

# $K$ -theoretic Catalan functions

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- Schubert calculus: connecting geometry and combinatorics
- Catalan functions: a new approach to old problems
- $K$ -theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

# Algebra of Symmetric Functions

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$$e_1 = x_1 + x_2 + x_3 \quad h_1 = x_1 + x_2 + x_3$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$



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- Bases indexed by integer partitions.

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

# Partitions

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

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- Schubert varieties  $X_\lambda = \overline{\Omega_\lambda}$ .

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .

# Classical Schubert Calculus

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## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  indexed by partitions such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

## Example

*Semistandard tableaux*: columns increasing and rows non-decreasing.

$$T =$$

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
3	4		
1	2	5	6

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$$\text{wt}(T) = (1, 3, 2, 1, 2) \quad (1, 1, 1, 1, 1, 1, 1, 1, 1)$$

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$$\text{wt}(T) = (1, 3, 2, 1, 2) \quad (1, 1, 1, 1, 1, 1, 1, 1, 1)$$

$$x^{\text{wt}(T)} = x_1^1 x_2^3 x_3^2 x_4^1 x_5^2 \quad x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$$

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$\text{SSYT}(\lambda) =$  all semistandard tableaux of shape  $\lambda$ .

$$\begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array} \end{array}$$



# Schur functions $s_\lambda$

Schur function  $s_\lambda$  is a “weight generating function” of semistandard tableaux:

2		3		3		2		3		3		2		3	
1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

$$s_{\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$s_\lambda(x)$  is homogeneous of degree  $\lambda_1 + \cdots + \lambda_\ell$ .

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}$$

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Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since  $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$ , subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients*  $c_{\lambda\mu}^\nu$ .

## Upshot

Let  $\{f_\lambda\}$  be a basis of  $\Lambda$  such that

- 1  $f_r = s_r$  and
- 2  $f_r f_\lambda$  satisfies the Pieri rule.

Then,  $f_\lambda = s_\lambda$ .

# Schur functions $s_\lambda$ (cont.)

## Upshot

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## Looking Ahead

This type of technique will be useful for establishing the equivalence of new formulas for other bases.



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- Does it have a Pieri rule? ( $s_r s_\lambda = \sum s_\nu$ )
- Does it have a direct formula? ( $s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T$ )

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# Schubert Calculus Variations

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Theory	$f_\lambda$
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
$K$ -homology of affine Grassmannian	$K$ - $k$ -Schur functions

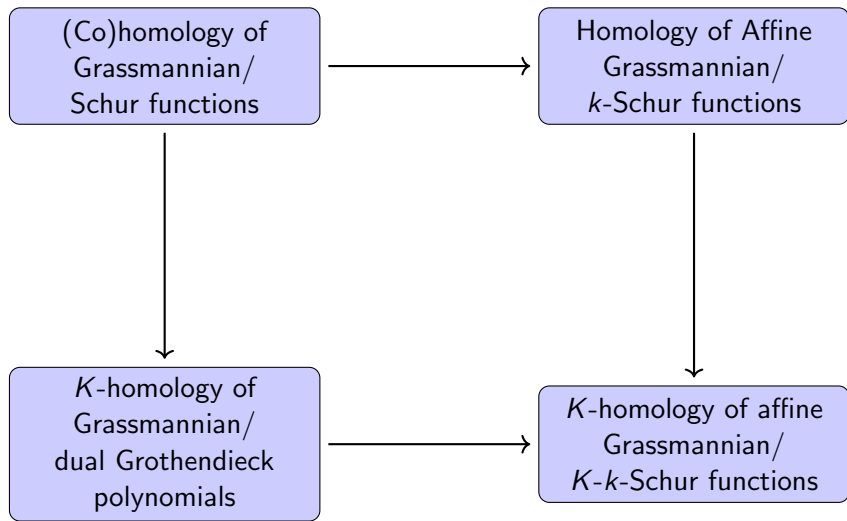
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$K$ -homology of affine Grassmannian	$K$ - $k$ -Schur functions

And many more!

# Big Picture



# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).



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- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\lambda} + s_{\lambda}$$

The diagram illustrates the branching rule for  $k=2$ . The left side shows  $s_{\lambda}^{(2)}$  with a Young diagram of shape  $(2,2)$ . The right side shows the sum of three terms, each with a Young diagram of shape  $(2,2)$ . The first term is  $s_{\lambda}^{(3)}$  with a Young diagram of shape  $(2,2)$ . The second and third terms are  $s_{\lambda}^{(3)}$  with a Young diagram of shape  $(2,2)$ . The Young diagrams are arranged in a row, with the first term on the left, followed by a plus sign, then the second term, followed by a plus sign, then the third term. The Young diagrams are arranged in a row, with the first term on the left, followed by a plus sign, then the second term, followed by a plus sign, then the third term.

