

# Diagonal Harmonics and Shuffle Theorems

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joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

UVA Graduate Seminar

29 March 2021

- 1 Symmetric functions,  $S_n$ -representations, and Frobenius characteristic
- 2 Diagonal harmonics and shuffle conjectures
- 3 Stable series approach
- 4 Application: extended Delta conjecture

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Based off of slides from

- Mark Haiman: “A Shuffle Theorem for Paths Under Any Line”  
<https://www.math.uwaterloo.ca/~opecheni/2020-06-12-AlCoVE.pdf>
- Jennifer Morse: “Hey Series, Tell Me About the Extended Delta Conjecture” (ICERM, March 22, 2021)

# Multivariate Polynomials

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- $\sigma \in S_n$  acts as  $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

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$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$  forms a vector space,  $\Lambda_{\mathbb{Q}}$ .



# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of  $\Lambda_{\mathbb{Q}}$ ?

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\hline\hline\hline\hline\hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|}\hline\hline\hline\hline\hline\end{array}$$

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Collection is called  $\text{SSYT}(\lambda)$ .

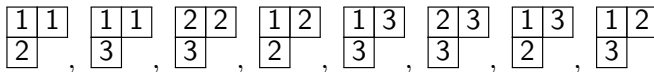
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For  $\lambda = (2, 1)$ ,

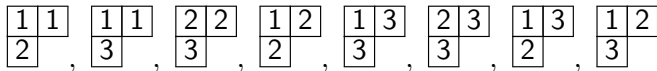


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1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2
2		3		3		2		3		3		2		3	

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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For  $\lambda$  a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for  $\Lambda_{\mathbb{Q}}$



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$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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Schur basis expansion counts multiplicity of irreducible  $S_n$ -representations!

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- 1 Schur functions  $\leftrightarrow$  irreducible  $S_n$ -representations.
- 2 Via Frobenius characteristic map, questions about  $S_n$ -action on vector spaces get translated to questions about Schur expansion coefficients in symmetric functions.

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Break  $M$  up into smallest  $S_n$  fixed subspaces

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Solution: irreducible  $S_n$ -representation of polynomials of degree  $d \mapsto q^d s_\lambda$   
(graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .

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- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu$

# An example of bi-degree

Capturing even more information...

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
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$$\tilde{H}_\mu = qts \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} + ts \begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} + qs \begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} + s \begin{smallmatrix} \square & \square & \square \end{smallmatrix}$$

- ① Symmetric functions,  $S_n$ -representations, and Frobenius characteristic
- ② **Diagonal harmonics and shuffle conjectures**
- ③ Stable series approach
- ④ Application: extended Delta conjecture

- $DH_n = \left\{ f \in \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}.$

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- E.g., Frobenius characteristic for  $DH_3$ :

$$(q^3 + q^2t + qt^2 + t^3 + qt)s_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} + (q^2 + qt + t^2 + q + t)s_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + s_{\square \square \square}$$

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## Question

What symmetric function gives the Frobenius characteristic of  $DH_n$ ?

Frobenius characteristic of  $DH_3$ :

# Diagonal Harmonics

Frobenius characteristic of  $DH_3$ :

$$\frac{t^3 \tilde{H}_{111}}{-qt^2 + t^3 + q^2 - qt} + \frac{(-q^2t - qt^2 - qt) \tilde{H}_{21}}{-q^2t^2 + q^3 + t^3 - qt} + \frac{-q^3 \tilde{H}_3}{-q^3 + q^2t + qt - t^2}$$

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## Definition

Define  $\nabla: \Lambda \rightarrow \Lambda$  via

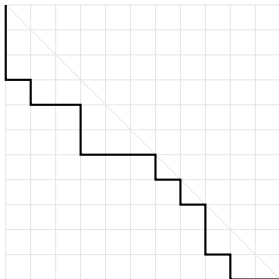
$$\nabla(\tilde{H}_\mu) = q^{n(\mu)} t^{n(\mu')} \tilde{H}_\mu$$

Nice, but not combinatorial...

