K-theoretic Catalan functions

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Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$\begin{array}{cccc} 5 & & & \\ 3 & 4 & & \\ 2 & 3 & & \\ 1 & 2 & 2 & 5 \end{array} \qquad \underbrace{\begin{array}{cccc} 8 & & & \\ 7 & 9 & & \\ 3 & 4 & & \\ 1 & 2 & 5 & 6 \end{array}}_{\text{standard } = \text{ no repeated letters}}$$

Schur functions s_λ

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Semistandard tableaux: columns increasing and rows non-decreasing.

$$\begin{array}{|c|ccc}\hline 5 & \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{4} & & \\ \hline 2 & \multicolumn{1}{|c|}{3} & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \underbrace{\begin{array}{|c|ccc}\hline 8 & \multicolumn{3}{c}{} \\ \hline 7 & \multicolumn{1}{|c|}{9} & & \\ \hline 3 & \multicolumn{1}{|c|}{4} & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \hline \end{array}}_{\text{standard = no repeated letters}}$$

Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}, \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

$$s_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2 x_1 x_2 x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\ydiagram{1}} s_{\ydiagram{2,1}} = s_{\ydiagram{3,1}} + s_{\ydiagram{2,2}} + s_{\ydiagram{2,1,1}}$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c^\nu_{\lambda\mu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$.

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$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$ have no tableaux description.

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
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where $s_\lambda^{(k)}$ is a k-Schur symmetric function and $\mathrm{Gr}_{SL_{k+1}}$ is the "affine Grassmannian."

Upshot

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Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

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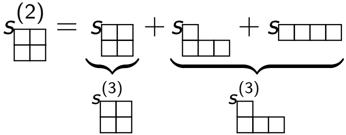

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- Many conjecturally equivalent definitions.

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Key: Catalan functions = large class of symmetric functions.

Ingredients for Catalan functions

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- Root ideals

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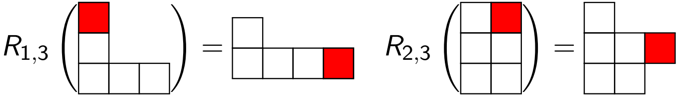

Raising Operators on Symmetric Functions

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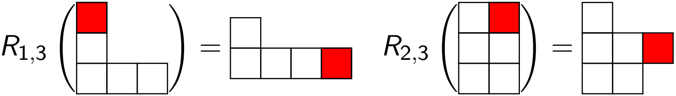

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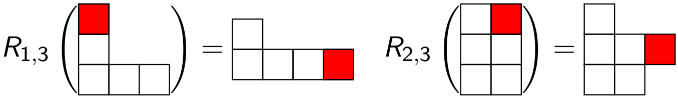

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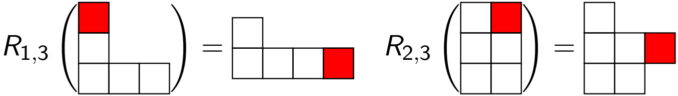

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$$\begin{aligned} s_{22} &= (1-R_{12})h_{22} = h_{22} - h_{31} \\ s_{211} &= (1-R_{12})(1-R_{23})(1-R_{13})h_{211} \\ &= h_{211} - h_{301} - h_{220} {\color{red}-h_{310} + h_{310}} + \underbrace{h_{32\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\text{-}1}}_{=0} \end{aligned}$$

some terms cancel

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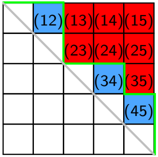
$$s_{1^r}^\perp s_\lambda = \sum_{S\subseteq[1,\ell],|S|=r} s_{\lambda-\epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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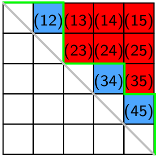

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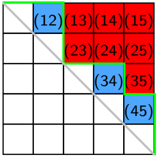

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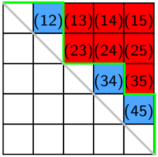

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- $\Psi = \varnothing \implies H(\varnothing;\gamma) = s_\gamma$
- Ψ = all roots $\implies H(\Psi;\gamma) = h_\gamma$

Catalan functions

Intuition

Catalan functions interpolate between h_λ and s_λ.

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi;\lambda)$ is Schur positive!

Catalan functions

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
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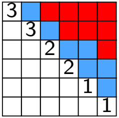
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

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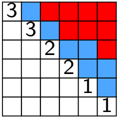
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

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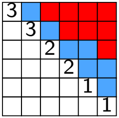
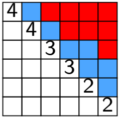
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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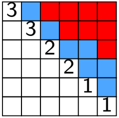
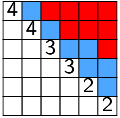
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Pieri:

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

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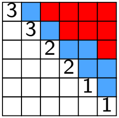
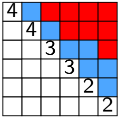
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

$$s_{\lambda}^{(k)} = s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- **K-theoretic Catalan functions**

Dual Grothendieck polynomials

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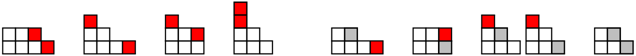

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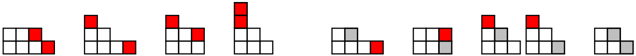

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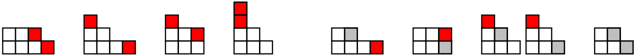

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- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m,n))$

