

# A Raising Operator Formula for Macdonald Polynomials (and The Journey There)

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joint work with J. Blasiak, M. Haiman, J. Morse, and A. Pun

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- ① **Background on symmetric functions and Macdonald polynomials**
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ A new formula for Macdonald polynomials

# Symmetric Polynomials

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## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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- E.g. for  $n = 3$ ,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

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- $\Lambda$  is a  $\mathbb{Q}(q, t)$ -algebra.

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$\implies$  any basis of symmetric functions is indexed by partitions.

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For  $\lambda = (2, 1)$ ,

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- $\{s_\lambda\}_\lambda$  forms a basis for  $\Lambda_{\mathbb{Q}}$ .

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## Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in  $\mathbb{N}$ ) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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Remark:  $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]^{S_3})$  is a “regular representation.”

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Solution: irreducible  $S_n$ -representation of polynomials of degree  $d \mapsto q^d s_\lambda$   
(graded Frobenius)

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Answer: Hall-Littlewood polynomial  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .
- Does there exist a family of  $S_n$ -regular representations whose bigraded Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  with  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .

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# Garsia-Haiman modules

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

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- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ .

# Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible $V_\lambda$	$\text{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman $M_\lambda$	??

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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## Question

What symmetric function is the bigraded Frobenius characteristic of  $DH_n$ ?

Frobenius characteristic of  $DH_3$



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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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Operator  $\nabla$

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X; q, t),$$

where  $n(\lambda) = \sum_i (i-1)\lambda_i$  and  $\lambda^*$  is the transpose partition to  $\lambda$ .

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Theorem (Haiman, 2002)

*The bigraded Frobenius characteristic of  $DH_n$  is given by  $\nabla e_n$ .*

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$s_\lambda(X)$	Irreducible $V_\lambda$	SSYT( $\lambda$ )
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman $M_\lambda$	??
$\nabla e_n$	$DH_n$	Shuffle theorem

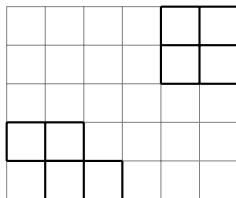
# Outline

- ① Background on symmetric functions and Macdonald polynomials
- ② **Shuffle theorems, combinatorics, and LLT polynomials**
- ③ A new formula for Macdonald polynomials

# LLT Polynomials

Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape =  $\lambda \setminus \mu$ )

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-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5



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- A pair  $(a, b) \in \nu$  is *attacking* if  $a$  precedes  $b$  in reading order and
  - $\text{content}(b) = \text{content}(a)$ , or
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$$\nu = \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

				$b_3$	$b_6$
				$b_5$	$b_8$
$b_1$	$b_2$				
	$b_4$	$b_7$			

Attacking pairs:  $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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Let  $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$  be a tuple of skew shapes. (Skew shape  $= \lambda \setminus \mu$ )

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The *LLT polynomial* indexed by a tuple of skew shapes  $\nu$  is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

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$$\text{inv}(T) = 4, \quad \mathbf{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

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- $\mathcal{G}_\nu$  is Schur-positive for any tuple of skew shapes  $\nu$  [Grojnowski-Haiman, 2007].

# A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
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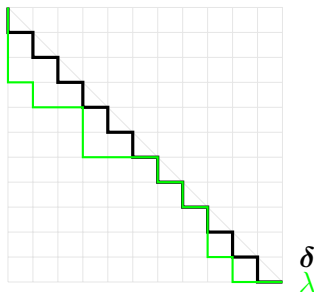
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- $\omega$  an automorphism of symmetric functions:  $\omega(s_{\lambda}) = s_{\lambda^*}$  for  $\lambda^* =$  transpose of  $\lambda$ .

# Dyck paths

## Dyck paths

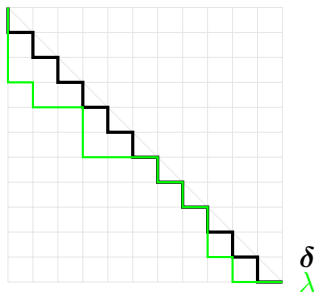
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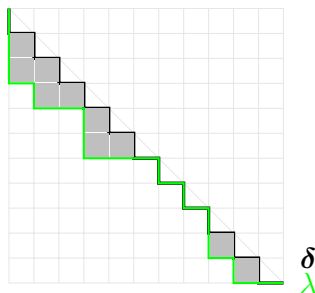


