Building Mathematical Bridges Between Symmetric Functions

George H. Seelinger

Jefferson Scholars Foundation

ghs9ae@virginia.edu

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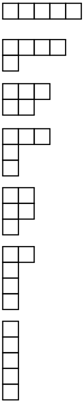

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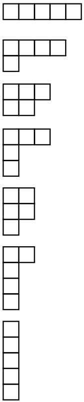

We will use these diagrams to describe a type of symmetric function called a "Schur function."

Raising Operators

To do this, we will need functions that change partition diagrams called "raising operators."

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If the result "does not make sense", we get 0:

$$R_{1,4}\left(\begin{ytableau} \ \\ \ \end{ytableau}\right) = 0$$

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$$\begin{aligned} s_\lambda = &(1 - R_{1,2}) \\ &(1 - R_{1,3})(1 - R_{2,3}) \\ &\ldots \\ &(1 - R_{1,\ell})(1 - R_{2,\ell}) \cdots (1 - R_{\ell-2,\ell})(1 - R_{\ell-1,\ell})\lambda \end{aligned}$$

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Example

$$s_{\ydiagram{3,2,1}} = (1-R_{1,2})(1-R_{1,3})(1-R_{2,3})\ydiagram{3,2,1}$$

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Recall the foil method from high school:

$$\begin{aligned}
&(1 - R_{1,2})(1 - R_{1,3})(1 - R_{2,3}) \\
=&(1 - R_{1,2} - R_{1,3} + R_{1,2}R_{1,3})(1 - R_{2,3}) \\
=&1 - R_{1,2} - R_{1,3} - R_{2,3} + R_{1,2}R_{1,3} + R_{1,2}R_{2,3} + R_{1,3}R_{2,3} - R_{1,2}R_{1,3}R_{2,3}
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\end{aligned}$$

So, we must compute $s_{\ydiagram{3,2,1}} =$

$$(1 - R_{1,2} - R_{1,3} - R_{2,3} + R_{1,2}R_{1,3} + R_{1,2}R_{2,3} + R_{1,3}R_{2,3} - R_{1,2}R_{1,3}R_{2,3})\ydiagram{3,2,1}$$

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$$\ydiagram{3,2,1}$$

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$$+R_{1,2}R_{1,3}(\ydiagram{3,2,1}) \qquad +R_{1,2}R_{2,3}(\ydiagram{3,2,1}) \qquad +R_{1,3}R_{2,3}(\ydiagram{3,2,1})$$

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-R_{1,2}(\ydiagram{3,2,1}) & -R_{1,3}(\ydiagram{3,2,1}) & -R_{2,3}(\ydiagram{3,2,1}) & & -\ydiagram{4,1,1} & -\ydiagram{4,2} \quad -\ydiagram{3,3} \\
+R_{1,2}R_{1,3}(\ydiagram{3,2,1}) & +R_{1,2}R_{2,3}(\ydiagram{3,2,1}) & +R_{1,3}R_{2,3}(\ydiagram{3,2,1}) & = & +\ydiagram{5,1} & +\ydiagram{4,2} \quad +0 \\
 & -R_{1,2}R_{1,3}R_{2,3}(\ydiagram{3,2,1}) & & & & -0
\end{array}$$

Example

$$s_{\ydiagram{3,2,1}} = (1 - R_{1,2})(1 - R_{1,3})(1 - R_{2,3})\ydiagram{3,2,1}$$

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Adding it all together, we get

Solution

$$s_{\ydiagram{3,2,1}} = \ydiagram{3,2,1} - \ydiagram{4,1,1} - \ydiagram{3,3} + \ydiagram{5,1}$$

Why Schur functions?

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Problem

However, the formula for Schur functions is complicated. If we have another formula for Schur functions, how can we prove they give the same result?

Multiplication for Symmetric Functions

Let us introduce a rule for multiplication of partition diagrams by "stacking."

Rule for Multiplication (Example)

$\ydiagram{3} \cdot \ydiagram{2} = \ydiagram{3,2} = \ydiagram{2,3} = \ydiagram{2} \cdot \ydiagram{3}$

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Example

$$\begin{aligned}\ydiagram[*(red)]{3} \cdot s_{\ydiagram{3,2,1}} &= \ydiagram[*(red)]{3} \cdot \left(\ydiagram{3,2,1} - \ydiagram{4,1,1} - \ydiagram{3,3} + \ydiagram{5,1}\right) \\ &= \ydiagram[*(red)]{3}*[*(white)]{3,3,2,1} - \ydiagram[*(red)]{0,3}*[*(white)]{4,3,1,1} - \ydiagram[*(red)]{3}*[*(white)]{3,3,3} + \ydiagram[*(red)]{0,3}*[*(white)]{5,3,1}\end{aligned}$$

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Problem

Result is in terms of partition diagrams, but we would like a result in terms of Schur functions.

The Pieri Rule

Example

$$\ydiagram[*(red)]{4} \cdot s_{\ydiagram{2,1}} = s_{\ydiagram[*(red)]{2+2,1+1,1}*[*(white)]{4,2,1}} + s_{\ydiagram[*(red)]{2+3,0,1}*[*(white)]{5,1,1}} + s_{\ydiagram[*(red)]{2+3,1+1}*[*(white)]{5,2}} + s_{\ydiagram[*(red)]{2+4}*[*(white)]{6,1}}$$

The Pieri Rule

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- In general, we get the result in terms of Schur functions by finding all ways to add the red boxes such that we only add at most one box to each column.

