

Diagonal Harmonics and Shuffle Theorems

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on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
arXiv:2102.07931

Capsule Research Talk

23 August 2021

Outline

- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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- E.g. for $n = 3$,

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$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

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- Λ is a $\mathbb{Q}(q, t)$ -algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

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Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

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- Algebraic LHS: ∇e_k doubly graded character of diagonal coinvariants for S_k ((Haiman, 2002) via Hilbert Scheme connection).

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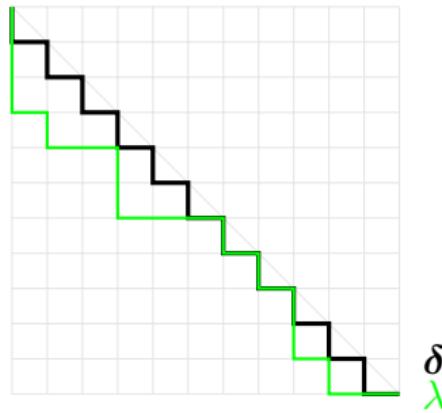
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- Summation over all k -by- k Dyck paths.
- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.
- $G_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

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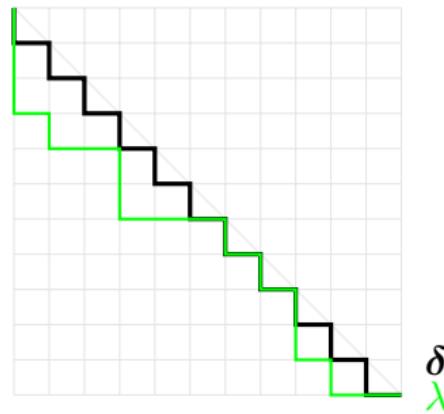
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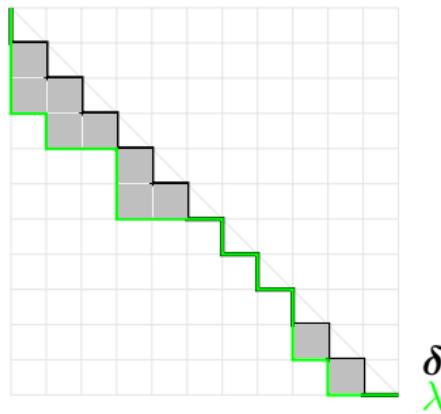


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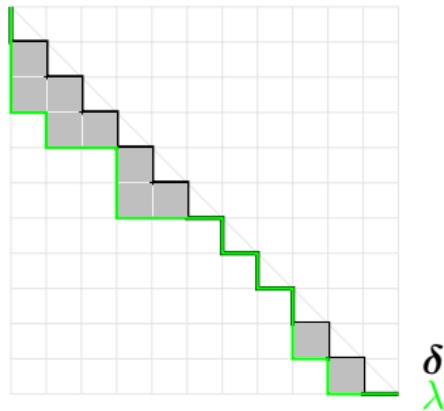


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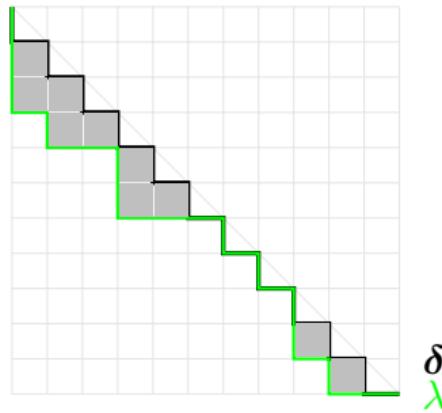


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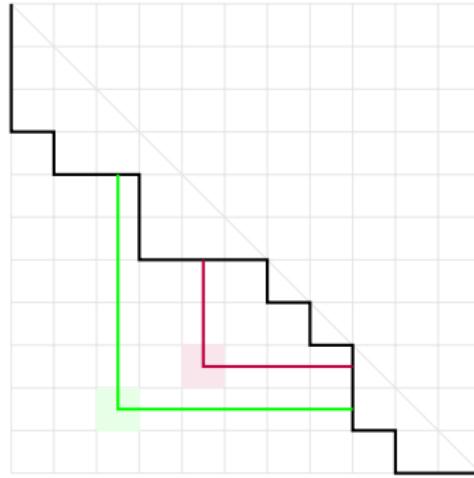
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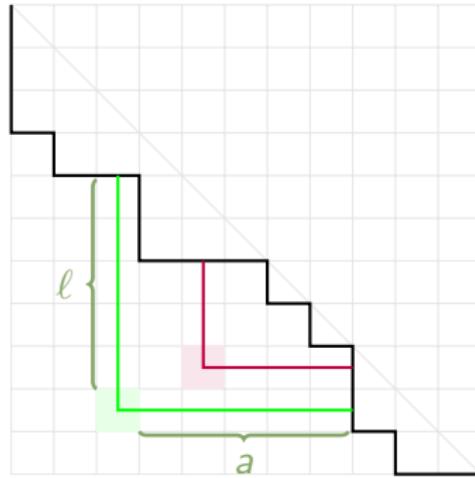
dinv

