

# Diagonal Harmonics and Shuffle Theorems

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on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun  
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OIST Representation Theory Seminar

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- Symmetric polynomials and diagonal harmonics
- The Shuffle Theorem and its generalizations
- Proof techniques and new progress

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$  for all  $\sigma \in S_n$ .

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## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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- E.g. for  $n = 3$ ,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

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- $\Lambda$  is a  $\mathbb{Q}(q, t)$ -algebra.

# Schur Polynomials

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left( \frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$



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## Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in  $\mathbb{N}[q, t]$ ) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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Remark:  $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_+^{S_3})$ .

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Solution: irreducible  $S_n$ -representation of polynomials of degree  $d \mapsto q^d s_\lambda$   
(graded Frobenius)

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Answer: Hall-Littlewood polynomial  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

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- Does there exist a family of  $S_n$ -regular representations whose bigraded Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .

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 $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$



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Irreducible  $S_n$ -representation with bidegree  $(a, b) \mapsto q^a t^b s_\lambda$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\deg=(0,0)}$$

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$$\tilde{H}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = qts \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + ts \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + qs \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + s \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

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- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ .



## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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## Question

What symmetric function is the bigraded Frobenius characteristic of  $DH_n$ ?

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Theorem (Haiman, 2002)

*The bigraded Frobenius characteristic of  $DH_n$  is given by  $\nabla e_n$ .*

## Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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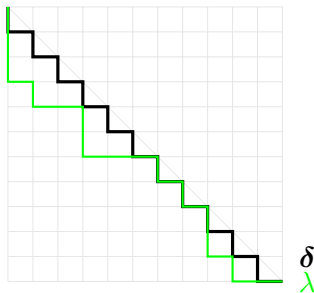
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# Dyck paths

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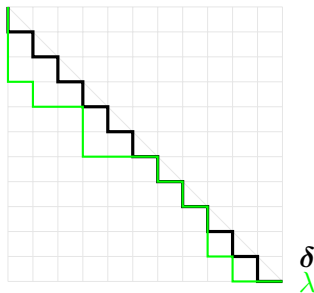
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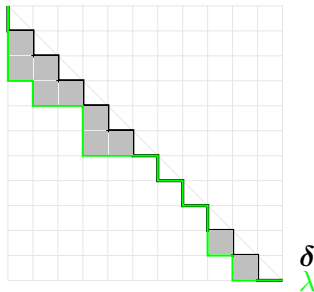


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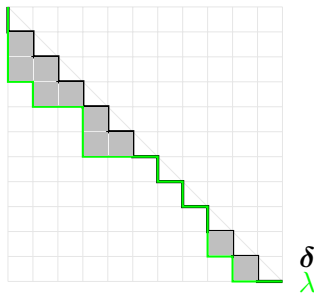


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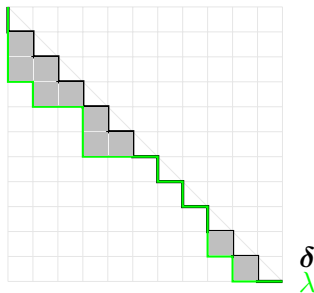
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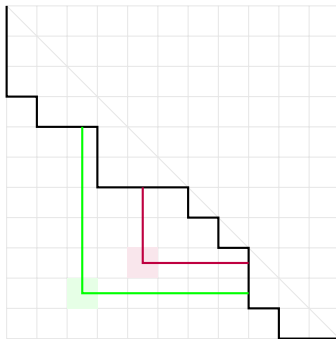
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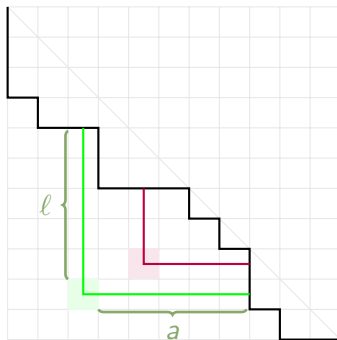
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Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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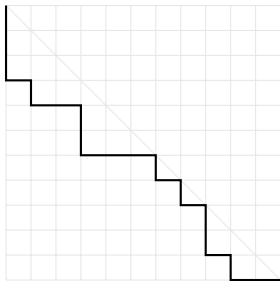
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- $\mathcal{G}_\nu$  is Schur-positive for any tuple of skew shapes  $\nu$  [Grojnowski-Haiman, 2007].

