

A Window into Symmetric Function Theory

George H. Seelinger

ghs9ae@virginia.edu

UVA Math Club
Lightning Round

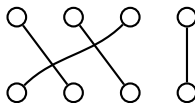
2 March 2021

Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

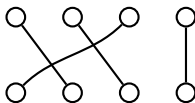
Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


Symmetric Group

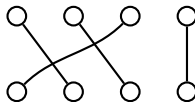
- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


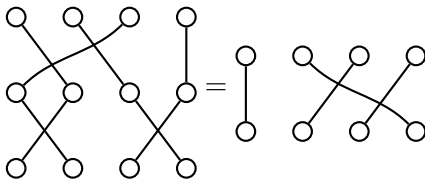
- Stacking = composition

Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

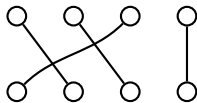
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


- Stacking = composition

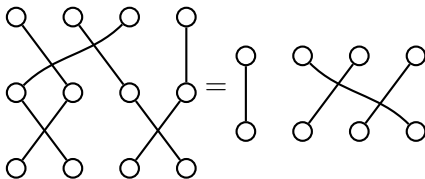


Symmetric Group

- Permutations $\sigma: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


- Stacking = composition



- S_n is a “group”

- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}[x_1, \dots, x_n]$ satisfying $\sigma.f = f$?

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}[x_1, \dots, x_n]$ satisfying $\sigma.f = f$?
- Symmetric polynomials ($n = 3$)

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}[x_1, \dots, x_n]$ satisfying $\sigma.f = f$?
- Symmetric polynomials ($n = 3$)

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric functions are polynomials in the e_1, e_2, \dots , or in the h_1, h_2, \dots

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric functions are polynomials in the e_1, e_2, \dots , or in the h_1, h_2, \dots

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

Basis of $\Lambda_{\mathbb{Q}}$?

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Partitions

Definition


$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

5 \rightarrow

--	--	--	--	--

$4 + 1 \rightarrow$

$3 + 2 \rightarrow$

$3 + 1 + 1 \rightarrow$ 

$2 + 2 + 1 \rightarrow$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

Partitions by themselves are interesting!

Partitions by themselves are interesting!

- ① How many partitions of n ? No known closed-form formula!

Partitions by themselves are interesting!

- ① How many partitions of n ? No known closed-form formula!
- ② Many interesting connections to number theory (Ramanujan).

Partitions by themselves are interesting!

- ① How many partitions of n ? No known closed-form formula!
- ② Many interesting connections to number theory (Ramanujan).
- ③ Generating function for $p(n)$ = number of partitions of n is inverse of Euler ϕ function.

Definition

Filling of partition diagram of λ with numbers such that

Definition

Filling of partition diagram of λ with numbers such that

- 1 strictly increasing down columns

Definition

Filling of partition diagram of λ with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Definition

Filling of partition diagram of λ with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Collection is called $\text{SSYT}(\lambda)$.

Definition

Filling of partition diagram of λ with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Collection is called $\text{SSYT}(\lambda)$.

For $\lambda = (2, 1)$,

1	1	1	1	1	1	1	1
2		3		3		3	

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

1	1								
2									

,

1	1								
3									

,

2	2								
3									

,

1	2								
2									

,

1	3								
3									

,

2	3								
3									

,

1	3								
2									

,

1	2								
3									

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function
- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break M up into smallest S_n fixed subspaces

Harmonic polynomials

- 1 S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Harmonic polynomials

- 1 S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- 3 How many times does an S_n fixed subspace occur?

Harmonic polynomials

- 1 S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- 3 How many times does an S_n fixed subspace occur? Frobenius:

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ③ How many times does an S_n fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ③ How many times does an S_n fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

Upshot

Upshot

- 1 Schur functions $\leftrightarrow S_n$ -invariant subspaces.

Upshot

- 1 Schur functions $\leftrightarrow S_n$ -invariant subspaces.
- 2 Via Frobenius characteristic map, questions about S_n -action on vector spaces get translated to questions about Schur expansion coefficients in symmetric functions.

