Artin-Wedderburn Theory
Notes inspired by a class taught by Brian Parshall in Fall 2017

George H. Seelinger

1. Krull-Schmidt

We start with an underated lemma.

1.1. Theorem (Fitting's Lemma). *Let $0 \neq M$ be a left R-module with a composition series and $f\colon M \to M$ a module homomorphism. Since M has a composition series, there exists an n such that*

$$\operatorname{im} f \supseteq \operatorname{im} f^2 \supseteq \cdots \supseteq \operatorname{im} f^n = \operatorname{im} f^{n+1} = \cdots$$

and

$$\ker f \subseteq \ker f^2 \subseteq \cdots \subseteq \ker f^n = \ker f^{n+1} = \cdots$$

Define $f^{\infty}(M) := \operatorname{im} f^n$ and $f^{-\infty}(M) := \ker f^n$. Then,

$$M \cong f^{\infty}(M) \oplus f^{-\infty}(M)$$

Proof. Let $x \in f^{\infty}(M) \cap f^{-\infty}(M)$. Then, $x = f^n(y) = f^{2n}(z)$ and $f^n(x) = 0$. Then, $0 = f^n(x) = f^{2n}(y)$. Thus, $y \in \ker f^{2n} = \ker f^n$, so $x = f^n(y) = 0$.

Now, let $x \in M$. Then, $x = [x - f^n(y)] + f^n(y)$ where $f^n(y) \in f^{\infty}(M)$ and y such that $f^{2n}(y) = f^n(x)$. Then,

$$f^n(x - f^n(y)) = f^n(x) - f^{2n}(y) = f^n(x) - f^n(x) = 0$$

Thus, $x - f^n(y) \in f^{-\infty}(M)$ and $f^n(y) \in f^{\infty}(M)$. $\square$

Now, we seek to prove the Krull-Schmidt theorem.

1.2. Theorem (Krull-Schmidt Theorem). *Let $0 \neq M$ be an R-module which is both Artinian and Noetherian. Suppose*

$$\begin{aligned} M &= M_1 \oplus \cdots \oplus M_h \\ &= N_1 \oplus \cdots \oplus N_k \end{aligned}$$

where M_i, N_j are indecomposable. Then, $h = k$ and, up to rearranging terms, $M_i \cong N_i$ for $i = 1, \ldots, h$.

To prove this theorem, we need the following lemmas.

1.3. Lemma. *Let M, N be R-modules with N decomposable and homomorphisms $\nu\colon M \to N$ and $\mu\colon N \to M$. If $\mu\nu\colon M \to M$ is an automorphism, then μ and ν are isomorphisms.*

Proof. Let $\tau = \mu\nu$ be an automorphism. Now, replace μ with $\tau^{-1}\mu$. Then, we have split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\nu"'] & N \arrow[l, bend right, "\mu"'] \arrow[r] & N/M \arrow[r] & 0 \end{tikzcd}$$

and so $N \cong M \oplus N/M$. However, N is indecomposable, so $N = M$. $\square$

1.4. Lemma. *Let $0 \neq M$ be indecomposable, Artinian, and Noetherian. Let $\tau_1, \dots, \tau_r \in \operatorname{End}_R M$ satisfy $\tau_1 + \cdots + \tau_r \in \operatorname{Aut}_R(M)$. Then, at least one τ_i is an automorphism of M*

Proof. Since $\tau = \tau_1 + (\tau_2 + \cdots + \tau_r)$, it suffices to prove for the sum of 2 endomorphisms. Let $\psi = \tau_1 + \tau_2$ be an automorphism. Then, $\phi_1 = \psi^{-1}\tau_1$ and $\phi_2 = \psi^{-1}\tau_2$. We then have $\phi_1 + \phi_2 = 1_M$ and so, since $\phi_1 = 1_M - \phi_2$, we get $\phi_1\phi_2 = \phi_2\phi_1$. Now, we can apply the binomial theorem to get

$$0 \neq 1_M^m = (\phi_1 + \phi_2)^m = \sum_{k=1}^{m} \binom{m}{k} \phi_1^k \phi_2^{m-k}$$

So, at least one of the ϕ_i's cannot be nilpotent. Without loss of generality, assume ϕ_1 is not nilpotent. Since ϕ_1 is not nilpotent and M has a composition series, we can apply Fitting's Lemma to get

$$M \cong \phi_1^{\infty}(M) \oplus \phi_1^{-\infty}(M)$$

However, M is indecomposable and $\phi_1^{\infty}(M) \neq 0$, so $M = \phi_1^{\infty}(M)$ and $\ker \phi_1 \subseteq \phi_1^{-\infty}(M) = 0$, so ϕ_1 is an automorphism. Therefore, τ_1 is an automorphism. $\square$

We are now ready to prove the Krull-Schmidt theorem.

Proof of Krull-Schmidt. Let $0 \neq M$ be an Artinian and Noetherian R-module. Suppose

$$\begin{aligned} M &= M_1 \oplus \cdots \oplus M_h \\ &= N_1 \oplus \cdots \oplus N_k \end{aligned}$$

where the M_i, N_j are indecomposable. We will proceed by induction on h. If $h = 1$, then M is already indecomposable and we are done. Now, for the inductive step, let $\mu \colon M \to M_i$ be the projection onto the ith summand and, similarly, let $\nu_j \colon M \to N_j$ be a projection. Then, we have $\mu_i\mu_j = \delta_{ij}\mu_i$ and $\nu_i\nu_j = \delta_{ij}\nu_j$. Now, we have the following identity

$$\mu_1 = \mu_1 \circ 1_M = \mu_1(\nu_1 + \cdots + \nu_k)$$

Now, let $\overline{\nu}_j = \nu_j|_{M_i} \colon M_i \to M$. If we restrict the above identity to M_1, we get

$$1_{M_1} = \mu_1\overline{\nu}_1 + \cdots + \mu_1\overline{\nu}_k \in \operatorname{Aut} M_1$$

