Diagonal Harmonics and Shuffle Theorems

George H. Seelinger

ghseeli@umich.edu

on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
arXiv:2102.07931

OIST Representation Theory Seminar

26 October 2021

Outline

- Symmetric polynomials and diagonal harmonics
- The Shuffle Theorem and its generalizations
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \text{ or } h_r = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \text{ or } h_r = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

- E.g. for $n=3$,

$$\begin{aligned} e_1 &= x_1+x_2+x_3 = h_1 \\ e_2 &= x_1x_2+x_1x_3+x_2x_3 \quad h_2 = x_1^2+x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3+x_1^2x_2+x_1^2x_3+x_1x_2^2+\cdots \end{aligned}$$

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \text{ or } h_r = \sum_{i_1\le i_2\le\cdots\le i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

- E.g. for $n=3$,

$$\begin{aligned} e_1 = x_1+x_2+x_3 =& h_1 \\ e_2 = x_1x_2+x_1x_3+x_2x_3 \quad & h_2 = x_1^2+x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2 \\ e_3 = x_1x_2x_3 \quad & h_3 = x_1^3+x_1^2x_2+x_1^2x_3+x_1x_2^2+\cdots \end{aligned}$$

- Let $\Lambda = \mathbb{Q}(q,t)[e_1,e_2,\ldots] = \mathbb{Q}(q,t)[h_1,h_2,\ldots]$. Call these "symmetric functions."

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 = h_1 \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 &= x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

- Let $\Lambda = \mathbb{Q}(q,t)[e_1, e_2, \ldots] = \mathbb{Q}(q,t)[h_1, h_2, \ldots]$. Call these "symmetric functions."
- Λ is a $\mathbb{Q}(q,t)$-algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \ldots, x_l) = \sum_{w \in S_l} w\left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i<j}(1 - x_j/x_i)}\right)$$

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \ldots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i<j}(1 - x_j/x_i)} \right)$$

- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \ldots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \cdots \geq \mu_l \geq 0$.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \ldots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i<j}(1 - x_j/x_i)} \right)$$

- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \ldots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \cdots \geq \mu_l \geq 0$.
- $s_\lambda = \sum_\mu \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu$ for irreducible S_n-character χ_λ.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \ldots, x_l) = \sum_{w \in S_l} w\left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i<j}(1 - x_j/x_i)}\right)$$

- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \ldots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \cdots \geq \mu_l \geq 0$.
- $s_\lambda = \sum_\mu \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu$ for irreducible S_n-character χ_λ.

Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

Harmonic polynomials

Harmonic polynomials

$M =$ polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

2. How many times does an irreducible S_n-representation occur?

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}} \oplus \underbrace{\mathrm{sp}\{1\}}$$

2. How many times does an irreducible S_n-representation occur?
Frobenius:

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

2. How many times does an irreducible S_n-representation occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\ydiagram{1,1,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\yng(1,1,1)} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\yng(2,1)} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\yng(2,1)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\yng(3)}$$

2. How many times does an irreducible S_n-representation occur?
   Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\yng(1,1,1)} + s_{\yng(2,1)} + s_{\yng(2,1)} + s_{\yng(3)}$$

Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_+^{S_3})$.

Getting more information

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\substack{\ydiagram{2,1}\\ \deg=2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\substack{\ydiagram{2,1}\\ \deg=1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\substack{\ydiagram{2,1} \\ \deg=2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\substack{\ydiagram{2,1} \\ \deg=1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\substack{\ydiagram{2,1} \\ \deg=2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\substack{\ydiagram{2,1} \\ \deg=1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{2,1}} + q s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Answer: Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X; q)$.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}(q, t) s_\mu \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}(q, t) s_\mu \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q,t)$ generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X;q,t) = \sum_\mu \tilde{K}(q,t) s_\mu \text{ conjecturally satisfies } \tilde{K}(q,t) \in \mathbb{N}[q,t]$$

- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): M_μ = span of partial derivatives of $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{\ydiagram{2,1}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{\ydiagram{2,1}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\mathrm{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\deg=(0,0)}$$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$$

$$M_{2,1} = \underbrace{\mathrm{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\deg=(0,0)}$$

Irreducible S_n-representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu =$ span of partial derivatives of $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{\ydiagram{2,1}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$$

$$M_{2,1} = \underbrace{\mathrm{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\deg=(0,0)}$$

Irreducible S_n-representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\ydiagram{2,1}} = qts_{\ydiagram{1,1,1}} + ts_{\ydiagram{2,1}} + qs_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via connection to the Hilbert Scheme $Hilb^n(\mathbb{C}^2)$.

