A Raising Operator Formula for Macdonald Polynomials and other related families

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University of Illinois Algebra-Geometry-Combinatorics Seminar

20 February 2025

Outline

1. **Background on symmetric functions and Macdonald polynomials**
2. Shuffle theorems, combinatorics, and LLT polynomials
3. A new formula for Macdonald polynomials

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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- E.g. for $n=3$,

$$\begin{aligned} e_1 = x_1+x_2+x_3 &= h_1 \\ e_2 = x_1x_2+x_1x_3+x_2x_3 &\quad h_2 = x_1^2+x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2 \\ e_3 = x_1x_2x_3 &\quad h_3 = x_1^3+x_1^2x_2+x_1^2x_3+x_1x_2^2+\cdots \end{aligned}$$

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- Λ is a $\mathbb{Q}(q,t)$-algebra.

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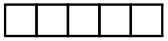
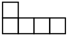
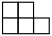
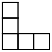
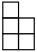
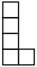
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

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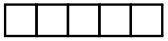
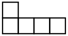
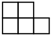
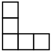
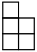
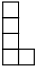
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

$1 + 1 + 1 + 1 + 1 \to$ 

$\Longrightarrow$ any basis of symmetric functions is indexed by partitions.

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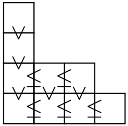

For $\lambda = (2, 1)$,

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

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Associate a polynomial to SSYT(λ).

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$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \boldsymbol{x}^T \text{ for } \boldsymbol{x}^T = \prod_{i \in T} x_i$$

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- $\{s_\lambda\}_\lambda$ forms a basis for $\Lambda_{\mathbb{Q}}$.

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

An Explicit Example: Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

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$M =$ polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

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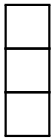

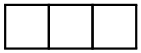

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Remark: M is a "regular representation."

Getting more information

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Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\deg=2} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\deg=1} \oplus \underbrace{\mathrm{sp}\{1\}}$$

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Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{(1,1,1)} + q^2 s_{(2,1)} + q^1 s_{(2,1)} + q^0 s_{(3)}$$

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Answer: Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X;q)$.

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- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\mathrm{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

∇e_n

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$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Operator ∇

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X;q,t)\,,$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$ and λ^* is the transpose partition to λ.

∇e_n

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

Symmetric functions, representation theory, and combinatorics

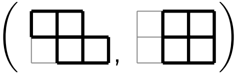
Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??
∇e_n	DH_n	Shuffle theorem

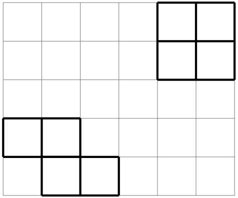
Outline

1. Background on symmetric functions and Macdonald polynomials
2. **Shuffle theorems, combinatorics, and LLT polynomials**
3. A new formula for Macdonald polynomials

Key Object: LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

$\boldsymbol{\nu} =$ 



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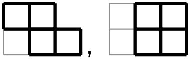
$\boldsymbol{\nu} = \Big($  $\Big)$

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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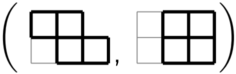
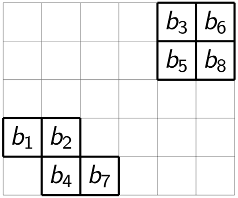
$\boldsymbol{\nu} =$ 



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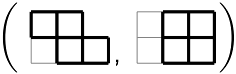
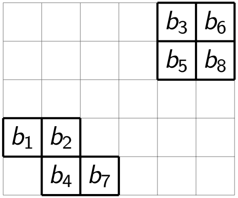
$\boldsymbol{\nu} =$ 



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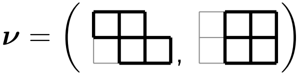

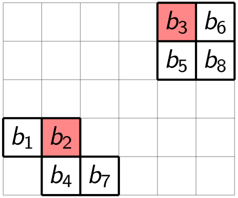

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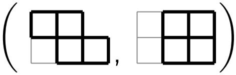
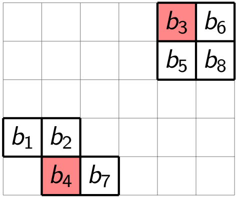
$\boldsymbol{\nu} =$ 



