Geometric Representation Theory: Lecture notes from a class taught by Weiqiang Wang

George H. Seelinger

1. Quivers and Quiver Algebras

(8/22/2017) Lecture 1. This class will cover some topics in geometric representation theory. Our general outline is

(1) Quiver representations, including Dynkin quivers and Gabriel's classification of indecomposables.
  - Non-simply laced [Geiss, Lederc, Schröer] (arXiv 1410.1403)
  - "Affine" (Euclidean) quivers

(2) Hall algebras, including "classical" Hall algebras which leads to Hall-Littlewood polynomials, and Ringel (over $\mathbb{F}_q$), which has connections to half quantum groups.

(2.5) Springer theory. This will be covered in a reading course and not included here. However, convolution algebra on "Steinberg varieties" leads to Weyl groups and representations.

(3) Quiver varieties. From this and the above, we get the Lusztig (semi)canonical basis, which leads to Nakajima's work on integrable modules of Kac-Moody semisimple Lie algebras. Type A is covered in Chriss-Ginzburg, chapter 4.

1.1. Definition. A *quiver* is a tuple $Q = (I, \Omega, s, t)$ (sometimes denoted (Q_0, Q_1, s, t)) where I is the vertex set, Ω is the edge set, and $s, t\colon \Omega \to I$ such that, for $h \in \Omega$, $h \mapsto s(h), t(h)$, respectively. We say h starts at $s(h)$ and terminates at $t(h)$.

$$s(h) \xrightarrow{h} t(h)$$

1.2. Definition. Given a quiver Q, we say $|Q|$ is the underlying graph with direction removed.

Throughout these lecture notes, unless otherwise stated, we will always assume Q is connected and that we are working of ground field $\Bbbk$.

1.3. Example. Examples of small quivers include:

(a)

$$\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$$

(b)

$$1 \longrightarrow 2$$

(b')

$$1 \longrightarrow 2 \longrightarrow 3$$

(c)

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] & 2 \end{tikzcd}$$

1.4. Definition. A *representation* of a quiver Q consists of the data

- To each $i \in I$, assign $\Bbbk$-vector space V_i.
- To each $h \in \Omega$, assign a linear map $x_h \colon V_{s(h)} \to V_{t(h)}$

1.5. Definition. A *morphism of representations* $f \colon V \to W$ consists of linear maps $f_i \colon V_i \to W_i$ and makes the following diagram commute.

$$\begin{array}{ccc} V_{s(h)} & \xrightarrow{x_h} & V_{t(h)} \\ \Big\downarrow{\scriptstyle f_{s(h)}} & & \Big\downarrow{\scriptstyle f_{t(h)}} \\ W_{s(h)} & \xrightarrow{y_h} & W_{t(h)} \end{array}$$

1.6. Definition. Given representations V, W, we define $\operatorname{Hom}(V, W) = \operatorname{Hom}_Q(V, W)$ as the set of all morphisms from V, W. We also let $\operatorname{Aut}_Q(V)$ be all invertible morphisms in $\operatorname{Hom}(V, V)$.

We will denote the category of (not necessarily finite-dimensional) representations of Q as $\operatorname{Rep}(Q)$. A basic problem of this course is to classify (simple, indecomposable) representations of Q, up to isomorphism.

1.7. Example.

(a) Consider 1.3(a). This is the same as asking the classification of all $f \colon \Bbbk^n \to \Bbbk^n$. If $\Bbbk = \mathbb{C}$, then this is well-known by the Jordan canonical form for square matrices.

(b) Consider 1.3(b) and assign V_1 to vertex 1 and V_2 to vertex 2. Then, if$x \colon V_1 \to V_2$ has rank r, it is of the form, up to a change of basis for V_1 and V_2,

$$x \approx \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$$

Remember that we can make sense of subrepresentations and quotient representations of Q.

1.8. Theorem. *$\operatorname{Rep}(Q)$ is closed under $\bigoplus$, taking kernel/cokernel, and subreprresentations and quotients. That is, $(\ker f)_i := \ker(f_i \colon V_i \to W_i)$. Hence, $\operatorname{Rep}(Q)$ is an abelian category.*

Proof. We must show that $\ker f$ is actually a representation. If $v \in (\ker f)_{s(h)}$, then $(f_{t(h)} \circ x_h)(v) = (y_h \circ f_{s(h)})(v) = 0$ by the definition of a representation morphism and the fact that v is in the kernel of $f_{s(h)}$. Thus $x_h(v) \in (\ker f)_{t(h)}$ and x_h induces a map from $(\ker f)_{s(h)} \to (\ker f)_{t(h)}$. We can apply a similar argument for cokernel.

If we define $\oplus$ in the obvious way, $(V \oplus W)_i = V_i \oplus W_i$, we get a direct sum representation. $\square$

1.9. Definition. A *path* in a quiver Q (of length ℓ) is a series of edges in Q, say $(h_1, \ldots, h_\ell)$ such that $t(h_i) = s(h_{i+1})$ for all i. We denote such a path as $h_\ell \cdots h_2 h_1 \colon s(h_1) \to t(h_\ell)$.

1.10. Definition. A *path algebra* $\Bbbk Q$ is given as the free module with basis given by paths in the quiver Q endowed with a multiplication operator on basis elements f, g given by

$$f \cdot g = \begin{cases} 0 & t(g) \neq s(f) \\ s(g) \xrightarrow{g} t(g) \xrightarrow{f} t(f) & t(g) = s(f) \end{cases}$$

1.11. Definition. A *path of length 0* in a quiver Q is of the form e_i for $i \in I$. For $p \in \Bbbk Q$, we say

$$e_i \cdot p = \begin{cases} p & t(p) = i \\ 0 & \text{else} \end{cases} \qquad p \cdot e_i = \begin{cases} p & s(p) = i \\ 0 & \text{else} \end{cases}$$

1.12. Proposition. *e_i is an orthogonal idempotent of $\Bbbk Q$.*

Proof. Consider that $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$ by definition of e_i. (Note, as stated, this is not actually true.) $\square$

1.13. Proposition. *$\Bbbk Q$ is associative with $1 := \sum_{i \in I} e_i$.*

1.14. Proposition. *$\Bbbk Q$ is $\mathbb{Z}_{\geq 0}$-graded by path lengths.*

1.15. Example. For quiver $Q = (I, \Omega, s, t)$, we get $(\Bbbk Q)_0 = \bigoplus_{i \in I} \Bbbk e_i$ and $(\Bbbk Q)_1 = \bigoplus_{h \in \Omega} \Bbbk h$.

1.16. Proposition. *$\Bbbk Q$ is finite-dimensional if and only if Q has no oriented cycles.*

Proof. If Q has an oriented cycle, then $\Bbbk Q$ may be infinite-dimensional. Consider Q as 1.3(a). Then, $\Bbbk Q$ is the algebra of all polynomials in 1 generator, $\Bbbk[x]$. For the forwards direction, ... $\square$

1.17. Example. Consider the path algebra of 1.3(b). This algebra has dimension 3: 2 paths of length 0 and 1 path of length 1. Similarly, 1.3(b') has 3 paths of length 0, 2 paths of length 1, and 1 path of length 2, giving us a total of 6 paths.

1.18. Proposition. *Let $A := \Bbbk Q$. Then,*

$$A = \bigoplus_{i \in I} A e_i$$

and Ae_i is a projective left A-module. Furthermore, given an A-module M, we get

$$\mathrm{Hom}_A(Ae_i, M) \cong e_i M$$

Proof. The decomposition of A is a standard fact for primitive orthogonal idempotents of an algebra. Ae_i is projective because it is a direct summand of the free A-module A.

To show the isomorphism, we first note that $f \in \mathrm{Hom}_A(Ae_i, M)$ is uniquely determined by $f(e_i)$. This is because f must be A-linear, so

$f(ae_i) = a.f(e_i)$ for all $a \in A$. Furthermore, $f(e_i) \in e_iM$ because $f(e_i) = f(e_i^2) = e_1.f(e_i)$. Thus, consider the A-module homomorphism given by $\phi\colon \mathrm{Hom}_A(Ae_i, M) \to e_iM$ given by $f \mapsto f(e_i)$. We already know that ϕ is injective since, if $\phi(f) = \phi(g)$, then $f(e_i) = g(e_i) \implies f = g$. Furthermore, ϕ is injective since, given $e_im \in e_iM$, there is an $f \in \mathrm{Hom}(Ae_i, M)$ such that $f(e_i) = m$. $\square$

1.19. Proposition. *Given $0 \neq f \in Ae_i$ and $0 \neq g \in e_iA$, it must be that $fg \neq 0$. This follows from the fact that fg must pass through i and has nonzero length.*

1.20. Proposition. *The e_i's are* primitive *idempotents, that is, Ae_i is indecomposable.*

Proof. We wish to show that the only idempotent in $\mathrm{End}_A(Ae_i)$ is e_i since, if $Ae_i \cong M \oplus N$ for non-trivial M, N, then $Ae_i \twoheadrightarrow M \hookrightarrow Ae_i$ would be an idempotent in $\mathrm{End}_A(Ae_i)$. By 1.18, $\mathrm{End}_A(Ae_i) \cong e_iAe_i$. Take idempotent $f \in e_iAe_i$. Then, $f^2 = f = fe_i$, so $f(f - e_i) = 0 \implies f = e_i$ by the preceding proposition. $\square$

1.21. Proposition. *$Ae_i \ncong Ae_j$ as A-modules for $i \neq j$*

Proof. If $Ae_i \cong Ae_j$, then take $f \in \mathrm{Hom}_A(Ae_i, Ae_j) \cong e_iAe_j$ (by previous proof) to be an isomorphism and $g \in e_jAe_i$ to be its inverse. Then, $fg = e_i \implies e_i \in Ae_jA$. This is only possible when $i = j$. $\square$

1.22. Proposition. *$\mathrm{Rep}(Q) \cong \Bbbk Q$-Mod.*

Proof. Consider the map $(V_i)_{i\in I} \mapsto V = \bigoplus_{i\in I} V_i$ where

$$h_\ell \cdots h_q \cdot v = \begin{cases} x_{h_\ell} \cdots x_{h_1}(v_i) & i = s(h_i) \\ 0 & \text{else} \end{cases}$$

This sends a representation (V_i) to a $\Bbbk Q$-module.

Conversely, given a $\Bbbk Q$-module M, we can construct an inverse to the map above by letting $M_i := e_iM$ and taking $M = \bigoplus_{i\in I} e_iM$ and letting $h \in \Omega$ be $h\colon M_{s(h)} \to M_{t(h)}$. We note that $h(M_i) \subseteq M_j$, so the collection (M_i) satisfies the definition of a representation of Q. $\square$

Due to this theorem, we will often use representations of Q and $\Bbbk Q$-modules interchangably.

(8/24/2017) Lecture 2.

1.23. Definition. Let Q be a quiver. For $i \in I$, we define the representation

$$S(i) := \begin{cases} \Bbbk & \text{at } i \\ 0 & \text{else} \end{cases}$$

1.24. THEOREM. *If a quiver Q has no oriented cycles, then $\{S(i)\}_{i\in I}$ is a full list of simple $\Bbbk Q$-modules.*

PROOF BY EXAMPLE. Let V be a simple representation of quiver

$$1 \longrightarrow 2$$

with $V_1 \neq 0 \neq V_2$. Then, consider the inclusion of the representation given below.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & V_2 \arrow[d] & = e_2 V \\ V_1 \arrow[r] & V_2 & \end{tikzcd}$$

Because there are no edges starting from 2, there can be no non-zero map starting at V_2 in V, so we have found a sub-representation of V, which is a contradiction. The full proof is of the same spirit, but replace 1 and 2 with i and j where, since Q is acyclic, there is a j such that no edges start at j. $\square$

1.25. DEFINITION. Let $A = \Bbbk Q$. Then, we define

$$P(i) := Ae_i$$

which is projective, as noted earlier.

Since $P(i)$ consists of all paths starting at i, then, if there is an edge going from i to j, $P(i)$ has a submodule isomorphic to $P(j)$.

1.26. PROPOSITION. *For an acyclic quiver Q, we get $P(i)/\operatorname{rad} P(i) \cong \operatorname{hd}(P(i)) \cong S(i)$ where $\operatorname{rad} P(i)$ is the intersection of all maximal submodules of $P(i)$.*

PROOF. $\square$

1.27. EXAMPLE. Let Q be the quiver

$$1 \longrightarrow 2 \longrightarrow 3$$

Then, we get that $P(1), P(2), P(3)$ are given respectively by

$$\Bbbk \xrightarrow{\sim} \Bbbk \xrightarrow{\sim} \Bbbk \ , \quad 0 \longrightarrow \Bbbk \xrightarrow{\sim} \Bbbk \ , \quad 0 \longrightarrow 0 \longrightarrow \Bbbk$$

One fast consequence of this is that $P(3) = S(3)$.

1.28. PROPOSITION. *Let $V \in \operatorname{Rep}(Q)$. Then,*

$$\operatorname{Hom}_Q(P(i), V) = V_i$$

PROOF. Let $A = \Bbbk Q$. Then,

$$\operatorname{Hom}_Q(P(i), V) \cong \operatorname{Hom}_{\Bbbk Q}(Ae_i, V) = e_i V$$

from above. Now, since, for $v \in V$,

$$e_i.v = x_{e_i}(v) = \begin{cases} v & \text{if } v \in V_i \\ 0 & \text{else} \end{cases}$$

it must be that $e_iV \subseteq V_i$. However, it is also clear that x_{e_i} is the identity on V_i, os $V_i \subseteq e_iV$. $\square$

1.29. PROPOSITION. *Assume Q has no oriented cycles. Then, $\{P(i)\}_{i\in I}$ is the full set of projective indecomposable modules (PIM) in* $\mathrm{Rep}(Q)$.

1.30. LEMMA. *If Q has no oriented cycles,* $\mathrm{End}(P(i)) = \Bbbk$.

PROOF OF LEMMA. Using previous results, we have that $\mathrm{Hom}(P(i), P(i)) \cong e_iP(i) \cong e_iAe_i$ since $P(i) = Ae_i$. However, since Q has no oriented cycles, the only path that begins and ends at i is the trivial path, so $e_iAe_i \cong \Bbbk$. $\square$

PROOF OF PROPOSITION. The lemma gives us that $P(i)$ is indecomposable. Now, consider a projective module P and let P' be a direct sum of $P(i)$'s with multiplicity $n_i = \dim \mathrm{Hom}(P, S(i))$. Then, we get that $\mathrm{Hom}(P, S(i)) \cong \mathrm{Hom}(P', S(i))(\cong \Bbbk^{n_i}$ I think) from the previous proposition. Thus, for any representation V, we can (somehow) show $\mathrm{Hom}(P, V) \cong \mathrm{Hom}(P', V)$. Thus, $P \cong P'$. $\square$

1.31. DEFINITION. The *Grothendieck* group of a category $\mathcal{C}$, denoted $K(\mathcal{C})$, is an abelian group generated by letters $[M]$ for $M \in \mathcal{C}$ and with the relations that $A - B + C = 0$ if there exists a short exact sequence

$$0 \to A \to B \to C \to 0$$

1.32. DEFINITION. For some quiver Q, we denote $K(Q) := K(\mathrm{Rep}(Q))$.

1.33. PROPOSITION. *Assume that Q has no oriented cycles. Then,*

(a) The Grothendieck group $K(Q) \cong \mathbb{Z}^I$ with isomorphism given by

$$[V] \mapsto \mathbf{dim}V$$

where $\mathbf{dim}V = (\dim V_i)_{i\in I}$. This is called the graded dimension.

(b) $\{[S(i)]\}_{i\in I}$ form a basis for $K(Q)$.

(c) $\{[P(i)]\}_{i\in I}$ form a basis for $K(Q)$.

The last two statements are similar to having a basis given by a diagonal matrix versus an upper triangular matrix.

We now move into some abstract nonsense (which we will not prove here) to lay the groundwork for the standard resolution.

