H^*(M;\mathbb{R}) \to H^*(F(E);\mathbb{R})$ is injective.

3.10. Example. Given a rank 2 complex vector bundle ξ with $\rho\colon E \to M$, we can take $F(E) = \mathbb{P}(E)$ because $\pi^*(\xi)$ decomposes as a direct sum of $S \oplus Q$, its universal sub and quotient bundles.

$$\begin{tikzcd} E \arrow[d, "\rho"] & \pi^*(E) \cong S \oplus Q \arrow[d] \\ M & \mathbb{P}(E) \arrow[l, "\pi"'] \end{tikzcd}$$

3.11. Example. Given a rank 3 complex vector bundle ξ with $\rho\colon E \to M$, we can iterate the construction above where the 2-dimensional quotient bundle Q_E can be split into a direct sum of line bundles when pulled back to $\mathbb{P}(Q_E)$.

$$\begin{tikzcd} E \arrow[d, "\rho"] & \alpha^*(E) \cong S_E \oplus Q_E \arrow[d] & \beta^*(S_E \oplus Q_E) \cong \beta^* S_e \oplus S_{Q_E} \oplus Q_{Q_E} \arrow[d] \\ M & \mathbb{P}(E) \arrow[l, "\alpha"'] & \mathbb{P}(Q_E) \arrow[l, "\beta"'] \end{tikzcd}$$

Thus, $\mathbb{P}(Q_E)$ is the split manifold $F(E)$.

This leads us to the general idea

3.12. Proposition (The Splitting Principle). [**BT82**, p 275] *To prove a polynomial identity in Chern classes, it suffices to assume that the vector bundles are direct sums of line bundles.*

Proof. Given a polynomial f, we want to show $f(c(E)) = 0$. Then, with setup

$$\begin{tikzcd} E \arrow[d] & \sigma^* E \arrow[d] \\ M & F(E) \arrow[l, "\sigma"'] \end{tikzcd}$$

we get $\sigma^*(f(c(E))) = f(c(\sigma^* E))$ and so, because σ^* is injective by definition of $F(E)$, $f(c(\sigma^*(E))) = 0 \Longrightarrow f(c(E)) = 0$. Since $\sigma^*(E)$ is a direct sum of line bundles by construction, we are done. $\square$

4. Chern class computations, flag manifolds, and the Grassmannian (presented by George H. Seelinger)

4.1. Proof of the Whitney Product Formula.

4.1. Lemma. *Let ξ be a direct sum of line bundles, that is $\xi = \lambda_1 \oplus \cdots \oplus \lambda_n$ where each λ_i is a line bundle over B. Then,*

$$c(\xi) = c(\lambda_1)c(\lambda_2)\cdots c(\lambda_n)$$

Proof. First, let us take vector bundle ξ to be a a direct sum of line bundles. Then, if we take pullback bundle ξ by the projectivization map (see 3.4) $\pi\colon \mathbb{P}(E) \to M$, $\pi^*\xi$ is a direct sum of line bundles, let us say $\pi^*\xi = L_1 \oplus \cdots \oplus L_n$.

$$\begin{tikzcd} S \arrow[r, hook] & \pi^*E \cong L_1 \oplus \ldots \oplus L_n \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho'"] & E \arrow[d, "\rho"] \\ & \mathbb{P}(E) \arrow[r, "\pi"] & M \end{tikzcd}$$

Now, let $s_i\colon S \to L_i$ be the projection of vector bundles. Thus, s_i induces a section of the bundle $\mathrm{Hom}(S, L_i) \to M$. [1] Furthermore, since the fiber of S over every point $y \in \mathbb{P}(E)$ is a 1-dimensional subspace of the fiber of y in $\pi^*\xi$, that is $(\rho')^{-1}(y)$, the projections $s_1, \ldots, s_n$ cannot be simultaneously zero and so the open sets

$$U_i := \{y \in \mathbb{P}(E) \mid s_i(y) \neq 0\}$$

form an open cover of $\mathbb{P}(E)$.

Now, it is useful to note that the bundle $\mathrm{Hom}(S, L_i) \cong S^* \otimes L_i$. Then, for each i, $c_1(S^* \otimes L_i) \in H^2(\mathbb{P}(E))$ restricts to zero in $H^2(U_i)$ by construction of U_i ($S^* \otimes L_i$ is a trivial line bundle over U_i), and so we can lift this element to $H^2(\mathbb{P}(E), U_i)$ by a long exact sequence argument. Thus, if we take the product,

$$\prod_{i=1}^{n} c_1(S^* \otimes L_i)$$

we can lift to $H^2(\mathbb{P}(E), U_1 \cup \cdots \cup U_n) = H^2(\mathbb{P}(E), \mathbb{P}(E)) = 0$ by using a relative cup product. Therefore,

$$0 = \prod_{i=1}^{n} c_1(S^* \otimes L_i) \overset{3.3(a)}{=} \prod_{i=1}^{n} (c_1(S^*) + c_1(L_i))$$

and if we set $x = c_1(S^*)$, we get

$$0 = \prod_{i=1}^{n} (x + c_1(L_i)) = x^n + e_1(c_1(L_1), \ldots, c_1(L_n))x^{n-1} + \cdots + e_n(c_1(L_1), \ldots, c_1(L_n))$$

where e_i is the ith symmetric polynomial. However, this is precisely how we defined $c_i(\xi)$, so we get

$$\pi^* c_i(\xi) = e_i(c_1(L_1), \ldots, c_1(L_n)) \Longrightarrow \pi^* c(E) = \prod (1 + c_1(L_i)) = \prod c(L_i)$$

and so $c(E) = \prod c(\lambda_i)$. $\square$

[1] This is a special case of a canonical fact. The map s_i induces a vector space isomorphism on each fiber over b, say $(s_i)_b$, and so the canonical associated section of the Hom bundle sends each b to the map $(s_i)_b$.

4.2. Corollary (Corollary of proof). *Given a complex vector bundle ξ of rank n,*

$$\pi^* c(E) = \prod_{i=1}^{n} (1 + c_1(L_i)) = \sum_{j=1}^{n} e_j(c_1(L_1), \dots, c_1(L_n))$$

where $\pi^*\xi = L_1 \oplus \cdots \oplus L_n$ is a direct sum of line bundles and, as before, $\pi \colon \mathbb{P}(E) \to M$.

Proof of 3.7(c). We now use the lemma above and the splitting principle 3.12 to prove the formula in general. Given complex vector bundles ξ, ξ' over M of rank n and m, respectively, we can iterate the splitting constructions to get a direct sum of line bundles for which the Chern class identites are equivalent.

$$\begin{tikzcd}
E \oplus E' \arrow[d] & L_1 \oplus \cdots \oplus L_n \oplus \pi^* E \arrow[d] & L_1 \oplus \cdots \oplus L_n \oplus L_1' \oplus \cdots \oplus L_m' \arrow[d] \\
M & F(E) \arrow[l, "\pi"'] & F(\pi^* E) \arrow[l, "\pi'"']
\end{tikzcd}$$

Now, if $\sigma = \pi' \circ \pi$, then

$$\begin{aligned}
\sigma^*(c(\xi \oplus \xi')) &= c(\sigma^*(\xi \oplus \xi')) && \text{By naturality of Chern classes 3.7(a)} \\
&= c(L_1 \oplus \cdots \oplus L_n \oplus L_1' \oplus \cdots \oplus L_m') && \text{By splitting principle construction above} \\
&= c(L_1) \cdots c(L_n) c(L_1') \cdots c(L_m') && \text{By Lemma 4.1} \\
&= c(L_1 \oplus \cdots \oplus L_n) c(L_1' \oplus \cdots \oplus L_m') && \text{By Lemma 4.1} \\
&= c(\sigma^*\xi) c(\sigma^*\xi') && \text{By splitting principle construction above} \\
&= \sigma^* c(\xi) c(\xi') && \text{By naturality of Chern classes}
\end{aligned}$$

However, also by construction (since we have a split manifold 3.9), σ^* is injective, and thus $c(\xi \oplus \xi') = c(\xi)c(\xi')$. □

4.3. Remark. (a) Given a short exact sequence of smooth complex vector bundles

$$0 \to A \to B \to C \to 0$$

one can prove that $c(B) = c(A)c(C)$ using the fact that $B \cong A \oplus C$ as smooth bundles and the Whitney product formula.

(b) Using the splitting principle and the Whitney product formula, one can directly prove 3.7(f), that is $c_n(\xi) = e(\xi_{\mathbb{R}})$. See [**BT82**, p 278] for a complete proof.

4.2. Computing some Chern classes. Recall from Corollary 4.2 that the Chern classes of a rank n complex vector bundle ξ are precisely the elementary symmetric functions in the first Chern classes of the line bundles into which ξ splits when pulled back to the splitting manifold $F(E)$. Thus, by the fundamental theorem of symmetric functions, any symmetric polynomial in these variables is a polynomial in $c_1(\xi), \dots, c_n(\xi)$.

4.4. Example. Given V a complex vector space with basis $\{v_1, \dots, v_n\}$,

(a) $\wedge^p V$ has a basis $\{v_{i_1} \wedge \cdots \wedge v_{i_p}\}_{1 \le i_1 < \cdots < i_p \le n}$. So, similarly, if we have a bundle $\xi = \lambda_1 \oplus \cdots \oplus \lambda_n$ where λ_i's are all line bundles, then

$$\wedge^p \xi = \bigoplus_{1 \le i_1 < \cdots < i_p \le n} (L_{i_1} \otimes \cdots \otimes L_{i_p})$$

and thus

$$\begin{aligned} c(\wedge^p \xi) &= \prod (1 + c_1(\lambda_{i_1} \otimes \cdots \otimes \lambda_{i_p})) \quad \text{by the Whitney product formula} \\ &= \prod (1 + x_{i_1} + \cdots + x_{i_p}) \qquad\qquad x_i := c_1(\lambda_i) \end{aligned}$$

Note, however, that this is symmetric in the x_i's! Thus, by the Fundamental Theorem of Symmetric Polynomials and the fact that $c_i(\xi) = e_i(x_1, \dots, x_n)$, we can write

$$c\left(\wedge^p \xi\right) = Q(c_1(\xi), \dots, c_n(\xi))$$

where Q is some polynomial. Note that this result holds for any ξ by the splitting principle and Q depends only on n and p.

