TWO ELEMENTARY EXAMPLES OF EXTREME CHARACTERS OF $U(\infty)$
INTEGRABLE PROBABILITY READING SEMINAR

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1. Introduction

First, we recall some definitions.

1.1. **Definition.** An $N \times N$ matrix U is *unitary* if $UU^* = I_N$ where U^* is the conjugate transpose of U. Then, $U(N)$ is the compact Lie group of all $N \times N$ unitary matrices. Since $U(N-1) \hookrightarrow U(N)$ via a canonical embedding, we also define

$$U(\infty) := \bigcup_{N=1}^{\infty} U(N)$$

that is, $U(\infty)$ are all infinite $\mathbb{N} \times \mathbb{N}$ unitary matrices that differ from the identity matrix only in a fixed number of positions.

1.2. **Definition.** A *normalized character* of $U(N)$ is a function $\chi \colon U(N) \to \mathbb{C}$ such that

(a) $\chi(e) = 1$ (normalized),
(b) $\chi(ab) = \chi(ba)$ (constant on conjugacy classes),
(c) $\left(\sum c_i \chi(a_i)\right) \overline{\left(\sum c_j \chi(a_j)\right)} = \sum c_i \bar{c}_j \chi(a_i a_j^{-1}) \geq 0$ (nonnegative definite),
(d) χ is continuous.

Normalized characters form a convex set since $t\chi_1 + (1-t)\chi_2$ meets all the axioms of a normalized character for all $t \in [0,1]$. Then, we can discuss the following notion.

1.3. **Definition.** An *extreme character* $\chi \colon U(N) \to \mathbb{C}$ is a normalized character such that $\chi \neq t\chi_1 + (1-t)\chi_2$ for any $t \in (0,1)$ for normalized characters $\chi_1, \chi_2 \neq \chi$.

1.4. **Definition.** The N*-dimensional* torus is

$$\mathbb{T}^N := \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid |x_i| = 1\}$$

and lies in $U(N)$ as diagonal matrices. The *finitary torus* is $\mathbb{T}^{\infty}_{fin} := \bigcup_{N=1}^{\infty} \mathbb{T}^N$.

Recall one of our main goals is to understand the following theorem.

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1.5. **Theorem** (Edrei-Voiculescu)**.** *Extreme characters of $U(\infty)$ are functions $\chi\colon T^{\infty}_{fin} \to \mathbb{C}$ depending on countably many parameters*

$$\begin{cases} \alpha^{\pm} = (\alpha^{\pm}_1 \geq \alpha^{\pm}_2 \geq \cdots \geq 0); \\ \beta^{\pm} = (\beta^{\pm}_1 \geq \beta^{\pm}_2 \geq \cdots \geq 0); \\ \gamma^{\pm} \geq 0 \end{cases}$$

such that

$$\sum_i \alpha^+_i + \sum_i \alpha^-_i + \sum_i \beta^+_i + \sum_i \beta^-_i < \infty, \quad \beta^+_1 + \beta^-_1 \leq 1$$

Furthermore, these functions have the form

$$\chi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \Phi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}(x_j)$$

where $\Phi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}\colon \mathbb{T} \to \mathbb{C}$ is the continuous function

$$\Phi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}(x) := e^{\gamma^+(x-1)+\gamma^-(x^{-1}-1)} \prod_{i=1}^{\infty} \left(\frac{1+\beta^+_i(x-1)}{1-\alpha^+_i(x-1)} \cdot \frac{1+\beta^-_i(x^{-1}-1)}{1-\alpha^-_i(x^{-1}-1)} \right).$$

1.6. **Goal.** In this presentation, we will outline two very special examples of this parameterization, namely when

(a) $\beta^+ = (\beta, 0, 0, \ldots), \beta^- = \alpha^{\pm} = (0, 0, \ldots), \gamma^{\pm} = 0$ for $\beta \in [0, 1]$ so that

$$\Phi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}(x) = 1 + \beta(x-1) \Longrightarrow \chi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty}(1 + \beta(x_j - 1))$$

(b) $\alpha^+ = (\alpha, 0, 0, \ldots), \beta^{\pm} = \alpha^+ = (0, 0, \ldots), \gamma^{\pm} = 0$ for $\alpha \in [0, 1]$ so that

$$\Phi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}(x) = \frac{1}{1-\alpha(x-1)} \Longrightarrow \chi_{\alpha^{\pm},\beta^{\pm},\gamma^{\pm}}(x_1, x_2, \ldots) = \prod_{j=1}^{\infty} \frac{1}{1-\alpha(x_j - 1)}$$

2. Symmetric Functions

In the last lecture, we introduced the following.

