

# A Raising Operator Formula for Macdonald Polynomials (and The Journey There)

George H. Seelinger

joint work with J. Blasiak, M. Haiman, J. Morse, and A. Pun

[ghseeli@umich.edu](mailto:ghseeli@umich.edu)

Università di Pisa Seminar on Combinatorics, Lie Theory, and Topology

12 October 2023

- ① **Background on symmetric functions and Macdonald polynomials**
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ A new formula for Macdonald polynomials

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$  for all  $\sigma \in S_n$ .

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$  for all  $\sigma \in S_n$ .

## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$  for all  $\sigma \in S_n$ .

## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for  $n = 3$ ,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$  for all  $\sigma \in S_n$ .

## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for  $n = 3$ ,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

- Let  $\Lambda = \mathbb{Q}(q, t)[e_1, e_2, \dots] = \mathbb{Q}(q, t)[h_1, h_2, \dots]$ . Call these “symmetric functions.”

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$  for all  $\sigma \in S_n$ .

## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for  $n = 3$ ,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

- Let  $\Lambda = \mathbb{Q}(q, t)[e_1, e_2, \dots] = \mathbb{Q}(q, t)[h_1, h_2, \dots]$ . Call these “symmetric functions.”
- $\Lambda$  is a  $\mathbb{Q}(q, t)$ -algebra.

# Bases for symmetric functions

Dimension of degree  $d$  symmetric functions?



# Bases for symmetric functions

Dimension of degree  $d$  symmetric functions? Number of partitions of  $d$ .

# Bases for symmetric functions

Dimension of degree  $d$  symmetric functions? Number of partitions of  $d$ .

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

# Bases for symmetric functions

Dimension of degree  $d$  symmetric functions? Number of partitions of  $d$ .

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition* of  $n$  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline \square & \square & \square & \square & \square \\ \hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline\end{array}$$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline \square & \square & \square \\ \hline\end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|}\hline \square \\ \hline\end{array}$$

# Bases for symmetric functions

Dimension of degree  $d$  symmetric functions? Number of partitions of  $d$ .

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition* of  $n$  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|}\hline\Box \\ \hline\end{array}$$

$\implies$  any basis of symmetric functions is indexed by partitions.

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- 1 strictly increasing up columns

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- 1 strictly increasing up columns
- 2 weakly increasing along rows

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- 1 strictly increasing up columns
- 2 weakly increasing along rows

Collection is called  $\text{SSYT}(\lambda)$ .



# Young Tableaux

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- ① strictly increasing up columns
- ② weakly increasing along rows

Collection is called  $\text{SSYT}(\lambda)$ .

For  $\lambda = (2, 1)$ ,

2		3		3		2		3		3		2		3	
1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

# Polynomials from tableaux

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

# Polynomials from tableaux

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

2		3		3		2		3		3		2		3	
1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

# Polynomials from tableaux

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

2		3		3		2		3		3		2		3	
1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

# Polynomials from tableaux

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

$$\begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array} \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 2 \end{array}, \begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 2 \end{array}, \begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

## Definition

For  $\lambda$  a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \mathbf{x}^T \text{ for } \mathbf{x}^T = \prod_{i \in T} x_i$$

# Polynomials from tableaux

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

2		3		3		2		3		3		2		3	
1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

## Definition

For  $\lambda$  a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \mathbf{x}^T \text{ for } \mathbf{x}^T = \prod_{i \in T} x_i$$

- $s_\lambda$  is a symmetric function.

# Polynomials from tableaux

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

$$\begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array} \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 2 \end{array}, \begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 2 \end{array}, \begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array}, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

## Definition

For  $\lambda$  a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \mathbf{x}^T \text{ for } \mathbf{x}^T = \prod_{i \in T} x_i$$

- $s_\lambda$  is a symmetric function.
- $\{s_\lambda\}_\lambda$  forms a basis for  $\Lambda_{\mathbb{Q}}$ .

# Representation theory and Schur functions

Frobenius characteristc,  $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$ .



# Representation theory and Schur functions

Frobenius characteristc,  $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$ .

- Irreducible representations of  $S_n$  are labeled by partitions of  $n$ .

# Representation theory and Schur functions

Frobenius characteristc,  $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$ .

- Irreducible representations of  $S_n$  are labeled by partitions of  $n$ .
- Irreducible  $S_n$ -representation  $V_\lambda$  has  $\text{Frob}(V_\lambda) = s_\lambda$

# Representation theory and Schur functions

Frobenius characteristc,  $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$ .

- Irreducible representations of  $S_n$  are labeled by partitions of  $n$ .
- Irreducible  $S_n$ -representation  $V_\lambda$  has  $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$

# Representation theory and Schur functions

Frobenius characteristc,  $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$ .

- Irreducible representations of  $S_n$  are labeled by partitions of  $n$ .
- Irreducible  $S_n$ -representation  $V_\lambda$  has  $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$
- $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$

# Representation theory and Schur functions

Frobenius characterstic,  $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$ .

