

A Raising Operator Formula for Macdonald Polynomials and other related families

George H. Seelinger

joint work with J. Blasiak, M. Haiman, J. Morse, and A. Pun

ghseeli@umich.edu

York University Applied Algebra Seminar

5 February 2024

- ① **Background on symmetric functions and Macdonald polynomials**
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ A new formula for Macdonald polynomials

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

$$e_3 = x_1 x_2 x_3 \quad h_3 = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + \dots$$

- Let $\Lambda = \mathbb{Q}(q, t)[e_1, e_2, \dots] = \mathbb{Q}(q, t)[h_1, h_2, \dots]$. Call these “symmetric functions.”
- Λ is a $\mathbb{Q}(q, t)$ -algebra.

Bases for symmetric functions

Dimension of degree d symmetric functions? Number of partitions of d .

Definition

$n \in \mathbb{Z}_{>0}$, a *partition* of n is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\Box & \Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\Box & \Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\Box & \Box & \Box \\ \hline\end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|}\hline\Box \\ \hline\end{array}$$

\implies any basis of symmetric functions is indexed by partitions.

Young Tableaux

Definition

Filling of partition diagram of λ with numbers such that

- ① strictly increasing up columns
- ② weakly increasing along rows

Collection is called $\text{SSYT}(\lambda)$.

For $\lambda = (2, 1)$,

2		3		3		2		3		3		2		3	
1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

Polynomials from tableaux

Associate a polynomial to $\text{SSYT}(\lambda)$.

$$\begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array} 1, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array} 1, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 2 \end{array} 2, \begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array} 2, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array} 3, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 2 \end{array} 3, \begin{array}{|c|c|}, \\ \hline 2 \\ \hline 1 \end{array} 3, \begin{array}{|c|c|}, \\ \hline 3 \\ \hline 1 \end{array} 2$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} \mathbf{x}^T \text{ for } \mathbf{x}^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function.
- $\{s_\lambda\}_\lambda$ forms a basis for $\Lambda_{\mathbb{Q}}$.

Representation theory and Schur functions

Frobenius character map, $\text{Frob}: \text{Rep}(S_n) \rightarrow \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n .
- Irreducible S_n -representation V_λ has $\text{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \text{Frob}(U) = \text{Frob}(V) + \text{Frob}(W)$
- $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \text{Frob}(V) \cdot \text{Frob}(W)$
- Upshot: S_n -representations go to symmetric functions in structure preserving way.

Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in \mathbb{N}) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

Harmonic polynomials

Harmonic polynomials

M = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

Harmonic polynomials

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ① Break M up into irreducible S_n -representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ② How many times does an irreducible S_n -representation occur?
Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]^{S_3})$ is a “regular representation.”

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: irreducible S_n -representation of polynomials of degree $d \mapsto q^d s_\lambda$
(graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Answer: Hall-Littlewood polynomial $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

A Problem

- In 1988, Macdonald introduces one basis of symmetric polynomials to rule them all!
- Coefficients in $\mathbb{Q}(q, t)$, specializations give Hall-Littlewood polynomials, Schur polynomials, and many other famous bases.
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n -regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

Irreducible S_n -representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = qts \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + ts \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + qs \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + s \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$

- Proved via connection to the Hilbert Scheme $\text{Hilb}^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}_{\lambda\mu}(q, t) s_\mu$ satisfies $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\text{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

Question

What symmetric function is the bigraded Frobenius characteristic of DH_n ?

Frobenius characteristic of DH_3

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Operator ∇

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X; q, t),$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$ and λ^* is the transpose partition to λ .

Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n .

