Raising operators in Schubert Calculus

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Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Overview of Schubert Calculus Combinatorics (cont.)

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Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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- $f_\lambda = s_\lambda$, the Schur functions
- $s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$ (Jacobi-Trudi)
- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda+\epsilon_i-\epsilon_j}$

$$R_{1,3}\left(\begin{array}{c}\blacksquare\\\square\square\end{array}\right) = \square\square\blacksquare \quad R_{2,3}\left(\begin{array}{c}\blacksquare\\\square\end{array}\right) = \square\blacksquare$$

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Focus

K-theory and K-homology of the affine Grassmannian

Schubert Calculus Variations

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Focus

K-theory and K-homology of the affine Grassmannian

- Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

K-Theory of Affine Grassmannian

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1. K-theory classes of Grassmannian (not affine!) represented by "Grothendieck polynomials." We are interested in their dual:

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for Kh_γ an inhomogeneous analogue of h_γ.

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2. Homology classes of affine Grassmannian represented by k-Schur functions ($t = 1$)

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3. (Lam et al., 2010) leave open the question: what is a direct formulation of the K-homology representatives of the affine Grassmannian (K-k-Schur functions)?

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Lowering Operators and Root Ideals

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$$L_3\left(\begin{array}{c}\blacksquare\\ \end{array}\right) = \quad \quad L_1\left(\begin{array}{c}\\ \blacksquare\end{array}\right) = $$

- Root ideal Ψ: given by Dyck path.

$$\Psi = $$

(12)(13)(14)(15)
(23)(24)(25)
(34)(35)
(45)

Roots above Dyck path
Non-roots below

Affine K-Theory Representatives with Raising Operators

Definition

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) K h_\gamma$$

for $K h_\gamma$ an inhomogeneous analogue of h_γ.

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for Kh_γ an inhomogeneous analogue of h_γ.

Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$K(\Psi; \mathcal{L}; 54332)$

$= (1 - L_4)^2 (1 - L_5)^2$

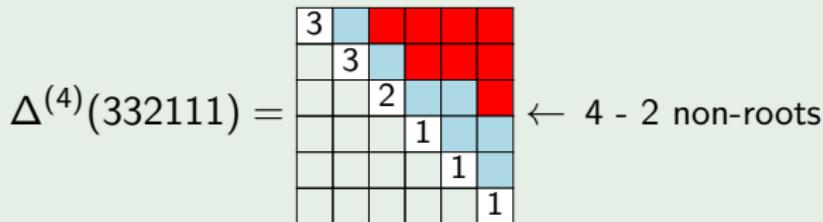$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) Kh_{54332}$

Affine K-Theory Representatives with Raising Operators

Definition

The *k-Schur root ideal*, $\Delta^{(k)}(\lambda)$ is the unique root ideal with $\lambda_i + \#$non-roots in row $i = k$.

Example

$k = 4, \lambda = 332111$

$$\Delta^{(4)}(332111) = \begin{array}{|c|c|c|c|c|c|}\hline 3 & & & & & \\\hline & 3 & & & & \\\hline & & 2 & & & \\\hline & & & 1 & & \\\hline & & & & 1 & \\\hline & & & & & 1 \\\hline\end{array}$$ ← 4 - 2 non-roots

Affine *K*-Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2019)

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For K-homology of affine Grassmannian,
$f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$ since this family satisfies the correct Pieri rule.

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Example

$$g_{332111}^{(4)} =$$

$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$

Property and Further Work

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Further work

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Dual Pieri rule: $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\mu^{(k)}$, $1 \le r \le k$
2. Branching rule: $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$

References

Thank you!

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Morse, Jennifer. 2011. *Combinatorics of the K-theory of affine Grassmannians*, Advances in Mathematics.