ROOT SYSTEMS

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1. Introduction

A root system is a configuration of vectors in some Euclidean space that satisfy nice properties concerning reflections [Wik17]. In particular, root systems are acted upon by associated Weyl groups of reflections. While root systems have some interesting features in their own right, they are intimately related to the structure of semisimple Lie algebras. However, this write-up seeks to simply talk about what is known about root systems in their own right. A lot of the definitions and treatments are lifted from [Hum72] and thus I do not lay any claim to the originality of statements of definitions, theorems, etc. I view these notes as mainly a good companion to be read along with [Hum72].

2. Definitions and Basic Concepts

Given a vector $\alpha \in E$, we can define a function $\sigma_\alpha \colon E \to E$ that reflects every vector over the hyperplane perpendicular to α. Explicitly, we get

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

where $\beta \in E$ and $(\cdot, \cdot)$ is the standard inner product on E. To see this works, note that $\sigma_\alpha(\alpha) = -\alpha$ as desired and if $(\beta, \alpha) = 0$, then $\sigma_\alpha(\beta) = \beta$, leaving all the vectors in the hyperplane fixed.

2.1. **Definition.** For notational convenience, we will say

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

It is important to note that $\langle \cdot, \cdot \rangle$ is linear only in the first variable.

2.2. **Definition.** [Hum72, p 42] A *root system* is a set of vectors Φ in Euclidean space E such that

- Φ is finite, $\operatorname{span} \Phi = E$, and $0 \notin \Phi$.
- If $\alpha \in \Phi$, then the only multiples of α in Φ are $\pm\alpha$.
- If $\alpha \in \Phi$, then $\sigma_\alpha(\Phi) = \Phi$
- If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

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2.3. **Example.** Consider the collection of vectors in $\mathbb{R}^2$ of equal length with angle from the origin given in multiples of 60 degress. Such a collection is a root system, as can be easily checked.

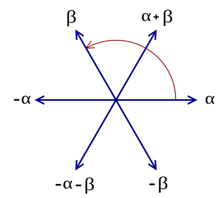


FIGURE 1. Root system A_2 from [Wik17]

These hyperplane reflections form a "nice" action of the root system. Furthermore, they form a group, since the composition of two reflections is a reflection and each of these described reflections is their own inverse. Thus, we have the following definition.

2.4. **Definition.** The *Weyl group* of a root system is the collection of these reflections σ_α with group operation as composition. Explicitly,

$$\mathscr{W} = \langle \{\sigma_\alpha \mid \forall \alpha \in \Phi\} \mid \{\sigma_\alpha^2 = 1,\ \forall \alpha \in \Phi\} \rangle$$

It is a straightfoward exercise to show the following fact

2.5. **Lemma.** [Hum72, p 43] *If $\sigma \in GL(E)$ leaves Φ invariant and $\sigma_\alpha \in \mathscr{W}$ is a simple reflection, then*

$$\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$$

and also

$$\langle \sigma(\beta), \sigma(\alpha) \rangle = \langle \beta, \alpha \rangle$$

2.6. **Definition.** We say a root system is *reducible* if there exists root systems $\Phi_1, \Phi_2 \subseteq \Phi$ such that $\Phi = \Phi_1 \sqcup \Phi_2$ such that $\text{span}(\Phi) = \text{span}(\Phi_1) \oplus \text{span}(\Phi_2)$. Any root system that is not reducible is called *irreducible.*

In general, we restrict our attention to irreducible root systems, since they are the basic building blocks of root systems. Now, we also have a convenience of notation as follows

2.7. **Definition.** For $\alpha \in \Phi$, a root system, we say a *coroot* $\alpha^\vee$ is given by

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$$

It is worth noting that $\Phi^\vee$ is also a root system with an associated Weyl group that is isomorphic to the Weyl group of Φ.

2.8. **Definition.** We define the *rank* of a root system Φ to be the dimension of Euclidean space it spans.

The axioms for a root system are fairly restrictive, and so it is relatively easy to classify all the possible root systems that span $\mathbb{R}$ and $\mathbb{R}^2$. Namely, since $(\alpha, \beta) = \|\alpha\| \, \|\beta\| \cos\theta$,

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2\theta \in \mathbb{Z}_{\geq 0}$$

Thus, we have the following

2.9. **Lemma.** *Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$, then*

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}$$

Proof. This follows immediately from $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ being integers and $4\cos^2\theta \leq 4$. □

Thus, roots can only ever have certain angles between them. To see this, examine Table 1 on page 45 of [Hum72].

However, these restrictions lead us to a useful application for determining when sums and differences of roots are roots.

2.10. **Lemma.** [Hum72, p 45] *Let $\alpha, \beta \in \Phi$ be nonproportional roots.*

- *If $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$.*
- *If $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.*

Proof. Note $(\alpha, \beta) > 0 \iff \langle \alpha, \beta \rangle > 0$. By the table of values, the acuteness of the angle requires that either $\langle \alpha, \beta \rangle$ or $\langle \beta, \alpha \rangle$ is 1. Thus, by the axioms of the root system, $\langle \alpha, \beta \rangle = 1 \implies \sigma_\beta(\alpha) = \alpha - \beta \in \Phi$ or $\langle \beta, \alpha \rangle = 1 \implies \beta - \alpha \in \Phi \implies \alpha - \beta \in \Phi$. □

An application of this lemma is to justify the existence of unbroken "strings" of roots of the form $\beta + n\alpha, n \in \mathbb{Z}, q \leq n \leq r$. Write this in more formally.

2.11. **Definition.** [Hum72, p 47] Given $\Delta \subseteq \Phi$, we say that Δ is a *simple system* or *base* if

- Δ is a basis for $\operatorname{span}\Phi$
- Every $\beta \in \Phi$ can be written as $\beta = \sum k_\alpha \alpha$ with $\alpha \in \Delta$ and all $k_\alpha \in \mathbb{Z}_{\geq 0}$ or all $k_\alpha \in \mathbb{Z}_{\leq 0}$.

We note that Δ need not be unique and so whenever any definition depends on Δ, there is a choice being made that must be accounted for.

2.12. **Definition.** The *height* of $\beta \in \Phi$ relative to Δ is given by

$$\operatorname{ht}(\beta) = \sum_{\alpha \in \Delta} k_\alpha$$

where $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$.

Using this notion of height, we can define positive roots relative to Δ as those with $k_\alpha \geq 0, \forall \alpha \in \Delta$ and a similar notion for negative roots. Note that this partitions $\Phi = \Phi^+ \sqcup \Phi^-$. We also note that we can define a partial order on the roots in Φ with a fixed Δ.

2.13. **Definition.** [Hum72, p 47] We say $\mu \prec \lambda$ if and only if $\lambda - \mu$ is a sum of positive roots (equivalently, of simple roots) or $\lambda = \mu$.

We note also the following relationship between simple roots.

2.14. **Lemma.** [Hum72, p 47] *If Δ is a base of Φ, then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ in Δ, and $\alpha - \beta$ is not a root.*

Proof. For $\alpha, \beta \in \Delta$, let $(\alpha, \beta) > 0$. Then, $\alpha - \beta \in \Phi$, which violates the second axiom of a base. $\square$

In essence, this lemma gives us that the angle between two simple roots is obtuse.

2.15. **Definition.** Let us define the *half sum of positive roots* as

$$\rho := \frac{1}{2} \sum_{\beta \succ 0} \beta.$$

This vector (not necessarily a root) has some important properties, such as the following

2.16. **Proposition.** [Hum72, p 50] $\sigma_\alpha(\rho) = \rho - \alpha$ *for all* $\alpha \in \Delta$.

Now, we wish to show the following existence theorem.

2.17. **Theorem.** *Every root system has a base.*

To do this, we will make use of the fact

2.18. **Proposition.** *The union of finitely many hyperplanes cannot be all of $\mathbb{R}^n$ for $n \geq 2$.*

As a result of this proposition, we may choose a vector in our Euclidean space E which does not lie in any hyperplane perpendicular to a root.

2.19. **Definition.** We call a vector $z \in E$ *regular* if $z \in E \setminus \bigcup_{\alpha \in \Phi} P_\alpha$, where P_α is the hyperplane orthogonal to the root α, and *singular* otherwise.

2.20. **Definition.** Given a vector $z \in E$, we define

$$\Phi^+(z) := \{\alpha \in \Phi \mid (z, \alpha) > 0\}$$

We also say $\alpha \in \Phi^+(z)$ is *decomposable* if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(z)$. Otherwise, it is *indecomposable*.

