Diagonal Harmonics and Shuffle Theorems

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joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

ISU Algebra Seminar

27 October 2022

Symmetric Group

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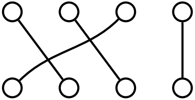

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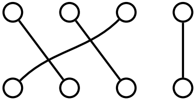

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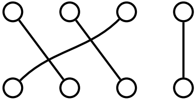

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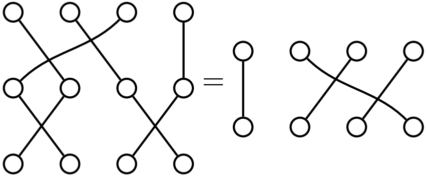

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- $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$

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$$\begin{aligned} e_1 = x_1 + x_2 + x_3 = & h_1 \\ e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad & h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 = x_1x_2x_3 \quad & h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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$$

- $\{f \in \mathbb{Q}[x_1, \ldots, x_n] \mid \sigma.f = f\ \forall \sigma \in S_n\}$ forms a vector space, $\Lambda_{\mathbb{Q}}$.

Combinatorics of Symmetric Polynomials

Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the $e_1, e_2, \ldots$, or in the $h_1, h_2, \ldots$

$$3h_2 h_1^2 - h_2^2 + 6h_3 h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of $\Lambda_{\mathbb{Q}}$?

Partitions

Definition

$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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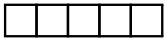
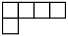
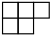
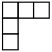
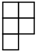
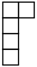
$5 \to$ 

$4 + 1 \to$ 

$3 + 2 \to$ 

$3 + 1 + 1 \to$ 

$2 + 2 + 1 \to$ 

$2 + 1 + 1 + 1 \to$ 

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For $\lambda = (2, 1)$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Schur functions

Associate a polynomial to SSYT(λ).

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- Schur functions form a basis for $\Lambda_{\mathbb{Q}}$

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Harmonic polynomials

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Explicitly, for

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$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

Harmonic polynomials

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$$e_1^3 = (x_1+x_2+x_3)^3 = s_{\ydiagram{1,1,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

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Schur basis expansion counts multiplicity of irreducible S_n fixed subspaces!

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Upshot

Via Frobenius characteristic map, questions about S_n-action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible S_n-fixed vector subspaces.

Upshot

Via Frobenius characteristic map, questions about S_n-action on vector spaces (representations) get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

Getting more information

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Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\substack{\ydiagram{2,1} \\ \deg=2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\substack{\ydiagram{2,1} \\ \deg=1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

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Solution: minimal S_n-fixed subspace of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

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Answer: "Hall-Littlewood polynomial" $H_{\ydiagram{1,1,1}}(X; q)$.

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- Does there exist a family of S_n-representations whose (bigraded) Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ satisfying $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$. (Still open!)

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^{r} \partial_{y_j}^{s}\right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2 t+qt^2+qt)\tilde{H}_{2,1}}{-q^2 t^2+q^3+t^3-qt} - \frac{q^3 \tilde{H}_3}{-q^3+q^2 t+qt-t^2}$$

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Operator ∇

$$\nabla \tilde{H}_\lambda(X;q,t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X;q,t)$$

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The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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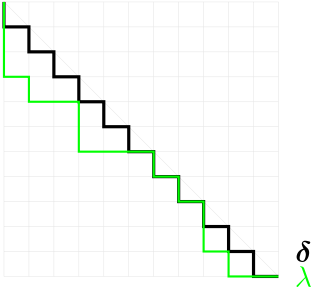
Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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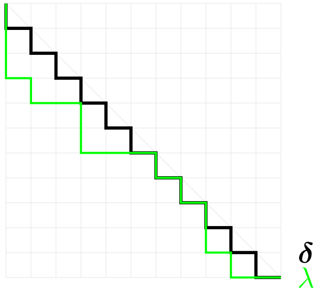