Thus, we can use the lemma above to assume that $\mu_1\overline{nu}_1 \in \operatorname{Aut} M_1$. Furthermore, by the other lemma, since $\mu_1\overline{\nu}_1$ is an automorphism, we get that $\overline{nu}_1$ is an isomorphism, as is $\mu_1 \colon N_1 \to M_1$. Now, consider

$$N_1 + M_2 + \cdots + M_n \leq M$$

We wish to show this sum is actually direct. Assume there is $n_1 \in N_1$ and $m_2 \in M_2, \dots, m_h \in M_h$ such that

$$n_1 + m_2 + \cdots + m_h = 0.$$

We can just apply μ_1 to this sum above to get

$$0 = \mu_1(n_1 + m_2 + \cdots + m_h) = \mu_1(n_1)$$

However, μ_1 is an isomorphism between N_1 and M_1, so $\mu_1(n_1) = 0 \Longrightarrow n_1 = 0$. So, furthermore, it must be that $m_2 = \cdots = m_h = 0$ by the direct sum construction, so

$$M' := N_1 + M_2 + \cdots + M_h = N_1 \oplus M_2 \oplus \cdots \oplus M_h \leq M$$

Thus, we wish to finally show that $M' = M$. Let

$$\rho \colon M_1 \oplus \cdots \oplus M_h \to M'$$

be given by $\rho = \overline{\nu}_1 + \mu_2 + \cdots + \mu_h$. Note that ρ is an isomorphism since $\overline{\nu}_1$ is an isomorphism from $M_1 \to M$. Since M is Artinian, there is some $a \gg 0$ such that

$$\rho^{a+1}(M) = \rho^a(M)$$

Hence, given $m \in M$, there is some $m' \in M$ such that

$$\begin{aligned} \rho^{a+1}(m') = \rho^a(m) &\Longrightarrow \rho^a(m - \rho(m')) = 0 \\ &\Longrightarrow m - \rho(m') \in \ker \rho^a = \{0\} \end{aligned}$$

Thus, $M' = M$ and ρ is an automorphism of M. Furthermore, $\rho(M_i) = M_i$ for $i = 2, \ldots, h$, $\rho(M_1) = N_1$, and

$$N_1 \oplus M_2 \oplus \cdots \oplus M_h = \rho(M) = M = M_1 \oplus \cdots \oplus M_h$$

Hence,

$$\begin{aligned} M_2 \oplus \cdots \oplus M_h &\cong M/M_1 \\ &\cong \rho(M)/\rho(M_1) \\ &\cong \rho(M)/N_1 \\ &\cong (N_1 \oplus \cdots \oplus N_k)/N_1 \\ &\cong N_2 \oplus \cdots \oplus N_k \end{aligned}$$

Thus, by induction, $h - 1 = k - 1 \Longrightarrow h = k$ and $M_i \cong N_i$ for $i = 2, \ldots, h$, and we already know $M_1 \cong N_1$ by above. $\square$

1.5. Corollary. *Let N be a direct summand of M as in the Krull-Schmidt theorem. Then, there exist a subset of $\{M_i\}$ indecomposable submodules $M_{i_1}, \cdots, M_{i_t}$ such that*

$$N \cong M_{i_1} \oplus \cdots \oplus M_{i_t}$$

2. Nakayama's Lemma: A Hard to Remember Lemma

While one statement of Nakayama's Lemma may be easy to remember, there are actually many special cases of this lemma and so it has a reputation of being hard to remember. To understand and prove Nakayama's Lemma, we must first lay the groundwork of Jacobson radicals.

2.1. Definition. The *Jacobson radical* (often just called the radical) R of a ring A is

$$R = \{x \in A \mid xM = 0, \forall \text{ simple } A\text{-modules } M\}$$

2.2. Definition. A ring A is called *semisimple* when its radical $R = 0$.

2.3. Theorem. *The radical R of A is a two-sided ideal of A and A/R is semisimple.*

Proof. If M is a simple A-module, then M is also an A/R-module. Conversely, any A/R-modules are naturally A-modules. Thus, if there were an $x+R \in A/R$ such that $(x+R)M = xM = 0$ for all simple A/R-modules M, then x would also annihilate all simple A-modules and thus $x \in R$, so $x + R = 0 + R$ and thus the radical of A/R is trivial. $\square$

2.4. Theorem. *Let R be the radical of a ring A. Then,*

$$R = \bigcap_{M \text{ simple}} \operatorname{Ann}_A(M) = \bigcap_{\substack{I \trianglelefteq A \\ \text{Maximal left ideal}}} I$$

That is, R is the intersection of all maximal left ideals of A.

Proof. Let $x \in R$ and let I be a maximal left ideal. We want to show that $x \in I$. Since I is maximal, A/I is a simple left A-module. thus, $x \in \operatorname{Ann}(A/I)$. Thus,

$$\begin{aligned} 0 = x(1+I) &\Longrightarrow x \in I \\ &\Longrightarrow x \in \bigcap I \\ &\Longrightarrow R \subseteq \bigcap I \end{aligned}$$

Conversely, let $x \in \bigcap I$. Let M be a simple module $0 \neq m \in M$. We show $xm = 0$. We have

$$\begin{aligned} A &\overset{\theta}{\twoheadrightarrow} M \\ a &\mapsto am \end{aligned}$$

Then, $M \cong A/I$ where $I = \operatorname{Ann}_A(m)$. Thus, $xm = 0$. $\square$

2.5. Theorem. *Given a ring A,*

$$\operatorname{rad} A = \{x \in A \mid 1 - axb \in A^\times, \forall a, b \in A\}$$

where $A^\times$ is the group of units of A.

