

Dens, nests, and Catalan animals: a walk through the zoo of shuffle theorems

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- $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial

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$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

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$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the e_1, e_2, \dots , or in the h_1, h_2, \dots

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

Partitions

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\hline\hline\hline\hline\hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|}\hline\hline\hline\hline\hline\end{array}$$

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Collection is called $\text{SSYT}(\lambda)$.

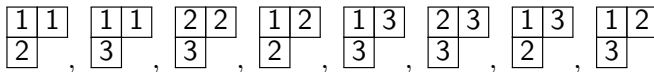
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For $\lambda = (2, 1)$,

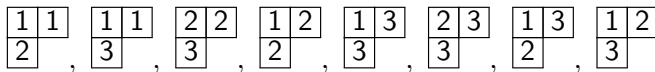


Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

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Weight: $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$, $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$
 $(2,1,0)$ $(2,0,1)$ $(0,2,1)$ $(1,2,0)$ $(1,0,2)$ $(0,1,2)$ $(1,1,1)$ $(1,1,1)$

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

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Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

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$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

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- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.

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Via Frobenius characteristic map, questions about S_n -representations get translated to questions about Schur expansion coefficients in symmetric functions.

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Upshot

Via Frobenius characteristic map, questions about S_n -representations get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

Getting more information

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Break M up into smallest S_n fixed subspaces

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Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Answer: "Hall-Littlewood polynomial" $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n -representations whose (bigraded) Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$. (Still open!)

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n ?

Frobenius characteristic of DH_3

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n .

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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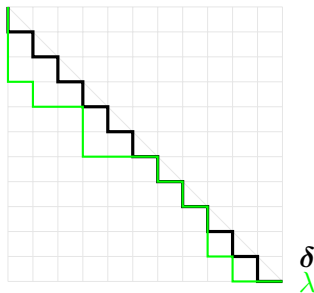
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- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

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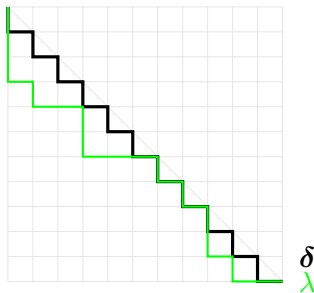
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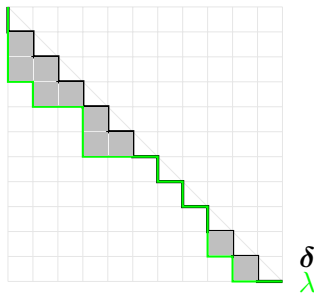


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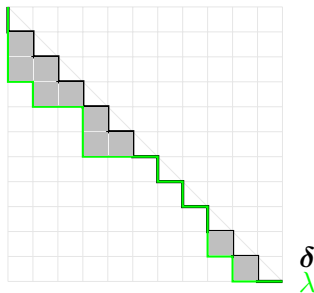


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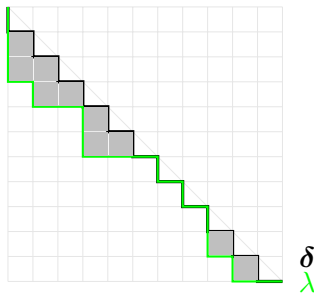


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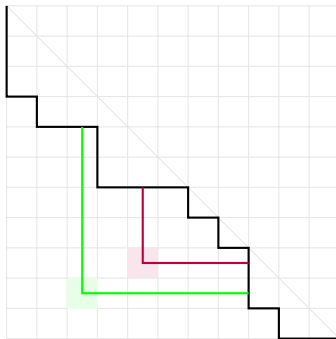
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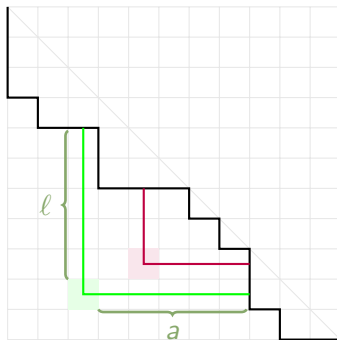
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dinv