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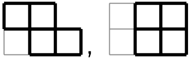
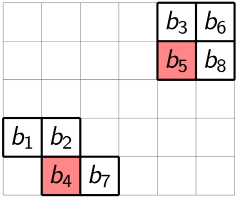
$\boldsymbol{\nu} = \Big($  $\Big)$



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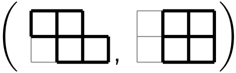
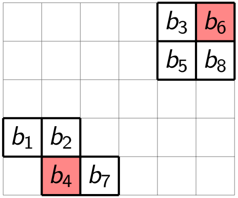
$\boldsymbol{\nu} =$ 



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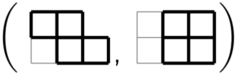
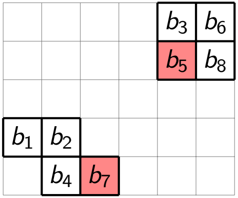
$\boldsymbol{\nu} =$ 



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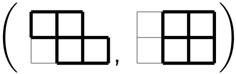
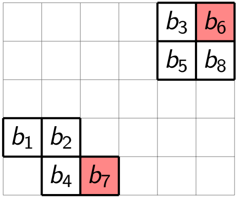
$\boldsymbol{\nu} =$ 



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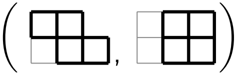
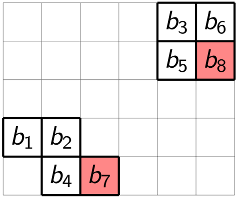
$\boldsymbol{\nu} =$ 



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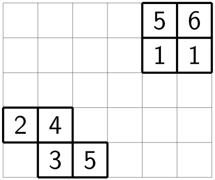

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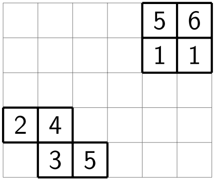

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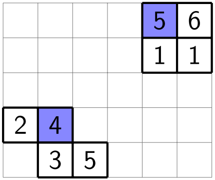

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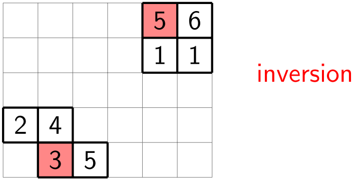

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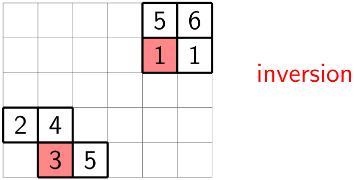

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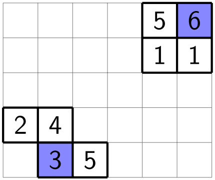

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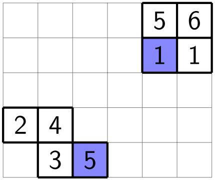

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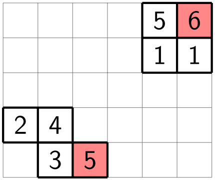

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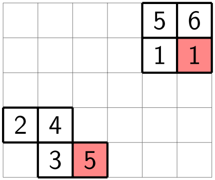
$T =$ 

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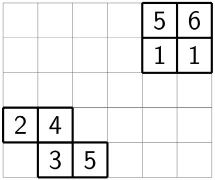

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- $\mathcal{G}_{\boldsymbol{\nu}}$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.
- $\mathcal{G}_{\boldsymbol{\nu}}$ is Schur-positive for any tuple of skew shapes $\boldsymbol{\nu}$ [Grojnowski-Haiman, 2007].

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} (q, t \ monomial)(LLT \ polynomial)$$

- Summation over all k-by-k Dyck paths.

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Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} (\mathit{LLT\ polynomial})$$

- Summation over all k-by-k Dyck paths.
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$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Summation over all k-by-k Dyck paths.
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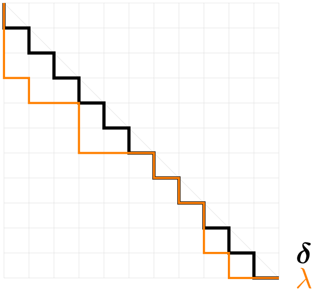
- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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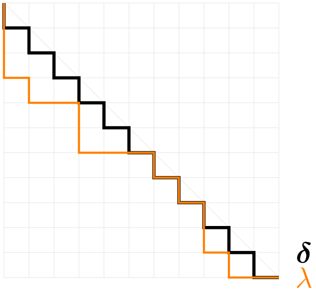

