

# TWO ELEMENTARY EXAMPLES OF EXTREME CHARACTERS OF $U(\infty)$ INTEGRABLE PROBABILITY READING SEMINAR

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## 1. INTRODUCTION

First, we recall some definitions.

**1.1. Definition.** An  $N \times N$  matrix  $U$  is *unitary* if  $UU^* = I_N$  where  $U^*$  is the conjugate transpose of  $U$ . Then,  $U(N)$  is the compact Lie group of all  $N \times N$  unitary matrices. Since  $U(N-1) \hookrightarrow U(N)$  via a canonical embedding, we also define

$$U(\infty) := \bigcup_{N=1}^{\infty} U(N)$$

that is,  $U(\infty)$  are all infinite  $\mathbb{N} \times \mathbb{N}$  unitary matrices that differ from the identity matrix only in a fixed number of positions.

**1.2. Definition.** A *normalized character* of  $U(N)$  is a function  $\chi: U(N) \rightarrow \mathbb{C}$  such that

- (a)  $\chi(e) = 1$  (normalized),
- (b)  $\chi(ab) = \chi(ba)$  (constant on conjugacy classes),
- (c)  $(\sum c_i \chi(a_i)) (\sum \bar{c}_j \chi(a_j)) = \sum c_i \bar{c}_j \chi(a_i a_j^{-1}) \geq 0$  (nonnegative definite),
- (d)  $\chi$  is continuous.

Normalized characters form a convex set since  $t\chi_1 + (1-t)\chi_2$  meets all the axioms of a normalized character for all  $t \in [0, 1]$ . Then, we can discuss the following notion.

**1.3. Definition.** An *extreme character*  $\chi: U(N) \rightarrow \mathbb{C}$  is a normalized character such that  $\chi \neq t\chi_1 + (1-t)\chi_2$  for any  $t \in (0, 1)$  for normalized characters  $\chi_1, \chi_2 \neq \chi$ .

**1.4. Definition.** The  $N$ -dimensional torus is

$$\mathbb{T}^N := \{(x_1, \dots, x_N) \in \mathbb{C}^N \mid |x_i| = 1\}$$

and lies in  $U(N)$  as diagonal matrices. The *finitary torus* is  $\mathbb{T}_{fin}^\infty := \bigcup_{N=1}^\infty \mathbb{T}^N$ .

Recall one of our main goals is to understand the following theorem.

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**1.5. Theorem** (Edrei-Voiculescu). *Extreme characters of  $U(\infty)$  are functions  $\chi: T_{fin}^\infty \rightarrow \mathbb{C}$  depending on countably many parameters*

$$\begin{cases} \alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \dots \geq 0); \\ \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \dots \geq 0); \\ \gamma^\pm \geq 0 \end{cases}$$

such that

$$\sum_i \alpha_i^+ + \sum_i \alpha_i^- + \sum_i \beta_i^+ + \sum_i \beta_i^- < \infty, \quad \beta_1^+ + \beta_1^- \leq 1$$

Furthermore, these functions have the form

$$\chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \dots) = \prod_{j=1}^{\infty} \Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_j)$$

where  $\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}: \mathbb{T} \rightarrow \mathbb{C}$  is the continuous function

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) := e^{\gamma^+(x-1) + \gamma^-(x^{-1}-1)} \prod_{i=1}^{\infty} \left( \frac{1 + \beta_i^+(x-1)}{1 - \alpha_i^+(x-1)} \cdot \frac{1 + \beta_i^-(x^{-1}-1)}{1 - \alpha_i^-(x^{-1}-1)} \right).$$

**1.6. Goal.** In this presentation, we will outline two very special examples of this parameterization, namely when

(a)  $\beta^+ = (\beta, 0, 0, \dots), \beta^- = \alpha^\pm = (0, 0, \dots), \gamma^\pm = 0$  for  $\beta \in [0, 1]$  so that

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = 1 + \beta(x-1) \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \dots) = \prod_{j=1}^{\infty} (1 + \beta(x_j - 1))$$

(b)  $\alpha^+ = (\alpha, 0, 0, \dots), \beta^\pm = \alpha^+ = (0, 0, \dots), \gamma^\pm = 0$  for  $\alpha \in [0, 1]$  so that

$$\Phi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x) = \frac{1}{1 - \alpha(x-1)} \implies \chi_{\alpha^\pm, \beta^\pm, \gamma^\pm}(x_1, x_2, \dots) = \prod_{j=1}^{\infty} \frac{1}{1 - \alpha(x_j - 1)}$$

## 2. SYMMETRIC FUNCTIONS

In the last lecture, we introduced the following.