(b) $c\left(\mathrm{Sym}^p \xi\right) = \prod_{1 \le i_1 \le \cdots \le i_p \le n} (1 + x_{i_1} + \cdots + x_{i_p})$

(c) $c(\xi \otimes \xi') = \prod_{1 \le i \le n, 1 \le j \le m} (1 + x_i + y_j)$n

(d) Since $c_1(L^*) = -c_1(L)$ for a line bundle L by definition,

$$\begin{aligned} \xi = \lambda_1 \oplus \cdots \oplus \lambda_n &\Longrightarrow c(\xi) = (1 + c_1(\lambda_1) \cdots (1 + c_1(\lambda_n))) \\ \xi^* = \lambda_1^* \oplus \cdots \oplus \lambda_n^* &\Longrightarrow c(\xi) = (1 - c_1(\lambda_1) \cdots (1 - c_1(\lambda_n))) \end{aligned}$$

and so, setting the homogeneous parts equal to each other, $c_q(\xi^*) = (-1)^q c_q(\xi)$.

4.3. Flag manifolds.

4.5. Definition. (a) Given a complex vector space V of dimension n, a *(complete) flag* in V is a sequence of subspaces

$$A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_n = V, \dim_{\mathbb{C}} A_i = i$$

Note that a flag on V induces a $\mathbb{C}$ basis on V by picking basis vector $b_i \in A_i / A_{i-1}$ (take $A_0 = \{0\}$).

(b) Let $\mathcal{F}\ell(V)$ be the collection of all flags of V, called the *flag manifold*.

4.6. Remark. In fact, $\mathcal{F}\ell(V)$ is a manifold because $GL(n, \mathbb{C})$ acts transitively on $\mathcal{F}\ell(V)$ and the stabilizer of a flag is the closed subgroup of all upper triangular matrices (with respect to the induced basis), say H, and so $\mathcal{F}\ell(V) \cong GL(n, \mathbb{C})/H$ as a G-space, but $GL(n, \mathbb{C})/H$ can be given a manifold structure since H is a closed subgroup and $GL(n, \mathbb{C})$ is a Lie group. For those with some Lie theory knowledge, H is a "Borel subgroup."

4.7. Definition. Given a vector bundle ξ with $\rho\colon E \to M$, one can form the *associated flag bundle* $\mathcal{F}\ell(\xi)$ with $f\colon \mathcal{F}\ell(E) \to M$ where $f^{-1}(x) = \mathcal{F}\ell(\rho^{-1}(x))$ with the natural transition maps.

Since the fiber is not a vector space, the associated flag bundle is not a vector bundle, but it is still a fiber bundle.

4.8. Proposition. [**BT82**, Proposition 21.15] *The associated flag bundle $\mathcal{F}\ell(\xi)$ of a vector bundle ξ is the split manifold constructed in the examples after 3.9.*

Proof. Let $V \to \{*\}$ be a rank 3 vector bundle over a point (ie a 3 dimensional vector space.) Then, we have

$$\begin{tikzcd} V \arrow[d] & S_V \oplus Q_V \arrow[d] & S_V \oplus S_{Q_V} \oplus Q_{Q_V} \arrow[d] \\ \{*\} & \mathbb{P}(V) \arrow[l] & \mathbb{P}(Q_V) = F(V) \arrow[l] \end{tikzcd}$$

However, we have correspondences

A point in $x \in \mathbb{P}(V)$	a line ℓ_x in V
A point in $S_V \oplus Q_V$	$(x, (v_1, v_2))$ such that $x \in \mathbb{P}(V), v_1 \in \ell_x, v_2 \in V/\ell_x$
A point in $\mathbb{P}(Q_V)$	a line ℓ in V and a line ℓ' in V/ℓ.

However, this means that we can similarly regard ℓ' as a 2-dimensional plane in V containing ℓ, so

$$F(V) = \mathbb{P}(Q_V) = \{\ell \subseteq \ell \oplus \ell' \subseteq V\} = \mathcal{F}\ell(V)$$

More generally, given rank n vector bundle $E \to M$, the split manifold $F(E)$ is obtained by a sequence of $n-1$ projectivizations (see 3.4). Then,

A point in $\mathbb{P}(E)$	(p, ℓ) where $p \in M$ and ℓ is a line in $\rho^{-1}(p)$
A point in $\mathbb{P}(Q_1)$ over $(p, \ell_1) \in \mathbb{P}(E)$	(p, ℓ_1, ℓ_2) where $\ell_2 \in \rho^{-1}(p)/\ell_1$
A point in $\mathbb{P}(Q_2)$ over $(p, \ell_1, \ell_2) \in \mathbb{P}(Q_1)$	$(p, \ell_1, \ell_2, \ell_3)$ where $\ell_3 \in \rho^{-1}(p)/(\ell_1 \oplus \ell_2)$
$\vdots$	$\vdots$
A point in $F(E) = \mathbb{P}(Q_{n-1})$ over $(p, \ell_1, \ell_2, \ldots, \ell_{n-1})$	$(p, \ell_1, \ldots, \ell_n)$ where $\ell_n \in \rho^{-1}(p)/(\ell_1 \oplus \cdots \oplus \ell_{n-1})$

But this last point naturally identifies with the flag

$$(p, \ell_1 \subseteq \ell_1 \oplus \ell_2 \subseteq \cdots \subseteq \rho^{-1}(p))$$

Thus, $F(E)$ and $\mathcal{F}\ell(E)$ are the same under this equivalence. □

Recall from before (3.6) that we essentially constructed the Chern classes of M such that

$$H^*(\mathbb{P}(E)) \cong H^*(M)[x]/(x^n + c_1(\xi)x^{n-1} + \cdots + c_n(\xi)), \quad x = c_1(S^*)$$

as rings. However, we have another useful characterization given by

4.9. Proposition. [**BT82**, Proposition 21.16] *Given the setup*

$$\begin{tikzcd} E \arrow[d] & 0 \to S \to \pi^*E \to Q \to 0 \arrow[d] \\ M & \mathbb{P}(E) \arrow[l, "\pi"'] \end{tikzcd}$$

we have

$$H^*(\mathbb{P}(E);\mathbb{R}) = H^*(M;\mathbb{R})[c_1(S), c_1(Q), \ldots, c_{n-1}(Q)]/(c(S)c(Q) = \pi^*c(E))$$

Proof. We simply eliminate the generators $c_1(Q), \ldots, c_{n-1}(Q)$ using the relation $c(S)c(Q) = \pi^*c(E)$ by equating terms of equal degrees in

$$(1 - c_1(S^*))(1 + c_1(Q) + \cdots + c_{n-1}(Q)) = 1 + \pi^*c_1(E) + \cdots + \pi^*c_n(E)$$

gives us

$$\begin{aligned} c_1(Q) - c_1(S^*) &= \pi^*c_1(E) \\ c_2(Q) - c_1(S^*)c_1(Q) &= \pi^*c_2(E) \\ c_3(Q) - c_1(S^*)c_2(Q) &= \pi^*c_3(E) \\ &\vdots \\ c_{n-1}(Q) - c_1(S^*)c_{n-2}(Q) &= \pi^*c_{n-1}(E) \\ -c_1(S^*)c_{n-1}(Q) &= \pi^*c_n(E) \end{aligned}$$

This shows that $c_i(Q), 1 \leq i \leq n-1$ can be expressed in terms of $c_1(S^*)$ and elements of $H^*(M)$ and so they can be eliminated as generators. The one remaining relation transformed by the other relations will give

$$-c_1(S^*)c_{n-1}(Q) = \pi^*c_n(E) \iff c_1(S^*)^n + \pi^*c_1(E)c_1(S^*)^{n-1} + \cdots + \pi^*c_n(E) = 0$$

and so we have established the equivalence. □

4.10. Proposition. [**BT82**, Proposition 21.17 (enhanced)] *Let $E \to M$ be a complex rank n vector bundle. Then, cohomology ring of the flag manifold $\mathcal{F}\ell(E)$ is*

$$\begin{aligned} H^*(\mathcal{F}\ell(E);\mathbb{R}) &= \mathbb{R}[x_1, \ldots, x_n]/\left(\prod_{i=1}^{n}(1 + x_i) = c(E)\right) \\ &= \mathbb{R}[x_1, \ldots, x_n]/\left(e_j(x_1, \ldots, x_n) = c_j(E) \quad \forall j\right) \end{aligned}$$

where $x_i = c_1(S_i)$ for $1 \leq i \leq n-1$ and $x_n = c_1(Q_{n-1})$ where the S_i's and Q_i's come from iterating the projectivization construction to acheive the splitting principle.

Furthermore, in the special case where our vector bundle is $V \to \{*\}$, we get the identity above with $c(E) = 1$, which says that all the non-trivial elementary symmetric polynomials in $x_1, \ldots, x_n$ would be zero.

