Diagonal Harmonics and Shuffle Theorems

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on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
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Capsule Research Talk

23 August 2021

Outline

- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1, \ldots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \text{ or } h_r = \sum_{i_1\leq i_2\leq\cdots\leq i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

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- E.g. for $n = 3$,

$$\begin{aligned} e_1 = x_1 + x_2 + x_3 &= h_1 \\ e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad & h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 = x_1x_2x_3 \quad & h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

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- Let $\Lambda = \mathbb{Q}(q,t)[e_1, e_2, \ldots] = \mathbb{Q}(q,t)[h_1, h_2, \ldots]$. Call these "symmetric functions."

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- Λ is a $\mathbb{Q}(q,t)$-algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_\mu(x_1, \ldots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i<j}(1 - x_j/x_i)} \right)$$

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- Basis of symmetric polynomials indexed by integer partitions $\mu = (\mu_1, \ldots, \mu_l) \in \mathbb{Z}^l$ where $\mu_1 \geq \cdots \geq \mu_l \geq 0$.

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Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting.

Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega G_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

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- Algebraic LHS: ∇e_k doubly graded character of diagonal coinvariants for S_k ((Haiman, 2002) via Hilbert Scheme connection).

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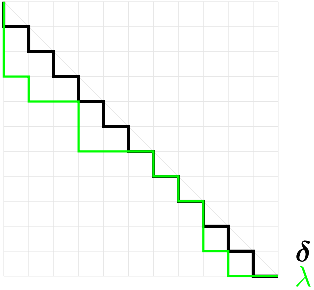
- Summation over all k-by-k Dyck paths.
- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

Dyck paths

A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



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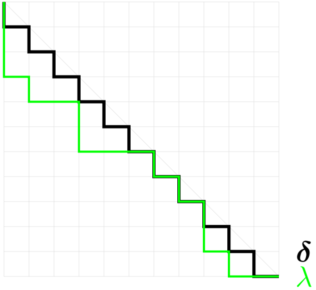

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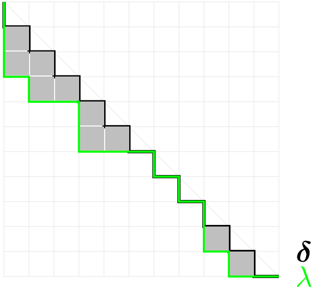

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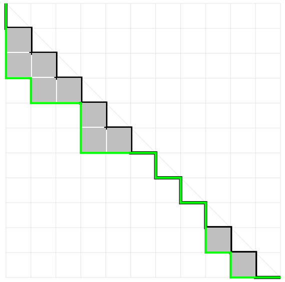

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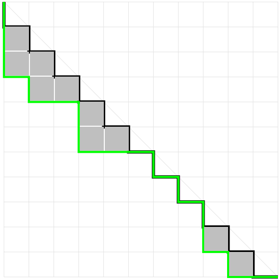

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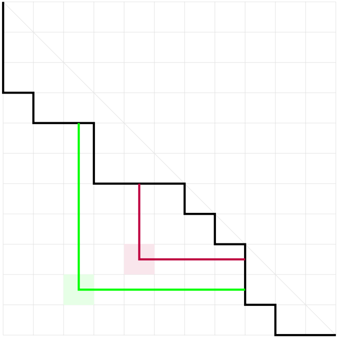
$\operatorname{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ.



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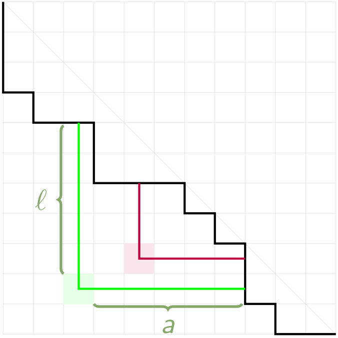

Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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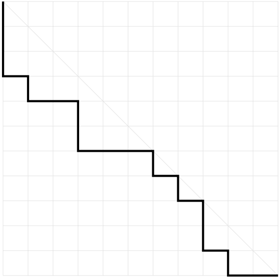

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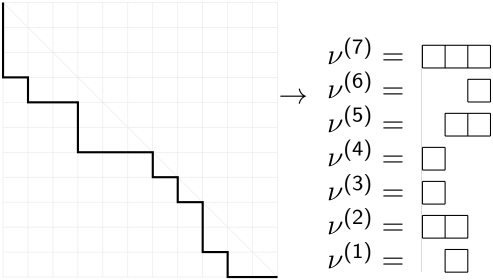

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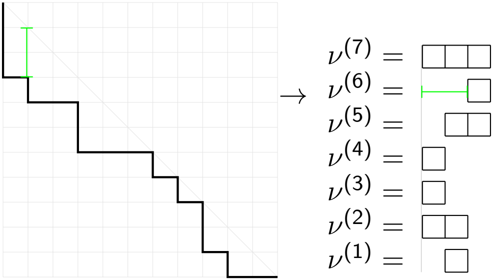

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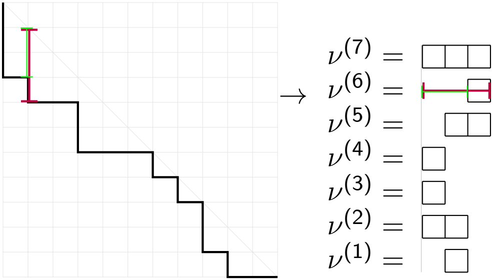

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\boxed{1} & \boxed{2} & \boxed{3} & \boxed{3} & \boxed{5} & & & & \\
\boxed{2} & \boxed{4} & \boxed{4} & \boxed{7} & \boxed{8} & \boxed{9} & \boxed{9} & & \\
 & & & \boxed{1} & \boxed{1} & \boxed{6} & \boxed{7} & \boxed{7} & \boxed{7}
\end{array}$$