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??
∇e_n	DH_n	Shuffle theorem

- ① Background on symmetric functions and Macdonald polynomials
- ② **Shuffle theorems, combinatorics, and LLT polynomials**
- ③ A new formula for Macdonald polynomials

Key Object: LLT Polynomials

Let $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape $= \lambda \setminus \mu$)

- The *content* of a box in row y , column x is $x - y$.
- *Reading order*: label boxes b_1, \dots, b_n by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \nu$ is *attacking* if a precedes b in reading order and
 - $\text{content}(b) = \text{content}(a)$, or
 - $\text{content}(b) = \text{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$$\nu = \left(\begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$$

-4	-3	-2	-1	b_3	b_6
-3	-2	-1	0	b_5	b_8
-2	-1	0	1	2	3
b_1	b_2	1	2	3	4
0	b_4	b_7	3	4	5

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

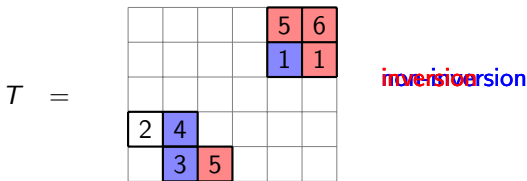
LLT Polynomials

- A *semistandard tableau* on ν is a map $T: \nu \rightarrow \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where $\text{inv}(T)$ is the number of attacking inversions in T and $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$.



$$\text{inv}(T) = 4, \quad \mathbf{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

LLT Polynomials $\mathcal{G}_\nu(X; q)$

- $\mathcal{G}_\nu(X; q)$ is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu(1)} \cdots s_{\nu(r)}$
- \mathcal{G}_ν were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazhdan-Luzstig polynomials.
- \mathcal{G}_ν is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

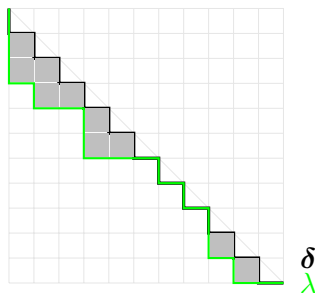
$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k -by- k Dyck paths.
- $\text{area}(\lambda)$ and $\text{dinv}(\lambda)$ statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset (skew) rows.
- ω an automorphism of symmetric functions: $\omega(s_{\lambda}) = s_{\lambda^*}$ for $\lambda^* =$ transpose of λ .

Dyck paths

Dyck paths

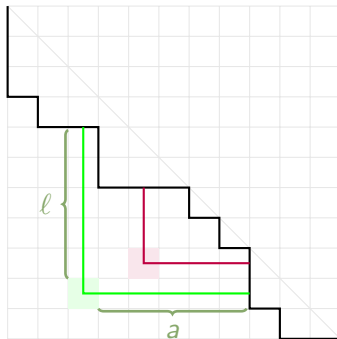
A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



- $\text{area}(\lambda)$ = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above $\text{area}(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k . $(1, 2, 5, 14, 42, \dots)$

dinv






$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Example ∇_{e_3}

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + qs_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qts_{2,1} + q^2 ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number” $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

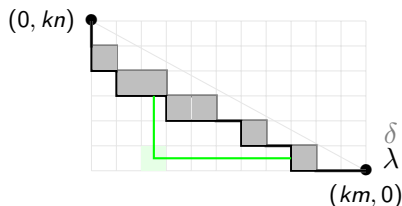
Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2021)

For $m, n > 0$ coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$



Elliptic Hall Algebra

When one has a family of operators, can they be realized by an action of an algebra?

Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p .

The *elliptic Hall algebra* \mathcal{E} is generated by subalgebras $\Lambda(X^{a,b})$ isomorphic to the ring of symmetric functions Λ over $\mathbb{k} = \mathbb{Q}(q, t)$, one for each coprime pair $(a, b) \in \mathbb{Z}^2$, along with an additional central subalgebra.

E.g., $e_k[-MX^{m,n}] \in \Lambda(X^{m,n})$.

\mathcal{E} acts on symmetric functions and $e_k[-MX^{1,1}] \cdot 1 = \nabla e_k$.

Can be difficult to work with in general. Can we make it more explicit?

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n , where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$\Psi = \text{Roots above Dyck path}$

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$, set

$$s_\gamma = \det(h_{\gamma_i+j-i})_{1 \leq i, j \leq n}$$

Then, $s_\gamma = \pm s_\lambda$ or 0 for some partition λ .

Precisely, for $\rho = (n-1, n-2, \dots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β ,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\operatorname{sort}(\beta)$.

