

# $K$ -theoretic Catalan functions

George H. Seelinger (joint work with J. Blasiak and J. Morse)

Junior Mathematician Research Archive

[ghs9ae@virginia.edu](mailto:ghs9ae@virginia.edu)

Based on [arXiv:2010.01759](https://arxiv.org/abs/2010.01759)

November 2020

- ① Schubert calculus
- ② Catalan functions
- ③ K-theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}$  for  $H^*(X)$  with property  $\sigma_\lambda \smile \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}$  for  $H^*(X)$  with property  $\sigma_\lambda \smile \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$



## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

# Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

# Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .



# Classical Schubert Calculus

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}_{\lambda \subseteq (n^m)}$  for  $H^*(X)$  with property  $\sigma_\lambda \smile \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$

# Classical Schubert Calculus

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .



## Cohomology

Schubert basis  $\{\sigma_\lambda\}_{\lambda \subseteq (n^m)}$  for  $H^*(X)$  with property  $\sigma_\lambda \smile \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$



## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for combinatorially understood Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

- Complete homogeneous symmetric function: for  $r \in \mathbb{Z}$ ,  
$$h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}.$$

# Schur polynomials and raising operators

- Complete homogeneous symmetric function: for  $r \in \mathbb{Z}$ ,  
$$h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}.$$
- For  $\lambda \in \mathbb{Z}^\ell$ ,  $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ .

# Schur polynomials and raising operators

- Complete homogeneous symmetric function: for  $r \in \mathbb{Z}$ ,  
$$h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}.$$
- For  $\lambda \in \mathbb{Z}^\ell$ ,  $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ .
- Raising operators  $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

$$R_{1,3} \left( \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \blacksquare & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \blacksquare \\ \hline \square & & & \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \blacksquare \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \\ \hline \square & \blacksquare \\ \hline \end{array}$$

# Schur polynomials and raising operators

- Complete homogeneous symmetric function: for  $r \in \mathbb{Z}$ ,  
$$h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}.$$
- For  $\lambda \in \mathbb{Z}^\ell$ ,  $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ .
- Raising operators  $R_{i,j}(h_\lambda) = h_{\lambda + \epsilon_i - \epsilon_j}$

$$R_{1,3} \left( \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \blacksquare \\ \hline \square & & & \\ \hline & & & \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \blacksquare \\ \hline \end{array}$$

- Schur function  $s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$  (Jacobi-Trudi)

# Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

# Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	$f_\lambda$
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
$K$ -homology of affine Grassmannian	$K$ - $k$ -Schur functions



# Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	$f_\lambda$
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
$K$ -homology of affine Grassmannian	$K$ - $k$ -Schur functions

## Focus

$K$ -theory and  $K$ -homology of the affine Grassmannian

# Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	$f_\lambda$
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
$K$ -homology of affine Grassmannian	$K$ - $k$ -Schur functions

## Focus

$K$ -theory and  $K$ -homology of the affine Grassmannian

Simultaneously generalizes  $K$ -theory of Grassmannian and (co)homology of affine Grassmannian.

What is known?

## What is known?

- ①  $K$ -theory classes of Grassmannian (not affine!) represented by “Grothendieck polynomials.” We are interested in their dual:

$$g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$$

for  $k_\gamma$  an inhomogeneous analogue of  $h_\gamma$ .

## What is known?

- ①  $K$ -theory classes of Grassmannian (not affine!) represented by “Grothendieck polynomials.” We are interested in their dual:

$$g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$$

for  $k_\gamma$  an inhomogeneous analogue of  $h_\gamma$ .

- ② Homology classes of affine Grassmannian represented by  $k$ -Schur functions ( $t = 1$ ).

# $K$ -Theory of Affine Grassmannian

## What is known?

- ①  $K$ -theory classes of Grassmannian (not affine!) represented by “Grothendieck polynomials.” We are interested in their dual:

$$g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$$

for  $k_\gamma$  an inhomogeneous analogue of  $h_\gamma$ .