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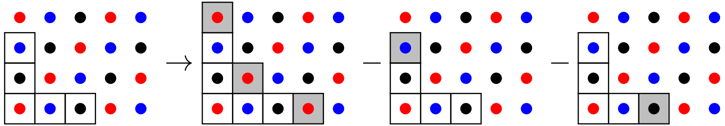
$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions $\leftrightarrow$ 3-cores



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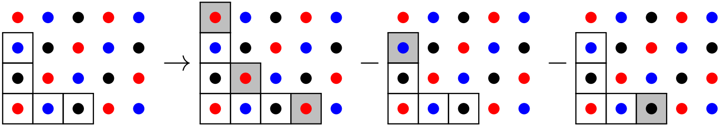

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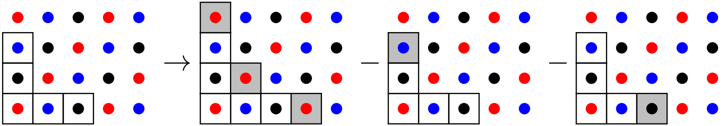

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Problem

No direct formula for $g_\lambda^{(k)}$

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Solution

Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

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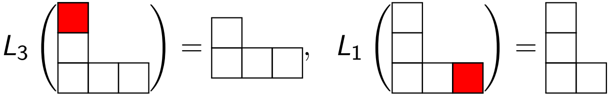
Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) k_\gamma$$

Affine K-Theory Representatives with Raising Operators

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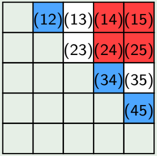
Example

non-roots of Ψ, roots of $\mathcal{L}$



$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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Answer (Blasiak-Morse-S., 2020)

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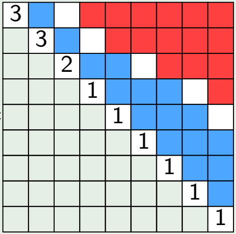
Example

$g^{(4)}_{332111111} =$ 

$\Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$

Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

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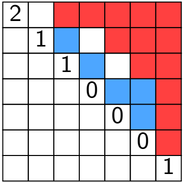

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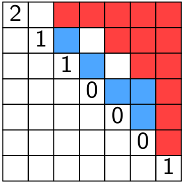
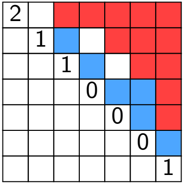
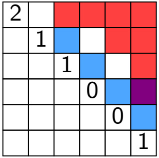
$g_1 g_{211}^{(2)}$ $=$ 

$=$ 

$+$ 

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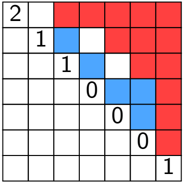

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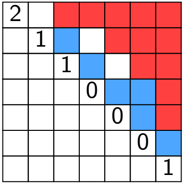

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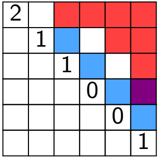

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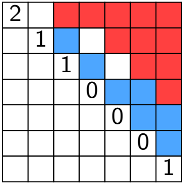

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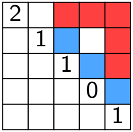

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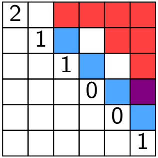

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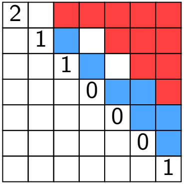
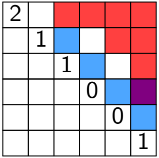
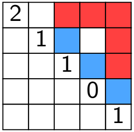
$$g_1 g_{211}^{(2)} =$$  $+$  $+$ 

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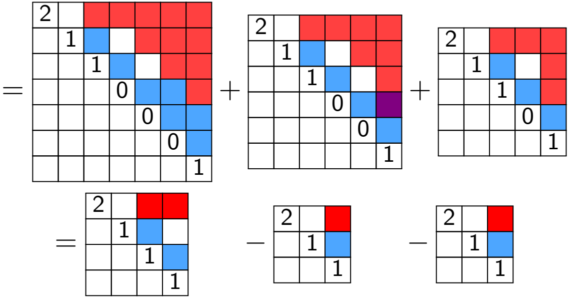

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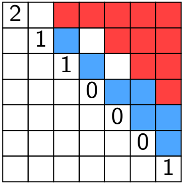
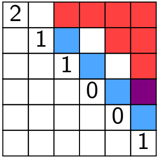
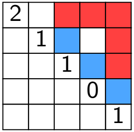
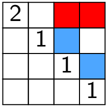
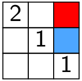
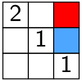
$g_1 g^{(2)}_{211}$ =  +  + 

=  −  − 

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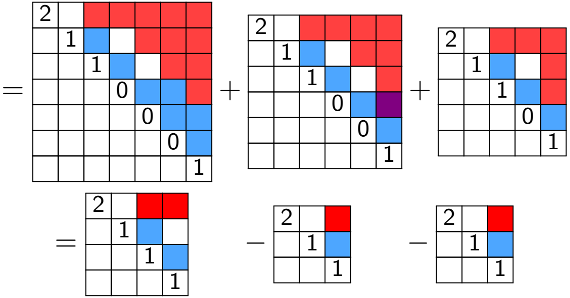

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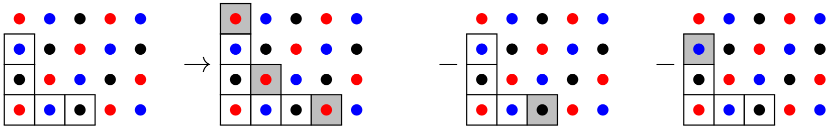
3-core perspective:



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Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

K-theoretic Peterson isomorphism

$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

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Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothtendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

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References

Thank you!

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