PLETHYSTIC SUBSTITUTION

GEORGE H. SEELINGER

1. Introduction

In [Mac79, p 135], a new type of product on symmetric functions is introduced called “plethysm,” which allows one to take $f \in \Lambda^m$ and $g \in \Lambda^n$ to get a product $f[g] \in \Lambda^{mn}$ (denoted $f \circ g$ in [Mac79]). This notion has become increasingly prevalent in algebraic combinatorics research, and this monograph seeks to give an outline of some of the essentials.

2. Definition and properties

Departing from [Mac79], we define the following.

2.1. **Definition.** Given a Laurent series A in indeterminates $a_1, a_2, a_3, \ldots$, we define $p_n[A]$ to be the series where each a_i is changed to a_i^n. In other words, each indeterminate is raised to the nth power. In particular, given a symmetric function $g \in \Lambda$, $p_n[g(x_1, x_2, \ldots)] = g(x_1^n, x_2^n, \cdots)$.

Furthermore, it is a common convention to let $X = x_1 + x_2 + x_3 + \cdots$ and then write things such as

$$p_n[X] = p_n(x_1, x_2, x_3, \ldots)$$

2.2. **Example.**
(a) If $A = a_1+a_2+a_3+\cdots$, then $p_n[A] = a_1^n+a_2^n+a_3^n+\cdots$.
(b) In particular, $p_n[p_m] = (x_1^n)^m + (x_2^n)^m + \cdots = p_{nm} = p_m[p_n]$. Thus, $p_n[1] = 1$.

2.3. **Proposition.** [Mac79, p 135] *For $n \geq 1$, the mapping $g \mapsto p_n[g]$ is an endomorphism of the ring Λ.*

Next, since any $f \in \Lambda$ can be written as a (rational) linear combination of p_λ's and each p_λ is a product of p_n's, we extend the definition of plethysm to say

2.4. **Definition.** Given a Laurent series A,

(a) we say $p_\lambda[A] = p_{\lambda_1}[A]p_{\lambda_2}[A]\cdots p_{\lambda_\ell}[A]$ and
(b) $(f+g)[A] = f[A] + g[A]$ for any $f, g \in \Lambda$, and

Thus, we can compute $f[A]$ for any symmetric function $f \in \Lambda$ by writing it as a linear combination of p_λ's and evaluating the plethysm on each term.

Date: July 2018.

2.5. **Example.** (a) Given $A = 1+t+t^2+t^3+\cdots = \frac{1}{1-t}$, we get $f[\frac{1}{1-t}] = f(1, t, t^2, t^3, \ldots)$ since

$$p_n\left[\frac{1}{1-t}\right] = 1 + t^n + t^{2n} + \cdots = p_n(1, t, t^2, \ldots)$$

(b) Recall $p_1(x) = x_1 + x_2 + \cdots =: X$. Then, $f[X + a]$ adds a variable a to our set of variables. Similarly, $f[X - x_i]$ removes x_i from the set of variables.

(c) Combining the ideas above, $f[X - (1-t)x_i]$ removes variable x_i but replaces it with variable tx_i.

(d) Finally, $f[\frac{1}{1-t}] = f(1, t, t^2, \ldots$ and $f[\frac{X}{1-t}] = f(x_1, tx_1, t^2x_1, \ldots, x_2, tx_2, t^2x_2, \ldots)$ since $\frac{X}{1-t} = x_1 + tx_1 + t^2x_1 + \cdots + x_2 + tx_2 + t^2x_2 + \cdots$.

2.6. **Proposition.** *Given $c \in \mathbb{Q}$, we get, by definition, that $f[cA] = cf[A]$ for all $f \in \Lambda$ and Laurent series A. However, given an indeterminate t, we get $p_n[tA] = t^n p_n[A]$. In other words, plethysm and variable evaluation do* not *commute.*

Proof. This follows since plethysm affects indeterminates but not constants. □

2.7. **Remark.** The proposition above can be the source of much confusion. On way to distinguish between these two different kinds of values is to call the constants *binomial variables*. So, in the proposition above, we say that c is a binomial variable but t is not.

2.8. **Definition.** It can be convenient to introduce a minus sign to each variable in the plethystic substitution. So, we define the variable ϵ such that

$$p_r[\epsilon X] := p_r(-x_1, -x_2, -x_3, \ldots) = (-1)^r p_r[X]$$

where $X = x_1 + x_2 + x_3 + \cdots$.

2.9. **Remark.** Notice that $p_r[\epsilon X]$ is not necessarily equal to $p_r[-X]$ in our notation. In particular, for a binomial variable $c \in \mathbb{Q}$, we have

$$p_r[cX] = cp_r[X] \text{ but } p_r[\epsilon X] = (-1)^r p_r[X]$$

Furthermore, authors are often not careful with this distinction, so one needs to use context.