Proof. Let $x \in \operatorname{rad} A$ with $a, b \in A$. Then, $\operatorname{rad} A \trianglelefteq A$ by 2.3, so $axb \in \operatorname{rad} A$. Now, write $y := axb$. We wish to show $1 - y$ is a unit.

If $A(1-y) \neq A$, let I be a maximal left ideal containing $A(1-y)$. Then, by 2.4, $\operatorname{rad} A \subseteq I$, so $y \in \operatorname{rad} A \subseteq I$ and $1 - y \in A$. This would give that $1 = y + (1-y) \in I \Longrightarrow I = A$, which is a contradiction of the maximality of I. So, $A(1-y) = A$ for all $y \in \operatorname{rad} A$.

Now, since $A(1-y) = A$, there is a t such that $t(1-y) = 1$. So, we wish to show $(1-y)t = 1$. Rearranging our expression $t(1-y) = 1$, we get

$$\begin{aligned}
1 - t = -ty \in \operatorname{rad} A &\implies A(1-(1-t)) = A && \text{by above since } A(1-y) = A, \forall y \in \operatorname{rad} A \\
&\implies At = A \\
&\implies \exists u \in A \text{ such that } ut = 1 \\
&\implies u = ut(1-y) = 1 - y && \text{since } t(1-y) = 1 \text{ and } ut = 1 \\
&\implies 1 - y \in A^{\times}
\end{aligned}$$

Thus, we have shown one containment.

Now, let $x \in A$ be such that $1 - axb \in A^{\times}$ for all $a, b \in A$. We want to prove $x \in \operatorname{rad} A$. We show $xM = 0$ if M is a simple A-module. Let $0 \neq m \in M$. Assume $xm \neq 0$. Then, $M = Axm$ since M is simple so $(xm) = M$. Thus, $m = axm$ for some $a \in A$ and so $(1 - ax)m = 0$. However, $1 - ax \in A^{\times}$, so $m = 0$, which is a contradiction. Thus, $xM = 0$ and so $x \in \operatorname{rad} A$. □

2.6. Theorem (Nakayama's Lemma). [**Jac89**, p 415] *Let A be a ring and let M be a finitely-generated A-module. If $(\operatorname{rad} A)M = M$, then $M = 0$.*

Proof. Let $m_1, m_2, \ldots, m_k$ be a minimum generating set of M. Given $m \in M$, there exists $a_1, \ldots, a_k \in A$ such that

$$m = a_1 m_1 + \cdots + a_k m_k$$

Take $m = m_1$. Then,

$$(1 - a_1)m_1 = a_2 m_2 + \cdots + a_k m_k$$

If $(\operatorname{rad} A)M = M$, we can assume each $a_i \in \operatorname{rad} A$, we can assume each $a_i \in \operatorname{rad} A$ since $\operatorname{rad} A$ is a 2-sided ideal by theorem 2.3. Thus, by 2.5, $1 - a_1$ is a unit, so

$$m_1 = (1 - a_1)^{-1} a_2 m_2 + \cdots + (1 - a_2)^{-1} a_k m_k \in \langle m_2, \ldots, m_k \rangle$$

However, this contradicts the minimality of the the generating set. □

2.7. Corollary. *Let M be a finitely generated left A-module and let $N \leq M$ be a submodule such that*

$$N + (\operatorname{rad} A)M = M$$

Then, $N = M$

Proof. Given the above, $\operatorname{rad} A(M/N) = M/N$ simply by quotienting both sides of the equality by N. Thus, by Nakayama's lemma, $M/N = 0$ and so $M = N$. □

2.8. Corollary. *Let J be a maximal ideal in a ring A. Then,* $\operatorname{rad} A \subseteq J$.

Proof. Assume not. Then $\operatorname{rad} A + J = A \Longrightarrow J = A$ by the corollary above, which is a contradiction to J being a maximal ideal. □

3. Completely Reducible Modules

Understanding all the modules of a ring, in general, is an incredibly difficult problem. However, usually the first step in such a program is understanding the simple (or irreducible) modules. Understanding these modules would then give a complete understanding of completely reducible modules.

3.1. Definition. Let A be a ring. A left A-module M is called *completely reducible* if, given any submodule $N \subseteq M$, there exists a submodule N' such that

$$M = N \oplus N'$$

However, there are equivalent notions of completely reducible.

3.2. Theorem. *Assume M is a left A-module. Then, the following are equivalent.*

(a) M is completely reducible.
(b) M is a direct sum of irreducible submodules, that is

$$M \cong \bigoplus_{i \in I} L_i$$

where I is some indexing set and L_i is simple.
(c) M is a sum of irreducible submodules, that is

$$M \cong \sum_{i \in I} L_i$$

where I is some indexing set and L_i is simple.

To prove this result, we use the following lemma.

3.3. Lemma. *If an A-module M is completely reducible, so is any submodule $N \leq M$.*

Proof of Lemma. Consider submodule N of completely reducible a A-module M. Then, by definition, there exists an N' such that

$$M \cong N \oplus N'.$$

Now, let $S \leq N$. Then, we also have $S \leq M$ and so there is a submodule T such that

$$M \cong S \oplus T.$$

Thus, we have

$$N = (S \oplus T) \cap N$$

$$= S \oplus (T \cap N)$$

since $S \leq N$. Since $T \cap N \leq N$, then N is completely irreducible. $\square$

3.4. Lemma. *Let M be a completely reducible A-module. Then, every nontrivial submodule N of M contains an irreducible submodule.*

Proof of Lemma. Let $N \leq M$ and $0 \neq n \in N$. Consider the collection

$$S = \{N' \leq N \mid n \notin N'\}$$

We note that S is nonempty since $(0) \in S$. Thus, since S is nonempty, it must contain a maximal element, say N_0. By the lemma above, we know that N is completely reducible and so $N = N_0 \oplus N_1$ for some submodule $N_1 \leq N$. N_1 must be irreducible because, if not, then there would be a proper submodule $N_2 \leq N_1$ and $N_1 = N_2 \oplus N_3$ for some submodule $N_3 \leq N_1$. However, this would give us

$$N = N_0 \oplus N_2 \oplus N_3 \implies \text{ either } n \notin N_0 + N_2 \text{ or } n \notin N_0 + N_3$$

since $(N_0 + N_2) \cap (N_0 + N_3) = N_0$. Such a result contradicts the maximality of N_0 in S, and so it must be that N_1 is irreducible. $\square$

Proof of Theorem. We first assume that $M \neq (0)$, otherwise the theorem is immediately true.