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via connection to the Hilbert Scheme $Hilb^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X; q, t) = \tilde{K}_{\lambda\mu}(q, t)s_\mu$ *satisfies* $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via connection to the Hilbert Scheme $Hilb^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X; q, t) = \tilde{K}_{\lambda\mu}(q, t) s_\mu$ *satisfies* $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^{r} \partial_{y_j}^{s}\right) f = 0, \forall r + s > 0\}$$

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

∇e_n

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2 t+qt^2+qt)\tilde{H}_{2,1}}{-q^2 t^2+q^3+t^3-qt} - \frac{q^3 \tilde{H}_3}{-q^3+q^2 t+qt-t^2}$$

∇e_n

Frobenius characteristic of DH_3

$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

∇e_n

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Operator ∇

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X;q,t)$$

∇e_n

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2 t+qt^2+qt)\tilde{H}_{2,1}}{-q^2 t^2+q^3+t^3-qt} - \frac{q^3 \tilde{H}_3}{-q^3+q^2 t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2 t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2 t+qt-t^2}$$

Operator ∇

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X;q,t)$$

Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathsf{area}(\lambda)} q^{\mathsf{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathsf{area}(\lambda)} q^{\mathsf{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathsf{area}(\lambda)} q^{\mathsf{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k-by-k Dyck paths.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k-by-k Dyck paths.
- $\mathrm{area}(\lambda)$ and $\mathrm{dinv}(\lambda)$ statistics of Dyck paths.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

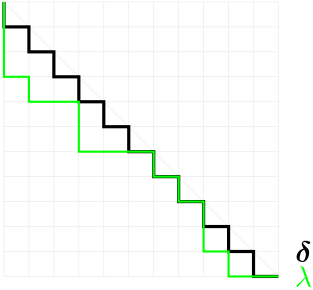
$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k-by-k Dyck paths.
- $\mathrm{area}(\lambda)$ and $\mathrm{dinv}(\lambda)$ statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

Dyck paths

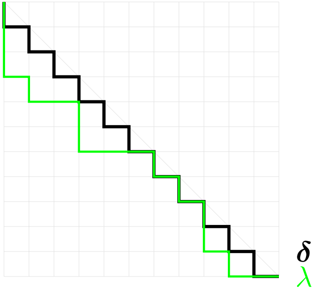
A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

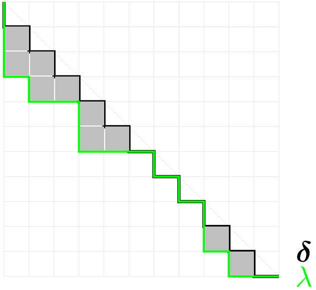


- $\text{area}(\lambda) =$ number of squares above λ but below the path δ of alternating S-E steps.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

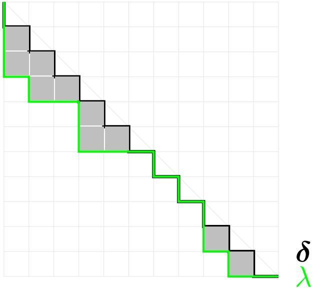


- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above area$(\lambda) = 10$.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.

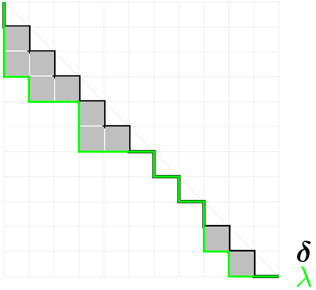


- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above area$(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k.

Dyck paths

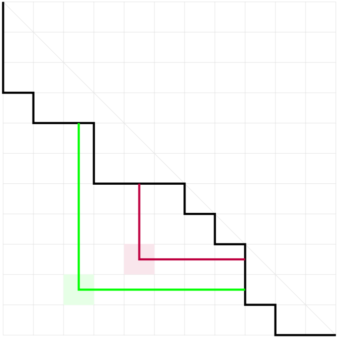
Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



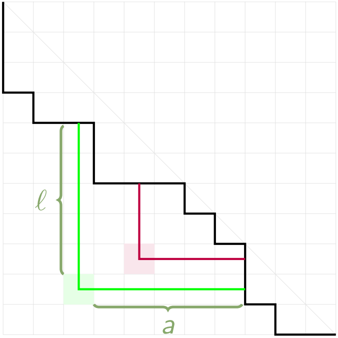
- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k. (1,2,5,14,42,...)

$\mathrm{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ.



dinv

dinv(λ) $=\#$ of balanced hooks in diagram below λ.



Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

LLT Polynomials

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$

LLT Polynomials

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function

LLT Polynomials

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$

LLT Polynomials

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_\nu$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$

LLT Polynomials

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_\nu$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.

LLT Polynomials

Defined in general for a tuple of skew shapes $\nu = (\nu^{(1)}, \ldots, \nu^{(r)})$

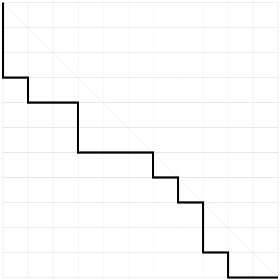
- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_\nu$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.
- $\mathcal{G}_\nu$ is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

LLT Polynomials

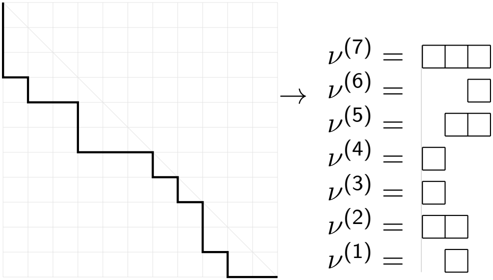
$G_{\nu(\lambda)}(X;q)$ is an LLT polynomial for a tuple of rows,
$\nu(\lambda) = (\nu^{(1)}, \ldots, \nu^{(r)})$.