2.10. **Proposition.** [Mac79, p 135] *Plethysm is associative. That is,*

$$(f[g])[h] = f[g[h]]$$

Proof. Because the p_n generate Λ over $\mathbb{Q}$, we need only verify the associativity for p_n's, which we already did in 2.2. □

2.11. **Lemma.** *Given Laurent series A and B, we get*

$$p_k[A + B] = p_k[A] + p_k[B]$$

Proof. By definition, $p_k[A+B]$ raises all the indeterminates from A and B to the kth power, which is the same effect as $p_k[A]$ and $p_k[B]$. $\square$

Now, recall the Cauchy kernel

$$\Omega(x,y) = \prod_{i,j} \frac{1}{1-x_iy_j}$$

We seek to generalize this notion as follows. Let us define

$$\Omega := \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k}\right)$$

which gives us that

2.12. **Proposition.** *(a)*

$$\Omega[x] = \exp\left(\sum_{k=1}^{\infty} \frac{x^k}{k}\right) = \exp\left(\log(x-1)\right) = \exp\left(\log((1-x)^{-1})\right) = \frac{1}{1-x}$$

(b) $\Omega[A+B] = \Omega[A]\Omega[B]$ *and* $\Omega[-A] = \frac{1}{\Omega[A]}$ *for any Laurent series A and B*

(c)

$$\Omega[X] = \prod_{i\geq 1} \frac{1}{1-x_i} \text{ and } \Omega[XY] = \Omega(x,y)$$

for formal power series $X = \sum x_i$ and $Y = \sum y_j$.

Proof. By definition, $p_k[x] = x^k$ and so the first part follows. For part (b), using the lemma above, we have

$$\exp\left(p_k[A+B]\right) = \exp\left(p_k[A] + p_k[B]\right) = \exp\left(p_k[A]\right)\exp\left(p_k[B]\right)$$

and so $\Omega[A+B] = \Omega[A]\Omega[B]$. Similarly,

$$\exp\left(p_k[-A]\right) = \exp\left(-p_k[A]\right) = \frac{1}{\exp\left(p_k[A]\right)}$$

Finally, part (c) follows from repeated iteration of part (a). $\square$

2.13. **Corollary.** *(a)* $e_r[X] = h_r[-\epsilon X] = (-1)^r h_r[-X]$.

(b) The involution on symmetric functions $\omega\colon \Lambda \to \Lambda$ corresponds to the plethystic substitution $X \mapsto -\epsilon X$.

Proof. To start, we note $\Omega[tX] = \prod_{i\geq 1} \frac{1}{1-tx_i} = \sum_{r\geq 0} h_r[X]t^r$ and

$$\sum_{r\geq 0} h_r[-\epsilon X]t^r = \Omega[-t\epsilon X] = \prod_{i\geq 1} 1 + x_i = \sum_{r\geq 0} e_r[X]t^r.$$

Then, the first part follows immediately. The second part follows from the first since one definition of ω is precisely that $\omega(h_r) = e_r$. $\square$

3. Examples with Schur Functions

Some examples

3.1. **Proposition.** [Sta99, Cor 7.21.3] *We have*

$$s_\lambda\left[\frac{1}{1-t}\right] = s_\lambda(1,t,t^2,t^3,\ldots) = \frac{t^{n(\lambda)}}{\prod_{x\in\lambda}(1-t^{h(x)})}$$

for $h(x)$ the hook-length of cell $x \in \lambda$ and $n(\lambda) = \sum_i (i-1)\lambda_i$.

Proof. As discussed above, $f\left[\frac{1}{1-t}\right] = f(1,t,t^2,t^3,\ldots)$ for any symmetric function f. Now, we observe that

$$s_\lambda(1,t,t^2,t^3,\ldots,t^{n-1}) = \frac{t^{n(\lambda)+n(n-1)(n-2)/6}\prod_{i<j}(1-t^{\lambda_i-\lambda_j-i+j})}{t^{n(n-1)(n-2)/6}\prod_{i<j}(1-t^{-i+j})}$$

However, one can show using the combinatorics of tableaux (see [Man98, Exercise 1.4.9] and proof of [Man98, Proposition 1.4.10]) that

$$\prod_{x\in\lambda}(1-t^{h(x)})\prod_{i<j}(1-t^{\lambda_i-\lambda_j-i+j}) = \prod_{i=1}^{n}\prod_{k=1}^{\lambda_i+n-i}(1-t^k)$$

and so, plugging this in, we get

$$s_\lambda(1,t,t^2,\ldots,t^{n-1}) = t^{n(\lambda)}\frac{\prod_{i=1}^n\prod_{k=1}^{\lambda_i+n-i}(1-t^k)}{\prod_{x\in\lambda}(1-t^{h(x)})\prod_{i=1}^n\prod_{k=1}^{n-i}(1-t^k)} = t^{n(\lambda)}\frac{\prod_{i=1}^n\prod_{k=n-i+1}^{\lambda_i+n-i}(1-t^k)}{\prod_{x\in\lambda}(1-t^{h(x)})}$$