2.1. **Definition.** Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, the *Schur polynomial* is given by

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det(x_j^{\lambda_i+N-i})^N_{i,j=1}}{\det(x_j^{N-i})^N_{i,j=1}}$$

Also, if λ has $\lambda_N \geq 0$, we can use "Littlewood's Combinatorial Description" of Schur functions

2.2. **Proposition.** *Given a sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$,*

$$s_\lambda(x_1, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(T)}$$

where $x^{\mathrm{wt}(T)} = \prod_{j=1}^{\sum \lambda_i} x_j^{\#\text{ of } j\text{'s in } T}$.

2.3. **Example.**

$$s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

We also proved that

2.4. **Theorem.** *The irreducible representations of $U(N)$ are in one-to-one correspondence with $\{\lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \cdots \geq \lambda_N\}$ where the character of representation T_λ of $U(N)$ corresponding to λ has character given by*

$$\mathrm{Tr}\left(T_\lambda\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix}\right) = s_\lambda(x_1, \ldots x_N)$$

We will work with two special cases of the Schur polynomials.

2.5. **Definition.** Let $e_m(x_1, \ldots, x_N) := s_{(1^m)}(x_1, \ldots, x_N)$ be the *elementary symmetric polynomials.*

2.6. **Example.** Using the semistandard Young tableaux formula for Schur functions (Littlewood's combinatorial description), we compute

(a)

$$e_2(x_1, x_2) = x_1 x_2$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

(b)

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} + \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} + \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$$

(c)

$$e_3(x_1, x_2, x_3) = x_1 x_2 x_3$$

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

2.7. **Remark.** $e_N(x_1, \ldots, x_N)$ encodes character of the "determinant representation" of $U(N)$, that is

$$T(U)v = (\det U)v = x_1 x_2 \cdots x_N v$$

since the determinant is just the product of the eigenvalues. More generally, $e_m(x_1, \ldots, x_N)$ encodes the representation induced by the $U(N)$-action on $\bigwedge^m \mathbb{C}^N$:

$$U \cdot (v_1 \wedge \cdots \wedge v_m) = (Uv_1 \wedge \cdots \wedge Uv_m)$$

Importantly, we also compute, generalizing our example above

2.8. **Proposition.** *For $0 < m \leq n$,*

$$e_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}((1^m)) \textit{ filled with elements of } \{1,\ldots,n\}} x^{\mathrm{wt}(T)} = \sum_{I \subseteq \{1,\ldots,n\}, |I|=m} x^I$$

where $x^I := \prod_{i \in I} x_i$ and consequently,

$$e_m(\underbrace{1, \ldots, 1}_{n}) = \binom{n}{m}$$

Proof. To see this, we simply observe that a single column semistandard tableau with m rows filled with letters $\{1, \ldots, n\}$ is a choice of m distinct elements of $\{1, \ldots, n\}$ since columns must be strictly increasing. $\square$

2.9. **Definition.** Let $h_m(x_1, \ldots, x_N) := s_{(m)}(x_1, \ldots, x_N)$ be the *complete homogeneous symmetric polynomials.*

2.10. **Example.** Using again our tableaux formula for Schur functions, we compute

(a)

$$h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$$
$$\boxed{1 \mid 1} + \boxed{1 \mid 2} + \boxed{2 \mid 2}$$

(b)

$$h_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$
$$\boxed{1 \mid 1} + \boxed{1 \mid 2} + \boxed{1 \mid 3} + \boxed{2 \mid 2} + \boxed{2 \mid 3} + \boxed{3 \mid 3}$$

2.11. **Proposition.** *For $0 < m \leq n$,*

$$h_m(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}((m)) \textit{ filled with elements of } \{1,\ldots,n\}} x^{\mathrm{wt}(T)} = \sum_{I \textit{ multiset of } \{1,\ldots,n\}, |I|=m} x^I$$

where $x^I := \prod_{i \in I} x_i$ and consequently,

$$\begin{aligned} &h_m(\underbrace{1, \ldots, 1}_{n}) \\ &= \textit{Number of ways to choose a multiset of size } m \textit{ from } n \textit{ things} \\ &= \binom{n+m-1}{m} = \binom{n+m-1}{n-1} \end{aligned}$$

2.12. **Remark.** The combinatorics of the identity above follow by considering a "stars and bars" approach, namely, both expressions are in bijection with the number of ways to place $n-1$ bars among m stars, allowing bars to be consecutive with each other.

$$\{1,1,1,2,4,5\} \to \star\star\star|\star||\star|\star$$

2.13. **Definition.** Let

$$\left(\!\!\binom{n}{m}\!\!\right) := \binom{n+m-1}{m}$$

be the number of ways to choose a multiset of size m from n things.