Proof. Since the flag bundle is obtained by a sequence of $n-1$ projectivizations, we can compute first

$$\begin{aligned} H^*(\mathbb{P}(Q_1);\mathbb{R}) &= H^*(\mathbb{P}(E);\mathbb{R})[c(S_2),c(Q_2)]/(c(S_2)c(Q_2)=c(Q_1)) \\ &= H^*(M;\mathbb{R})[c(S_1),c(Q_1),c(S_2),c(Q_2)]/(c(S_1)c(Q_1)=c(E),c(S_2)c(Q_2)=c(Q_1)) \\ &= H^*(M;\mathbb{R})[c(S_1),c(S_2),c(Q_2)]/(c(S_1)c(S_2)c(Q_2)=c(E)) \end{aligned}$$

Then, using induction,

$$H^*(\mathbb{P}(Q_{n-2});\mathbb{R}) = H^*(M;\mathbb{R})[c(S_1),\ldots,c(S_{n-1}),c(Q_{n-1})]/(c(S_1)\cdots c(S_{n-1})c(Q_{n-1})=c(E))$$

Then, since $c(S_i)=(1+x_i)$ and $c(Q_{n-1})=(1+x_n)$, we are done. □

4.11. Definition. Given a manifold M, the *Poincaré series* of a manifold M is

$$P_t(M) := \sum_{i=0}^{\infty} \dim H^i(M;\mathbb{R})t^i$$

More generally, if $A=\bigoplus_{i=0}^{\infty} A_i$ is a graded algebra over a field K, then the Poincaré series is given by

$$P_t(A) := \sum_{i=0}^{\infty} (\dim_K A_i)t^i$$

4.12. Remark. $P_t(A)$ is sometimes usually called the "graded dimension of A" by algebraists and so the Poincaré series of a manifold is the graded dimension of its cohomology ring.

4.13. Example. Since $H^*(\mathbb{CP}^{n-1};\mathbb{R}) = \mathbb{R}[x]/(x^n)$ with $\deg x = 2$, we get immediately that

$$P_t(\mathbb{CP}^{n-1}) = 1+t^2+\cdots+t^{2(n-1)} = \frac{1-t^{2n}}{1-t^2}$$

4.14. Lemma. *Since*

$$H^*(\mathbb{P}(E);\mathbb{R}) \cong H^*(M;\mathbb{R}) \otimes H^*(\mathbb{CP}^{n-1};\mathbb{R})$$

as modules, we have that

$$P_t(\mathbb{P}(E)) = P_t(M)\frac{1-t^{2n}}{1-t^2}$$

Proof. The result follows immediately from dimension counting of the tensor product and the example above. □

4.15. Corollary. *Let V be a complex complex vector space of dimension n. Then,*

$$P_t(\mathcal{F}\ell(V)) = \frac{(1-t^2)(1-t^4)\cdots(1-t^{2n})}{(1-t^2)^n}$$

and, more generally, if $E \to M$ is a rank n vector bundle, then

$$P_t(\mathcal{F}\ell(E)) = P_t(M)\frac{(1-t^2)(1-t^4)\cdots(1-t^{2n})}{(1-t^2)^n}$$

Proof. The flag manifold is constructed by a sequence of projectivizations and, since each time we projectivize a rank k vector bundle we multiply the Poincaré polynomial by $(1-t^{2k})/(1-t^2)$, we get

$$P_t(\mathcal{F}\ell(E)) = \frac{1-t^{2n}}{1-t^2}P_t(\mathbb{P}(Q_{n-3})) = \cdots = \frac{1-t^{2n}}{1-t^2}\frac{1-t^{2n-2}}{1-t^2}\cdots\frac{1-t^2}{1-t^2}P_t(M)$$

□

4.16. Remark. One can reformulate the above using notions of "quantum factorial" and "quantum binomial" as follows.

(a) $P_t(\mathbb{CP}^{n-1}) = \frac{1-t^{2n}}{1-t^2} =: [n]_{t^2}$
(b) $P_t(\mathcal{F}\ell(V)) = [n]_{t^2}[n-1]_{t^2}\cdots[1]_{t^2} =: [n]_{t^2}!$

4.4. The Grassmannian.

4.17. Definition. The *Grassmannian* of a complex vector space V, denoted $G_k(V)$ is the set of all subspaces of codimension k in V.

4.18. Example. Note that for V a complex vector space of dimension n, $G_{n-1}(V) = \mathbb{P}(V) \cong \mathbb{CP}^{n-1}$.

4.19. Remark. The Grassmannian is a manifold because it can be represented as the space

$$G_k(V) = \frac{U(n)}{U(k)\times U(n-k)}$$

The argument follows a similar one to 4.6. In Lie theoretic language, a Grassmannian can be thought of as $GL(V)/P$ for P a parabolic subgroup of $GL(V)$.

4.20. Definition. Given a complex vector space V of dimension n, we define the following vector bundles over $G_k(V)$.

(a) The *universal subbundle* which has total space

$$S := \{(x,v) \in G_k(V)\times V \mid v \in x\}$$

and thus the fiber of $x \in G_k(V)$ is the plane x defines in V.
(b) The *product bundle* $\hat{V} := G_k(V)\times V$.
(c) The *universal quotient bundle* Q defined by the exact sequence

$$0 \to S \to \hat{V} \to Q \to 0$$

4.21. Proposition. [**BT82**, Proposition 23.1] *The cohomology of the complex Grassmannian $G_k(V)$has Poincaré polynomial*

$$P_t(G_k(V)) = \frac{(1-t^2)\cdots(1-t^{2n})}{(1-t^2)\cdots(1-t^{2k})(1-t^2)\cdots(1-t^{2(n-k)})}$$

Proof. Consider the setup over the Grassmannian

$$\begin{tikzcd} S \oplus Q \arrow[d] & f^*Q \arrow[d] & \\ G_k(V) & F(S) \arrow[l, "f"'] & F(f^*Q) \arrow[l] \end{tikzcd}$$

Then, we consider

A point in $F(S)$	$(P, L_1 \subseteq \cdots \subseteq P)$ for $(n-k)$ plane $P \subseteq V$ and flag in P
A point in $F(f^*Q)$	A point $(P, L_1 \subseteq \cdots \subseteq P)$ in $F(S)$ together with a flag in V/P ie $(P, L_1 \subseteq \cdots \subseteq L_{n-k-1} \subseteq P \subseteq L_{n-k+1} \subseteq \cdots \subseteq V)$

Thus, $F(f^*Q)$ is the flag manifold $F(V)$ and is obtained from the Grassmannian by 2 flag constructions. Thus, by our

computations of the flag manfold Poincaré polynomials 4.15,

$$P_t(\mathcal{F}\ell(V)) = P_t(F(f^*Q)) = P_t(G_k(V))\frac{(1-t^2)\cdots(1-t^{2(n-k)})(1-t^2)\cdots(1-t^{2k})}{(1-t^2)^n}$$

□

4.22. Remark. Thus, once again using the "quantum factorial language",

$$P_t(G_k(V)) = \frac{[n]_{t^2}!}{[n-k]_{t^2}![k]_{t^2}!} =: \binom{n}{k}_{t^2}$$

4.23. Proposition. [**BT82**, Proposition 23.2] *Let V be a complex vector space of dimension n.*

(a) As a ring,

$$H^*(G_k(V);\mathbb{R}) = \frac{\mathbb{R}[c_1(S),\ldots,c_{n-k}(S),c_1(Q),\ldots,c_k(Q)]}{(c(S)c(Q)=1)}$$

(b) The Chern classes $c_1(Q),\ldots,c_k(Q)$ of the quotient bundle generate the cohomology ring $H^*(G_k(V))$.

(c) For a fixed k and a fixed i, there are no polynomial relations of degree i among $c_1(Q),\ldots,c_k(Q)$ if the dimension of V is large enough.

To prove this, we will need the following lemma

4.24. Lemma. [**BT82**, p 297] *If I is an ideal in $A = \mathbb{R}[x_1,\ldots,x_{n-k},y_1,\ldots,y_k]$ generated by the homogeneous terms of*

$$(1+x_1+\cdots+x_{n-k})(1+y_1+\cdots+y_k)-1$$

where $\deg x_i = 2i$ and $\deg y_i = 2i$, then the Poincaré series of A/I is given by

$$P_t(A/I) = \frac{(1-t^2)\cdots(1-t^{2n})}{(1-t^2)\cdots(1-t^{2(n-k)})(1-t^2)\cdots(1-t^{2k})}$$

Proof of Proposition. Consider once again the setup

$$\begin{tikzcd} S \oplus Q \arrow[d] & f^*Q \arrow[d] & \\ G_k(V) & F(S) \arrow[l, "f"'] & F(f^*Q) = \mathcal{F}\ell(V) \arrow[l] \end{tikzcd}$$

Then, we have by repeated application of 4.10 for $H^*(F(f^*Q))$ that

$$\begin{aligned} H^*(\mathcal{F}\ell(V)\mathbb{R}) &= H^*(F(S);\mathbb{R})[y_1,\dots,y_k]/\left(\prod(1+y_j) = c(Q)\right) \\ &= H^*(G_k(V);\mathbb{R})[x_1,\dots,x_{n-k},y_1,\dots,y_k]/\left(\prod(1+x_i) = c(S), \prod(1+y_j) = c(Q)\right) \end{aligned}$$

but also by the special case of 4.10, we have directly that

$$H^*(\mathcal{F}\ell(V);\mathbb{R}) = \mathbb{R}[x_1,\dots,x_{n-k},y_1,\dots,y_k]/\left(\left(\prod(1+x_i)\prod(1+y_j) = 1\right)\right)$$

and so $c(S), c(Q)$ have no other relations besides $c(S)c(Q) = 1$ in $H^*(G_k(V;\mathbb{R}))$, otherwise they would be present in the presentation above. Then, it follows that there is an injection of algebras where the left algebra is thought of as a formal polynomial ring with relations.

$$\begin{aligned} \frac{\mathbb{R}[a_1,\dots,a_{n-k},b_1,\dots,b_k]}{\left(\left(1+\sum_{i=1}^{n-k} a_i\right)\left(1+\sum_{j=1}^{k} b_j\right) = 1\right)} &\hookrightarrow H^*(G_k(V);\mathbb{R}) \\ a_i &\mapsto c_i(S) \\ b_j &\mapsto c_j(Q) \end{aligned}$$

However, by the lemma above,

$$P_t\left(\frac{\mathbb{R}[a_1,\dots,a_{n-k},b_1,\dots,b_k]}{\left(1+\sum_{i=1}^{n-k} a_i\right)\left(1+\sum_{j=1}^{k} b_j\right) = 1}\right) = \frac{(1-t^2)^n}{(1-t^2)\cdots(1-t^{2(n-k)})(1-t^2)\cdots(1-t^{2k})} = P_t(G_k(V))$$

Thus, the injection is an isomorphism by dimension counting, proving part (a).