# Dyck paths

## Dyck paths

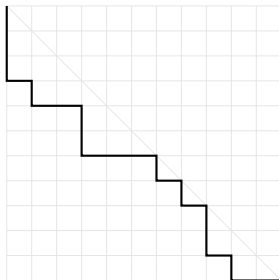
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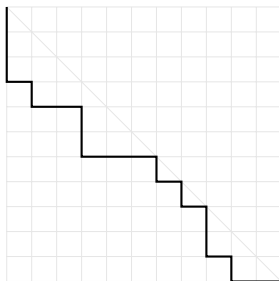


- $\text{area}(\lambda) =$  number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.

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- $\text{area}(\lambda)$  = number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.
- E.g., above  $\text{area}(\lambda) = 10$ .

# Shuffle Conjecture

Conjecture (Haglund-Haiman-Loehr-Remmel-Ulyanov, 2005)

$$\nabla e_n = \sum_{\lambda \in \mathbf{DP}_n} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(x; q^{-1}).$$

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- $\omega \mathcal{G}_{\nu(\lambda)}$  an “LLT polynomial” associated to  $\lambda$  given as a  $q$ -weight generating function over tuples of row SSYTs.

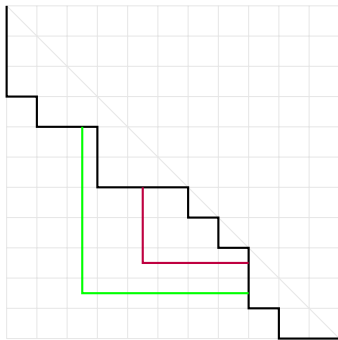
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- $\omega \mathcal{G}_{\nu(\lambda)}$  an “LLT polynomial” associated to  $\lambda$  given as a  $q$ -weight generating function over tuples of row SSYTs.
- $\text{dinv}(\lambda) =$  number of balanced hooks.



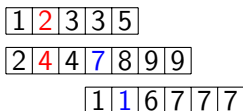


Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}.$$

$$\mathcal{G}_\nu(x; q^{-1}) = \sum_{T \in \text{SSYT}(\nu)} q^{-i(T)} x^T$$

for  $i(T)$  the number of attacking inversions:



- $\mathcal{G}_\nu$  is symmetric and Schur positive.

## Representation Theory: Diagonal Harmonics

$$DH_n = \left\{ f \in \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \sum_{1 \leq j \leq n} \partial_{x_j}^a \partial_{y_j}^b f(x, y) = 0 \right\}$$

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Frobenius characteristic  $\nabla e_n$ .

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## Combinatorics: Shuffle Theorem (Carlsson-Mellit, 2018)

$$\nabla e_n = \sum_{\lambda \in \mathbf{DP}_n} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(x; q).$$

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# Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

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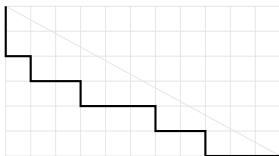
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Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

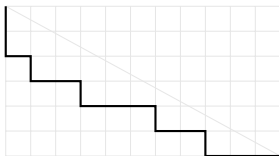
$$e_k[-MX^{m,n}] \cdot 1 = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega_{\mathcal{G}_{\nu(\lambda)}}(X; q^{-1})$$

where summation is over all  $(kn, km)$ -Dyck paths.

# Rational Path Combinatorics

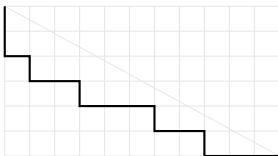


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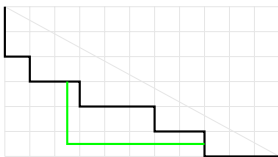


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$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \quad p = \frac{n}{m} - \epsilon$$

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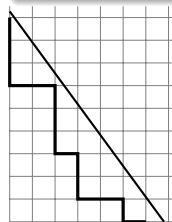


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## Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \dots, x_l) = H_{q,t} \left( \frac{x_1^{b_1} \dots x_l^{b_l}}{\prod_{i=1}^{l-1} (1 - qtx_i/x_{i+1})} \right)_{\text{pol}}$$

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- $\chi_\lambda \leftrightarrow s_\lambda$  when  $\lambda_I \geq 0$ .