1.34. PROPOSITION. *Let A be an associative algebra with 1 and $M \in A$-mod. Then, we have*

$$M \cong A \otimes_A M \cong (A \otimes_{\Bbbk} M)/I$$

where I is the $\Bbbk$-subspace spanned by $ab \otimes m - a \otimes bm$ for all $a \in A, m \in M, b \in A$.

This proposition comes in two variations.

1.35. Proposition (Variation 1). *Replace "$\mathfrak{b} \in A$" by "$\mathfrak{b} \in L$" where L is a generating set of A. Reformulated as an exact sequence in A-mod, we get*

$$A \otimes L \otimes_{\Bbbk} M \xrightarrow{d_1} A \otimes_{\Bbbk} M \xrightarrow{d_0} M \to 0$$

where d_1 sends $a \otimes \ell \otimes m \mapsto a\ell \otimes m - a \otimes \ell m$ and d_0 sends $a \otimes m \mapsto am$.

Proof. The proof follows from the identity

$$ab_1b_2 \otimes m - a \otimes b_1b_2m = (ab_1b_2 \otimes m - ab_1 \otimes b_2m) + (ab_1 \otimes b_2m - a \otimes b_1b_2m) \in I$$

$\square$

1.36. Proposition (Variation 2). *Take subalgebra $A_0 \subseteq A$ and $\Bbbk$-subspace $L \subseteq A$ such that $A_0L \subseteq L$, $LA_0 \subseteq L$, and $\langle A_0, L\rangle = A$. Then,*

$$A \otimes_{A_0} L \otimes_{A_0} M \to A \otimes_{A_0} M \to M \to 0$$

is exact.

1.37. Theorem (The Standard Resolution). *Let $V \in \Bbbk Q$-Mod. Then, there exists an exact sequence of $\Bbbk Q$-modules*

$$0 \to \bigoplus_{h \in \Omega} P(t(h)) \otimes \Bbbk h \otimes V_{s(h)} \xrightarrow{d_1} \bigoplus_{i \in I} P(i) \otimes V_i \xrightarrow{d_0} V \to 0$$

given by maps $p \otimes h \otimes v \mapsto ph \otimes v - p \otimes x_h(v)$ and $p \otimes v \mapsto x_p(v)$.

Proof. Using variation 2, we take $A = \Bbbk Q$, $A_0 = \bigoplus_{i \in I} \Bbbk e_i$, and $L = \bigoplus_{h \in \Omega} \Bbbk h$. Then,

$$A \otimes_{A_0} V = (\bigoplus_{i \in I} Ae_i) \otimes V = \bigoplus_{i \in I} (Ae_i \otimes V) = \bigoplus_{i \in I} Ae_i \otimes e_iV = \bigoplus_{i \in I} Ae_i \otimes V_i$$

It remains to show that d_1 is injective. Assume

$$\sum_n p_n \otimes h_n \otimes v_n \overset{d_1}{\mapsto} \sum_n (p_nh_n \otimes v_n - p_n \otimes x_{h_n}(v_n)) = 0$$

We show that $\ker d_1$ is trivial by contradiction. Let $\ell =$ the maximum length of the p_n. We will then show that all such $p_nh_n \otimes v_n$ are actually zero. Take the terms in the sum (image of d_1) with length $ell + 1$, say

$$\sum_{\ell(p_n) = \ell} p_nh_n \otimes v_n = 0$$

Since none of these collected p_nh_n are equal to each other, the collection $\{p_nh_n\}_{\ell(p_n)=\ell}$ is linearly independent. Thus, it must be that $v_n = 0$, which means $p_nh_n \otimes v_n = 0$, so we have contradicted the maximality of ℓ and thus d_1 is injective. $\square$

1.38. Corollary. *(a) $\mathrm{Rep}(Q)$ has enough projectives, and so Ext^i makes sense.*

(b) For all $V, W \in \mathrm{Rep}(Q)$, we have

$$\mathrm{Ext}^i_Q(V,W) = 0, \quad i > 1$$

Proof. Given an arbitrary representation V, the standard resolution is a projective resolution of the form

$$0 \to P_1 \to P_0 \to V \to 0$$

since $V_i \cong \Bbbk^n$ for some n and is thus always free. Then computing long exact sequence

$$\cdots \to \mathrm{Ext}^{i-1}(P_1, W) \to \mathrm{Ext}^i_Q(V,W) \to \mathrm{Ext}^i_Q(P_0, W) \to \cdots$$

However, for $i \geq 1$, $\mathrm{Ext}^i(P, W) = 0$ for any projective module P by definition of a projective module. Thus, starting at $i = 2$, our long exact sequence degenerates to

$$\cdots \to 0 \to \mathrm{Ext}^i_Q(V,W) \to 0 \to \cdots$$

and thus, by exactness, $\mathrm{Ext}^i_Q(V,W) = 0$ for $i > 1$. □

1.39. Example. The standard resolution for $S(i)$ is given by

$$0 \to \bigoplus_{h\colon i \to j} P(j) \to P(i) \to S(i) \to 0$$

since $(S(i))_j = 0$ for $j \neq i$.

1.40. Proposition. *For simple representations $S(i), S(j)$ of a quiver Q,*

$$\dim \mathrm{Hom}(S(i), S(j)) = \delta_{ij}$$
$$\mathrm{Ext}^1(S(i), S(j)) = \#\{\textit{edges } i \to j\}$$

Proof. From the projective resolution of $S(i)$, we get the long exact sequence

Prove first part of proposition

$$0 \to \mathrm{Hom}(S(i), S(j)) \longrightarrow \mathrm{Hom}(P(i), S(j)) \longrightarrow \mathrm{Hom}(\textstyle\bigoplus_{h\colon i\to j} P(j), S(j)) \to \mathrm{Ext}^1(S(i), S(j)) \longrightarrow \mathrm{Ext}^1(P(i), P(j)) = 0$$

Now, if $i \neq j$, then $\mathrm{Hom}(P(i), S(j)) = S(j)_i = 0$ by earlier proposition. If $i = j$, then $\mathrm{Hom}(P(i), S(i)) = (S(i))_i \cong \Bbbk$ and, by the first part of the proposition, $\mathrm{Hom}(S(i), S(i)) \cong \Bbbk$, so an injection between them must be an isomorphism. Thus, we get

$$0 \to \mathrm{Hom}(S(i), S(j)) \xrightarrow{\sim} \mathrm{Hom}(P(i), S(j)) \to 0 \to \mathrm{Hom}(\bigoplus_{h\colon i\to j} P(j), S(j)) \xrightarrow{\sim} \mathrm{Ext}^1(S(i), S(j)) \to 0$$

thereby proving the second part of the propositon. □

1.41. Proposition. *Any submodule of a projective module $P \in \mathrm{Rep}(Q)$ is projective.*

PROOF. It suffices to show that $\mathrm{Ext}^{i\geq 1}(S,-)=0$. $\square$

1.42. REMARK. There exist "injective" counterparts using $Q(i) := (e_iA)^*$, the graded dual.

1.43. EXAMPLE. Let Q be the quiver

I do not see how this example is relevant

$$1 \longrightarrow 2 \longleftarrow 3$$

Then, we get indecomposables $S(1), S(2), S(3)$ as well as

$$\Bbbk \xrightarrow{\sim} \Bbbk \longleftarrow 0 \ , \ 0 \longrightarrow \Bbbk \xleftarrow[\sim]{} \Bbbk \ , \ \Bbbk \xrightarrow{\sim} \Bbbk \xleftarrow[\sim]{} \Bbbk$$

(8/29/2017) Lecture 3. Since we have 1.41, then $\mathrm{Rep}(Q)$ is hereditary. In particular, it is quasi-hereditary, but the standard and costandard modules are boring and, since we do not have any duality, BGG reciprocity does not yield anything terribly interesting.

1.1. Variety of representions.

For this section, let Q be a quiver and $\overline{\Bbbk} = \Bbbk$ (eg $\mathbb{C}$).

1.44. DEFINITION. Let $V \in \Bbbk Q\text{-Mod}$ and $(w_i)_{i\in I} = \vec{w} = \mathbf{dim}V$, that is to say, for $h\colon i \to j$, we have $V_i = \Bbbk^{w_i} \overset{x_h}{\to} \Bbbk^{w_j} = V_j$. Then, we define *the representation space of* $\vec{w}$ to be

$$\begin{aligned}\mathrm{Rep}(\vec{w}) &:= \{ \text{ all representations of } Q \text{ with } \mathbf{dim} = \vec{w}\} \\ &= \bigoplus_{h\in\Omega} \mathrm{Hom}_{\Bbbk}(\Bbbk^{w_{s(h)}}, \Bbbk^{w_{t(h)}}) \\ &= \bigoplus_{h\in\Omega} \mathrm{Hom}_{\Bbbk}(V_{s(h)}, V_{t(h)})\end{aligned}$$

1.45. DEFINITION. For $\vec{w} = (w_i)_{i\in I}$, let us define

$$GL(\vec{w}) := \prod_{i\in I} GL(w_i, \Bbbk)$$

1.46. PROPOSITION. *$GL(\vec{w})$ acts on $\mathrm{Rep}(\vec{w})$ by conjugation as follows. Let $(g_i) = g \in GL(\vec{w})$. Then, for all $h\colon i \to j$, g sends*

$$x_h \mapsto g_j x_h g_i^{-1}$$

1.47. PROPOSITION. *As vector spaces*

$$\dim \mathrm{Rep}(\vec{w}) = \sum_{h\in\Omega} w_{s(h)} w_{t(h)}$$

and

$$\dim GL(\vec{w}) = \sum_{i\in I} w_i^2$$

PROOF. This is clear given the decompositions of $\mathrm{Rep}(\vec{w})$ and $GL(\vec{w})$ when considering Hom spaces as matrices since these are linear transformations. $\square$

1.48. Proposition. *For $x, x' \in \operatorname{Rep}(\vec{w})$, we get*

$$\{\Bbbk Q\text{-isomorphisms } x \to x'\} \leftrightarrow \{g \in GL(\vec{w}) \mid gx = x'\}$$

Proof. $\square$

Not sure why this is true. Write out the actual isomorphism diagrams.

1.49. Corollary. *Given $x \in \operatorname{Rep}(\vec{w})$,*

$$\operatorname{Aut}_{\Bbbk Q}(x) \cong \operatorname{Stab}_G(x) =: G_x$$

Proof. It is clear that an element in the stabalizer of x is a $\Bbbk Q$ automorphism of x. By above, take $x' = x$. Then, the set on the right is $\operatorname{Stab}_G(x)$ and the set on the left is all $\Bbbk Q$-automorphisms of x. $\square$

1.50. Proposition. *Let $x, x' \in \operatorname{Rep}(\vec{w})$. Then, $x \cong x'$ if and only if x and x' are in the same G-orbit.*

Proof. This follows directly from the correspondance between $\Bbbk Q$-isomorphisms between $x \to x'$ and the set $\{g \in GL(\vec{w}) \mid gx = x'\}$. $\square$

Now, we list some facts about these orbits that follow from the fact that $GL(\vec{w})$ is a linear algebraic group (in particular, it is a variety) acting on a finite-dimensional vector space. In the following few propositions, $G := GL(\vec{v})$.

1.51. Proposition. *Each $GL(\vec{w})$-orbit of $\operatorname{Rep}(\vec{w})$ is a variety.*

1.52. Proposition. *Given $x \in \operatorname{Rep}(\vec{w})$, G_x is closed in G in the Zariski topology.*

1.53. Proposition. *The orbit $\mathcal{O}_x = G/G_x$. Furthermore,*

$$T_x\mathcal{O}_x = T_1G/T_1G_x, \text{ and } \dim \mathcal{O}_x = \dim GL(\vec{w}) - \dim G_x$$

where $\dim$ *is the dimension as a variety.*

1.54. Proposition. *There exists at most one orbit of maximum dimension equal to* $\dim \operatorname{Rep}(\vec{w})$.

1.55. Corollary. *There exists at most one orbit that is dense and open in* $\dim \operatorname{Rep}(\vec{w})$.

1.56. Proposition. *G_x is connected.*

1.57. Proposition. *For $V, W \in \operatorname{Rep}(Q)$, there exists exact sequence*

$$0 \to \operatorname{Hom}_Q(V, W) \to \bigoplus_{i \in I} \operatorname{Hom}_\Bbbk(V_i, W_i) \to \bigoplus_{h \in \Omega} \operatorname{Hom}_\Bbbk(V_{s(h)}, W_{t(h)}) \to \operatorname{Ext}^1_Q(V, W) \to 0$$

Proof. Apply $\operatorname{Hom}_Q(-, W)$ to the standard resolution, use the fact that $\operatorname{Hom}_Q(P(i), W) = W_i$ and $\operatorname{Ext}_Q(P(i), W) = 0$, as well as tensor-hom adjointness, to get the final result. $\square$

1.58. Corollary. *For $V \in \mathrm{Rep}(Q)$, and $\vec{w} = \mathbf{dim} V$, there exists exact sequence of vector spaces*

$$0 \to \mathrm{End}_Q(V) \to \bigoplus_{i \in I} \mathrm{End}(V_i) \to \mathrm{Rep}(\vec{w}) \to \mathrm{Ext}^i_Q(V, V) \to 0$$

Proof. Using the above proposition, take $W = V$. $\square$

1.59. Proposition. *Given $V \in \mathrm{Rep}(Q)$ with $\vec{w} = \mathbf{dim} V$, we get*

(a) $\dim \mathrm{Ext}^1_Q(V, V) = \dim \mathrm{Rep}(\vec{w}) - \dim \mathcal{O}_V$

(b) $\dim \mathrm{Ext}^1_Q(V, V) = \dim \mathrm{End}_Q(V) - q(\vec{w})$ *where* $q(\vec{w}) := \sum_{i \in I} w_i^2 - \sum_{h \in \Omega} w_{s(h)} w_{t(h)}$

Proof. Given that the alternating sum of the dimensions of objects in an exact sequence must be 0, we get from the short exact sequence in the proposition above

$$\dim \mathrm{Ext}^1_Q(V, V) - \dim \mathrm{Rep}(\vec{w}) + \dim \bigoplus_{i \in I} \mathrm{End}(V_i) - \dim \mathrm{End}_Q(V) = 0$$

Now, by 1.53, we know that $\dim \mathcal{O}_V = \dim GL(\vec{w}) - \dim G_V = \dim \mathrm{End}_Q(V) - \dim \bigoplus_{i \in I} \mathrm{End}(V_i)$ and thus the first equality is proven.

understand why $\dim G_V = \dim \bigoplus_{i \in I} \mathrm{End}(V_i)$

From 1.47, we get that $q(\vec{w}) = \dim GL(\vec{w}) - \dim \mathrm{Rep}(\vec{w}) = \dim \mathrm{End}_Q(V) - \dim \mathrm{Rep}(\vec{W})$ and thus the second equality is proven. $\square$

1.60. Proposition. *Let $V \in \mathrm{Rep}(\vec{w})$. Then, $\mathcal{O}_V$ is open if and only if* $\mathrm{Ext}^1_Q(V, V) = 0$.

Proof. This follows from the first equality in the above proposition. From 1.55, if $\dim \mathcal{O}_V = \dim \mathrm{Rep}(\vec{w})$, then $\mathcal{O}_V$ is open and dense in $\mathrm{Rep}(\vec{w})$. If $\mathcal{O}_V$ is open, then $\dim \mathcal{O}_V = \dim \mathrm{Rep}(\vec{w})$ by virtue of being open in the Zariski topology. Thus by 1.59(a), we are done. $\square$

1.61. Proposition. *Let $\vec{w} \neq 0$ and $q(\vec{w}) \leq 0$. Then, there are infinity many orbits in* $\mathrm{Rep}(\vec{w})$

Proof. The proof follows from 1.59 and the fact the orbits are not open, but will not be reproduced here. $\square$ reference?