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Balanced hook is given by a cell below λ satisfying

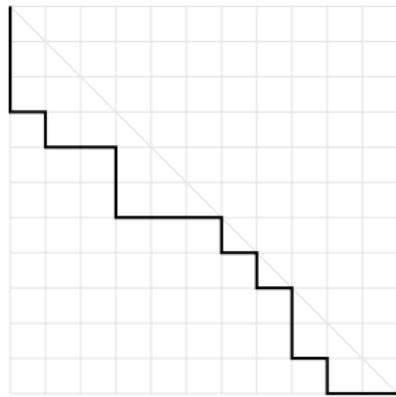
$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.

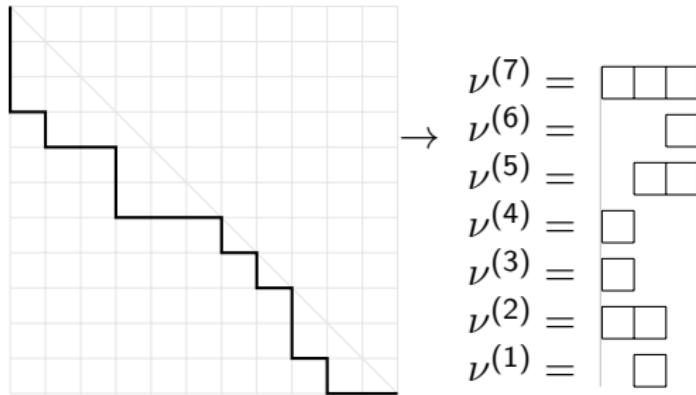
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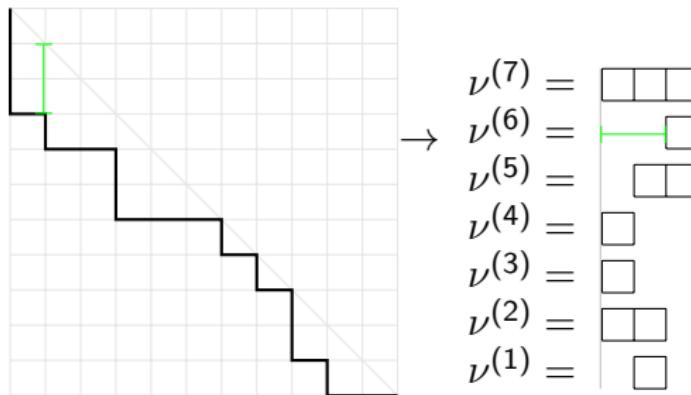
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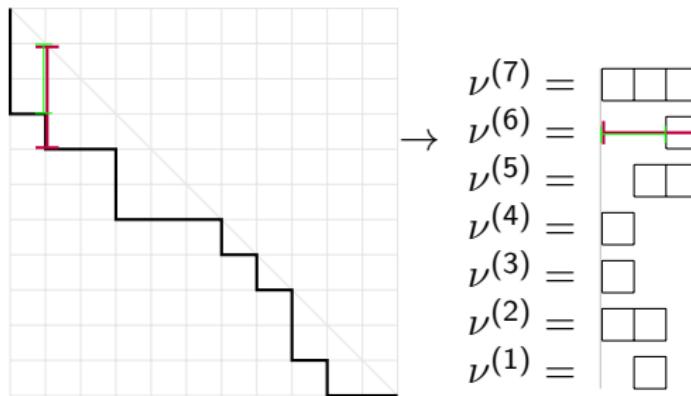
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for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

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---	---	---	---	---

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1	1	1	2	1	2	2	1	1	2	2
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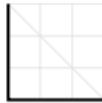
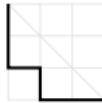
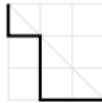
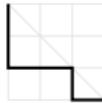
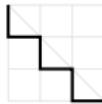
- \mathcal{G}_ν is symmetric and Schur positive.