**2.1. Definition.** Given a sequence of integers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ , the *Schur polynomial* is given by

$$s_\lambda(x_1, \dots, x_N) = \frac{\det(x_j^{\lambda_i + N - i})_{i,j=1}^N}{\det(x_j^{N-i})_{i,j=1}^N}$$

Also, if  $\lambda$  has  $\lambda_N \geq 0$ , we can use “Littlewood’s Combinatorial Description” of Schur functions

**2.2. Proposition.** *Given a sequence of integers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ ,*

$$s_\lambda(x_1, \dots, x_N) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}$$

where  $x^{\text{wt}(T)} = \prod_{j=1}^{\sum \lambda_i} x_j^{\# \text{ of } j\text{'s in } T}$ .

### 2.3. Example.

$$s_{(2,1)}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$$

We also proved that

**2.4. Theorem.** *The irreducible representations of  $U(N)$  are in one-to-one correspondence with  $\{\lambda \in \mathbb{Z}^N \mid \lambda_1 \geq \dots \geq \lambda_N\}$  where the character of representation  $T_\lambda$  of  $U(N)$  corresponding to  $\lambda$  has character given by*

$$\text{Tr} \left( T_\lambda \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix} \right) = s_\lambda(x_1, \dots, x_N)$$

We will work with two special cases of the Schur polynomials.

**2.5. Definition.** Let  $e_m(x_1, \dots, x_N) := s_{(1^m)}(x_1, \dots, x_N)$  be the *elementary symmetric polynomials*.

**2.6. Example.** Using the semistandard Young tableaux formula for Schur functions (Littlewood's combinatorial description), we compute

(a)

$$e_2(x_1, x_2) = x_1 x_2$$

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}$$

(b)

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}$$

(c)

$$e_3(x_1, x_2, x_3) = x_1 x_2 x_3$$

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

**2.7. Remark.**  $e_N(x_1, \dots, x_N)$  encodes character of the “determinant representation” of  $U(N)$ , that is

$$T(U)v = (\det U)v = x_1 x_2 \cdots x_N v$$

since the determinant is just the product of the eigenvalues. More generally,  $e_m(x_1, \dots, x_N)$  encodes the representation induced by the  $U(N)$ -action on  $\bigwedge^m \mathbb{C}^N$ :

$$U \cdot (v_1 \wedge \dots \wedge v_m) = (Uv_1 \wedge \dots \wedge Uv_m)$$

Importantly, we also compute, generalizing our example above

**2.8. Proposition.** For  $0 < m \leq n$ ,

$$e_m(x_1, x_2, \dots, x_n) = \sum_{T \in \text{SSYT}((1^m)) \text{ filled with elements of } \{1, \dots, n\}} x^{\text{wt}(T)} = \sum_{I \subseteq \{1, \dots, n\}, |I|=m} x^I$$

where  $x^I := \prod_{i \in I} x_i$  and consequently,

$$e_m(\underbrace{1, \dots, 1}_n) = \binom{n}{m}$$

*Proof.* To see this, we simply observe that a single column semistandard tableau with  $m$  rows filled with letters  $\{1, \dots, n\}$  is a choice of  $m$  distinct elements of  $\{1, \dots, n\}$  since columns must be strictly increasing.  $\square$

**2.9. Definition.** Let  $h_m(x_1, \dots, x_N) := s_{(m)}(x_1, \dots, x_N)$  be the *complete homogeneous symmetric polynomials*.

**2.10. Example.** Using again our tableaux formula for Schur functions, we compute

(a)

$$h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array}$$

(b)

$$h_2(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 3 & 3 \\ \hline \end{array}$$

**2.11. Proposition.** For  $0 < m \leq n$ ,

$$h_m(x_1, x_2, \dots, x_n) = \sum_{T \in \text{SSYT}((m)) \text{ filled with elements of } \{1, \dots, n\}} x^{\text{wt}(T)} = \sum_{I \text{ multiset of } \{1, \dots, n\}, |I|=m} x^I$$

where  $x^I := \prod_{i \in I} x_i$  and consequently,

$$\begin{aligned} & h_m(\underbrace{1, \dots, 1}_n) \\ &= \text{Number of ways to choose a multiset of size } m \text{ from } n \text{ things} \\ &= \binom{n+m-1}{m} = \binom{n+m-1}{n-1} \end{aligned}$$

**2.12. Remark.** The combinatorics of the identity above follow by considering a “stars and bars” approach, namely, both expressions are in bijection with the number of ways to place  $n - 1$  bars among  $m$  stars, allowing bars to be consecutive with each other.

$$\{1, 1, 1, 2, 4, 5\} \rightarrow \star \star \star | \star || \star | \star$$

**2.13. Definition.** Let

$$\left(\left(\begin{matrix} n \\ m \end{matrix}\right)\right) := \binom{n+m-1}{m}$$

be the number of ways to choose a multiset of size  $m$  from  $n$  things.