3. Two Examples of $U(\infty)$ characters

Now, we wish to take a sequence of $U(N)$ characters to get a character of $U(\infty)$.

3.1. **Definition.** We say that a sequence of central functions f_N (i.e. f_N only depends on the eigenvalues of the input) on $U(N)$ *converge* to a central function f on $U(\infty)$ if, for every fixed K, we have

$$f_N(x_1,\dots,x_K,1,1,\dots,1) \to f(x_1,\dots,x_K,1,1,\dots)$$

uniformly on the K-torus $\mathbb{T}^K$ of diagonal matrices.

3.2. **Proposition.** *Let $L\colon \mathbb{N} \to \mathbb{N}$ be a sequence such that $L(N)/N \to \beta \in [0,1]$ as $N \to \infty$. Then,*

$$\frac{e_{L(N)}(x_1,\dots,x_N)}{e_{L(N)}(1,\dots,1)} \to \prod_{i=1}^{\infty}(1+\beta(x_i-1)), \quad (x_1,x_2,\dots) \in \mathbb{T}^\infty_{fin}$$

Proof. Fix $K \le N$. Then,

$$\begin{aligned}
&e_{L(N)}(x_1,\dots,x_K,1,\dots,1)\\
&= \sum_{T\in \mathrm{SSYT}((1^{L(N)}))\text{ labelled with }\{1,\dots,N\}} x^{\mathrm{wt}(T|_{\le K})}\\
&= \sum_{\text{binary } K \text{ sequences } \epsilon} \#\{N \text{ sequences with sum } L(N) \text{ that start with } (\epsilon_1,\dots,\epsilon_K)\}x^{(\epsilon_1,\dots,\epsilon_K)}\\
&= \sum_{\text{binary } K \text{ sequences } \epsilon} \binom{N-K}{L(N)-\sum_{i=1}^K \epsilon_i} x_1^{\epsilon_1}\cdots x_K^{\epsilon_K}
\end{aligned}$$

where the last equality comes from considering how to fill tableaux of the form

$$\begin{array}{l}\left.\begin{array}{c}\square\\\square\\\square\\\square\end{array}\right\}\text{Fill } \textstyle\sum_{i=1}^K \epsilon_i \text{ boxes with } \{1,\dots,K\}\\ \left.\begin{array}{c}\square\\\square\\\square\\\square\end{array}\right\}\text{Fill } L(N)-\textstyle\sum_{i=1}^K \epsilon_i \text{ boxes with } \{K+1,\dots,N\}\end{array}$$

$$\Longrightarrow \frac{e_{L(N)}(x_1,\dots,x_K,1,\dots,1)}{e_{L(N)}(1,\dots,1)} = \sum_{\text{binary } K \text{ sequences } \epsilon} \left(\binom{N-K}{L(N)-\sum_{i=1}^K \epsilon_i}\Big/\binom{N}{L(N)}\right) x_1^{\epsilon_1}\cdots x_K^{\epsilon_K}$$

where

$$\binom{N-K}{L(N)-\sum_{i=1}^K \epsilon_i}\Big/\binom{N}{L(N)} = \frac{(N-K)!}{N!}\times\frac{(L(N))!}{(L(N)-\sum_{i=1}^K \epsilon_i)!}\times\frac{(N-L(N))!}{(N-L(N)-(K-\sum_{i=1}^K \epsilon_i))!}$$
$$\xrightarrow{N\to\infty} \beta^{\sum_{i=1}^K \epsilon_i}(1-\beta)^{\sum_{i=1}^K \epsilon_i} \text{ since } L(N)/N\to\beta$$

Thus, taking the limit as $N\to\infty$ on our ratio, we get

$$\sum_{\text{binary } K \text{ sequences } \epsilon} x_1^{\epsilon_1}\cdots x_K^{\epsilon_K}\beta^{\sum_{i=1}^K \epsilon_i}(1-\beta)^{K-\sum_{i=1}^K \epsilon_i} = \prod_{i=1}^K((1-\beta)+\beta x_i)$$

and so, taking $K\to\infty$ completes the proof. $\square$

3.3. **Remark.** An astute reader may notice that $(1-\beta)^{K-\sum_{i=1}^K \epsilon_i}\beta^{\sum_{i=1}^K \epsilon_i}$ represenents the probability of $\sum_{i=1}^K \epsilon_i$ successes in K trials where each attempt has probability of success β. One can use "de Finetti's theorem" in order to derive the proposition directly from this observation. See [Pet12]§4.1.10 for this approach.