Example ∇e_3

$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

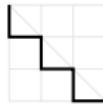
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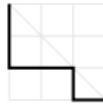


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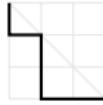
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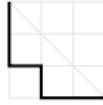
$$q^3$$



$$q^2 t$$



$$q t$$



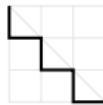
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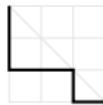
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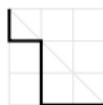
$$q^3$$

$$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$$



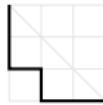
$$q^2 t$$

$$qts_{2,1} + q^2 ts_{1,1,1}$$



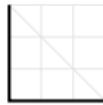
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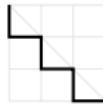


$$t^3$$

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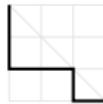
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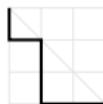
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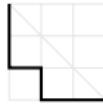
$$q^2 t$$

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$$qt^2$$

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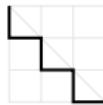
$$t^3$$

$$t^3 s_{1,1,1}$$

- Entire quantity is q, t -symmetric

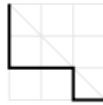
Example ∇e_3

$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$



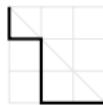
$$q^3$$

$$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$$



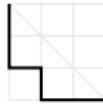
$$q^2 t$$

$$qts_{2,1} + q^2 ts_{1,1,1}$$



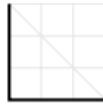
$$qt$$

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$$qt^2$$

$$t^2 s_{2,1} + qt^2 s_{1,1,1}$$



$$t^3$$

$$t^3 s_{1,1,1}$$

- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number”
 $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Outline

- Symmetric polynomials and The Shuffle Theorem
- **Generalizations of The Shuffle Theorem**
- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra \mathcal{E}

- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

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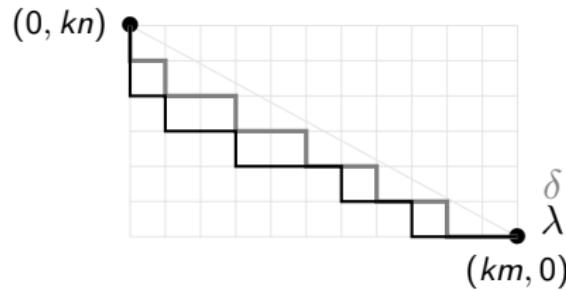
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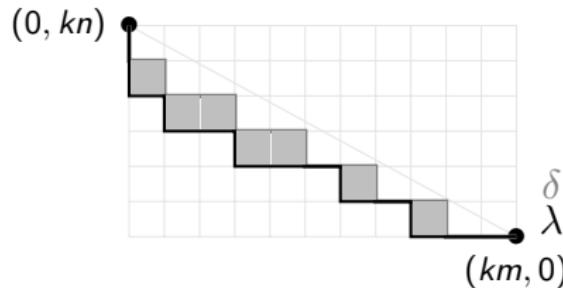
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Rational Path Combinatorics

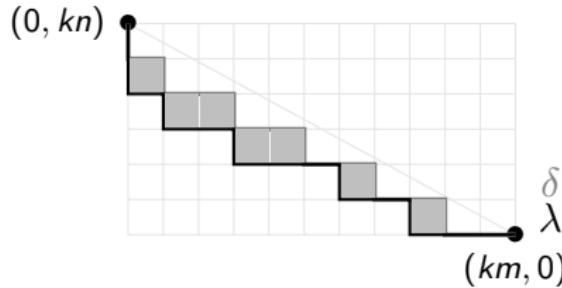


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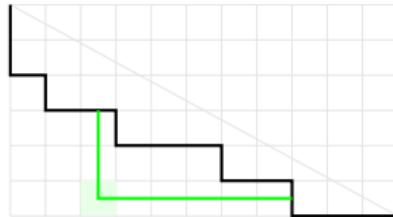


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$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \quad p = \frac{n}{m} - \epsilon$$

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Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \dots, b_I) \in \mathbb{Z}^I$ to be the south step sequence of highest path δ under the line $y + px = s$.

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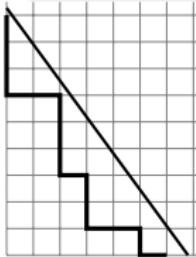
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Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1 < j} (1 - qtx_i/x_{i+1})}{\prod_{i < j} ((1 - \frac{x_j}{x_i})(1 - q \frac{x_i}{x_j})(1 - t \frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

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for infinite formal sum $\mathcal{L}_{\beta/\alpha}^\sigma$ a “series LLT.” (Grojnowski-Haiman, 2007).

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- (Twisted) non-symmetric Hall-Littlewood polynomials
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Loehr-Warrington Conjecture

$$\nabla s_\mu = \text{sgn}(\mu) \sum_{(G, R) \in LNDP_\mu} t^{\text{area}(G, R)} q^{\text{dinv}(G, R)} x^R$$

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- Coefficient of $s_{1,\dots,1}$ coincides with (q, t) -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

References

Thank you!

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