### 3. TWO EXAMPLES OF $U(\infty)$ CHARACTERS

Now, we wish to take a sequence of  $U(N)$  characters to get a character of  $U(\infty)$ .

**3.1. Definition.** We say that a sequence of central functions  $f_N$  (i.e.  $f_N$  only depends on the eigenvalues of the input) on  $U(N)$  *converge* to a central function  $f$  on  $U(\infty)$  if, for every fixed  $K$ , we have

$$f_N(x_1, \dots, x_K, 1, 1, \dots, 1) \rightarrow f(x_1, \dots, x_K, 1, 1, \dots)$$

uniformly on the  $K$ -torus  $\mathbb{T}^K$  of diagonal matrices.

**3.2. Proposition.** *Let  $L: \mathbb{N} \rightarrow \mathbb{N}$  be a sequence such that  $L(N)/N \rightarrow \beta \in [0, 1]$  as  $N \rightarrow \infty$ . Then,*

$$\frac{e_{L(N)}(x_1, \dots, x_N)}{e_{L(N)}(1, \dots, 1)} \rightarrow \prod_{i=1}^{\infty} (1 + \beta(x_i - 1)), \quad (x_1, x_2, \dots) \in \mathbb{T}_{fin}^{\infty}$$

*Proof.* Fix  $K \leq N$ . Then,

$$\begin{aligned} & e_{L(N)}(x_1, \dots, x_K, 1, \dots, 1) \\ &= \sum_{T \in \text{SSYT}((1^{L(N)})) \text{ labelled with } \{1, \dots, N\}} x^{\text{wt}(T)|_{\leq K}} \\ &= \sum_{\text{binary } K \text{ sequences } \epsilon} \#\{N \text{ sequences with sum } L(N) \text{ that start with } (\epsilon_1, \dots, \epsilon_K)\} x^{(\epsilon_1, \dots, \epsilon_K)} \\ &= \sum_{\text{binary } K \text{ sequences } \epsilon} \binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \end{aligned}$$

where the last equality comes from considering how to fill tableaux of the form

$$\begin{aligned} & \left. \begin{array}{l} \text{Fill } \sum_{i=1}^K \epsilon_i \text{ boxes with } \{1, \dots, K\} \\ \text{Fill } L(N) - \sum_{i=1}^K \epsilon_i \text{ boxes with } \{K+1, \dots, N\} \end{array} \right\} \\ \implies & \frac{e_{L(N)}(x_1, \dots, x_K, 1, \dots, 1)}{e_{L(N)}(1, \dots, 1)} = \sum_{\text{binary } K \text{ sequences } \epsilon} \left( \binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} / \binom{N}{L(N)} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \end{aligned}$$

where

$$\binom{N-K}{L(N)-\sum_{i=1}^K \epsilon_i} / \binom{N}{L(N)} = \frac{(N-K)!}{N!} \times \frac{(L(N))!}{(L(N)-\sum_{i=1}^K \epsilon_i)!} \times \frac{(N-L(N))!}{(N-L(N)-(K-\sum_{i=1}^K \epsilon_i))!}$$

$$\xrightarrow{N \rightarrow \infty} \beta^{\sum_{i=1}^K \epsilon_i} (1-\beta)^{K-\sum_{i=1}^K \epsilon_i} \text{ since } L(N)/N \rightarrow \beta$$

Thus, taking the limit as  $N \rightarrow \infty$  on our ratio, we get

$$\sum_{\text{binary } K \text{ sequences } \epsilon} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \beta^{\sum_{i=1}^K \epsilon_i} (1 - \beta)^{K - \sum_{i=1}^K \epsilon_i} = \prod_{i=1}^K ((1 - \beta) + \beta x_i)$$

and so, taking  $K \rightarrow \infty$  completes the proof.