2.21. **Proposition.** *If $z \in E$ is regular, then*

$$\Phi = \Phi^+(z) \cup \Phi^+(-z)$$

Thus, following [Hum72], we rewrite our theorem above in a more explicit manner.

2.22. **Theorem.** *Let $z \in E$ be regular. Then, the set $\Delta(z)$ of all indecomposable roots in $\Phi^+(z)$ is a base of Φ and every base of Φ has the form $\Delta(z')$ for some regular $z' \in E$.*

Proof. Let $\dim E \geq 2$, otherwise the theorem is immediate. Now,$\beta \in \Phi \Longrightarrow \beta \in \Phi^+(z)$ or $-\beta \in \Phi^+(z)$ by the proposition above, so we need only show

$$\beta = \sum_{\alpha \in \Delta^+(z)} k_\alpha \alpha, \; k_\alpha \in \mathbb{Z}_{\geq 0}$$

Assume that is not the case. Then, we may pick a $\beta \in \Phi^+(z)$ such that (z, β) is as small as possible. Since $\beta \notin \Delta(z)$, there is a $\beta_1, \beta_2 \in \Phi^+(z)$ such that

$$\beta = \beta_1 + \beta_2 \Longrightarrow (z, \beta) = (z, \beta_1) + (z, \beta_2) > 0$$

but then either β_1 or β_2 is not a positive integral linear combination of elements in $\Delta(z)$, contradicting $\beta_i \in \Phi^+(z)$.

Now, we need only show the elements of $\Delta(z)$ are linearly independent. By the lemma 2.14, $(\alpha, \beta) \leq 0$ for $\alpha, \beta \in \Delta(z)$ with $\beta \neq \pm\alpha$. Thus, for $c_\alpha \in \mathbb{R}$,

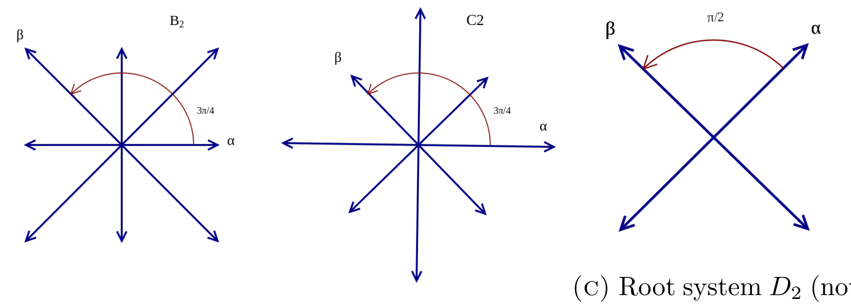
$$\begin{aligned}
\sum_{\alpha \in \Delta(z)} c_\alpha \alpha = 0 &\Longrightarrow x := \sum_{\alpha | r_\alpha > 0} r_\alpha \alpha = \sum_{\beta | r_\beta < 0} (-r_\beta)\beta \\
&\Longrightarrow (x, x) = \sum_{\substack{\alpha | r_\alpha > 0 \\ \beta | r_\beta < 0}} -r_\alpha r_\beta (\alpha, \beta) \leq 0 \\
&\Longrightarrow x = 0 \\
&\Longrightarrow 0 = (x, z) = \sum_{\alpha | r_\alpha > 0} r_\alpha \cdot (\alpha, z)
\end{aligned}$$

where $(\alpha, z) > 0$ since $\alpha \in \Phi^+(z)$ by definition, so it must be that all the $r_\alpha = 0$ and similarly for all r_β, thus giving linear independence.

Thus, $\Delta(z)$ is a base of $\Phi = \Phi^+(z) \cup \Phi^+(-z)$ since $\Delta(z)$ spans $\Phi^+(z)$ and thus all of Φ, and it meets the base axioms by above.

To show every base of Φ has the form $\Delta(z)$ for some regular $z \in E$, we start with a given base Δ and pick a $z \in E$ so that $(z, \alpha) > 0$ for all $\alpha \in \Delta$, which is possible since the intersection of the positive open half-spaces associated with any basis will be non-empty. Thus, z is regular, $\Phi^+ \subseteq \Phi^+(z)$, and $\Phi^- \subseteq \Phi^+(-z)$, so $\Phi^+ = \Phi^+(z)$. Since Δ contains only indecomposable elements, $\Delta \subseteq \Delta(z)$, but $|\Delta| = |\Delta(z)| = \ell \Longrightarrow \Delta = \Delta(z)$. $\square$

2.23. **Remark.** In plain geometric language, the linear independence argument in the above proof shows that any set of vectors lying strictly on one side of a hyperplane in Euclidean space and forming pairwise obstuse angles must be linearly independent.



(A) Root system B_2 (B) Root system C_2 (C) Root system D_2 (not irreducible.)

FIGURE 2. Classical irreducible root systems when $n = 2$ from [Wik17]

3. Standard Examples

Above, we saw the root system A_2. Classically, some standard root systems are of types A_n, B_n, C_n, and D_n, all of rank n. Whe $n = 2$, each of these has simple system $\Delta = \{\alpha, \beta\}$, which is suggestively given by the labeling in the diagrams. Note that in B_2, $|\beta| = \sqrt{2}|\alpha|$ and in C_2, $|\alpha| = \sqrt{2}|\beta|$. More formally, we define these systems for any $n \in \mathbb{N}$.

3.1. **Definition.** Let V be the subspace of $\mathbb{R}^{n+1}$ for which the coordinates sum to 0 and let Φ be the set of vectors in V that have length $\sqrt{2}$ and integer coordinates in $\mathbb{R}^{n+1}$. We call Φ the A_n *root system.*

3.2. **Definition.** Let Φ be all integers in $\mathbb{R}^n$ having length 1 or $\sqrt{2}$ and integer coordinates. We call Φ the B_n *root system.*

3.3. **Definition.** Let Φ be all vectors in $\mathbb{R}^n$ having length $\sqrt{2}$ and integer coordinates, as well as all vectors $2v$ where v is vector having length 1 and integer coordinates. Then, we call Φ the C_n *root system.*

3.4. **Definition.** Let Φ be all vectors in $\mathbb{R}^n$ having length $\sqrt{2}$ and integer coordinates. We call Φ the D_n *root system.*

It turns out that all these root systems are irreducible, but they are not the only ones. There are other so-called "exceptional" types, such as

3.5. **Definition.** The following configuration of roots in figure 3 is called G_2.

3.6. **Remark.** Note that the short roots in G_2 form the system A_2, and similarly for the long roots.

Using Dynkin diagrams, we will arrive at *all* the irreducible root systems. While it might take a little work to derive directly at this stage, the root

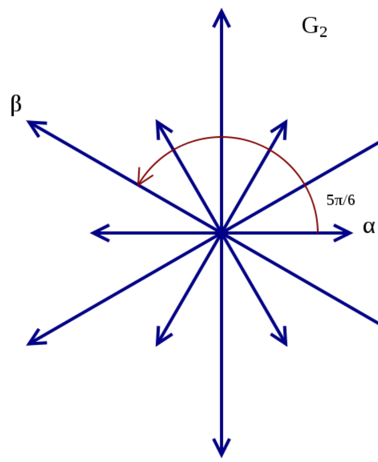


FIGURE 3. G_2 Root System from [Wik17]

A_n	S_{n+1}
B_n	$(\mathbb{Z}/2)^n \rtimes S_n$
C_n	$(\mathbb{Z}/2)^n \rtimes S_n$
D_n	$(\mathbb{Z}/2)^{n-1} \rtimes S_n$
G_2	D_{12}, the Dihedral group of order 12.

TABLE 1. Weyl Groups for Standard Irreducible Root Systems

systems have the following Weyl groups, presented in Table 1, which we will derive later.

4. WEYL GROUPS

Weyl groups have some interesting and useful properties as groups. Furthermore, an arbitrary Weyl group may have group actions on sets of "weights" (discussed later), not just root systems.

We start with a useful theorem for how Weyl groups interact with root systems and their bases

4.1. **Theorem.** [Hum72, Thm 10.3] *Let Δ be a base of Φ.*

(a) If $z \in E$ is regular, then there exists $\sigma \in \mathscr{W}$ such that $(\sigma(z), \alpha) > 0$ for all $\alpha \in \Delta$.
(b) If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathscr{W}$. In other words, $\mathscr{W}$ acts transitively on bases.
(c) If $\alpha \in \Phi$, there exists $\sigma \in \mathscr{W}$ such that $\sigma(\alpha) \in \Delta$

(d) $\mathscr{W}$ *is generated by* $\{\sigma_\alpha \mid \alpha \in \Delta\}$.