For part (b), we write

$$c(S) \overset{(a)}{=} \frac{1}{c(Q)} = \frac{1}{\prod(1+y_j)} = \prod_j \left(\sum_{r=0}^{N} y_j^r\right)$$

since there exists an N such that $y_j^N = 0$ for all j since the Grassmannian is finite dimensional. Thus, $c(S)$ can be written in terms of $c_i(Q)$'s and it follows that the $c_i(S)$ terms are unnecessary generators.

However, this process also induces polynomial relations of degrees $2(n-k+1),\dots,2n$ among the $c_1(Q),\dots,c_k(Q)$. So, if $2(n-k+1) > i$, there exists no polynomial relations of degree i among the $c_j(Q)$'s. $\square$

4.25. Remark. (a) A more careful proof of statement (a) for the cohomology of $G_k(V)$ with $\mathbb{Z}$ coefficeints can be found in [**Hat01**, pp 435–437].

(b) In these notes, we have chosen to look at Grassmannians with codimension k planes. The whole theory translates into Grassmannians with dimension k planes by dualizing the whole picture. Then, the roles of the universal sub-bundle and universal quotient bundle are reversed.

(c) The Grassmannian plays a special role in the theory of fiber bundles because the universal quotient bundle over the Grassmannian is a "universal vector bundle" in the sense that every vector bundle is a pullback for sufficiently large n in $G_k(\mathbb{C}^n)$. More precisely, if ξ is a rank k complex vector bundle over a compact base space, then there exists an $n \in \mathbb{N}$ and a map $f\colon B \to G_k(\mathbb{C}^n)$ such that ξ is the pullback under f of the universal quotient bundle. This is essentially [**BT82**, Proposition 23.9]. This idea is also covered in [**MS74**, Section 5], although for real vector bundles.

5. Equivariant cohomology and Chern classes (presented by Liron Speyer)

In this talk, we will introduce notions of equivariant cohomology and equivariant Chern classes while simultaneously abandoning the "differential" assumptions of the earlier chapters since we will move away from [**BT82**] as a reference and move instead to lecture notes by Anderson and Fulton.[2]

If a Lie group G acts on a space X, we want to construct a cohomology theory for X that reflects the G-action. A natural guess might be that $H^*(X/G)$ would reflect this, but if the G-action is not free, X/G could be quite ugly (eg not Hausdorff). So, instead, we need to modify our picture to get a free G-action.

5.1. Definition. Let G be a Lie group acting on the left on a space X. Then, we define abstractly spaces

(a) EG a contractible space on which G acts freely (on the right) and

(b) BG, the *classifying space*, which is given by

$$BG := EG/G = EG/(x = x.g, \forall g \in G)$$

(c)

$$EG \times^G X := EG \times X/((e.g, x) \sim (e, g.x))$$

Since EG is contractible, $EG \times^G X$ is homotopy-equivalent to X on which G acts freely, which it may not have done on just X.

[2]Author did not attend this talk.

5.2. Definition. The *equivariant cohomology* of X with respect to a Lie group G is given by

$$H^i_G(X) := H^i(EG \times^G X)$$

where $H^i(-)$ is ordinary (singular) cohomology.

5.3. Proposition. *(a) Since $BG = EG \times^G \{pt\}$, $H^*_G(\{pt\}) = H^*(BG)$.*

(b) The (unique) map $X \to \{pt\}$ induces pullback on cohomology $H^*_G(\{pt\}) \to H^*_G(X)$, so the equivariant cohomology of X has the structure of a $H^*_G(\{pt\})$-module.

5.4. Example. (a) Let $G = \mathbb{C}^\times = S^1$. Then, G acts freely on $\mathbb{C}^\infty \setminus \{0\}$, which is contractible, so we will take $EG = \mathbb{C}^\infty \setminus \{0\}$. (Remember we do not yet have a way to construct EG in general). Then, $BG = \mathbb{CP}^\infty$ and so

$$H^*_{\mathbb{C}^\times}(\{pt\}) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[t]$$

where $t = c_1(L)$ for L the tautological line bundle over $\mathbb{CP}^\infty$. Thus, $H^*_{\mathbb{C}^\times}(X)$ is always a $\mathbb{Z}[t]$-module.

(b) Let $G = GL_n(\mathbb{C})$. Then, $BG = Gr(n, \mathbb{C}^\infty) = \{W \subseteq \mathbb{C}^\infty \mid \dim W = n\}$ (we will prove this later). So, then, $H^*_{GL_n(\mathbb{C})}(\{pt\}) = H^*(Gr(n, \mathbb{C}^\infty)) = \mathbb{Z}[t_1, \ldots, t_n]$ for $t_i = c_i(S)$ where S is the universal sub-bundle over $Gr(n, \mathbb{C}^\infty)$ (see 4.23).

5.5. Remark. Equivariant cohomology can recover ordinary cohomology by taking G to be the trivial group. Then, $H^*(\{pt\}) = \mathbb{Z}$ and we recover the fact that $H^*(X; \mathbb{Z})$ always has the structure of a $\mathbb{Z}$-module.

5.6. Definition. (a) A (right) *G-bundle* is a fiber bundle $\rho\colon E \to B$ with G acting on E on the right preserving fibers. In otherwords, for $x \in B, y \in \rho^{-1}(x)$, then $y.g \in \rho^{-1}(x)$ for all $g \in G$.

(b) A *principal G-bundle* is a G-bundle on which G acts freely and transitively on fibers.

5.7. Remark. It is immediate that a principal G-bundle has fiber $\rho^{-1}(x) = G.x \subseteq E$ since the G-action is transitive and $G.x$ is isomorphic to G as a G-set since the G-action is free.

5.8. Proposition. *The bundle $EG \to BG$ is a principal G-bundle with the universal property that, if $E \to B$ is a principal G-bundle, then there exists a map $f\colon B \to BG$ such that $E \to B$ is the pullback bundle of $EG \to BG$.*

$$\begin{tikzcd} f^*(EG) = E \arrow[d] & EG \arrow[d] \\ B \arrow[r, "f"] & BG \end{tikzcd}$$

In general, EG is infinite-dimensional, so it is not necessarily an algebraic variety. However, we will find finite-dimensional approximation spaces

by constructing principal G-bundles $EG_m \to BG_m$ for $m \in \mathbb{N}$ such that $\pi_i(EG_m) = 0 = H^i(EG_m)$ for $0 < i < k(m)$ where $k(m) \to \infty$ as $m \to \infty$. For this to work, we need a corollary to Leray-Hirsch 2.9.

5.9. Corollary. *(a) If $H^i(F) = 0$ for $0 < i \leq m$, then $H^i(B) \to H^i(E)$ is an isomorphism for $i \leq m$.*

(b) If $E \to B$ and $E' \to B'$ are two principal right G-bundles with $H^i(E) = H^i(E') = 0$ for $0 < i \leq k$, then there is a canonical isomorphism

$$H^i(E \times^G X) \cong H^i(E' \times^G X) \quad i < k$$

Proof. We prove the second part granting the first. If we let G act diagonally on $E \times E'$, we get the diagram

$$\begin{tikzcd}
E \times X & E \times E' \times X \arrow[l] \arrow[r] \arrow[d] & E' \times X \arrow[d] \\
E \times^G X \arrow[u] & (E \times E') \times^G X \arrow[l] \arrow[r] & E' \times^G X
\end{tikzcd}$$

where each vertical map is a G-bundle, the horizontal maps on the left are bundles with fiber E', and the horizontal maps on the right are bundles with fiber E. Then, the induced maps

$$H^i(E \times^G X) \longrightarrow H^i((E \times E') \times^G X) \longleftarrow H^i(E' \times^G X)$$

are isomorphisms for $i < k$ by the first part of the corollary. □

5.10. Definition. (a) We say ξ is a *G-equivariant complex vector bundle* if the total space and base spaces have G-actions and the G-action commutes with the projection map. In other words, for $\rho\colon E \to B$, we have $\rho(g.x) = g.\rho(x)$ and $g.\rho^{-1}(x) \subseteq \rho^{-1}(g.x)$.

(b) To a G-equivariant complex vector bundle ξ given by $\rho\colon E \to B$, we associate a complex vector bundle ξ' given by $\rho'\colon EG \times^G E \to EG \times^G B$. This is nonstandard terminology.

(c) If ξ is a G-equivariant complex vector bundle over B, then it has *equivariant Chern classes*

$$c_i^G(\xi) \in H_G^{2i}(B)$$

where $c_i^G(\xi) = c_i(\xi')$ for ξ' the associated complex vector bundle to ξ.

5.11. Remark. (a) Note, we can replace EG with EG_m for m sufficiently large.

(b) If we have a G-equivariant vector bundle $E \to \{pt\}$, then $E \cong \mathbb{C}^n$ is an n-dimensional representation of G. In other words, the G-action on $E \cong \mathbb{C}^n$ can be encoded by a group homomorphism $\rho\colon G \to GL_n(\mathbb{C})$ such that, for $x \in E = \mathbb{C}^n$,

$$g.x = \rho(g)x$$

Thus, we will often write $E_\rho \to \{pt\}$ to indicate that G acts on E via ρ or that E_ρ is the vector space corresponding to representation ρ.

