

**Quantum Groups**  
**Notes from a class taught by Weiqiang**  
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## 1. $q$ -Numbers

Let  $q$  be an indeterminate. Then, we will work in any of the following rings

$$\mathbb{Z}[q] \subseteq \mathbb{Z}[q, q^{-1}] \subseteq \mathbb{Q}(q) \subseteq \mathbb{C}(q)$$

1.1. DEFINITION. For an indeterminate  $q$  and  $n \in \mathbb{Z}$ , we define

- (a)  $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$
- (b)  $[0]_q! := 1$  and  $[n]_q! := [n]_q [n-1]_q \cdots [1]_q$  for  $n \in \mathbb{Z}_{\geq 0}$
- (c) If  $m \in \mathbb{Z}, n \geq 0$ , then

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q [m-1]_q \cdots [m-n+1]_q}{[n]_q!} \text{ if } m \geq 0 \quad \frac{[m]_q!}{[n]_q! [m-n]_q!}$$

1.2. REMARK. When the  $q$  is clear, we will drop the  $q$  from the notation and say  $[n] := [n]_q$ , etc.

1.3. EXAMPLE. We compute some examples of  $q$ -numbers.

- (a)  $[0] = 0$
- (b)  $[1] = 1$
- (c)  $[2] = q + q^{-1}$

1.4. PROPOSITION. *We have the following simple identities on  $q$ -numbers.*

- (a)  $[-n] = -[n]$  for any  $n \in \mathbb{Z}$ .
- (b)  $\begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} m \\ m \end{bmatrix}_q$  for all  $m \in \mathbb{Z}$ .
- (c)  $\begin{bmatrix} m \\ n \end{bmatrix}_q = 0$  for  $0 \leq m < n$ .

1.5. PROPOSITION. *We have the identity*

$$\begin{bmatrix} m+1 \\ n \end{bmatrix}_q = q^{-n} \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{m-n+1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q$$

and also that both  $[n]_q$  and  $\begin{bmatrix} m \\ n \end{bmatrix}_q$  are elements of  $\mathbb{Z}[q, q^{-1}]$

PROOF. We compute directly that

$$\begin{aligned} q^{-n} \begin{bmatrix} m \\ n \end{bmatrix}_q + q^{m-n+1} \begin{bmatrix} m \\ n-1 \end{bmatrix}_q &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[m][m-1] \cdots [m-n+2]}{[n-1]_q!} \\ &= q^{-n} \frac{[m][m-1] \cdots [m-n+1]}{[n]_q!} + q^{m-n+1} \frac{[n][m][m-1] \cdots [m-n+2]}{[n][n-1]_q!} \\ &= (q^{-n}[m-n+1] + q^{m-n+1}[n]) \frac{[m][m-1] \cdots [m-n+2]}{[n]_q!} \\ &= \left( \frac{q^{m-2n+1} - q^{-m-1}}{q - q^{-1}} + \frac{q^{m+1} - q^{m-2n+1}}{q - q^{-1}} \right) \frac{[m] \cdots [m-n+2]}{[n]_q!} \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{q^{m+1} - q^{-(m+1)}}{q - q^{-1}} \right) \frac{[m] \cdots [m - n + 2]}{[n]_q!} \\
&= \frac{[m+1][m] \cdots [(m+1) - n + 1]}{[n]_q!} = \left[ \begin{matrix} m+1 \\ n \end{matrix} \right]_q
\end{aligned}$$

Now, observe that

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{q}{q^n} \cdot \frac{q^{2n} - 1}{q^2 - 1} = \frac{1}{q^{n-1}} (q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1) \in \mathbb{Z}[q, q^{-1}]$$

This immediately gives that  $[n]_q! \in \mathbb{Z}[q, q^{-1}]$ . To show  $\left[ \begin{matrix} m \\ n \end{matrix} \right]_q \in \mathbb{Z}[q, q^{-1}]$ ,

we proceed by induction on  $m$ . Namely,  $\left[ \begin{matrix} m \\ 0 \end{matrix} \right]_q = 1 \in \mathbb{Z}[q, q^{-1}]$  for all  $m \in \mathbb{Z}$ . Then,

$$\left[ \begin{matrix} m+1 \\ n \end{matrix} \right]_q = q^n \underbrace{\left[ \begin{matrix} m \\ n \end{matrix} \right]_q}_{\in \mathbb{Z}[q, q^{-1}]} + q^{-m+n-1} \underbrace{\left[ \begin{matrix} m \\ n-1 \end{matrix} \right]_q}_{\in \mathbb{Z}[q, q^{-1}]} \in \mathbb{Z}[q, q^{-1}]$$

□

1.6. THEOREM. *q-Binomial Theorem For an indeterminate  $z$  and  $r \geq 0$ ,*

$$\prod_{j=0}^{r-1} (1 + q^{2j} z) = \sum_{i=0}^{r-1} q^{i(r-1)} \left[ \begin{matrix} r \\ i \end{matrix} \right]_q z^i$$