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Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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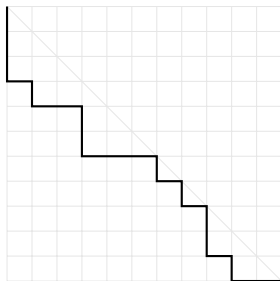
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- \mathcal{G}_ν is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
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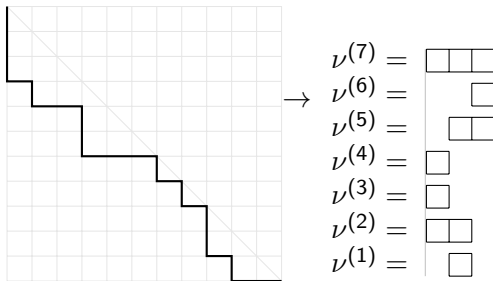
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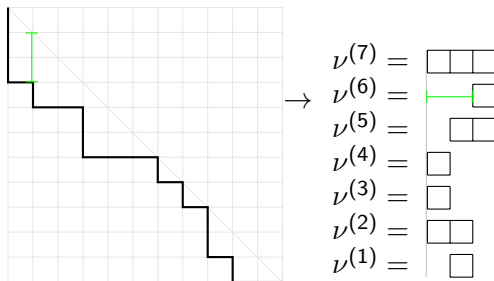
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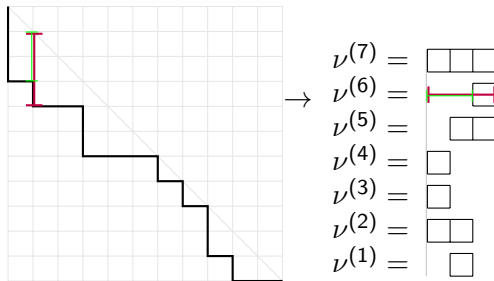
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for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

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$$\mathcal{G}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

Example ∇e_3






$$\lambda \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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




Example ∇e_3

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	$q^2 t$	
	qt	
	qt^2	
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




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- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number”
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For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

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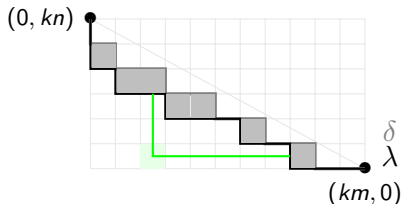
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- \mathcal{E} contains subalgebra $\Lambda(X^{m,n}) \cong \Lambda$ for each coprime pair $(m, n) \in \mathbb{Z}^2$.
- In general, \mathcal{E} -action can be a pain to compute in a nice way, but sometimes it is nice!

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- We can take “polynomial part” (restrict to only polynomial GL_l -characters) to get a symmetric function.

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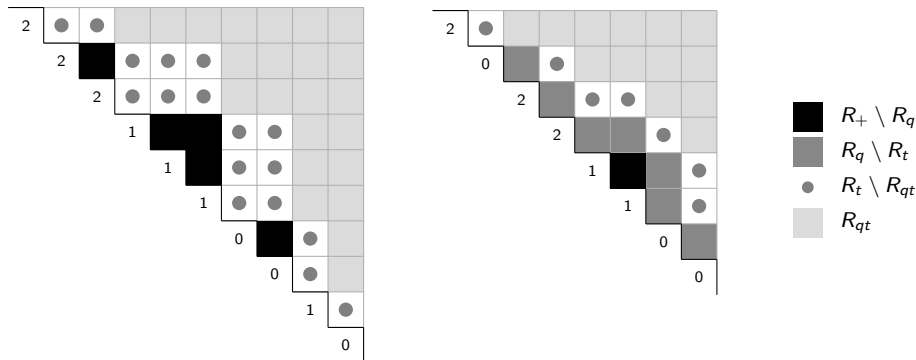
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- In this case, we set $\operatorname{cub}(H) = f$.
- The cuddly conditions allow a nice coproduct formula for $f[X + Y]$ in terms of cubs of “restrictions” of H .

Cuddly Catalananimals with cub e_k

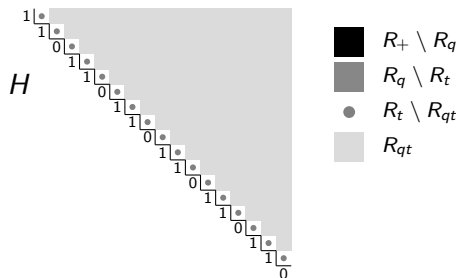
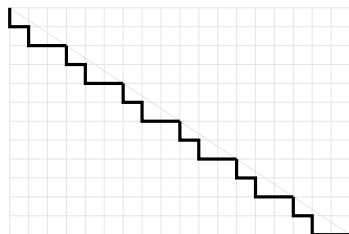
- $H(R_+, R_+, [R_+, R_+], (1^k))$ is $(1, 1)$ -cuddly with cub e_k .