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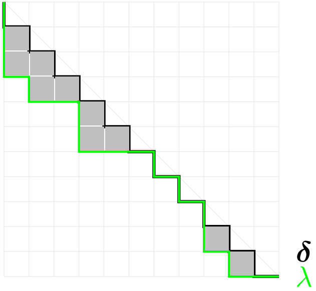

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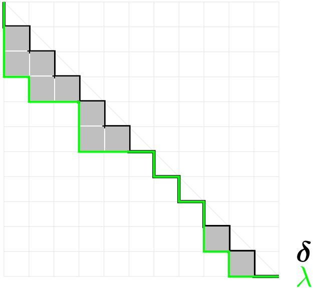

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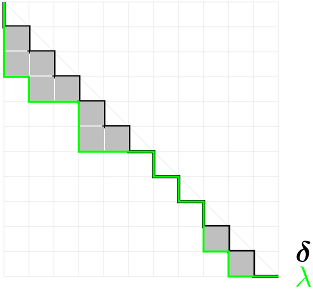

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dinv

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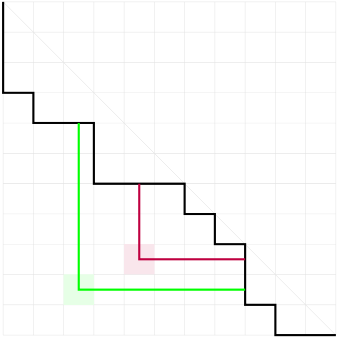

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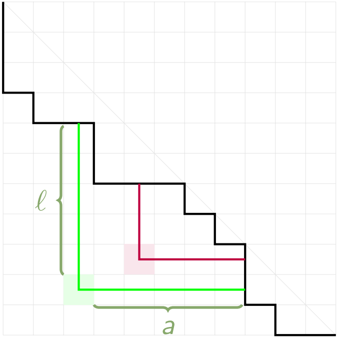

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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- $\mathcal{G}_\nu$ is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

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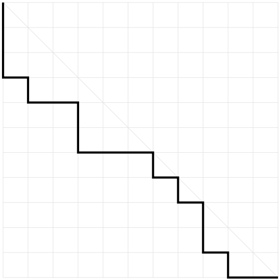

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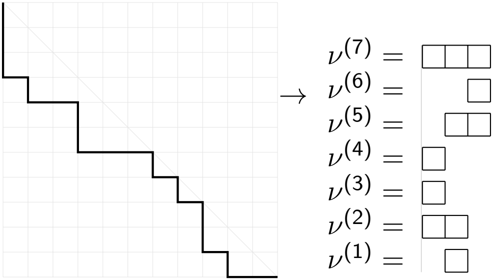

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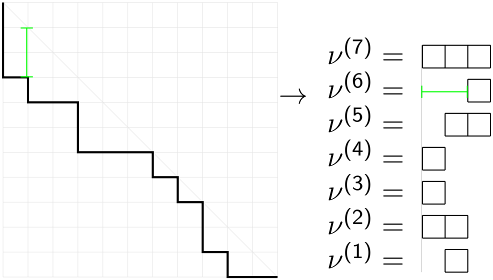

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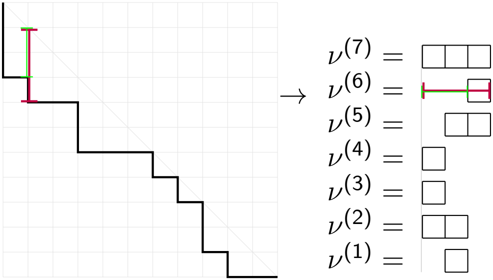

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for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

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$$T = \begin{array}{l} \boxed{1}\boxed{\color{red}2}\boxed{3}\boxed{3}\boxed{5} \\ \boxed{2}\boxed{\color{red}4}\boxed{4}\boxed{\color{blue}7}\boxed{8}\boxed{9}\boxed{9} \\ \phantom{\boxed{0}\boxed{0}\boxed{0}}\boxed{1}\boxed{\color{blue}1}\boxed{6}\boxed{7}\boxed{7}\boxed{7} \end{array}$$