PROOF. This follows by induction. If  $r = 0$ , then we simply have  $1 = 1$ . Now, proceed by induction. Then,

$$\prod_{j=0}^r (1 + q^{2j} z) = (1 + q^{2r} z) \left( \sum_{i=0}^{r-1} q^{i(r-1)} \left[ \begin{matrix} r \\ i \end{matrix} \right]_q z^i \right) = \sum_{i=0}^{r-1} q^{i(r-1)} \left[ \begin{matrix} r \\ i \end{matrix} \right]_q z^i + \sum_{i=0}^{r-1} q^{i(r-1)+2r} \left[ \begin{matrix} r \\ i \end{matrix} \right]_q z^{i+1}$$

Then, if we fix the  $z$  power for some  $1 \leq k \leq r-1$ , we get coefficient

$$\begin{aligned}
q^{k(r-1)} \left[ \begin{matrix} r \\ k \end{matrix} \right]_q + q^{(k-1)(r-1)+2r} \left[ \begin{matrix} r \\ k-1 \end{matrix} \right]_q &= q^{k(r-1)} \left[ \begin{matrix} r \\ k \end{matrix} \right]_q + q^{k(r-1)+r+1} \left[ \begin{matrix} r \\ k-1 \end{matrix} \right]_q \\
&= q^{kr} \left( q^{-k} \left[ \begin{matrix} r \\ k \end{matrix} \right]_q + q^{-k+r+1} \left[ \begin{matrix} r \\ k-1 \end{matrix} \right]_q \right) \\
&= q^{kr} \left[ \begin{matrix} r+1 \\ k \end{matrix} \right]_q
\end{aligned}$$

where the last equality follows from 1.5. □

1.7. COROLLARY. *As consequences to 1.6, we get*

(a) For  $r \geq 1$ ,

$$\sum_{i=0}^r (-1)^i q^{-i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_q = 0$$

(b) Assume  $xy = q^2 yx$ . Then,

$$(x + y)^n = \sum_{t=0}^n q^{t(n-t)} \begin{bmatrix} n \\ t \end{bmatrix}_q y^t x^{n-t}$$

Sometimes in the literature,  $q$ -numbers are encoded slightly differently. We present the alternate definition here.

1.8. DEFINITION.  $\{n\}_v := 1 + v + v^2 + \cdots + v^{n-1} = \frac{v^n - 1}{v - 1}$

Then, the two definitions are related as follows.

1.9. PROPOSITION. Setting  $v = q^2$ ,

$$\{n\}_v = q^{n-1} [n]_q$$

## 2. The Quantum Group $\mathcal{U}_q(\mathfrak{sl}_2)$

Throughout this section, we will let  $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$ . Let  $\mathbb{k}$  be a field of characteristic 0 with  $q \in \mathbb{k}$ ,  $q \neq 0$ , and  $q$  is not a root of 1.

2.1. DEFINITION. We define the *quantum group*  $\mathcal{U} := \mathcal{U}_q(\mathfrak{sl}_2)$  to be the  $\mathbb{k}$ -algebra generated by elements  $E, F, K, K^{-1}$  with relations

- (a)  $KK^{-1} = 1 = K^{-1}K$
- (b)  $KE = q^2 EK$
- (c)  $KF = q^{-1} FK$
- (d)  $EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$

2.2. DEFINITION. We define the *Drinfeld double*  $\tilde{\mathcal{U}} = \langle E, F, K, K' \rangle$  to be the  $\mathbb{k}$ -algebra with relations

- (a)  $K'E = q^{-2} EK'$
- (b)  $K'F = q^2 EK'$
- (c)  $EF - FE = \frac{K - K'}{q - q^{-1}}$

2.3. REMARK. Note that  $\tilde{\mathcal{U}} / \langle KK' - 1 \rangle \cong \mathcal{U}$  and that  $KK'$  is central in  $\tilde{\mathcal{U}}$ .

2.4. DEFINITION. We define the following maps.

(a) The  $\mathbb{k}$ -linear involution  $\omega$  acts on  $\mathcal{U}$  by

$$\omega(E) = F, \omega(F) = E, \omega(K) = K^{-1}$$

(b) The  $\mathbb{k}$ -linear anti-involution  $\tau$  (ie  $\tau(xy) = \tau(y)\tau(x)$ ) acts on  $\mathcal{U}$  by

$$\tau(E) = E, \tau(F) = F, \tau(K) = K^{-1}$$

2.5. DEFINITION. For making computations more compact, we define the short hand

- (a)  $[K; n] = \frac{q^n K - q^{-n} K^{-1}}{q - q^{-1}}$
- (b) For  $n \in \mathbb{Z}_{\geq 0}$ ,  $E^{(n)} = \frac{E^n}{[n]_q!}$  and  $F^{(n)} = \frac{F^n}{[n]_q!}$ .

2.6. THEOREM (PBW Theorem).  $\{F^s K^n E^r \mid s, r \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}\}$  forms a basis for  $\mathcal{U}$ .

SKETCH OF PROOF. (a) Use the commutation relations of  $\mathcal{U}$  to show that this is a spanning set; when commuting an  $E$  past an  $F$ , one only picks up lower degree correction terms.

