

# A Window into Symmetric Function Theory

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UVA Math Club  
Lightning Round

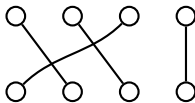
2 March 2021

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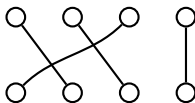
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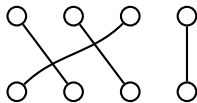
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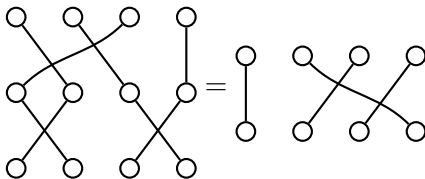
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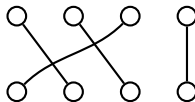
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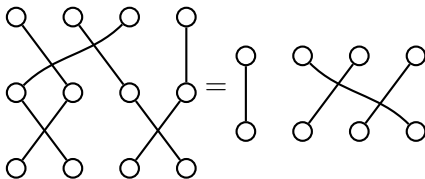


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- $\sigma \in S_n$  acts as  $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$  forms a vector space,  $\Lambda_{\mathbb{Q}}$ .

# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of  $\Lambda_{\mathbb{Q}}$ ?

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .



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5 → 


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|--|--|--|--|--|
|  |  |  |  |  |
|--|--|--|--|--|

$4 + 1 \rightarrow$ 

|  |  |  |  |
|--|--|--|--|
|  |  |  |  |
|  |  |  |  |

$3 + 2 \rightarrow$ 

|  |  |  |
|--|--|--|
|  |  |  |
|  |  |  |

$3 + 1 + 1 \rightarrow$  

$2 + 2 + 1 \rightarrow$ 

|  |  |
|--|--|
|  |  |
|  |  |
|  |  |

$$2 + 1 + 1 + 1 \rightarrow$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

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- ② Many interesting connections to number theory (Ramanujan).
- ③ Generating function for  $p(n)$  = number of partitions of  $n$  is inverse of Euler  $\phi$  function.

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For  $\lambda = (2, 1)$ ,

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 |   | 3 |   | 3 |   | 3 |   |

# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

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|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
| 1 | 1 |  |  |  |  |  |  |  |  |
| 2 |   |  |  |  |  |  |  |  |  |

, 

|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
| 1 | 1 |  |  |  |  |  |  |  |  |
| 3 |   |  |  |  |  |  |  |  |  |

, 

|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
| 2 | 2 |  |  |  |  |  |  |  |  |
| 3 |   |  |  |  |  |  |  |  |  |

, 

|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
| 1 | 2 |  |  |  |  |  |  |  |  |
| 2 |   |  |  |  |  |  |  |  |  |

, 

|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
| 1 | 3 |  |  |  |  |  |  |  |  |
| 3 |   |  |  |  |  |  |  |  |  |

, 

|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
| 2 | 3 |  |  |  |  |  |  |  |  |
| 3 |   |  |  |  |  |  |  |  |  |

, 

|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
| 1 | 3 |  |  |  |  |  |  |  |  |
| 2 |   |  |  |  |  |  |  |  |  |

, 

|   |   |  |  |  |  |  |  |  |  |
|---|---|--|--|--|--|--|--|--|--|
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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda = \sum_{T \in \text{SSYT}} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for  $\Lambda_{\mathbb{Q}}$



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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

# Harmonic polynomials

- ①  $S_3$  action on  $M$  fixes vector subspaces!

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- ③ How many times does an  $S_n$  fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ③ How many times does an  $S_n$  fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Schur basis expansion counts multiplicity of irreducible  $S_n$  fixed subspaces!

## Upshot

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- 2 Via Frobenius characteristic map, questions about  $S_n$ -action on vector spaces get translated to questions about Schur expansion coefficients in symmetric functions.

Interesting algebraic combinatorics questions

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- 1 Is a symmetric function Schur positive?



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- 1 Is a symmetric function Schur positive?
- 2 What do the Schur expansion coefficients count?

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Solution: minimal  $S_n$ -fixed subspace of degree  $d \mapsto q^d s_\lambda$  (graded Frobenius)

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Minimal  $S_n$ -invariant subspace with bidegree  $(a, b) \mapsto q^a t^b s_\lambda$

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# Diagonal harmonics

- Define  $\nabla$  by  $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$  for eigenvalue  $B_\mu(q, t) \in \mathbb{Q}[q, t]$ .

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What is the Schur expansion of  $\nabla e_n$ ?

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Recover earlier story by taking  $t = 0$  and  $y_i = 1$  for all  $y_i$ 's.