Raising operators in Schubert Calculus

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Overview of Schubert Calculus Combinatorics

Geometric problem

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Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

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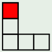
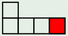
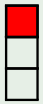
- $f_\lambda = s_\lambda$, the Schur functions
- $s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$ (Jacobi-Trudi)
- Raising operators $R_{i,j}(h_\lambda) = h_{\lambda+\epsilon_i-\epsilon_j}$

$R_{1,3}$ () =  $R_{2,3}$ () = 

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Focus

K-theory and K-homology of the affine Grassmannian

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Focus

K-theory and K-homology of the affine Grassmannian

- Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

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3. (Lam et al., 2010) leave open the question: what is a direct formulation of the K-homology representatives of the affine Grassmannian (K-k-Schur functions)?

Remember?

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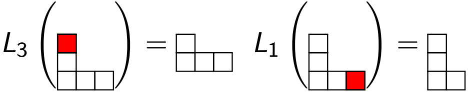
Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Lowering Operators and Root Ideals

- Lowering Operators $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$



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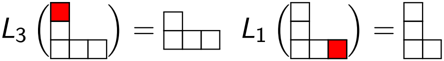
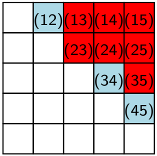


- Root ideal Ψ: given by Dyck path.

$\Psi =$ 

Roots above Dyck path
Non-roots below

Affine K-Theory Representatives with Raising Operators

Definition

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) K h_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} &K(\Psi; \mathcal{L}; 54332) \\ &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) Kh_{54332} \end{aligned}$$

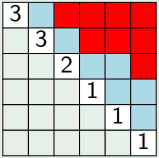
Affine K-Theory Representatives with Raising Operators

Definition

The *k-Schur root ideal*, $\Delta^{(k)}(\lambda)$ is the unique root ideal with $\lambda_i + \#\text{non-roots in row } i = k$.

Example

$k = 4, \lambda = 332111$

$\Delta^{(4)}(332111) =$ 

$\leftarrow$ 4 - 2 non-roots

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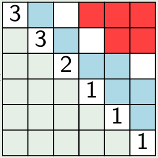
Example

$g_{332111}^{(4)} =$ 

$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$

Property and Further Work

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The $g_\lambda^{(k)}$ are "shift invariant", ie for $\ell = \ell(\lambda)$

$$G_{1^\ell}^{\perp} g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

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Further work

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Dual Pieri rule: $G_{1^r}^{\perp} g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ??G_\lambda^{(k)}$, $1 \leq r \leq k$
2. Branching rule: $g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k+1)}$

References

Thank you!

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