(e) If $\sigma(\Delta) = \Delta$ *for* $\sigma \in \mathscr{W}$*, then* $\sigma = 1$*, that is,* $\mathscr{W}$ *acts simply transitively on bases.*

Proof. Let $\mathscr{W}' \leq \mathscr{W}$ be the subgroup generated by $\{\sigma_\alpha \mid \alpha \in \Delta\}$. For (a)–(c), we will show the result using $\mathscr{W}'$, then show $\mathscr{W} = \mathscr{W}'$. □

Actually write this proof or just cite it.

We can define a length function on the elements of a Weyl group $\mathscr{W}$ by writing all the elements of $\mathscr{W}$ in reduced form. If $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ with n minimal, then we say that $\ell(w) = n$. We can also characterize this as follows.

4.2. **Lemma.** [Hum72, p 52] *Let* $n(\sigma)$ *be the number of positive roots for which* $\sigma(\alpha) \prec 0$*. Then, for all* $\sigma \in W$*, we have that* $\ell(\sigma) = n(\sigma)$*.*

This length function has many results.

4.3. **Proposition.** *Let* Φ *be a root system. Then,*

(a) The number of $\alpha \in \Phi^+$ *for which* $w\alpha \prec 0$ *is precisely* $\ell(w)$*. In particular, when* $\alpha \in \Delta$*, we have that* $\sigma_\alpha\beta \succ 0$ *for all* $\beta \neq \alpha$ *in* Φ^+*. Moreover,* w *is uniquely determined by the set of* $\alpha \succ 0$ *for which* $w\alpha \prec 0$*.*

(b) If $w \in W$*, then* $\ell(w) = \ell(w^{-1})$*. Thus,* $\ell(w) = |\Phi^+ \cap w(\Phi^-)|$*.*

4.1. **Chevalley-Bruhat Ordering (Optional).** Now, we can also define a partial order on $\mathscr{W}$ called the "Chevalley-Bruhat Ordering" following the approach in [Hum08, p 5]. Let

$$T := \bigcup_{w \in \mathscr{W}} wSw^{-1}$$

where S is the set of all simple reflections (s_α) in W. Then, T is the set of all reflections. Now, if $w, w' \in W$, then we can say $w' \to w$ if $w = tw'$ for some $t \in T$ and $\ell(w') < \ell(w)$. We extend this relation to a partial ordering of W, saying $w' < w$ if there is some sequence of relations $w' = w_0 \to w_1 \to \cdots \to w_n = w$ for some $w_i \in \mathscr{W}$. This partial order satisfies some useful properties.

4.4. **Proposition.** [Hum08, p 5] *Let* $w, w' \in \mathscr{W}$*.*

(a) $w' < w \Longrightarrow \ell(w') < \ell(w)$

(b) $w' \leq w$ *if and only if* w' *occurs as a subexpression in one (hence any) reduced expression* $s_1 \cdots s_n$*,* $s_i \in S$*, for* w*.*

(c) Adjacent elements in the Bruhat ordering differ in length by 1*.*

(d) If $w' < w$ *and* $s \in S$*, then* $w's \leq w$ *or* $w's \leq ws$ *(or both).*

(e) If $\ell(w_1) + 2 = \ell(w_2)$*, the number of elements* $w \in \mathscr{W}$ *satisfying* $w_1 < w < w_2$ *is* 0 *or* 2*.*

4.5. **Remark.** When $\mathscr{W} = S_n$, then the Chevalley-Bruhat order is the strong order on permutations.

5. Cartan Matrices

If we fix an ordering of our simple roots, we can encode the relationships between our simple roots using a matrix.

5.1. **Definition.** [Hum72, p 55] Given a root system Φ and an ordered set of simple roots, $(\alpha_1, \ldots, \alpha_n)$, we can construct a matrix $M = (\langle \alpha_i, \alpha_j \rangle)_{i,j}$. Such a matrix is called a *Cartan matrix* of Φ and it entries are called *Cartan integers*.

Cartan matrices are useful because they reveal some structure about root systems that may not be apparent from the original definition. Furthermore, Cartan matrices are actually unique up to ordering of the simple roots and the Cartan integers allow us to reconstruct Φ. Note, too, that all the diagonal entries must necessarily be 2's, since $\langle \alpha, \alpha \rangle = \frac{2(\alpha,\alpha)}{(\alpha,\alpha)} = 2$.

5.2. **Example.** For A_2 with simple system $\Delta = \{\alpha, \beta\}$, we compute that $\langle \alpha, \beta \rangle = \frac{2|\alpha||\beta| \cos(\frac{2\pi}{3})}{|\alpha|^2} = -1$. We get the same result for $\langle \beta, \alpha \rangle$ since $\cos(\theta)$ is an even function and $|\alpha| = |\beta|$. Thus, we get our Cartan matrix as

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

5.3. **Example.** For B_2 with simple system $\Delta = \{\alpha, \beta\}$, we compute that

$$\langle \alpha, \beta \rangle = \frac{2|\alpha||\beta| \cos(\frac{3\pi}{4})}{|\alpha|^2} = 2\sqrt{2} \cos(\frac{3\pi}{4}) = -2$$

We also compute that

$$\langle \beta, \alpha \rangle = \frac{2|\beta||\alpha| \cos(-\frac{3\pi}{4})}{|\beta|^2} = \frac{2(-\frac{\sqrt{2}}{2})}{\sqrt{2}} = -1$$

and so we get our Cartan matrix as

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

5.4. **Example.** Similar computations for C_2 reveal that the Cartan matrix is given by

$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

From these examples, we can see some important common facts about the structure of Cartan matrices.

5.5. **Proposition.** *Given a Cartan matrix A of a root system Φ, we see it has the following properties.*

(a) $A_{i,i} = 2$ *for all* i.
(b) $A_{i,j} \in \{0, -1, -2, -3\}$ *if* $i \neq j$.
(c) If $A_{i,j} = -2$ *or* -3*, then* $A_{j,i} = -1$.

(d) $A_{i,j} = 0 \Longleftrightarrow A_{j,i} = 0$.

Proof. (a) follows since $\langle \alpha_i, \alpha_i \rangle = 2\frac{(\alpha_i,\alpha_i)}{(\alpha_i,\alpha_i)} = 2$. (b) and (c) follow immediately from lemma 2.9 and the fact that any two distinct roots in Δ form an obtuse angle. Finally, (d) follows since $\langle \alpha_i, \alpha_j \rangle = 0$ tells us that α_i and α_j are orthogonal, and thus the product in the other order will be 0 as well. □

Now, we prove the important proposition

5.6. **Proposition.** [Hum72, Prop 11.1] *Let $\Phi \subseteq E$ and $\Phi' \subseteq E'$ be root systems with bases $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\Delta' = \{\alpha'_1, \ldots, \alpha'_\ell\}$ respectively. If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for $1 \le i, j \le \ell$, then*

- *The bijection $\alpha_i \mapsto \alpha'_i$ extends (uniquely) to an isomorphism $\phi\colon E \to E'$,*
- *ϕ maps Φ onto Φ',*
- *and $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$.*

Therefore, the Cartan matrix of Φ determines Φ up to isomorphism

Proof. Since Δ and Δ' are bases of E, E', there is a unique vector space isomorphism $\phi\colon E \to E'$ sending α_i to α'_i. If $\alpha, \beta \in \Delta$, then

$$\begin{aligned}
\sigma_{\phi(\alpha)}(\phi(\beta)) &= \sigma_{\alpha'}(\beta') \\
&= \beta' - \langle \beta', \alpha' \rangle \alpha' \\
&= \phi(\beta) - \langle \beta, \alpha \rangle \phi(\alpha) \qquad \text{by hypothesis } \langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle \\
&= \phi(\beta - \langle \beta, \alpha \rangle \alpha) \\
&= \phi(\sigma_\alpha(\beta))
\end{aligned}$$

that is to say, the following diagram commutes for any $\alpha \in \Delta$

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, "\sigma_\alpha"] & E' \arrow[d, "\sigma_{\phi(\alpha)}"] \\
E \arrow[r, "\phi"] & E'
\end{tikzcd}$$