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$\delta = (1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0)$ and $e_6[-MX^{3,2}] \cdot 1 = \omega \operatorname{pol}_X H$

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 \rightarrow

+	.	.	-
+	.	-	
+	.	-	

 \rightarrow

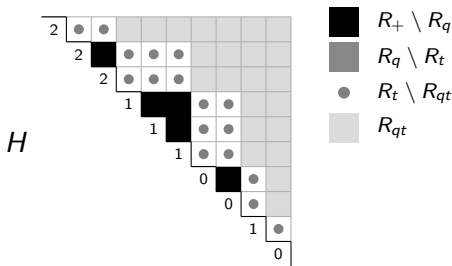
1	0	1	0
2	1	0	
2	2	1	

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Theorem (Blasiak-Haiman-Morse-Pun-S. (2021⁺))

For every partition μ and coprime positive integers m, n , we have

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- Conjectured by Loehr-Warrington (2008) when $n = 1$ with different combinatorics (but bijectively related).

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For given partition λ

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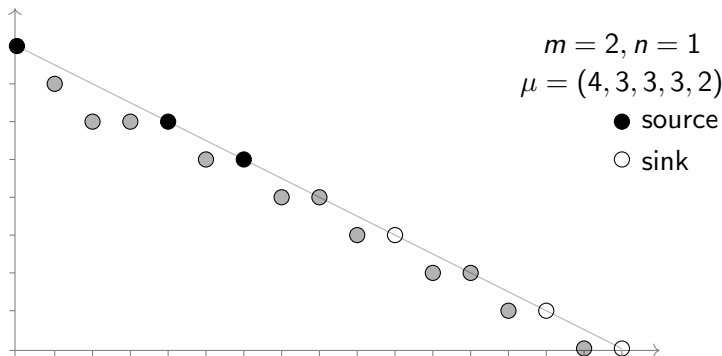
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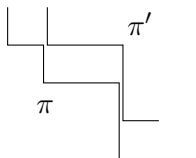
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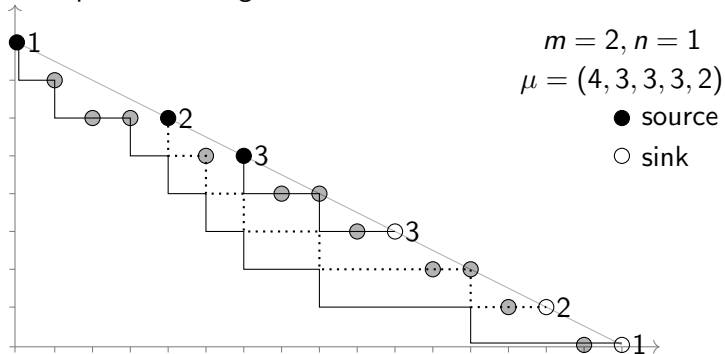
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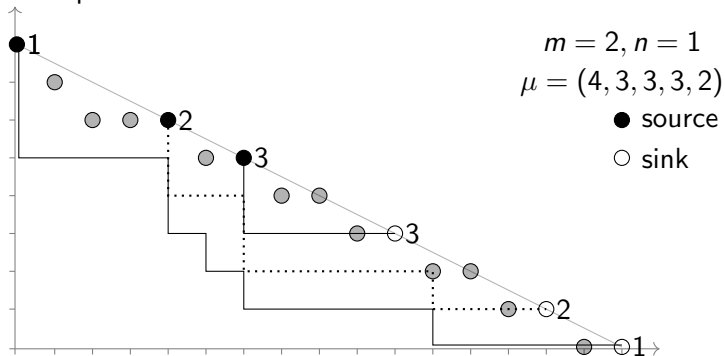
Dens and nests

Example of the “highest nest” π^0



Dens and nests

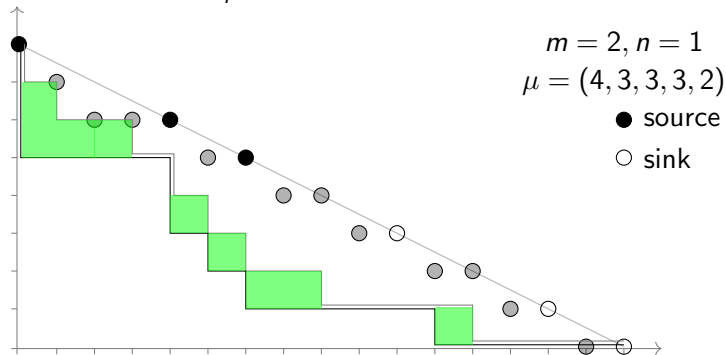
Example of another nest.