1.62. Example. Let Q be given by

$$1 \longrightarrow 2$$

and let $\vec{w} = (1, 1)$. Then, we have the following representations in $\mathrm{Rep}(\vec{w})$.

$$\Bbbk \xrightarrow{\sim} \Bbbk \qquad \Bbbk \xrightarrow{0} \Bbbk$$

So, $\mathrm{Rep}(\vec{w}) \cong \mathrm{Hom}(\Bbbk, \Bbbk) \cong \Bbbk$ by definition and $GL(\vec{w}) \cong GL(1, \Bbbk) \times GL(1, \Bbbk) \cong \Bbbk^\times \times \Bbbk^\times$ by definition. Now, an element $(\lambda, \mu) \in GL(\vec{w})$ acts on $x \in \mathrm{Rep}(\vec{w})$ by

$$(\lambda, \mu)(x) = \lambda x \mu^{-1}$$

So, it is clear that $\{0\}$ is its own orbit and then $\Bbbk^\times$ is its own orbit, since $(yx^{-1}, 1)(x) = y$ for any $x, y \in \Bbbk^\times$. Thus, there are two orbits corresponding to the isomorphism classes of representations given above.

1.2. Closed orbits.

1.63. Proposition. *Let*

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of $\Bbbk Q$*-modules. Then,* $\mathcal{O}_{V'\oplus V''} \subseteq \overline{\mathcal{O}}_V$

Proof. Identify $V = V' \oplus V''$ as a vector space and extend a basis for V' to a basis for V. Then, we can view

$$x_h = \begin{pmatrix} x' & x_{12} \\ 0 & x'' \end{pmatrix}$$

Now, consider a 1 parameter subgroup $g(t) \in GL(\vec{w})$ with $t \in \Bbbk^\times$ where

$$g(t)_i = \begin{pmatrix} tI_{N'} & 0 \\ 0 & tI_{N''} \end{pmatrix}$$

Then, $g(t)$ acts on x_h via

$$x_h \mapsto g(t)_{t(h)} x_h g(t)^{-1}_{s(h)} = \begin{pmatrix} x' & tx_{12} \\ 0 & x'' \end{pmatrix}$$

From here, we may take the limit $t \to 0$ to see that the closure $\overline{\mathcal{O}}_V$ contains the matrix

$$\begin{pmatrix} x' & 0 \\ 0 & x'' \end{pmatrix}$$

This matrix exactly corresponds to the representations of $V' \oplus V''$, and so we are done. $\square$

1.64. Definition. Given a filtration F of a $\Bbbk Q$-module V, say

$$V = V^0 \supseteq V^1 \supseteq \cdots \supseteq V^\ell = 0$$

we define the *associated graded representation* by

$$\operatorname{gr}_F V = \bigoplus_{m=0}^{\ell-1} V^m / V^{m+1}$$

1.65. Proposition. *Given a representation* V*,* $\mathcal{O}_{\operatorname{gr} V} \subseteq \overline{\mathcal{O}}_V$*.*

Proof. This follows by the proposition above and induction on ℓ. $\square$ write the actual proof.

1.66. Proposition. *If closed* $\mathcal{O}_{V'} \leq \overline{\mathcal{O}}_V$*, where* $\mathbf{dim} V' = \mathbf{dim} V$*, then* $V' \cong \operatorname{gr}_F V$ *for some filtration* F*.*

1.67. Proposition. *Let* $V \in \operatorname{Rep}(\vec{w})$*. Then,* $\mathcal{O}_V$ *is closed if and only if* V *is semisimple, that is to say* V *is a direct sum of simple representations.*

Proof. The proof is by the propositions above. $\mathcal{O}_V$ is closed if and only if $V \cong \operatorname{gr} V$ for every filtration if and only if V is semisimple. $\square$

1.68. Corollary. *The closure of every orbit $\mathcal{O}_V$ contains a unique closed orbit.*

1.69. Definition. We denote the unique closed orbit of $\mathcal{O}_V$ by $O_{V^{ss}}$

1.70. Corollary. *Let Q be a quiver with no oriented cycles. Then, the only closed orbit of* $\operatorname{Rep}(\vec{w})$ *is* $\{0\}$.

Proof. This follows from the fact that the complete list of simples is $\{S(i)\}$ and so any semisimple representation has $x_h = 0$. $\square$

1.71. Example. Let Q be

$$\begin{tikzcd} 1 \arrow[loop above] \end{tikzcd}$$

and let $\vec{w} = (n)$. Then, $\operatorname{Rep}(\vec{w}) \cong M_n(\Bbbk)$ with action given by conjugation by $GL(\vec{w}) = GL_n(\Bbbk)$ on x_h (the unique edge). Thus, the orbits are conjugacy classes of matrices. By the above result, an orbit $\mathcal{O}_V$ is closed if and only if V is semisimple, which in this case is the same thing as x_h being given by a diagonalizable matrix. (The only simple matrices are 1 by 1). Thus, given an arbitrary matrix $A \in M_n(\Bbbk)$, the unique closed orbit contained in $\overline{O}_A$ is $\mathcal{O}_{A^{ss}}$ where A^{ss} is the diagonalizable matrix with the same eigenvalues as A.

(8/31/2017) Lecture 4. In this lecture, we will discuss the Euler form and Dynkin quivers.

1.72. Definition. For $V, W \in \operatorname{Rep}(Q)$ define

$$\langle V, W \rangle := \sum_i (-1)^i \dim \operatorname{Ext}^i_Q(V, W) = \dim \operatorname{Hom}_Q(V, W) - \dim \operatorname{Ext}^1_Q(V, W)$$

Note how this resembles the Euler characteristic. Also, the second equality is due to the fact that higher Ext groups vanish in this category.

1.73. Proposition.
- *(a) $\langle P(i), S(j) \rangle = \delta_{ij}$ and $\langle P(i), W \rangle = \dim W_i$ for an arbitrary representation $W \in \operatorname{Rep}(Q)$.*
- *(b) $\langle -, - \rangle$ is biadditive with respect to exact sequences, that is to say, if we have exact sequence*

$$0 \to V' \to V \to V'' \to 0$$

then $\langle V, W \rangle = \langle V', W \rangle + \langle V'', W \rangle$. Note, this is called the "Euler-Poincare" principle.

Proof. We know that $\mathrm{Hom}(P(i), W) = W_i$ by 1.18 and $\mathrm{Ext}^1_Q(P(i), W)$ is trivial for any W since $P(i)$ is projective. Thus,

$$\langle P(i), W\rangle = \dim \mathrm{Hom}(P(i), W) - \dim \mathrm{Ext}^1_Q(P(i), W) = \dim W_i - 0$$

If we take $W = S(i)$, then we get

$$\langle P(i), S(i)\rangle = \dim(S(j))_i - 0 = \delta_{ij}.$$

For the second part, this follows by getting the long exact sequence by applying $\mathrm{Hom}(-, W)$ to $0 \to V' \to V \to V'' \to 0$. □

1.74. Theorem. *Let $V, W \in \mathrm{Rep}(Q)$, then*

(a) $\langle V, W\rangle$ only depends on $\vec{v} := \mathbf{dim} V$ and $\vec{w} := \mathbf{dim} W$. So, we have a (Euler) form

$$\langle \cdot, \cdot\rangle \colon \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$$

(b)

$$\langle \vec{v}, \vec{w}\rangle = \langle V, W\rangle = \sum_{i\in I} v_i w_i - \sum_{h\in\Omega} v_{s(h)} w_{t(h)}$$

Proof. For the second part, recall the standard resolution

$$0 \to P_1 \to P_0 \to V \to 0$$

Then, by the proposition above, $\langle V, W\rangle = \langle P_0, W\rangle - \langle P_1, W\rangle$. Moreover,

$$\begin{aligned}
\dim \mathrm{Hom}(P_1, W) &= \dim \mathrm{Hom}\Big(\bigoplus_{h\in\Omega} P(t(h)) \otimes V_{s(h)}, W\Big) \\
&= \sum_{h\in\Omega} \dim \mathrm{Hom}(P(t(h)) \otimes V_{s(h)}, W) \\
&= \sum_{h\in\Omega} \dim \mathrm{Hom}\Big(\bigoplus_{j=1}^{v_{s(h)}} P(t(h)), W\Big) \\
&= \sum_{h\in\Omega} v_{s(h)} \dim \mathrm{Hom}(P(t(h)), W) \\
&= \sum_{h\in\Omega} v_{s(h)} w_{t(h)}
\end{aligned}$$

and similarly for $\dim \mathrm{Hom}(P_0, W)$. Finally, since $\dim \mathrm{Ext}^1(P_0, W) = 0 = \dim \mathrm{Ext}^1(P_1, W)$, we get our desired identity. □

1.75. Definition. The associated quadratic form (the so-called Tits form) is given by

$$q(\vec{v}) := \langle \vec{v}, \vec{v}\rangle = \sum_{i\in I} v_i^2 - \sum_{h\in\Omega} v_{s(h)} v_{t(h)}$$

1.76. Definition. We define a symmetric bilinear form $(\cdot, \cdot) \colon \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ by

$$(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{w}\rangle + \langle \vec{w}, \vec{v}\rangle$$

1.77. Example. Consider the quiver

$$1 \longrightarrow 2$$

with representations given by

$$P(1) = \ \Bbbk \xrightarrow{\sim} \Bbbk \ , \quad P(2) = S(2) = \ 0 \longrightarrow \Bbbk$$

Then, $\langle P(1), P(2)\rangle = \dim(P(2))_1 = 0$ and $\langle P(2), P(1)\rangle = \dim(P(1))_2 = 1$. Thus, we see that $\langle \cdot, \cdot \rangle$ is not symmetric and $(P(1), P(2)) = 1$

1.78. Theorem. *Let n_{ij} be the number of edges between i and j in $|Q|$. Then,*

$$(\vec{v}, \vec{w}) = \sum_i v_i w_i (2 - 2n_{ij}) - \sum_{i \neq j} n_{ij} v_i w_j$$

Now, equipped with these forms, we move on to a discussion of Dynkin quivers.

1.79. Theorem. *Let Q be a quiver and assume $|Q|$ is connected. Then*

(a) q_Q is positive definite if and only if $|Q|$ is Dynkin, that is, it is of types A, D, or E.
(b) q_Q is positive semidefinite if and only if $|Q|$ is Euclidean, that is, it is an "untwisted affine ADE".

Understand why this theorem is true.

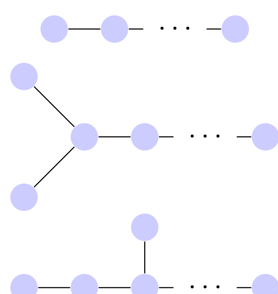


Figure 1. Dynkin graphs A_n, D_n, and $E_{6,7,8}$

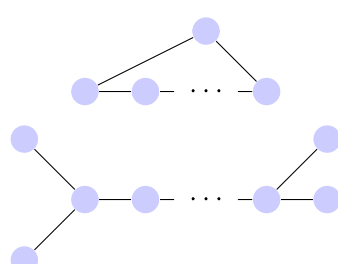
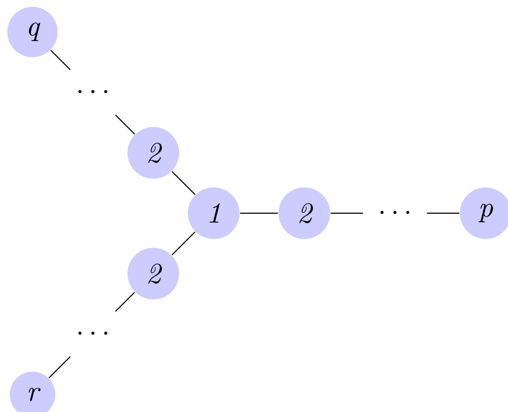


Figure 2. Euclidean Dynkin graphs $\hat{A}_n$ and $\hat{D}_n$

1.80. Remark. The difference from the “Lie” setting is here $n_{ij} \in \mathbb{Z}$ where in the “Lie” setting, $n_{ij} = \sqrt{a_{ij}a_{ji}}$.

Proof of Theorem. The second part of the theorem is a “boundary” case. Any graph containing $\widehat{ADE}$ has q_Q indefinite and any graph not containing $\widehat{ADE}$ is covered by the first case. $\square$

1.81. Corollary. *Consider the “star” graph* $\Gamma(p,q,r)$



Then, $\Gamma(p,q,r)$ *has positive-definite* q_Q *if and only if*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

which happens if and only if (p,q,r) *is of the following forms*

$$\begin{cases} (p,q,1) & A_{p+q-1} \\ (p,2,2) & D_{p+2} \\ (p,3,2) & p = 3,4,5 \quad E_{p+3} \end{cases}$$

Understand why this is a corollary and why it is true.

A natural question one may ask is about types B, C, F, G. To do this, we will have to do some more work. Due to the resemblence of these constructions to Lie theory, we will use some similar definitions.

1.82. Definition. Assume $|Q|$ has no edge loop, that is $n_{ii} = 0$. Then, we define the *Cartan matrix* to be

$$C_Q := C_{|Q|} = 2I - A$$

where $A = (n_{ij})$ is the adjacency matrix.

This matrix can be used to define Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}_Q$ (or simple Lie algebra if Q is Dynkin). Then, we get the following.

1.83. Definition. We define the *root latice* $L = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \cong \mathbb{Z}^I$ where $\alpha_i = \mathbf{dim} S(i)$ are the *simple roots*. This gives rise to *simple reflections* $s_i \colon L \to L$ defined by $\alpha \mapsto \alpha - (\alpha, \alpha_i)\alpha_i$ and thus *Weyl group* $W = \langle s_i \mid i \in I \rangle \leq GL(L \otimes_{\mathbb{Z}} \mathbb{R}$

1.84. Proposition. *(a)* W *preserves* $(\cdot, \cdot)$.

(b) *For Dynkin (or Euclidean), the root system of* $\mathfrak{g}$, $R = \{\alpha \in L \setminus 0 \mid (\alpha, \alpha) \leq 2\}$ *(or* $= 2$ *if Dynkin).*

Now, we are ready to discuss Gabriel's Theorem. We require $\overline{\Bbbk} = \Bbbk$.

1.85. Definition. A quiver Q is of *finite type* if the number of indecomposable representations of Q of any fixed $\mathbf{dim} = \vec{w}$ is finite.

1.86. Theorem (Gabriel's Theorem). *A connected quiver Q is of finite type if and only if $|Q|$ is Dynkin (ie types ADE). Furthermore, if Q is Dynkin then there is a one-to-one correspondance between indecomposables and positive roots in the root system R.*

Proof. For the forward direction, assume Q is of finite type. Then, for $0 \neq \vec{v} \in \mathbb{N}^I$, there exists finitely many representations of $\mathbf{dim} = \vec{v}$. So, $GL(\vec{v})$ acting on $\operatorname{Rep}(\vec{v})$ has finitely many orbits. Therefore, $\operatorname{Rep}(\vec{v}) = \overline{\mathcal{O}_x}$ for some $x \neq 0$. Now, $\dim \operatorname{Rep}(\vec{v}) = \dim \mathcal{O}_x = \dim GL(\vec{v}) - \dim G_x$ by . Thus, we have

why?

why?

$$1 \leq \dim G_x = \dim GL(\vec{v}) - \dim \operatorname{Rep}(\vec{v}) = q(\vec{v})$$

since the identity matrix, $I \in G_x \implies \dim G_x \geq 1 \implies q(\vec{v}) \geq 1$. This completes the forward direction. We still must lay more groundwork to prove the rest of this theorem. $\square$

1.87. Proposition. *Given $\vec{u} \in \mathbb{Z}^I$ and $\vec{v}$ such that $v_i = |u_i|$ for all i, then $q(\vec{v}) \leq q(\vec{u})$.*

Proof.

$$q(\vec{v}) = \sum |u_i|^2 - \sum_h |u_{s(h)}||u_{t(h)}| \leq \sum u_i^2 - \sum u_{s(h)}u_{t(h)} = q(\vec{u})$$

$\square$

1.88. Example. Consider the quiver

$$1 \longrightarrow 2$$

finish this example

(9/5/2017) Lecture 5.