Now, since $\mathscr{W}, \mathscr{W}'$ are generated by simple reflections,

$$\sigma \mapsto \phi \circ \sigma \circ \phi^{-1}$$

is an isomorphism between $\mathscr{W}$ and $\mathscr{W}'$ that sends $\sigma_\alpha \mapsto \sigma_{\phi(\alpha)}$ for $\alpha \in \Delta$. Furthermore, any $\beta \in \Phi$ is conjugate under $\mathscr{W}$ to a simple root, that is, there exists $\alpha \in \Delta$ such that $\beta = \sigma(\alpha)$. Thus,

$$\phi(\beta) = \underbrace{(\phi \circ \sigma \circ \phi^{-1})}_{\in \mathscr{W}'}\underbrace{(\phi(\alpha))}_{\in \Delta'} \Longrightarrow \phi(\beta) \in \Phi'$$

Thus, $\phi(\Phi) \subseteq \Phi' \Longrightarrow \phi(\Phi) = \Phi'$. Finally, since $\mathscr{W}, \mathscr{W}'$ are generated by reflections of simple roots, we have a canonical isomorphism between $\mathscr{W}$ and $\mathscr{W}'$, say ϕ_*. Thus, since any root can be obtained from a simple root by applying an element of the Weyl group, it must be that all Cartan integers are preserved under the isomorphism. $\square$

We now conclude by computing Cartan matrices for various classes of root systems we discussed earlier.

5.7. **Example.** Consider the root system A_n. All vectors are the same length and any choice of simple system Δ will require an angle of $\frac{2\pi}{3}$ with some other simple vector. Thus, up to reordering of simple roots, we get the following

$$A_n : \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

where everything not filled in represents the appropriate number of 0's. These computations are completely analogous for the A_2 example worked out above.

For B_n, the situation is slightly different because our simple system Δ must have at least one short root and at least one long root. We will take the simple system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_n\}$. Then, by direct computation, the angle between $e_i - e_{i+1}$ and $e_{i+1} - e_{i+2}$ is $\frac{2\pi}{3}$, and so

$$\langle e_i - e_{i+1}, e_{i+1} - e_{i+2} \rangle = -1$$

just like in the A_n case. However, $e_{n-1} - e_n$ forms an angle of $\frac{3\pi}{4}$ with e_n, thus getting

$$\langle e_{n-1} - e_n, e_n \rangle = 2\frac{\sqrt{2}\cos\frac{3\pi}{4}}{1} = 2\sqrt{2} \cdot -\frac{\sqrt{2}}{2} = -2$$

and thus $\langle e_n, e_{n-1} - e_n \rangle = -1$ and we arrive at the Cartan matrix

$$B_n : \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & -1 & \\ & -1 & 2 & -2 \\ & & -1 & 2 \end{pmatrix} = \begin{pmatrix} A_{n-2} & -E_{n-2,1} \\ -E_{1,n-2} & B_2 \end{pmatrix}$$

The C_n situation is symmetric with the choice $\Delta = \{e_i - e_{i+1} \mid 1 \le i \le n-1\} \cup \{2e_n\}$, giving us

$$C_n : \begin{pmatrix} A_{n-2} & -E_{n-2,1} \\ -E_{1,n-2} & C_2 \end{pmatrix}$$

For D_n, we make the choice $\Delta = \{e_i - e_{i+1} \mid 1 \le i \le n-1\} \cup \{e_{n-1} + e_n\}$. As above, the first $n-3$ roots will behave like the A case, but we compute the following

$$\begin{aligned} \langle e_{n-2} - e_{n-1}, e_{n-1} + e_n \rangle &= 2 \cdot -\frac{1}{\sqrt{2} \cdot \sqrt{2}} = -1 \\ \langle e_{n-1} + e_n, e_{n-2} - e_{n-1} \rangle &= 2 \cdot -\frac{1}{\sqrt{2} \cdot \sqrt{2}} = -1 \\ \langle e_{n-1} - e_n, e_{n-1} + e_n \rangle &= 0 \end{aligned}$$

Thus, our Cartan matrix has the form

$$D_n : \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & -1 \\ & & -1 & 2 & 0 \\ & & -1 & 0 & 2 \end{pmatrix}$$

6. Dynkin Diagrams

In general, it can be hard to determine the Weyl group of a root system by inspection. Instead, we can encode the root system using the Cartan matrix as a Dynkin diagram to reveal symmetries of roots. We let every simple root α_i have a vertex and connect each vertex by $\langle \alpha_i, \alpha_j \rangle \cdot \langle \alpha_j, \alpha_i \rangle$ edges. Note that the axioms on root systems limit the result of this product to $0, 1, 2, 3$. This gives us a *Coxeter graph*.

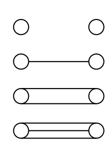
6.1. **Example.** Consider the root systems $A_1 \times A_1$, A_2, B_2, and G_2. Then we have the following Coxeter graphs:

$A_1 \times A_1$

A_2

B_2

G_2



For $A_1 \times A_1$, we note that any two simple roots must be orthogonal, since the two A_1's form a 90 degree angle. Thus, if $\Delta = \{\alpha, \beta\}$, then $\langle \alpha, \beta \rangle = 0$ and $\langle \beta, \alpha \rangle = 0$. So, there is no edge connecting the two vertices.

In contrast, consider A_2. Taking $\Delta = \{\alpha, \beta\}$ as denoted where A_2 was defined, we check

$$\langle \alpha, \beta \rangle = 2(\alpha, \beta) = 2\cos\frac{2\pi}{3} = -1 = 2\cos(\frac{2\pi}{3}) = 2(\beta, \alpha) = \langle \beta, \alpha \rangle$$

and thus $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 1$.

Next, consider B_2 taking $\Delta = \{\alpha, \beta\}$ as denoted where B_2 was defined. We check

$$\begin{aligned}\langle \alpha, \beta \rangle &= 2\frac{(\alpha, \beta)}{(\beta, \beta)} = 2\frac{|\alpha|\sqrt{2}|\alpha| \cos\frac{3\pi}{4}}{2|\alpha|^2} = \sqrt{2}\cos\frac{3\pi}{4} = -1 \\ \langle \beta, \alpha \rangle &= 2\sqrt{2}\cos\frac{3\pi}{4} = -2\end{aligned}$$

Thus, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 2$.

Do G_2

However, two different root systems can have the same Coxeter graph. Luckily, with a slight fix, we can use diagrams such as these to distinguish different root systems.

6.2. **Definition.** We define the *Dynkin diagram* of a root system Φ to be the Coxeter graph with directed edges such that, given double or triple edges between two roots, the edges will point away from the longer root and towards the shorter root.

We then have the following theorem.

6.3. **Theorem.** *Up to ordering of simple roots, every Cartan matrix has a unique Dynkin diagram and vice versa.*

Proof. □

Add proof or delete environment

6.4. **Theorem.** Φ *is irreducible if and only if its Coxeter graph is connected.*

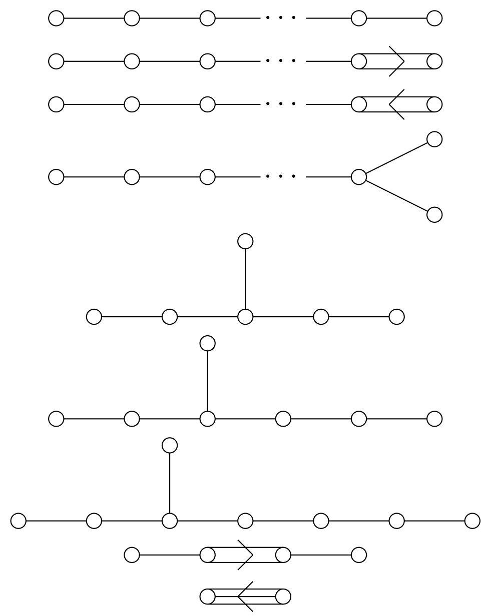
6.5. **Corollary.** *A root system* Φ *in Euclidean space* E *decomposes (uniquely) as the union of irreducible root systems* Φ_i *(in subspaces* $E_i \subseteq E$*) such that* $E = E_1 \oplus E_2 \oplus \cdots \oplus E_k$.

7. Irreducible Root Systems

Now that we have a structure theorem for root systems, we wish to arrive at a classification of irreducible root systems. To do this, one first figures out all the possible connected Coxeter graphs using elementary geometry/graph theory. We will only summarise here.