**3.3. Remark.** An astute reader may notice that  $(1-\beta)^{K-\sum_{i=1}^K \epsilon_i} \beta^{\sum_{i=1}^K \epsilon_i}$  represents the probability of  $\sum_{i=1}^K \epsilon_i$  successes in  $K$  trials where each attempt has probability of success  $\beta$ . One can use “de Finetti’s theorem” in order to derive the proposition directly from this observation. See [Pet12]§4.1.10 for this approach.

**3.4. Proposition.** *Let  $L: \mathbb{N} \rightarrow \mathbb{N}$  be a sequence such that  $L(N)/N \rightarrow \alpha \in [0, 1]$  as  $N \rightarrow \infty$ . Then,*

$$\frac{h_{L(N)}(x_1, \dots, x_N)}{h_{L(N)}(1, \dots, 1)} \rightarrow \prod_{i=1}^{\infty} \frac{1}{1 - \alpha(x_i - 1)}, \quad (x_1, x_2, \dots) \in \mathbb{T}_{fin}^{\infty}$$

*Proof.* We proceed much as in the proposition above. For a fixed  $K \leq N$ , we have

$$\begin{aligned} & h_{L(N)}(x_1, \dots, x_K, 1, \dots, 1) \\ &= \sum_{\epsilon \in \mathbb{N}_0^K} \#\{N \text{ sequences with sum } L(N) \text{ starting with } (\epsilon_1, \dots, \epsilon_K)\} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \\ &= \sum_{\epsilon \in \mathbb{N}_0^K} \left( \binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \end{aligned}$$

where the last line comes from thinking about

Fill  $\sum_{i=1}^K \epsilon_i$  boxes with  $\{1, \dots, K\}$

Fill  $N - \sum_{i=1}^K \epsilon_i$  boxes with  $\{K+1, \dots, N\}$

and so

$$\begin{aligned} & \frac{h_{L(N)}(x_1, \dots, x_K, 1, \dots, 1)}{h_{L(N)}(1, \dots, 1)} \\ &= \sum_{\epsilon \in \mathbb{N}_0^K} \left[ \left( \binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) / \left( \binom{N}{L(N)} \right) \right] x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \end{aligned}$$

Consider that, for fixed  $K \leq N$ , we have

$$\begin{aligned} & \left( \binom{N-K}{L(N) - \sum_{i=1}^K \epsilon_i} \right) / \left( \binom{N}{L(N)} \right) \\ &= \binom{N-K+L(N) - \sum_{i=1}^K \epsilon_i - 1}{L(N) - \sum_{i=1}^K \epsilon_i} / \binom{N+L(N)-1}{L(N)} \\ &= \frac{(N+L(N)-K - \sum_{i=1}^K \epsilon_i - 1)!}{(N+L(N)-1)!} \times \frac{(L(N))!}{(L(N) - \sum_{i=1}^K \epsilon_i)!} \times \frac{(N-1)!}{(N-K-1)!} \\ &\approx \frac{(L(N))^{\sum \epsilon_i} N^K}{(N+L(N))^{K+\sum \epsilon_i}} \\ &= \left( \frac{L(N)}{N} \right)^{\sum \epsilon_i} \left( \frac{1}{1 + \frac{L(N)}{N}} \right)^{K+\sum \epsilon_i} \\ &\xrightarrow{N \rightarrow \infty} \left( \frac{\alpha}{1+\alpha} \right)^{\sum \epsilon_i} \left( \frac{1}{1+\alpha} \right)^K \end{aligned}$$

Thus,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{h_{L(N)}(x_1, \dots, x_K, 1, \dots, 1)}{h_{L(N)}(1, \dots, 1)} &= \sum_{\epsilon} \left( \frac{1}{1+\alpha} \right)^K \left( \frac{\alpha}{1+\alpha} \right)^{\sum \epsilon_i} x_1^{\epsilon_1} \cdots x_K^{\epsilon_K} \\ &= \prod_{i=1}^K \left( \frac{1}{1+\alpha} \right) \left( 1 + \frac{\alpha}{1+\alpha} x_i + \left( \frac{\alpha}{1+\alpha} \right)^2 x_i^2 + \cdots \right) \\ &= \prod_{i=1}^K \frac{1}{1+\alpha} \times \frac{1}{1 - \frac{\alpha}{1+\alpha} x_i} \\ &= \prod_{i=1}^K \frac{1}{1+\alpha - \alpha x_i} \end{aligned}$$

So, taking  $K \rightarrow \infty$  completes the proof.  $\square$

## REFERENCES

- [Pet12] L. Petrov, *Representation Theory of Big Groups and Probability* (2012). Accessed as a draft from <https://lpetrov.cc/reading-2019/> on January 31, 2019.