

K -theoretic Catalan functions

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UVA Algebra Seminar

April 5, 2021

- Schubert calculus
- Catalan functions: a new approach to old problems
- K -theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of subvarieties in a variety X .

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Cohomology

Schubert basis $\{\sigma_\lambda\}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda \subseteq (n^m)}$ in variety $X = \text{Gr}(m, n)$.

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$.

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
3	4		
1	2	5	6

standard = no repeated letters

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Schur function s_λ is a “weight generating function” of semistandard tableaux:

<table><tr><td>2</td></tr><tr><td>1 1</td></tr></table>	2	1 1	<table><tr><td>3</td></tr><tr><td>1 1</td></tr></table>	3	1 1	<table><tr><td>3</td></tr><tr><td>2 2</td></tr></table>	3	2 2	<table><tr><td>2</td></tr><tr><td>1 2</td></tr></table>	2	1 2	<table><tr><td>3</td></tr><tr><td>1 3</td></tr></table>	3	1 3	<table><tr><td>3</td></tr><tr><td>2 3</td></tr></table>	3	2 3	<table><tr><td>2</td></tr><tr><td>1 3</td></tr></table>	2	1 3	<table><tr><td>3</td></tr><tr><td>1 2</td></tr></table>	3	1 2
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$$s_{\square\square}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}$$

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$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$, subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu$.

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$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$ have no tableaux description.

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Theory	f_λ
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- P and Q functions
(Co)homology of affine Grassmannian	(dual) k -Schur functions
K -theory of Grassmannian	Grothendieck polynomials
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And many more!

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$$\begin{aligned}\Phi: QH^*(Fl_{k+1}) &\rightarrow H_*(Gr_{SL_{k+1}})_{loc} \\ \mathfrak{S}_w^Q &\mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}\end{aligned}$$

where $s_\lambda^{(k)}$ is a k -Schur symmetric function and $Gr_{SL_{k+1}}$ is the “affine Grassmannian.”

Upshot

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Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

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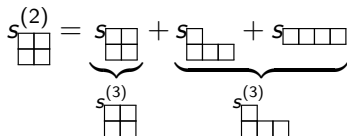
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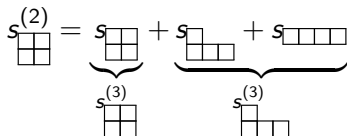
The diagram shows the branching of the 2-Schur function $s_{(2)}^{(2)}$ into 3-Schur functions. On the left is a 2x2 square representing $s_{(2)}^{(2)}$. On the right is the sum of three 3-Schur functions: $s_{(2)}^{(3)}$ (a 2x2 square), $s_{(1,1)}^{(3)}$ (a 2x1 rectangle), and $s_{(1,1,1)}^{(3)}$ (a 1x3 horizontal row). Brackets below the right side group the terms as $s_{(2)}^{(3)} + s_{(1,1)}^{(3)} + s_{(1,1,1)}^{(3)}$.

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$$s_{\lambda}^{(2)} = s_{\lambda}^{(3)} + s_{\lambda}^{(3)}$$

The diagram shows the equation $s_{\lambda}^{(2)} = s_{\lambda}^{(3)} + s_{\lambda}^{(3)}$ with Young diagrams. On the left is the Young diagram for $s_{(2)}^{(2)}$, which is a 2x2 square. This is equal to the sum of two Young diagrams for $s_{\lambda}^{(3)}$. The first is the Young diagram for $s_{(2)}^{(3)}$, which is a 2x2 square. The second is the Young diagram for $s_{(1,1)}^{(3)}$, which is a 2x1 rectangle. Braces indicate that the two terms on the right are both $s_{\lambda}^{(3)}$.

- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.
- Branching with t important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- K -theoretic Catalan functions

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Key:

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Key: Catalan functions = large class of symmetric functions.

Ingredients for Catalan functions

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Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

$$R_{1,3} \left(\begin{array}{|c|c|c|} \hline \text{red} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \text{red} \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left(\begin{array}{|c|c|} \hline & \text{red} \\ \hline & \\ \hline & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \text{red} \\ \hline & & \\ \hline \end{array}$$

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$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \text{red } h_{310} + \text{red } h_{310} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^+ \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

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- $\Psi = \emptyset \implies H(\emptyset; \gamma) = s_{\gamma}$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



Ψ = Roots above Dyck path
 $\Delta_{\ell}^{+} \setminus \Psi$ = Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

- $\Psi = \emptyset \implies H(\emptyset; \gamma) = s_{\gamma}$
- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_{\gamma}$

Intuition

Catalan functions interpolate between h_λ and s_λ .