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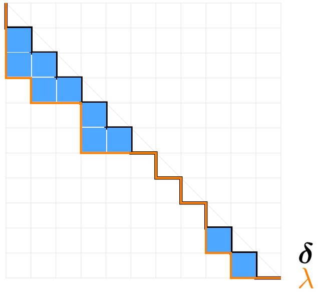

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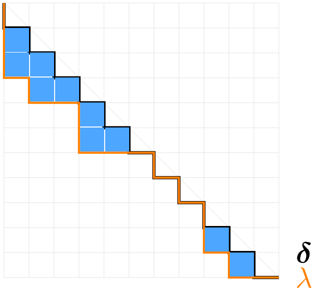

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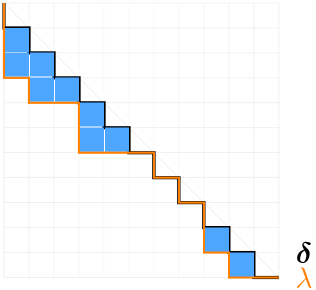

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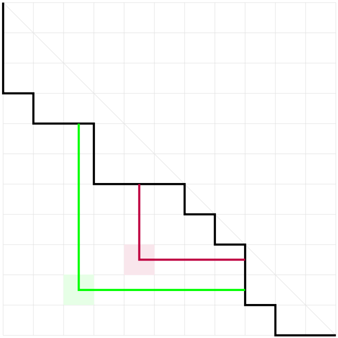
dinv

dinv(λ) $=\#$ of balanced hooks in diagram below λ.



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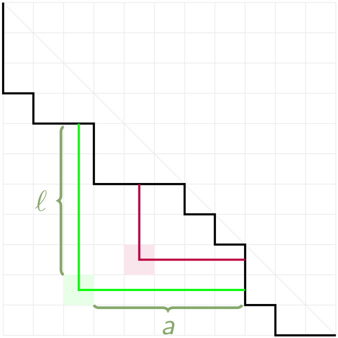

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1-\epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

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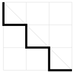		
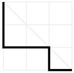		
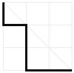		
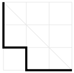		
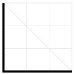		

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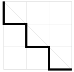
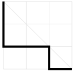
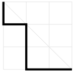
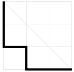
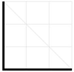
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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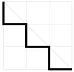
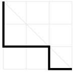
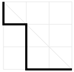
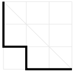
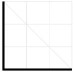
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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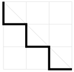
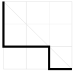
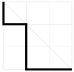
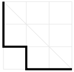
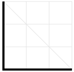
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qt s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + qt s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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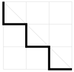
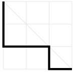
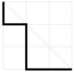
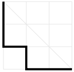
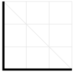
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + q t s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q, t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

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Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2021)

For $m, n > 0$ coprime, the operator $e_k^{(m,n)}$ acting on Λ satisfies

$$e_k^{(m,n)} \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$

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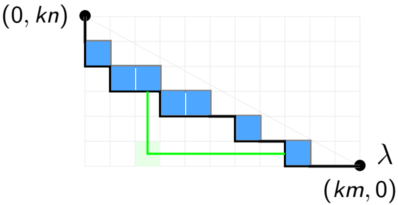

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Algebra $\mathcal{E} \curvearrowright \Lambda =$ symmetric polynomials

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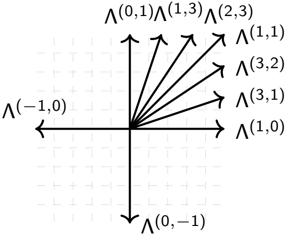

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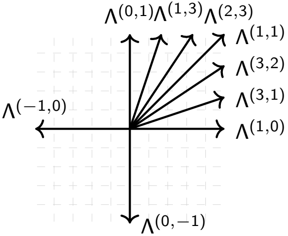

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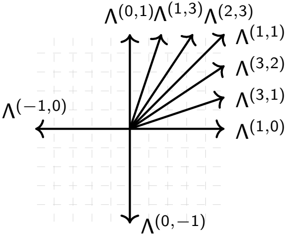

$\Lambda^{(m,n)} \cong$ symmetric polynomials

LHS of Shuffle Theorem $= e_k^{(1,1)} \in \Lambda^{(1,1)}$ acting on $1 \in \Lambda$.
LHS of Rational Shuffle Theorem $= e_k^{(m,n)} \in \Lambda^{(m,n)}$ acting on $1 \in \Lambda$.