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For  $\mathbf{b} \in \mathbb{Z}^I$  corresponding to some choice of coprime  $m, n$ ,

$$\begin{aligned} H_{q,t} \left( \frac{x_1^{b_1} \cdots x_I^{b_I}}{\prod_{i=1}^{I-1} (1 - qtx_i/x_{i+1})} \right) \\ = \sum_{a_1, \dots, a_{I-1} \geq 0} t^{|\mathbf{a}|} \mathcal{L}_{((b_I, \dots, b_1) + (0, a_{I-1}, \dots, a_1)) / (a_{I-1}, \dots, a_1, 0)}^\sigma(x; q) \end{aligned}$$

- Infinite series of  $GL_I$ -characters  $\chi_\lambda$  where  $\lambda \in \mathbb{Z}^I$  satisfies  $\lambda_1 \geq \cdots \geq \lambda_I$ .
- $\chi_\lambda \leftrightarrow s_\lambda$  when  $\lambda_I \geq 0$ .
- Under polynomial truncation,  $\mathcal{L}_{\beta/\alpha}^\sigma \rightarrow q^{\text{dinv}_p(\lambda)} \mathcal{G}_{\nu(\lambda)}$

- (Twisted) non-symmetric Hall-Littlewood polynomials  $E_{\lambda}^{\sigma}(x; q)$  defined via Demazure-Lusztig operators.

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- $\mathcal{L}_{\beta/\alpha} = H_q(w_0(F_{\beta}^{\sigma^{-1}}(x; q) \overline{E_{\alpha}^{\sigma^{-1}}(x; q)}))$

# What have we learned?

## Shuffle Theorem for any path

$$D_{\mathbf{b}} \cdot 1 = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}$$

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- ① Symmetric functions,  $S_n$ -representations, and Frobenius characteristic
- ② Diagonal harmonics and shuffle conjectures
- ③ Stable series approach
- ④ **Application: extended Delta conjecture**

# Another family of symmetric function operators

Changing the eigenvalues of Macdonald polynomials:

$$\Delta_f H_\mu = f[B_\mu] H_\mu \quad \Delta'_f H_\mu = f[B_\mu - 1] H_\mu$$

for any  $f \in \Lambda$  and  $B_\mu = \sum_{(i,j) \in \mu} q^{i-1} t^{j-1}$ . (Note  $\Delta'_{e_{n-1}} e_n = \nabla e_n$ ).

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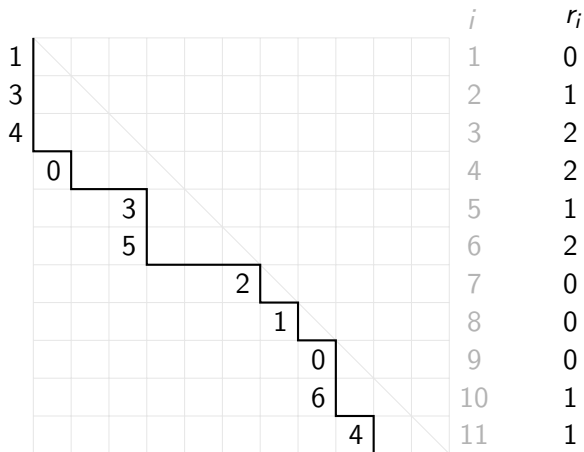
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## Extended Delta Conjecture (Haglund-Remmel-Wilson, 2018)

