

# $K$ -theoretic Catalan functions

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- Schubert calculus: connecting geometry and combinatorics
- Catalan functions: a new approach to old problems
- $K$ -theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

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$$e_1 = x_1 + x_2 + x_3 \quad h_1 = x_1 + x_2 + x_3$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$



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- Bases indexed by integer partitions.

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

# Partitions

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\hline\hline\hline\hline\hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

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- Schubert varieties  $X_\lambda = \overline{\Omega_\lambda}$ .

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .

# Classical Schubert Calculus

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## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  indexed by partitions such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

## Example

*Semistandard tableaux*: columns increasing and rows non-decreasing.

$$T =$$

|   |   |   |   |
|---|---|---|---|
| 5 |   |   |   |
| 3 | 4 |   |   |
| 2 | 3 |   |   |
| 1 | 2 | 2 | 5 |

|   |   |   |   |
|---|---|---|---|
| 8 |   |   |   |
| 7 | 9 |   |   |
| 3 | 4 |   |   |
| 1 | 2 | 5 | 6 |

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$$\text{wt}(T) = (1, 3, 2, 1, 2) \quad (1, 1, 1, 1, 1, 1, 1, 1, 1)$$

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$$x^{\text{wt}(T)} = x_1^1 x_2^3 x_3^2 x_4^1 x_5^2 \quad x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$$

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$\text{SSYT}(\lambda) =$  all semistandard tableaux of shape  $\lambda$ .

$$\begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array} \end{array}$$



# Schur functions $s_\lambda$

Schur function  $s_\lambda$  is a “weight generating function” of semistandard tableaux:

$$\begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$$

$$s_{\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$s_\lambda(x)$  is homogeneous of degree  $\lambda_1 + \cdots + \lambda_\ell$ .

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}$$

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Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since  $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$ , subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients*  $c_{\lambda\mu}^\nu$ .

## Upshot

Let  $\{f_\lambda\}$  be a basis of  $\Lambda$  such that

- ①  $f_r = s_r$  and
- ②  $f_r f_\lambda$  satisfies the Pieri rule.

Then,  $f_\lambda = s_\lambda$ .

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## Upshot

Let  $\{f_\lambda\}$  be a basis of  $\Lambda$  such that

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- 2  $f_r f_\lambda$  satisfies the Pieri rule.

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## Looking Ahead

This type of technique will be useful for establishing the equivalence of new formulas for other bases.



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- Does it have a Pieri rule? ( $s_r s_\lambda = \sum s_\nu$ )
- Does it have a direct formula? ( $s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T$ )

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# Schubert Calculus Variations

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| Theory                                 | $f_\lambda$                  |
|--|------------------------------|
| (Co)homology of Grassmannian           | Schur functions              |
| (Co)homology of flag variety           | Schubert polynomials         |
| Quantum cohomology of flag variety     | Quantum Schuberts            |
| (Co)homology of Types BCD Grassmannian | Schur- $P$ and $Q$ functions |
| (Co)homology of affine Grassmannian    | (dual) $k$ -Schur functions  |
| $K$ -theory of Grassmannian            | Grothendieck polynomials     |
| $K$ -homology of affine Grassmannian   | $K$ - $k$ -Schur functions   |

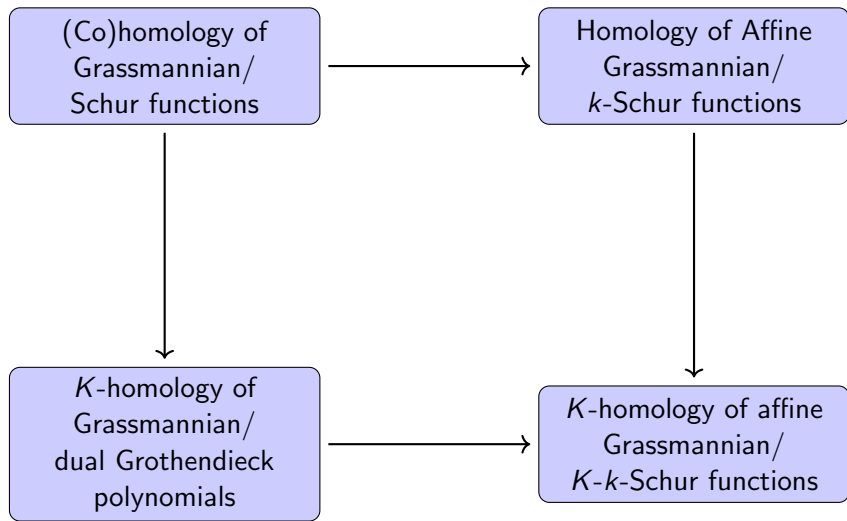
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And many more!

# Big Picture



# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).