1.3. Reflection functors. Starting off, this section is completely general and not restricted to quivers of type ADE.

1.89. Definition. If a vertex i in a quiver Q has no edges leaving i, we call i a *sink*.

$$\begin{tikzcd} \cdot \arrow[dr] & & \\ & i & \cdot \arrow[l] \\ \cdot \arrow[ur] & & \end{tikzcd}$$

Similarly, if a vertex i in a quiver Q has no edges entering i, we call i a *source*.

$$\begin{tikzcd} {} & & \\ & i \arrow[ul] \arrow[dl] \arrow[r] & {} \\ {} & & \end{tikzcd}$$

1.90. Remark. Note that

$$\longrightarrow i \longrightarrow$$

is neither a sink nor a source.

1.91. Definition. We define reflections $s_i^{\uparrow}$ and $s_i^{\downarrow}$ to flip all arrows incident to i and are defined as follows

$$\begin{tikzcd} \text{source} \arrow[r, bend left, "s_i^{\uparrow}"] & \text{sink} \arrow[l, bend left, "s_i^{\downarrow}"] \end{tikzcd}$$

1.92. Proposition. $s_i^{\downarrow} s_i^{\uparrow}(Q) = Q$ *for* i *a sink in* Q.

Prove this or at least do a basic example.

1.93. Proposition. *Assume* $|Q|$ *is a tree. Let* Ω, Ω' *be 2 orientations of* $|Q|$. *Then,* Q' *is obtained from* Q *via a sequence of* $s_i^{\downarrow}$ *and* $s_i^{\uparrow}$ *operations.*

Proof. The proof proceeds by indcution on the number of vertices. Locate a sink or a source with one incident edge, which must exist since $|Q|$ is a tree. Call it k. Then, by the inductive hypothesis, one can obtain the orientation on the other vertices consistent wiht Q'. At this point, either the edge incident to k is in the correct orientation or it is not. If not, apply s_k and you are done. $\square$

1.94. Definition. For a sink i of Q, let $Q' = s_i^{\downarrow}(Q)$. Define

$$\Phi_i^{\downarrow} \colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q')$$

and take $V \in \operatorname{Rep}(Q)$. Then, if $V' = \Phi_i^{\downarrow}(V)$, we have that $V'_j = V_j$ when $j \neq i$ and

$$V'_i = \Phi_i^{\downarrow}(V)_i = \ker\left(\bigoplus_{k \to i} V_k \to V_i\right)$$

Thus, $\Phi_i^{\downarrow}$ only changes the vector space at i and the maps to and from i.

1.95. Definition. For source i at Q, let $Q' = s_i^{\uparrow}(Q)$. Similar to the above, we can define

$$\Phi_i^{\uparrow} \colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q')$$

where $\Phi_i^{\uparrow}(V)_i = \operatorname{coker}\left(V_i \to \bigoplus_{i \to k} V_k\right)$

1.96. Remark. These functors are called the *BGP reflection functors*.

1.97. Example. We note that $\Phi_i^{\downarrow}(S(i)) = 0$ where i is a sink in Q. This happens because $\ker\left(\bigoplus_{k\to i} V_k \to V_i\right)$ but, $\bigoplus_{k\to i} V_k = 0$. Similarly $\Phi_i^{\uparrow}(S(i)) = 0$ since $V_i \to \bigoplus_{i\to k} V_k = 0$ so the codomain is trivial and thus $\operatorname{coker}(V_i \to \bigoplus_{i\to k} V_k)$.

1.98. Example. Let Q be the quiver

$$1 \longrightarrow 2$$

and consider the representation

$$V = \ \Bbbk \longrightarrow 0$$

Then, $\Phi_2^{\downarrow}(V)_2 = \ker\left(\bigoplus_{k\to 2} V_k \to V_2\right) = \ker\left(V_1 \to V_2\right) = \ker(\Bbbk \to 0)$. So, we get

$$V' := \Phi_2^{\downarrow}(V) = \ \Bbbk \longleftarrow \Bbbk$$

Now, if we apply $\Phi_2^{\uparrow}(V')$, we see that we get $\operatorname{coker}(\Bbbk \to \Bbbk) = 0$, so we actually recover V. Now, take $\Phi_1^{\uparrow}(V')$. Since $\ker\left(\bigoplus_{k\to 1} V_k \to V_1\right) = \ker(\Bbbk \to \Bbbk) = 0$. So,

$$W := \Phi_2^{\downarrow}(V') = \ 0 \longrightarrow \Bbbk$$

Similarly, $\Phi_1^{\uparrow}(W) = V'$ since we are taking $\operatorname{coker}(0 \to \Bbbk) \cong \Bbbk$. Now, notice that these are the 3 indecomposables and the correspond to the 3 positve roots (2 simple). To see this, note that $|Q|$ is of type A_2, so it has 2 simple roots $\beta = (0,1)$ and $\alpha = (1,0)$. Then, we get the correspondance using graded dimension

$$(\Bbbk \longrightarrow 0) \longmapsto (1,0) \text{ is simple}$$

$$(\Bbbk \longleftarrow \Bbbk) \longmapsto (1,1)$$

$$(0 \longrightarrow \Bbbk) \longmapsto (0,1) \text{ is simple}$$

1.99. Example. Take the quiver Q and V, a representation, to be

$$1 \longrightarrow 2 \longleftarrow 3\ , \qquad 0 \longrightarrow 0 \longleftarrow \Bbbk$$

Then, we can apply $\Phi_2^{\downarrow}$ to V to get V'. We compute that

$$V_2' = \ker\left(\bigoplus_{k\to 2} V_k \to V_2\right) = \ker(\Bbbk \to 0) = \Bbbk$$

So,

$$V' = \ 0 \longleftarrow \Bbbk \longrightarrow \Bbbk$$

Doing similar computations, we get the following diagram

$$\begin{tikzcd}
 & & 0 & \Bbbk \arrow[l] \arrow[r] \arrow[dll, dashed, "\Phi_1^\downarrow"'] \arrow[drr, dashed, "\Phi_3^\downarrow"] & \Bbbk & & \\
\Bbbk \arrow[r] & \Bbbk \arrow[r] \arrow[drr, dashed, "\Phi_3^\downarrow"'] & \Bbbk & & \Bbbk & \Bbbk \arrow[l] \arrow[dll, dashed, "\Phi_1^\downarrow"] & 0 \arrow[l] \\
 & & \Bbbk \arrow[r] & \Bbbk \arrow[d, dashed, "\Phi_2^\downarrow"] & 0 \arrow[l] & & \\
 & & \Bbbk & 0 \arrow[l] \arrow[r] & 0 & &
\end{tikzcd}$$

1.100. EXAMPLE. Let Q be a quiver and V, a representation of Q, given by

$$1 \longrightarrow 2 \longleftarrow 3 \qquad 0 \longrightarrow \Bbbk \longleftarrow 0$$

If we apply $\Phi_1^\uparrow(V) =: V'$, we see that

$$V_1' = \operatorname{coker}\left(V_1 \to \bigoplus_{1\to k} V_k\right) = \operatorname{coker}(0 \to \Bbbk) = \Bbbk$$

So, we get the following

$$\Phi_1^\uparrow(V) = \ \Bbbk \longleftarrow \Bbbk \longleftarrow 0 \qquad \Phi_3^\uparrow(V) = \ 0 \longrightarrow \Bbbk \longrightarrow \Bbbk$$

where the second equality follows from symmetry. However, note that

$$\Phi_2^\downarrow(V)_2 = \ker\left(\bigoplus_{k\to 2} V_k \to V_2\right) = \ker(0 \to \Bbbk) = 0 \Longrightarrow \Phi_2^\downarrow(V) = \ 0 \longleftarrow 0 \longrightarrow 0$$

Thus, we have found a situation where $\Phi_i^\downarrow$ and $\Phi_i^\uparrow$ are not inverses!

Finish these computations.

1.101. PROPOSITION. *For sink i of Q, $s_i^\downarrow$ is left exact. Furthermore, the right derived functor $R^n\Phi_i^\downarrow = 0$ for $n > 1$. Finally,*

$$R^1\Phi_i^\downarrow(V) = 0 \Longleftrightarrow \bigoplus_{k\to i} V_k \to V_i \text{ is surjective. Call this property } \begin{pmatrix}\downarrow\\ i\end{pmatrix}$$

1.102. DEFINITION. In the case of the above proposition, we define $\mathrm{Rep}^{i\leftarrow}$ is the full subcategory of $\mathrm{Rep}(Q)$ consisting of objects V that satisfy $\begin{pmatrix}\downarrow\\ i\end{pmatrix}$.

1.103. PROPOSITION. *For source i of Q, $\Phi_i^\uparrow$ is right exact. Furthermore, the left derived functor $L^n\Phi_i^\uparrow = 0$ for $n > 1$. Finally,*

$$L^1\Phi_i^\uparrow(V) = 0 \Longleftrightarrow V_i \to \bigoplus_{i\to k} V_k \text{ is injective. Call this property } \begin{pmatrix}\uparrow\\ i\end{pmatrix}.$$

1.104. Definition. In the case of the above proposition, we define $\mathrm{Rep}^{i\to}$ is the full subcategory of $\mathrm{Rep}(Q)$ consisting of objects V that satisfy $\begin{pmatrix}\uparrow\\ i\end{pmatrix}$.

1.105. Remark. We will end up showing that $\mathrm{Rep}^{i\leftarrow}(Q)$ and $\mathrm{Rep}^{i\to}(Q)$ contain all indecomposables of Q except for $S(i)$. In a sense, this encodes the same fact that a reflection s_i on a positive set of roots R_+ takes the set $R_+ \setminus \{\alpha_i\}$ to $R_+ \setminus \{\alpha_i\}$. This becomes explicitly manifest in **??**.

I understand this, but get a much better handle on it. And what is this reference?

Proof of Above Propositions. $\square$

Understand this proof...

1.106. Proposition. *For i a sink of Q, let $Q' = s^{\downarrow}(Q)$. Then, $\Phi_i^{\downarrow}$ and $\Phi_i^{\uparrow}$ are an adjoint pair. That is, for $W \in \mathrm{Rep}(Q)$ and $V' \in \mathrm{Rep}(Q')$, we get*

$$\mathrm{Hom}_Q(\Phi_i^{\uparrow}(V'), W) \cong \mathrm{Hom}_{Q'}(V', \Phi_i^{\downarrow}(W))$$

Proof. Consider that the cokernel is the universal object such that the following diagram commutes

$$\begin{tikzcd}
\bigoplus_{i\to k} V'_k \arrow[dr] & \\
V'_i \arrow[u] \arrow[r, "0"] & \mathrm{coker}(V'_i \to \bigoplus_{i\to k} V_k)
\end{tikzcd}$$

Similarly, the kernel is the universal object such that the following diagram commutes

$$\begin{tikzcd}
\bigoplus_{k\to i} W_k \arrow[dr] & \\
\ker(\bigoplus_{k\to i} W_k \to W_i) \arrow[u] \arrow[r, "0"] & W_i
\end{tikzcd}$$

So, let $f \in \mathrm{Hom}_Q(\Phi_i^{\uparrow}(V'), W)$ map $\mathrm{coker}(V'_i \to \bigoplus_{i\to k} V_k) \to W_i$. Then, we get

$$\begin{tikzcd}
 & & \bigoplus_k W_k \arrow[d, "pr"] \\
\bigoplus_{i\to k} V'_k \arrow[urr, bend left, "\bigoplus_{i\to k} f_k"] & \ker(\bigoplus_k W_k \to W_i) \arrow[ur] \arrow[r, "0"] & W_i \\
V'_i \arrow[u] \arrow[ur, dashed, "\exists!"] \arrow[r, "0"] \arrow[urr, bend right, "0"'] & \mathrm{coker}(V'_i \to \bigoplus_{i\to k} V_k) \arrow[ur, "f_i"] &
\end{tikzcd}$$

Thus, a map $f \in \mathrm{Hom}_Q(\Phi_i^{\uparrow}(V'), W)$ induces a unique map in $\mathrm{Hom}_{Q'}(V', \Phi_i^{\downarrow}(W))$. Similarly, given a $g \in \mathrm{Hom}_{Q'}(V', \Phi_i^{\downarrow}(W))$, we get

$$\begin{tikzcd}
\bigoplus_{i\to k} V'_k \arrow[dr] \arrow[drr, bend left, "\bigoplus_k g_k"] & & \\
V'_i \arrow[u] \arrow[r, "0"] \arrow[dr, "g_i"'] \arrow[drr, bend right, "0"'] & \mathrm{coker}(V'_i \to \bigoplus_{i\to k} V_k) \arrow[dr, dashed, "\exists!"] & \bigoplus_k W_k \arrow[d] \\
& \ker(\bigoplus_k W_k \to W_i) \arrow[r, "0"] & W_i
\end{tikzcd}$$

So, a map $g \in \mathrm{Hom}_{Q'}(V', \Phi_i^{\downarrow}(W))$ corresponds to a unique map in $\mathrm{Hom}_Q(\Phi_i^{\uparrow}(V'), W)$. Therefore, $\Phi_i^{\uparrow}$ and $\Phi_i^{\downarrow}$ form an adjoint pair. $\square$

1.107. Proposition. *With the same situation as above, $\Phi_i^{\downarrow}(W) \in \mathrm{Rep}^{i\to}(Q')$ and $\Phi_i^{\uparrow}(V') \in \mathrm{Rep}^{i\leftarrow}(Q)$.*

Proof. By definition, $\mathrm{Rep}^{i\to}(Q')$ is all representations that have $V_i \hookrightarrow \bigoplus_{i\to k} V_k$. However, $\Phi_i^{\downarrow}(W)_i = \ker(\bigoplus_{k\to i} W_k \to W_i) \hookrightarrow \bigoplus_{i\to k} W_k$, by definition. Note that the reflection functor changes all edges that were $k \to i$ into $i \to k$ edges.