$((a) \implies (b))$. Let $\{M_i \mid i \in I\}$ be the collection of all irreducible submodules of M and let

$$T = \left\{ J \subseteq I \ \mid \ \sum_{j \in J} M_j \text{ is direct} \right\}$$

T is nonempty since it contains at least singleton sets are in T and any union of an ascending chain of elements in T is in T. Thus, we can apply Zorn's lemma to get a maximal element of T, say J_0. Now, let

$$M' := \bigoplus_{j \in J_0} M_j$$

be properly contained in M. Then, by complete reducibility of M, we get

$$M = M' \oplus M''$$

for $(0) \neq M'' \leq M$. However, since M'' must be completely reducible, by the lemma above, M'' must contain an an irreducible submodule, say M_{i_0}. Then,

$$\left(\bigoplus_{j \in J_0} M_j \right) + M_{i_0} \text{ is direct}$$

and thus we have violated the maximality of J_0.

$((b) \implies (c))$. This result is immediate by the definition of direct sum.

$((c) \implies (a))$. Let $N \leq M$ and let N' be a maximal submodule with respect to the property that $N \cap N' = (0)$. Assume $M \neq N \oplus N'$. Then, there is an $m \in M, m \notin N \oplus N'$. However, since M is a sum of irreducible submodules, $m = m_1 + \cdots + m_k$ where each m_i belonds to an irreducible summand. Thus, at least one $m_i \notin N \oplus N'$ since $m \notin N \oplus N'$ and thus $M_i \nsubseteq N \oplus N'$. However, M_i is irreducible and so $M_i \cap (N \oplus N') = (0)$. Thus, $N' \subsetneqq N' + M_i$ and $(N' + M_i) \cap N = (0)$, thus violating the maximality of N'. Thus, $M = N \oplus N'$.

□

3.5. Proposition. *Let M be a completely reducible module for a ring A. Then, M satisfies the ascending chain condition if and only if M satisfies the descending chain condition.*

4. Nilpotent and non-Nilpotent Ideals in Artinian Rings

To get to the Artin-Wedderburn Theorem, we must have an understanding of idempotents in Artinian rings, which are intimately (non)-related to nilpotent ideals. We wish to culminate in a theorem that says any non-nilpotent left ideal in a left-artinian ring must contain an idempotent. First, we present a theorem similar to Schur's Lemma, but for Noetherian rings.

4.1. Theorem. *Let M be a (left) Noetherian A-module for ring A. If $f \in \mathrm{End}_A(M)$ is surjective, then f is an isomorphism*

Proof. If f is surjective, then $\mathrm{im}\, f = M$ and so f is not nilpotent. Thus, by Fitting's Lemma

$$M = f^{\infty}(M) \oplus f^{-\infty}(M)$$

we seek to show $\ker f = 0$. For some integer n, $\ker f^n = \ker f^{n+1}$. Let $x \in \ker f$. Then, since f is surjective,

$$f^n(M) = f^{n+1}(M) = M$$

Thus, $x = f^n(y)$ for some $y \in M$. We then see

$$0 = f(x) = f^{n+1}(y) \implies y \in \ker f^{n+1} = \ker f^n \implies 0 = f^n(y) = x$$

So, we have that $\ker f = 0$ and thus f is injective. □

4.2. Theorem. *Let A be a left Noetherian ring. If $a, b \in A$ be such that $ab = 1$, then $ba = 1$ and $a, b \in A^{\times}$.*

Proof. Let $M = {}_AA$. Then, M is a (left) Noetherian A-module. Then, since $ab = 1$,

$$A = Aab \subseteq Ab \subseteq A$$

and so $Ab = A$. Thus, we have a map

$$\begin{aligned} f \colon M &\twoheadrightarrow M \\ x &\mapsto xb \end{aligned}$$

and so, by the theorem above, f is an isomorphism. However,

$$\begin{aligned} f(1-ba) &= (1-ba)b \\ &= b - bab \\ &= b - b && \text{since } ab = 1 \\ &= 0 \end{aligned}$$

Thus, $1 - ba = 0 \Longrightarrow ba = 1 = ab$. $\square$

4.3. Definition. An *idempotent* of a ring A is an element $0 \neq e \in A$ such that $e^2 = e$.

One advantage of idempotents is that they allow us to "project" the ring onto "orthogonal" compotents, that is, given a ring A with idempotent e, then, as a module over itself

$$_AA \cong {}_AAe \oplus {}_AA(1-e)$$

This also tells us the following

4.4. Remark. $_AAe$ is projective as an A-module since it is the direct summand of a free A-module.

4.5. Definition. We call a left or right ideal, I, *nilpotent* if there is an $m \in \mathbb{N}$ such that $I^m = \{0\}$.

4.6. Example. Consider $R = \mathbb{Z}/p^n\mathbb{Z}$ where p is a prime number. Then, since R is a PID, every proper ideal is generated by some $p^k + (p^n)$ and is nilpotent. The only idempotent of R is $1 + (p^n)$.

