

# $K$ -theoretic Catalan functions

George H. Seelinger (joint with J. Blasiak and J. Morse)

*ghs9ae@virginia.edu*

arXiv:2010.01759

UVA Algebra Seminar

April 5, 2021

- Schubert calculus
- Catalan functions: a new approach to old problems
- $K$ -theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}$  for  $H^*(X)$  with property  $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}$  for  $H^*(X)$  with property  $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$



## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

# Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .

# Classical Schubert Calculus

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}_{\lambda \subseteq (n^m)}$  for  $H^*(X)$  with property  $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$



# Classical Schubert Calculus

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}_{\lambda \subseteq (n^m)}$  for  $H^*(X)$  with property  $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$



## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

# Schur functions $s_\lambda$

## Example

*Semistandard tableaux*: columns increasing and rows non-decreasing.

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
3	4		
1	2	5	6

standard = no repeated letters

# Schur functions $s_\lambda$

## Example

*Semistandard tableaux*: columns increasing and rows non-decreasing.

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
3	4		
1	2	5	6

standard = no repeated letters

Schur function  $s_\lambda$  is a “weight generating function” of semistandard tableaux:

2	3	3	2	3	3	2	3
1	1	1	1	1	2	3	1
1	1	2	2	3	3	3	2

$$s_{\square\square}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$$

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square & \end{smallmatrix}}$$

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_\mu s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square & \end{smallmatrix}}$$

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

Since  $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$ , subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients*  $c_{\lambda\mu}^\nu$ .

# Next Step: Flag Variety

## Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$



## Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by  $w \in S_n$ .

## Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by  $w \in S_n$ .
- $H^*(Fl_n(\mathbb{C}))$  supported by Schubert polynomials  $\mathfrak{S}_w \in \mathbb{Z}[x_1, \dots, x_n]$   
(Not necessarily symmetric!)

## Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by  $w \in S_n$ .
- $H^*(Fl_n(\mathbb{C}))$  supported by Schubert polynomials  $\mathfrak{S}_w \in \mathbb{Z}[x_1, \dots, x_n]$   
(Not necessarily symmetric!)

$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

### Open Problem

Structure constants  $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$  have no tableaux description.

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

# Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	$f_\lambda$
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
$K$ -homology of affine Grassmannian	$K$ - $k$ -Schur functions

# Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	$f_\lambda$
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
$K$ -homology of affine Grassmannian	$K$ - $k$ -Schur functions

And many more!

# Peterson Isomorphism

- $QH^*(Fl_{k+1})$  quantum deformation of  $H^*(Fl_{k+1})$  by  $q_1, \dots, q_k$ .

# Peterson Isomorphism

- $QH^*(Fl_{k+1})$  quantum deformation of  $H^*(Fl_{k+1})$  by  $q_1, \dots, q_k$ .
- Supported by quantum Schubert polynomials  $\mathfrak{S}_w^Q$  (Fomin et al., 1997).



# Peterson Isomorphism

- $QH^*(Fl_{k+1})$  quantum deformation of  $H^*(Fl_{k+1})$  by  $q_1, \dots, q_k$ .
- Supported by quantum Schubert polynomials  $\mathfrak{S}_w^Q$  (Fomin et al., 1997). ( $\mathfrak{S}_w^Q \rightarrow \mathfrak{S}_w$  when  $q_i = 0$ .)

# Peterson Isomorphism

- $QH^*(Fl_{k+1})$  quantum deformation of  $H^*(Fl_{k+1})$  by  $q_1, \dots, q_k$ .
- Supported by quantum Schubert polynomials  $\mathfrak{S}_w^Q$  (Fomin et al., 1997). ( $\mathfrak{S}_w^Q \rightarrow \mathfrak{S}_w$  when  $q_i = 0$ .)
- Peterson isomorphism

$$\Phi: QH^*(Fl_{k+1}) \rightarrow H_*(Gr_{SL_{k+1}})_{loc}$$

# Peterson Isomorphism

- $QH^*(Fl_{k+1})$  quantum deformation of  $H^*(Fl_{k+1})$  by  $q_1, \dots, q_k$ .
- Supported by quantum Schubert polynomials  $\mathfrak{S}_w^Q$  (Fomin et al., 1997). ( $\mathfrak{S}_w^Q \rightarrow \mathfrak{S}_w$  when  $q_i = 0$ .)
- Peterson isomorphism