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$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
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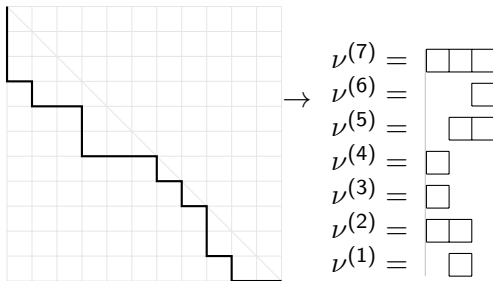
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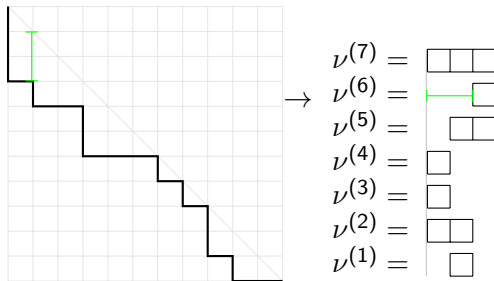
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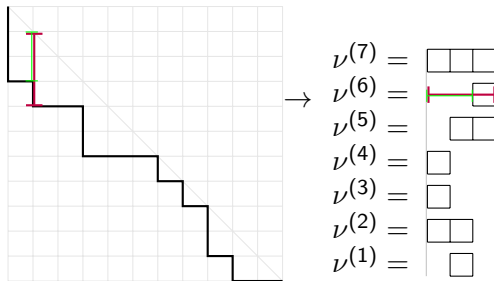
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$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

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$$\mathcal{G}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

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$$= s_3 + q s_{2,1}$$

## Example $\nabla e_3$






$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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




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- Symmetric polynomials and diagonal harmonics
- **The Shuffle Theorem and its generalizations**
- Proof techniques and new progress

# Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

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- $\mathcal{E}$  contains, for every coprime  $m, n \in \mathbb{Z}$ , subalgebra  $\Lambda(X^{m,n}) \cong \Lambda$ , with relations between them. (Burban-Schiffmann, 2012)

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- $\mathcal{E}$  acts on  $\Lambda$ , e.g., for  $M = (1 - q)(1 - t)$  and automorphism  $\omega$ ,

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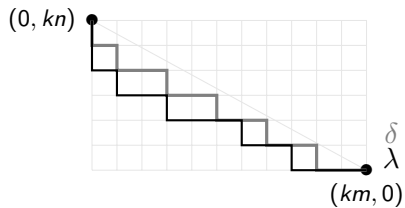
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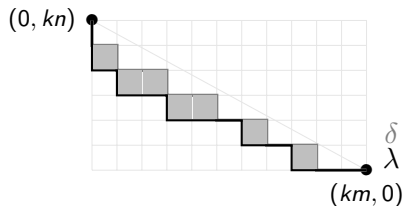
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# Rational Path Combinatorics

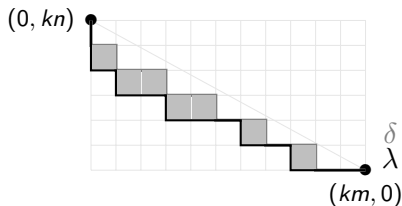


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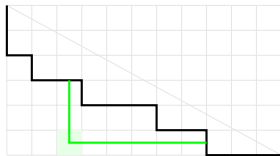


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$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \quad p = \frac{n}{m} - \epsilon$$

## Negut Elements

For  $\mathbf{b} \in \mathbb{Z}^I$ , special elements  $D_{\mathbf{b}} \in \mathcal{E}$  generalizing  $e_k[-MX^{m,n}]$ .

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Given  $r, s \in \mathbb{R}_{>0}$  such that  $p = s/r$  irrational, take  $(b_1, \dots, b_I) \in \mathbb{Z}^I$  to be the south step sequence of highest path  $\delta$  under the line  $y + px = s$ .

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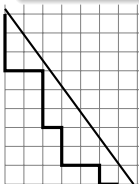
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# Proof Idea

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## Stable Shuffle Theorem

For  $\mathbf{b} \in \mathbb{Z}^l$  corresponding to some choice of highest path under line of slope  $-r/s$ ,

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For  $\mathbf{b} = (b_1, \dots, b_I)$  the south steps of highest path under a convex curve, the Schur expansion of  $D_{\mathbf{b}} \cdot 1$  has coefficients in  $\mathbb{N}[q, t]$ .



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- Experimental computation suggests this is “tight.”
- Coefficient of  $s_{1, \dots, 1}$  coincides with  $(q, t)$ -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

## Loehr-Warrington Conjecture (2008)

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- $S_I$ -representation theory interpretations?

# References

Thank you!

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