Example: $s_{201} = 0$, $s_{2-11} = -s_{200}$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$ by linearly extending

$$\mathbf{z}^\gamma \mapsto s_\gamma(X)$$

where $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

Example

$$\sigma(\mathbf{z}^{111} + \mathbf{z}^{201} + \mathbf{z}^{210} + \mathbf{z}^{3-11}) = s_{111} + s_{201} + s_{210} + s_{3-11} = s_{111} + s_{210} - s_{300}$$

Definition

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left(\frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right),$$

where $z^{\alpha_{ij}} = z_i/z_j$ and $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \dots$.

With $n = 3$,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left(\frac{z^{111} (1 - qt z_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - q z_i/z_j) (1 - t z_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i+1 < j\}$.

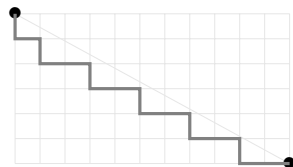
Proposition

For $(m, n) \in \mathbb{Z}^2$ coprime,

$$e_k[-MX^{m,n}] \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

for $\mathbf{b} = (b_0, \dots, b_{km-1})$ satisfying $b_i =$ the number of south steps on vertical line $x = i$ of highest lattice path under line $y + \frac{n}{m}x = n$.

δ = highest Dyck path.



δ

$$\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

Results

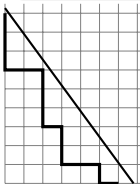
Manipulating Catalanimal \implies a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



$\text{area}(\lambda)$ as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks}$ $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

A Question

Why stop at $e_k[-MX^{m,n}]$?

For which symmetric functions f can we find a Catalanoid such that $f[-MX^{m,n}] \cdot 1 =$ a Catalanoid?

Answer: for f equal to any LLT polynomial!

Special case: $\mathcal{G}_\nu[-MX^{1,1}] \cdot 1 = \nabla \mathcal{G}_\nu(X; q)$.

LLT Catalananimals

For a tuple of skew shapes ν , the *LLT Catalananimal* $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

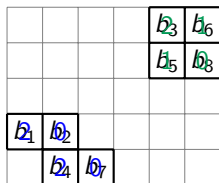
- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q$ = pairs of boxes in the same diagonal in the same shape,
- $R_q \setminus R_t$ = the attacking pairs,
- $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- λ : fill each diagonal D of ν with $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
Listing this filling in reading order gives λ .

LLT Catalanimals

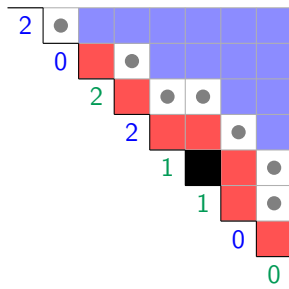
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal,
- $R_q \setminus R_t =$ the attacking pairs,
- $R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,
- $R_{qt} =$ all other pairs,

λ : fill each diagonal D of ν with

$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.



λ , as a filling of ν



Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let ν be a tuple of skew shapes and let $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega H_\nu \\ &= c_\nu \omega \sigma \left(\frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

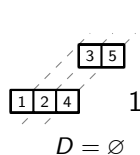
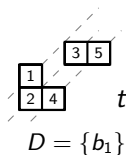
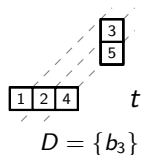
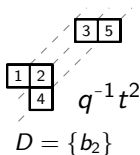
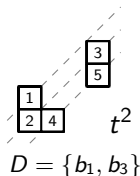
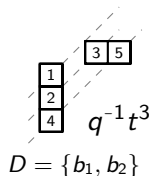
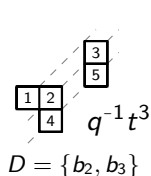
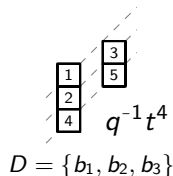
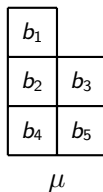
What about Macdonald polynomials?!

- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.
- We have a formula for $\nabla \mathcal{G}_\nu$.
- Does there exist formula $\tilde{H}_\mu = \sum_\nu a_{\mu\nu}(q, t) \mathcal{G}_\nu$? Yes!