$$\begin{aligned}\Phi: QH^*(Fl_{k+1}) &\rightarrow H_*(Gr_{SL_{k+1}})_{loc} \\ \mathfrak{S}_w^Q &\mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}\end{aligned}$$

where  $s_\lambda^{(k)}$  is a  $k$ -Schur symmetric function and  $Gr_{SL_{k+1}}$  is the “affine Grassmannian.”

Upshot

# Peterson Isomorphism

- $QH^*(Fl_{k+1})$  quantum deformation of  $H^*(Fl_{k+1})$  by  $q_1, \dots, q_k$ .
- Supported by quantum Schubert polynomials  $\mathfrak{S}_w^Q$  (Fomin et al., 1997). ( $\mathfrak{S}_w^Q \rightarrow \mathfrak{S}_w$  when  $q_i = 0$ .)
- Peterson isomorphism

$$\begin{aligned}\Phi: QH^*(Fl_{k+1}) &\rightarrow H_*(Gr_{SL_{k+1}})_{loc} \\ \mathfrak{S}_w^Q &\mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}\end{aligned}$$

where  $s_\lambda^{(k)}$  is a  $k$ -Schur symmetric function and  $Gr_{SL_{k+1}}$  is the “affine Grassmannian.”

## Upshot

Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_{1^r} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_1 s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .



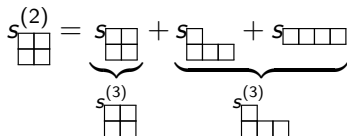
# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_1 s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda}^{(3)} + s_{\lambda}^{(3)}$$

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_1 s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda}^{(3)} + s_{\lambda}^{(3)}$$


- (Lam et al., 2010) gives geometric interpretation,

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_1 r s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\lambda} + s_{\lambda}$$

The diagram shows the branching of the 2-Schur function  $s_{\lambda}^{(2)}$  into three 3-Schur functions. On the left,  $s_{\lambda}^{(2)}$  is represented by a 2x2 square. On the right, the sum of three terms is shown:  $s_{\lambda}^{(3)}$  (a 2x2 square),  $s_{\lambda}^{(3)}$  (a 2x1 rectangle), and  $s_{\lambda}^{(3)}$  (a 1x3 horizontal row). Brackets below the terms on the right identify them as  $s_{(2)}^{(3)}$ ,  $s_{(1,1)}^{(3)}$ , and  $s_{(1,1,1)}^{(3)}$  respectively.

- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_1 r s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}$$

$\underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}^{(3)}} \quad \underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}^{(3)}}$

- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.
- Branching with  $t$  important for Macdonald polynomial positivity.

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_1 r s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda}^{(3)} + s_{\mu}^{(3)} + s_{\nu}^{(3)}$$

The diagram shows the branching of the Schur function  $s_{\lambda}^{(2)}$  into three Schur functions of degree 3. On the left,  $s_{\lambda}^{(2)}$  is represented by a 2x2 square. On the right, it is equal to the sum of three terms:  $s_{\lambda}^{(3)}$  (a 3x2 rectangle),  $s_{\mu}^{(3)}$  (a 2x2 rectangle with an extra cell to the right), and  $s_{\nu}^{(3)}$  (a horizontal row of three cells). Brackets group the three terms on the right under  $s_{(3)}^{(3)}$  and  $s_{(2,1)}^{(3)}$ .

- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.
- Branching with  $t$  important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- $K$ -theoretic Catalan functions

# Why a new definition of $k$ -Schur?

# Why a new definition of $k$ -Schur?

## Answer

- 1 (Blasiak et al., 2019) gives a new definition of  $s_{\lambda}^{(k)}$  and shows it is equivalent to many other previous definitions.