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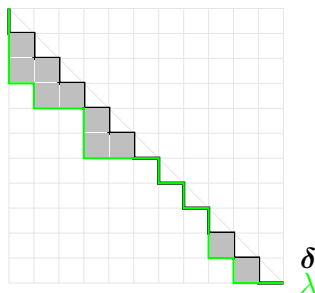


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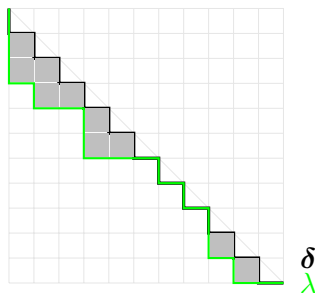


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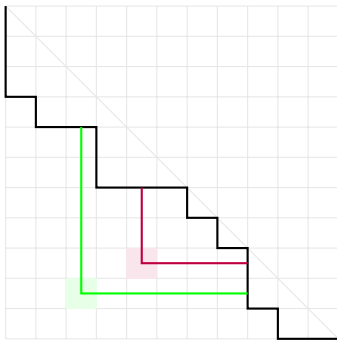
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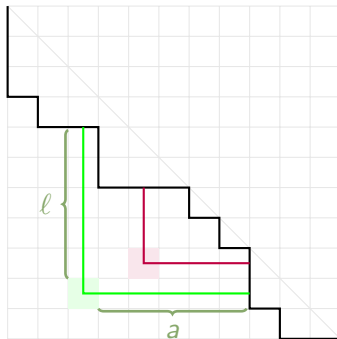
# dinv

$\text{dinv}(\lambda) = \#$  of balanced hooks in diagram below  $\lambda$ .





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Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{l}{a+1} < 1 - \epsilon < \frac{l+1}{a}, \quad \epsilon \text{ small.}$$

## Example $\nabla e_3$

$$\lambda \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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






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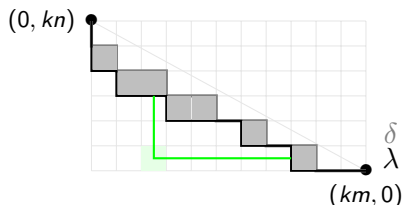
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# Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra  $\mathcal{E}$  of the Hall algebra of coherent sheaves on an elliptic curve over  $\mathbb{F}_p$ .

The *elliptic Hall algebra*  $\mathcal{E}$  is generated by subalgebras  $\Lambda(X^{a,b})$  isomorphic to the ring of symmetric functions  $\Lambda$  over  $\mathbb{k} = \mathbb{Q}(q, t)$ , one for each coprime pair  $(a, b) \in \mathbb{Z}^2$ , along with an additional central subalgebra.

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Can be difficult to work with in general. Can we make it more explicit?

# Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$  denotes the set of positive roots for  $GL_n$ , where  $\alpha_{ij} = \epsilon_i - \epsilon_j$ .

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A root ideal  $\Psi \subseteq R_+$  is an upper order ideal of positive roots.

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		(23)	(24)	(25)
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$\Psi = \text{Roots above Dyck path}$



# Schur functions revisited

- Convention:  $h_0 = 1$  and  $h_d = 0$  for  $d < 0$ .
- For any  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$ , set

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Precisely, for  $\rho = (n-1, n-2, \dots, 1, 0)$ ,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$  = weakly decreasing sequence obtained by sorting  $\beta$ ,
- $\operatorname{sgn}(\beta)$  = sign of the shortest permutation taking  $\beta$  to  $\operatorname{sort}(\beta)$ .

Example:  $s_{201} = 0$ ,  $s_{2-11} = -s_{200}$ .

# Weyl symmetrization

Define the *Weyl symmetrization operator*  $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$  by linearly extending

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## Example

$$\sigma(\mathbf{z}^{111} + \mathbf{z}^{201} + \mathbf{z}^{210} + \mathbf{z}^{3-11}) = s_{111} + s_{201} + s_{210} + s_{3-11} = s_{111} + s_{210} - s_{300}$$

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where  $z^{\alpha_{ij}} = z_i/z_j$  and  $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \dots$ .

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With  $n = 3$ ,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left( \frac{z^{111} (1 - qtz_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - qz_i/z_j) (1 - tz_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$



# Why?

Let  $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$  and  $R_+^0 = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$ .