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The diagram illustrates the branching of the Schur function  $s_{(2)}^{(2)}$  into three copies of  $s_{(2)}^{(3)}$ . On the left,  $s_{(2)}^{(2)}$  is represented by a Young diagram with two rows of two boxes. This is equal to the sum of three terms. The first term is  $s_{(2)}^{(3)}$ , represented by a Young diagram with two rows of two boxes. The second and third terms are also  $s_{(2)}^{(3)}$ , represented by Young diagrams with two rows of two boxes. Brackets underneath the three terms on the right group them together, with the label  $s_{(2)}^{(3)}$  placed below each group.

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$\underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}^{(3)}} \quad \underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}^{(3)}}$

- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.

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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\lambda} + s_{\lambda}$$

The diagram shows the branching of the 2nd Schur function  $s_{\lambda}^{(2)}$  into three 3rd Schur functions. The left side shows  $s_{\lambda}^{(2)}$  as a 2x2 square. The right side shows the sum of three 3x3 squares:  $s_{\lambda}^{(3)}$  (a 2x2 square with an extra cell at the bottom),  $s_{\lambda}^{(3)}$  (a 2x2 square with an extra cell at the bottom right), and  $s_{\lambda}^{(3)}$  (a 2x2 square with an extra cell at the bottom right). Brackets below the right side group the three terms under  $s_{\lambda}^{(3)}$  and  $s_{\lambda}^{(3)}$ .

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- but no combinatorial interpretation of branching coefficients.
- Branching with  $t$  important for Macdonald polynomial positivity.

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).
- Schubert representatives for  $H_*(Gr_{SL_{k+1}})$  (Lam, 2008).
- Has a tableaux formulation and Pieri rule:  $s_1 r s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$
- $s_{\lambda}^{(k)} = s_{\lambda}$  as  $k \rightarrow \infty$ .
- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}$$

$\underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}^{(3)}} \quad \underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}^{(3)}}$

- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.
- Branching with  $t$  important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.



- Schubert calculus
- **Catalan functions: a new approach to old problems**
- $K$ -theoretic Catalan functions

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Key:  $\{s_{\lambda}^{(k)}\}_{\lambda} \subseteq \text{Catalan functions} = \text{large class of symmetric functions.}$

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- Raising operators  $R_{i,j}$  act on diagrams

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$$s_{22} = (1 - R_{12})h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13})h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \text{red } h_{310} + \text{red } h_{310} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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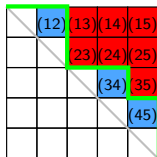
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# Root Ideals

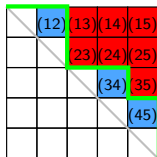
A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi =$  Roots above Dyck path  
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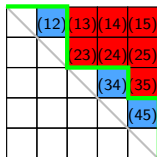
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^+ \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

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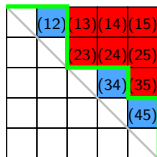
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## Intuition

Catalan functions interpolate between  $h_\lambda$  and  $s_\lambda$ .

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## Theorem (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive!  
Precisely,  $H(\Psi; \lambda) = \sum_{\nu} c_{\Psi, \lambda}^{\nu} s_{\nu}$  satisfies  $c_{\Psi, \lambda}^{\nu} \in \mathbb{Z}_{\geq 0}$ .

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$



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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots

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$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

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3					
	3				
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4					
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		3			
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				2	
					2

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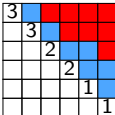
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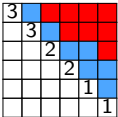
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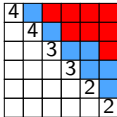
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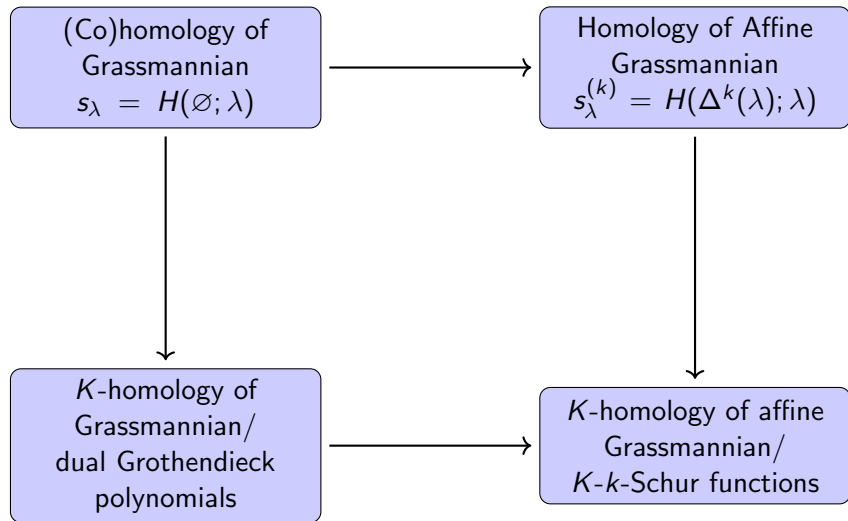
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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$



# Big Picture



- Schubert calculus
- Catalan functions: a new approach to old problems
- ***K*-theoretic Catalan functions**

# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda + \text{lower degree terms}$ .