LLT Polynomials

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$$\begin{array}{cccccc} \begin{smallmatrix}\boxed{1}\boxed{1}\\ \;\;\;\boxed{1}\end{smallmatrix} & \begin{smallmatrix}\boxed{1}\boxed{2}\\ \;\;\;\boxed{1}\end{smallmatrix} & \begin{smallmatrix}\boxed{1}\boxed{2}\\ \;\;\;\boxed{2}\end{smallmatrix} & \begin{smallmatrix}\boxed{2}\boxed{2}\\ \;\;\;\boxed{2}\end{smallmatrix} & \begin{smallmatrix}\boxed{1}\boxed{1}\\ \;\;\;\boxed{2}\end{smallmatrix} & \begin{smallmatrix}\boxed{2}\boxed{2}\\ \;\;\;\boxed{1}\end{smallmatrix} \end{array}$$

$$= s_3 + qs_{2,1}$$

Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

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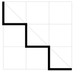		
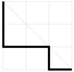		
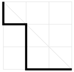		
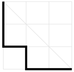		
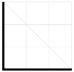		

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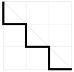
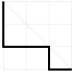
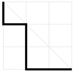
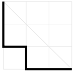
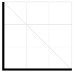
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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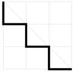
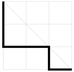
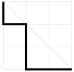
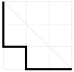
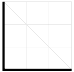
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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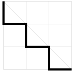
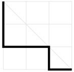
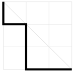
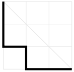
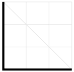
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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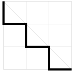
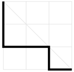
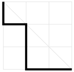
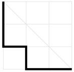
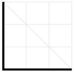
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q,t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q,t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

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When a problem is too difficult, try generalizing!

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For m, n coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

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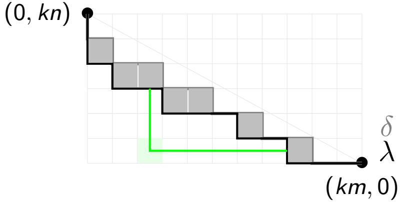

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

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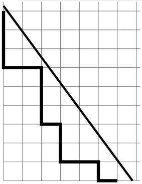

area(λ) as before
$\mathrm{dinv}_p(\lambda) = \#p$-balanced hooks $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

Proof Overview (algebraic side)

- $\psi\colon \mathcal{E}^+ \cong S$
- $\mathcal{E}^+$ is the "positive half" of $\mathcal{E}$
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Key relationship

For $\xi \in \mathcal{E}^+$,

$$\omega(\xi \cdot 1) = \mathrm{pol}_X(\psi(\xi))$$

for automorphism $\omega\colon \Lambda \to \Lambda$ and $\mathrm{pol}_X\colon S \to \Lambda$ a "polynomial truncation" operation.

Proof Overview (combinatorial side)

- For $\xi = D_{\mathbf{b}}$, we get

$$\mathrm{pol}_X\, \mathbf{H}_q \left(\frac{z^{\mathbf{b}} \prod_{i<j+1}(1 - qtz_i/z_j)}{\prod_{i<j}(1 - tz_i/z_j)} \right) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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Need an "infinite series" version of LLT polynomials!

Cauchy identity

- For a fixed $\sigma \in S_l$, there exists a basis of $\mathbb{Q}(q)[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$ called "non-symmetric Hall-Littlewood polynomials", denoted $E_\lambda^\sigma = E_\lambda^\sigma(z_1, \ldots, z_l; q)$ for $\lambda \in \mathbb{Z}^l$.