- Irreducible representations of  $S_n$  are labeled by partitions of  $n$ .
- Irreducible  $S_n$ -representation  $V_\lambda$  has  $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$
- $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$
- Upshot:  $S_n$ -representations go to symmetric functions in structure preserving way.

# Representation theory and Schur functions

Frobenius character map,  $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$ .

- Irreducible representations of  $S_n$  are labeled by partitions of  $n$ .
- Irreducible  $S_n$ -representation  $V_\lambda$  has  $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$
- $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$
- Upshot:  $S_n$ -representations go to symmetric functions in structure preserving way.

## Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in  $\mathbb{N}$ ) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

# Harmonic polynomials

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

# Harmonic polynomials

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$



# Harmonic polynomials

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$M$  is the vector space given by

# Harmonic polynomials

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$M$  is the vector space given by

$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

# Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

# Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 1 Break  $M$  up into irreducible  $S_n$ -representations.

# Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

① Break  $M$  up into irreducible  $S_n$ -representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

# Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ① Break  $M$  up into irreducible  $S_n$ -representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ② How many times does an irreducible  $S_n$ -representation occur?

# Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ① Break  $M$  up into irreducible  $S_n$ -representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ② How many times does an irreducible  $S_n$ -representation occur?  
Frobenius:

# Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ① Break  $M$  up into irreducible  $S_n$ -representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ② How many times does an irreducible  $S_n$ -representation occur?  
Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$



# Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ① Break  $M$  up into irreducible  $S_n$ -representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ② How many times does an irreducible  $S_n$ -representation occur?  
Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Remark:  $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]^{S_3})$  is a “regular representation.”

## Getting more information

# Getting more information

Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

# Getting more information

Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: irreducible  $S_n$ -representation of polynomials of degree  $d \mapsto q^d s_\lambda$   
(graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

# Getting more information

Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: irreducible  $S_n$ -representation of polynomials of degree  $d \mapsto q^d s_\lambda$   
(graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Answer: Hall-Littlewood polynomial  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials (and many other famous bases).

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials (and many other famous bases).
- Defined by orthogonality and triangularity under a certain inner-product.

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials (and many other famous bases).
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$



# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials (and many other famous bases).
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials (and many other famous bases).
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .
- Does there exist a family of  $S_n$ -regular representations whose bigraded Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  with  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  with  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of}$   
 $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  with  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of}$   
 $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  with  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\deg=(0,0)}$$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  with  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

Irreducible  $S_n$ -representation with bidegree  $(a, b) \mapsto q^a t^b s_\lambda$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  with  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

Irreducible  $S_n$ -representation with bidegree  $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = qts \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + ts \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + qs \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + s \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$



# Garsia-Haiman modules

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

- Proved via connection to the Hilbert Scheme  $\text{Hilb}^n(\mathbb{C}^2)$ .

# Garsia-Haiman modules

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

- Proved via connection to the Hilbert Scheme  $\text{Hilb}^n(\mathbb{C}^2)$ .

## Corollary

$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}_{\lambda\mu}(q, t) s_\mu$  satisfies  $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$ .

# Garsia-Haiman modules

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

- Proved via connection to the Hilbert Scheme  $\text{Hilb}^n(\mathbb{C}^2)$ .

## Corollary

$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}_{\lambda\mu}(q, t) s_\mu$  satisfies  $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$ .

- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ .

# Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible $V_\lambda$	$\text{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman $M_\lambda$	??

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

# Garsia-Haiman modules

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

## Question

What symmetric function is the bigraded Frobenius characteristic of  $DH_n$ ?

Frobenius characteristic of  $DH_3$



Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Operator  $\nabla$

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X; q, t),$$

where  $n(\lambda) = \sum_i (i-1)\lambda_i$  and  $\lambda^*$  is the transpose partition to  $\lambda$ .

Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Operator  $\nabla$

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X; q, t),$$

where  $n(\lambda) = \sum_i (i-1)\lambda_i$  and  $\lambda^*$  is the transpose partition to  $\lambda$ .

Theorem (Haiman, 2002)

*The bigraded Frobenius characteristic of  $DH_n$  is given by  $\nabla e_n$ .*

# Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible $V_\lambda$	SSYT( $\lambda$ )
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman $M_\lambda$	??
$\nabla e_n$	$DH_n$	Shuffle theorem

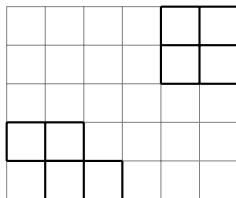
# Outline

- ① Background on symmetric functions and Macdonald polynomials
- ② **Shuffle theorems, combinatorics, and LLT polynomials**
- ③ A new formula for Macdonald polynomials

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape =  $\lambda \setminus \mu$ )

$$\nu = \left( \begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$



# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape =  $\lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5



# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape =  $\lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.

$$\nu = \left( \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape =  $\lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$

					$b_3$	$b_6$
					$b_5$	$b_8$
$b_1$	$b_2$					
	$b_4$	$b_7$				

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

					$b_3$	$b_6$
					$b_5$	$b_8$
$b_1$	$b_2$					
	$b_4$	$b_7$				

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

					$b_3$	$b_6$
					$b_5$	$b_8$
$b_1$	$b_2$					
	$b_4$	$b_7$				

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$



# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

- The *content* of a box in row  $y$ , column  $x$  is  $x - y$ .
- *Reading order*: label boxes  $b_1, \dots, b_n$  by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
  - $\text{content}(b) = \text{content}(a) + 1$  and  $a \in \nu_{(i)}, b \in \nu_{(j)}$  with  $i > j$ .