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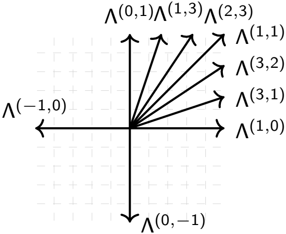

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Can be difficult to work with in general. Can we make it more explicit?

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \le i < j \le n\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

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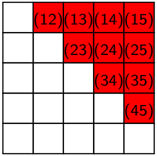
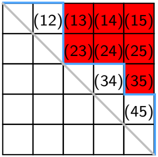


A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.



Ψ = Roots above Dyck path

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, set

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Then, $s_\gamma = \pm s_\lambda$ or 0 for some partition λ.
Precisely, for $\rho = (n-1, n-2, \ldots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\mathrm{sort}(\gamma+\rho)-\rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\mathrm{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\mathrm{sort}(\beta)$.

Example: $s_{201} = 0, s_{2\text{-}11} = -s_{200}$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma}\colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$\boldsymbol{z}^\gamma \mapsto s_\gamma(X)$$

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Example

$$\boldsymbol{\sigma}(\boldsymbol{z}^{111} + \boldsymbol{z}^{201} + \boldsymbol{z}^{210} + \boldsymbol{z}^{3\text{-}11}) = s_{111} + s_{201} + s_{210} + s_{3\text{-}11} = s_{111} + s_{210} - s_{300}$$

Catalanimals

Definition

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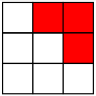
With $n = 3$, $R_+ =$ 

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) =$$

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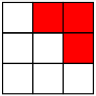
With $n = 3$, $R_+ =$ 

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1 \le i < j \le 3}(1 - qz_i/z_j)(1 - tz_i/z_j)} \right)$$

$$= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3$$

$$= \omega \nabla e_3.$$

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$.

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Proposition

For $(m,n) \in \mathbb{Z}_+^2$ *coprime,*

$$e_k^{(m,n)} \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

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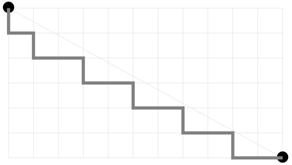
for $\mathbf{b} = (b_0, \ldots, b_{km-1})$ satisfying $b_i =$ the number of south steps on vertical line $x = i$ of highest lattice path under line $y + \frac{n}{m}x = n$.

$\delta =$ highest Dyck path.



δ

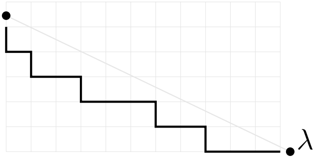
$$\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

Results

Manipulating Catalanimal $\Longrightarrow$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.



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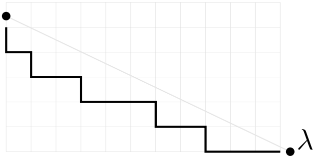
$$H(R_+, R_+, R^0_+, \mathbf{b})$$



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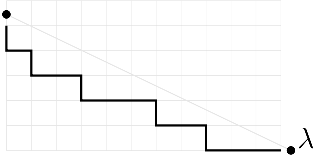
$$H(R_+, R_+, R_+^0, \mathbf{b}) =$$



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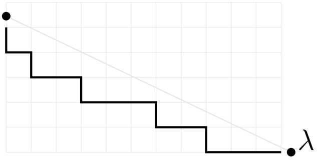
$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} \qquad\qquad \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



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$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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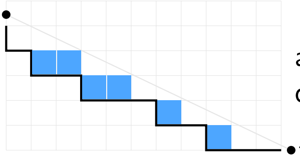

area(λ) as before

$\mathrm{dinv}_p(\lambda) = \#p$-balanced hooks $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

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Special case: $\mathcal{G}_{\boldsymbol{\nu}}^{(1,1)} \cdot 1 = \nabla \mathcal{G}_{\boldsymbol{\nu}}(X; q)$.

LLT Catalanimals

For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,

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  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in reading order gives λ.