7.1. **Theorem.** *If Φ is an irreducible root system of rank ℓ, its Dynkin diagram is one of the following with ℓ vertices:*

A_ℓ *with* $\ell \geq 1$
B_ℓ *with* $\ell \geq 2$
C_ℓ *with* $\ell \geq 3$
D_ℓ *with* $\ell \geq 4$
E_6
E_7
E_8
F_4
G_2



where the restrictions on ℓ are to avoid duplication of diagrams.

Proof. See [Hum72, pp 58–63] for a full proof. In summary, we rely on the concept of an *admissible configuration* in $\mathbb{R}^n$, namely, a collection of unit vectors in an open half-space such that any two vectors form an angle of $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}$, or $\frac{5\pi}{6}$.

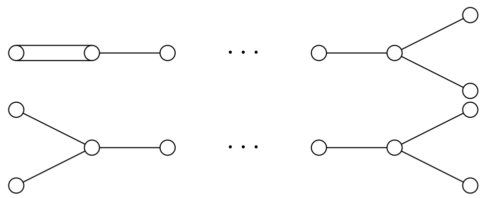
7.2. **Lemma.** *If $v_1, \ldots, v_n$ form an admissible onfiguration, then they are linearly independent.*

7.3. **Lemma.** *The Coxeter graph of an admissible configuration is acyclic, ignoring multiplicity of edgegs.*

7.4. **Lemma.** *Any vertex in the Coxeter graph has degree at most* 3*, counting multiplicity.*

7.5. **Lemma.** *If $v_1, \ldots, v_n$ form an admissible configuration with v_i, v_j connected by a single edge, then the configuration formed by removing v_i and v_j and replacing them with a single vector $v_i + v_j$, corresponding to deleting the two nodes and replacing them with one in the Coxeter diagram, forms another admissible configuration.*

7.6. **Lemma.** *A Coxeter graph of an admissible configuration cannot contain any of the following subgraphs:*



$\square$

Thus, we have a complete classification of Dynkin diagrams, which completely determines all possible Cartan matrices. Furthermore, since we can recover a root system from its Cartan matrices, we have effectively classified all irreducible root systems. Thus, in a sense, root systems are a very well understood object.

8. Automorphisms of a Root System

As discussed before, the Weyl group is the group generated by all the reflections across hyperplanes orthogonal to the roots of a root system. However, this is typically only a subgroup of all automorphisms of a root system.

8.1. **Lemma.** *Given a root system, Φ, $\mathscr{W} \trianglelefteq \operatorname{Aut}\Phi$, that is the group of all isomorphisms of Φ onto itself.*

Proof. It was left as an exercise to prove 2.5, that is, for $\sigma \in \operatorname{Aut}\Phi$ and $\alpha \in \Phi$,

$$\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$$

Thus, since the conjugate of any generator of the Weyl group is in the Weyl group, and $\mathscr{W} \leq \operatorname{Aut}\Phi$, it must be that $\mathscr{W} \trianglelefteq \operatorname{Aut}\Phi$. $\square$

Next, we define the following subgroup of $\operatorname{Aut}\Phi$.

8.2. **Definition.** Let Φ be a root system with simple system Δ. Then, we define

$$\Gamma := \{\sigma \in \operatorname{Aut}\Phi \mid \sigma(\Delta) = \Delta\}$$

8.3. **Proposition.** *Let $\mathscr{W}$ be the Weyl group of a root system Φ. Then,*

$$\operatorname{Aut}\Phi \cong \mathscr{W} \rtimes \Gamma$$

Proof. Let $\tau \in \Gamma \cap \mathscr{W}$. Thus, $\tau(\Delta) = \Delta$ because $\tau \in \Gamma$. Then, by 4.1, since $\tau \in \mathscr{W}$, $\tau(\Delta) = \Delta \implies \tau = 1$.

Next, consider $\tau \in \operatorname{Aut}\Phi$. Then, $\tau(\Delta)$ is another simple system of Φ, so there exists a $\sigma \in \mathscr{W}$ such that $\sigma\tau(\Delta) = \Delta$ by 4.1. Thus, $\sigma\tau \in \Gamma$ and $\sigma \in \mathscr{W}$ together gives $\tau \in \mathscr{W}\Gamma$. $\square$

Now, recall that for all $\tau \in \operatorname{Aut}\Phi$, $\langle \alpha, \beta \rangle = \langle \tau(\alpha), \tau(\beta) \rangle$ for all $\alpha, \beta \in \Phi$ by definition.

8.4. **Proposition.** *For a root system Φ, every $\tau \in \Gamma$ corresponds to a diagram automorphism of the Dynkin diagram associated to Φ.*

Proof. As noted above, for $\tau \in \Gamma$, $\langle \alpha, \beta \rangle = \langle \tau(\alpha), \tau(\beta) \rangle$, and $\tau(\Delta) = \Delta$, so τ will permute the vertex labels of the Dynkin diagram while still preserving the number of edges between them. Similarly, a diagram automorphism will permute Δ while preserving the number of edges, thus giving an element of Γ. Thus, we have the desired correspondance. $\square$

Thus, given that we know all the possible Dynkin diagrams, to fully understand the automorphisms, we need only understand the Weyl group structure of our root systems. To summarise our understanding, so far, we have the following presentation of the Weyl group

8.5. **Proposition.** *Let Φ be a root system with Weyl group $\mathscr{W}$, simple system $\Delta = \{\alpha_1, \dots, \alpha_n\}$, and Cartan matrix A. Then,*

$$\mathscr{W} = \left\langle \{\sigma_\alpha \mid \alpha \in \Delta\} \mid \sigma_\alpha^2 = 1,\ \textit{for } \beta \neq \pm\alpha, \sigma_\alpha\sigma_\beta = \begin{cases} (\sigma_\alpha\sigma_\beta)^2 = 1 & \textit{if } \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0 \\ (\sigma_\alpha\sigma_\beta)^3 = 1 & \textit{if } \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 1 \\ (\sigma_\alpha\sigma_\beta)^4 = 1 & \textit{if } \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 2 \\ (\sigma_\alpha\sigma_\beta)^6 = 1 & \textit{if } \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 3 \end{cases} \right\rangle$$

that is to say, $\mathscr{W}$ is generated by reflections of the simple roots with relations given by the Dynkin diagram of Φ.

Proof. The generators come from 4.1. Since $\alpha, \beta \in \Delta$, this means we can the following implications for θ, the angle between α and β.

$$\begin{cases} \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0 & \implies \theta = \frac{\pi}{2} \\ \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 1 & \implies \theta = \frac{2\pi}{3} \\ \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 2 & \implies \theta = \frac{3\pi}{4} \\ \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 3 & \implies \theta = \frac{5\pi}{6} \end{cases}$$

Thus, from a geometric perspective, $\sigma_\alpha\sigma_\beta$ is a rotation by 2θ on the root system. Thus, the relations immediately follow. $\square$

This immediately tells us the information we need to understand the Weyl groups of irreducible root systems.

8.6. **Example.** Let $\Phi = A_n$ with simple system $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$. Then, the Weyl group generators act on the set $\{e_1, \dots, e_n\}$ by

$$\sigma_{e_i - e_{i+1}}(e_k) = e_k - 2\frac{(e_k, e_i - e_{i+1})}{(e_i - e_{i+1}, e_i - e_{i+1})}(e_i - e_{i+1}) = e_k - (\delta_{k,i} - \delta_{k,i+1})(e_i - e_{i+1})$$

and so $\sigma_{e_i - e_{i+1}}$ sends

$$e_i \mapsto e_{i+1}, e_{i+1} \mapsto e_i, e_k \mapsto e_k \text{ for } k \neq i, i+1$$

Thus, $f \colon \mathscr{W} \twoheadrightarrow S_n$ via $\sigma_{e_i - e_{i+1}} \mapsto (i, i+1)$, but if some $\sigma \in \mathscr{W}$ has $\sigma(e_k) = e_k$ for all k, then $\sigma(\Delta) = \Delta$ and thus $\sigma = id$, so $\ker f = \{id\}$ and thus $\mathscr{W} \cong S_n$.