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- (Grojnowski-Haiman 2007) defines a (symmetric) "series LLT" polynomial $\mathcal{L}_{\beta/\alpha}^\sigma(x_1, \ldots, x_l; q) = H_q(w_0(F_\beta^{\sigma^{-1}}\overline{E_\alpha^{\sigma^{-1}}}))$

Proof Idea

Stable Shuffle Theorem (BHMPS 21a)

For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to highest path under a line of slope $-r/s$,

$$\psi D_{\mathbf{b}} = \sum_{a_1,\ldots,a_{l-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}^{\sigma}_{((b_l,\ldots,b_1)+(0,a_{l-1},\ldots,a_1))/(a_{l-1},\ldots,a_1,0)}(x_1,\ldots,x_l;q)$$

Under polynomial truncation,
$\mathcal{L}^{\sigma}_{\beta/\alpha}(x_1,\ldots,x_l;q) \to q^{\mathrm{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}(x_1,\ldots,x_l;q^{-1})$

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$$\implies \omega(D_{\mathbf{b}}\cdot 1)(x_1,\ldots,x_l) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)}\mathcal{G}_{\nu(\lambda)}(x_1,\ldots,x_l;q^{-1}).$$

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$$\Delta_{h_r}\Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda,P} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(\lambda)} x^P \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} \left(1 + zt^{-r_i(\lambda)}\right).$$

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Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \mathrm{sgn}(\mu) \sum_{(G,R)\in LNDP_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} x^R$$

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Generalizing our methods further, we arrive at the following.

Theorem (BHMPS21c)

$$s_\mu[-MX^{m,n}] \cdot 1 = \sum_\pi t^{a(\pi)} q^{\operatorname{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

For a sum over all "nests" π in a "den" corresponding to s_μ.

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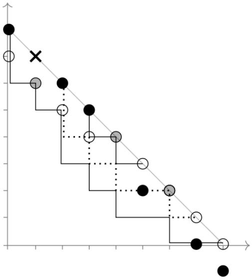

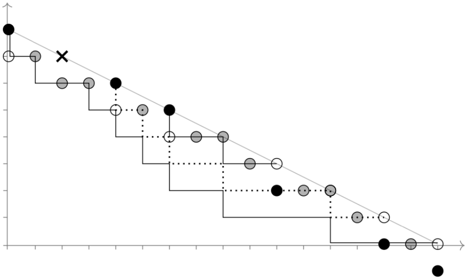

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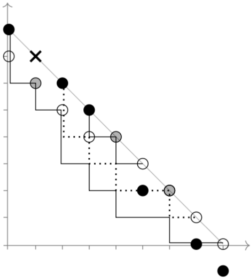

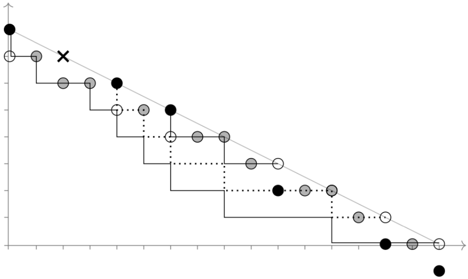

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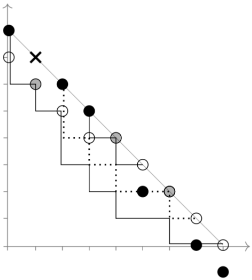

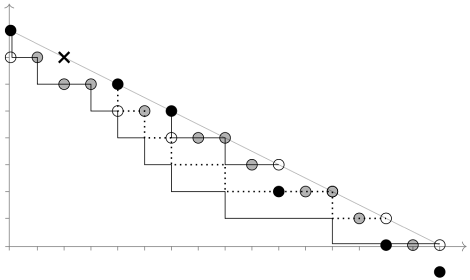

- Implies the Loehr-Warrington Conjecture as a special case.
- Also proves $\mathrm{sgn}(\mu)\nabla s_\mu$ is Schur-positive.

Generalizations

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Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

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- Experimental computation suggests this is "tight."
- Coefficient of $s_{1,\ldots,1}$ coincides with (q, t)-polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

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- S_l-representation theory interpretations?

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Thank you!

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