Interesting algebraic combinatorics questions

Interesting algebraic combinatorics questions

- 1 Is a symmetric function Schur positive?

Interesting algebraic combinatorics questions

- 1 Is a symmetric function Schur positive?
- 2 What do the Schur expansion coefficients count?

Getting more information

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

An example of bi-degree

Capturing even more information...

An example of bi-degree

Capturing even more information...

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

An example of bi-degree

Capturing even more information...

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman: $M_\mu = \text{span of partial derivatives of } \Delta_\mu$

An example of bi-degree

Capturing even more information...

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman: $M_\mu = \text{span of partial derivatives of } \Delta_\mu$

$$\Delta_{\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

An example of bi-degree

Capturing even more information...

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman: $M_\mu = \text{span of partial derivatives of } \Delta_\mu$

$$\Delta_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

An example of bi-degree

Capturing even more information...

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman: $M_\mu = \text{span of partial derivatives of } \Delta_\mu$

$$\Delta_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

Minimal S_n -invariant subspace with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

An example of bi-degree

Capturing even more information...

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman: $M_\mu = \text{span of partial derivatives of } \Delta_\mu$

$$\Delta_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

Minimal S_n -invariant subspace with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_\mu = qts \begin{array}{|c|c|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + ts \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + qs \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + s \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$ for eigenvalue $B_\mu(q, t) \in \mathbb{Q}[q, t]$.

$$\nabla \tilde{H}_{2,1} = qt \tilde{H}_{2,1}$$

Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$ for eigenvalue $B_\mu(q, t) \in \mathbb{Q}[q, t]$.

$$\nabla \tilde{H}_{2,1} = qt \tilde{H}_{2,1}$$

- $\hat{M} = \left\{ f \in \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}.$

Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$ for eigenvalue $B_\mu(q, t) \in \mathbb{Q}[q, t]$.

$$\nabla \tilde{H}_{2,1} = qt \tilde{H}_{2,1}$$

- $\hat{M} = \left\{ f \in \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}.$
- $\hat{M} \rightarrow \nabla e_n$

$$\nabla e_3 = (q^3 + q^2t + qt^2 + t^3 + qt) s_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} + (q^2 + qt + t^2 + q + t) s_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + s_{\square \square \square}$$

Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$ for eigenvalue $B_\mu(q, t) \in \mathbb{Q}[q, t]$.

$$\nabla \tilde{H}_{2,1} = qt \tilde{H}_{2,1}$$

- $\hat{M} = \left\{ f \in \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}.$
- $\hat{M} \rightarrow \nabla e_n$

$$\nabla e_3 = (q^3 + q^2t + qt^2 + t^3 + qt) s_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} + (q^2 + qt + t^2 + q + t) s_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \end{smallmatrix}}$$

Open question

Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$ for eigenvalue $B_\mu(q, t) \in \mathbb{Q}[q, t]$.

$$\nabla \tilde{H}_{2,1} = qt \tilde{H}_{2,1}$$

- $\hat{M} = \left\{ f \in \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}.$
- $\hat{M} \rightarrow \nabla e_n$

$$\nabla e_3 = (q^3 + q^2t + qt^2 + t^3 + qt) s_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} + (q^2 + qt + t^2 + q + t) s_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + s_{\square \square \square}$$

Open question

What is the Schur expansion of ∇e_n ?

Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$ for eigenvalue $B_\mu(q, t) \in \mathbb{Q}[q, t]$.

$$\nabla \tilde{H}_{2,1} = qt \tilde{H}_{2,1}$$

- $\hat{M} = \left\{ f \in \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}.$
- $\hat{M} \rightarrow \nabla e_n$

$$\nabla e_3 = (q^3 + q^2t + qt^2 + t^3 + qt) s_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} + (q^2 + qt + t^2 + q + t) s_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + s_{\square \square \square}$$

Open question

What is the Schur expansion of ∇e_n ?

Recover earlier story by taking $t = 0$ and $y_i = 1$ for all y_i 's.