Similarly, $\mathrm{Rep}^{i\leftarrow}(Q)$ is all representations that have $\bigoplus_{k\to i} V_k \twoheadrightarrow V_i$ and $\Phi_i^{\uparrow}(V')_i = \mathrm{coker}(V'_i \to \bigoplus_{k\to i} V_k) \twoheadrightarrow V'_i$ by definition, and since $\bigoplus_{i\to k} V'_k \twoheadrightarrow \mathrm{coker}(V'_i \to \bigoplus_{k\to i} V'_k) \twoheadrightarrow V'_i$, we are done. $\square$

1.108. Proposition. *As functors, $\Phi_i^{\uparrow}$ and $\Phi_i^{\downarrow}$ are inverses between $\mathrm{Rep}^{i\to}(Q)$ and $\mathrm{Rep}^{i\leftarrow} Q$*

Proof. We want to show that $\Phi_i^{\uparrow}\Phi_i^{\downarrow}(W) \cong W$ for $W \in \mathrm{Rep}^{i\leftarrow}(Q)$. Now, since $W \in \mathrm{Rep}^{i\leftarrow}(Q)$, then, by definition, we get

$$\bigoplus_{k\to i} W_k \twoheadrightarrow W_i$$

Thus, it must be that

$$\begin{aligned}
\Phi_i^{\uparrow}(\Phi_i^{\downarrow}(W))_i &= \mathrm{coker}\left(\ker\left(\bigoplus_{k\to i} W_k \twoheadrightarrow W_i\right) \to \bigoplus_k W_k\right) \\
&= \left(\bigoplus_k W_k\right) / \ker\left(\bigoplus_{k\to i} W_k \twoheadrightarrow W_i\right)
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{im} \bigoplus_{k\to i} W_k \twoheadrightarrow W_i \\
&= W_i
\end{aligned}
$$

$\square$

1.109. Proposition. *Let Q be arbitrary quiver. Let $s_i = s_{\alpha_i} \colon \mathbb{Z}^I \to \mathbb{Z}^I$ be the simple reflection sending $\alpha \mapsto \alpha - (\alpha, \alpha_i)\alpha_i$, see 1.83.*

(a) For $W \in \mathrm{Rep}^{i\leftarrow}(Q)$,

$$\mathbf{dim}\Phi_i^{\downarrow}(W) = s_i(\mathbf{dim}W)$$

(b) For $V' \in \mathrm{Rep}^{i\rightarrow}(Q)$,

$$\mathbf{dim}\Phi_i^{\uparrow}(V') = s_i(\mathbf{dim}V')$$

Proof. For the first part, we know that W has the property that $\bigoplus_{k\to i} W_k \twoheadrightarrow W_i$ so $\dim(\Phi_i^{\downarrow}(W))_i = \sum_{k\to i}(\dim W_k) - \dim W_i$. First, we compute that, if j is not incident to i, then $s_i(\alpha_j) = \alpha_j$, if $k \to i$, then

$$(\alpha_k, \alpha_i) = \left(\sum_{j\in I}(\alpha_k)_j(\alpha_i)_j - \sum_{h\in\Omega}(\alpha_k)_{s(h)}(\alpha_i)_{t(h)}\right) + \left(\sum_{j\in I}(\alpha_i)_j(\alpha_k)_j - \sum_{h\in\Omega}(\alpha_i)_{s(h)}(\alpha_k)_{t(h)}\right)$$

However, since $(\alpha_i)_j = \delta_{ij}$, $(\alpha_k)_j = \delta_{kj}$, and $i \neq k$, then the vertex indexed sums will be 0. Furthermore, there is only one edge $k \to i$, so the first sum will be 1 and the other will be 0. Thus, we get

$$(\alpha_k, \alpha_i) = -1 \implies s_i(\alpha_k) = \alpha_k - (\alpha_k, \alpha_i)\alpha_i = \alpha_k + \alpha_i$$

We also compute that

$$(\alpha_i, \alpha_i) = 2\sum_{j\in I}(\alpha_i)_j^2 - 2\sum_{h\in\Omega}(\alpha_i)_{s(h)}(\alpha_i)_{t(h)} = 2$$

and so $s_i(\alpha_i) = \alpha_i - (\alpha_i, \alpha_i)\alpha_i = -\alpha_i$. Now, we compute that

$$
\begin{aligned}
s_i(\mathbf{dim}W) &= s_i\left(\sum_{j \text{ not incident to } i} w_j\alpha_j + \sum_{k\to i}(w_k\alpha_k) + w_i\alpha_i\right) \\
&= \sum_{j \text{ not incident to } i} w_j\alpha_j + \sum_{k\to i}(w_k(\alpha_k + \alpha_i)) - w_i\alpha_i \\
&= \sum_{j\neq i} w_j\alpha_j + \sum_{k\to i}(w_k\alpha_i) - w_i\alpha_i \\
&= \mathbf{dim}\Phi_i^{\downarrow}(W)
\end{aligned}
$$

A similar computation works for part (b). $\square$

1.110. Remark. Let $\mathcal{D}(Q) := \mathcal{D}(\mathrm{Rep}(Q))$. Then, $\mathcal{D}(Q) \cong \mathcal{D}(Q')$.

Cite lecture notes by Milicic.

Always? Or Q, Q' as above?

(9/7/2017) Lecture 6.

1.111. Proposition. *Let Q be a quiver.*

(a) For i a sink for Q and $Q' = s_i^{\downarrow}Q$, if $V \in \mathrm{Rep}^{i\leftarrow}(Q)$, $W \in \mathrm{Rep}(Q)$, then

$$\mathrm{Hom}_Q(V, W) \cong \mathrm{Hom}_{Q'}(\Phi_i^{\downarrow}V, \Phi_i^{\downarrow}W)$$

(b) For i a source for Q, if $V \in \mathrm{Rep}(Q), W \in \mathrm{Rep}^{i\rightarrow}(Q)$, then

$$\mathrm{Hom}_Q(V, W) \cong \mathrm{Hom}_{Q'}(\Phi_i^{\uparrow}V, \Phi_i^{\uparrow}W)$$

Proof. For part (a), consider that $V = \Phi_i^{\uparrow}(\Phi_i^{\downarrow}V)$ by 1.108, then we can apply the adjointness (1.106) to get

$$\mathrm{Hom}_Q(\Phi_i^{\uparrow}(\Phi_i^{\downarrow}V), W) \cong \mathrm{Hom}_{Q'}(\Phi_i^{\downarrow}V, \Phi_i^{\downarrow}W)$$

A dual argument applies for part (b). $\square$

1.112. Definition. Let $\vec{v} \in \mathbb{Z}^{\geq 0}$. Then, we say $(v_i)_{i\in I} = \vec{v} > 0 \iff v_i \geq 0$ for all $i \in I$.

1.113. Proposition. *Let $0 \neq V \in \mathrm{Rep}(Q)$ be indecomposable, i a sink/source. Then,*

(a) If $V \cong S(i)$, then $\Phi_i^{\downarrow\uparrow}(V) = 0$.

(b) If $V \not\cong S(i)$, then $V \in \mathrm{Rep}^{i\leftrightarrows}(Q)$.

(c) $\Phi_i^{\downarrow\uparrow}(V)$ is nontrivial and indecomposable if and only if $s_i(\mathbf{dim}V) \geq 0$.

Proof. Part (a) follows from computations like in example 1.100. For part (b), if $V \notin \mathrm{Rep}^{i\leftarrow}(Q)$, then $\bigoplus_{k\to i} V_k \xrightarrow{f} V_i$ is *not* surjective. So, we can take $V_i = \mathrm{im}(f) \oplus V_i''$ with $V_i'' \neq 0$. Define $V', V'' \in \mathrm{Rep}(Q)$ by

$$V_j' = \begin{cases} V_j & j \neq i \\ \mathrm{im}(f) & j = i \end{cases}, \quad V_j'' = \begin{cases} 0 & j \neq i \\ V_i'' & j = i \end{cases}$$

Then, $V = V' \oplus V''$, which contradicts the indecomposability of V. To show that nontrivial, indecomposable $V \in \mathrm{Rep}^{i\rightarrow}$ for i a source, assume, as above, that $\bigoplus_{k\to i} V_k \xrightarrow{f} V_i$ is not surjective.

Finish this part of the proof. Maybe ask for a hint!

For part (c), the proof follows from (a), (b), and 1.109. $\square$

Write the actual proof. I do not see quite how to connect these facts to get what we want. Help!

1.114. Proposition. *Let Q be a quiver and $Q' = s^{\uparrow\downarrow}Q$.*

(a) For sink i of Q, if $V \in \mathrm{Rep}(Q), W \in \mathrm{Rep}^{i\leftarrow}(Q)$, then

$$\mathrm{Ext}^1_Q(V, W) \cong \mathrm{Ext}^1_{Q'}(\Phi_i^{\downarrow}V, \Phi_i^{\downarrow}W).$$

(b) For source i of Q, if $V \in \mathrm{Rep}^{i\rightarrow}(Q), W \in \mathrm{Rep}(Q)$, then

$$\mathrm{Ext}^1_Q(V, W) = \mathrm{Ext}^1_{Q'}(\Phi_i^{\uparrow}V, \Phi_i^{\uparrow}W).$$

1.115. Example. One could have assumed $V, W \in \mathrm{Rep}^{i\leftarrow}$ to get more restrictive theorems, but we can check that $S(i)$, which is not in $\mathrm{Rep}^{i\leftarrow}$, will work if put in the right spots. Namely,

$$\mathrm{Hom}_Q(V, S(i)) = 0$$
$$\mathrm{Ext}_Q(S(i), W) = 0$$

where the second equality follows because $S(i) = P(i)$ when i is a sink. Thus, we have found a useful way to remember that we can enlarge the possible category for W in the Hom case and enlarge the possible category of V in the Ext case.

Proof of Proposition. For part (a), if $V \cong S(i)$, we just showed in the above example that the left side is trivial and the right side is trivial using part (a) of 1.113. Furthermore, by general homological algebra, we know that

$$\mathrm{Ext}^1_Q(\bigoplus_\alpha V_\alpha, W) \cong \prod_\alpha \mathrm{Ext}^1_Q(V_\alpha, W)$$

and so it suffices to consider V as indecomposable. By part (b) of 1.113, since $V \ncong S(i)$, then $V \in \mathrm{Rep}^{i\leftarrow}(Q)$. Now, consider the short exact sequence in $\mathrm{Rep}(Q)$ given by

$$0 \to W \to M \to V \to 0$$

representing an element in $\mathrm{Ext}^1_Q(V, W)$. Using the right derived functor $R\Phi_i^\downarrow$, we get long exact sequence

$$0 \to \Phi_i^\downarrow W \to \Phi_i^\downarrow M \to \Phi_i^\downarrow V \to R^1\Phi_i^\downarrow W \to R^1\Phi_i^\downarrow M \to R^1\Phi_i^\downarrow V \to 0$$

However, $R^1\Phi_i^\downarrow V = 0 = R^1\Phi_i^\downarrow W$ since $V, W \in \mathrm{Rep}^{i\leftarrow}(Q)$ (1.101). So, this forces $R^1\Phi_i^\downarrow M = 0 \implies M \in \mathrm{Rep}^{i\leftarrow}(Q)$. Furthermore, we are left with the short exact sequence

$$0 \to \Phi_i^\downarrow W \to \Phi_i^\downarrow M \to \Phi_i^\downarrow V \to 0$$

which represents an element in $\mathrm{Ext}^1_Q(\Phi_i^\downarrow V, \Phi_i^\downarrow W)$. Thus, we have a map $\mathrm{Ext}^1_Q(V, W) \to \mathrm{Ext}_{Q'}(\Phi_i^\downarrow V, \Phi_i^\downarrow W)$. We can then make a similar argument with $\mathrm{Ext}^i_{Q'}(\Phi_i^\downarrow V, \Phi_i^\downarrow W) \to \mathrm{Ext}^1_Q(\Phi_i^\uparrow\Phi_i^\downarrow V, \Phi_i^\uparrow\Phi_i^\downarrow W) = \mathrm{Ext}^1_Q(V, W)$. These two maps are inverse to each other $\square$

Actually check this?

1.116. Proposition. *Assume i is a sink (or source) of Q and $Q' = s^{\uparrow\downarrow}Q$. If, for $a = 1, 2$, $0 \neq I_a \in \mathrm{Rep}(Q)$ is indecomposable and $\Phi_i^{\downarrow\uparrow}(I_a) \neq 0$ (thus indecomposable), then*

$$\mathrm{Hom}_Q(I_1, I_2) \cong \mathrm{Hom}_{Q'}(\Phi_i^{\downarrow\uparrow} I_1, \Phi_i^{\downarrow\uparrow} I_2)$$
$$\mathrm{Ext}^1_Q(I_1, I_2) \cong \mathrm{Ext}^1_{Q'}(\Phi_i^{\downarrow\uparrow} I_1, \Phi_i^{\downarrow\uparrow} I_2)$$

We now return back to the program of proving Gabriel's theorem (1.86).

1.117. Remark. For all quivers Q without edge loops, there exists a correspondance between the indecomposable representations of Q and the positive roots of its root system. In fact, the real positive roots have a one-to-one correspondance, but the imaginary roots correspond to "a family" of indecomposables. Counting these families gives rise to the Kac polynomials $K_\alpha(q) = a_0 q^N + \cdots + a_N$ where N is the dimension of the "family", a_N is the multiplicity of α as a root, and $a_i \in \mathbb{N}$. This is due to a conjecture by Kac that has been subsequently proven.

1.118. Lemma. *Assume Q is a Dynkin quiver and $0 \neq \vec{v} \in \mathbb{Z}_{\geq 0}^{|I|}$. Then, there exists a sequence $i_1, \ldots, i_{k+1} \in I$ such that*

$$\begin{gathered} i_1 \text{ is a sink for } Q \\ i_2 \text{ is a sink for } s_{i_1}^{\downarrow}(Q) \\ i_3 \text{ is a sink for } s_{i_2}^{\downarrow} s_{i_1}^{\downarrow}(Q) \\ \vdots \\ i_{k+1} \text{ is a sink for } s_{i_k}^{\downarrow} \cdots s_{i_1}^{\downarrow}(Q) \end{gathered}$$

and $\vec{v} \geq 0, s_{i_1}(\vec{v}) \geq 0, \ldots, s_{i_k} \cdots s_{i_1}(\vec{v}) \geq 0$, but $s_{i_{k+1}} s_{i_k} \cdots s_{i_1}(\vec{v}) \ngeq 0$.

1.119. Definition. We say a sequence $i_1, \ldots, i_{k+1} \in I$ meeting only the sink conditions (not necessarily the positivity ones), is *adapted* to an orientation Ω of Q with no oriented cycles.

Now, with this lemma in hand (whose proof we will postpone), we seek to prove the second part of Gabriel's theorem.

Proof of Gabriel's Theorem (continued). Given any indecomposable $V \neq 0$ and $\vec{v} = \mathbf{dim} V$, we can construct a sequence $i_1, \ldots, i_{k+1} \in I$ that is adapted to Ω. Then,

$$\begin{aligned} \Phi_{i_1}^{\downarrow}(V) &\neq 0 \\ \Phi_{i_2}^{\downarrow}\Phi_{i_1}^{\downarrow}(V) &\neq 0 \\ &\vdots \\ V' := \Phi_{i_k}^{\downarrow} \cdots \Phi_{i_1}^{\downarrow}(V) &\neq 0 \end{aligned}$$

and $\Phi_{i_{k+1}}^{\downarrow}(V') = 0$. Thus, it must be that $V' = S(i_{k+1})$ and so, we can work backwards to get $V = \Phi_{i_1}^{\uparrow} \cdots \Phi_{i_k}^{\uparrow}(S(i_{k+1}))$ and thus, by 1.109,

$$\mathbf{dim} V = s_{i_1} \cdots s_{i_k}(\alpha_{i_{k+1}}) \in R_+$$

where the root is positive the lemma above. So, we have

$$\mathrm{Ext}^1(V, V) = \cdots = \mathrm{Ext}^1(S(i), S(i)) = 0$$

since $S(i) = P(i)$, and so, by 1.59, $\mathcal{O}_V$ is open in $\mathrm{Rep}(\vec{v})$. $\square$

Thus, our proof is complete modulo proving the lemma, which will be our program for the next lecture.

(9/12/2017) Lecture 7. For a quiver Q, recall that we can take $L := \mathbb{Z}^I$ to be a root lattice and take a set of simple roots $\{\alpha_i \mid i \in I\}$ and then get a Weyl group $W = \langle s_i \rangle_{i \in I}$ that acts on the roots. Using this Weyl group, we can define a total order on $I = \{i_1, \ldots, i_r\}$.

1.120. Definition. Let $C := s_{i_r} \cdots s_{i_1} \in W$. C is a *Coxeter element of* W. A Coxeter element depends on a choice of simple roots and the ordering of the simple roots.

1.121. Example. Take $W = S_n$ and simple reflections given by $\{(i, i+1)\}_{i \leq n-1}$. Then a Coxeter element is $(1, 2, \ldots, n)$. So, if $n = 4$, then there are 6 4-cycles, which are all Coexeter elements.