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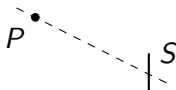


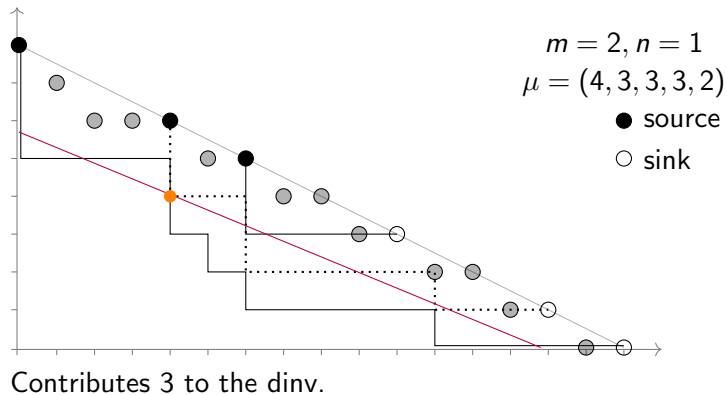
$$\text{area}(\pi_1) = 9$$

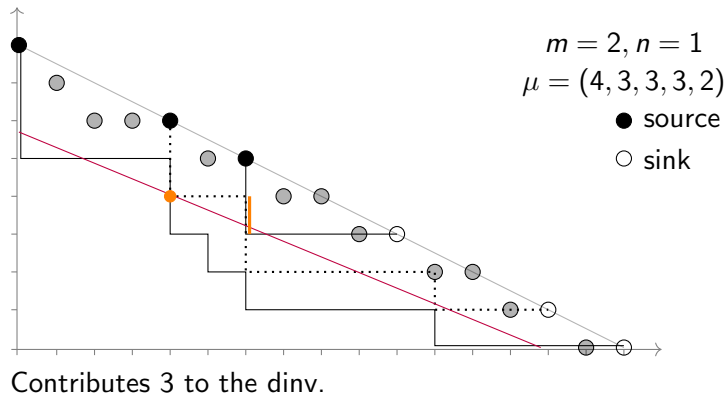
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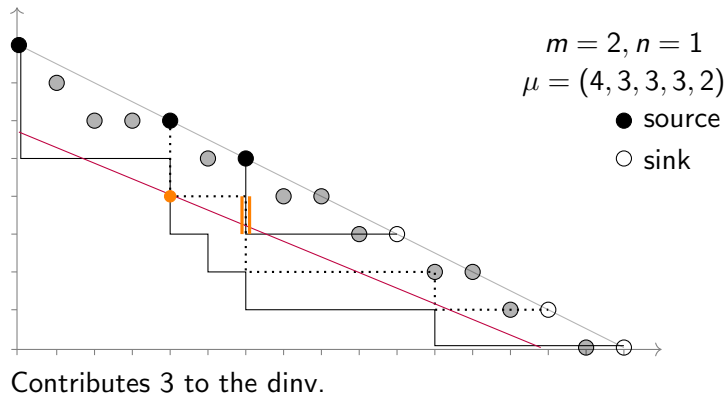
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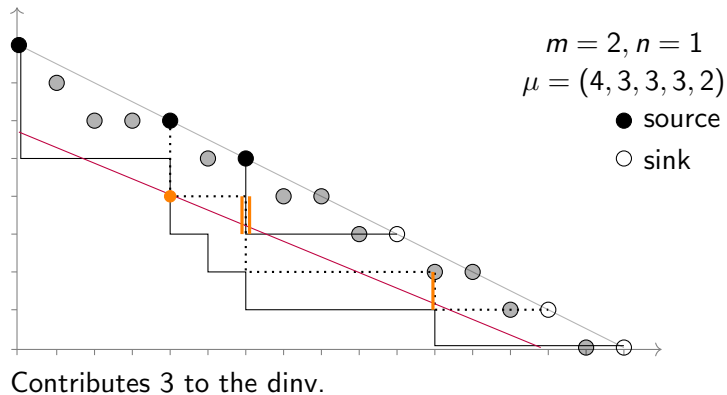
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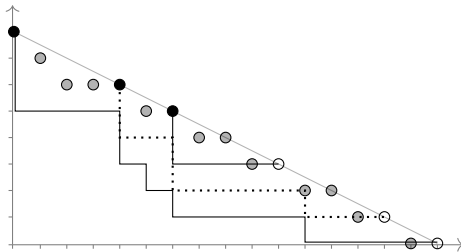
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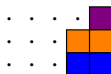
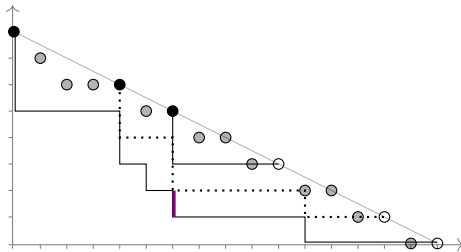
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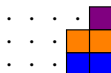
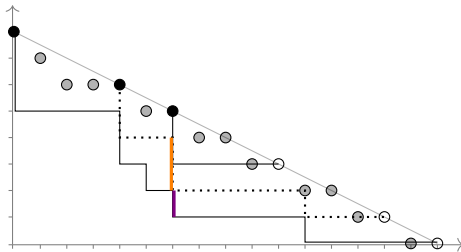
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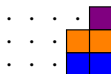
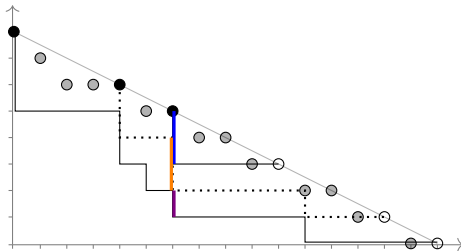
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- To each den we can associate a tame Catalan animal H and give a corresponding shuffle theorem as a sum over the nests of the den.
- These results hold “stably.” In other words, a stronger result is proven before applying polynomial truncation.
- This allows us to simultaneously generalize the $s_\lambda[-MX^{m,n}]$ formula and our “shuffle theorem for paths under any line” formula (BHMPs).