8.7. **Example.** Let $\Phi = B_n$ with simple system $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_n\}$. Then, $\Phi = \{e_i \pm e_j\} \cup \{\pm e_i\}$. Let $\sigma \in \mathscr{W}$ act on the set of pairs of vectors $\{\pm e_1, \dots, \pm e_n\}$. Then, σ induces a permutation on this set, and so we have a homomorphism $f\colon \mathscr{W} \to S_n$. Consider that $\sigma_{e_i} \in \ker f$ since all the e_i are orthogonal. Furthermore, if $\tau \in \ker f$, then, by the orthogonality of e_i's, we can factor (not necessarily uniquely) $\tau = \sigma_{e_{i_1}} \cdots \sigma_{e_{i_k}}$. Thus, $\ker f = \langle \sigma_{e_i} \mid 1 \leq i \leq n \rangle$ with all generators commuting. Thus, $\ker f \cong (\mathbb{Z}/2\mathbb{Z})^n$ and, since f must be surjective by virtue of the fact that we could have picked any short root in Δ and the Weyl group acts transitively on the set of simple systems, we have short exact sequence of groups

I am skeptical about this step.

$$1 \to (\mathbb{Z}/2\mathbb{Z})^n \to \mathscr{W} \xrightarrow{f} S_n \to 1$$

Furthermore, using a geometric argument, one sees that $f(\sigma_{e_i - e_{i+1}}) = (i, i+1)$. From the A_n example above, we also know that $\langle \sigma_{\alpha_1}, \dots, \sigma_{\alpha_n} \rangle \cong S_n \leq \mathscr{W}$, so we have a splitting $g\colon S_n \to \mathscr{W}$ given by

$$(i, i+1) \mapsto \sigma_{e_i - e_{i+1}}$$

Thus, our split short exact sequence gives us $\mathscr{W} \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. The C_n case is analogous by symmetry.

8.8. **Example.** For $\Phi = D_n = \{\pm e_i \pm e_j \mid 1 \leq i, j < n, i \neq j\}$ with $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$, consider the action of $\sigma_{e_i - e_{i+1}}$ on the set of roots. By direct computation

$$\begin{cases} e_i \mapsto e_{i+1} \\ e_{i+1} \mapsto e_i \\ e_k \mapsto e_k & k \neq i, j \end{cases}$$

and thus we have a straightforward embedding of the symmetric group $S_n \hookrightarrow \mathscr{W}$ via $(i, i+1) \mapsto \sigma_{e_i - e_{i+1}}$. We also compute that, for $\sigma_{e_{n-1} + e_n}$,

$$\begin{cases} e_{n-1} \mapsto -e_n \\ e_n \mapsto -e_{n-1} \end{cases}$$

Thus, $\sigma_{e_{n-1} - e_n} \sigma_{e_{n-1} + e_n}$ will send $e_{n-1} \mapsto -e_{n-1}$ and $e_n \mapsto -e_n$. From this element, we are able to create elements which change (only) an even number of signs on the standard basis vectors of $\mathbb{R}^n$ but that do not permute them. Because of this even-ness restriction, there can only be 2^{n-1} such possible transformations, all of which are realized. Thus, looking at $f\colon \mathscr{W} \to S_n$ where $f(w)$ is sent to the permutation w induces on the set $\{\pm e_1, \dots, \pm e_n\}$, we get (split) short exact sequence

$$1 \to \ker f \to \mathscr{W} \to S_n \to 1$$

where $\ker f$ must consist of those 2^{n-1} sign changing permutations that do not permute the basis vectors. Since these actions commute, we have $\ker f \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$ and so $\mathscr{W} \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n$ by the splitness of the short exact sequence.

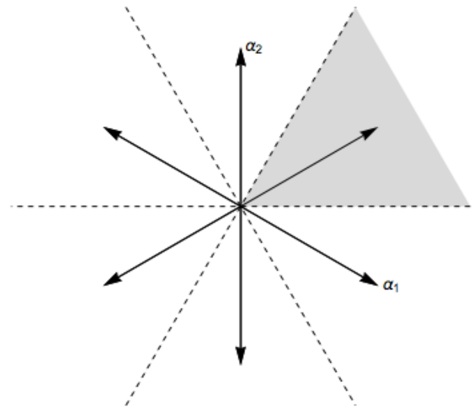


FIGURE 4. Weyl Chambers for Root System A_2 from [WMM18]

9. Weyl Chambers

In understanding the geometry of root systems, we have made great use of the set of hyperplanes orthogonal to each root. However, it is worth considering the complement to this collection of hyperplanes. Such a complement will be disconnected, and so we can consider the connected components of this complement.

9.1. **Definition.** Given a root system Φ of Euclidean space E, we call the connected components of $E \backslash \bigcup_{\alpha\in\Phi} P_\alpha$ the (open) *Weyl chambers* of E, where P_α is the hyperplane orthogonal to α.

9.2. **Example.** In Figure 4, the root system A_2 is drawn with associated hyperplanes given by the dashed lines. Thus, there are 6 Weyl chambers. The shaded chamber above corresponds to the choice of simple system $\{\alpha_1, \alpha_2\}$ indicated in the figure.

9.3. **Proposition.** *Given a root system Φ of Euclidean space E*

(a) *Each regular $\gamma \in E$ belongs to precisely one Weyl chamber.*
(b) *Weyl chambers are in natural one-to-one correspondence with bases.*
(c) *The Weyl group acts naturally on the set of Weyl chambers, and this action is free and transitive.*

Proof. Part (a) is because $\gamma \in E$ is regular precisely when it does not lie in a hyperplane P_α. Thus, if two regular vectors γ, γ' are in the same Weyl chamber, they are both on the same sides of all the P_α's. Therefore, recalling the notation in the proof of 2.22, $\Delta(\gamma) = \Delta(\gamma')$. Thus, each Weyl chamber associates to a unique base and, furthermore, by 2.22, each base is of the form $\Delta(z)$ for some regular $z \in E$ which must lie in precisely one Weyl chamber. Thus, we have established our correspondance.

For part (c), the action of the Weyl group is given by reflecting the Weyl chamber across the appropriate hyperplane. The transitivity and free-ness of this action comes from the correspondance with a choice of base along with theorem 4.1. □

9.4. **Definition.** Given a choice of Δ, we call the Weyl chamber corresponding to this choice consisting of all $v \in E$ such that $(v, \alpha) > 0$ for all $\alpha \in \Delta$ the *fundamental Weyl chamber relative to* Δ. Geometrically, this says that the fundamental Weyl chamber consists of those vectors forming an acute angle with all of the simple roots.

At first glance, it is not apparent what value the Weyl chamber perspective gives over the choice of base persepctive, but there are some useful techniques that use closure arguments, such as

9.5. **Theorem.** *Fix a Weyl chamber C. Then, for all $v \in E$, the Weyl group orbit of v contains exactly one point in the closure $\overline{C}$ of C.*

Proof. Consider the closure $\overline{C}$ of C. By construction, every vector $v \in E$ must lie in the closure of some chamber and, by 9.3(c), $\mathscr{W}$ acts transitively on the chambers. Thus, each orbit of $\mathscr{W}$ on V intersects $\overline{C}$.

Now, let $v_1, v_2 \in \overline{C}$ with $w(v_1) = v_2$ for some $w \in \mathscr{W}$. If $\ell(w) = 0$, then $w = 1$ and so $v_1 = v_2$. Now, let us proceed by assuming that, if $v_1, v_2 \in \overline{C}$ and $w(v_1) = v_2$ for some $w \in \mathscr{W}$ with $\ell(w) < n$, then $v_1 = v_2$. Then, if $w(v_1) = v_2$ for $\ell(w) = n$, we know there is at least one $\alpha_i \in \Delta$ such that $w(\alpha_i) \in \Phi^-$. Therefore, we have

$$0 \leq \langle v_1, \alpha_i \rangle = \langle v_2, w(\alpha_i) \rangle \leq 0 \Longrightarrow \langle v_i, \alpha_i \rangle = 0$$

Therefore, $\sigma_{\alpha_i}(v_1) = v_1$ and thus $w\sigma_{\alpha_i}(v_1) = v_2$. However, the only positive root that becomes negative under the action of σ_{α_i} is α_i. Therefore, the positive roots made negative by w and $w\sigma_{\alpha_i}$ are the same except for α_i, which is made negative only by w. Therefore,

$$\ell(w) = \ell(ws_{\alpha_i}) + 1 \Longrightarrow \ell(ws_{\alpha_i}) = \ell(w) - 1$$

and so $v_1 = v_2$ by the inductive hypothesis. □

An immediate consequence of the second part of this proof is the following

9.6. **Lemma.** [Hum72, Lemma 10.3B p 52] *If $v_1, v_2 \in \overline{C}$ such that $w(v_1) = v_2$ for some $w \in \mathscr{W}$, then w is a product of simple reflections which fix v_1. In particular, $v_1 = v_2$.*

10. Weights

Let Λ_W be a collection of vectors $\lambda \in E$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$. We call the elements of Λ_W *weights*. We note that Λ_W is endowed with a subgroup structure with vector addition as the operation since $\langle \lambda_1 + \lambda_2, \alpha \rangle = \langle \lambda_1, \alpha \rangle + \langle \lambda_2, \alpha \rangle$.