$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6
				1	1
2	4				
	3	5			

# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6
				1	1
2	4				
	3	5			

non-inversion

# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6	
				1	1	
2	4					
	3	5				

inversion

# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6
				1	1
2	4				
	3	5			

inversion

# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6	
				1	1	
2	4					
	3	5				

non-inversion

# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6	
				1	1	
2	4					
	3	5				

non-inversion

# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6	
				1	1	
2	4					
	3	5				

inversion



# LLT Polynomials

- A *semistandard tableau* on  $\nu$  is a map  $T: \nu \rightarrow \mathbb{Z}_+$  which restricts to a semistandard tableau on each  $\nu_{(i)}$ .
- An *attacking inversion* in  $T$  is an attacking pair  $(a, b)$  such that  $T(a) > T(b)$ .

The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where  $\text{inv}(T)$  is the number of attacking inversions in  $T$  and  $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$ .

$T =$

				5	6	
				1	1	
2	4					
	3	5				

inversion

$$\text{inv}(T) = 4, \quad \mathbf{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

# LLT Polynomials $\mathcal{G}_\nu(X; q)$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function

# LLT Polynomials $\mathcal{G}_\nu(X; q)$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu(1)} \cdots s_{\nu(r)}$

# LLT Polynomials $\mathcal{G}_\nu(X; q)$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu(1)} \cdots s_{\nu(r)}$
- $\mathcal{G}_\nu$  were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of  $U_q(\hat{\mathfrak{sl}}_r)$

# LLT Polynomials $\mathcal{G}_\nu(X; q)$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu(1)} \cdots s_{\nu(r)}$
- $\mathcal{G}_\nu$  were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of  $U_q(\hat{\mathfrak{sl}}_r)$
- When  $\nu^{(i)}$  are partitions, the Schur-expansion coefficients are essentially parabolic Kazhdan-Luzstig polynomials.

# LLT Polynomials $\mathcal{G}_\nu(X; q)$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu(1)} \cdots s_{\nu(r)}$
- $\mathcal{G}_\nu$  were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of  $U_q(\hat{\mathfrak{sl}}_r)$
- When  $\nu^{(i)}$  are partitions, the Schur-expansion coefficients are essentially parabolic Kazhdan-Luzstig polynomials.
- $\mathcal{G}_\nu$  is Schur-positive for any tuple of skew shapes  $\nu$  [Grojnowski-Haiman, 2007].

# A Combinatorial Connection: Shuffle Theorem

## Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all  $k$ -by- $k$  Dyck paths.

# A Combinatorial Connection: Shuffle Theorem

## Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all  $k$ -by- $k$  Dyck paths.
- $\text{area}(\lambda)$  and  $\text{dinv}(\lambda)$  statistics of Dyck paths.



# A Combinatorial Connection: Shuffle Theorem

## Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all  $k$ -by- $k$  Dyck paths.
- $\text{area}(\lambda)$  and  $\text{dinv}(\lambda)$  statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$  a symmetric LLT polynomial indexed by a tuple of offset (skew) rows.

# A Combinatorial Connection: Shuffle Theorem

## Theorem (Carlsson-Mellit, 2018)

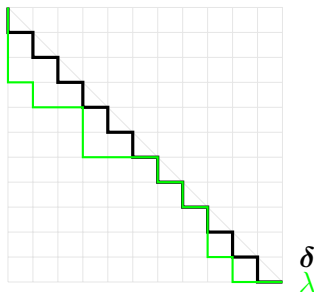
$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all  $k$ -by- $k$  Dyck paths.
- $\text{area}(\lambda)$  and  $\text{dinv}(\lambda)$  statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$  a symmetric LLT polynomial indexed by a tuple of offset (skew) rows.
- $\omega$  an automorphism of symmetric functions:  $\omega(s_{\lambda}) = s_{\lambda^*}$  for  $\lambda^* =$  transpose of  $\lambda$ .

# Dyck paths

## Dyck paths

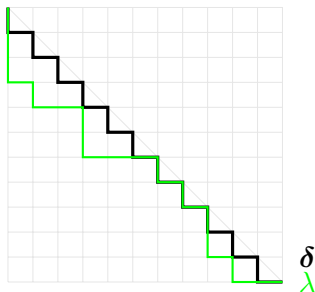
A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .



# Dyck paths

## Dyck paths

A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .

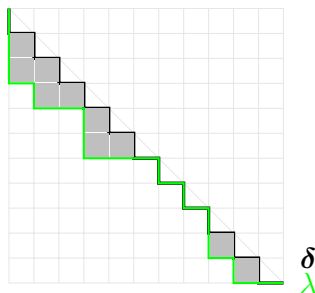


- $\text{area}(\lambda) = \text{number of squares above } \lambda \text{ but below the path } \delta \text{ of alternating S-E steps.}$

# Dyck paths

## Dyck paths

A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .

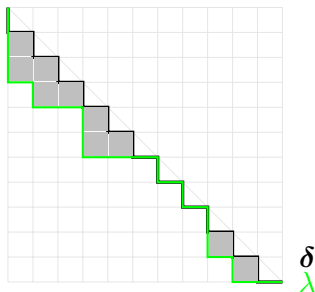


- $\text{area}(\lambda)$  = number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.
- E.g., above  $\text{area}(\lambda) = 10$ .

# Dyck paths

## Dyck paths

A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .

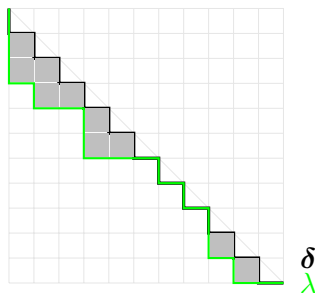


- $\text{area}(\lambda) =$  number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.
- E.g., above  $\text{area}(\lambda) = 10$ .
- Catalan-number many Dyck paths for fixed  $k$ .

# Dyck paths

## Dyck paths

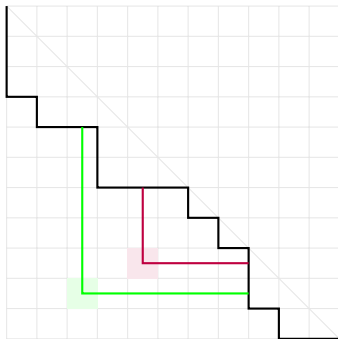
A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .



- $\text{area}(\lambda)$  = number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.
- E.g., above  $\text{area}(\lambda) = 10$ .
- Catalan-number many Dyck paths for fixed  $k$ .  $(1, 2, 5, 14, 42, \dots)$

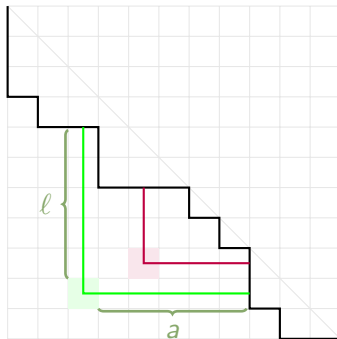
# dinv

$\text{dinv}(\lambda) = \#$  of balanced hooks in diagram below  $\lambda$ .





$\text{dinv}(\lambda) = \#$  of balanced hooks in diagram below  $\lambda$ .



Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

## Example $\nabla e_3$

$$\lambda \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

# Example $\nabla e_3$

$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$



# Example $\nabla e_3$

$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$



$$q^3$$



$$q^2 t$$



$$qt$$








$$qt^2$$



$$t^3$$

# Example $\nabla e_3$

$\lambda$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	$q^3$	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	$qt$	$ts_{2,1} + qts_{1,1,1}$
	$qt^2$	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	$t^3$	$t^3 s_{1,1,1}$

# Example $\nabla e_3$

$\lambda$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	$q^3$	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	$qt$	$ts_{2,1} + qts_{1,1,1}$
	$qt^2$	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	$t^3$	$t^3 s_{1,1,1}$

- Entire quantity is  $q, t$ -symmetric

# Example $\nabla e_3$

$\lambda$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	$q^3$	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	$qt$	$ts_{2,1} + qts_{1,1,1}$
	$qt^2$	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	$t^3$	$t^3 s_{1,1,1}$

- Entire quantity is  $q, t$ -symmetric
- Coefficient of  $s_{1,1,1}$  in sum is a “ $(q, t)$ -Catalan number”  $(q^3 + q^2 t + qt + qt^2 + t^3)$ .

# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!



# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For  $m, n > 0$  coprime, the operator  $e_k[-MX^{m,n}]$  acting on  $\Lambda$  satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$

# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

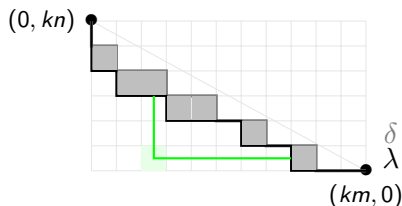
Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For  $m, n > 0$  coprime, the operator  $e_k[-MX^{m,n}]$  acting on  $\Lambda$  satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$



# Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra  $\mathcal{E}$  of the Hall algebra of coherent sheaves on an elliptic curve over  $\mathbb{F}_p$ .

The *elliptic Hall algebra*  $\mathcal{E}$  is generated by subalgebras  $\Lambda(X^{a,b})$  isomorphic to the ring of symmetric functions  $\Lambda$  over  $\mathbb{k} = \mathbb{Q}(q, t)$ , one for each coprime pair  $(a, b) \in \mathbb{Z}^2$ , along with an additional central subalgebra.

# Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra  $\mathcal{E}$  of the Hall algebra of coherent sheaves on an elliptic curve over  $\mathbb{F}_p$ .