# Why a new definition of $k$ -Schur?

## Answer

- 1 (Blasiak et al., 2019) gives a new definition of  $s_{\lambda}^{(k)}$  and shows it is equivalent to many other previous definitions.
- 2 From a new definition, (Blasiak et al., 2019) shows the branching coefficients  $b_{\lambda\mu}$  in the expansion  $s_{\lambda}^{(k)} = \sum_{\mu} b_{\lambda\mu} s_{\mu}^{(k+1)}$  have combinatorial interpretation!

Key:

# Why a new definition of $k$ -Schur?

## Answer

- 1 (Blasiak et al., 2019) gives a new definition of  $s_{\lambda}^{(k)}$  and shows it is equivalent to many other previous definitions.
- 2 From a new definition, (Blasiak et al., 2019) shows the branching coefficients  $b_{\lambda\mu}$  in the expansion  $s_{\lambda}^{(k)} = \sum_{\mu} b_{\lambda\mu} s_{\mu}^{(k+1)}$  have combinatorial interpretation!

Key: Catalan functions = large class of symmetric functions.

# Ingredients for Catalan functions

- Raising operators

# Ingredients for Catalan functions

- Raising operators
- Symmetric functions indexed by integer vectors

# Ingredients for Catalan functions

- Raising operators
- Symmetric functions indexed by integer vectors
- Root ideals

# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

$$R_{1,3} \left( \begin{array}{|c|c|c|} \hline \text{red} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \text{red} \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|c|} \hline & \text{red} \\ \hline & \\ \hline & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \text{red} \\ \hline & & \\ \hline \end{array}$$

# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

$$R_{1,3} \left( \begin{array}{|c|c|c|} \hline \text{red} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \text{red} \\ \hline & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|c|} \hline & \text{red} \\ \hline & \\ \hline & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \text{red} \\ \hline & \\ \hline \end{array}$$

- Extend action to a symmetric function  $f_\lambda$  by  $R_{i,j}(f_\lambda) = f_{\lambda + \epsilon_i - \epsilon_j}$ .

# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

$$R_{1,3} \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \text{red} \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|c|c|} \hline & & \text{red} \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \text{red} \\ \hline & & \\ \hline \end{array}$$

- Extend action to a symmetric function  $f_\lambda$  by  $R_{i,j}(f_\lambda) = f_{\lambda + \epsilon_i - \epsilon_j}$ .
- For  $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$ , we have the *Jacobi-Trudi identity*

$$s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$$



# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

$$R_{1,3} \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \text{red} \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|c|c|} \hline & & \text{red} \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \text{red} \\ \hline & & \\ \hline \end{array}$$

- Extend action to a symmetric function  $f_\lambda$  by  $R_{i,j}(f_\lambda) = f_{\lambda+\epsilon_i-\epsilon_j}$ .
- For  $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$ , we have the *Jacobi-Trudi identity*

$$s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$$

$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \text{red } h_{310} + \text{red } h_{310} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

# Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector  $\alpha \in \mathbb{Z}^\ell$ .

# Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector  $\alpha \in \mathbb{Z}^\ell$ . Straightening:

$$s_\alpha = \prod_{i < j} (1 - R_{ij}) h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

# Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector  $\alpha \in \mathbb{Z}^\ell$ . Straightening:

$$s_\alpha = \prod_{i < j} (1 - R_{ij}) h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

Simplifies formulas. E.g., for  $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$  (note  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ ),

$$s_{1^r}^\perp s_\lambda =$$

# Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector  $\alpha \in \mathbb{Z}^\ell$ . Straightening:

$$s_\alpha = \prod_{i < j} (1 - R_{ij}) h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

Simplifies formulas. E.g., for  $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$  (note  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ ),

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi =$  Roots above Dyck path  
 $\Delta_{\ell}^+ \setminus \Psi =$  Non-roots below

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi$  = Roots above Dyck path  
 $\Delta_{\ell}^{+} \setminus \Psi$  = Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi$  = Roots above Dyck path  
 $\Delta_{\ell}^{+} \setminus \Psi$  = Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

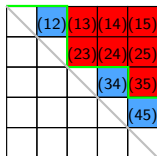
$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

- $\Psi = \emptyset \implies H(\emptyset; \gamma) = s_{\gamma}$



# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi$  = Roots above Dyck path  
 $\Delta_{\ell}^{+} \setminus \Psi$  = Non-roots below

## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

- $\Psi = \emptyset \implies H(\emptyset; \gamma) = s_{\gamma}$
- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_{\gamma}$

## Intuition

Catalan functions interpolate between  $h_\lambda$  and  $s_\lambda$ .