4.7. Theorem. *Let N be a nilpotent left ideal of a ring A. Let $x \in A$ be non-nilpotent such that $x^2 - x \in N$. Then, the left ideal Ax has an idempotent y.*

The idea to proving this theorem is to take A and factor out N. Then, one can find an idempotent y such that, under the quotient map $q\colon A \to A/N$, $q(x) = q(y)$.

Proof. Assume $N^k = 0$ for some positive integer k. Let $m_1 := x^2 - x \in N$. If $m_1 = 0$, then $x^2 - x = 0$ so x is an idempotent itself and we can take $y = x$.

Assume $m_1 \neq 0$. Then, let

$$x_1 := x + m_1 - 2xm_1 \in Ax \text{ since } m_1 \in Ax$$

Note that x_1, x, m_1 all commute. Then, note that x_1 is not nilpotent as well and $x^2 = x + m_1$. Consider

$$\begin{aligned} x_1^2 - x_1 &= (x + m_1 - 2xm_1)^2 - (x + m_1 - 2xm_1) \\ &= x^2 + xm_1 - 2x^2m_1 + m_1x + m_1^2 - 2xm_1^2 - 2x^2m_1 - 2xm_1^2 + 4x^2m_1^2 - x - m_1 + 2xm_1 \end{aligned}$$

$$
\begin{aligned}
&= 4x^2m_1^2 - 4x^2m_1 - 4xm_1^2 + x^2 + 4xm_1 + m_1^2 - x - m_1 \\
&= 4x^2m_1^2 - 4x^2m_1 - 4xm_1^2 + 4xm_1 + m_1^2 \\
&= (4x^2 - 4x)m_1^2 + (-4x^2 + 4x)m_1 + m_1^2 \\
&= (4x^2 - 4x - 4m_1)m_1^2 + (-4x^2 + 4x + 4m_1)m_1 + m_1^2 + 4m_1^3 - 4m_1^2 \\
&= 4m_1^3 - 3m_1^2
\end{aligned}
$$

Then, take

$$m_2 = 4m_1^3 - 3m_1^2 \; x_2 = x_1 + m_2 - 2x_1m_2.$$

and note that m_2 contains m_1^2 as a factor. Thus, we can successively construct non-nilpotent elements $x_1, x_2, \ldots$ in Ax such that $x_i^2 - x_i$ contains $m_1^{2^i}$ as a factor and commutes with x. Since m_1 is nilpotent, then $m_1^{2^i} = 0$ for sufficiently large i and so $x_i^2 - x_i = 0$ for some sufficiently large i. Therefore, for that sufficiently large i, x_i is nilpotent. $\square$

4.8. Remark. Note that any nilpotent ideal cannot contain an idempotent element. However, the following theorem gives us a (useful!) converse to that fact.

4.9. Theorem. *Let L be a non-nilpotent left ideal in a left-artinian ring A. Then, L contains an idempotent e.*

Proof. We seek to use the theorem above to get such an idempotent by finding a non-nilpotent $x \in A$ such that $x^2 - x$ is in a nilpotent ideal.

Choose a minimal left ideal $L_1 \subseteq L$ which is not nilpotent. Then, $0 \neq L_1L_1 \subseteq L_1$ is not nilpotent, so $L_1L_1 = L_1$ by the minimality of L_1.

Let I be a left ideal contained in L_1 such that $L_1I \neq 0$ and minimal with respect to this property. Let $a \in I$ be such that $L_1a \neq 0$. Then,

$$L_1L_1a = L_1^2a = L_1a \neq 0 \text{ and } L_1a \subseteq I$$

Hence, $I = L_1a$ by the minimality of I. Thus, there is an $x \in L_1$ such that $a = xa$. Hence,

$$0 \neq a = xa = x^2a = \cdots = x^ka = \cdots$$

Therefore, x is not nilpotent.

Let $N = \{b \in L_1 \mid ba = 0\}$. This is a left ideal contained in L_1. Since $xa = x^2a$,

$$(x - x^2)a = 0 \Longrightarrow x - x^2 \in N$$

Also, $L_1a \neq 0$ so $N \subsetneq L_1$. Hence, N is nilpotent by the minimality of L_1 as a non-nilpotent ideal. Thus, we now have x, N as in the theorem above and so there is an idempotent $e \in Ax \subseteq L_1 \subseteq L$. $\square$

4.10. Theorem. *Let A be a left artinian ring. Then* rad A *is the largest nilpotent left ideal.*

PROOF. $\square$

4.11. PROPOSITION. *If $f: A \to B$ is a surjective homomorphism of rings, then*

$$f(\operatorname{rad} A) \subseteq \operatorname{rad} B$$

5. The Radical of an Artinian Ring

In this section, we seek to understand the radical or an artinian ring A and show that when it is trivial, we gain a huge amount of insight into A-modules. Namely, for semisimple A, every A-module is completely reducible and every irreducible A-module is isomorphic to a non-zero minimum left ideal of A. Perhaps most surprisingly of all, this tells us that a semisimple A has only a finite number of irreducible modules, which is not a typical phenemenon! We start by finding a deeper insight into the relationship between $\operatorname{rad} A$ and the nilpotent ideals of A.

5.1. THEOREM. *Let A be a left artinian ring. Then, the sum of all left nilpotent ideals of A is a nilpotent ideal, say N. It contains every nilpotent right ideal of A. Also, the quotient ring A/N has no nontrivial nilpotent ideal.*

To show this, we use the following lemma

5.2. LEMMA. *Any finite sum of nilpotent ideals is nilpotent.*

PROOF OF LEMMA. If we assume the result is true for a sum of $r-1$ nilpotent ideals, then it is true for r nilpotent ideals. Consider nilpotent ideals $N_1, \ldots, N_r$. Then,

$$N_1 + \cdots + N_r = \underbrace{(N_1 + \cdots + N_{r-1})}_{\text{Nilpotent ideal by assumption}} + N_r \text{ is nilpotent.}$$

Thus, it suffices to show the result is true for the sum of two nilpotent ideals.