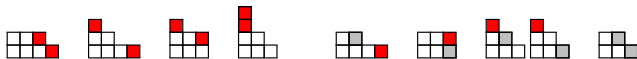
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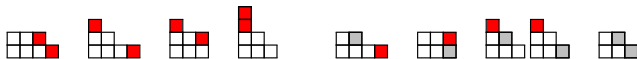
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- Dual to Grothendieck polynomials  $G_\lambda$ : Schubert representatives for  $K^*(Gr(m, n))$

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2-bounded partitions  $\leftrightarrow$  3-cores

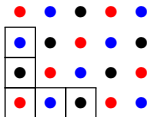
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Problem

No direct formula for  $g_{\lambda}^{(k)}$



## Solution

Find a formula for  $g_{\lambda}^{(k)}$  analogous to raising operator formula for  $s_{\lambda}^{(k)}$ .

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Requires an inhomogeneous refinement of Catalan functions.

# An Extra Ingredient: Lowering Operators

Lowering Operators  $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned}
 &K(\Psi; \mathcal{L}; 54332) \\
 &= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332}
 \end{aligned}$$

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For  $K$ -homology of affine Grassmannian,  $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$  since this family satisfies the Pieri rule.

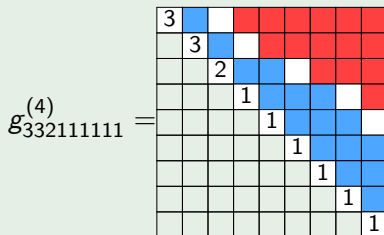


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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

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$$=$$

2							
	1						
		1					
			0				
				0			
					0		
						1	

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$$=$$

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$$=$$

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$$+$$

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# Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

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		1				
			0			
				0		
					0	
						1

 $+$ 

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						1

 $+$ 

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$$=$$

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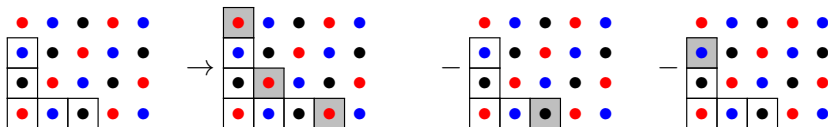
$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
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3-core perspective:



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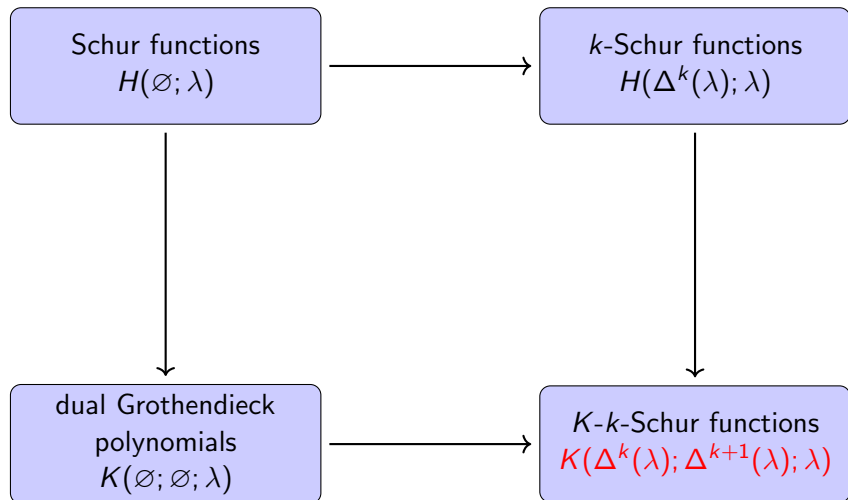
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# Big Picture



# $K$ -theoretic Peterson isomorphism

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What can be said about  $K$ -theoretic Catalan functions in general?



# Positivity of $K$ -theoretic Catalan functions

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Raising operator techniques extend beyond Schubert calculus! Work by Blasiak-Haiman-Morse-Pun-S.:

- Shuffle theorems (Blasiak et al., 2021a; 2021b).
- Macdonald polynomials and LLT polynomials (Blasiak et al., 2021c).
- Much more work to be done!

## Thank you!

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$$k_m^{(r)} = \sum_{i=0}^m \binom{r+i-1}{i} h_{m-i} = s_m(X+r),$$

a specialization of “multiSchur functions.” See, e.g., Lascoux-Naruse (2014).

$$k_{\gamma} = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_{\ell}}^{(\ell-1)}$$