LLT Polynomials

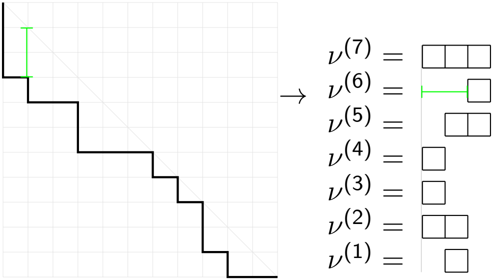
$G_{\nu(\lambda)}(X;q)$ is an LLT polynomial for a tuple of rows, $\nu(\lambda) = (\nu^{(1)}, \ldots, \nu^{(r)})$.



$G_{\nu(\lambda)}(X;q)$ is an LLT polynomial for a tuple of rows, $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$.

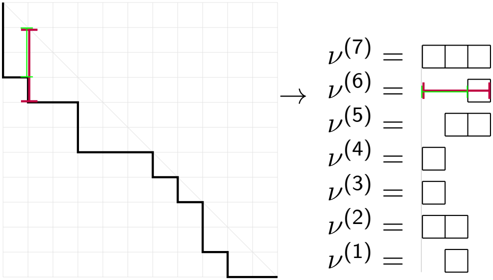


$G_{\nu(\lambda)}(X;q)$ is an LLT polynomial for a tuple of rows, $\nu(\lambda) = (\nu^{(1)}, \ldots, \nu^{(r)})$.



LLT Polynomials

$G_{\nu(\lambda)}(X;q)$ is an LLT polynomial for a tuple of rows, $\nu(\lambda) = (\nu^{(1)}, \ldots, \nu^{(r)})$.



LLT Polynomials

$$\mathcal{G}_\nu(X;q) = \sum_{T \in \mathrm{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

LLT Polynomials

$$\mathcal{G}_\nu(X;q) = \sum_{T \in \mathrm{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 3 & 5 & & & & \\
\hline
2 & 4 & 4 & 7 & 8 & 9 & 9 & & \\
\hline
 & & & 1 & 1 & 6 & 7 & 7 & 7 \\
\hline
\end{array}$$

LLT Polynomials

$$\mathcal{G}_\nu(X;q) = \sum_{T \in \mathrm{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

$$\begin{array}{l} \boxed{1}\boxed{\color{red}{2}}\boxed{3}\boxed{3}\boxed{5} \\ \boxed{2}\boxed{\color{red}{4}}\boxed{4}\boxed{\color{blue}{7}}\boxed{8}\boxed{9}\boxed{9} \end{array}$$

$$T = \qquad \boxed{1}\boxed{\color{blue}{1}}\boxed{6}\boxed{7}\boxed{7}\boxed{7} \to q^{i(T)}x^T = q^{18}x_1^3x_2^2x_3^2x_4^2x_5x_6x_7^4x_8x_9^2$$

$$\mathcal{G}_\nu(X;q) = \sum_{T \in \mathrm{SSYT}(\nu)} q^{i(T)} x^T$$

for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

$$T = \begin{array}{l} \boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{5} \\ \boxed{2}\boxed{4}\boxed{4}\boxed{7}\boxed{8}\boxed{9}\boxed{9} \\ \qquad\qquad \boxed{1}\boxed{1}\boxed{6}\boxed{7}\boxed{7}\boxed{7} \end{array} \to q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

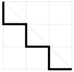
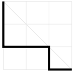
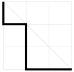
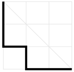
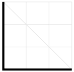
-

$$\begin{aligned} \mathcal{G}_{\substack{\square\square \\ \square}}(x_1, x_2; q) =& x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2 \\ & \begin{array}{l} \boxed{1}\boxed{1} \\ \quad\ \boxed{1} \end{array} \quad \begin{array}{l} \boxed{1}\boxed{2} \\ \quad\ \boxed{1} \end{array} \quad \begin{array}{l} \boxed{1}\boxed{2} \\ \quad\ \boxed{2} \end{array} \quad \begin{array}{l} \boxed{2}\boxed{2} \\ \quad\ \boxed{2} \end{array} \quad \begin{array}{l} \boxed{1}\boxed{1} \\ \quad\ \boxed{2} \end{array} \quad \begin{array}{l} \boxed{2}\boxed{2} \\ \quad\ \boxed{1} \end{array} \\ =& s_3 + q s_{2,1} \end{aligned}$$

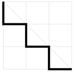
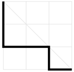
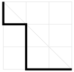
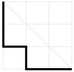
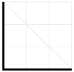
Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

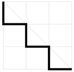
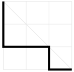
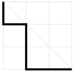
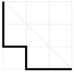
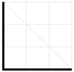
Example ∇e_3

λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
		
		
		
		
		

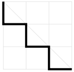
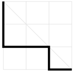
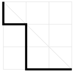
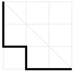
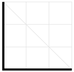
Example ∇e_3

λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

Example ∇e_3

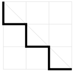
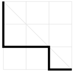
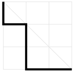
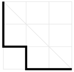
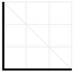
λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

Example ∇e_3

λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric

Example ∇e_3

λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$q t s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + q t s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + q t^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q,t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Outline

- Symmetric polynomials and diagonal harmonics
- **The Shuffle Theorem and its generalizations**
- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

For an abelian category $\mathcal{A}$, the *Hall algebra* of $\mathcal{A}$ has basis $\{[A]\}_{A \in ob(\mathcal{A})}$ and product

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

For an abelian category $\mathcal{A}$, the *Hall algebra* of $\mathcal{A}$ has basis $\{[A]\}_{A\in ob(\mathcal{A})}$ and product

$$[A]\cdot[B] = \langle A, B\rangle \sum_{C\in ob(\mathcal{A})} \frac{P^C_{A,B}}{a_A a_B}[C]$$

where $P^C_{A,B} = \#\{0 \to B \to C \to A \to 0\}$ and
$a_A = |\operatorname{Aut}(A)|, a_B = |\operatorname{Aut}(B)|$

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

For an abelian category $\mathcal{A}$, the *Hall algebra* of $\mathcal{A}$ has basis $\{[A]\}_{A \in ob(\mathcal{A})}$ and product

$$[A] \cdot [B] = \langle A, B \rangle \sum_{C \in ob(\mathcal{A})} \frac{P_{A,B}^{C}}{a_A a_B} [C]$$

where $P_{A,B}^{C} = \#\{0 \to B \to C \to A \to 0\}$ and
$a_A = |\operatorname{Aut}(A)|$, $a_B = |\operatorname{Aut}(B)|$

- Burban and Schiffmann studied a subalgebra $\mathcal{E}$ of the Hall algebra of coherent sheaves on an elliptic curve over $\mathbb{F}_p$

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

For an abelian category $\mathcal{A}$, the *Hall algebra* of $\mathcal{A}$ has basis $\{[A]\}_{A\in ob(\mathcal{A})}$ and product

$$[A]\cdot[B] = \langle A, B\rangle \sum_{C\in ob(\mathcal{A})} \frac{P_{A,B}^{C}}{a_A a_B}[C]$$

where $P_{A,B}^{C} = \#\{0 \to B \to C \to A \to 0\}$ and $a_A = |\operatorname{Aut}(A)|, a_B = |\operatorname{Aut}(B)|$

- Burban and Schiffmann studied a subalgebra $\mathcal{E}$ of the Hall algebra of coherent sheaves on an elliptic curve over $\mathbb{F}_p$
- $\mathcal{E}$ contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

- $\mathcal{E}$ acts on Λ, e.g., for $M = (1-q)(1-t)$ and automorphism ω,

$$e_k[-MX^{m,1}] \cdot 1 = \omega \nabla^m e_k$$

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

- $\mathcal{E}$ acts on Λ, e.g., for $M = (1-q)(1-t)$ and automorphism ω,

$$e_k[-MX^{m,1}] \cdot 1 = \omega \nabla^m e_k$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

$$e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all (kn, km)-Dyck paths.

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

- $\mathcal{E}$ acts on Λ, e.g., for $M = (1-q)(1-t)$ and automorphism ω,

$$e_k[-MX^{m,1}] \cdot 1 = \omega \nabla^m e_k$$

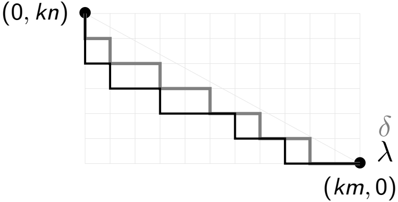
Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

$$e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

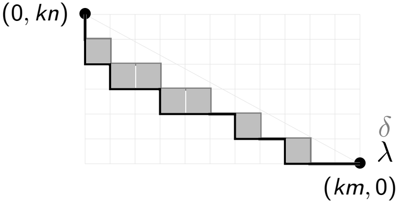
where summation is over all (kn, km)-Dyck paths.

- Coefficient of $s_{1,\ldots,1}$ is "rational (q,t)-Catalan number"

Rational Path Combinatorics

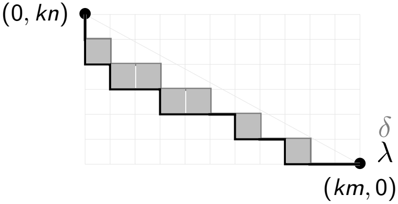


Rational Path Combinatorics

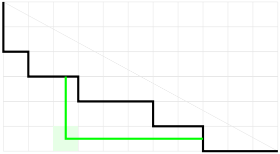


- area(λ) as before; number of boxes between λ and highest path δ below $y + \frac{n}{m}x = kn$.