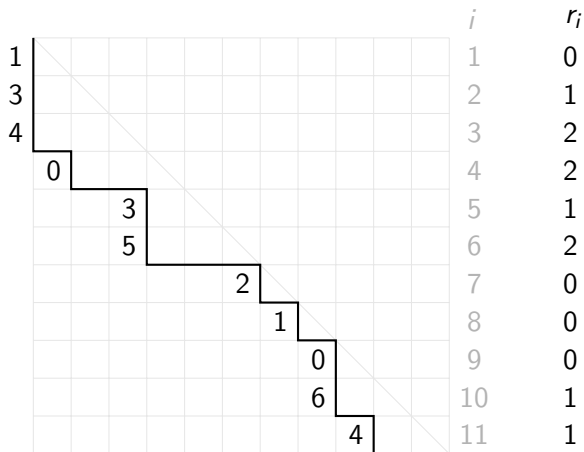
$$\Delta_{h_l} \Delta'_{e_{k-1}} e_n = \langle z^{n-k} \rangle \sum_{\lambda \in \mathbf{DP}_{n+l}} \sum_{P \in LD_{n+l,l}(\lambda)} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^{\text{wt}_+(P)} \prod_{r_i(\lambda)=r_{i-1}(\lambda)+1} \left(1 + z t^{-r_i(\lambda)}\right)$$

# Delta Combinatorics



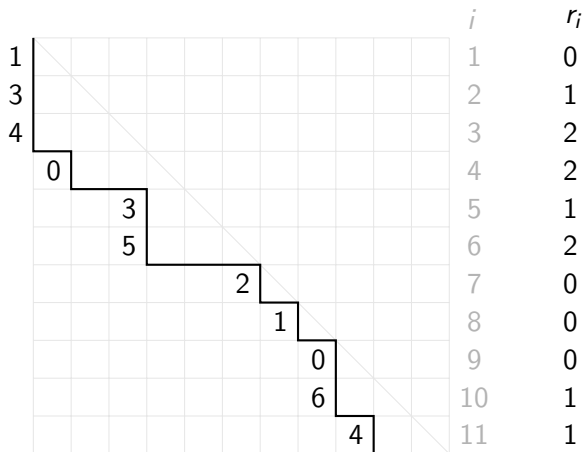
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- $\text{wt}_+ = x_1^2 x_2 x_3^2 x_4^2 x_5 x_6$
- $\text{dinv} \leftrightarrow i(T)$  under suitable translation.

## Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S.)

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# Application of previous program

## Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S.)

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 &= \sum_{\substack{J \subseteq [k+l-1] \\ |J|=l}} \sum_{\substack{(0, \mathbf{a}), \tau \in \mathbb{N}^{k+l} \\ |\tau|=n-k}} t^{|\mathbf{a}|} q^{d(\mathbf{a}, \tau, J)} (\mathcal{L}_{\beta/\alpha}^{w_0})_{\text{pol}}
 \end{aligned}$$

## Stable Extended Delta Theorem

$$\begin{aligned}
 & H_q \left( \frac{\prod_{i+1 \leq j} (1 - qtx_i/x_j)}{\prod_{i < j} (1 - tx_i/x_j)} (x_1 \cdots x_{k+l}) h_{n-k}(x_1, \dots, x_{k+l}) \overline{e_l(x_2, \dots, x_{k+l})} \right) \\
 &= \sum_{\substack{J \subseteq [k+l-1] \\ |J|=l}} \sum_{\substack{(0, \mathbf{a}), \tau \in \mathbb{N}^{k+l} \\ |\tau|=n-k}} t^{|\mathbf{a}|} q^{d(\mathbf{a}, \tau, J)} \mathcal{L}_{\beta/\alpha}^{w_0}
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# What next?

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- ② Combinatorial description of Schur expansion coefficients for  $D_{\mathbf{b}} \cdot 1$ ?
- ③ Loehr-Warrington conjecture for  $\nabla s_{\lambda}$ .

# References

Thank you!

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