Other exhibits for next time

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- Special cases include Schur functions and Hall-Littlewood polynomials.
- Unicorn Catalan animals (or Catalan functions) where $R_t = R_{qt} = \emptyset$ also have a rich (older) results and combinatorics, but served as inspiration. (Chen-Haiman, Blasiak-Morse-Pun-Summers, Blasiak-Morse-Pun)

Future work: exit through the gift shop

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- What connections do Catalan animals have with machinery used to prove other shuffle theorems, such as work by Carlsson-Mellit?

Thank you for visiting!

Bergeron, Francois, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. 2016. *Compositional (km, kn) -shuffle conjectures*, Int. Math. Res. Not. IMRN **14**, 4229–4270, DOI 10.1093/imrn/rnv272. MR3556418

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021a. *A Shuffle Theorem for Paths Under Any Line*, arXiv e-prints, available at arXiv:2102.07931.

———. 2021b. *Dens, nests and the Loehr-Warrington conjecture*, arXiv e-prints, available at arXiv:2112.07070.

———. 2021c. *LLT polynomials in the Schiffmann algebra*, arXiv e-prints, available at arXiv:2112.07063.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Anton Mellit. 2018. *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31**, no. 3, 661–697, DOI 10.1090/jams/893. MR3787405

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

References continued

- Grojnowski, Ian and Mark Haiman. 2007. *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, Unpublished manuscript.
- Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.
- Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919
- _____. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676
- Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sémin. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754
- Loehr, Nicholas A. and Gregory S. Warrington. 2008. *Nested quantum Dyck paths and $\nabla(s_\lambda)$* , Int. Math. Res. Not. IMRN **5**, Art. ID rnm 157, 29, DOI 10.1093/imrn/rnm157. MR2418288
- Mellit, Anton. 2016. *Toric braids and (m, n) -parking functions*, arXiv e-prints, arXiv:1604.07456, available at arXiv:1604.07456.
- Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004