1.122. Proposition. *All Coxeter elements are* W*-conjugate.*

Proof. (Lifted from Humphrey's "Reflection Groups and Coxeter Groups" 74–75). From basic Lie algebras, we know that all simple systems are W-conjugate. So, it suffices to show that, for a fixed set of simples, the Coxeter elements resulting from different orderings of the roots are conjugate. Any cyclic permutation of indices will give a conjugate element

$$s_n s_1 \cdots s_{n-1} = s_n(s_1 \cdots s_n)s_n$$

Also, an interchange of an adjacent commuting pair of s_i, s_j will not change the Coxeter element. Using an inductive argument, one can show that all permutations of the indices can be acheived using combinations of these two types of permutations. $\square$

1.123. Definition. We define $|C| = h$ (the order of C in W) to be the *Coxeter number.*

1.124. Example. When $W = S_n$, we can take $s_i = (i, i+1)$ and thus $C = (1, 2, \ldots, n)$ and so the Coxeter number is just n. When $W = D_{2m}$, then C is a product of two generating reflections, and thus a rotation by $\frac{2k\pi}{m}$ where k is relatively prime to m, and so it has order m, which is thus the Coxeter number.

1.125. Proposition. *Let* $x \in L_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} L$*. Then,* $Cx = x$ *if and only if* $x \in \mathrm{Rad}(\cdot, \cdot)$*, the radical of the form* $(\cdot, \cdot)$*.*

Proof. Let $Cx = x$. Then,

$$\begin{aligned} s_{i_r} \cdots s_{i_1} x = x &\Longrightarrow s_{i_{r-1}} \cdots s_{i_1} x = s_{i_r} x \\ &\Longrightarrow s_{i_{r-1}} \cdots s_{i_1} x - x = s_{i_r} x - x = x - (x, \alpha_{i_r})\alpha_{i_r} - x = (x, \alpha_{i_r})\alpha_{i_r} \end{aligned}$$

However, the right hand side must be a multiple of α_{i_r} and the left hand side must be a linear combination of $\{\alpha_{i_1}, \ldots, \alpha_{i_{r-1}}\}$, so the only way this can happen is if both sides are 0. Thus, $(x, \alpha_{i_r}) = 0$. Now, once can repeat this process to get

$$s_{i_{r-1}} \cdots s_{i_1} x = x \Longrightarrow \cdots \Longrightarrow \begin{cases} (x, \alpha_{i_{r-1}}) = 0 \\ \vdots \\ (x, \alpha_{i_1}) = 0 \end{cases}$$

and thus $x \in \operatorname{Rad}(\cdot, \cdot)$.

If $x \in \operatorname{Rad}(\cdot, \cdot)$, then $s_i(x) = x - (x, \alpha_i)\alpha_i = x$ by definition, so $Cx = s_{i_r} \cdots s_{i_1} x = x$. □

1.126. Example. One non-trivial example is with quivers of type $\widehat{ADE}$. There is only one imaginary root, δ and $s_i(\delta_i) = \delta_i$ for all i.

1.127. Proposition. *Let Q be a Dynkin quiver. Then, $C-1$ is invertible in $L_{\mathbb{R}}$.*

Proof. When Q is Dynkin, $\operatorname{Rad}(\cdot, \cdot) = 0$ since $(x, x) = 2$ for all $x \in L$ (see 1.84). Thus, by 1.125, C has no fixed points in $L_{\mathbb{R}}$, so 1 is not an eigenvalue of C. Thus, $C - 1$ is invertible as an endomorphism of $L_{\mathbb{R}}$. □

1.128. Proposition. *Let Q be a Dynkin quiver. For all $0 \neq \alpha \in L$, there exists a $k \neq 0$ such that $C^k\alpha \not\geq 0$.*

Proof. Let h be the order of C. Then, consider

$$\frac{C^h - 1}{C - 1}(\alpha) = 0 \Longrightarrow \alpha + C\alpha + \cdots + C^{n-1}\alpha = 0$$

Then, $C^i\alpha \not\geq 0$ for some i, otherwise the equality could not hold. □

1.129. Lemma. *Given a quiver Q with no oriented cycles, there always exist an ordering $I = \{i_1, \ldots, i_r\}$ such that the ordering is adapted to Q. (See 1.119).*

Proof. One can take the ordering of listing j before i if there exists an edge $i \to j$. Since the quiver is acyclic, such an ordering exists. To show it is adapted, □ Show it is adapted

1.130. Example. Consider the quiver with the ordering

$$\begin{tikzcd} & & 2 & & & \\ & 1 & 4 \arrow[l] \arrow[u] \arrow[r] & 3 & 5 \arrow[l] & 6 \arrow[l] \end{tikzcd}$$

Then, we can apply a series of reflections to get

$$s_1^{\downarrow} Q = \begin{tikzcd} & 2 & & & \\ 1 \arrow[r] & 4 \arrow[u] \arrow[r] & 3 & 5 \arrow[l] & 6 \arrow[l] \end{tikzcd}$$

$$s_2^{\downarrow}s_1^{\downarrow}Q = \begin{tikzcd} & 2 \arrow[d] & & & \\ 1 \arrow[r] & 4 \arrow[r] & 3 & 5 \arrow[l] & 6 \arrow[l] \end{tikzcd}$$

$$s_3^{\downarrow}s_2^{\downarrow}s_1^{\downarrow}Q = \begin{tikzcd} & 2 \arrow[d] & & & \\ 1 \arrow[r] & 4 & 3 \arrow[l] \arrow[r] & 5 & 6 \arrow[l] \end{tikzcd}$$

$$s_4^{\downarrow}s_3^{\downarrow}s_2^{\downarrow}s_1^{\downarrow}Q = \begin{tikzcd} & 2 & & & \\ 1 & 4 \arrow[l] \arrow[u] \arrow[r] & 3 \arrow[r] & 5 & 6 \arrow[l] \end{tikzcd}$$

$$s_5^{\downarrow}s_4^{\downarrow}s_3^{\downarrow}s_2^{\downarrow}s_1^{\downarrow}Q = \begin{tikzcd} & 2 & & & \\ 1 & 4 \arrow[l] \arrow[u] \arrow[r] & 3 & 5 \arrow[l] \arrow[r] & 6 \end{tikzcd}$$

So, the ordering is adapted.

1.131. Proposition. *Using the choice of ordering from the lemma above, $C = s_{i_r} \cdots s_{i_1}$ satisfies*

$$C^{\downarrow}Q = s_{i_r}^{\downarrow} \cdots s_{i_1}^{\downarrow}Q = Q$$

1.132. Example. Apply C to the example above and one recovers $CQ = Q$.

Proof. For any edge $h \in \Omega$, its orientation gets reversed exactly twice when applying $s_{i_r}^{\downarrow} \cdots s_{i_1}^{\downarrow}$. $\square$

Proof of Lemma 1.118. We use the above proposition. Consider the sequence

$$\begin{aligned} &\vec{v},\ s_{i_1}^{\downarrow}\vec{v},\ s_{i_2}^{\downarrow}s_{i_1}^{\downarrow}\vec{v},\ \ldots,\ s_{i_r}^{\downarrow}\cdots s_{i_1}^{\downarrow}\vec{v} = C\vec{v},\\ &s_{i_1}^{\downarrow}C\vec{v},\ s_{i_2}^{\downarrow}s_{i_1}^{\downarrow}C\vec{v}, \ldots, C^2\vec{v},\\ &\vdots \end{aligned}$$

By 1.128, $C^{\ell}v \not\geq 0$ for some ℓ. Thus, the lemma is proven. $\square$

Why? This is not clear to me since the lemma states that $Cv \not\geq 0$, but we proved there is some ℓ such that $C^{\ell}v \not\geq 0$.

1.4. Longest Element and Ordering of Positive Roots.

1.133. Definition. Let $w_\circ \in W$ be the largest element $w_\circ - s_{i_N} \cdots s_{i_1}$ reduced, where $N = \ell(w_\circ) = |R_+|$.

1.134. Remark. $w_\circ^2 = 1$ since $w_\circ$ takes all positive roots to all negative roos and vice-versa, so applying it twice takes the positive roots back to the positive roots.

Why should a given positive root go back to the **same** positive root?

1.135. Definition. Let $\gamma_1 := \alpha_{i_1}$, $\gamma_2 := s_{i_1}\alpha_{i_2}$, $\gamma_3 := s_{i_1}s_{i_2}\alpha_{i_3}$, $\ldots$, $\gamma_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$.

1.136. Proposition. *$R_+ = \{\gamma_1, \dots, \gamma_N\}$. This is a special case of $R_+(W)$.*

Why? Any intuition at least?

1.137. Proposition. *A Dynkin graph is bipartite, that is, there is an indexing $I = I_0 \sqcup I_1$ such that each edge connects a vertex in I_0 to a vertex in I_1. (Note that Dynkin graphs are always trees, so this definition coincides with the more general notion of a bipartite graph.)*

1.138. Remark. Note that s_i, s_j commute for all $i, j \in I_0$.

1.139. Definition. Define

$$c_0 := \prod_{i \in I_0} s_i, \quad c_1 := \prod_{i \in I_1} s_i$$

By definition, $c_1 c_0$ is a Coxeter element.

1.140. Proposition. *Let h be the order of $w_\circ$. Then,*

$$w_\circ = \underbrace{c_1 c_0 c_1 c_0 \cdots}_{h \text{ factors}} = \underbrace{c_0 c_1 c_0 c_1 \cdots}_{h \text{ factors}}$$

.

Proof. The proof of this can be found somewhere in Humphreys "Reflection groups and Coxeter groups" section 3.17 (but where?!) by looking at the action of c_0 and c_1 on a special two-dimensional real plane. □

1.141. Corollary. *Given a root system R, $|R| = h \cdot r$ where $r = |I|$, the rank of the quiver, and h is the Coxeter number.*

1.142. Example. This makes sense with given types. SL_n has $|R| = n(n-1)$, but we computed earlier that $h = n$ and since $\mathfrak{sl}_n$ corresponds to Dynkin graph A_{n-1}, we get $r = n - 1$.

Given type D_n, we compute

Do this!

1.143. Proposition. *Given an orientation Ω of Dynkin $|Q|$, there exists a reduced expression $w_\circ = s_{i_N} \cdots s_{i_1}$ such that $\{i_1, \dots, i_N\}$ is adapted to Ω.*

Proof. Given $w_\circ = s_{i_N} \cdots s_{i_1}$ is adapted to Ω, then for some j, $w_\circ = s_j s_{i_N} \cdots s_{i_1}$ is adapted to $s_i^{\downarrow}\Omega$. Thus,

$$s_j w_\circ = w_\circ s_{i_1} \Longrightarrow s_j = w_\circ s_{i_1} w_\circ^{-1}$$

Thus, $j = -w_\circ(i_1)$. So, it suffices to find one reduced word of $w_\circ$ adapted to one distinguished $\Omega_\circ$. In fact, $w_\circ = \cdots c_0 c_1 c_0$ is adapted to $\Omega_\circ$ with I_0 being the sinks and I_1 being the sources. □

why the minus sign?

What about vertices that are neither sink nor source? Like in A_3.

1.144. Corollary. *Let Q be a Dynkin quiver. Then, a complete list of indecomposable representations is given by*

$$I_1 = S(i_1)$$

$$I_2 = \Phi_{i_1}^{\uparrow}(S(i_2)) \qquad S(i_2) \in \mathrm{Rep}(s_{i_1}^{\downarrow} Q)$$

$$
\begin{aligned}
I_3 =& \Phi^{\uparrow}_{i_1}\Phi^{\uparrow}_{i_2}(S(i_3)) && S(i_3) \in \mathrm{Rep}(s^{\downarrow}_{i_2}s^{\downarrow}_{i_1}Q) \\
&\vdots \\
I_n =& \Phi^{\uparrow}_{i_1}\cdots\Phi^{\uparrow}_{i_{n-1}}(S(i_n))
\end{aligned}
$$

1.145. Example. Consider A_3 given by

$$2 \longrightarrow 1 \longleftarrow 3$$

Then,

$$
\begin{aligned}
I_1 &= S(1) = 0 \to \Bbbk \leftarrow 0 \\
I_2 &= \Phi^{\uparrow}_1(S(2)) && S(2) = \Bbbk \leftarrow 0 \to 0 \\
&= \Bbbk \to \Bbbk \leftarrow 0 \\
I_3 &= \Phi^{\uparrow}_1\Phi^{\uparrow}_2(S(3)) && S(3) = 0 \to 0 \to \Bbbk \\
&= \Phi^{\uparrow}_1(0 \leftarrow 0 \to \Bbbk) \\
&= 0 \to \Bbbk \leftarrow \Bbbk
\end{aligned}
$$

2. Hall Algebras

(9/14/2017) Lecture 8.

2.1. Classical Hall Algebras. This section is based primarily on ([**Mac95**]). Consider the quiver

$$
\begin{tikzcd}
1 \arrow[loop above]
\end{tikzcd}
$$

and take $O = \Bbbk[t]$ where $\Bbbk$ is a finite field with $|\Bbbk| = q$. Then, we have $t\Bbbk[t] = \mathfrak{p} \trianglelefteq O$. We can then constuct M to be a finite dimensional $\Bbbk$-vector space with nilpotent T a linear transformation. Then, we turn M into a finite O-module via the action $t \mapsto T$.

2.1. Remark. More generally, let $\hat{O} = \Bbbk[[t]] \supseteq \mathfrak{p}$ is a discrete valuation ring over the p-adic numbers Z_p.

2.2. Theorem. *Since O is a PID, we have, by the classification of finitely generated modules over a PID, that*

$$M = \bigoplus_{i=1}^{r} O/\mathfrak{p}^{\lambda_i}$$

for natural numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. In other words

$$t = \begin{pmatrix} J_{\lambda_1}(0) & & & \\ & J_{\lambda_2}(0) & & \\ & & \ddots & \\ & & & J_{\lambda_r}(0) \end{pmatrix}$$

Note that we have tableau $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.

2.3. Definition. We define the *transpose* of a tableau λ to be... , denoted λ'. fill in

2.4. Proposition. *Let* $\mu_i = \dim_{\Bbbk}(\mathfrak{p}^{i-1}M/\mathfrak{p}^i M)$. *Then,* $\mu = (\mu_1, \mu_2, \ldots) = \lambda'$

Proof. Let x_j be a generator of $O/\mathfrak{p}^{\lambda_j}$. Then $\mathfrak{p}^{i-1}M$ is generated by those $t^{i-1}x \neq 0$, i.e. $\lambda_j \geq i$. Thus,

$$\lambda'_i = \#\{j \mid \lambda_j \geq i\} \quad \frac{\mathfrak{p}^{i-1}(O/\mathfrak{p}^{\lambda_j})}{\mathfrak{p}^i(O/\mathfrak{p}^{\lambda_j})} = \begin{cases} \Bbbk & \lambda_j \geq i \\ 0 & \text{otherwise} \end{cases} \implies \dim_{\Bbbk}\left(\frac{\mathfrak{p}^{i-1}(O/\mathfrak{p}^{\lambda_j})}{\mathfrak{p}^i(O/\mathfrak{p}^{\lambda_j})}\right) = \lambda'_i$$

□

2.5. Definition. Call λ the *type* of M.

2.6. Definition. Define

$$|\lambda| := \sum_i \lambda_i$$

to be the *length* of M for M of type λ. Note that $|\lambda|$ is the length of the composition series of M. We will sometimes denote this $\ell(M) = |\lambda|$.

2.7. Definition. Let $N \leq M$. Then, the *cotype of* N *in* M is the type of M/N.

2.8. Example. If $\lambda = (r) = \square\square\cdots\square$, then $M = O/\mathfrak{p}^r$ is cyclic.

If $\lambda = (1^r) = \begin{array}{c}\square\\ \square\\ \vdots\\ \square\end{array}$, then $M \cong \Bbbk^r$ as a vectorspace. This is referred to as an *elementary module*.