Let N_1, N_2 be nilpotent left ideals. Then, there exist positive integers q, r such that

$$N_1^q = N_2^r = 0.$$

And element in $(N_1 + N_2)^{q+r}$ is a finite sum of products

$$y_1 y_2 \cdots y_{q+r},\ y_i \in N_1 \cup N_2, s \in \mathbb{N}$$

We can assume at least q of the y_i belong to N_1 or at least r of the y_i belong to N_2. Without loss of generality, we assume the former. Then, the product above may be written

$$(x_1 x_2 \cdots x_{i_1})(x_{i_1+1} \cdots x_{i_2}) \cdots (x_{i_{s-1}+1} \cdots x_{i_s}) \cdots ,$$

where $x_{i_1}, x_{i_2}, \ldots, x_{i_s} \in N_1$ and $s \geq q$. Since N_1 is a left ideal, each "segment" belongs to N_1. Thus,

$$y_1 y_2 \cdots y_{q+r} \in N_1^q = 0 \Longrightarrow y_1 y_2 \cdots y_{q+r} = 0$$

However, $y_1 y_2 \cdots y_{q+r}$ was an arbitrary element of $(N_1 + N_2)^{q+r}$, so $(N_1 + N_2)^{q+r} = 0$ and thus the sum of 2 nilpotent (left) ideals is nilpotent. $\square$

Proof of Theorem. Let $\{N_i\}$ be a set of nilpotent left ideals. Then, the elements of N are finite sums, i.e.

$$n \in N \Longrightarrow n = x_1 + \cdots + x_r, x_i \in \text{ some } N_i$$

Since the sum of left ideals is a left ideal, N is a left ideal.

Assume N is not nilpotent, then since A is left-artinian, N contains an idempotent, e (see 4.9). Thus, there is a finite sum of nilpotent ideals such that

$$e \in N_1 + \cdots + N_r$$

However, $N_1 + \cdots + N_r$ is nilpotent by the lemma above, and so there is a k such that

$$e = e^k \in (N_1 + \cdots + N_r)^k = 0$$

which is a contradiction.

If N_i is a nilpotent left ideal, then $N_i A$ is a nilpotent (two-sided) ideal since, for large enough $t \in \mathbb{N}$,

$$(N_i A)^t = N_i (A N_i) \cdots (A N_i) A \subseteq N_i (N_i)^{t-1} A = 0$$

Thus, N is an ideal. If I is a nilpotent right ideal, AI is a nilpotent left ideal by the same argument. Thus, $I \subseteq AI \subseteq N$.

How does this follow exactly?

Since N is a two-sided ideal, then every left ideal of R/N is of the form I/N for some left ideal I of R, by the correspondance theorem. Thus,

$$(I/N)^k = 0 \Longleftrightarrow I^k \subseteq N \Longleftrightarrow I \text{ is nilpotent} \Longleftrightarrow I \subseteq N$$

since N is the sum of all nilpotent left ideals. Thus, R/N contains no nilpotent left ideal except 0, and by the discussion above, also contains no nontrivial nilpotent right or two-sided ideals. $\square$

5.3. Theorem. *Let A be a left artinian ring. If is N the sum of all left nilpotent ideals, then $N = \operatorname{rad} A$.*

Proof. We already have that $N \subseteq \operatorname{rad} A$ by (4.10) since N is nilpotent by the above theorem. However, consider the sequence of ideals

Is this not the same result as the theorem we are proving? Did we show something weaker, earlier?

$$\operatorname{rad} A \supseteq (\operatorname{rad} A)^2 \supseteq \cdots \supseteq (\operatorname{rad} A)^n \supseteq \cdots$$

Since A is artinian, there is some k such that

$$(\operatorname{rad} A)^k = (\operatorname{rad} A)^{k+1} = \cdots$$

and thus $(\operatorname{rad} A)(\operatorname{rad} A)^n = (\operatorname{rad} A)^n$. Thus, by Nakayama's lemma (2.6), $(\operatorname{rad} A)^n = 0$. Thus, $\operatorname{rad} A$ is nilpotent. $\square$

5.4. Theorem. *Let A be a left artinian ring. Then, ${}_AA$ is completely reducible if and only if A is semisimple.*

Proof. ($\Longleftarrow$). Let A be semisimple. Then, $\operatorname{rad} A = 0$. Now, Let I_1 be minimal among nontrivial non-nilpotent left ideals of A. Then, I_1 has an idempotent, say e_1. Thus, $e_1 \in I_1 \Longrightarrow Ae_1 \subseteq I_1$, but I_1 is minimal among non-nilpotent ideals, so $I_1 = Ae_1$. Now, notice that

$$A = Ae_1 \oplus A(1 - e_1) \Longrightarrow {}_AA = I_1 \oplus I_1',$$

where $I_1' = A(1 - e_1)$ is a non-nilpotent left ideal if $I_1' \neq 0$. Then, we can repeat this process with I_1' to get

Why is I_1' minimal?

$${}_AA = I_1 \oplus I_2 \oplus I_2', \;\; I_2 = Ae_2$$

Continuing, we get

There is more work to do here.

$${}_AA = I_1 \oplus I_2 \oplus \cdots \oplus I_r$$

where $I_i = Ae_i, e_i$ idempotent. Each I_i is irreducible as a left A-module since A is semisimple.