However, since $\lambda_i = 0$ for $i > \ell(\lambda)$, we can remove one dependence on n to get:

$$s_\lambda(1,t,t^2,\ldots,t^{n-1}) = t^{n(\lambda)}\frac{\prod_{i=1}^{\ell}\prod_{k=n-i+1}^{\lambda_i+n-i}(1-t^k)}{\prod_{x\in\lambda}(1-t^{h(x)})}$$

At this point, [Sta99] claims that $\lim_{n\to\infty}(1-t^n) = 1$, so we are done. □

Of course, I do not understand why this should be true; certainly, it would not work for my calculus students. Perhaps since we are using an expansion where t ¡ 1 anyways, this follows.

3.2. **Proposition.** *Let λ be a partition. Then,*

$$s_\lambda[X+a] = \sum_k a^k \sum_{\lambda=\nu+\textit{horizontal } k\textit{-strip}} s_\nu(x)$$

for $X = x_1 + x_2 + x_3 + \cdots$.

Proof. Using Littlewood's combinatorial description of Schur functions, we get

$$s_\lambda(x_1,\ldots,x_n) = \sum_{T\in\mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})}$$

However, since semistandard tableaux must have strictly increasing columns, all the boxes labelled n must form a (possibly empty) horizontal strip. Thus, if we break up the sum based on how many boxes labelled n there are, we get

$$s_\lambda(x_1,\dots,x_n) = \sum_{k\geq 0} x_n^k \sum_{\lambda=\nu+\text{ horizontal } k\text{-strip}} \sum_{\mathsf{T}\in\mathrm{SSYT}(\nu)} x^{\mathrm{wt}(\mathsf{T})}$$

where $\mathrm{SSYT}(\nu)$ are labelled with letters $\{1,\dots,n-1\}$.

$$= \sum_{k\geq 0} x_n^k \sum_{\lambda=\nu+\text{ horizontal } k\text{-strip}} s_\nu(x_1,\dots,x_{n-1})$$

Thus, we see how to write a Schur function in terms of Schur functions with one fewer variable. $\square$

3.3. **Proposition.** [Mac79, 8.8] *Given a partition λ and symmetric functions g, h,*

$$s_\lambda[g+h] = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu[g] s_\nu[h]$$

where $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.

Proof. This follows from [Mac79, 5.9] which states

$$s_\lambda(x,y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(y) s_\nu(x)$$

following from formal manipulations of skew-Schur functions. $\square$

3.4. **Remark.** One can actually take this as the definition of a skew-Schur function. That is, we can define the skew-Schur function $s_{\lambda/\mu}$ to be such that

$$s_\lambda[X+Y] = \sum_\mu s_{\lambda/\mu}[X] s_\mu[Y]$$

for $X = x_1 + x_2 + \cdots$ and $Y = y_1 + y_2 + \cdots$.

Finally, we state without proof

3.5. **Theorem.** *Given partitions λ, μ, we get*

$$s_\lambda[s_\mu] = \sum_\nu a^\nu_{\lambda\mu} s_\nu$$

with $a^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

3.6. **Remark.** While one can prove that these coefficients are non-negative (see [Mac79, Appendix I.A]), actually describing these coefficients is an old and difficult problem in general, sometimes referred to as the "plethysm problem."

References

[Hai03] M. Haiman, *Combinatorics, Symmetric Functions, and Hilbert Schemes* (2003). https://math.berkeley.edu/~mhaiman/ftp/cdm/cdm.pdf.

[Mac79] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 1979. 2nd Edition, 1995.

[Man98] L. Manivel, *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci*, 1998. Translated by John R. Swallow; 2001.

[Sch14] A. Schilling, *Plethystic notation*, 2014. UCD MAT 280: Macdonald Polynomials and Crystal Bases; https://math.libretexts.org/LibreTexts/University_of_California%2C_Davis/UCD_MAT_280%3A_Macdonald_Polynomials_and_Crystal_Bases/Plethystic_notation..

[See18] G. H. Seelinger, *Algebraic Combinatorics*, 2018. [Online] https://ghseeli.github.io/grad-school-writings/class-notes/algebraic-combinatorics.pdf.

[Sta99] R. P. Stanley, *Endumerative Combinatorics, Vol 2*, 1999.