LLT Catalanimals

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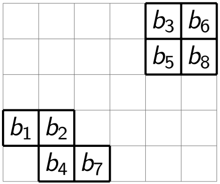
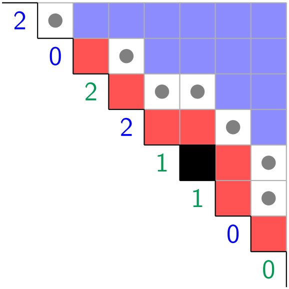


$\boldsymbol{\nu}$



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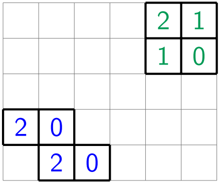
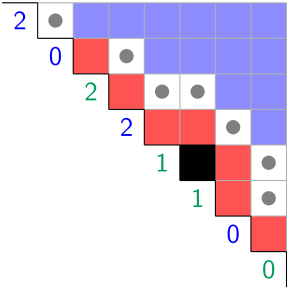


λ, as a filling of $\boldsymbol{\nu}$



LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let $\boldsymbol{\nu}$ be a tuple of skew shapes and let $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) &= c_{\boldsymbol{\nu}}\, \omega H_{\boldsymbol{\nu}} \\ &= c_{\boldsymbol{\nu}}\, \omega \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} (1 - qt\, \boldsymbol{z}^{\alpha})}{\prod_{\alpha \in R_q} (1 - q\, \boldsymbol{z}^{\alpha}) \prod_{\alpha \in R_t} (1 - t\, \boldsymbol{z}^{\alpha})} \right)\end{aligned}$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

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Outline

1. Background on symmetric functions and Macdonald polynomials
2. Shuffle theorems, combinatorics, and LLT polynomials
3. **A new formula for Macdonald polynomials**

Haglund-Haiman-Loehr formula example

$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u\in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}\right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X;q)$

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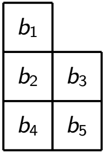
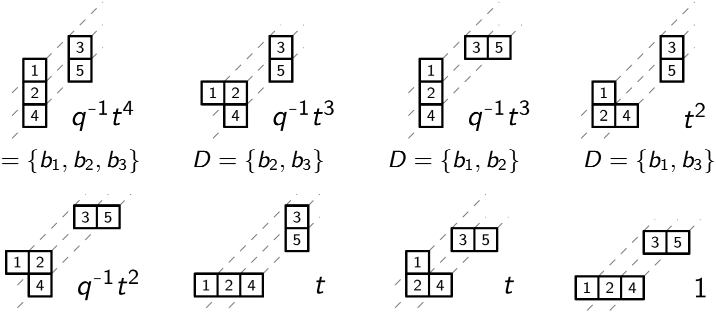


μ



$q^{-1}t^4$ — $D = \{b_1, b_2, b_3\}$

$q^{-1}t^3$ — $D = \{b_2, b_3\}$

$q^{-1}t^3$ — $D = \{b_1, b_2\}$

t^2 — $D = \{b_1, b_3\}$

$q^{-1}t^2$ — $D = \{b_2\}$

t — $D = \{b_3\}$

t — $D = \{b_1\}$

1 — $D = \varnothing$

Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.

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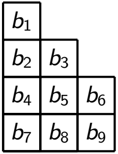
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{(b_i,b_j)\in V(\mu)}(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j)$ factor for $V(\mu)$ the set of vertical dominoes (b_i, b_j) in μ.

$$\tilde{H}_\mu = \omega\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod\limits_{\alpha_{ij}\in V(\mu)} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod\limits_{\alpha\in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha\in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

The root ideal R_μ



row reading order

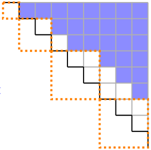
$b_1 \prec b_2 \prec \cdots \prec b_n$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\},$$

$$\widehat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \prec b_j\},$$

$$R_\mu \setminus \widehat{R}_\mu \leftrightarrow V(\mu) = \text{vertical dominoes in } \mu$$

Example:

$R_{3321} =$ 

The root ideal R_μ

$$\begin{array}{|c|c|c|}\hline b_1 \\ \cline{1-1} b_2 & b_3 \\ \cline{1-2} b_4 & b_5 & b_6 \\ \hline b_7 & b_8 & b_9 \\ \hline\end{array}$$

row reading order

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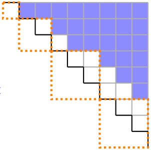
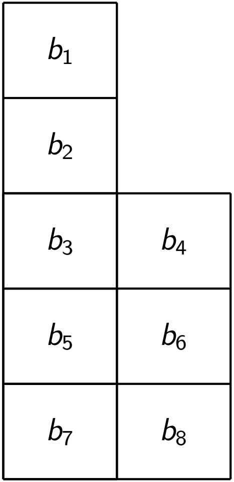
$R_{3321} =$ 