10.1. **Definition.** We say the *root lattice* is the subgroup of Λ_W generated by Φ. We denote this Λ_R.

Note that Λ_R is actually a lattice since it is a $\mathbb{Z}$-span of a basis of E.

10.2. **Definition.** Fix a base $\Delta \subseteq \Phi$. We say that $\lambda \in \Lambda_W$ is *dominant* if all the integers $\langle \lambda, \alpha \rangle$ are nonnegative for all $\alpha \in \Delta$. If they are all positive, then λ is *strongly dominant*.

We denote by Λ_W^+ the set of all dominant weights.

10.3. **Definition.** Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Then, we know that $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ forms a basis of E. Then, if we have the dual basis to this basis, $\{\omega_1, \ldots, \omega_n\}$, we can check that the ω_i are all dominant weights which we will call the *fundamental dominant weights (relative to Δ)*. In otherwords, the fundamental dominant weights are such that $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$.

Check this definition with the coroots.

10.4. **Definition.** We say that Λ_W/Λ_R is the *fundamental group* of Φ.

To see that Λ_W/Λ_R is a finite group, we can view both Λ_W and Λ_R as $\mathbb{Z}$-modules. Then, $\Lambda_W \cong \mathbb{Z}^m$ where m is the dimension of Λ_W. By definition, Λ_R is a submodule of Λ_W but has the same dimension. Both are free. So, the quotient will be some finite group. More concretely, one notes that the Cartan matrix is a change of basis matrix, so to write the ω_i in terms of the α_i, we can invert the Cartan matrix. Thus,

10.5. **Proposition.** *Given Cartan matrix M of a root system Φ,*

(a) The transpose of the Cartan matrix expresses the simple roots as linear combinations of the fundamental weights.
(b) The index of Λ_R in Λ_W is given by $\det C$.

Proof. Write $\alpha_i = \sum_j c_{i,j}\omega_j$ for $c_{i,j} \in \mathbb{Z}$. Then, we see

$$M_{i,k} = \langle \alpha_i, \alpha_k \rangle = \sum_j c_{i,j} \langle \omega_j, \alpha_k \rangle = c_{i,k} \tag{10a}$$

giving us

$$\alpha_i = \sum_j c_{i,j}\omega_j = \sum_j M_{i,j}\omega_j$$

Thus, the Cartan matrix is a change of basis matrix to write the α_i's in terms of the ω_i's. To go the other direction, we simply invert M, which is a matrix with entries in $\mathbb{Z}$, so $M^{-1} = \frac{1}{\det M}\operatorname{adj}(M)$, and thus $\det M \cdot \omega_i \in \Lambda_R$. $\square$

10.6. **Example.** Consider the root system A_2 with $\Delta = \{\alpha, \beta\}$. Then, since the Cartan matrix is

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

We have

$$\begin{cases} \alpha &= 2\omega_1 - \omega_2 \\ \beta &= -\omega_1 + 2\omega_2 \end{cases} \implies \begin{cases} \omega_1 &= \frac{2}{3}\alpha + \frac{1}{3}\beta \\ \omega_2 &= \frac{1}{3}\alpha + \frac{2}{3}\beta \end{cases}$$

Compute the index for each type.

Recall the partial order on roots 2.13. We can extend this to a partial order on weights as follows.

10.7. **Definition.** Given $\lambda, \mu \in \Lambda_W$, we say $\mu \prec \lambda$ if and only if $\lambda - \mu$ is a sum of positive roots.

In some ways, dominant weights serve as a good notion for a base of the weights, by virtue of the following proposition.

10.8. **Proposition.** [Hum72, p 68] *Each weight is conjugate under $\mathscr{W}$ to one and only one dominant weight. If λ is dominant, then $w\lambda \prec \lambda$ for all $w \in \mathscr{W}$, and if λ is strongly dominant, then $w\lambda = \lambda$ only when $w = 1$.*

Proof. This result follows from 9.5 and 9.6 since all the fundamental weights lie in the closure of the fundamental Weyl chamber. $\square$

However, dominant weights are not not necessarily maximal with respect to the partial order in general.

10.9. **Example.** Consider the weight lattice for A_2 with $\Delta = \{\alpha, \beta\}$, the standard choice. Then, $\lambda = 3\alpha + \beta$ has

$$\langle \lambda, \beta \rangle = -3 + 2 = -1$$

so λ is not a dominant weight. However, $\lambda - \alpha$ has

$$\langle \lambda - \alpha, \beta \rangle = -2 + 2 = 0, \quad \langle \lambda - \alpha, \alpha \rangle = 4 - 1 = 3$$

and thus is a dominant weight. Therefore, $\lambda - \alpha \prec \lambda$ since $\lambda - (\lambda - \alpha) = \alpha \in \Delta$, but λ is not dominant.

However, given a fixed dominant weight, the space of all such examples is finite.

10.10. **Lemma.** [Hum72, p 70] *Let λ be a dominant weight. Then, the number of dominant weights μ such that $\mu \prec \lambda$ is finite.*

Proof. Observe that, since $\lambda - \mu$ is a sum of positive roots,

$$0 \leq (\lambda + \mu, \lambda - \mu) = (\lambda, \lambda) - (\mu, \mu)$$

so $\mu \in \{x \in E \mid (x, x) \leq (\lambda, \lambda)\}$, which is compact. Furthermore, the dominant weights is a discrete set, so there can only be a finite number of such μ. $\square$

10.11. **Proposition.** [Hum72, p 70] *Recall $\rho = \frac{1}{2} \sum_{\alpha \in \Phi, \alpha \succ 0} \alpha$ (2.15). As discussed before, $\rho \in \Lambda_W$. Furthermore,*

(a) ρ is a (strongly) dominant weight and, in fact,

$$\rho = \sum_{i=1}^{n} \omega_i$$

(b) For $\mu \in \Lambda_W^+$ and $\nu = w^{-1}(\mu)$ for $w \in \mathscr{W}$ we have

$$(\nu + \rho, \nu + \rho) \leq (\mu + \rho, \mu + \rho)$$

with equality only if $\nu = \mu$.

Proof. For (a), observe

$$\sigma_i \rho = \rho - \alpha_i \implies (\rho - \alpha_i, \alpha_i) = (\sigma_i^2 \rho, \sigma_i \alpha_i) = (\rho, -\alpha_i) \implies 2(\rho, \alpha_i) = (\alpha_i, \alpha_i) = 2$$

Thus, $\langle \rho, \alpha_i \rangle = 1$, so

$$\rho \overset{10a}{=} \sum_i \langle \rho, \alpha_i \rangle \omega_i = \sum_i \omega_i$$

For (b), observe

$$\begin{aligned}
(\nu + \rho, \nu + \rho) &= (w(\nu + \rho), w(\nu + \rho)) \\
&= (\mu + w\rho, \mu + w\rho) \\
&= (\mu + \rho, \mu + \rho) - 2(\mu, \rho - w\rho) \\
&\leq (\mu + \rho, \mu + \rho) && \text{since } \mu \in \Lambda_W^+ \text{ and } \rho - w\rho \text{ is a sum of positive roots}
\end{aligned}$$

with equality only if $(\mu, \rho - w\rho) = 0$ by 10.8. However, this gives us

Why is the last equality true?

$$(\mu, \rho) = (\mu, w\rho) = (\nu, \rho) \implies (\mu - \nu, \rho) = 0$$

but $\mu - \nu$ is a sum of positive roots and ρ is strongly dominant, so it must be that $\mu = \nu$. $\square$

Finally, we give a discussion of the notion of saturated sets of weights. This idea is important for those interested in understanding how [Hum72] proves results on irreducible finite dimensional semisimple Lie algebra representations over an algebraically closed field of characteristic 0 in section 21.

10.12. **Definition.** We call a subset $S \subseteq \Lambda_W$ *saturated* if, for all $\lambda \in S$, $\alpha \in \Phi$, and i between 0 and $\langle \lambda, \alpha \rangle$, the weight $\lambda - i\alpha \in S$. We say a saturated set of weights has a *highest weight* λ if $\lambda \in S$ and $\mu \prec \lambda$ for all $\mu \in S$.