## Intuition

Catalan functions interpolate between  $h_\lambda$  and  $s_\lambda$ .

## Theorem (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive!

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots

# Catalan functions

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots

$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

## Shift Invariance (Blasiak et al., 2019)

For partition  $\lambda$  of length  $\ell$  with  $\lambda_1 \leq k$ ,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$



# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

## Shift Invariance (Blasiak et al., 2019)

For partition  $\lambda$  of length  $\ell$  with  $\lambda_1 \leq k$ ,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$

Proof:  $k - \lambda_i = (k + 1) - (\lambda_i + 1)$

# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

## Shift Invariance (Blasiak et al., 2019)

For partition  $\lambda$  of length  $\ell$  with  $\lambda_1 \leq k$ ,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$

Proof:  $k - \lambda_i = (k + 1) - (\lambda_i + 1)$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

# Key ingredient of branching proof

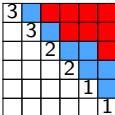
Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .


## Shift Invariance (Blasiak et al., 2019)

For partition  $\lambda$  of length  $\ell$  with  $\lambda_1 \leq k$ ,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$

Proof:  $k - \lambda_i = (k + 1) - (\lambda_i + 1)$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$


$$\Delta^5(4, 4, 3, 3, 2, 2) =$$


Pieri:

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

# Key ingredient of branching proof

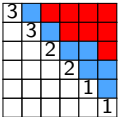
Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

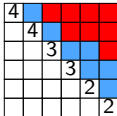
## Shift Invariance (Blasiak et al., 2019)

For partition  $\lambda$  of length  $\ell$  with  $\lambda_1 \leq k$ ,

$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$

Proof:  $k - \lambda_i = (k + 1) - (\lambda_i + 1)$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$


$$\Delta^5(4, 4, 3, 3, 2, 2) =$$


Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

- Schubert calculus
- Catalan functions: a new approach to old problems
- ***K*-theoretic Catalan functions**

# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda + \text{lower degree terms}$ .

# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda +$  lower degree terms.
- Satisfies Pieri rule on “set-valued strips”

# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda +$  lower degree terms.
- Satisfies Pieri rule on “set-valued strips”

$$g_{1^2}g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{31}$$



Add (addable) or mark (removable) in any combination of  $r$  boxes, but only once per row.



# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda + \text{lower degree terms}$ .
- Satisfies Pieri rule on “set-valued strips”

$$g_{1^2}g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{31}$$



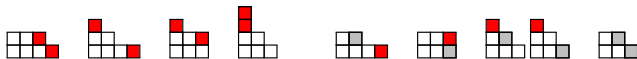
Add (addable) or mark (removable) in any combination of  $r$  boxes, but only once per row.

- $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$  for  $k_\lambda$  and inhomogeneous analogue of  $h_\lambda$ .

# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda + \text{lower degree terms}$ .
- Satisfies Pieri rule on “set-valued strips”

$$g_{1^2}g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{31}$$



Add (addable) or mark (removable) in any combination of  $r$  boxes, but only once per row.

- $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$  for  $k_\lambda$  and inhomogeneous analogue of  $h_\lambda$ .
- Dual to Grothendieck polynomials  $G_\lambda$ : Schubert representatives for  $K^*(Gr(m, n))$

# $K$ - $k$ -Schur functions

- Inhomogeneous basis:  $g_{\lambda}^{(k)} = s_{\lambda}^{(k)} + \text{lower degree terms}$

# $K$ - $k$ -Schur functions

- Inhomogeneous basis:  $g_{\lambda}^{(k)} = s_{\lambda}^{(k)} + \text{lower degree terms}$
- Satisfies Pieri rule on “affine set-valued strips”