Rational Path Combinatorics



- $\text{area}(\lambda)$ as before; number of boxes between λ and highest path δ below $y + \frac{n}{m}x = kn$.
- $\text{dinv}_p(\lambda) =$ number of p-balanced hooks:



$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \qquad p = \frac{n}{m} - \epsilon$$

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1,\ldots,b_l)} \cdot 1$$

Any Line

Negut Elements

For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1,\ldots,b_l)} \cdot 1 = \sum_{\lambda} \qquad\qquad \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,

Any Line

Negut Elements

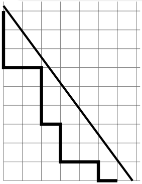
For $\mathbf{b} \in \mathbb{Z}^l$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $(b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$D_{(b_1,\ldots,b_l)} \cdot 1 = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



area(λ) as before

$\mathrm{dinv}_p(\lambda) = \#p$-balanced hooks $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

Outline

- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- **Proof techniques and new progress**

Schiffmann to Shuffle isomorphism

- $\mathcal{E}^+ \cong S$, the Shuffle algebra.

Schiffmann to Shuffle isomorphism

- $\mathcal{E}^+ \cong S$, the Shuffle algebra.
- Define $\sigma \colon \mathbb{Q}(q,t)(x_1, \dots, x_l) \to \mathbb{Q}(q,t)(x_1, \dots, x_l)^{S_l}$,

$$\phi \mapsto \sum_{w \in S_l} w \left(\phi \prod_{i<j} \frac{(1 - qtz_i/z_j)}{(1 - z_j/z_i)(1 - qz_i/z_j)(1 - tz_i/z_j)} \right) .$$

Schiffmann to Shuffle isomorphism

- $\mathcal{E}^+ \cong S$, the Shuffle algebra.
- Define $\sigma\colon \mathbb{Q}(q,t)(x_1,\dots,x_l) \to \mathbb{Q}(q,t)(x_1,\dots,x_l)^{S_l}$,

$$\phi \mapsto \sum_{w \in S_l} w \left(\phi \prod_{i<j} \frac{(1 - qtz_i/z_j)}{(1 - z_j/z_i)(1 - qz_i/z_j)(1 - tz_i/z_j)} \right) .$$

- $S = \sigma(k[z_1^{\pm 1}, \dots, z_l^{\pm 1}])$.

Schiffmann to Shuffle isomorphism

- $\mathcal{E}^+ \cong S$, the Shuffle algebra.
- Define $\sigma\colon \mathbb{Q}(q,t)(x_1,\dots,x_l) \to \mathbb{Q}(q,t)(x_1,\dots,x_l)^{S_l}$,

$$\phi \mapsto \sum_{w \in S_l} w \left(\phi \prod_{i<j} \frac{(1 - qtz_i/z_j)}{(1 - z_j/z_i)(1 - qz_i/z_j)(1 - tz_i/z_j)} \right).$$

- $S = \sigma(k[z_1^{\pm 1}, \dots, z_l^{\pm 1}])$.
- However, σ applied to a rational function f could satisfy $\sigma(f) = \sigma(p)$ for some Laurent polynomial p.

Schiffmann to Shuffle isomorphism

- $\mathcal{E}^+ \cong S$, the Shuffle algebra.
- Define $\sigma \colon \mathbb{Q}(q,t)(x_1,\dots,x_l) \to \mathbb{Q}(q,t)(x_1,\dots,x_l)^{S_l}$,

$$\phi \mapsto \sum_{w \in S_l} w \left(\phi \prod_{i<j} \frac{(1 - qtz_i/z_j)}{(1 - z_j/z_i)(1 - qz_i/z_j)(1 - tz_i/z_j)} \right).$$

- $S = \sigma(k[z_1^{\pm 1}, \dots, z_l^{\pm 1}])$.
- However, σ applied to a rational function f could satisfy $\sigma(f) = \sigma(p)$ for some Laurent polynomial p.
- Under isomorphism

$$\mathcal{E}^+ \ni D_{\mathbf{b}} \leftrightarrow \sigma \left(\frac{z_1^{b_1} \cdots z_l^{b_l}}{\prod_{i=1}^{n-1}(1 - qtz_i/z_{i+1})} \right) \in S$$

Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \ldots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1<j} (1 - qtx_i/x_{i+1})}{\prod_{i<j} ((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1<j}(1 - qtx_i/x_{i+1})}{\prod_{i<j}((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\text{pol}}$$

- Let $\psi D_{\mathbf{b}}$ be RHS without applying pol. Easier to prove a "shuffle theorem-like" result on infinite series: $\psi D_{\mathbf{b}} = \sum$ over something

Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \ldots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1<j} (1 - qtx_i/x_{i+1})}{\prod_{i<j}((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\mathrm{pol}}$$

- Let $\psi D_{\mathbf{b}}$ be RHS without applying pol. Easier to prove a “shuffle theorem-like” result on infinite series: $\psi D_{\mathbf{b}} = \sum$ over something
- Need an “infinite series” version of LLT polynomials!