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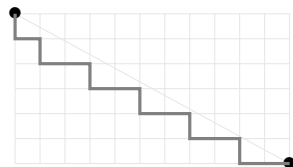
## Proposition

For  $(m, n) \in \mathbb{Z}^2$  coprime,

$$e_k[-MX^{m,n}] \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

for  $\mathbf{b} = (b_0, \dots, b_{km-1})$  satisfying  $b_i =$  the number of south steps on vertical line  $x = i$  of highest lattice path under line  $y + \frac{n}{m}x = n$ .

$\delta$  = highest Dyck path.



$\delta$

$$\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

# Results

Manipulating Catalanimal  $\implies$  a proof of the Rational Shuffle Theorem + a generalization.

## Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given  $r, s \in \mathbb{R}_{>0}$  such that  $p = s/r$  irrational, take  $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$  to be the south step sequence of highest path  $\delta$  under the line  $y + px = s$ .

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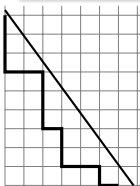
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$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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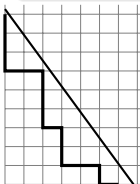
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$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line  $y + px = s$ ,



$\text{area}(\lambda)$  as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks}$   $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

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For which symmetric functions  $f$  can we find a Catalan animal such that  $f[-MX^{m,n}] \cdot 1 = \text{a Catalan animal}$ ?

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Answer: for  $f$  equal to any LLT polynomial!

Special case:  $\mathcal{G}_\nu[-MX^{1,1}] \cdot 1 = \nabla \mathcal{G}_\nu(X; q)$ .

# LLT Catalanimals

For a tuple of skew shapes  $\nu$ , the *LLT Catalanimal*  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  is determined by

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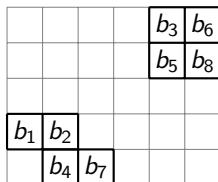
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- $\lambda$ : fill each diagonal  $D$  of  $\nu$  with  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$ .  
Listing this filling in reading order gives  $\lambda$ .

# LLT Catalanimals

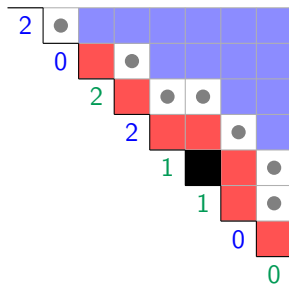
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$\nu$

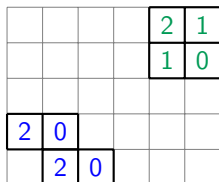


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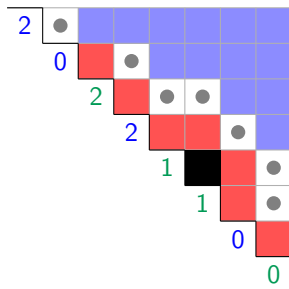
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$\lambda$ , as a filling of  $\nu$





## Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let  $\nu$  be a tuple of skew shapes and let  $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$  be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega H_\nu \\ &= c_\nu \omega \sigma \left( \frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some  $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$ .

# What about Macdonald polynomials?!

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# Outline

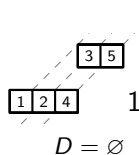
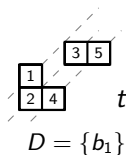
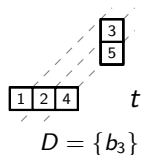
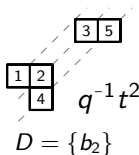
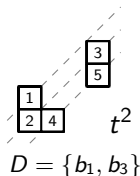
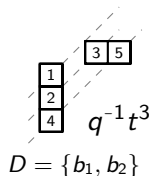
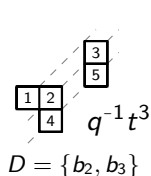
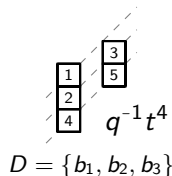
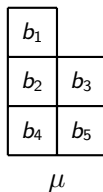
- ① Background on symmetric functions and Macdonald polynomials
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ **A new formula for Macdonald polynomials**

## Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X; q, t) = \sum_D \left( \prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q)$$

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- Collect terms to get  $\prod_{(b_i, b_j) \in V(\mu)} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$  factor for  $V(\mu)$  the set of vertical dominoes  $(b_i, b_j)$  in  $\mu$ .