The *elliptic Hall algebra*  $\mathcal{E}$  is generated by subalgebras  $\Lambda(X^{a,b})$  isomorphic to the ring of symmetric functions  $\Lambda$  over  $\mathbb{k} = \mathbb{Q}(q, t)$ , one for each coprime pair  $(a, b) \in \mathbb{Z}^2$ , along with an additional central subalgebra.

E.g.,  $e_k[-MX^{m,n}] \in \Lambda(X^{m,n})$ .

$\mathcal{E}$  acts on symmetric functions and  $e_k[-MX^{1,1}] \cdot 1 = \nabla e_k$ .

# Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra  $\mathcal{E}$  of the Hall algebra of coherent sheaves on an elliptic curve over  $\mathbb{F}_p$ .

The *elliptic Hall algebra*  $\mathcal{E}$  is generated by subalgebras  $\Lambda(X^{a,b})$  isomorphic to the ring of symmetric functions  $\Lambda$  over  $\mathbb{k} = \mathbb{Q}(q, t)$ , one for each coprime pair  $(a, b) \in \mathbb{Z}^2$ , along with an additional central subalgebra.

E.g.,  $e_k[-MX^{m,n}] \in \Lambda(X^{m,n})$ .

$\mathcal{E}$  acts on symmetric functions and  $e_k[-MX^{1,1}] \cdot 1 = \nabla e_k$ .

Can be difficult to work with in general. Can we make it more explicit?

# Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$  denotes the set of positive roots for  $GL_n$ , where  $\alpha_{ij} = \epsilon_i - \epsilon_j$ .

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

# Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$  denotes the set of positive roots for  $GL_n$ , where  $\alpha_{ij} = \epsilon_i - \epsilon_j$ .

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

A root ideal  $\Psi \subseteq R_+$  is an upper order ideal of positive roots.

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$\Psi = \text{Roots above Dyck path}$



# Schur functions revisited

- Convention:  $h_0 = 1$  and  $h_d = 0$  for  $d < 0$ .
- For any  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$ , set

$$s_\gamma = \det(h_{\gamma_i + j - i})_{1 \leq i, j \leq n}$$

# Schur functions revisited

- Convention:  $h_0 = 1$  and  $h_d = 0$  for  $d < 0$ .
- For any  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$ , set

$$s_\gamma = \det(h_{\gamma_i+j-i})_{1 \leq i, j \leq n}$$

Then,  $s_\gamma = \pm s_\lambda$  or 0 for some partition  $\lambda$ .

# Schur functions revisited

- Convention:  $h_0 = 1$  and  $h_d = 0$  for  $d < 0$ .
- For any  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$ , set

$$s_\gamma = \det(h_{\gamma_i+j-i})_{1 \leq i, j \leq n}$$

Then,  $s_\gamma = \pm s_\lambda$  or 0 for some partition  $\lambda$ .

Precisely, for  $\rho = (n-1, n-2, \dots, 1, 0)$ ,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$  = weakly decreasing sequence obtained by sorting  $\beta$ ,
- $\operatorname{sgn}(\beta)$  = sign of the shortest permutation taking  $\beta$  to  $\operatorname{sort}(\beta)$ .

Example:  $s_{201} = 0$ ,  $s_{2-11} = -s_{200}$ .

# Weyl symmetrization

Define the *Weyl symmetrization operator*  $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$  by linearly extending

$$\mathbf{z}^\gamma \mapsto s_\gamma(X)$$

where  $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$ .

# Weyl symmetrization

Define the *Weyl symmetrization operator*  $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$  by linearly extending

$$\mathbf{z}^\gamma \mapsto s_\gamma(X)$$

where  $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$ .

## Example

$$\sigma(\mathbf{z}^{111} + \mathbf{z}^{201} + \mathbf{z}^{210} + \mathbf{z}^{3-11}) = s_{111} + s_{201} + s_{210} + s_{3-11} = s_{111} + s_{210} - s_{300}$$

## Definition

The *Catalanimal* indexed by  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\lambda \in \mathbb{Z}^n$  is

## Definition

The *Catalanimal* indexed by  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\lambda \in \mathbb{Z}^n$  is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qtz^\alpha)}{\prod_{\alpha \in R_q} (1 - qz^\alpha) \prod_{\alpha \in R_t} (1 - tz^\alpha)} \right),$$

where  $z^{\alpha_{ij}} = z_i/z_j$  and  $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \cdots$ .

## Definition

The *Catalanimal* indexed by  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\lambda \in \mathbb{Z}^n$  is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right),$$

where  $z^{\alpha_{ij}} = z_i/z_j$  and  $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \dots$ .

With  $n = 3$ ,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left( \frac{z^{111} (1 - qt z_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - q z_i/z_j) (1 - t z_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$



# Why?

Let  $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$  and  $R_+^0 = \{\alpha_{ij} \in R_+ \mid i+1 < j\}$ .

# Why?

Let  $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$  and  $R_+^0 = \{\alpha_{ij} \in R_+ \mid i+1 < j\}$ .