# $K$ - $k$ -Schur functions

- Inhomogeneous basis:  $g_{\lambda}^{(k)} = s_{\lambda}^{(k)} + \text{lower degree terms}$
- Satisfies Pieri rule on “affine set-valued strips”

$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions  $\leftrightarrow$  3-cores

The diagram illustrates the Pieri rule for  $K$ - $k$ -Schur functions. It shows the product of a 1-strip ( $g_1$ ) and a 2-bounded partition ( $g_{211}^{(2)}$ ) resulting in the difference of two 2-bounded partitions ( $g_{2111}^{(2)} - 2g_{211}^{(2)}$ ). The partitions are represented as 5x5 grids of dots colored red, blue, or black, with some cells shaded gray to indicate the addition or subtraction of strips.

# $K$ - $k$ -Schur functions

- Inhomogeneous basis:  $g_{\lambda}^{(k)} = s_{\lambda}^{(k)} + \text{lower degree terms}$
- Satisfies Pieri rule on “affine set-valued strips”

$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions  $\leftrightarrow$  3-cores

The diagram illustrates the Pieri rule for  $K$ -Schur functions. It shows the product of a 1-strip (represented by a 5x5 grid of colored dots) and a 2-bounded partition (represented by a 5x5 grid of colored dots). The result is the difference of two 2-bounded partitions (represented by 5x5 grids of colored dots). The partitions are labeled as 2-bounded partitions  $\leftrightarrow$  3-cores.

- Conjecture:  $g_{\lambda}^{(k)}$  have positive branching into  $g_{\mu}^{(k+1)}$  (Lam et al., 2010; Morse, 2011).

# $K$ - $k$ -Schur functions

- Inhomogeneous basis:  $g_{\lambda}^{(k)} = s_{\lambda}^{(k)} + \text{lower degree terms}$
- Satisfies Pieri rule on “affine set-valued strips”

$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions  $\leftrightarrow$  3-cores

- Conjecture:  $g_{\lambda}^{(k)}$  have positive branching into  $g_{\mu}^{(k+1)}$  (Lam et al., 2010; Morse, 2011).

## Problem

No direct formula for  $g_{\lambda}^{(k)}$

## Solution

Find a formula for  $g_{\lambda}^{(k)}$  analogous to raising operator formula for  $s_{\lambda}^{(k)}$ .



## Solution

Find a formula for  $g_{\lambda}^{(k)}$  analogous to raising operator formula for  $s_{\lambda}^{(k)}$ .

Requires an inhomogeneous refinement of Catalan functions.

# An Extra Ingredient: Lowering Operators

Lowering Operators  $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

# Affine $K$ -Theory Representatives with Raising Operators

## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ = (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332} \end{aligned}$$

Answer (Blasiak-Morse-S., 2020)

# Affine $K$ -Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

For  $K$ -homology of affine Grassmannian,  $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$  since this family satisfies the Pieri rule.

# Affine $K$ -Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

For  $K$ -homology of affine Grassmannian,  $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$  since this family satisfies the Pieri rule.

Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$



# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

$$=$$

2							
	1						
		1					
			0				
				0			
					0		
						1	

# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$g_1 g_{211}^{(2)}$

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

# Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$ 

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$ 

2						
	1					
		1				
			0			
				0		
					0	
						1

# Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			1			
				0		
					0	
						1

$$=$$

2			
	1		
		1	
			1

$$-$$

2		
	1	
		1

$$-$$

2		
	1	
		1

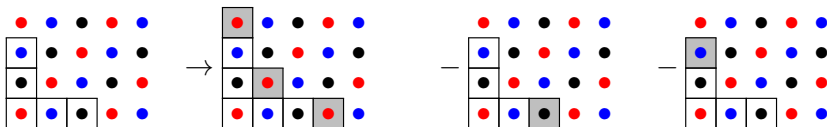
# Pieri Rule Illustrated (Straightening)

$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
 &= g_{2111}^{(2)} - g_{211}^{(2)} - g_{211}^{(2)}
 \end{aligned}$$

# Pieri Rule Illustrated (Straightening)

$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
 &= g_{2111}^{(2)} - g_{211}^{(2)} - g_{211}^{(2)}
 \end{aligned}$$

