

A Window into Symmetric Function Theory

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UVA Math Club
Lightning Round

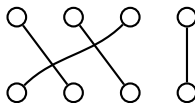
2 March 2021

Symmetric Group

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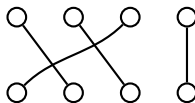
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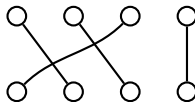
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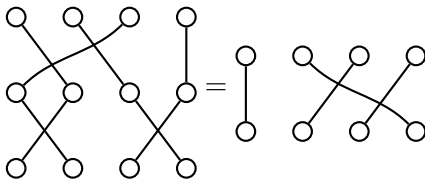
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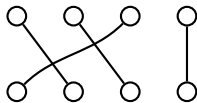
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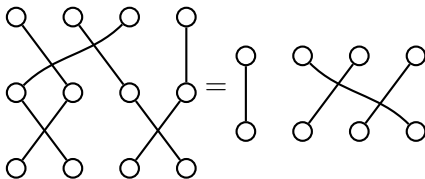


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- S_n is a “group”

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

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$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the e_1, e_2, \dots , or in the h_1, h_2, \dots

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

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- ② Many interesting connections to number theory (Ramanujan).
- ③ Generating function for $p(n)$ = number of partitions of n is inverse of Euler ϕ function.

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Filling of partition diagram of λ with numbers such that

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For $\lambda = (2, 1)$,

1	1	1	1	1	1	1	1
2	3	3	2	3	3	2	3

Schur functions

Associate a polynomial to $\text{SSYT}(\lambda)$.

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1	1								
2									

,

1	1								
3									

,

2	2								
3									

,

1	2								
2									

,

1	3								
3									

,

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

Why Schur functions?

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

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Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

Harmonic polynomials

- ① S_3 action on M fixes vector subspaces!

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Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

Upshot

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- 1 Schur functions $\leftrightarrow S_n$ -invariant subspaces.

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- 2 Via Frobenius characteristic map, questions about S_n -action on vector spaces get translated to questions about Schur expansion coefficients in symmetric functions.

Interesting algebraic combinatorics questions

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- 2 What do the Schur expansion coefficients count?

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Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal S_n -fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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An example of bi-degree

Capturing even more information...

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- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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Minimal S_n -invariant subspace with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

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Diagonal harmonics

- Define ∇ by $\nabla \tilde{H}_\mu = B_\mu(q, t) \tilde{H}_\mu$ for eigenvalue $B_\mu(q, t) \in \mathbb{Q}[q, t]$.

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Open question

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What is the Schur expansion of ∇e_n ?

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Recover earlier story by taking $t = 0$ and $y_i = 1$ for all y_i 's.