10.13. **Proposition.** *Given a saturated set of weights S, we have the following*

(a) S is stable under $\mathscr{W}$, that is, $\sigma_\alpha \lambda = \lambda - \langle \lambda, \alpha \rangle \alpha \in S$ for all $\lambda \in S, \alpha \in \Phi$.
(b) If S has a highest weight, then S must be finite.
(c) If S has highest weight λ, then if $\mu \in \Lambda_W^+$ and $\mu \prec \lambda$, then $\mu \in S$.
(d) (Optional) If S has highest weight λ, then if $\mu \in S$, then

$$(\mu + \rho, \mu + \rho) \leq (\lambda + \rho, \lambda + \rho)$$

with equality only if $\mu = \lambda$.

Proof. Part (a) is immediate from the definition of S being saturated. For part (b), we note that the number of dominant weights less than the highest weight, say $\lambda \in S$, must be finite by 10.10. Thus, since our root system is also finite, each dominant weight will only have a finite number of other weights to make S saturated.

For (c), we take an arbitrary $\mu' = \mu + \sum_{\alpha\in\Delta} k_\alpha \alpha \in S$ with $k_\alpha \in \mathbb{Z}^+$. One reduces one of the k_α while still remaining in S until only $\mu \in S$.

For (d), we can reduce to μ being dominant using part (c). If $\mu = \lambda - \pi$ for π a sum of positive roots, we have

$$\begin{aligned}(\lambda+\rho,\lambda+\rho)-(\mu+\rho,\mu+\rho) &= (\lambda+\rho,\lambda+\rho)-(\lambda+\rho-\pi,\lambda+\rho-\pi)\\ &= (\lambda+\rho,\pi)+(\pi,\mu+\rho)\\ &\geq (\lambda+\rho,\pi)\\ &\geq 0\end{aligned}$$

where the inequalities follow because $\mu+\rho$ and $\lambda+\rho$ are dominant. Because $\lambda+\rho$ is strongly dominant, equality holds only if $\pi = 0$. □

The proposition above gives us a good picture of what a saturated set of weights S looks like if it has a highest weight. Namely, S consists of all dominant weights lower than or equal to highest weight λ and the conjugates of all these weights under $\mathscr{W}$

10.14. **Corollary.** [Hum72, p 71] *Given $\lambda \in \Lambda_W^+$, there exists precisely one set of saturated weights S such that λ is a highest weight of S.*

Proof. Given the exposition above, for $\lambda \in \Lambda_W^+$, we see that at most one saturated set of weights can exist with λ the highest weight. Conversely, given $\lambda \in \Lambda_W^+$, let us construct a saturated set of weights where λ is a highest weight.

(a) Take S to be the set of all dominant weights below λ and their $\mathscr{W}$-conjugates.
(b) Observe that S is stable under the $\mathscr{W}$. Then, $w\mu \prec \mu$ for all dominant weights μ and all $w \in \mathscr{W}$ by 10.8. Thus,
$$S = \{\mu \in \Lambda_W \mid w\mu \prec \lambda \text{ for all } w \in \mathscr{W}\}$$
Take $\alpha \in \Phi$. For $\mu' = \mu - k\alpha$ for k between 0 and $\langle \mu, \alpha\rangle$, we wish to show $\mu' \in S$.
(c) Since μ' is on the α-chain of μ, $w(\mu')$ is on the $w(\alpha)$-chain from $w(\mu)$ to $w(\mu - \langle\mu,\alpha\rangle\alpha) = w(\mu) - \langle\mu,\alpha\rangle w(\alpha)$. Thus, we have
$$w\mu \prec w\mu' \prec w\mu - \langle\mu,\alpha\rangle w(\alpha) \text{ or } w\mu - \langle\mu,\alpha\rangle w(\alpha) \prec w\mu' \prec w\mu$$
(d) Note, howver, $w\mu$ and $w\mu - \langle\mu,\alpha\rangle w(\alpha)$ are both in the $\mathscr{W}$-orbit of μ, so by the description of S above, they are both lower than λ.

Therefore, $w\mu' \prec \lambda$ for all $w \in \mathscr{W}$, thus giving us $\mu' \in S$ by our description of S.

$\square$

11. Appendix: Summary Information

Throughout, we denote ϵ_i for the ith standard basis vector of $\mathbb{R}^n$. Refer to Section 3 for definitions of these root systems.

11.1. Type A_n.

11.1. Proposition. *The root system $A_n \subseteq \mathbb{R}^{n+1}$ has*

(a) $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n+1\}$ *and* $|\Phi| = n(n+1)$.
(b) The canonical choice of simple roots is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n+1\}$$

and thus $\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n+1\}$.
(c) From the action on $\{e_1, \ldots, e_n\}$, *one can infer* $\mathscr{W} = S_n$.
(d) The fundamental weights are given by $\omega_i = \epsilon_1 + \cdots + \epsilon_i$ *for* $1 \leq i \leq n$ *and*

$$\rho = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n$$

11.2. Type B_n.

11.2. Proposition. *The root system $B_n \subseteq \mathbb{R}^n$ has*

(a) $\Phi = \{\pm(\epsilon_i+\epsilon_j) \mid 1 \leq i < j \leq n\} \cup \{\pm(\epsilon_i-\epsilon_j) \mid 1 \leq i < j \leq n\} \cup \{\pm\epsilon_i \mid 1 \leq i \leq n\}$ *and so* $|\Phi| = 2n(n-1) + 2n = 2n^2$.
(b) The canonical choice of simple roots is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i < n\} \cup \{\epsilon_n\}$$

and thus $\Phi^+ = \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{\epsilon_i \mid 1 \leq i \leq n\}$.
(c) Since the Dynkin diagram has no graph automorphisms, it must be that $\mathscr{W} = \operatorname{Aut}\Phi$. *Furthermore, automorphisms can permute* $\{\epsilon_1, \ldots, \epsilon_n\}$ *and change any of the signs, thus giving* $\mathscr{W} = (\mathbb{Z}/2)^n \rtimes S_n$.
(d) The fundamental weights are given by

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i, (i < n) \quad \omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$$

and

$$\rho = \frac{1}{2}((2n-1)\epsilon_1 + (2n-3)\epsilon_2 + \cdots + \epsilon_n)$$

11.3. **Type C_n.**

11.3. **Proposition.** *The root system $C_n \subseteq \mathbb{R}^n$ has*

(a) $\Phi = \{\pm(\epsilon_i+\epsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm(\epsilon_i-\epsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i\}$ *and so* $|\Phi| = 2n(n-1) + 2n = 2n^2$ *just as in* B_n.

(b) The canonical choice of simple roots is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} \mid 1 \le i < n\} \cup \{2\epsilon_n\}$$

and thus $\Phi^+ = \{\epsilon_i + \epsilon_j \mid 1 \le i < j \le n\} \cup \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\} \cup \{2\epsilon_i \mid 1 \le i \le n\}$.

(c) C_n *presents a somewhat dual situation to* B_n *and thus also has Weyl group* $\mathscr{W} = (\mathbb{Z}/2)^n \rtimes S_n$.

(d) The fundamental weights are given by

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i, (i \le n)$$

and

$$\rho = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n$$

11.4. **Type D_n.**

11.4. **Proposition.** *The root system $D_n \subseteq \mathbb{R}^n$ has*

(a) $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n\} \cup \{\pm(\epsilon_i + \epsilon_j) \mid 1 \le i < j \le n\}$ *and so* $|\Phi| = n(n-1) + 2\frac{n(n-1)}{2} = 2n(n-1)$.

(b) The canonical choice of simple roots is

$$\Delta = \{\epsilon_i - \epsilon_{i+1} \mid 1 \le i < n\} \cup \{\epsilon_{n-1} + \epsilon_n\}$$

and thus $\Phi^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le n\}$.

(c) D_n *has Weyl group* $\mathscr{W} = (\mathbb{Z}/2)^{n-1} \rtimes S_n$.

(d) The fundamental weights are given by

$$\begin{cases} \omega_i = \epsilon_1 + \cdots + \epsilon_i & i < n-1 \\ \omega_{n-1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n) & \\ \omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n) & \end{cases}$$

and thus

$$\rho = (n-1)\epsilon_1 + (n-2)\epsilon_2 + \cdots + \epsilon_{n-1}$$

References

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