Cauchy Identity

- Let Hecke algebra of S_l act on $\mathbb{Q}(q)[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ via
$$T_i = qs_i + (1-q)\frac{s_i - 1}{1 - x_{i+1}/x_i}$$

Cauchy Identity

- Let Hecke algebra of S_l act on $\mathbb{Q}(q)[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ via
$$T_i = qs_i + (1-q)\frac{s_i - 1}{1 - x_{i+1}/x_i}$$
- (Twisted) non-symmetric Hall-Littlewood polynomials $E_\lambda^\sigma(x_1, \ldots, x_l; q)$ for $\lambda \in \mathbb{Z}^l$ and $\sigma \in S_l$ defined via

Cauchy Identity

- Let Hecke algebra of S_l act on $\mathbb{Q}(q)[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ via
$$T_i = qs_i + (1-q)\frac{s_i - 1}{1 - x_{i+1}/x_i}$$
- (Twisted) non-symmetric Hall-Littlewood polynomials $E_\lambda^\sigma(x_1, \ldots, x_l; q)$ for $\lambda \in \mathbb{Z}^l$ and $\sigma \in S_l$ defined via
$$E_\lambda(x_1, \ldots, x_l; q) = q^{-\ell(w)} T_w x^{\lambda_+}, \text{ for } w(\lambda_+) = \lambda$$

Cauchy Identity

- Let Hecke algebra of S_l act on $\mathbb{Q}(q)[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ via
$$T_i = qs_i + (1-q)\frac{s_i - 1}{1 - x_{i+1}/x_i}$$
- (Twisted) non-symmetric Hall-Littlewood polynomials $E_\lambda^\sigma(x_1, \ldots, x_l; q)$ for $\lambda \in \mathbb{Z}^l$ and $\sigma \in S_l$ defined via
$$E_\lambda(x_1, \ldots, x_l; q) = q^{-\ell(w)} T_w x^{\lambda_+}, \text{ for } w(\lambda_+) = \lambda$$
$$E_\lambda^\sigma(x_1, \ldots, x_l; q) = q^{|\operatorname{Inv}(\sigma^{-1}) \cap \operatorname{Inv}(\lambda + \epsilon\rho)|} T_{\sigma^{-1}}^{-1} E_{\sigma^{-1}(\lambda)}(x_1, \ldots, x_l; q)$$

Cauchy Identity

- Let Hecke algebra of S_l act on $\mathbb{Q}(q)[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ via
$$T_i = qs_i + (1-q)\frac{s_i - 1}{1 - x_{i+1}/x_i}$$

- (Twisted) non-symmetric Hall-Littlewood polynomials $E_\lambda^\sigma(x_1, \ldots, x_l; q)$ for $\lambda \in \mathbb{Z}^l$ and $\sigma \in S_l$ defined via
$$E_\lambda(x_1, \ldots, x_l; q) = q^{-\ell(w)} T_w x^{\lambda_+}, \text{ for } w(\lambda_+) = \lambda$$
$$E_\lambda^\sigma(x_1, \ldots, x_l; q) = q^{|\operatorname{Inv}(\sigma^{-1}) \cap \operatorname{Inv}(\lambda + \epsilon\rho)|} T_{\sigma^{-1}}^{-1} E_{\sigma^{-1}(\lambda)}(x_1, \ldots, x_l; q)$$

- Dual basis: $F_\lambda^\sigma = \overline{E_{-\lambda}^{\sigma w_0}}$.

Cauchy Identity

- Let Hecke algebra of S_l act on $\mathbb{Q}(q)[x_1^{\pm 1}, \ldots, x_l^{\pm 1}]$ via
$$T_i = q s_i + (1-q)\frac{s_i - 1}{1 - x_{i+1}/x_i}$$
- (Twisted) non-symmetric Hall-Littlewood polynomials $E_\lambda^\sigma(x_1, \ldots, x_l; q)$ for $\lambda \in \mathbb{Z}^l$ and $\sigma \in S_l$ defined via
$$E_\lambda(x_1, \ldots, x_l; q) = q^{-\ell(w)} T_w x^{\lambda_+}, \text{ for } w(\lambda_+) = \lambda$$
$$E_\lambda^\sigma(x_1, \ldots, x_l; q) = q^{|\operatorname{Inv}(\sigma^{-1}) \cap \operatorname{Inv}(\lambda + \epsilon\rho)|} T_{\sigma^{-1}}^{-1} E_{\sigma^{-1}(\lambda)}(x_1, \ldots, x_l; q)$$
- Dual basis: $F_\lambda^\sigma = \overline{E_{-\lambda}^{\sigma w_0}}$.

Cauchy identity

$$\frac{\prod_{i<j}(1 - q\, t\, x_i\, y_j)}{\prod_{i \le j}(1 - t\, x_i\, y_j)} = \sum_{\mathbf{a} \ge 0} t^{|\mathbf{a}|} E_{\mathbf{a}}^\sigma(x_1, \ldots, x_l; q^{-1}) F_{\mathbf{a}}^\sigma(y_1, \ldots, y_l; q),$$

Proof Idea

Let $H_q(f) = \sigma\left(\frac{f}{\prod_{i<j}(1-qx_i/x_j)}\right)$.

Note $\psi D_{\mathbf{b}} = H_q\left(x^{\mathbf{b}} \frac{\prod_{i+1<j}(1-qtx_i/x_j)}{\prod_{i<j}(1-tx_i/x_j)}\right)$

Proof Idea

Let $H_q(f) = \sigma\left(\frac{f}{\prod_{i<j}(1-qx_i/x_j)}\right)$.