The Pieri Rule

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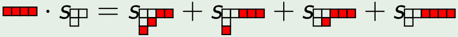

- In general, we get the result in terms of Schur functions by finding all ways to add the red boxes such that we only add at most one box to each column.
- We call this method *the Pieri rule* and it is a fundamental property of Schur functions.

Proof Technique

One approach to show two formulas for Schur functions are the same:

Proof technique

$$\begin{tikzcd}
\text{Base cases are equal} \arrow[dr] & & \text{Pieri rules are the same} \arrow[dl] \\
& \text{Linear algebra} \arrow[d] & \\
& \text{Functions are the same!} &
\end{tikzcd}$$

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What do I think about?

- Most problems about Schur functions are solved.
- Instead, I think about a class of functions called "type C dual affine Stanley symmetric functions" which have similar properties to Schur functions.
- However, the current formula for these functions is not as concrete as the formula I gave you for Schur functions.

Type C dual affine Stanley symmetric functions

Start with "word" with letters given by colors, $\{{\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{green}\blacksquare}\}$. For example, let's use $w = {\color{green}\blacksquare}{\color{blue}\blacksquare}{\color{green}\blacksquare}$.

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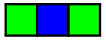
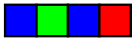
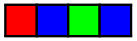
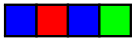
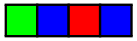
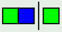
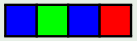
We must find all "subword decompositions" of w that are also subwords of $\rho = \textcolor{blue}{\blacksquare}\textcolor{green}{\blacksquare}\textcolor{blue}{\blacksquare}\textcolor{red}{\blacksquare}$ or any of its "rotations" $\textcolor{red}{\blacksquare}\textcolor{blue}{\blacksquare}\textcolor{green}{\blacksquare}\textcolor{blue}{\blacksquare}, \textcolor{blue}{\blacksquare}\textcolor{red}{\blacksquare}\textcolor{blue}{\blacksquare}\textcolor{green}{\blacksquare}, \textcolor{green}{\blacksquare}\textcolor{blue}{\blacksquare}\textcolor{red}{\blacksquare}\textcolor{blue}{\blacksquare}$.

Type C dual affine Stanley symmetric functions

Start with "word" with letters given by colors, {, , }. For example, let's use $w =$ .

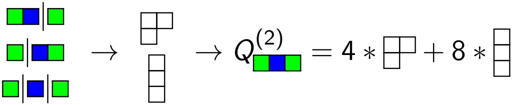
We must find all "subword decompositions" of w that are also subwords of $\rho =$  or any of its "rotations" , , .

Example

 is a subword decomposition of w where each part appears as a subword of $\rho =$ , but  is not a subword of ρ or any of its rotations.

Example continued

Then, you take all such subword decompositions to get a formula



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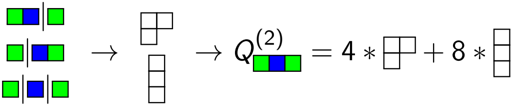

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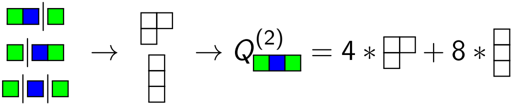

But, unfortunately, you are not done!

Problem

You then have to take the "dual" of this function to get the Type C dual affine Stanley symmetric function, $P^{(2)}$. This process is not direct and not computationally straightforward.

What have I done?

- I have a conjectured formula that describes type C dual affine Stanley symetric functions ($P_w^{(n)}$) directly using raising operators.

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What have I done?

- I have a conjectured formula that describes type C dual affine Stanley symetric functions ($P_w^{(n)}$) directly using raising operators.
- Computational evidence suggests my conjecture is correct.
- However, proving the formulas are the same directly would be quite hard, so instead I am seeking to use the Pieri rule approach

Base cases are equal

Pieri rules are the same

Linear algebra

Functions are the same!

Thank you for your support and for listening!

Jefferson Scholars Foundation

Symmetric Functions?

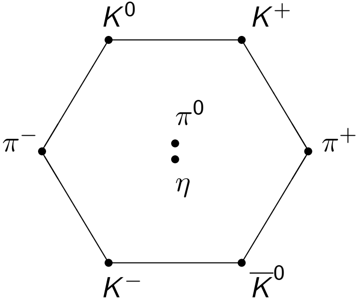
I pulled the wool over your eyes. Our partition diagrams represent polynomial functions with an infinite number of variables and an infinite number of terms.

Dictionary

$$
\begin{aligned}
\ydiagram{1} \quad &\to h_{\ydiagram{1}}(x_1, x_2, x_3, \ldots) = & &x_1 + x_2 + x_3 + \cdots \\
\ydiagram{2} \quad &\to h_{\ydiagram{2}}(x_1, x_2, x_3, \ldots) = & &x_1^2 + x_1x_2 + x_1x_3 + \cdots \\
& & &+x_2^2 + x_2x_3 + \cdots \\
& & &+x_3^2 + x_3x_4 + \cdots \\
\ydiagram{3} \quad &\to h_{\ydiagram{3}}(x_1, x_2, x_3, \ldots) = & &x_1^3 + x_1^2x_2 + x_1^2x_3 + \cdots \\
&\vdots
\end{aligned}
$$

$$
s_{\ydiagram{3,2,1}} = h_{\ydiagram{3,2,1}} - h_{\ydiagram{4,1,1}} - h_{\ydiagram{3,3}} + h_{\ydiagram{5,1}}
$$

Applications?



The “eightfold way” from particle physics is encoded in Schur functions by

$$s_{\ydiagram{2,1}}(e_{\epsilon_1}, e_{\epsilon_2}, e_{\epsilon_3})$$