3.4. **Proposition.** *Let $L\colon \mathbb{N}\to\mathbb{N}$ be a sequence such that $L(N)/N\to\alpha\in[0,1]$ as $N\to\infty$. Then,*

$$\frac{h_{L(N)}(x_1,\dots,x_N)}{h_{L(N)}(1,\dots,1)} \to \prod_{i=1}^\infty \frac{1}{1-\alpha(x_i-1)}, \quad (x_1,x_2,\dots)\in\mathbb{T}^\infty_{fin}$$

Proof. We proceed much as in the proposition above. For a fixed $K\le N$, we have

$$\begin{aligned} &h_{L(N)}(x_1,\dots,x_K,1,\dots,1)\\ &=\sum_{\epsilon\in\mathbb{N}_0^K} \#\{N \text{ sequences with sum } L(N) \text{ starting with } (\epsilon_1,\dots,\epsilon_K)\}x_1^{\epsilon_1}\cdots x_K^{\epsilon_K}\\ &=\sum_{\epsilon\in\mathbb{N}_0^K}\left(\!\!\binom{N-K}{L(N)-\sum_{i=1}^K \epsilon_i}\!\!\right) x_1^{\epsilon_1}\cdots x_K^{\epsilon_K}\end{aligned}$$

where the last line comes from thinking about

$$\underbrace{\square\square\square\cdots\square}_{\substack{\text{Fill } \sum_{i=1}^K \epsilon_i \text{boxes}\\ \text{with } \{1,\dots,K\}}}\underbrace{\square\square\square\cdots\square}_{\substack{\text{Fill } N-\sum_{i=1}^K \epsilon_i \text{ boxes}\\ \text{with } \{K+1,\dots,N\}}}$$

and so

$$\begin{aligned}
&\frac{h_{L(N)}(x_1,\ldots,x_K,1,\ldots,1)}{h_{L(N)}(1,\ldots,1)} \\
&= \sum_{\epsilon\in\mathbb{N}_0^K}\left[\left(\!\!\binom{N-K}{L(N)-\sum_{i=1}^K\epsilon_i}\!\!\right)\Big/\left(\!\!\binom{N}{L(N)}\!\!\right)\right]x_1^{\epsilon_1}\cdots x_K^{\epsilon_K}
\end{aligned}$$

Consider that, for fixed $K\le N$, we have

$$\begin{aligned}
&\left(\!\!\binom{N-K}{L(N)-\sum_{i=1}^K\epsilon}\!\!\right)\Big/\left(\!\!\binom{N}{L(N)}\!\!\right) \\
&= \binom{N-K+L(N)-\sum_{i=1}^K\epsilon_i-1}{L(N)-\sum_{i=1}^K\epsilon_i}\Big/\binom{N+L(N)-1}{L(N)} \\
&= \frac{(N+L(N)-K-\sum\epsilon_i-1)!}{(N+L(N)-1)!}\times\frac{(L(N))!}{(L(N)-\sum\epsilon_i)!}\times\frac{(N-1)!}{(N-K-1)!} \\
&\approx \frac{(L(N))^{\sum\epsilon_i}N^K}{(N+L(N))^{K+\sum\epsilon_i}} \\
&= \left(\frac{L(N)}{N}\right)^{\sum\epsilon_i}\left(\frac{1}{1+\frac{L(N)}{N}}\right)^{K+\sum\epsilon_i} \\
&\xrightarrow{N\to\infty}\left(\frac{\alpha}{1+\alpha}\right)^{\sum\epsilon_i}\left(\frac{1}{1+\alpha}\right)^{K}
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{N\to\infty}\frac{h_{L(N)}(x_1,\ldots,x_K,1,\ldots,1)}{h_{L(N)}(1,\ldots,1)} &= \sum_{\epsilon}\left(\frac{1}{1+\alpha}\right)^K\left(\frac{\alpha}{1+\alpha}\right)^{\sum\epsilon_i}x_1^{\epsilon_1}\cdots x_K^{\epsilon_K} \\
&= \prod_{i=1}^K\left(\frac{1}{1+\alpha}\right)\left(1+\frac{\alpha}{1+\alpha}x_i+\left(\frac{\alpha}{1+\alpha}\right)^2x_i^2+\cdots\right) \\
&= \prod_{i=1}^K\frac{1}{1+\alpha}\times\frac{1}{1-\frac{\alpha}{1+\alpha}x_i} \\
&= \prod_{i=1}^K\frac{1}{1+\alpha-\alpha x_i}
\end{aligned}$$

So, taking $K\to\infty$ completes the proof. $\square$

References

[Pet12] L. Petrov, *Representation Theory of Big Groups and Probability* (2012). Accessed as a draft from https://lpetrov.cc/reading-2019/ on January 31, 2019.