5.12. Example. (a) Let $G = \mathbb{C}^\times$ act on the line bundle $\mathbb{C} = L \to \{pt\}$ via multiplication: $g.z = gz$. Then, we have the picture

$$\begin{tikzcd}
L \arrow[r, squiggly] \arrow[d] & EG \times^{\mathbb{C}^\times} L \arrow[d] & (\mathbb{C}^\infty \setminus \{0\}) \times^{\mathbb{C}^\times} L \arrow[d] \\
\{pt\} & EG \times^{\mathbb{C}^\times} \{pt\} \arrow[r, equal] & \mathbb{C}\mathbb{P}^\infty
\end{tikzcd}$$

Now, we know we have a map on the total spaces

$$\begin{aligned} EG_m \times L &\to \mathbb{C}\mathbb{P}^{m-1} \times \mathbb{C}^m \\ (z_1, \dots, z_m) \times z &\mapsto [z_1, \dots, z_m] \times (z_1 z, \dots, z_m z) \end{aligned}$$

whose image gives the tautological bundle since $(z_1 z, \dots, z_m z) \in [z_1, \dots, z_m]$. Now, if we quotient by the $\mathbb{C}^\times$ action, we have

$$\begin{tikzcd}
(z_1 g, \dots, z_m g) \times z \arrow[r, mapsto] & [z_1, \dots, z_m] \times (z_1 z, \dots, z_m z) \arrow[d, equal] \\
(z_1, \dots, z_m) \times gz \arrow[r, mapsto] & [z_1, \dots, z_m] \times (z_1 gz, \dots, z_m gz)
\end{tikzcd}$$

and so our map passes nicely to the quotient, giving us an isomorphism of vector bundles

$$\begin{tikzcd}
EG \times^{\mathbb{C}^\times} L \arrow[r, equal] \arrow[d] & (\mathbb{C}^\infty \setminus \{0\}) \times^{\mathbb{C}^\times} L \arrow[d] \\
EG \times^{\mathbb{C}^\times} \{pt\} \arrow[r, equal] & \mathbb{C}\mathbb{P}^\infty
\end{tikzcd}$$

Thus, $c_i^{\mathbb{C}^\times}(L) = c_i(EG \times^{\mathbb{C}^\times} L) \in H^{2i}(\mathbb{C}\mathbb{P}^\infty) = \mathbb{Z}[t]$ where $t = c_1(S)$. So, by pulling back over our isomoprhism, we get that $c_i^{\mathbb{C}^\times}(L) = t \in \mathbb{Z}[t]$.

(b) More generally, if we let $G = \mathbb{C}^\times$ act on the line bundle $\mathbb{C} = L \to \{pt\}$ via $g.z = g^a z$ for $a \in \mathbb{Z}$. Let us denote such a bundle $L_a \to \{pt\}$. Then we have the picture

$$\begin{tikzcd}
L_a \arrow[r, squiggly] \arrow[d] & EG \times^{\mathbb{C}^\times} L \arrow[d] & (\mathbb{C}^\infty \setminus \{0\}) \times^{\mathbb{C}^\times} L \arrow[d] \\
\{pt\} & EG \times^{\mathbb{C}^\times} \{pt\} = \mathbb{C}\mathbb{P}^\infty \arrow[r, "f"] & \mathbb{C}\mathbb{P}^\infty
\end{tikzcd}$$

where f is the map such that we have a pullback. In particular, we have that

Come back to this example!

(c) If we let $G = (\mathbb{C}^\times)^n$, we have that $EG_m = (\mathbb{C}^m \setminus \{0\})^n$ and $BG_m = (\mathbb{CP}^{m-1})^n$. Furthermore, we let L_{ξ_i} be the 1-dimensional representation of G with character $\chi_i(z_1, \ldots, z_n) = z_i$. Then, $C_1^G(L_{\xi_i}) := c_i(EG_m \times^G L_{\xi_i})$ and so, on bundles,

$$\begin{tikzcd} EG \times^G L_{\xi_i} \arrow[r, "\sim"] \arrow[d] & p_i^*(S) \arrow[r] \arrow[d] & S \arrow[d] \\ BG_m \arrow[r, "\sim"] & (\mathbb{CP}^{m-1})^n \arrow[r, "p_i"] & \mathbb{CP}^{m-1} \end{tikzcd}$$

where p_i is the projection onto the i-th coordinate. Thus, we get $c_1(EG_m \times^G L_{\xi_i}) = c_1(p_i^*(S)) = t_i \in \mathbb{Z}[t_1, \ldots, t_n] = H_G^*(\{pt\})$.

(d) Let $G = GL_n\mathbb{C}$. Then, for $m \geq n$, we take $EG_m = M_{m,n}^0 = \{A \in M_{m,n} \mid A \text{ has rank } n\}$ with G acting on the right by matrix multiplication. We will take as given that $\pi_i(M_{m,n}^0) = 0$ for $0 < i \leq 2(m-n)$. Then, $BG_m = M_{m,n}^0/G \cong Gr(n, \mathbb{C}^m)$ by mapping a matrix A to its image, which defined an n-plane in $\mathbb{C}^m$.

We already know that $H^*(Gr(n, V))$ is generated by the Chern classes of the tautological subbundle of rank n with relations in degrees $m-n+1, \ldots, m$. Thus, $H_G^*(\{pt\}) = \mathbb{Z}[c_1, \ldots, c_n]$ since these relations disappear as $m \to \infty$.

Now, we have a G-equivariant vector bundle $V \to \{pt\}$ and this yields isomorphisms

$$\begin{tikzcd} EG_m \times^G V \arrow[r, "\sim"] \arrow[d] & S \arrow[d] \\ BG_m \arrow[r, "\sim"] & Gr(n, \mathbb{C}^m) \end{tikzcd}$$

where the map is given by $\phi \times v \mapsto \phi(v) \in \mathrm{im}(\phi)$, which all works out since $\phi \cdot g \times v$ and $\phi \times gv$ both map to $\phi(gv)$.

(e) For subgroups $G \leq GL_n\mathbb{C}$, we can use the same $EG_m = M_{m,n}^0$. So, if we take the torus $G = (\mathbb{C}^\times)^n \subseteq GL_n\mathbb{C}$, then we get

$$BG_m = M_{m,n}^0/(\mathbb{C}^\times)^n = \{(L_1, \ldots, L_n) \mid \dim L_k = 1, L_i \cap L_j = \{0\}, \forall i \neq j\}$$

comes equipped with tautological line bundles $L_1, \ldots, L_n$. Furthermore, $c_1(L_i) = t_i \in \mathbb{Z}[t_1, \ldots, t_n] = H_G^*(\{pt\})$.

why?

Similary, if $G = B^+$, the upper-triangular matrices in $GL_n\mathbb{C}$, then

$$M_{m,n}^0/G = \{0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \subseteq \mathbb{C}^m \mid \dim L_i = i\} = \mathcal{F}\ell(1, \ldots, n; \mathbb{C}^m)$$

where the map is given by sending a matrix A to $(L_1 \subseteq \cdots \subseteq L_n)$ where L_i is the span for the first i columns of A. Thus, we can construct a tautological sequence of bundles $S_1 \subseteq \cdots \subseteq S_n$ and get that the cohomology ring of $\mathcal{F}\ell(1, \ldots, n; \mathbb{C}^m)$ is generated by $t_i = c_1(S_i/S_{i-1})$.

why? Connect to earlier lecture.

5.13. Proposition. *Let Lie group G act on X and Lie group G' act on X'. Furthermore, let $\phi\colon G \to G'$ be a continuous Lie group homomorphism and let $f\colon X \to X'$ be continuous and equivariant with respect to ϕ, ie*

$$f(g \cdot x) = \phi(g) \cdot f(x), \quad x \in X, g \in G$$

Then, there is a degree-preserving ring homomorphism

$$H^*_{G'}X' \to H^*_G X$$

and this map is functorial.

Proof. One can find a continuous map $EG \to EG'$ that is equivariant for the right actions of G and G' so that we get commutative diagram

$$\begin{tikzcd} EG \arrow[r] \arrow[d] & EG' \arrow[d] \\ BG \arrow[r] & BG' \end{tikzcd}$$

In fact, these maps are well-defined up to homotopy, and so we can get an induced map

$$EG \times^G X \to EG' \times^{G'} X$$

and thus we get our map on cohomology via functoriality. □

6. Localization in Equivariant Cohomology (presented by Chris Chung)

6.1. Background.

6.1. Definition. Throughout this talk, we will denote the tautological line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and its dual $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, dropping the $\mathbb{P}^{n-1}$ when it is clear by context.

6.2. Remark. This notation is identifying line bundles with the structure sheaf of $\mathbb{P}^{n-1}$, denoted $\mathcal{O}_{\mathbb{P}^{n-1}}$, combined with a grading shift or "twisting" given by the number in parentheses. It is not important to understand this notion for this talk.

6.3. Example. Let $T = (\mathbb{C}^\times)^n$ act on $\mathbb{C}^n$ via the standard action. Then, $H^*_T(\{pt\}) = \mathbb{Z}[t_1, \ldots, t_n]$ where $t_i = c_1(\mathcal{O}(-1))$. This induces an action of T on $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$.