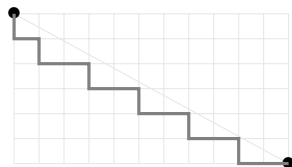
## Proposition

For  $(m, n) \in \mathbb{Z}^2$  coprime,

$$e_k[-MX^{m,n}] \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

for  $\mathbf{b} = (b_0, \dots, b_{km-1})$  satisfying  $b_i =$  the number of south steps on vertical line  $x = i$  of highest lattice path under line  $y + \frac{n}{m}x = n$ .

$\delta$  = highest Dyck path.



$\delta$

$$\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

# Results

Manipulating Catalanimal  $\implies$  a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given  $r, s \in \mathbb{R}_{>0}$  such that  $p = s/r$  irrational, take  $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$  to be the south step sequence of highest path  $\delta$  under the line  $y + px = s$ .

# Results

Manipulating Catalanimal  $\implies$  a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given  $r, s \in \mathbb{R}_{>0}$  such that  $p = s/r$  irrational, take  $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$  to be the south step sequence of highest path  $\delta$  under the line  $y + px = s$ .

$$H(R_+, R_+, R_+^0, \mathbf{b})$$

# Results

Manipulating Catalanimal  $\implies$  a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given  $r, s \in \mathbb{R}_{>0}$  such that  $p = s/r$  irrational, take  $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$  to be the south step sequence of highest path  $\delta$  under the line  $y + px = s$ .

$$H(R_+, R_+, R_+^0, \mathbf{b}) =$$

# Results

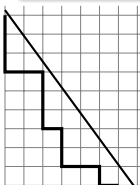
Manipulating Catalanimal  $\implies$  a proof of the Rational Shuffle Theorem + a generalization.

## Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given  $r, s \in \mathbb{R}_{>0}$  such that  $p = s/r$  irrational, take  $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$  to be the south step sequence of highest path  $\delta$  under the line  $y + px = s$ .

$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line  $y + px = s$ ,



# Results

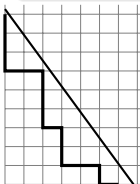
Manipulating Catalanimal  $\implies$  a proof of the Rational Shuffle Theorem + a generalization.

## Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given  $r, s \in \mathbb{R}_{>0}$  such that  $p = s/r$  irrational, take  $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$  to be the south step sequence of highest path  $\delta$  under the line  $y + px = s$ .

$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line  $y + px = s$ ,



$\text{area}(\lambda)$  as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks}$   $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

# A Question

Why stop at  $e_k[-MX^{m,n}]$ ?



# A Question

Why stop at  $e_k[-MX^{m,n}]$ ?

For which symmetric functions  $f$  can we find a Catalan animal such that  $f[-MX^{m,n}] \cdot 1 = \text{a Catalan animal}$ ?

## A Question

Why stop at  $e_k[-MX^{m,n}]$ ?

For which symmetric functions  $f$  can we find a Catalanimal such that  $f[-MX^{m,n}] \cdot 1 = \text{a Catalanimal}$ ?

Answer: for  $f$  equal to any LLT polynomial!

## A Question

Why stop at  $e_k[-MX^{m,n}]$ ?

For which symmetric functions  $f$  can we find a Catalanoid such that  $f[-MX^{m,n}] \cdot 1 =$  a Catalanoid?

Answer: for  $f$  equal to any LLT polynomial!

Special case:  $\mathcal{G}_\nu[-MX^{1,1}] \cdot 1 = \nabla \mathcal{G}_\nu(X; q)$ .

# LLT Catalananimals

For a tuple of skew shapes  $\nu$ , the *LLT Catalananimal*  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$ ,

# LLT Catalananimals

For a tuple of skew shapes  $\nu$ , the *LLT Catalananimal*  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$ ,
- $R_+ \setminus R_q$  = pairs of boxes in the same diagonal,
- $R_q \setminus R_t$  = the attacking pairs,
- $R_t \setminus R_{qt}$  = pairs going between adjacent diagonals,

# LLT Catalananimals

For a tuple of skew shapes  $\nu$ , the *LLT Catalananimal*  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  is determined by

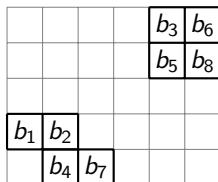
- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$ ,
- $R_+ \setminus R_q =$  pairs of boxes in the same diagonal,
- $R_q \setminus R_t =$  the attacking pairs,
- $R_t \setminus R_{qt} =$  pairs going between adjacent diagonals,
- $\lambda$ : fill each diagonal  $D$  of  $\nu$  with  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .  
Listing this filling in reading order gives  $\lambda$ .

# LLT Catalanimals

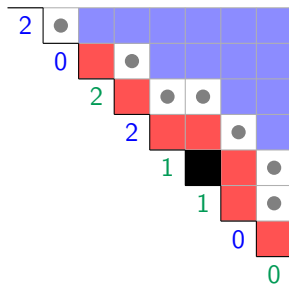
- $R_+ \setminus R_q =$  pairs of boxes in the same diagonal,
- $R_q \setminus R_t =$  the attacking pairs,
- $R_t \setminus R_{qt} =$  pairs going between adjacent diagonals,
- $R_{qt} =$  all other pairs,

$\lambda$ : fill each diagonal  $D$  of  $\nu$  with

$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .



$\nu$

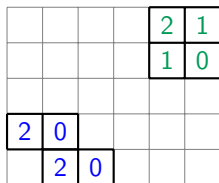


# LLT Catalanimals

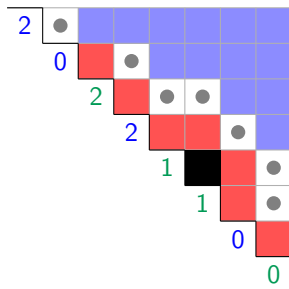
- $R_+ \setminus R_q =$  pairs of boxes in the same diagonal,
- $R_q \setminus R_t =$  the attacking pairs,
- $R_t \setminus R_{qt} =$  pairs going between adjacent diagonals,
- $R_{qt} =$  all other pairs,

$\lambda$ : fill each diagonal  $D$  of  $\nu$  with

$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .



$\lambda$ , as a filling of  $\nu$





## Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let  $\nu$  be a tuple of skew shapes and let  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega H_\nu \\ &= c_\nu \omega \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some  $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$ .

# What about Macdonald polynomials?!

- Remember  $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$ .

# What about Macdonald polynomials?!

- Remember  $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$ .
- We have a formula for  $\nabla \mathcal{G}_\nu$ .

# What about Macdonald polynomials?!

- Remember  $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$ .
- We have a formula for  $\nabla \mathcal{G}_\nu$ .
- Does there exist formula  $\tilde{H}_\mu = \sum_\nu a_{\mu\nu}(q, t) \mathcal{G}_\nu$  ?

# What about Macdonald polynomials?!

- Remember  $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$ .
- We have a formula for  $\nabla \mathcal{G}_\nu$ .
- Does there exist formula  $\tilde{H}_\mu = \sum_\nu a_{\mu\nu}(q, t) \mathcal{G}_\nu$  ? Yes!

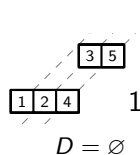
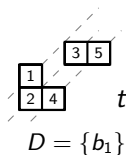
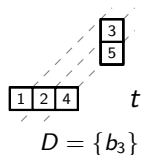
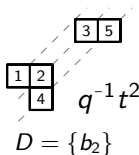
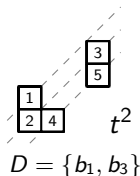
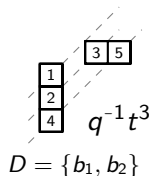
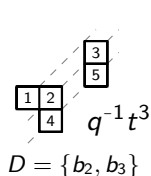
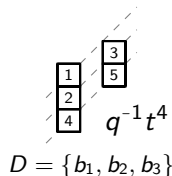
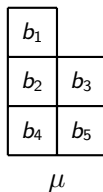
- ① Background on symmetric functions and Macdonald polynomials
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ **A new formula for Macdonald polynomials**

## Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X; q, t) = \sum_D \left( \prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q)$$

# Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X; q, t) = \sum_D \left( \prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q)$$





## Putting it all together

- Take HHL formula  $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$  and apply  $\omega \nabla$ .

# Putting it all together

- Take HHL formula  $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$  and apply  $\omega \nabla$ .
- By construction, all the LLT Catalan animals  $H_{\nu(\mu,D)}$  appearing on the RHS will have the same root ideal data  $(R_q, R_t, R_{qt})$ .

# Putting it all together

- Take HHL formula  $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$  and apply  $\omega \nabla$ .
- By construction, all the LLT Catalan animals  $H_{\nu(\mu,D)}$  appearing on the RHS will have the same root ideal data  $(R_q, R_t, R_{qt})$ .
- Collect terms to get  $\prod_{(b_i, b_j) \in V(\mu)} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$  factor for  $V(\mu)$  the set of vertical dominoes  $(b_i, b_j)$  in  $\mu$ .

$$\tilde{H}_\mu = \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\substack{\alpha_{ij} \in V(\mu)}} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

# The root ideal $R_\mu$

$b_1$		
$b_2$	$b_3$	
$b_4$	$b_5$	$b_6$
$b_7$	$b_8$	$b_9$

row reading order

$$b_1 \prec b_2 \prec \cdots \prec b_n$$

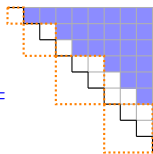
$$R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \},$$

$$\hat{R}_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j \},$$

$$R_\mu \setminus \hat{R}_\mu \leftrightarrow V(\mu)$$

Example:

$$R_{3321} =$$



# The root ideal $R_\mu$

$b_1$		
$b_2$	$b_3$	
$b_4$	$b_5$	$b_6$
$b_7$	$b_8$	$b_9$

row reading order

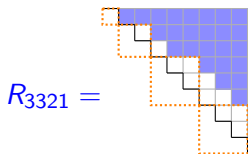
$$b_1 \prec b_2 \prec \cdots \prec b_n$$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j\},$$

$$\hat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j\},$$

$$R_\mu \setminus \hat{R}_\mu \leftrightarrow V(\mu)$$

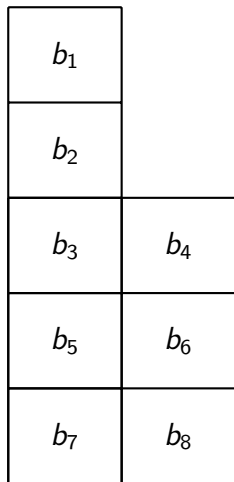
Example:



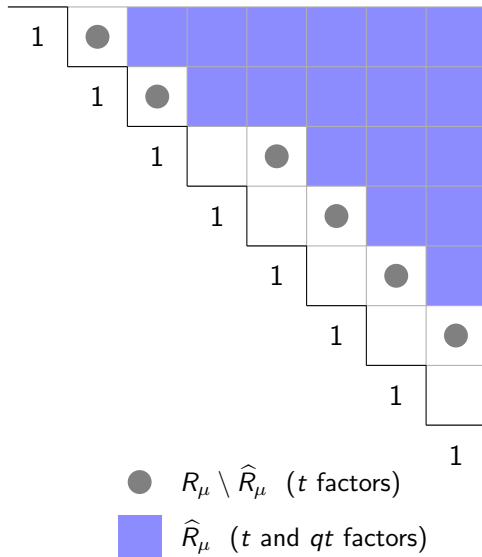
Remark

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right)$$

# Example



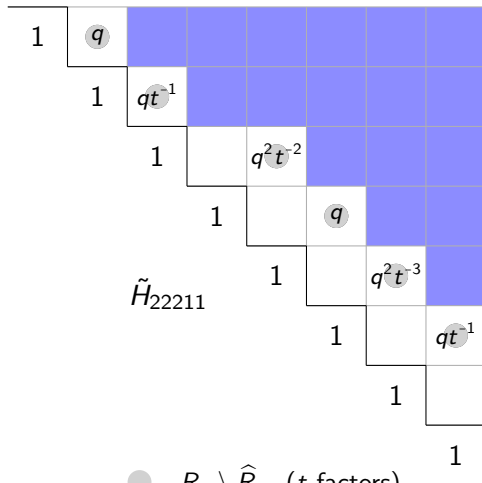
partition  $\mu = 22211$



# Example

$1 - q^{\frac{z_1}{z_2}}$	
$1 - qt^{-1} \frac{z_2}{z_3}$	
$1 - q^2 t^{-2} \frac{z_3}{z_5}$	$1 - q^{\frac{z_4}{z_6}}$
$1 - q^2 t^{-3} \frac{z_5}{z_7}$	$1 - qt^{-1} \frac{z_6}{z_8}$

numerator factors  $1 - q^{\text{arm}+1} t^{-\text{leg}} z_i/z_j$



●  $R_\mu \setminus \hat{R}_\mu$  ( $t$  factors)

■  $\hat{R}_\mu$  ( $t$  and  $qt$  factors)

$q = t = 1$  specialization

$$\begin{aligned}
 & \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=t=1} \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \hat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\
 & = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\
 & = \omega h_1^n \\
 & = e_1^n
 \end{aligned}$$



## A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

# A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega \sigma \left( (z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right)$$

## Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition  $\mu$  and positive integer  $s$ , the symmetric function  $\tilde{H}_\mu^{(s)}$  is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_\nu(X)$$

satisfy  $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$ .

# Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible $V_\lambda$	SSYT( $\lambda$ )
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman $M_\lambda$	HHL
$\nabla e_n$	$DH_n$	Shuffle theorem
$\tilde{H}_\lambda^{(s)}(X; q, t)$	??	??

# Grazie mille!

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2023/ed. *A Shuffle Theorem for Paths Under Any Line*, Forum of Mathematics, Pi **11**, e5, DOI 10.1017/fmp.2023.4.

———. 2021. *LLT Polynomials in the Schiffmann Algebra*, arXiv e-prints, arXiv:2112.07063.

———. 2023. *A Raising Operator Formula for Macdonald Polynomials*, arXiv e-prints, arXiv:2307.06517.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Mellit, Anton. 2018. *A Proof of the Shuffle Conjecture* **31**, no. 3, 661–697, DOI 10.1090/jams/893.

Feigin, B. L. and Tsymbaliuk, A. I. 2011. *Equivariant K-theory of Hilbert Schemes via Shuffle Algebra*, Kyoto J. Math. **51**, no. 4, 831–854.

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

Haglund, J., M. Haiman, and N. Loehr. 2005. *A Combinatorial Formula for Macdonald Polynomials* **18**, no. 3, 735–761 (electronic).

Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.

Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919

———. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676

Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sémin. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754

Mellit, Anton. 2021. *Toric Braids and  $(m,n)$ -Parking Functions*, Duke Math. J. **170**, no. 18, 4123–4169, DOI 10.1215/00127094-2021-0011.

Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004

Schiffmann, Olivier and Vasserot, Eric. 2013. *The Elliptic Hall Algebra and the K-theory of the Hilbert Scheme of  $A_2$* , Duke Mathematical Journal **162**, no. 2, 279–366, DOI 10.1215/00127094-1961849.