- ① Background on symmetric functions and Macdonald polynomials
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ **A new formula for Macdonald polynomials**

Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X; q, t) = \sum_D \left(\prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q)$$



Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$ and apply $\omega \nabla$.
- By construction, all the LLT Catalan animals $H_{\nu(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}) .
- Collect terms to get $\prod_{(b_i, b_j) \in V(\mu)} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$ factor for $V(\mu)$ the set of vertical dominoes (b_i, b_j) in μ .

$$\tilde{H}_\mu = \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\substack{\alpha_{ij} \in V(\mu)}} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

The root ideal R_μ

b_1		
b_2	b_3	
b_4	b_5	b_6
b_7	b_8	b_9

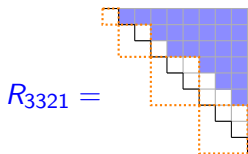
row reading order
 $b_1 \prec b_2 \prec \cdots \prec b_n$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j\},$$

$$\hat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j\},$$

$$R_\mu \setminus \hat{R}_\mu \leftrightarrow V(\mu) = \text{vertical dominoes in } \mu$$

Example:



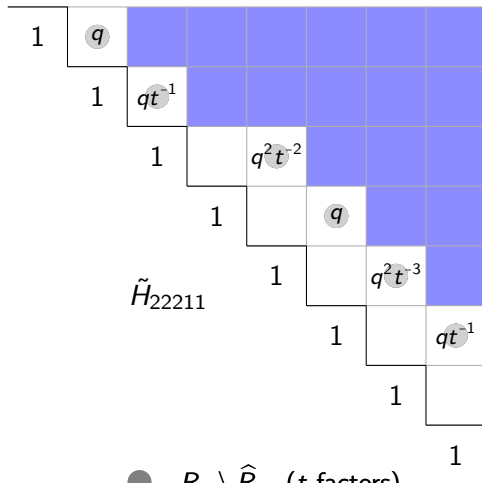
Remark

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right)$$

Example

$1 - b_1 \frac{z_1}{z_2}$	
$1 - b_2 \frac{z_2}{z_3}$	
$1 - b_3 \frac{z_3}{z_5}$	$1 - b_4 \frac{z_4}{z_6}$
$1 - b_5 \frac{z_5}{z_7}$	$1 - b_6 \frac{z_6}{z_8}$
b_7	b_8

numerator partition $\mu = 22211$ $1 = q^{22211} t^{-\text{leg}} z_i/z_j$



● $R_\mu \setminus \hat{R}_\mu$ (t factors)

■ \hat{R}_μ (t and qt factors)

$q = t = 1$ specialization

$$\begin{aligned}
 & \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=t=1} \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \hat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\
 & = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\
 & = \omega h_1^n \\
 & = e_1^n
 \end{aligned}$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega \sigma \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s , the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_\nu(X)$$

satisfy $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	HHL
∇e_n	DH_n	Shuffle theorem
$\tilde{H}_\lambda^{(s)}(X; q, t)$??	??

Thank you!

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2023/ed. *A Shuffle Theorem for Paths Under Any Line*, Forum of Mathematics, Pi **11**, e5, DOI 10.1017/fmp.2023.4.

———. 2021. *LLT Polynomials in the Schiffmann Algebra*, arXiv e-prints, arXiv:2112.07063.

———. 2023. *A Raising Operator Formula for Macdonald Polynomials*, arXiv e-prints, arXiv:2307.06517.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Mellit, Anton. 2018. *A Proof of the Shuffle Conjecture* **31**, no. 3, 661–697, DOI 10.1090/jams/893.

Feigin, B. L. and Tsymbaliuk, A. I. 2011. *Equivariant K-theory of Hilbert Schemes via Shuffle Algebra*, Kyoto J. Math. **51**, no. 4, 831–854.

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

Haglund, J., M. Haiman, and N. Loehr. 2005. *A Combinatorial Formula for Macdonald Polynomials* **18**, no. 3, 735–761 (electronic).

Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.

Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919

———. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676

Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sémin. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754

Mellit, Anton. 2021. *Toric Braids and (m,n) -Parking Functions*, Duke Math. J. **170**, no. 18, 4123–4169, DOI 10.1215/00127094-2021-0011.

Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004

Schiffmann, Olivier and Vasserot, Eric. 2013. *The Elliptic Hall Algebra and the K-theory of the Hilbert Scheme of A_2* , Duke Mathematical Journal **162**, no. 2, 279–366, DOI 10.1215/00127094-1961849.