K-theoretic Catalan functions

George H. Seelinger (joint work with J. Blasiak and J. Morse)

Junior Mathematician Research Archive

ghs9ae@virginia.edu

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Overview

1. Schubert calculus
2. Catalan functions
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

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Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for combinatorially understood Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Schur polynomials and raising operators

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$$R_{1,3}\left(\begin{ytableau} \ & \ & \ \\ \ \\ *(red) \end{ytableau}\right) = \begin{ytableau} \ & \ & \ & *(red) \\ \ \end{ytableau} \quad R_{2,3}\left(\begin{ytableau} \ \\ \ \\ *(red) \end{ytableau}\right) = \begin{ytableau} \ \\ \ & *(red) \end{ytableau}$$

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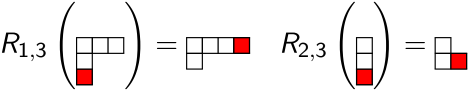

- Schur function $s_\lambda = \prod_{i<j}(1 - R_{ij}) h_\lambda$ (Jacobi-Trudi)

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
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K-theory and K-homology of the affine Grassmannian

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Focus

K-theory and K-homology of the affine Grassmannian

Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

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1. K-theory classes of Grassmannian (not affine!) represented by "Grothendieck polynomials." We are interested in their dual:

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2. Homology classes of affine Grassmannian represented by k-Schur functions ($t = 1$).
3. (Lam et al., 2010) leave open the question: what is a direct formulation of the K-homology representatives of the affine Grassmannian (K-k-Schur functions)?

Remember?

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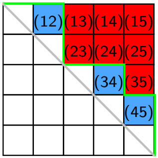
Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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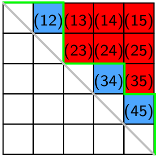

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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

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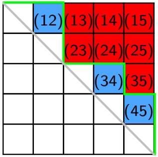

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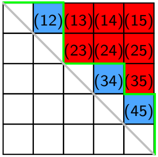

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Catalan functions

k-Schur root ideal for λ

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$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i,j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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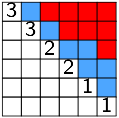
$\Delta^4(3,3,2,2,1,1) =$ 

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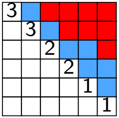
$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

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Remark

(Blasiak et al., 2019) show results for k-Schur functions with parameter t, but $t = 1$ specialization is necessary for Schubert calculus.

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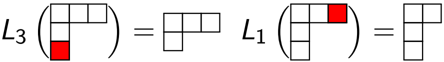
- Extra ingredient: lowering operators $L_j(h_\lambda) = h_{\lambda-\epsilon_j}$



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$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) k_\gamma$$

for k_γ an inhomogeneous analogue of h_γ.

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Example

non-roots of Ψ in blue, roots of $\mathcal{L}$ marked with $\bullet$

	(12)		$\bullet$	$\bullet$
			$\bullet$	$\bullet$
			(34)	
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2$$
$$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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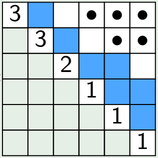
$g^{(4)}_{332111} =$



$\Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$

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Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ *"branching coefficients" are alternating by degree, i.e. the* $b_{\lambda\mu}^{(k)}$ *in*

$$g_\lambda^{(k)} = \sum_\mu b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

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3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ??s_\mu^{(k)}$.

References

Thank you!

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