$$\tilde{H}_\mu = \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\substack{\alpha_{ij} \in V(\mu)}} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

# The root ideal $R_\mu$

$b_1$		
$b_2$	$b_3$	
$b_4$	$b_5$	$b_6$
$b_7$	$b_8$	$b_9$

row reading order

$$b_1 \prec b_2 \prec \cdots \prec b_n$$

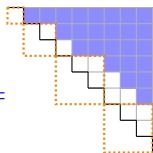
$$R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \},$$

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$$R_\mu \setminus \hat{R}_\mu \leftrightarrow V(\mu)$$

Example:

$$R_{3321} =$$



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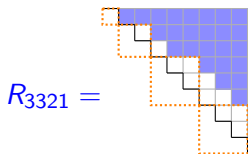
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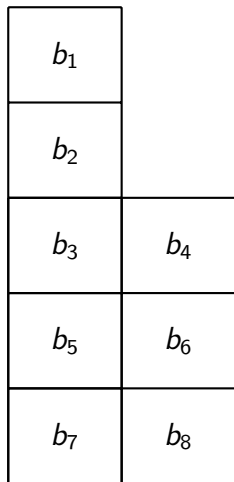
Example:



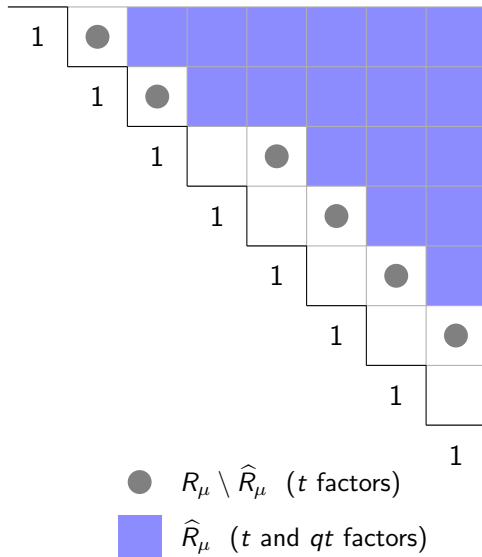
Remark

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right)$$

# Example



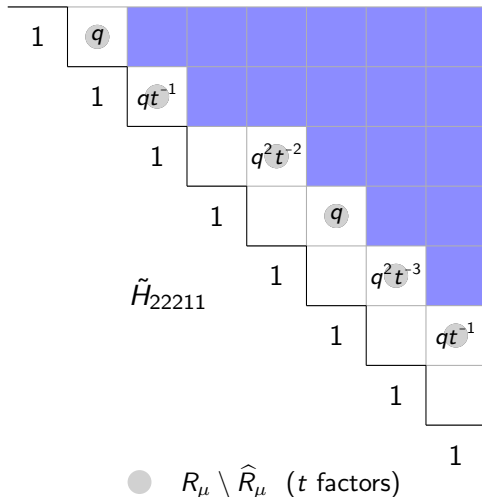
partition  $\mu = 22211$



# Example

$1 - q^{\frac{z_1}{z_2}}$	
$1 - qt^{-1} \frac{z_2}{z_3}$	
$1 - q^2 t^{-2} \frac{z_3}{z_5}$	$1 - q^{\frac{z_4}{z_6}}$
$1 - q^2 t^{-3} \frac{z_5}{z_7}$	$1 - qt^{-1} \frac{z_6}{z_8}$

numerator factors  $1 - q^{\text{arm}+1} t^{-\text{leg}} z_i/z_j$



●  $R_\mu \setminus \hat{R}_\mu$  ( $t$  factors)

■  $\hat{R}_\mu$  ( $t$  and  $qt$  factors)

$q = t = 1$  specialization

$$\begin{aligned}
 & \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=t=1} \omega \sigma \left( z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \hat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\
 & = \omega \sigma \left( \frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\
 & = \omega h_1^n \\
 & = e_1^n
 \end{aligned}$$



## A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

# A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega \sigma \left( (z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right)$$

## Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition  $\mu$  and positive integer  $s$ , the symmetric function  $\tilde{H}_\mu^{(s)}$  is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_\nu(X)$$

satisfy  $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$ .

# Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible $V_\lambda$	SSYT( $\lambda$ )
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman $M_\lambda$	HHL
$\nabla e_n$	$DH_n$	Shuffle theorem
$\tilde{H}_\lambda^{(s)}(X; q, t)$	??	??

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