K-theoretic Catalan functions

George H. Seelinger (joint work with J. Blasiak and J. Morse)

Junior Mathematician Research Archive

ghs9ae@virginia.edu

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Overview

1. Schubert calculus
2. Catalan functions
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

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Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu$ for combinatorially understood Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$.

Schur polynomials and raising operators

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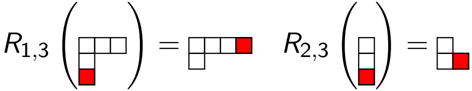

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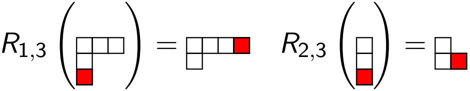

- Schur function $s_\lambda = \prod_{i<j}(1 - R_{ij}) h_\lambda$ (Jacobi-Trudi)

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
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K-theory and K-homology of the affine Grassmannian

Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

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2. Homology classes of affine Grassmannian represented by k-Schur functions ($t = 1$).
3. (Lam et al., 2010) leave open the question: what is a direct formulation of the K-homology representatives of the affine Grassmannian (K-k-Schur functions)?

Remember?

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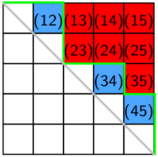
Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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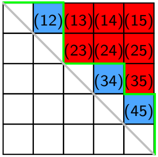

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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

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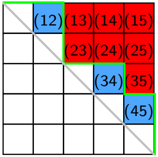

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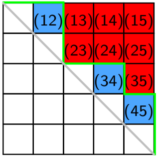

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Catalan functions

k-Schur root ideal for λ

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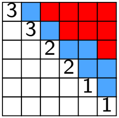
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

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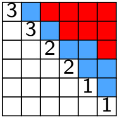
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

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Remark

(Blasiak et al., 2019) show results for k-Schur functions with parameter t, but $t = 1$ specialization is necessary for Schubert calculus.

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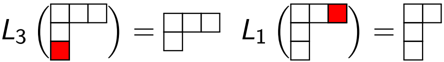
- Extra ingredient: lowering operators $L_j(h_\lambda) = h_{\lambda-\epsilon_j}$



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Example

non-roots of Ψ in blue, roots of $\mathcal{L}$ marked with $\bullet$

	(12)		$\bullet$	$\bullet$
			$\bullet$	$\bullet$
			(34)	
				(45)

$$\begin{aligned} &K(\Psi;\mathcal{L};54332) \\ &= (1-L_4)^2(1-L_5)^2 \\ &\cdot (1-R_{12})(1-R_{34})(1-R_{45})k_{54332} \end{aligned}$$

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Example

$$g_{332111}^{(4)} = \begin{array}{|c|c|c|c|c|c|}\hline 3 & & & \bullet & \bullet & \bullet \\ \hline & 3 & & & \bullet & \bullet \\ \hline & & 2 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \\ \hline \end{array} \quad \Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$$

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Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ *"branching coefficients" are alternating by degree, i.e. the* $b_{\lambda\mu}^{(k)}$ *in*

$$g_\lambda^{(k)} = \sum_\mu b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

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3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ??s_\mu^{(k)}$.

References

Thank you!

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