6.4. Proposition. *Both $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ are T-equivariant line bundle.*

6.5. Definition. Let $\zeta := c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$.

6.6. Proposition.

$$\begin{aligned} H^*_T(\mathbb{P}^{n-1}) &\cong \mathbb{Z}[t_1, \ldots, t_n][\zeta]/(\zeta^n + e_1(t)\zeta^{n-1} + \cdots + e_r(t)) \\ &\cong \mathbb{Z}[t_1, \ldots, t_n][\zeta]/\left(\prod_{i=1}^n \zeta + t_i\right) \end{aligned}$$

Proof. Let T act on $\mathbb{C}^n \to \{pt\}$. Then, we have associated approximation space $E = ET_m \times^T \mathbb{C}^n \to BT_m$ where $ET_m = (S^m)^n$ and $BT_m = (\mathbb{CP}^{m-1})^n$. Thus, we have $ET_m \times^T \mathbb{P}^{n-1} = \mathbb{P}(E)$ and so, abusing notation slightly, we have $ET_m \times^T \mathcal{O}_{\mathbb{P}^{n-1}}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$.

Next, recall that

$$H^*(\mathbb{P}(E)) = H^*(BT)[\zeta]/(\zeta^n + c_1(E)\zeta^{n-1} + \cdots + c_n(E))$$

Then, $\zeta := c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. Thus, we get $c_i^T(\mathbb{C}^n) = c_i(E) = e_i(t)$. $\square$

6.7. Theorem (A Generalization). *Let $T = (\mathbb{C}^\times)^n$ act on vector space V of dimension r by characters (weights) $\chi_1, \ldots, \chi_r$, ie*

$$t.(v_1, \ldots, c_r) = (\chi_1(t)v_1, \ldots, \chi_r(t)v_r)$$

Then, we have that

$$H_T^*(\mathbb{P}(V)) = \mathbb{Z}[t_1, \ldots, t_n][\zeta]/\left(\prod_{i=1}^{r}(\zeta + \chi_i)\right)$$

where, in the expression above, we think of χ_i as the pullback of $c_1^T(L_{\chi_i})$ across the splitting map.

So, if $r = n$ and $\chi_i = t_i = c_1^T(\mathcal{O}(1))$, we recover the standard action in the proposition above. Now, consider the setup

$$\begin{tikzcd} X \arrow[r] \arrow[d] & EG \times^G X \arrow[d] \\ \{pt\} \arrow[r] & BG \end{tikzcd}$$

where G acts on X. Then,

(a) We have a canonical map $H_G^*(X) \to H^*(X)$ compatible with $H_G^*(\{pt\}) \to \mathbb{Z}$. Furthermore, under nice conditions, we get $H_G^*(X) \to H^*X$ is surjective with kernel given by $\ker(H_G^*(\{pt\}) \to \mathbb{Z})$.
(b) Let X^G be the G fixed points of X. Then, the "fixed locus" $\iota\colon X^G \hookrightarrow X$ induces a "restriction" $\iota^*\colon H_G^*X \to H_G^*X^G$. Furthermore, under nice conditions, ι^* is injective.

6.8. Example. We give an example where these nice condtions are not met. Let $G = \mathbb{C}^\times = X$ with G acting on X via left multiplication. Then,

$$H^1_{C^\times}(\mathbb{C}^\times) = H^1(S^\infty \times^{\mathbb{C}^\times} \mathbb{C}^\times) = H^1(S^\infty) = 0$$

but $H^1(\mathbb{C}^\times) = H^1(S^1) = \mathbb{Z}$. Also, $X^G = \varnothing$. Thus, both nice situations fail.

6.2. Restriction and Gysin Maps. Throughout this section, let $T = (\mathbb{C}^\times)^n$ and $\Lambda_T = H_T^*(\{pt\}) = \mathbb{Z}[t_1, \ldots, t_n]$.

6.9. Definition. For T acting on X and $p \in X^T$ a T-action fixed point, let $\iota_p \colon \{p\} \hookrightarrow X$ be the inclusion map of this point.

Thus, we have induced map $\iota_p^* \colon H_T^*(X) \to \Lambda_T$.

6.10. Example. (a) Let $E \to X$ be a rank r T-equivariant vector bundle with $p \in X^T$ and fiber at p denoted $E_p \to \{p\}$. Such a fiber is a representation of T with weights, say $\chi_1, \ldots, \chi_r$. Then,

$$\iota_p^*(c_i^T(E)) = c_i^T(E_p) = e_i(\chi_1, \ldots, \chi_r)$$

In particular, if $i = r$, then

$$\iota_p^*(c_r^T(E)) = \chi_1 \cdots \chi_r$$

(b) If $X = \mathbb{CP}^{n-1}$ and T acts on X by the standard action

$$(z_1, \ldots, z_n).[x_1, \ldots, x_n] = [z_1 x_1, \ldots, z_n x_n]$$

The fixed points of this action are $p_i = [0, \ldots, 0, 1, 0, \ldots, 0]$ where the 1 is in the ith coordinate for $i = 1, \ldots, n$. The fiber of $\mathcal{O}(-1)$ at p_i is given by $\mathbb{C}\epsilon_i$, the "coordinate line". One can check that T acts on $\mathcal{O}(-1)_{p_i}$ by the character t_i and on $\mathcal{O}(1)_{p_i}$ by the character $-t_i$ (where we write characters additively, ie $(z_1, \ldots, z_n).x = xz_i^{-1}$). Thus, if $\zeta = c_1^T(\mathcal{O}(1))$, then $\iota_{p_i}^* \zeta = -t_i$. Hence, on cohomology, the map

$$\begin{aligned} \iota^* \colon H_T^* \mathbb{P}^{n-1} = \Lambda_T[\zeta] / \left(\prod (\zeta + t_i) \right) \to & H_T^* X^T = \bigoplus_{p_i \in X^T} H_T^*(p_i) = \Lambda_T^{\oplus n} \\ \zeta \mapsto & (-t_1, \ldots, -t_n) \end{aligned}$$

is injective. This idea generalized to T acting on $\mathbb{P}^{n-1}$ by characters $\chi_1, \ldots, \chi_n$ and the map above would then send $\zeta \mapsto (-\chi_1, \ldots, -\chi_n)$.

(c) Let $T = \mathbb{C}^\times$ act on $\mathbb{P}^2$ with characters $0, z, 2z$, ie

$$z.[x_1, x_2, x_3] = [x_1, zx_1, z^2 x_2]$$

(remember that we are using additive notation for characters.) Then, the fixed points are $[1, 0, 0], [0, 1, 0]$, and $[0, 0, 1]$, which we will call p_1, p_2, and p_3, respectively. Furthermore, every closed T-invariant curve is isomorphic to P^1 and contains two of $\{p_1, p_2, p_3\}$.

So, for instance, if $x_1 = 0$, then

$$z.[0, 1, b] = [0, z, z^2 b] = [0, 1, zb]$$

has character $\pm z$. Similarly, $x_2 = 0$ has character $\pm 2z$ and $x_3 = 0$ has character $\pm z$. Also, for $\lambda \neq 0$, we have a family of conics passing through p_1, p_3 via

$$x_2^2 - \lambda x_1 x_3 = 0$$

with character $\pm t$. Now, from above work, we see

$$H_T^*\mathbb{P}^2 = \Lambda_T[\zeta]/(\zeta(\zeta+z)(\zeta+2z)) \hookrightarrow \Lambda_T^{\oplus 3}$$
$$\zeta \mapsto (0, -z, -2z)$$

Furthermore, one can work out that this image consists of triples (u_1, u_2, u_3) satisfying conditions
(i) $x \mid u_2 - u_1$, $2z \mid u_3 - u_1$, $z \mid u_3 - u_2$
(ii) $2z^2 \mid u_1 - 2u_2 + u_3$

6.11. Definition. For X, Y non-singular G-varieties and a proper equivariant map $f\colon Y \to X$, we can define

$$f_*\colon H_G^i(Y) \to H_G^{i+2d}(X)$$

where $d = \dim X - \dim Y$ (not necessarily non-negative) called the *Gysin pushforward*.

We will not actually construct this map here. For the curious, see [**Ful07**, Appendix A].

6.12. Proposition (Properties of Gysin maps). *(a) Functoriality: given $q\colon Z \to Y$ another proper equivariant map,*

$$(f \circ q)_* = f_* \circ q_*$$

(b) Naturality: given $g\colon Y' \to X$ another proper equivariant map and $g(Y') \cap f(Y) = \varnothing$, then $g^* \circ f_* = 0$.
(c) Closed embedding: Given a G-invariant closed embedding $\iota\colon Y \hookrightarrow X$ of codimension d, then we have induced map

$$\iota_*\colon H_G^j(Y) \to H_G^{j+2d}(X)$$

that satisfies
(i) $\iota_*(1) = [Y]^G = [EG \times^G Y]$
(ii) (self-intersection), for $\alpha \in H_G^*(Y)$ we have $\iota^*\iota_*(\alpha) = c_d^G(N_{Y|X})\alpha$. In other words, $\iota^*\iota_*$ is multiplication by the Euler class of the normal bundle.

normal or conormal?

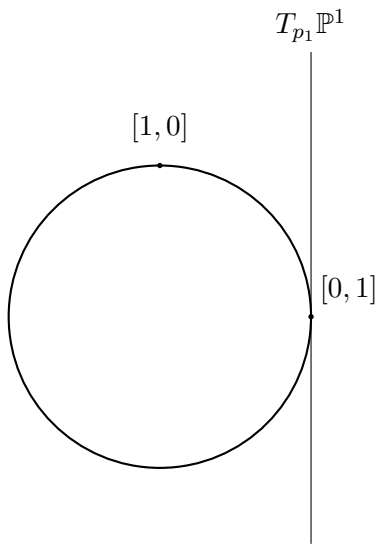
(d) Equivariant integration: given X a complete nonsingular variety of dimension n, then for $\rho\colon X \to \{pt\}$,

$$\rho_*\colon H_T^j(X) \to H_T^{j-2n}(\{pt\})$$

6.13. Example. Let T act on $\mathbb{P}^1$ with weights $\chi_1 \neq \chi_2$ so

$$z.[x_1, x_2] = [\chi_1(z)x_1, \chi_2(z)x_2]$$

Set $\chi := \chi_1 - \chi_2$, $p_1 = [0, 1]$, and $p_2 = [1, 0]$.



$$\begin{aligned} z.[a,1] &= [\chi_1(z)a, \chi_2(z)] \\ &= [(\chi_1 - \chi_2)(z)a, 1] \\ &= [\chi(z)a, 1] \end{aligned}$$

So, $T_{p_1}\mathbb{P}^1$ has weight χ. Similarly, $T_{p_2}\mathbb{P}^1$ has weight $-\chi$.