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$$\mathcal{G}_{\substack{\square\square \\ \square}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{r} \boxed{1}\boxed{1} \\ \boxed{1} \end{array} \quad \begin{array}{r} \boxed{1}\boxed{2} \\ \boxed{1} \end{array} \quad \begin{array}{r} \boxed{1}\boxed{2} \\ \boxed{2} \end{array} \quad \begin{array}{r} \boxed{2}\boxed{2} \\ \boxed{2} \end{array} \quad \begin{array}{r} \boxed{1}\boxed{1} \\ \boxed{2} \end{array} \quad \begin{array}{r} \boxed{2}\boxed{2} \\ \boxed{1} \end{array}$$

$$= s_3 + q s_{2,1}$$

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$$\begin{aligned}\mathcal{G}_{\begin{smallmatrix}\square&\square\\ &\square\end{smallmatrix}}(x_1,x_2;q) =& x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + qx_1^2x_2 + qx_1x_2^2 \\ & \begin{array}{cccccc} \begin{array}{cc}\boxed{1}&\boxed{1}\\ &\boxed{1}\end{array} & \begin{array}{cc}\boxed{1}&\boxed{2}\\ &\boxed{1}\end{array} & \begin{array}{cc}\boxed{1}&\boxed{2}\\ &\boxed{2}\end{array} & \begin{array}{cc}\boxed{2}&\boxed{2}\\ &\boxed{2}\end{array} & \begin{array}{cc}\boxed{1}&\boxed{\color{red}1}\\ &\boxed{\color{red}2}\end{array} & \begin{array}{cc}\boxed{\color{blue}2}&\boxed{2}\\ &\boxed{\color{blue}1}\end{array} \end{array} \\ =& s_3 + qs_{2,1}\end{aligned}$$

- $\mathcal{G}_\nu$ is symmetric and Schur positive.

Example ∇e_3

λ	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)}$	$q^{\mathsf{dinv}(\lambda)} t^{\mathsf{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$

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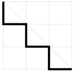		
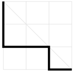		
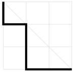		
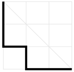		
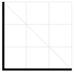		

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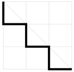
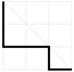
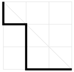
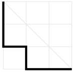
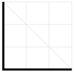
	q^3	
	$q^2 t$	
	qt	
	qt^2	
	t^3	

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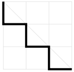
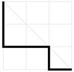
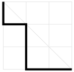
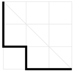
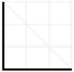
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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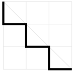
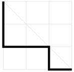
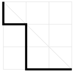
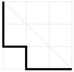
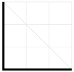
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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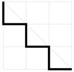
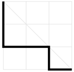
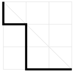
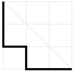
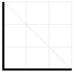
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

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- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra $\mathcal{E}$

- $\mathcal{E}$ contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

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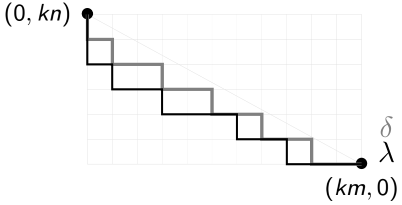

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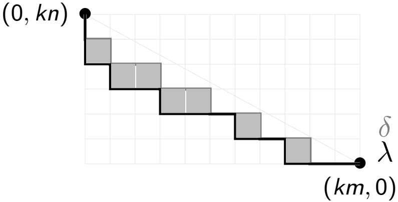

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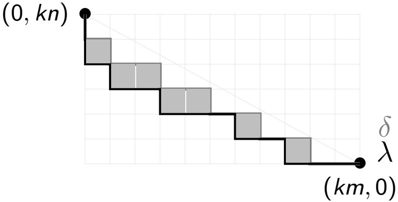

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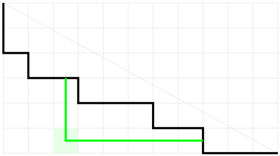

$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \qquad p = \frac{n}{m} - \epsilon$$

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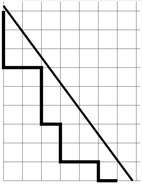

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Proof Idea

Key Relationship

$$\omega(D_{\mathbf{b}} \cdot 1)(x_1, \ldots, x_l) = \left(\sum_{w \in S_l} w \left(\frac{x_1^{b_1} \cdots x_l^{b_l} \prod_{i+1<j}(1 - qtx_i/x_{i+1})}{\prod_{i<j}((1 - \frac{x_j}{x_i})(1 - q\frac{x_i}{x_j})(1 - t\frac{x_i}{x_j}))} \right) \right)_{\mathrm{pol}}$$

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for infinite formal sum $\mathcal{L}^{\sigma}_{\beta/\alpha}$ a "series LLT." (Grojnowski-Haiman, 2007).

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- $\mathcal{L}^\sigma_{\beta/\alpha} = H_q(w_0(F^{\sigma^{-1}}_\beta(x;q)\overline{E^{\sigma^{-1}}_\alpha(x;q)}))$ for

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Loehr-Warrington Conjecture

$$\nabla s_\mu = \mathrm{sgn}(\mu) \sum_{(G,R)\in LNDP_\mu} t^{\mathrm{area}(G,R)} q^{\mathrm{dinv}(G,R)} x^R$$

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References

Thank you!

Bergeron, Francois, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. 2016. *Compositional (km, kn)-shuffle conjectures*, Int. Math. Res. Not. IMRN **14**, 4229–4270, DOI 10.1093/imrn/rnv272. MR3556418

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