3-core perspective:



## Theorem (Blasiak-Morse-S., 2020)



## Theorem (Blasiak-Morse-S., 2020)

The  $g_{\lambda}^{(k)}$  are “shift invariant”, i.e. for  $\ell = \ell(\lambda)$

$$G_{1^{\ell}}^{\perp} g_{\lambda+1^{\ell}}^{(k+1)} = g_{\lambda}^{(k)}$$

## Theorem (Blasiak-Morse-S., 2020)

The  $g_\lambda^{(k)}$  are “shift invariant”, i.e. for  $\ell = \ell(\lambda)$

$$G_{1^\ell}^\perp g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

## Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

# $K$ -theoretic Peterson isomorphism

$$\Phi: QK^*(Fl_{k+1}) \rightarrow K_*(Gr_{SL_{k+1}})_{loc}$$

# $K$ -theoretic Peterson isomorphism

$$\Phi: QK^*(Fl_{k+1}) \rightarrow K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For  $w \in S_{k+1}$  and  $\mathfrak{G}_w^Q$  a “quantum Grothendieck polynomial”,

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

# $K$ -theoretic Peterson isomorphism

$$\Phi: QK^*(Fl_{k+1}) \rightarrow K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For  $w \in S_{k+1}$  and  $\mathfrak{G}_w^Q$  a “quantum Grothendieck polynomial”,

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies  $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$  such that  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

# $K$ -theoretic Peterson isomorphism

$$\Phi: QK^*(Fl_{k+1}) \rightarrow K_*(Gr_{SL_{k+1}})_{loc}$$

## Conjecture (Ikeda et al., 2018)

For  $w \in S_{k+1}$  and  $\mathfrak{G}_w^Q$  a “quantum Grothendieck polynomial”,

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies  $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$  such that  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

## Theorem (Blasiak-Morse-S., 2020)

*If  $\lambda \subseteq (d^{k+1-d})$  for some  $1 \leq d \leq k$ , then  $g_\lambda^{(k)} = g_\lambda$ . Thus, conjecture is true for  $w$  a Grassmannian permutation (i.e.  $w$  has only one descent).*

# $K$ -theoretic Peterson isomorphism

$$\Phi: QK^*(Fl_{k+1}) \rightarrow K_*(Gr_{SL_{k+1}})_{loc}$$

## Conjecture (Ikeda et al., 2018)

For  $w \in S_{k+1}$  and  $\mathfrak{G}_w^Q$  a “quantum Grothendieck polynomial”,

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies  $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$  such that  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

## Theorem (Blasiak-Morse-S., 2020)

*If  $\lambda \subseteq (d^{k+1-d})$  for some  $1 \leq d \leq k$ , then  $g_\lambda^{(k)} = g_\lambda$ . Thus, conjecture is true for  $w$  a Grassmannian permutation (i.e.  $w$  has only one descent).*

## Conjecture (Blasiak-Morse-S., 2020)

$$\tilde{g}_w = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Definition (Blasiak-Morse-S., 2020)

For any partition  $\lambda$  with  $\lambda_1 \leq k$ , we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$



# Closed $K$ - $k$ -Schur functions

## Definition (Blasiak-Morse-S., 2020)

For any partition  $\lambda$  with  $\lambda_1 \leq k$ , we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

## Conjecture (Blasiak-Morse-S., 2020)

These  $\tilde{g}_\mu^{(k)}$  satisfy the following properties.

# Closed $K$ - $k$ -Schur functions

## Definition (Blasiak-Morse-S., 2020)

For any partition  $\lambda$  with  $\lambda_1 \leq k$ , we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

## Conjecture (Blasiak-Morse-S., 2020)

These  $\tilde{g}_\mu^{(k)}$  satisfy the following properties.

- The coefficients in  $G_{1^m}^\perp \tilde{g}_\mu^{(k)} = \sum_\nu c_{\mu\nu} \tilde{g}_\nu^{(k)}$  satisfy  $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$ .