From an earlier example 6.10, $\iota_{p_1}^*(\zeta) = -\chi_1 = \chi - \chi_2$. So, in $H_T^*(\mathbb{P}^1)$,

$$[p_1]^T = \zeta + \chi_2,\ [p_2]^T = \zeta + \chi_1$$

The general consequence (for $\mathbb{P}^{n-1}$) is that

$$c_n^T(T_{p_i}(\mathbb{P}^{n-1})) = \iota_{p_i}^*([p_i]^T) = \prod_{j \neq i}(\chi_j - \chi_i)$$

6.3. Localization. Let T act on X, a nonsingular variety with X^T finite. Consider

$$\bigoplus_{p \in X^T} \Lambda_T = H_T^*(X^T) \xrightarrow{\iota_*} H^*(X) \xrightarrow{\iota^*} \bigoplus_{p \in X^T} \Lambda_T$$

By naturality (6.12(b)), $\iota^* \circ \iota_* \colon \Lambda^{\oplus |X^T|} \to \Lambda^{\oplus |X^T|}$ is diagonal and is multiplication by $c_n^T(T_pX)$ on the summand for $p \in X^T$.

6.14. Theorem (First Locatlization Theorem). *Let S be a multiplicatively close set containing $c := \prod_{p \in X^T} c_n^T(T_pX)$.*

(a) The map $S^{-1}\iota^* \colon S^{-1}H_T^*X \to S^{-1}H_T^*X^T$ is surjective and $\operatorname{coker}(\iota^* \colon H_T^*X \to H_T^*X^T)$ *is annihilated by c.*
(b) If, in addition, H_T^*X is a free Λ_T-module with rank less than or equal to $|X^T|$, then $S^{-1}\iota^*$ is an isomorphism.

Proof. For the first part, observe $\det(\iota^* \circ \iota_*) = c$ and $S^{-1}(\iota^* \circ \iota_*) = (S^{-1}\iota^*) \circ (S^{-1}\iota_*)$. After localizing at S, which contains c, we get that c is a unit, giving us that $S^{-1}(\iota^* \circ \iota_*)$ is surjective. Thus, $S^{-1}\iota^*$ is surjective.

The second part follows since $S^{-1}\Lambda_T$ is Noetherian. □

One may wonder when the conditions in the 6.14(b) hold.

6.15. Definition. we say that X is *equivariantly formal* if $H_T^*(X)$ is a free Λ_T-module and has a basis that restricts to a $\mathbb{Z}$-basis for $H^*(X)$.

6.16. Proposition. *If X is a non-singular projective variety with X^T finite, then X is equivariantly formal.*

6.17. Corollary. *For such a T-variety, $H_T^*(X) \twoheadrightarrow H^*(X)$ and $H_T^*(X) \hookrightarrow H_T^*(X^T)$*

6.18. Remark. This corollary is peculiar to T-actions.

6.19. Theorem (Atiyah-Bott-Berline-Vergne, ABBV Integration Formula). *Let X be a compact nonsingular T-variety of dimension n, X^T finite. Let $\rho\colon X \to \{pt\}$ and thus*

$$\rho_*\colon H_T^j(X) \to H_T^{j-2n}(\{pt\})$$

Then, for any $\alpha \in H_T^*(X)$,

$$\rho_*\alpha = \sum_{p \in X^T} \frac{\iota_p^*\alpha}{c_n(T_pX)}$$

Proof.

$$\begin{tikzcd}
H_T^*(X^T) \arrow[r, "\iota_*"] \arrow[d, equal] & H_T^*(X) \arrow[r, "\iota^*"] \arrow[d, "\rho_*"] & H_T^*(X^T) = \bigoplus_{p\in X^T} H_T^*(\{p\}) \\
\bigoplus_{p\in X^T} H_T^*(\{p\}) \arrow[r, dashed, "\text{addition}"] & H_T^*(\{pt\}) &
\end{tikzcd}$$

where the square commutes by the functoriality of the Gysin map for the composition $\{p\} \hookrightarrow X \twoheadrightarrow \{pt\}$. It suffices to show the theorem after we invert $c = \prod_{p\in X^T} c_n^T(T_pX)$ so that ι^*, ι_* become isomorphisms by 6.14. So, we reduce to $\alpha = (\iota_p)_*(\beta)$ for fixed p. Then,

$$\rho_*(\alpha) = \rho_*(\iota_p)_*(\beta) = (\rho \circ \iota_p)_*(\beta) = \beta$$

On the other hand,

$$\sum_{q\in X^T} \frac{\iota_q^*(\iota_p)_*(\beta)}{c_n^T(T_qX)} = \frac{\iota_p^*(\iota_p)_*(\beta)}{c_n^T(T_pX)} = \beta$$

□

6.20. Example. Let T act on $\mathbb{P}^{n-1}$ with distinct characters $\chi_1, \ldots, \chi_n$ and fixed points $p_1, \ldots, p_n$. For $\zeta^k \in H_T^{2k}(X)$, we get

$$\rho_*(\zeta^k) = \begin{cases} 0 & \text{if } k < n-1 \\ 1 & \text{if } k = n-1 \end{cases}$$

However, by ABBV,

$$\rho_*(\zeta^k) = \sum_{i=1}^n \frac{(-\chi_i)^k}{\prod_{j\neq i}(\chi_j - \chi_i)} = \begin{cases} 0 & \text{if } k < n-1 \\ 1 & \text{if } k = n-1 \end{cases}$$

7. Schubert Calculus (presented by Weinan Zhang)

This presentation mostly follows the first section of [Bri05] with some additional proofs and examples.

7.1. The Grassmannian.

7.1. Definition. Let $V = \mathbb{C}^n$. Then, we will define *the Grassmannian of* V to be $Gr_d(V) := \{E \subseteq V \mid \dim W = d\}$.

7.2. Proposition. *There is an embedding of the Grassmannian, called the* Plücker embedding *given by*

$$Gr_d(V) \to \mathbb{P}\left(\wedge^d V\right)$$
$$E = \langle v_1, \ldots, v_d \rangle \mapsto [v_1 \wedge \cdots \wedge v_d]$$

Thus, $Gr_d(V)$ has the structure of a projective variety.

7.3. Definition. Let $(e_1, \ldots, e_n)$ be the standard basis of V. For $I = (i_1, \ldots, i_d)$, we define

$$E_I := \langle e_{i_1}, \ldots, e_{i_d} \rangle$$

7.4. Definition. Given the subgroup of upper triangular matrices $B \leq GL_n\mathbb{C}$ (a Borel subgroup),

(a) the *Schubert cells* in $Gr_d(V)$ are $C_I := BE_I$
(b) the *Schubert varieties* in $Gr_d(V)$ are $X_I := \overline{C_I}$.

7.5. Remark. $Gr_d(V)$ is a disjoint union of Schubert cells if one insists that $i_1 < i_2 < \cdots < i_d$. This is a straightforward consequence of the next proposition.

7.6. Example. (a) Let $d = 1$. Then $Gr_1(V) = \mathbb{P}(V)$, $C_j = Be_j$, and $X_j \cong \mathbb{P}^{j-1}$. So, we have

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n = \mathbb{P}(V)$$

(b) Notice that, for any $d \geq 1$, $X_{1,2,\ldots,d} = \overline{C_{1,2,\ldots,d}} = \overline{BE_{1,2,\ldots,d}} = \overline{\{E_{1,2,\ldots,d}\}} = \{E_{1,2,\ldots,d}\}$ is a single point in the Grassmannian. In an opposite scenario, $X_{n-d+1,n-d+2,\ldots,n} = Gr_d(V)$.

7.7. Proposition. *Let $I = (i_1, \ldots, i_d)$ with $i_1 < i_2 < \cdots < i_d$.*

(a) $C_I \cong \mathbb{C}^{|I|}$ are locally closed subsets of $Gr_d(V)$ where $|I| := \sum_{j=1}^{d}(i_j - j)$.
(b) The X_I are irreducible closed subvarieties of $Gr_d(V)$.
(c) $E \in C_I \iff \dim(E \cap \langle e_1, \ldots, e_j \rangle) = |\{k \mid 1 \leq k \leq d, i_k < j\}|$.
(d) $E \in X_I \iff \dim(E \cap \langle e_1, \ldots, e_j \rangle) \geq |\{k \mid 1 \leq k \leq d, i_k < j\}|$.
(e) $X_I = \bigcup_{J \leq I} C_J$ where "$\leq$" means $j_k \leq i_k$ for all k.

Proof. Take $B = UT$ where U is the subgroup of unipotent matrices and T is the subgroup of diagonal matrices. Then, the stabalizer of E_I is

$$U_{E_I} := \{A \in U \mid a_{ij} = 0 \text{ if } i \notin I \text{ and } j \in I\}$$

Let us also define its logical complement,

$$U^I := \{A \in U \mid a_{ij} = 0 \text{ if } i \in I \text{ or } j \notin I\}$$

So, $U = U^I \times U_{E_I}$. Both T and U_{E_I} fix E_I and since any $g \in GL_n\mathbb{C}$ can be decomposed in $U^I \times U_{E_I} \times T$, we need only concern ourselves with matrices lying in U^I. Now, the map

$$\begin{aligned} U^I &\to Gr_d(V) = G/P \\ g &\mapsto gE_I \end{aligned}$$

is a locally closed embedding with image C_I, so $C_I \cong \mathbb{C}^{|I|}$.