Note $\psi D_{\mathbf{b}} = H_q\left(x^{\mathbf{b}}\frac{\prod_{i+1<j}(1-qtx_i/x_j)}{\prod_{i<j}(1-tx_i/x_j)}\right)$ (looks related to $\frac{\prod_{i<j}(1-qt\,x_i\,y_j)}{\prod_{i\leq j}(1-t\,x_i\,y_j)}$)

Proof Idea

Let $H_q(f) = \sigma\left(\frac{f}{\prod_{i<j}(1-qx_i/x_j)}\right)$.

Note $\psi D_{\mathbf{b}} = H_q\left(x^{\mathbf{b}}\frac{\prod_{i+1<j}(1-qtx_i/x_j)}{\prod_{i<j}(1-tx_i/x_j)}\right)$ (looks related to $\frac{\prod_{i<j}(1-qt\,x_i\,y_j)}{\prod_{i\leq j}(1-t\,x_i\,y_j)}$)

- $\psi D_{\mathbf{b}} = H_q\left(\sum_{a_1,\ldots a_{l-1}\geq 0} t^{|\mathbf{a}|} w_0\left(F^{\sigma^{-1}}_{(b_l,\cdots,b_1)+(0,a_{l-1},\ldots,a_1)}\overline{E^{\sigma^{-1}}_{a_{l-1},\ldots,a_1,0}}\right)\right)$

Proof Idea

Let $H_q(f) = \sigma\left(\frac{f}{\prod_{i<j}(1-qx_i/x_j)}\right)$.
Note $\psi D_{\mathbf{b}} = H_q\left(x^{\mathbf{b}}\frac{\prod_{i+1<j}(1-qtx_i/x_j)}{\prod_{i<j}(1-tx_i/x_j)}\right)$ (looks related to $\frac{\prod_{i<j}(1-qt\,x_i\,y_j)}{\prod_{i\leq j}(1-t\,x_i\,y_j)}$)

- $\psi D_{\mathbf{b}} = H_q\left(\sum_{a_1,\ldots a_{l-1}\geq 0} t^{|\mathbf{a}|} w_0\left(F^{\sigma^{-1}}_{(b_l,\cdots,b_1)+(0,a_{l-1},\ldots,a_1)}\overline{E^{\sigma^{-1}}_{a_{l-1},\ldots,a_1,0}}\right)\right)$
- $\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q) = H_q(w_0(F^{\sigma^{-1}}_{\beta}\overline{E^{\sigma^{-1}}_{\alpha}}))$

Proof Idea

Let $H_q(f) = \sigma\left(\frac{f}{\prod_{i<j}(1-qx_i/x_j)}\right)$.

Note $\psi D_{\mathbf{b}} = H_q\left(x^{\mathbf{b}}\frac{\prod_{i+1<j}(1-qtx_i/x_j)}{\prod_{i<j}(1-tx_i/x_j)}\right)$ (looks related to $\frac{\prod_{i<j}(1-qt\,x_i\,y_j)}{\prod_{i\leq j}(1-t\,x_i\,y_j)}$)

- $\psi D_{\mathbf{b}} = H_q\left(\sum_{a_1,\ldots a_{l-1}\geq 0} t^{|\mathbf{a}|} w_0\left(F^{\sigma^{-1}}_{(b_l,\cdots,b_1)+(0,a_{l-1},\ldots,a_1)}\overline{E^{\sigma^{-1}}_{a_{l-1},\ldots,a_1,0}}\right)\right)$
- $\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q) = H_q(w_0(F^{\sigma^{-1}}_{\beta}\overline{E^{\sigma^{-1}}_{\alpha}}))$

Stable Shuffle Theorem

For $\mathbf{b}\in\mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1,\ldots,a_{l-1}\geq 0} t^{|\mathbf{a}|}\mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0,a_{l-1},\ldots,a_1))/(a_{l-1},\ldots,a_1,0)}(x_1,\ldots,x_l;q)$$

Proof Idea

Stable Shuffle Theorem

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1,\ldots,a_{l-1}\geq 0} t^{|\mathbf{a}|} \mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0,a_{l-1},\ldots,a_1))/(a_{l-1},\ldots,a_1,0)}(x_1,\ldots,x_l;q)$$

Proof Idea

Stable Shuffle Theorem

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1,\ldots,a_{l-1}\geq 0} t^{|\mathbf{a}|} \mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0,a_{l-1},\ldots,a_1))/(a_{l-1},\ldots,a_1,0)}(x_1,\ldots,x_l;q)$$

Under polynomial truncation,
$\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q) \to q^{\mathrm{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1,\ldots,x_l;q^{-1})$

Proof Idea

Stable Shuffle Theorem

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1,\ldots,a_{l-1}\geq 0} t^{|\mathbf{a}|} \mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0,a_{l-1},\ldots,a_1))/(a_{l-1},\ldots,a_1,0)}(x_1,\ldots,x_l;q)$$

Under polynomial truncation,
$\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q) \to q^{\mathrm{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1,\ldots,x_l;q^{-1})$

$$\Longrightarrow \omega(D_{\mathbf{b}} \cdot 1)(x_1,\ldots,x_l) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1,\ldots,x_l;q^{-1}).$$

Generalizations

Same paradigm works to show the following formula.