- (b) Construct a “regular representation”  $M = \mathbb{k}[\tilde{F}, \tilde{E}, \tilde{K}, \tilde{K}^{-1}]$  on which  $\mathcal{U}$  acts to show linear independence. This argument is more sophisticated, but since this is a faithful representation, you get that the map  $\theta: \mathcal{U} \rightarrow \text{End}_{\mathbb{k}}(M)$  is injective and since  $\theta(F^s K^n E^r)(1) = \tilde{F}^s \tilde{K}^n \tilde{E}^r$ , which is known to be linearly independent, then the set  $\{\theta(F^s K^n E^r)\}$  is linearly independent, thus giving us the desired linear independence by the injectivity of  $\theta$ . See [Jan95, Theorem 1.5].

□

2.7. LEMMA (Useful Identities). (a)  $[K; n]E = E[K; n + 2]$

(b)  $[K; n]F = F[K; n - 2]$

(c)  $EF^s = F^s E + [s]F^{s-1}[K; 1 - s]$  for  $s \geq 0$

(d)  $E^r F^s = \sum_{i=0}^{\min(r,s)} \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} s \\ i \end{bmatrix}_q [i]! F^{s-i} \left( \prod_{j=1}^i [K; i - (r + s) + j] \right) E^{r-i}$

d'  $E^{(r)} F^{(s)} = \sum_{i=0}^{\min(r,s)} F^{(s-i)} \begin{bmatrix} K; 2i - (r + s) \\ i \end{bmatrix}_q E^{(r-i)}$  where  $\begin{bmatrix} K; c \\ i \end{bmatrix}_q := \frac{[K; c][K; c-1] \cdots [K; c-i+1]}{[i]!}$ .

Identity (d') gives one reason why divided powers are sometimes more convenient; writing identities with them can sometimes be simpler.

2.8. REMARK.  $\mathcal{U}_q(\mathfrak{sl}_2)$  has no zero-divisors.

### 2.1. Finite-dimensional Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ .

2.9. EXAMPLE. Let  $M = \mathbb{k}m_0 \oplus \mathbb{k}m_1$  with  $Km_0 = qm_0$  and  $Km_1 = q^{-1}m_1$  and  $E, F$  actions given by

$$\begin{array}{ccccc} 0 & \xleftarrow{F} & m_1 & \begin{array}{c} \xrightarrow{E} \\ \xleftarrow{F} \end{array} & m_0 & \xrightarrow{E} & 0 \end{array}$$

2.10. LEMMA. Let  $M$  be a finite-dimensional  $\mathcal{U}$ -module. Then, there exists an  $r > 0$  such that  $E^r M = 0$  and  $F^r M = 0$ .

2.11. DEFINITION. For  $M \in \mathcal{U}\text{-}\mathbf{mod}$ ,  $\lambda \in \mathbb{k}^\times$ , let  $M_\lambda := \{m \in M \mid Km = \lambda m\}$  be called the  $\lambda$ -weight subspace of  $M$ .

2.12. LEMMA. (a)  $EM_\lambda \subseteq M_{q^2\lambda}$  and  $FM_\lambda \subseteq M_{q^{-2}\lambda}$ .  
(b) If  $M_\lambda \neq 0$  and  $M$  is simple, then

$$M = \bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}$$

2.13. PROPOSITION. Let  $M$  be a finite-dimensional  $\mathcal{U}$ -module. Then,

$$M = \bigoplus_{a \in \mathbb{Z}} M_{+q^a} \oplus M_{-q^a}$$

PROOF. It is equivalent to show that the minimal polynomial of  $K$  on  $M$  is of the form  $\prod_i (K - \lambda_i)$  with distinct  $\lambda_i \in \pm q^\mathbb{Z}$ . To do this, set

$$h_r := \prod_{j=1-r}^{r-1} [K; r-s+j], \quad r > 0, h_0 = 1$$

Now, for  $s > 0$ , if  $F^s M = 0$ , then  $F^{s-r} h_r M = 0$  for all  $0 \leq r \leq s$  because

$$\left( E^r F^s \prod_{j=1}^{r-1} [K; r-s+j] \right) M = \left( \sum_{i=0}^r a_i F^{s-i} h_i \prod_{j=0}^{r-i-1} [K; -s-j] E^{r-i} \right) M$$

for  $a_i = \begin{bmatrix} r \\ i \end{bmatrix}_q \begin{bmatrix} s \\ i \end{bmatrix}_q [i]!$  by 2.7(d) allows us to use induction. Then, we have

$$0 = h_s M = \prod_{j=1-s}^{s-1} \left[ \underbrace{(q - q^{-1})^{-1} q^j K^{-1}}_{\text{Invertible scalar}} \quad \underbrace{(K^2 - q^{-2j})}_{\text{Minimal polynomial divides this}} \right] M$$

and thus we have distinct  $\lambda_i \in \pm q^\mathbb{Z}$  □

## Bibliography

[Jan95] J. C. Jantzen, *Lectures on Quantum Groups*, 1995.