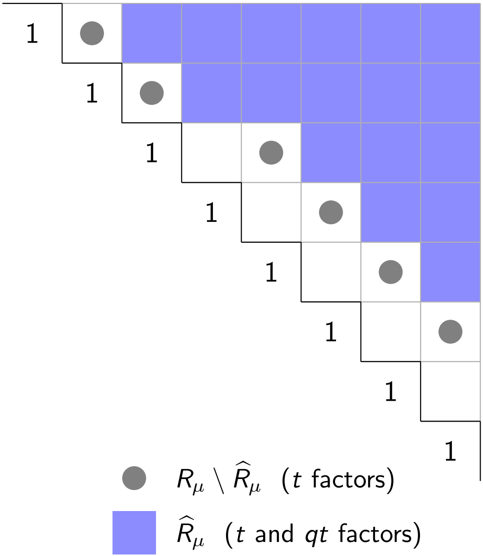
Remark

$$\tilde{H}_\mu(X;0,t) = \omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_\mu}(1 - t\boldsymbol{z}^\alpha)}\right)$$

Example



partition $\mu = 22211$



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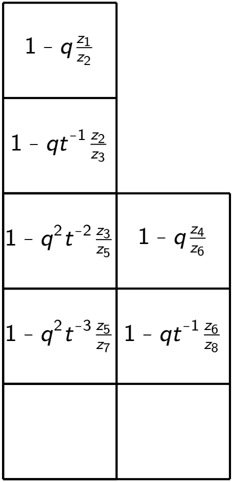
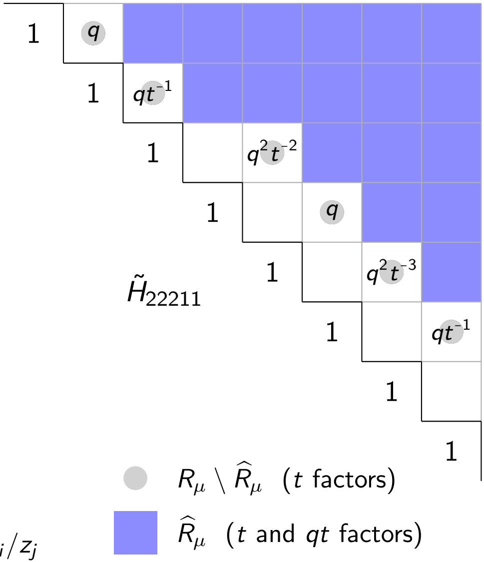


numerator factors $1 - q^{\mathrm{arm}+1}t^{-\mathrm{leg}}z_i/z_j$



$q = t = 1$ specialization

$$
\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha\in\widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha\in R_\mu}\left(1 - t\boldsymbol{z}^\alpha\right)} \right) \\
\overset{q=t=1}{\to} &\omega\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod_{\alpha\in R_\mu\setminus\widehat{R}_\mu}(1 - \boldsymbol{z}^\alpha) \prod_{\alpha\in\widehat{R}_\mu}(1 - \boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_+}(1 - \boldsymbol{z}^\alpha) \prod_{\alpha\in R_\mu}(1 - \boldsymbol{z}^\alpha)} \right) \\
= &\omega\boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha\in R_+}(1 - \boldsymbol{z}^\alpha)} \right) \\
= &\omega h_1^n \\
= &e_1^n
\end{aligned}
$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

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$$\tilde{H}_{\mu}^{(s)} := \omega \boldsymbol{\sigma} \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i / z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^{\alpha}\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^{\alpha}\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^{\alpha}\right)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}_{\mu}^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_{\mu}^{(s)} = \sum_{\nu} K_{\nu,\mu}^{(s)}(q,t)\, s_{\nu}(X)$$

satisfy $K_{\nu,\mu}^{(s)}(q,t) \in \mathbb{N}[q,t]$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	HHL
∇e_n	DH_n	Shuffle theorem
$\tilde{H}_\lambda^{(s)}(X; q, t)$	??	??

Thank you!

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