K-theoretic Catalan functions

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6 February 2020

Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Next Step: Flag Variety

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Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$ are combinatorially unknown.

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
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And many more!

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Upshot

Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

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  - Definition with t important for Macdonald polynomials.
- Many definitions. A new one makes proofs easier!

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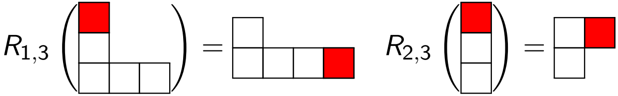
- K-theoretic Catalan functions

- Raising operators $R_{i,j}$ act on diagrams



Raising Operators on Symmetric Functions

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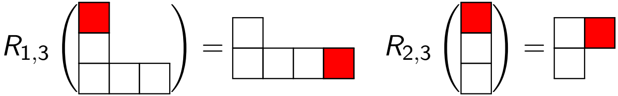

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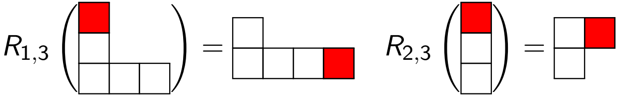

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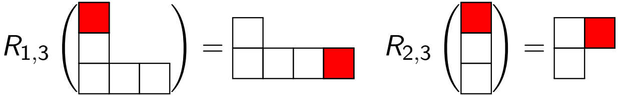

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$$\begin{aligned} s_{22} &= (1 - R_{12})h_{22} = h_{22} - h_{31} \\ s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13})h_{211} \\ &= h_{211} - h_{301} - h_{220} \textcolor{red}{- h_{310} + h_{310}} + \underbrace{h_{32\,\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\,\text{-}1}}_{=0} \end{aligned}$$

some terms cancel

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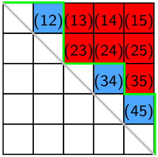
For $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



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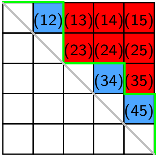

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For Ψ and $\gamma \in \mathbb{Z}^\ell$

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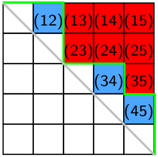

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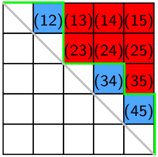

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- $\Psi =$ all roots $\implies H(\Psi;\gamma) = h_\gamma$

Catalan functions

k-Schur root ideal for λ

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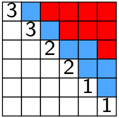
$\Delta^4(3,3,2,2,1,1) =$ 

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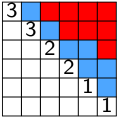
$\Delta^4(3,3,2,2,1,1) =$ 

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k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

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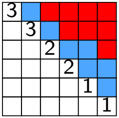
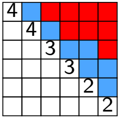
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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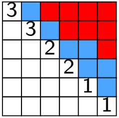
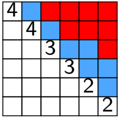
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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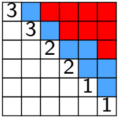
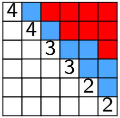
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

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- **K-theoretic Catalan functions**

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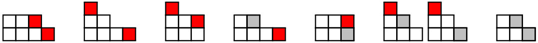

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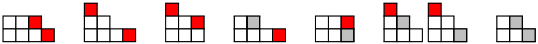

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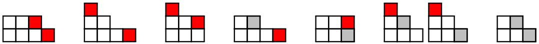

- $g_\lambda = \prod_{i<j}(1 - R_{ij}) k_\lambda$ for k_λ and inhomogeneous analogue of h_λ.
- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m,n))$

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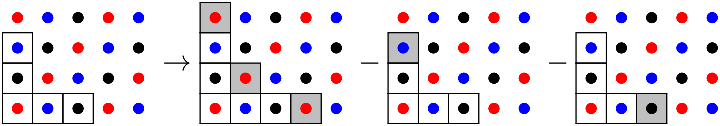
2-bounded partitions $\leftrightarrow$ 3-cores



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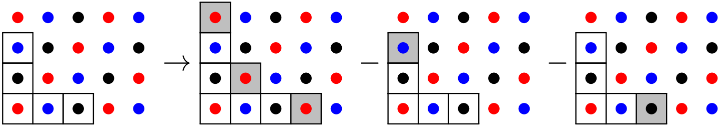

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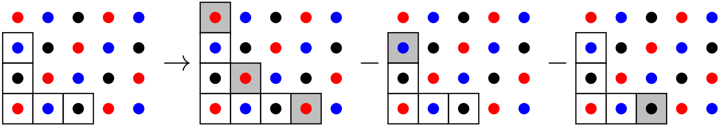

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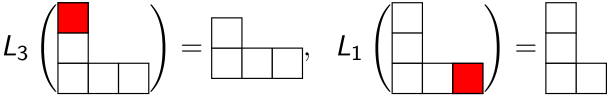
Problem

No direct formula for $g_\lambda^{(k)}$

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$

$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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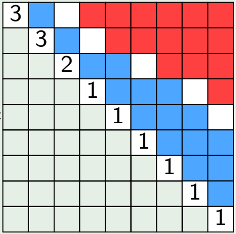
$g^{(4)}_{332111111} =$



$\Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$

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Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

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References

Thank you!

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