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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\mu} + s_{\nu}$$

The diagram shows the branching rule for  $k$ -Schur functions. On the left,  $s_{\lambda}^{(2)}$  is represented by a 2x2 square. This is equal to the sum of three terms. The first term is  $s_{\lambda}^{(3)}$ , represented by a 2x2 square. The second and third terms are grouped by a brace and labeled  $s_{\lambda}^{(3)}$ , represented by a 2x3 rectangle. The second term is  $s_{\mu}$  (a 2x2 square with the bottom-right cell missing) and the third term is  $s_{\nu}$  (a 1x3 horizontal row).

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- Branching with positive coefficients (Lam et al., 2010):

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$\underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}^{(3)}} \quad \underbrace{\hspace{10em}}_{s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}^{(3)}}$

- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.

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# $k$ -Schur functions

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- Schubert calculus
- **Catalan functions: a new approach to old problems**
- $K$ -theoretic Catalan functions

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Key:  $\{s_{\lambda}^{(k)}\}_{\lambda} \subseteq \text{Catalan functions} = \text{large class of symmetric functions.}$

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$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \text{red } h_{310} + \text{red } h_{310} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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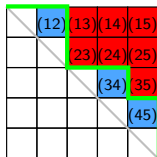
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# Root Ideals

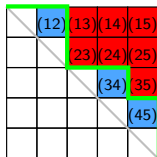
A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



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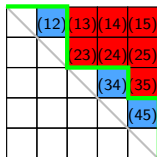
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

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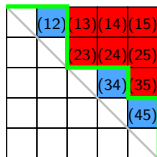
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Catalan functions interpolate between  $h_\lambda$  and  $s_\lambda$ .

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## Theorem (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive!  
Precisely,  $H(\Psi; \lambda) = \sum_\nu c_{\Psi, \lambda}^\nu s_\nu$  satisfies  $c_{\Psi, \lambda}^\nu \in \mathbb{Z}_{\geq 0}$ .

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$



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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 3 |   |   |   |   |   |
|   | 3 |   |   |   |   |
|   |   | 2 |   |   |   |
|   |   |   | 2 |   |   |
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$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

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|   |   |   |   |   |   |
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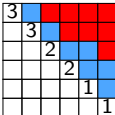
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Pieri:

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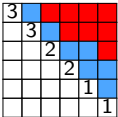
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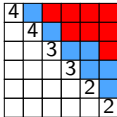
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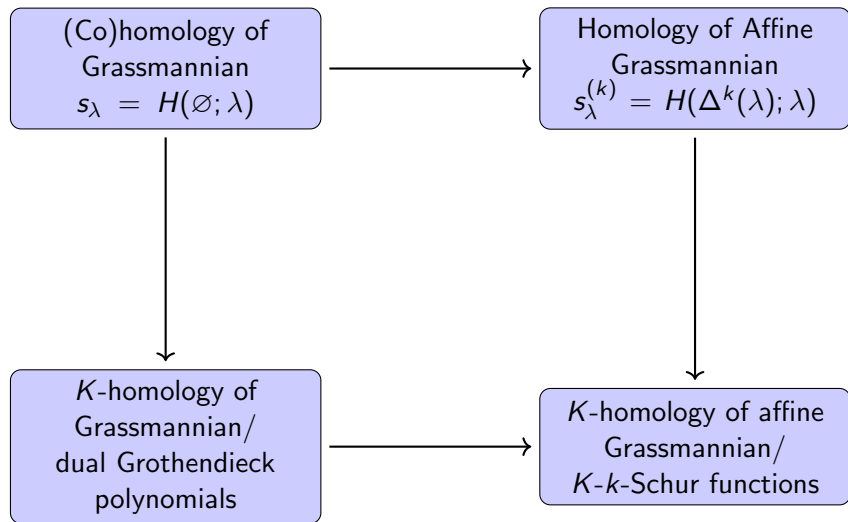
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Branching is a special case of Pieri:

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# Big Picture



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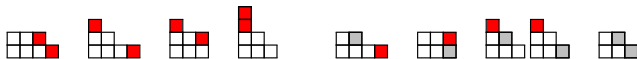
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2-bounded partitions  $\leftrightarrow$  3-cores

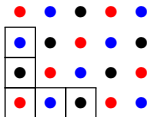
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$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions  $\leftrightarrow$  3-cores

# $K$ - $k$ -Schur functions

- Inhomogeneous basis:  $g_{\lambda}^{(k)} = s_{\lambda}^{(k)} + \text{lower degree terms}$
- Satisfies Pieri rule on “affine set-valued strips”
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Problem

No direct formula for  $g_{\lambda}^{(k)}$



## Solution

Find a formula for  $g_{\lambda}^{(k)}$  analogous to raising operator formula for  $s_{\lambda}^{(k)}$ .

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Requires an inhomogeneous refinement of Catalan functions.

# An Extra Ingredient: Lowering Operators

Lowering Operators  $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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“ $\Psi$  =raising ideal,  $\mathcal{L}$  =lowering ideal.”

# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

|  |      |      |      |      |
|--|------|------|------|------|
|  | (12) | (13) | (14) | (15) |
|  |      | (23) | (24) | (25) |
|  |      |      | (34) | (35) |
|  |      |      |      | (45) |
|  |      |      |      |      |
|  |      |      |      |      |

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ = (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332} \end{aligned}$$

Answer (Blasiak-Morse-S., 2020)

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For  $K$ -homology of affine Grassmannian,  $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$  since this family satisfies the Pieri rule.

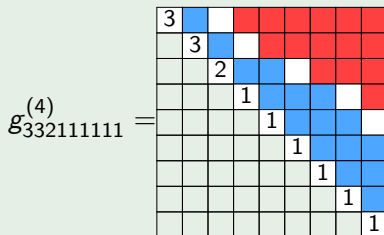


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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

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$$g_1 g_{211}^{(2)}$$

$$=$$

|   |   |   |   |   |   |   |  |
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| 2 |   |   |   |   |   |   |  |
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# Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

|   |   |   |   |   |   |   |
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| 2 |   |   |   |   |   |   |
|   | 1 |   |   |   |   |   |
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|   |   |   | 0 |   |   |   |
|   |   |   |   | 0 |   |   |
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|   | 1 |   |   |   |   |   |
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|   |   |   |   | 0 |   |   |
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$$=$$

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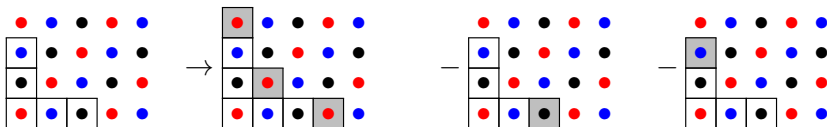
$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
 &= g_{2111}^{(2)} - g_{211}^{(2)} - g_{211}^{(2)}
 \end{aligned}$$



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3-core perspective:



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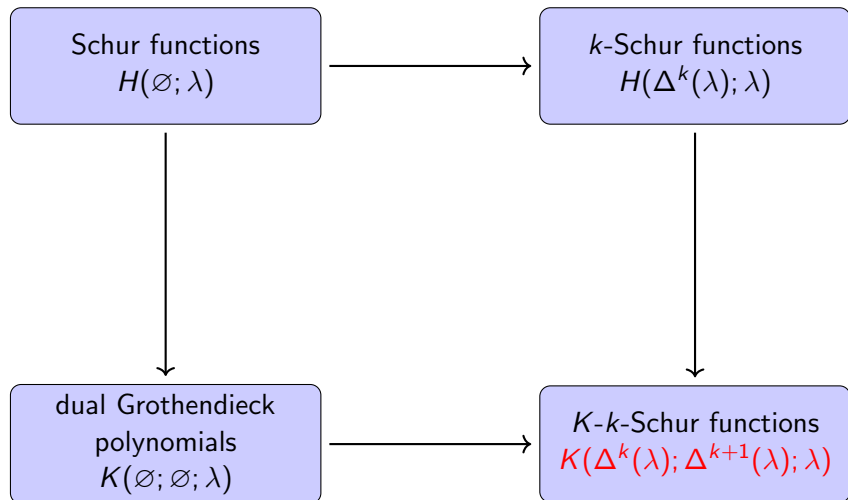
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The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

# Big Picture



# $K$ -theoretic Peterson isomorphism

$$\Phi: QK^*(Fl_{k+1}) \rightarrow K_*(Gr_{SL_{k+1}})_{loc}$$

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Conjecture (Ikeda et al., 2018)

For  $w \in S_{k+1}$  and  $\mathfrak{G}_w^Q$  a “quantum Grothendieck polynomial”,

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$$\tilde{g}_w = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Definition (Blasiak-Morse-S., 2020)

For any partition  $\lambda$  with  $\lambda_1 \leq k$ , we set

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# $k$ -Rectangle Property

## Theorem (S., 2021)

*For  $1 \leq d \leq k$ , set  $R_d = ((k + 1 - d)^d)$  to be the  $k$ -rectangle partition.*



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where  $\mu \cup R_d$  is the partition given by sorting  $(\mu, R_d)$ .

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- Must be true for geometric connection with Peterson isomorphism.

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$$\tilde{g}_{R_d}^{(k)} \tilde{g}_{\mu}^{(k)} = \tilde{g}_{\mu \cup R_d}^{(k)},$$

where  $\mu \cup R_d$  is the partition given by sorting  $(\mu, R_d)$ .

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# $k$ -Rectangle Property

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What can be said about  $K$ -theoretic Catalan functions in general?



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Raising operator techniques extend beyond Schubert calculus! Work by Blasiak-Haiman-Morse-Pun-S.:

- Shuffle theorems (Blasiak et al., 2021a; 2021b).
- Macdonald polynomials and LLT polynomials (Blasiak et al., 2021c).
- Much more work to be done!

## Thank you!

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$$k_m^{(r)} = \sum_{i=0}^m \binom{r+i-1}{i} h_{m-i} = s_m(X+r),$$

a specialization of “multiSchur functions.” See, e.g., Lascoux-Naruse (2014).

$$k_{\gamma} = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_{\ell}}^{(\ell-1)}$$