2.9. Definition. Let M be a module of type λ and let μ, ν be partitions. Then, we define the *Hall numbers* to be

$$G^{\lambda}_{\mu\nu} := \#\{0 \to N \to M \to M/N \to 0 \mid N \leq M \text{ has type } \nu \text{ and cotype } \mu\}$$

we can generalize this definition as follows. Let $\mu^{(1)}, \ldots, \mu^{(r)}$ be a sequence of partitions. Then

$$G^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}} := \#\{M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = 0 \mid M_{i-1}/M_i \text{ has type } \mu^{(i)}\}$$

2.10. Definition. The *Hall algebra* H is a free $\mathbb{Z}$-algebra with basis u_λ where λ is a partition such that, for partitions μ, ν,

$$u_\mu u_\nu = \sum_\lambda G^{\lambda}_{\mu\nu} u_\lambda$$

2.11. Proposition. *$G^{\lambda}_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$. Thus, the sum defined in the Hall algebra multiplication is a finite sum.*

Proof. To have a short exact sequence

$$0 \to N \to M \to M/N \to 0$$

with N having type ν and cotype μ and M having type λ, it is necessary for $|\mu| + |\nu| = |\lambda|$. This follows from the classification of $M = \bigoplus O/\mathfrak{p}^{\lambda_i}$ as a module over PID O. $\square$

2.12. Proposition. *A Hall algebra H is commutative (that is, $G^{\lambda}_{\mu\nu} = G^{\lambda}_{\nu\mu}$) and associative with $1 = u_{\varnothing}$.*

Proof. The commutativity follows from the duality (given $M, \hat{M}$ of type λ),

Where does this duality come from? Transposition of the JCF type argument?

$$\{N \mid N \leq M, N \text{ has type } \nu \text{ and cotype } \mu\} \xleftrightarrow{\text{1-1}} \{\hat{N} \mid \hat{N} \leq \hat{M} \mid \hat{N} \text{ has type } \mu \text{ and cotype } \nu\}.$$

The associativity follows from the fact that the coefficient of u_λ in $(u_\mu u_\nu)u_\rho$ and in $u_\mu(u_\nu u_\rho)$ will be $G^{\lambda}_{\mu\nu\rho}$. $\square$

2.13. Proposition. *As a $\mathbb{Z}$-algebra, the Hall algebra H is generated by $u_{(1^r)}, r \geq 1$ algebraically independently.*

Proof. Denote $v_r = u_{(1^r)}$. Then, for $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$,

$$v_{\lambda'} := v_{\lambda'_1} v_{\lambda'_2} \cdots v_{\lambda'_s} = \sum_{\mu} a_{\lambda\mu} u_\mu$$

where $a_{\lambda\mu} = \#\{M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_s = 0 \mid M \text{ fixed type } \mu, \text{type of } M_{i-1}/M_i = (1^{\lambda_i})\}$. Now, if $a_{\lambda\mu} \neq 0$,

$$\begin{aligned}
&\implies \text{Such } (M_i) \text{ exists with } \mathfrak{p}(M_{i-1}/M_i) = 0\\
&\implies \mathfrak{p}(M_{i-1}) \subseteq M_i, \forall i\\
&\implies \mathfrak{p}^i M \subseteq M_i, \forall i\\
&\implies \mu'_1 + \cdots + \mu'_i = \ell(M/\mathfrak{p}^i M) \geq \ell(M/M_i) = \lambda'_1 + \cdots + \lambda'_i\\
&\implies \lambda' \trianglelefteq \mu'\\
&\implies \mu \trianglelefteq \lambda
\end{aligned}$$

So, if $\mu = \lambda$, then there exists only one filtration with $M_i = \mathfrak{p}^i M$ and so $a_{\lambda\lambda} = 1$. Thus, $(a_{\lambda\mu})$ is strictly upper unitriangular, with $a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$. Therefore, it is invertible over $\mathbb{Z}$ and can be solved, thus giving u_μ as a $\mathbb{Z}$-linear combination of $v_{\lambda'}$. $\square$

2.14. Corollary. *As rings, $H \cong \Lambda$, the ring of symmetric function in infinitely many variables via the isomorphism*

$$u_{(1^r)} \mapsto q^{-\frac{r(r-1)}{2}} e_r$$

2.15. Definition. Let λ be a partition. Then, we define

$$n(\lambda) := \sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2}$$

2.16. Example. These equalities follow from just summing the entries of a partition over rows versus over colums. For example, take

$$\lambda = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}$$

Then, $n(\lambda) = 0*4+1*2+2*1 = (0+1+2)+(0+1)+(0)+(0) = 4$

Now, our goal is to understand these structure constants $G^{\lambda}_{\mu\nu}$. As a reminder, we are still working over field $\mathbb{k}$ with $|\mathbb{k}| = q$.

2.17. Theorem (Steinitz). *(a) There exists a polynomial $g^{\lambda}_{\mu\nu}(t) \in \mathbb{Z}[t]$ such that*

$$G^{\lambda}_{\mu\nu} = g^{\lambda}_{\mu\nu}(q)$$

(b) The degree of $g^{\lambda}_{\mu\nu}$ is less than or equal to $n(\lambda) - n(\mu) - n(\nu)$. Furthermore, the coefficient of $t^{n(\lambda)-n(\mu)-n(\nu)} = c^{\lambda}_{\mu\nu}$, the Littlewood-Richardson coefficients.
(c) $g^{\lambda}_{\mu\nu}(t) = g^{\lambda}_{\nu\mu}(t)$
(d) If $c^{\lambda}_{\mu\nu} = 0$, then $g^{\lambda}_{\mu\nu} = 0$.

Recall that $v_{\lambda_i'} = \sum_\mu a_{\lambda\mu} u_\mu$.

2.18. Proposition. *(a) There exists a combinatorial formula*

$$a_{\lambda\mu} = \sum_{A \text{ diagrams}} q^{d(A)} \in \mathbb{Z}[q]$$

(b) $\deg a_{\lambda\mu}(t) \leq n(\mu) - n(\lambda)$
(c) $\tilde{a}_{\lambda\mu}(t) = t^{n(\mu)-n(\lambda)} a_{\lambda\mu}(t^{-1})$ satisfies $\tilde{a}(0) = K_{\mu'\lambda'}$, the Kostka number.
(d) $a_{\lambda\mu}(t) = 0$ unless $\mu \trianglelefteq \lambda$ and $a_{\lambda\lambda} = 1$.

What kind of diagrams? Young diagrams?

Where does this t come from?

We already showed this part, or do we mean $a_{\lambda\lambda}(t) = 1$?

Note that part (a) above implies the following theorem

2.19. Theorem. *$(a_{\lambda\mu}(t))$ is a unitriangular matrix over $\mathbb{Z}[t]$ and is thus invertible.*

2.20. Definition. Given two partitions λ and μ, we define the *union* $\lambda \cup \mu$ to be a partition of $|\lambda| + |\mu|$ where each row is a row of λ or a row of μ.

2.21. Lemma. $v_{\lambda'} v_{\rho'} = v_{\lambda' \cup \mu'}$.

Proof. This follows from the commutativity of our algebra. $\square$

2.22. PROPOSITION. $g^\lambda_{\mu\nu} \in \mathbb{Z}[t]$.

PROOF. Given that $(a_{\lambda\mu}(t))$ is inverticle, this means that $u_\mu, u_\nu \in \sum_\lambda \mathbb{Z}[t]v_{\lambda'}$. Thus, the product $u_\mu u_\nu$ is also a $\mathbb{Z}$-linear combination of $v_{\lambda'}$'s, so $g^\lambda_{\mu\nu} \in \mathbb{Z}[t]$. □

2.23. DEFINITION. We define $P_\mu(x, ; t)$ to be the polynomials such that, for $e_{\lambda'} \in \Lambda_{\mathbb{Z}[t]}$,

$$e_{\lambda'} = \sum_\mu \tilde{a}_{\lambda\mu}(t) P_\mu(x; t)$$

Finish this lecture. The last page is hard to follow.

(9/19/2017) Lecture 9.

2.2. Generic Hall Algebras. Let Q be a quiver. Then, we can make sense of the category $\mathrm{Rep}(Q, \mathbb{F}_q)$.

2.24. DEFINITION. Let $M_1, M_2, L \in \mathrm{Rep}(Q, \mathbb{F}_q)$. Define

$$\mathcal{F}^L_{M_1,M_2} := \{\text{Subrepresentation } X \subseteq L \mid X \cong M_2, L/X \cong M_1\}$$

that is

$$\mathcal{F}^L_{M_1,M_2} = \{0 \to X \xrightarrow{f} L \xrightarrow{g} L/X \to 0 \mid X \cong M_2, L/X \cong M_1\} \cong \{(f,g)\}/\operatorname{Aut}_Q(M_2) \times \operatorname{Aut}_Q(M_1)$$

Then, we have generalization for $M_1, \ldots, M_k, L \in \mathrm{Rep}(Q, \mathbb{F}_q)$ given by

$$\mathcal{F}^L_{M_1,\ldots,M_k} := \{L = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_k = 0 \mid L_{i-1}/L_i \cong M_i, \forall i\}$$

2.25. DEFINITION. Define the structure constants

$$F^L_{M_1,M_2} := |\mathcal{F}^L_{M_1,M_2}|$$

similarly,

$$F^L_{M_1,\ldots,M_k} := |\mathcal{F}^L_{M_1,\ldots,M_k}|$$

Thus, we have defined a generalization of the structure constants $G^\lambda_{\mu\nu}$ for the classical Hall algebra and the program of this lecture will be to prove analogous theorems to those that we have for classical Hall algebras.

2.26. PROPOSITION. $F^\ell_{M_1,\ldots,M_k} = 0$ *unless* $\mathbf{dim}M_1 + \cdots + \mathbf{dim}M_k = L$.

2.27. DEFINITION. We define the *Hall algebra* $H(Q, \mathbb{F}_q)$ to be the $\mathbb{Z}$-algebra with basis given by $[M]$, the isomorphism classes in $\mathrm{Rep}(Q)$ such that

$$[M_1] * [M_2] = \sum_{[L]} F^L_{M_1,M_2}[L]$$

2.28. PROPOSITION. *$H(Q, \mathbb{F}_q)$ is an associative algebra with $1 = [0]$. Moreover, it is $\mathbb{N}^I$-graded, that is*

$$H(Q, \mathbb{F}_q) = \bigoplus_{\vec{v} \in \mathbb{N}^I} H_{\vec{v}}(Q, \mathbb{F}_q)$$

where $H_{\vec{v}}(Q, \mathbb{F}_q)$ is spanned by $[M]$ with $\mathbf{dim}M = \vec{v}$.

PROOF. To see associativity, note that

$$([M_1] * [M_2]) * [M_3] = F^L_{M_1,M_2,M_3} = [M_1] * ([M_2] * [M_3])$$

$\square$

2.29. REMARK. There is also a twisted product, given by

$$U \cdot V = q^{\frac{\langle \vec{u}, \vec{v} \rangle}{2}} U * V$$

for $U \in H_{\vec{u}}(Q)$ and $V \in H_{\vec{v}}(Q)$ where $\langle , \rangle$ is the Euler form. This product is also associative with $[0] = 1$.

2.30. DEFINITION. Let n be a natural number. Then, we define

$$[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$$

and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}$$

2.31. EXAMPLE. Take $Q = \bullet$ and let S be the 1-dimensional representation. Then, all representations are of the form $nS := S^{\oplus n}$, $n \geq 0$. We then know that

$$[(n-1)S] * [S] = \alpha[nS]$$

for some α. To figure out what α is, we must count the number of submodules of nS, which one can quickly check is equivalent to $|\mathbb{P}^{n-1}(\mathbb{F}_q)| = \frac{q^n-1}{q-1}$. Thus,

$$[(n-1)S] * [S] = \frac{q^n - 1}{q - 1}[nS]$$

In fact, we can show the following proposition.

2.32. PROPOSITION. *For $Q = \bullet$ and S as in the above example,*

$$[nS] * [mS] = \begin{bmatrix} m+n \\ m \end{bmatrix}_q [(n+m)S]$$

for natural numbers m, n.

PROOF. There are two proof methods for this proposition, either by induction or counting directly. We will go by the latter route. Note that

$$\#\{m\text{-dimensional subspaces of } \mathbb{F}_q^{m+n}\} = \# \operatorname{Gr}(m, n+m)$$

However, since $GL(n+m, \mathbb{F}_q)$ acts transitively on $\operatorname{Gr}(m, n_m)$, then

$$\operatorname{Gr}(m, n+m) = GL(n+m, \mathbb{F}_q)/P_{n,m}$$

for some block matrix $P_{n,m}$ of the form

$$P_{n,m} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

It is a common abstract algebra fact that

$$|GL_n(q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

$$= q^{\binom{n}{2}}(q-1)^n \frac{(q^n-1)\cdots(q-1)}{(q-1)\cdots(q-1)}$$
$$= q^{\binom{n}{2}}(q-1)^n [n]_q!$$

So,

$$\#GL(m+n,\mathbb{F}_q)/P_{n,m} = \frac{q^{\binom{m+n}{2}}(q-1)^{m+n}[n+m]_q!}{q^{\binom{n}{2}}(q-1)^n[n]_q! q^{\binom{m}{2}}(q-1)^m[m]_q! q^{mn}}$$
$$= \frac{q^{\binom{m+n}{2}}[n+m]_q!}{q^{\binom{n}{2}}[n]_q! q^{\binom{m}{2}}[m]_q! q^{mn}}$$

$\square$

2.33. Remark. This gives rise toa "generic" Hall algebra over $\mathbb{Z}[t]$, $H(\bullet)$ where q is replaced by t.

2.34. Proposition. *For $H(\bullet)$,*

$$[S]^n = [n]_q![nS] \Longrightarrow [nS] = \frac{[S]^n}{[n]_q!}$$

This suggests that the Hall algebra is isomorphic to the algebra $\mathbb{C}[x]$ of polynomials in one variable with isomorphism given by

$$[nS] \mapsto \frac{x^n}{[n]_q!}$$

2.35. Definition. Define

$$x^{(n)} := \frac{x^n}{[n]_t!}$$

2.36. Proposition. *$H(\bullet) \cong \mathbb{Z}[t][x, x^{(2)}, x^{(3)}, \ldots]$ and over $\mathbb{Q}(t)$,*

$$H(\bullet) \otimes \mathbb{Q}(t) \cong \mathbb{Q}(t)[x]$$

Understand where this is coming from.

2.37. Proposition. *Given a general quiver Q, if $S \in \mathrm{Rep}(Q,\mathbb{F}_q)$ such that*

$$\mathrm{Ext}^1_Q(S,S) = 0 \quad \mathrm{Hom}_Q(S,S) = \mathbb{F}_q$$

Then, $[nS] = \frac{[S]^n}{[n]_q!}$.