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive!

k -Schur root ideal for λ

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

\leftarrow row i has $4 - \lambda_i$ non-roots

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k -Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$ for $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$.

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3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

Key ingredient of branching proof

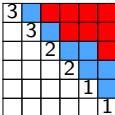
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Pieri:

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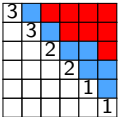
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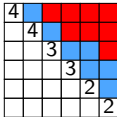
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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

- Schubert calculus
- Catalan functions: a new approach to old problems
- ***K*-theoretic Catalan functions**

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda + \text{lower degree terms}$.

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- Satisfies Pieri rule on “set-valued strips”

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Add (addable) or mark (removable) in any combination of r boxes, but only once per row.

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- Dual to Grothendieck polynomials G_λ : Schubert representatives for $K^*(Gr(m, n))$

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$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions \leftrightarrow 3-cores

The diagram illustrates the Pieri rule for K - k -Schur functions. It shows the product of a 1-strip (g_1) and a 2-bounded partition ($g_{211}^{(2)}$) resulting in the difference of two 2-bounded partitions ($g_{2111}^{(2)} - 2g_{211}^{(2)}$). The partitions are represented as 5x5 grids of colored dots (red, blue, black) with some cells shaded gray to indicate the addition or subtraction of strips.

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Problem

No direct formula for $g_{\lambda}^{(k)}$

Solution

Find a formula for $g_{\lambda}^{(k)}$ analogous to raising operator formula for $s_{\lambda}^{(k)}$.

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Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left(\begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

K -theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

Affine K -Theory Representatives with Raising Operators

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Example

non-roots of Ψ , roots of \mathcal{L}

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$

$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332}$$

Answer (Blasiak-Morse-S., 2020)

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For K -homology of affine Grassmannian, $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$ since this family satisfies the Pieri rule.

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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

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$$g_1 g_{211}^{(2)}$$

$$=$$

2							
	1						
		1					
			0				
				0			
					0		
						1	

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$$g_1 g_{211}^{(2)}$$

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

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$$g_1 g_{211}^{(2)}$$

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$

2						
	1					
		1				
			0			
				0		
					0	
						1

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$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$

2						
	1					
		1				
			0			
				0		
					0	
						1

 $=$

2			
	1		
		1	
			1

 $-$

2		
	1	
		1

 $-$

2		
	1	
		1

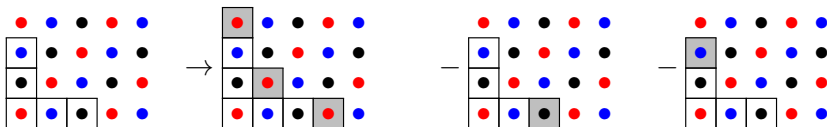
Pieri Rule Illustrated (Straightening)

$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
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3-core perspective:



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satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

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Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and \mathfrak{G}_w^Q a “quantum Grothendieck polynomial”,

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Conjecture (Blasiak-Morse-S., 2020)

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These $\tilde{g}_\mu^{(k)}$ satisfy the following properties.

- The coefficients in $G_{1^m}^\perp \tilde{g}_\mu^{(k)} = \sum_\nu c_{\mu\nu} \tilde{g}_\nu^{(k)}$ satisfy $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.
- The coefficients in $\tilde{g}_\mu^{(k)} = \sum_\nu a_{\mu\nu} \tilde{g}_\nu^{(k+1)}$ satisfy $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.
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k -Rectangle Property

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For $1 \leq d \leq k$, set $R_d = ((k + 1 - d)^d)$ to be the k -rectangle partition.

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- Corresponding result for $s_{\lambda}^{(k)}$ is known, but this gives a Catalan/Katalan-theoretic proof.
- k -Rectangle Property fails for $g_{\lambda}^{(k)}$.

Positivity of Catalan functions

Recall (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive.

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Thank you!

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$$k_m^{(r)} = \sum_{i=0}^m \binom{r+i-1}{i} h_{m-i} = s_m(X+r),$$

a specialization of “multiSchur functions.” See, e.g., Lascoux-Naruse (2014).

$$k_{\gamma} = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_{\ell}}^{(\ell-1)}$$