# Closed $K$ - $k$ -Schur functions

## Definition (Blasiak-Morse-S., 2020)

For any partition  $\lambda$  with  $\lambda_1 \leq k$ , we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

## Conjecture (Blasiak-Morse-S., 2020)

These  $\tilde{g}_\mu^{(k)}$  satisfy the following properties.

- The coefficients in  $G_{1^m}^\perp \tilde{g}_\mu^{(k)} = \sum_\nu c_{\mu\nu} \tilde{g}_\nu^{(k)}$  satisfy  $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$ .
- The coefficients in  $\tilde{g}_\mu^{(k)} = \sum_\nu a_{\mu\nu} \tilde{g}_\nu^{(k+1)}$  satisfy  $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$ .

# Closed $K$ - $k$ -Schur functions

## Definition (Blasiak-Morse-S., 2020)

For any partition  $\lambda$  with  $\lambda_1 \leq k$ , we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

## Conjecture (Blasiak-Morse-S., 2020)

These  $\tilde{g}_\mu^{(k)}$  satisfy the following properties.

- The coefficients in  $G_{1^m}^\perp \tilde{g}_\mu^{(k)} = \sum_\nu c_{\mu\nu} \tilde{g}_\nu^{(k)}$  satisfy  $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$ .
- The coefficients in  $\tilde{g}_\mu^{(k)} = \sum_\nu a_{\mu\nu} \tilde{g}_\nu^{(k+1)}$  satisfy  $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$ .
- The coefficients in  $\tilde{g}_\mu^{(k)} = \sum_\nu b_{\mu\nu} g_\nu^{(k)}$  satisfy  $(-1)^{|\mu|-|\nu|} b_{\mu\nu} \in \mathbb{Z}_{\geq 0}$ .

# $k$ -Rectangle Property

Theorem (S. (thesis), 2021)

*For  $1 \leq d \leq k$ , set  $R_d = ((k + 1 - d)^d)$  to be the  $k$ -rectangle partition.*

## Theorem (S. (thesis), 2021)

For  $1 \leq d \leq k$ , set  $R_d = ((k+1-d)^d)$  to be the  $k$ -rectangle partition. Then,

$$\tilde{g}_{R_d}^{(k)} \tilde{g}_{\mu}^{(k)} = \tilde{g}_{\mu \cup R_d}^{(k)},$$

where  $\mu \cup R_d$  is the partition given by sorting  $(\mu, R_d)$ .

# $k$ -Rectangle Property

## Theorem (S. (thesis), 2021)

For  $1 \leq d \leq k$ , set  $R_d = ((k+1-d)^d)$  to be the  $k$ -rectangle partition. Then,

$$\tilde{g}_{R_d}^{(k)} \tilde{g}_{\mu}^{(k)} = \tilde{g}_{\mu \cup R_d}^{(k)},$$

where  $\mu \cup R_d$  is the partition given by sorting  $(\mu, R_d)$ .

- Corresponding result for  $s_{\lambda}^{(k)}$  is known, but this gives a Catalan/Katalan-theoretic proof.

# $k$ -Rectangle Property

## Theorem (S. (thesis), 2021)

For  $1 \leq d \leq k$ , set  $R_d = ((k+1-d)^d)$  to be the  $k$ -rectangle partition. Then,

$$\tilde{g}_{R_d}^{(k)} \tilde{g}_{\mu}^{(k)} = \tilde{g}_{\mu \cup R_d}^{(k)},$$

where  $\mu \cup R_d$  is the partition given by sorting  $(\mu, R_d)$ .

- Corresponding result for  $s_{\lambda}^{(k)}$  is known, but this gives a Catalan/Katalan-theoretic proof.
- $k$ -Rectangle Property fails for  $g_{\lambda}^{(k)}$ .



# Positivity of Catalan functions

Recall (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive.

Conjecture (Blasiak-Morse-S., 2020)

For  $\Psi$  a root ideal and  $\lambda$  a partition,

# Positivity of Catalan functions

Recall (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive.

Conjecture (Blasiak-Morse-S., 2020)

For  $\Psi$  a root ideal and  $\lambda$  a partition,

- $K(\Psi; \Psi; \lambda) = \sum_{\mu} a_{\mu} g_{\mu}$  satisfies  $(-1)^{|\lambda| - |\mu|} a_{\mu} \in \mathbb{Z}_{\geq 0}$ .