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b)\in\mu} q^{a-1}t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|}\hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \cline{1-2}\end{array} \to B_\mu = 1 + q + q^2 + t + qt$

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b)\in\mu} q^{a-1}t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|}\hline 11 & 12 & 13 \\ \hline 21 & 22 \\ \cline{1-2}\end{array} \to B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu]\tilde{H}_\mu$

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b)\in\mu} q^{a-1}t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|}\hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \cline{1-2}\end{array} \to B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu]\tilde{H}_\mu$
- $\Delta'_f \tilde{H}_\mu = f[B_\mu - 1]\tilde{H}_\mu$

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b)\in\mu} q^{a-1}t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|}\hline 11 & 12 & 13 \\ \hline 21 & 22 \\ \cline{1-2}\end{array} \to B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu]\tilde{H}_\mu$
- $\Delta'_f \tilde{H}_\mu = f[B_\mu - 1]\tilde{H}_\mu$

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

$$\Delta_{h_r}\Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda,P} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} \left(1 + zt^{-r_i(\lambda)}\right).$$

Generalizations

Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b)\in\mu} q^{a-1}t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|}\hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \cline{1-2}\end{array} \to B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu]\tilde{H}_\mu$
- $\Delta'_f \tilde{H}_\mu = f[B_\mu - 1]\tilde{H}_\mu$

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

$$\Delta_{h_r}\Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda,P} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} \left(1 + zt^{-r_i(\lambda)}\right).$$

- $\Delta_{h_r}\Delta'_{e_{n-1}} e_k = \sum_{\substack{s\in\mathbb{N}^{k+r}:|s|=n-k \\ 1\in J\subseteq[k+r],|J|=k}} \left(D_{s+\epsilon_J}\cdot 1\right)$

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^l$. When is $D_{\mathbf{b}} \cdot 1$ nice?

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^l$. When is $D_{\mathbf{b}} \cdot 1$ nice?

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^l$. When is $D_{\mathbf{b}} \cdot 1$ nice?

Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \ldots, b_l)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^l$. When is $D_{\mathbf{b}} \cdot 1$ nice?

Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \ldots, b_l)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

- Experimental computation suggests this is "tight."

Generalizations

$D_{\mathbf{b}}$ defined for any $\mathbf{b} \in \mathbb{Z}^l$. When is $D_{\mathbf{b}} \cdot 1$ nice?

Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \ldots, b_l)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

- Experimental computation suggests this is "tight."
- Coefficient of $s_{1,\ldots,1}$ coincides with (q, t)-polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

Future Directions

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \mathrm{sgn}(\mu) \sum_{(G,R)\in LNDP_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} x^R$$

Future Directions

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R)\in LNDP_\mu} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?

Future Directions

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \mathrm{sgn}(\mu) \sum_{(G,R) \in LNDP_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} x^R$$

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?
- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)

Future Directions

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R)\in LNDP_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} x^R$$

- What are the Schur expansion coefficients of $D_{\mathbf{b}} \cdot 1$?
- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)
- S_l-representation theory interpretations?

References

Thank you!

Bergeron, Francois, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. 2016. *Compositional (km, kn)-shuffle conjectures*, Int. Math. Res. Not. IMRN **14**, 4229–4270, DOI 10.1093/imrn/rnv272. MR3556418

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H Seelinger. 2021a. *A Shuffle Theorem for Paths Under Any Line*, arXiv e-prints, available at arXiv:2102.07931.

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021b. *A proof of the Extended Delta Conjecture*, arXiv e-prints, available at arXiv:2102.08815.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Anton Mellit. 2018. *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31**, no. 3, 661–697, DOI 10.1090/jams/893. MR3787405

Galashin, Pavel and Thomas Lam. 2021. *Positroid Catalan numbers*, arXiv e-prints, arXiv:2104.05701, available at arXiv:2104.05701.

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

Gorsky, Eugene, Graham Hawkes, Anne Schilling, and Julianne Rainbolt. 2020. *Generalized q, t-Catalan numbers*, Algebr. Comb. **3**, no. 4, 855–886, DOI 10.5802/alco.120. MR4145982

References

Grojnowski, Ian and Mark Haiman. 2007. *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, Unpublished manuscript.

Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.

Haglund, J., J. B. Remmel, and A. T. Wilson. 2018. *The delta conjecture*, Trans. Amer. Math. Soc. **370**, no. 6, 4029–4057, DOI 10.1090/tran/7096. MR3811519

Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919

_______. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676

Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sém. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754

Loehr, Nicholas A. and Gregory S. Warrington. 2008. *Nested quantum Dyck paths and* $\nabla(s_\lambda)$, Int. Math. Res. Not. IMRN **5**, Art. ID rnm 157, 29, DOI 10.1093/imrn/rnm157. MR2418288

Mellit, Anton. 2016. *Toric braids and* (m,n)*-parking functions*, arXiv e-prints, arXiv:1604.07456, available at `arXiv:1604.07456`.

Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004