($\Longrightarrow$). Let ${}_AA$ be completely reducible. Consider that $N = \operatorname{rad} A$ is a left ideal of A and thus also a submodule of ${}_AA$ and thus

$${}_AA = N \oplus N'$$

for some left ideal N' of A. Then,

$$1 = x + x'$$

for some $x \in N, x' \in N'$, which then yields

$$x \cdot 1 = x(x + x') = x^2 + xx' \Longrightarrow x - x^2 = xx' \in N \cap N' = 0$$

Thus, $x - x^2 = 0 \Longrightarrow x = x^2 = \cdots = 0$ since N is nilpotent. Thus, $x' = 1$ and $N' = A \Longrightarrow \operatorname{rad} A = N = 0$ and thus A is semisimple. $\square$

5.5. Theorem. *Let A be a (left) artinian ring. Then, A is semisimple if and only if every A-module is completely reducible.*

Proof. To prove ($\Longleftarrow$), we simply note that ${}_AA$ is an A-module, and so the previous theorem tells us that A is semisimple.

For ($\Longrightarrow$), assume A is semisimple and M is an A-module. Then, it suffices to show M is a (not-necessarily direct) sum of irreducible A-modules by 3.2. By the theorem above, we know

$${}_AA \cong L_1 \oplus \cdots \oplus L_n$$

Now, given $0 \neq m \in M$, consider $\phi\colon A \to M$ given by $a \mapsto a.m$. Then, $\phi|_{L_i}\colon L_i \to M$ is either injective or the 0 map. Thus, we get

$$A.M = M \Longrightarrow (L_1 \oplus \cdots \oplus L_n).M = M \Longrightarrow \phi(L_1) + \cdots + \phi(L_n) = M$$

but since each $\phi(L_i)$ is irreducible or 0, it must be that M is completely reducible. $\square$

5.6. THEOREM. *Let A be artinian and semisimple. Then, every irreducible A-module L is isomorphic to a non-zero minimum left ideal of A. Thus, A has only a finite number of irreducible modules.*

PROOF. Similar to above, we know

$$_A A \cong L_1 \oplus \cdots \oplus L_n$$

and, for irreducible A-module M and $0 \neq m \in M$, we have a map $\phi\colon A \to M$ given by $a \mapsto a.m$. Then,

$$M = \phi(L_1) + \cdots + \phi(L_n)$$

and one of the summands must be nonzero, say $\phi(L_i)$. However, since M is irreducible, this means that $M = \phi(L_i)$ and thus $\phi|_{L_i}$ is an isomorphism. Since A has only a finite number of ideals up to isomorphism by the Krull-Schmidt theorem, this means that A has only a finite number of irreducible modules. □

5.7. THEOREM. *Let A be artinian and semisimple. Let Ae be a left ideal for idempotent e. Then, Ae is irreducible if and only if eAe is a division ring.*

PROOF. Let us assume Ae is irreducible and let L be any nonzero left ideal of eAe. Then,

$$AL \subseteq A \cdot eAe \subseteq Ae$$

and so AL is a left ideal of A contained in Ae, but Ae is irreducible, so $AL = Ae$. Furthermore, since e acts as the identity on eAe, then $eL = L$ and

$$eAe = eAL = eA(eL) = eAe \cdot L \subseteq L \Longrightarrow eAe = L$$

and so eAe has only (0) and itself as left ideals. Thus, eAe is a division ring.

Let us assume eAe is a division ring. Since A is semisimple, if Ae is not irreducible, then

$$Ae = L_1 \oplus L_2$$

where L_1, L_2 are proper submodules. Then, this gives us the decomposition $e = e_1 + e_2$ for idempotents $e_1 \in L_1, e_2 \in L_2$. Note that

$$\begin{aligned} e &= e^2 \\ &= (e_1 + e_2)(e_1 + e_2) \\ &= e_1^2 + e_2e_1 + e_1e_2 + e_2^2 \\ \Longrightarrow 0 &= e_1^2 - e_1 + e_2e_1 + e_1e_2 + e_2^2 - e_2 \end{aligned}$$

However, $e_1e_2 + e_2^2 - e_2 \in L_2$ and $-e_2e_1 - e_1^2 + e_1 \in L_1$ tells us that $e_1e_2 + e_2^2 - e_2 = -e_2e_1 - e_1^2 + e_1 = 0$ since $L_1 \cap L_2 = 0$. Furthermore, $e_1, e_2 \in eAe$ since $e_1, e_2 \in Ae \Longrightarrow e_1e = e_1$ and $e_2e = e_2$, as well as

$$ee_2 = (e_1 + e_2)e_2$$

$$
\begin{aligned}
&= e_1e_2 + e_2^2 \\
&= e_2
\end{aligned}
$$

and similarly $ee_1 = e_1$. Thus, $e_1 = ee_1e \in eAe$ and $e_2 = ee_2e \in eAe$. Finally,

$$
\begin{aligned}
e_2^2 + e_1e_2 &= e_2 \\
&= e_2(e_1 + e_2) \\
&= e_2e_1 + e_2^2 \\
\implies e_1e_2 &= e_2e_1
\end{aligned}
$$

But $e_1e_2 \in L_2$ and $e_2e_1 \in L_1$ and thus $e_1e_2 = 0$, which tells us that eAe has zero divisors. Thus, eAe is not a division ring.

□

6. The Structure of Semisimple (Left) Artinian Rings

6.1. Definition. A left artinian ring A is called *simple* if its only two-sided ideals are 0 and A.

6.2. Definition. A set of idempotents $\{e_1, \dots, e_n\}$ of a ring A is a *complete set of orthogonal idempotents* if

- Each e_i is an idempotent,
- $e_1 + \cdots + e_n = 1$
- and $e_ie_j = 0$ if $i \neq j$.

Note that, for an artinian ring A, we have

$$
{}_AA = L_1 \oplus \cdots \oplus L_n
$$

for L_i irreducible A-modules, also seen as minimal left ideals. Thus,

$$
1 = e_1 + \cdots + e_n, \quad e_i \in L_i
$$

with $e_je_i \in L_i$. Thus, $e_j^2 = e_j$ and $e_ie_j = 0$ if $i \neq j$, so $\{e_1, \dots, e_n\}$ is a complete set of orthogonal idempotents for A.