- ② Homology classes of affine Grassmannian represented by  $k$ -Schur functions ( $t = 1$ ).
- ③ (Lam et al., 2010) leave open the question: what is a direct formulation of the  $K$ -homology representatives of the affine Grassmannian ( $K$ - $k$ -Schur functions)?

# Remember?

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).

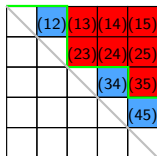


$\Psi =$  Roots above Dyck path  
 $\Delta_{\ell}^+ \setminus \Psi =$  Non-roots below



# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi$  = Roots above Dyck path  
 $\Delta_{\ell}^{+} \setminus \Psi$  = Non-roots below

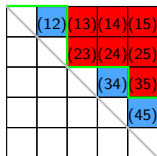
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi$  = Roots above Dyck path  
 $\Delta_{\ell}^{+} \setminus \Psi$  = Non-roots below

## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

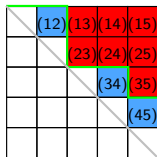
For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

- $\Psi = \emptyset \implies H(\emptyset; \gamma) = s_{\gamma}$

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi$  = Roots above Dyck path  
 $\Delta_{\ell}^{+} \setminus \Psi$  = Non-roots below

## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

- $\Psi = \emptyset \implies H(\emptyset; \gamma) = s_{\gamma}$
- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_{\gamma}$

## $k$ -Schur root ideal for $\lambda$

For  $k \in \mathbb{Z}_{\geq 0}$  and  $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell) \in \mathbb{Z}^\ell$ ,

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

# Catalan functions

## $k$ -Schur root ideal for $\lambda$

For  $k \in \mathbb{Z}_{\geq 0}$  and  $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell) \in \mathbb{Z}^\ell$ ,

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots

# Catalan functions

## $k$ -Schur root ideal for $\lambda$

For  $k \in \mathbb{Z}_{\geq 0}$  and  $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell) \in \mathbb{Z}^\ell$ ,

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots

$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

By realizing  $k$ -Schur functions in the framework of Catalan functions, (Blasiak et al., 2019) proves

By realizing  $k$ -Schur functions in the framework of Catalan functions, (Blasiak et al., 2019) proves

- The  $k$ -Schur functions are “shift invariant”, i.e. for  $\ell = \ell(\lambda)$ ,  
$$s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}.$$



By realizing  $k$ -Schur functions in the framework of Catalan functions, (Blasiak et al., 2019) proves

- The  $k$ -Schur functions are “shift invariant”, i.e. for  $\ell = \ell(\lambda)$ ,  
 $s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$ .
- This implies the  $k+1$ -Schur expansion of a  $k$ -Schur function has positive coefficients.

By realizing  $k$ -Schur functions in the framework of Catalan functions, (Blasiak et al., 2019) proves

- The  $k$ -Schur functions are “shift invariant”, i.e. for  $\ell = \ell(\lambda)$ ,  
 $s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)}$ .
- This implies the  $k+1$ -Schur expansion of a  $k$ -Schur function has positive coefficients.
- Also the Schur expansion of a  $k$ -Schur function has positive coefficients.

## Remark

(Blasiak et al., 2019) show results for  $k$ -Schur functions with parameter  $t$ , but  $t = 1$  specialization is necessary for Schubert calculus.

- Recall  $K$ -theory/homology of affine Grassmannian simultaneously generalizes:
  - $K$ -theory of Grassmannian:  $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$  and

# Lowering Operators

- Recall  $K$ -theory/homology of affine Grassmannian simultaneously generalizes:
  - $K$ -theory of Grassmannian:  $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$  and
  - Homology of affine Grassmannian:  $s_\lambda^{(k)} = \prod_{(i,j) \in \Delta^+ \setminus \Delta^k(\lambda)} (1 - R_{ij}) h_\lambda$

# Lowering Operators

- Recall  $K$ -theory/homology of affine Grassmannian simultaneously generalizes:
  - $K$ -theory of Grassmannian:  $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$  and
  - Homology of affine Grassmannian:  $s_\lambda^{(k)} = \prod_{(i,j) \in \Delta^+ \setminus \Delta^k(\lambda)} (1 - R_{ij}) h_\lambda$
- Extra ingredient: lowering operators  $L_j(h_\lambda) = h_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array} \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \color{red}\square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \square & \\ \hline \end{array}$$

## Definition

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$

## Definition

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

for  $k_\gamma$  an inhomogeneous analogue of  $h_\gamma$ .

