

Raising operators in Schubert Calculus

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Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of subvarieties in a variety X .

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

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Goal

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Classical Example

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- $X = \text{Gr}_{m,n}$
- $f_\lambda = s_\lambda$, the Schur functions
- $s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$ (Jacobi-Trudi)
- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

$$R_{1,3} \left(\begin{array}{|c|c|c|} \hline \text{red} & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline & & & \text{red} \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left(\begin{array}{|c|} \hline \text{red} \\ \hline \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \text{red} \\ \hline & \\ \hline \end{array}$$

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Focus

K -theory and K -homology of the affine Grassmannian

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Focus

K -theory and K -homology of the affine Grassmannian

- Simultaneously generalizes K -theory of Grassmannian and (co)homology of affine Grassmannian.

What is known?

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- ① K -theory classes of Grassmannian (not affine!) represented by “Grothendieck polynomials.” We are interested in their dual:

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for $K h_\gamma$ an inhomogeneous analogue of h_γ .

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- ② Homology classes of affine Grassmannian represented by k -Schur functions ($t = 1$)

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- ③ (Lam et al., 2010) leave open the question: what is a direct formulation of the K -homology representatives of the affine Grassmannian (K - k -Schur functions)?

Remember?

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Lowering Operators and Root Ideals

- Lowering Operators $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$

$$L_3 \left(\begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad L_1 \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

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- Root ideal Ψ : given by Dyck path.

$$\Psi = \begin{array}{|c|c|c|c|c|} \hline & (12) & (13) & (14) & (15) \\ \hline & & (23) & (24) & (25) \\ \hline & & & (34) & (35) \\ \hline & & & & (45) \\ \hline & & & & \\ \hline \end{array} \quad \begin{array}{l} \text{Roots above Dyck path} \\ \text{Non-roots below} \end{array}$$

Definition

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) Kh_\gamma$$

for Kh_γ an inhomogeneous analogue of h_γ .

Affine K -Theory Representatives with Raising Operators

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Example

non-roots of Ψ , roots of \mathcal{L}

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) Kh_{54332} \end{aligned}$$

Affine K -Theory Representatives with Raising Operators

Definition

The k -Schur root ideal, $\Delta^{(k)}(\lambda)$ is the unique root ideal with $\lambda_i + \#\text{non-roots in row } i = k$.

Example

$k = 4, \lambda = 332111$

$$\Delta^{(4)}(332111) =$$

3					
	3				
		2			
			1		
				1	
					1

← 4 - 2 non-roots

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Example

$$g_{332111}^{(4)} =$$

3						
	3					
		2				
			1			
				1		
					1	

$$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$$

Theorem (Blasiak-Morse-S., 2019)

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$$G_{1^{\ell}}^{\perp} g_{\lambda+1^{\ell}}^{(k+1)} = g_{\lambda}^{(k)}$$

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Further work

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

- 1 Dual Pieri rule: $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)},$
 $1 \leq r \leq k$
- 2 Branching rule: $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$

Thank you!

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