The second part follows because the closure of an irreducible set is irreducible and C_I is irreducible.

Since U^I provides an embedding of C_I with respect to the standard basis, we see that $E \in C_I$ (under this embedding) will take the form

$$E = \text{rowspace}\begin{pmatrix} * & \cdots & 1 & 0 & \cdots & 0 & \cdots & & & & \\ * & \cdots & 0 & * & \cdots & 1 & 0 & \cdots & & & \\ \vdots & & \vdots & \vdots & & \vdots & & & & & \\ * & \cdots & 0 & * & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

where the rightmost 1 in row j occurs in column i_j. Thus, this immediately yields our alternative characterization of the elements of C_I.

However, we need not be so stringent with respect to our embedding, so if we realize instead our $E \in C_I$ with the form

$$E = \text{rowspace}\begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 & \cdots & & & & \\ * & \cdots & * & * & \cdots & * & 0 & \cdots & & & \\ \vdots & & \vdots & \vdots & & \vdots & & & & & \\ * & \cdots & * & * & \cdots & * & \cdots & * & 0 & \cdots & 0 \end{pmatrix}$$

where the rightmost $*$ in row j occurs in column i_j and is nonzero but is not necessarily equal to 1. Then, it becomes clear that $C_J \in X_I = \overline{C_I}$ if $j_k \leq i_k$ for all k, which also gives us characterization (d) of X_I. $\square$

7.8. Example. Some elements of the proof are easier seen by example. For example, if $I = (2, 4)$ and $\dim V = 4$, then

$$U^{(2,4)} = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, to see the embedding $C_I \hookrightarrow Gr_2(\mathbb{C}^4)$,

$$\begin{pmatrix} 1 & a_{1,2} & 0 & a_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \langle e_2, e_4 \rangle = \left\{ \langle a_{1,2}e_1 + e_2, a_{1,4}e_1 + a_{3,4}e_3 + e_4 \rangle \in G_2(\mathbb{C}^4) \right\} \cong \mathbb{C}^3$$

We can rewrite this as

$$E \in C^{(2,4)} \iff E = \text{rowspace} \begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$$

Then, we see that

$$\begin{cases} \dim(E \cap \langle e_1 \rangle) = 0 \\ \dim(E \cap \langle e_1, e_2 \rangle) = 1 \\ \dim(E \cap \langle e_1, e_2, e_3 \rangle) = 1 \\ \dim(E \cap \langle e_1, e_2, e_3, e_4 \rangle) = 2 \end{cases}$$

From looking at this example, one may notice that the proposition forces these dimension intersections to jump by at most 1 and that those jumps happen precisely at the indices 2 and 4.

7.9. Remark. Note, there is a correspondence between the indices I with strictly increasing entries and partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$ where $\lambda_j = i_j - j$ (listed in increasing order). Furthermore, such a partition must have length $\leq d$ and each part must be bounded above by $n - d$.

Also, many texts take the third and fourth parts of this proposition to define the Schubert cells and varieties with respect to a fixed flag.

7.10. Definition. For a partition λ, we will define $X_\lambda := X_I$ under the correspondence described in the remark above.

7.2. Flag Varieties. Recall from earlier sections that $\mathcal{F}\ell(V)$ is the collection of all flags. In this section, we give $\mathcal{F}\ell(V)$ the structure of a projective variety (instead of a manifold) which is realized via

$$\mathcal{F}\ell(V) \subseteq \prod_{i=0}^{n} Gr_i(V) \subseteq \prod_{i=0}^{n} \mathbb{P}(\wedge^i V) \subseteq \mathbb{P}^N$$

GL_n acts on $\mathcal{F}\ell(V)$ (for $\dim V = n$) and, since the standard Borel subgroup of GL_n stabalizes the coordinate flag, $0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq V$, we can identify $\mathcal{F}\ell(V) = GL_n/B$.

7.11. Definition. For $w \in \mathfrak{S}_n$,

(a) We define the flag

$$F_w := 0 \subseteq \langle e_{w(1)} \rangle \subseteq \langle e_{w(1)}, e_{w(2)} \rangle \subseteq \cdots \subseteq V$$

(b) We define *Schubert cells* in $\mathcal{F}\ell(V)$ by $C_w := BF_w = BwB/B$.

(c) We define *Schuber varieties* in $\mathcal{F}\ell(V)$ by $X_w = \overline{C_w}$.

7.12. Remark. To state the second equality of part (b), we made use of the general "Bruhat decomposition"

$$G = \bigsqcup_{\sigma \in W} B\sigma B^-$$

for G any connected, reductive algebraic group (eg $G = GL_n$).

In fact, if $G = GL_n$, this is precisely the statement of the existence of an LU-decomposition.

7.13. Proposition. *(a) $C_w \cong \mathbb{C}^{\ell(w)}$ are locally closed subsets in $\mathcal{F}\ell(V)$.*
(b) X_w are irreducible subvarieties in $\mathcal{F}\ell(V)$.
(c) $X_w = \bigcup_{v \leq w} C_v$ where $\leq$ is the (strong) Bruhat order.

Proof. We want to show $F_v \in X_w$ if $v \leq w$. Without loss of generality, assume $v = w \cdot (j,k)$ such that $j < k$ and $w(j) > w(k)$. We then define two bases

$$\begin{cases} f_i := e_{w(i)} & \text{if } i \neq j,k \\ f_j := e_{w(j)} + \frac{1}{t} e_{w(k)} \\ f_k := e_{w(k)} \end{cases}, \begin{cases} \tilde{f}_i = e_{w(i)} = e_{v(i)} & \text{if } i \neq j,k \\ \tilde{f}_j = e_{w(k)} + t e_{w(j)} = e_{v(j)} + t e_{v(k)} \\ \tilde{f}_k = e_{w(j)} = e_{v(k)} \end{cases}$$

Let $F(t)$ be the flag spanned by $f_1, \ldots, f_n$ or equivalently spanned by $\tilde{f}_1, \ldots, \tilde{f}_n$. Then, we note $F(t) \in X_w$ by the first description. The second description gives us that

$$F(t) \xrightarrow{t \to 0} F_v \implies F_w \in X_w$$

since X_w is closed. Thus, $C_v \subseteq X_w$ if $v \leq w$.

Show at least the other direction.

□

7.14. Definition. For $w \in \mathfrak{S}_n$,

(a) we define the *opposite Schubert cell* $C^w := B^- F_w = w_\circ C_{w_\circ w}$ where $w_\circ$ is the longest element of $\mathfrak{S}_n$ and B^- is the group of all lower triangular matrices, ie $B^- = w_\circ B w_\circ$.
(b) we define the *opposite Schubert variety* $X^w := \overline{C^w} = w_\circ X_{w_\circ w}$.

7.15. Corollary. *(a) $X^w = \bigcup_{v \geq w} C^v$*
(b) $\operatorname{codim}(X^w) = \ell(w)$

7.16. Remark. Under these definitions, $X_{w_\circ} = \mathcal{F}\ell(V)$ and $X^{w_\circ} = \{*\}$.

7.17. Theorem. $X_w \cap X^v \neq \varnothing \iff v \leq w$.

Proof. If $v \leq w$, then $F_\sigma \in X_w \cap X^v$ for all $v \leq \sigma \leq w$.

Suppose $X_w \cap X^v \neq \varnothing$. Then, $X_w \cap X^v$ is stabalized by $B \cap B^- = T$. Then, since the flags F_w for $w \in \mathfrak{S}_n$ are precisely the fixed points of the action of T on $\mathcal{F}\ell(V)$ (by direct computation), it must be that there is a σ such that $F_\sigma \in X_w \cap X^v \implies v \leq \sigma \leq w$. □

7.18. Remark. In fact, $\dim(X_w \cap X^v) = \ell(w) - \ell(v)$. In particular, this means $X_w \cap X^w = \{F_w\}$ since this intersection is transverse.

7.19. Definition. Let M be an oriented compact conected n-dimensional real manifold. Then, the *fundamental class of* M (in homology) is a canonical generator of $H_n(M) \cong \mathbb{Z}$.

Recall that the fundamental class gives us an isomorphism between cohomology and homology via the cap product

$$\begin{aligned} H^j(X) &\to H_{n-j}(X) \\ \alpha &\mapsto \alpha \cap [X] \end{aligned}$$

This then allows us to have the following definition.

7.20. Definition. Let $m = \dim_{\mathbb{R}} \mathcal{F}\ell(V) = n(n-1)$ for $\dim V = n$.

(a) Given Schubert variety $X_w \subseteq \mathcal{F}\ell(V)$, we define its *Schubert class* $[X_w] \in H^{m-2\ell(w)}$ to be the Poincare dual to the representative in homology of X_w. In other words, it is given by sending the unique generator of $H_{2\ell(w)}(X_w)$ through these two maps:

$$H_{2\ell(w)}(X_w) \xrightarrow{\iota_*} H_{2\ell(w)}(\mathcal{F}\ell(V)) \to H^{m-2\ell(w)}(\mathcal{F}\ell(V))$$

(b) Given an opposite Schubert variety X^w, we define its *Schubert class* $[X^w] \in H^{2\ell(w)}$ to be its representative in cohomology following the same procedure as above.

7.21. Proposition. *Both* $\{[X_w]\}$ *and* $\{[X^w]\}$ *form* $\mathbb{Z}$*-bases of* $H^*(\mathcal{F}\ell(V))$.

Proof. Since $\mathcal{F}\ell(V)$ is a disjoint union of its Schubert classes and the real dimensions of all its cells are concentrated in even degree, its integral homology groups will not exhibit any torsion. Thus, the Schubert classes form a basis of the integral cohomology which is a free Z-module with rank $n!$. $\square$

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