2.38. Example. Simply take $S = S(i)$.

2.39. Remark. In general, $[M_1] * [M_2] \neq [M_1 \oplus M_2]$.

2.40. Proposition. *Assume $\mathrm{Hom}_Q(M_2, M_1) = 0 = \mathrm{Ext}^1_Q(M_1, M_2)$. Then,*

$$[M_1] * [M_2] = [M_1 \oplus M_2]$$

Proof. Consider a short exact sequence

$$0 \to M_2 \to L \to M_1 \to 0$$

Then, since $\operatorname{Ext}^1(M_1, M_2) = 0$, $L = M_1 \oplus M_2$. Thus,

$$[M_1] * [M_2] = \alpha[M_1 \oplus M_2]$$

for some α. Let $N \subseteq M_1 \oplus M_2$ be isomorphic to M_2 but $N \neq \{(0, m_2) \mid m_2 \in M_2\}$, that is, let N not be the standard embedding of M_2 into $M_1 \oplus M_2$. Then, we have

$$\begin{tikzcd} N \arrow[rr, "\iota"] \arrow[dr, dashed, "\neq 0"'] & & M_1 \oplus M_2 \arrow[dl, "\pi_1"] \\ & M_1 & \end{tikzcd}$$

However, this contradicts $\operatorname{Hom}_Q(M_2, M_1) = 0$, so the only possible embedding for N is $N = \{(0, m_2) \mid m_2 \in M_2\}$ and thus $\alpha = 1$. $\square$

From now on, assume Q is Dynkin. Recall that we then have

$$\{\text{Indecomposables}\} \xleftrightarrow{\text{1-1}} R_+ \text{ (Positive roots)}$$

$$I_\alpha \leftarrow\!\shortmid \alpha$$

$$\text{Isomorphism classes in } \operatorname{Rep}(Q) \leftrightarrow \mathbb{N}^{R_+} := \{\vec{n} \colon R_+ \to \mathbb{N}\}$$

$$M_n = \bigoplus_{\alpha \in R_+} n_\alpha I_\alpha \leftarrow\!\shortmid \vec{n} = (n_\alpha)_{\alpha \in R_+}$$

2.41. Proposition. *Let Q be Dynkin. Then, for all $\vec{n}, \vec{m}, \vec{k} \in \mathbb{N}^{R_+}$, there is a polynomial $\phi^{\vec{m}}_{\vec{n},\vec{k}} \in \mathbb{Z}[t]$ such that*

$$\phi^{\vec{m}}_{\vec{n},\vec{k}}(q) = F^{M_{\vec{m}}}_{M_{\vec{n}}, M_{\vec{k}}}$$

This gives rise to the "generic" Hall algebra $H_t(Q) = H(Q)_{\mathbb{Z}[t]}$ with multiplication

$$[M_{\vec{n}}] * [M_{\vec{k}}] = \sum_{\vec{m}} \phi^{\vec{m}}_{\vec{n},\vec{k}}(t) M_{\vec{m}}$$

We can also make use of specialization to get

$$H(Q)_{\mathbb{Z}[t]}|_{t=1} =: H_1(Q) \text{ is a } \mathbb{Z}\text{-algebra}$$

and also

$$H(Q)_{\mathbb{C}} = H(Q)_{\mathbb{Z}[t]} \otimes_{\mathbb{Z}[t]} \mathbb{C}$$

2.42. Theorem. *Let $\tilde{\mathcal{A}} = \mathbb{Z}[t][[n]_t^{-1}, \forall n \geq 1]$. Then, the (Ringel-)Hall algebra $H(Q)_{\tilde{\mathcal{A}}}$ is generated by $\theta_i = [S(i)]$ for $i \in I$.*

This theorem will be expanded upon in the next lecture.

(9/21/2017) Lecture 10. Recall that, by 1.143, if Q is Dynkin, there exists a reduced $w_\circ = s_{i_n} \cdots s_{i_1}$ such that $\{i_1, \dots, i_N\}$ is adapted to Q. Thus, we have positive roots $R_+ = \{\gamma_1, \dots, \gamma_N\}$ where

$$\gamma_1 = \alpha_1, \quad \gamma_2 = s_{i_1}(\alpha_2), \dots, \gamma_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_N)$$

and Gabriel's Theorem 1.86 says that if Q is Dynkin, there is a one-to-one correspondance with the indecomposable representations of Q given by $\gamma_i \mapsto I_i$. Also, recall that, in this setting,

How exactly does this follow?

$$\mathrm{Hom}_Q(I_a, I_b) = \delta_{a,b} \text{ for } a \geq b$$
$$\mathrm{Ext}^1_Q(I_b, I_a) = 0$$

2.43. Proposition. *For Q Dynkin, if $V \cong \bigoplus_{k=1}^N n_k I_k$, then*

$$[V] = [I_1]^{(n_1)} * \cdots * [I_N]^{(n_N)} \text{ in } H_t(Q)$$

Proof. It suffices to check in $H(Q, \mathbb{F}_q)$. By, 2.40 and above, we have

why does it suffice to check only this?

$$\left[\bigoplus_{k=1}^D n_k I_k\right] = [n_1 I_1] * [n_2 I_2] * \cdots * [n_D I_D]$$

$\square$

Proof of 2.42. We define a partial order on $\mathbb{N}^I$ by

$$\vec{v} \leq \vec{w} \Longleftrightarrow v_i \leq w_i$$

This order allows us to induce a partial order on the isomorphism classes in $\mathrm{Rep}(Q)$ by

$$V \leq W \Longleftrightarrow \mathbf{dim} V \leq \mathbf{dim} W \text{ or } \mathbf{dim} V = \mathbf{dim} W \text{ and } \mathcal{O}_V \subseteq \overline{\mathcal{O}}_W$$

Let $H' := \tilde{\mathcal{A}}\langle \mathcal{O}_i \mid i \in I\rangle$. Then, we proceed by induction on $\leq$ to show $[V] \in H'$. Assume for all $V' < V$, $[V'] \in H'$. We have two cases:

(i) V is decomposable, that is $V = V' \oplus V''$. By the proposition above, $[V] = [V'] * [V'']$, both of which are in H' by the inductive hypothesis since $\mathbf{dim} V' < \mathbf{dim} V$ and $\mathbf{dim} V'' < \mathbf{dim} V$. Thus, $V \in H'$.

(ii) V is indecomposable. Then, find an i such that $V_i \neq 0$ and, for all edges $i \to j$, $V_j = 0$. Let $v_i = \dim V_i$. Then, we have short exact sequence

$$0 \to v_i S(i) \to V \to V/(v_i S(i)) \to 0$$

So, $[V/(v_i S(i))] * [V'] = [V] + \sum_k c_k [V_k'']$, where V_k'' are other representations with $\mathbf{dim} V_k'' = \mathbf{dim} V$. However, since V is a unique indecomposable with $\mathbf{dim} V$ by the one-to-one correspondance, the V_k'' must be decomposable and thus $\sum c_k [V_k''] \in H'$ and $[V/(v_i S(i))] * [V'] \in H'$ give us that $[V] \in H'$.

$\square$

2.44. Corollary. *Over $\mathbb{C}$, $\{\theta_i \mid i \in I\}$ generates $H_1(Q)_{\mathbb{C}}$.*

We now move on to a discussion of the relations among the θ_i's.

2.45. Proposition. *Let Q be a Dynkin quiver. If i and j are not connected, then in the Hall algebra*

$$\theta_i * \theta_j = [S(i) \oplus S(j)] = \theta_j * \theta_i$$

2.46. Example. Let us do some computations with the quiver

$$Q = \; 1 \longrightarrow 2$$

with $\Bbbk = \mathbb{F}_q$. Then, we have representations

$$\begin{aligned} S(1) &= \Bbbk \to 0 \\ S(2) &= 0 \to \Bbbk \\ S_{12} &:= \Bbbk \xrightarrow{\sim} \Bbbk \\ \tilde{S}_{12} &:= \Bbbk \xrightarrow{0} \Bbbk \end{aligned}$$

Then,

$$\begin{aligned} \theta_1 * \theta_2 &= [S_{12}] + [\tilde{S}_1 2] \\ \theta_2 * \theta_1 &= [\tilde{S}_{12}] \end{aligned}$$

since θ_2 is a submodule of S_{12} and $\tilde{S}_{12}$ with multiplicity 1 and θ_1 is a submodule of $\tilde{S}_{12}$ with multiplicity 1.

$$\theta_2 \hookrightarrow S_{12} = \begin{array}{ccc} \Bbbk & \xrightarrow{\sim} & \Bbbk \\ \uparrow & & \uparrow \\ \Bbbk & \longrightarrow & 0 \end{array} \quad \theta_2 \hookrightarrow \tilde{S}_{12} = \begin{array}{ccc} \Bbbk & \xrightarrow{0} & \Bbbk \\ \uparrow & & \uparrow \\ \Bbbk & \longrightarrow & 0 \end{array} \quad \theta_1 \hookrightarrow \tilde{S}_{12} = \begin{array}{ccc} \Bbbk & \xrightarrow{0} & \Bbbk \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \Bbbk \end{array}$$

Note that there is no way to embed $\theta_1 \hookrightarrow S_{12}$ and have the diagram commute. Now, we wish to compute $\theta_1^2 * \theta_2, \theta_1 * \theta_2 * \theta_1$, and $\theta_2 * \theta_1^2$. To get the right quotients, we must first write down representations of Q of $\mathbf{dim} = (2,1)$:

$$S_{112} = \; \Bbbk^2 \longrightarrow\!\!\!\!\!\rightarrow \Bbbk \quad \tilde{S}_{112} = \; \Bbbk^2 \xrightarrow{\;0\;} \Bbbk$$

Then, we have

$$\begin{aligned} \theta_1^2 * \theta_2 &= (q+1)[2S(1)] * [S(2)] = (q+1)([S_{112}] + [\tilde{S}_{112}]) \\ \theta_2 * \theta_1^2 &= (q+1)[S(2)] * [2S(1)] = (q+1)[\tilde{S}_{112}] \\ \theta_1 * \theta_2 * \theta_1 &= \theta_1 * [\tilde{S}_{12}] = [S_{112}] + (q+1)[\tilde{S}_{112}] \end{aligned}$$

To see the first equality, we compute

$$\theta_1 \hookrightarrow 2S(1) = \begin{array}{ccc} \Bbbk^2 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \Bbbk & \longrightarrow & 0 \end{array}$$

and see that there are $q+1$ possible maps $\Bbbk \to \Bbbk^2$ that make the diagram commute, since the map could be encoded by a rank 1 2×1 matrix with entries in $\mathbb{F}_q$, up to non-zero scalar multiplication by $\mathbb{F}_q$, thus $\frac{q^2-1}{q-1} = q+1$. Then, we see

$$S(2) \hookrightarrow S_{112} = \begin{tikzcd} \Bbbk^2 \arrow[r, two heads] & \Bbbk \\ 0 \arrow[u] \arrow[r] & \Bbbk \arrow[u] \end{tikzcd} \quad S(2) \hookrightarrow \tilde{S}_{112} = \begin{tikzcd} \Bbbk^2 \arrow[r, "0"] & \Bbbk \\ 0 \arrow[u] \arrow[r] & \Bbbk \arrow[u] \end{tikzcd} \quad 2S(1) \hookrightarrow \tilde{S}_{112} = \begin{tikzcd} \Bbbk^2 \arrow[r, "0"] & \Bbbk \\ \Bbbk^2 \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

each only have 1 embedding and

$$\tilde{S}_{12} \hookrightarrow S_{112} = \begin{tikzcd} \Bbbk^2 \arrow[r, two heads] & \Bbbk \\ \Bbbk \arrow[u] \arrow[r, "0"] & \Bbbk \arrow[u] \end{tikzcd} \quad \tilde{S}_{12} \hookrightarrow \tilde{S}_{112} = \begin{tikzcd} \Bbbk^2 \arrow[r, "0"] & \Bbbk \\ \Bbbk \arrow[u] \arrow[r, "\sim"] & \Bbbk \arrow[u] \end{tikzcd}$$

have 1 and $\frac{q^2-1}{q-1} = q+1$ embeddings, respectively.

From this example, we can get the following proposition

2.47. Proposition. *Let $i \to j$ in a quiver Q. Then, in $H(Q)_{\mathbb{Z}[t]}$*

$$\theta_i^2 * \theta_j - (t+1)\theta_i * \theta_j * \theta_i + t\theta_2 * \theta_i^2 = 0$$

Proof. The proof is the same as the computations in the exercise above. An exercise for the reader is to find a similar relationship between $\theta_2^2 * \theta_1, \theta_2 * \theta_1 * \theta_2$, and $\theta_1 * \theta_2^2$. □

2.48. Proposition. *Recall the twisted product for the Hall algebra given in 2.29. Given $Q = 1 \to 2$, then in $H(Q, \mathbb{F}_q)$, we get*

$$\begin{aligned} \theta_1 \cdot \theta_2 &= q^{-\frac{1}{2}}\theta_1 * \theta_2 \\ \theta_2 \cdot \theta_1 &= \theta_2 * \theta_1 \\ \theta_1^2 \cdot \theta_2 &= q^{-\frac{1}{2}}\theta_1^{*2} * \theta_2 \\ \theta_2 \cdot \theta_1^2 &= q^{\frac{1}{2}}\theta_2 * \theta_1^{*2} \\ \theta_1 \cdot \theta_2 \cdot \theta_1 &= \theta_1 * \theta_2 * \theta_1 \end{aligned}$$

Proof. These computations follow from the computations that

$$\langle S(i), S(i)\rangle = 1,\ \langle S(1), S(2)\rangle = -1,\ \langle S(2), S(1)\rangle = 0$$

where $\langle V, W\rangle = \sum_i v_i w_i - \sum_h v_{s(h)} w_{t(h)}$ is the Euler form. □

2.49. Remark. From now on, we may choose to omit the $\cdot$ in the Hall algebra, and simply write

$$[V][W] := [V] \cdot [W]$$

2.50. Corollary. *For $Q = 1 \to 2$ and $v = t^{\frac{1}{2}}$, in $H(Q)_{\mathbb{Z}[t]}$, we get*

(a) $\theta_1^2\theta_2 - (v + v^{-1})\theta_1\theta_2\theta_1 + \theta_2\theta_1^2 = 0$
(b) $\theta_2^2\theta_1 - (v + v^{-1})\theta_2\theta_1\theta_2 + \theta_1\theta_2^2 = 0$

PROOF. For part (a),

$$\begin{aligned}\theta_1^2\theta_2 - (v+v^{-1})\theta_1\theta_2\theta_1 + \theta_2\theta_1^2 &= t^{-\frac12}\theta_1^{*2} * \theta_2 - (t^{\frac12}+t^{-\frac12})\theta_1 * \theta_2 * \theta_1 + t^{\frac12}\theta_2 * \theta_1^{*2} \\ &= t^{-\frac12}(\theta_1^{*2} * \theta_2 - (t+1)\theta_1 * \theta_2 * \theta_1 + t\theta_2 * \theta_1^{*2}) \\ &= t^{-\frac12} * (0) && \text{by 2.47} \\ &= 0\end{aligned}$$

□

2.51. THEOREM (Ringel's Theorem). *Let Q be a Dynkin quiver. Let*

$$\tilde{H}(Q,v^2) = H_t(Q) \otimes_{\mathbb{Z}[t]} Q(v) \quad (t \mapsto v^2)$$

be equipped with the twisted product. Then, there exists an algebra isomorphism

$$\psi\colon U_v\mathfrak{n}^+ \to \tilde{H}(Q,v^2), \quad E_i \mapsto \theta_i$$

where

$$U_v\mathfrak{n}^+ = Q(v)\langle E_i \ (i \in I) \mid {}_{E_i^2E_j - [2]_v E_iE_jE_i + E_jE_i^2 = 0 \text{ for } i \to j}^{E_iE_j = E_jE_i \text{ for } i \not\to j,}\rangle$$

2.52. COROLLARY. *There is an algebra isomorphism*

$$U\mathfrak{n}^+ \xrightarrow{\sim} H_t(Q) \otimes_{\mathbb{Z}[t]} \mathbb{C}$$

sending $e_i \to \theta_i$.

Get a handle on the quantum groups vs enveloping algebras here. When is E_i appropriate and when is e_i appropriate?

PROOF OF THEOREM. We must check the following

- ψ is a homomorphism. This follows from 2.50.
- ψ is onto.
- ψ respects the grading $(U_v\mathfrak{n}^+)_{\vec{d}} \to \tilde{H}_{\vec{d}}(Q,v^2)_{\vec{d} \in \mathbb{N}^{\pm}}$

We also check

$$\dim \tilde{H}_{\vec{d}}(Q,v^2) = \left|\left\{(n_\alpha)_{\alpha \in R^+} \mid \sum n_\alpha \alpha = \vec{d}\right\}\right| = \dim U\mathfrak{n}^+$$

by the PBW theorem. □

Bibliography

[CG00] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, 2000.

[DDPW08] B. Deng, J. Du, B. Parshall, and J. Wang, *Finite Dimensional Algebras and Quantum Groups*, 2008.

[Hum72] J. E. Humphreys, *Introduction to Lie Algebras and Representaton Theory*, 1972. Third printing, revised.

[Hum90] ———, *Reflection Groups and Coxeter Groups*, 1990.

[Jan96] J. C. Jantzen, *Lectures on Quantum Groups*, 1996.

[KJ16] A. Kirillov Jr., *Quiver Representations and Quiver Varieties*, 2016.

[Mac95] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 1995. Second edition.