We now seek to prove the Artin-Wedderburn theorem, that will tell us that a semisimple left artinian ring is isomorphic to a direct sum of matrix rings over division rings.

6.3. Lemma. *Let L, L' be minimal left ideals. Then, $L \cong L'$ as left A-modules if and only if there is an $a' \in L'$ such that $L' = La'$.*

Proof. For ($\implies$), let $\phi\colon L \xrightarrow{\sim} L'$. Then, for $x \in L = (e)$,

$$
\phi(x) = \phi(xe) = x\phi(e) = xa' \quad \text{where } a' = \phi(e)
$$

Then, $L' = La'$ as required.

For ($\Longleftarrow$), assume there is an $a' \in L'$ such that $L' = La'$. Define $\phi\colon L \to L'$ by $\phi(x) = xa'$. Thus, ϕ is surjective. Since L is irreducible, $\ker \phi = \{0\}$ and so ϕ is injective. Thus, ϕ is an isomorphism. $\square$

6.4. Theorem (Weddurburn). *Let A be a semisimple artinian ring. Let L be a minimal left ideal. The sum of all minimal left ideals $\{L' \trianglelefteq A \mid L' \cong L\}$, say B_L, is a simple subring and a 2-sided ideal in A. Also,*

$$A = \bigoplus_{i=1}^{r} B_{L_i}$$

where $L_1, \dots, L_r$ are representatives of distinct isomorphism classes of left ideals.

Proof. Let

$$A = L_1 \oplus \cdots \oplus L_r$$

where the L_i are minimal left ideals of A. There is an equivalence relation on $\{L_1, \dots, L_r\}$ given by

$$L_i \sim L_j \Longleftrightarrow L_i \cong L_j$$

Thus, if $L_1, \dots, L_s$ is an equivalence class, then for $i, j \leq s$, $L_i L_j = L_j$. Thus, let us define

$$B_{L_1} := L_1 \oplus \cdots \oplus L_s \subseteq A$$

Such a subset is closed under multiplication and is a 2-sided ideal since A. Let us do this analogously for every equivalence class to get $B_1, B_2, \dots, B_t$. Then,

$$i \neq j \Longrightarrow B_i B_j = 0$$

and also,

$$B_1 \oplus B_2 \oplus \cdots \oplus B_t = A$$

For a given B_i, we have

$$B_i = L_{i,1} \oplus L_{i,2} \oplus \cdots \oplus L_{i,s_i}$$

where $L_{i,j} = L_\ell$ for some ℓ. We note

$$1_i := e_{i,1} + \cdots + e_{i,s_i}$$

is the multiplicative identity in B_i. Thus, B_i is a subring of A. Also, B_i is a left ideal, so it is artinian.

It still remains to show that B_i is simple subring and a 2-sided ideal. Without loss of generality, consider

It is not clear to me where we show B_i is a 2-sided ideal.

$$B_1 = L_1 \oplus \cdots \oplus L_r$$
$$1_B = e_1 + \cdots + e_r$$

and let $0 \neq D \trianglelefteq B_1$, that is, let D be a nontrivial 2-sided ideal of B_i. Then, D is a left-ideal and thus contains a minimal left ideal L of A. Thus,

$$L \subseteq D \subseteq B_1 \Longrightarrow L \cong L_1$$

and so we conclude that $L \cong L_1$ since $L \subseteq B_1$. However, we also know that

$$Lx \subseteq D, \quad x \in A$$

since D is a right ideal, too. Thus, by 6.3, $\{Lx \mid x \in A\}$ gives all minimal left ideals of A isomorphic to L. Therefore, it must be that $D = B_1$, so B_1 contains no non-trivial two-sided ideals and is thus a simple subring. □

6.5. Theorem (Wedderburn). *Let A be a simple artinian ring. Then, there exists a unique division ring and a unique positive integer n such that*

$$A \cong M_n(D)$$

Conversely, if D is a division ring, then $M_n(D)$ is a simple artinian ring.

Thus, combining our results, we arrive at the landmark theorem.

6.6. Theorem (Artin-Wedderburn Theorem). *Let A be a semisimple left artinian ring. Then, up to reordering, there is a unique decomposition*

$$A \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r)$$

where each D_i is a division ring and each n_i is a positive integer.

7. The Double Centralizer Property

In order to finish our proofs, however, we will appeal to some other facts that make use of the following language.

7.1. Definition. Let A be a ring and M be a left A-module. Furthermore, let $A_L \subseteq \mathrm{End}_A(M)$ be the endomorphisms given by applying elements of A to M. If $A_L = \mathrm{End}_A(M)$, we sa that the pair (A, M) has the *double centralizer property.*

7.2. Remark. More generall, this notion can be extended to any ring R with subring S. We say (R, S) has the double centralizer property if $C_R(C_R(S)) = S$.

Thus, Wedderburn's theorem above can be reprhased as follows

7.3. Theorem (Wedderburn). *Let A be a simple Artinian ring. Then, for some minimal left ideal $M \trianglelefteq A$, (A, M) has the double centralizer property.*

7.4. Lemma. *Let A be a ring. Them $(A, {}_AA)$ has the double centralizer property.*

7.5. Lemma. *Let $V = M^{\bigoplus k}$ for a left A-module M, $k \in \mathbb{N}$. If (A, V) has the double centralizer property, then (A, M) also has the double centralizer property.*

7.6. Lemma. *Let $M = Ae$ for $e \in A$ an idempotent. Then,*

$$\mathrm{End}_A(M) \cong eAe$$

via isomorphism $f\colon \mathrm{End}_A(M) \to eAe$ and $md = m \cdot f(d)$ for all $m \in M, d \in \mathrm{End}_A(M)$.

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