# Positivity of Katalan functions

## Recall (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive.

## Conjecture (Blasiak-Morse-S., 2020)

For  $\Psi$  a root ideal and  $\lambda$  a partition,

- $K(\Psi; \Psi; \lambda) = \sum_{\mu} a_{\mu} g_{\mu}$  satisfies  $(-1)^{|\lambda| - |\mu|} a_{\mu} \in \mathbb{Z}_{\geq 0}$ .
- $K(\Psi; RC^a(\Psi); \lambda) = \sum_{\mu} b_{\mu} s_{\mu}$  satisfies  $b_{\mu} \in \mathbb{Z}_{\geq 0}$ .

For  $G_\lambda^{(k)}$  an affine Grothendieck polynomial (dual to  $g_\lambda^{(k)}$ ),

For  $G_\lambda^{(k)}$  an affine Grothendieck polynomial (dual to  $g_\lambda^{(k)}$ ),

- ① Combinatorially describe dual Pieri rule:

$$G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}, \quad 1 \leq r \leq k.$$

For  $G_\lambda^{(k)}$  an affine Grothendieck polynomial (dual to  $g_\lambda^{(k)}$ ),

- 1 Combinatorially describe dual Pieri rule:

$$G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}, \quad 1 \leq r \leq k.$$

- 2 Combinatorially describe branching coefficients:  $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$ .

For  $G_\lambda^{(k)}$  an affine Grothendieck polynomial (dual to  $g_\lambda^{(k)}$ ),

- 1 Combinatorially describe dual Pieri rule:

$$G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}, \quad 1 \leq r \leq k.$$

- 2 Combinatorially describe branching coefficients:  $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$ .

- 3 Combinatorially describe  $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$ .

## Thank you!

- Anderson, David, Linda Chen, and Hsian-Hua Tseng. 2017. *On the quantum  $K$ -ring of the flag manifold*, preprint. arXiv: 1711.08414.
- Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and  $k$ -Schur Positivity*, J. Amer. Math. Soc. **32**, no. 4, 921–963.
- Blasiak, Jonah, Jennifer Morse, and Anna Pun. 2020. *Demazure crystals and the Schur positivity of Catalan functions*, preprint. arXiv: 2007.04952.
- Blasiak, Jonah, Jennifer Morse, and George H. Seelinger. 2020.  *$K$ -theoretic Catalan functions*, preprint. arXiv: 2010.01759.
- Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for  $k$ -Schur functions*, Ph.D. thesis.
- Fomin, Sergey, Sergei Gelfand, and Alexander Postnikov. 1997. *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10**, no. 3, 565–596, DOI 10.1090/S0894-0347-97-00237-3. MR1431829
- Ikeda, Takeshi, Shinsuke Iwao, and Toshiaki Maeno. 2018. *Peterson Isomorphism in  $K$ -theory and Relativistic Toda Lattice*, preprint. arXiv: 1703.08664.
- Lam, Thomas. 2008. *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. **21**, no. 1, 259–281.
- Lam, Thomas, Luc Lapointe, Jennifer Morse, and Mark Shimozono. 2010. *Affine insertion and Pieri rules for the affine Grassmannian*, Mem. Amer. Math. Soc. **208**, no. 977.
- Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010.  *$K$ -theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.
- Lapointe, Luc, Alain Lascoux, and Jennifer Morse. 2003. *Tableau atoms and a new Macdonald positivity conjecture*, Duke Mathematical Journal **116**, no. 1, 103–146.
- Morse, Jennifer. 2011. *Combinatorics of the  $K$ -theory of affine Grassmannians*, Advances in Mathematics.
- Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.



$$k_m^{(r)} = \sum_{i=0}^m \binom{r+i-1}{i} h_{m-i} = s_m(X+r),$$

a specialization of “multiSchur functions.” See, e.g., Lascoux-Naruse (2014).

$$k_{\gamma} = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_{\ell}}^{(\ell-1)}$$