# Affine $K$ -Theory Representatives with Raising Operators

## Definition

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

for  $k_\gamma$  an inhomogeneous analogue of  $h_\gamma$ .

## Example

non-roots of  $\Psi$  in blue, roots of  $\mathcal{L}$  marked with •

	(12)		•	•
			•	•
		(34)		
			(45)	

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) \\ &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332} \end{aligned}$$



# Affine $K$ -Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

## Answer (Blasiak-Morse-S., 2020)

For  $K$ -homology of affine Grassmannian,  
 $f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$  since this family has the correct structure constants.

# Affine $K$ -Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

For  $K$ -homology of affine Grassmannian,

$f_\lambda = g_\lambda^{(k)} := K(\Delta^{(k)}(\lambda); \Delta^{(k+1)}(\lambda); \lambda)$  since this family has the correct structure constants.

Example

$$g_{332111}^{(4)} =$$

3				•	•	•
	3				•	•
			2			
				1		
					1	
						1

$$\Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$$

## Theorem (Blasiak-Morse-S., 2020)

## Theorem (Blasiak-Morse-S., 2020)

The  $g_{\lambda}^{(k)}$  are “shift invariant”, i.e. for  $\ell = \ell(\lambda)$

$$G_{1^{\ell}}^{\perp} g_{\lambda+1^{\ell}}^{(k+1)} = g_{\lambda}^{(k)}$$

# Property and Further Work

## Theorem (Blasiak-Morse-S., 2020)

The  $g_{\lambda}^{(k)}$  are “shift invariant”, i.e. for  $\ell = \ell(\lambda)$

$$G_{1^{\ell}}^{\perp} g_{\lambda+1^{\ell}}^{(k+1)} = g_{\lambda}^{(k)}$$

## Theorem (Blasiak-Morse-S., 2020)

The  $g_{\lambda}^{(k)}$  “branching coefficients” are alternating by degree, i.e. the  $b_{\lambda\mu}^{(k)}$  in

$$g_{\lambda}^{(k)} = \sum_{\mu} b_{\lambda\mu}^{(k)} g_{\mu}^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$ .

For  $G_{\lambda}^{(k)}$  an affine Grothendieck polynomial (dual to  $g_{\lambda}^{(k)}$ ),

For  $G_\lambda^{(k)}$  an affine Grothendieck polynomial (dual to  $g_\lambda^{(k)}$ ),

- 1 Combinatorially describe dual “Pieri rule”:

$$G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}, \quad 1 \leq r \leq k.$$



For  $G_\lambda^{(k)}$  an affine Grothendieck polynomial (dual to  $g_\lambda^{(k)}$ ),

- 1 Combinatorially describe dual “Pieri rule”:

$$G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}, \quad 1 \leq r \leq k.$$

- 2 Combinatorially describe branching coefficients:  $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$ .

For  $G_\lambda^{(k)}$  an affine Grothendieck polynomial (dual to  $g_\lambda^{(k)}$ ),

- 1 Combinatorially describe dual “Pieri rule”:

$$G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}, \quad 1 \leq r \leq k.$$

- 2 Combinatorially describe branching coefficients:  $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$ .
- 3 Combinatorially describe  $g_\lambda^{(k)} = \sum_\mu ?? s_\mu^{(k)}$ .

Thank you!

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and  $k$ -Schur Positivity*, Journal of the AMS.

Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for  $k$ -Schur functions*, Ph.D. thesis.

Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010.  *$K$ -theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.

Morse, Jennifer. 2011. *Combinatorics of the  $K$ -theory of affine Grassmannians*, Advances in Mathematics.

Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.