K-theoretic Catalan functions

George H. Seelinger (joint with J. Blasiak and J. Morse)

ghs9ae@virginia.edu

`arXiv:2010.01759`

LACIM at UQAM

19 March 2021

Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

$$\downarrow$$

Cohomology

Schubert basis $\{\sigma_\lambda\}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

$$\downarrow$$

Cohomology

Schubert basis $\{\sigma_\lambda\}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$

$$\downarrow$$

Representatives

Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^{\nu} f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Classical Schubert Calculus

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda \subseteq (n^m)}$ in variety $X = \mathrm{Gr}(m, n)$.

Classical Schubert Calculus

Geometric problem

Find $c^{\nu}_{\lambda\mu} = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda\subseteq(n^m)}$ in variety $X = \mathrm{Gr}(m, n)$.

$$\downarrow$$

Cohomology

Schubert basis $\{\sigma_\lambda\}_{\lambda\subseteq(n^m)}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^{\nu}_{\lambda\mu}\sigma_\nu$

Classical Schubert Calculus

Geometric problem

Find $c^{\nu}_{\lambda\mu} = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda\subseteq(n^m)}$ in variety $X = \mathrm{Gr}(m,n)$.

$$\downarrow$$

Cohomology

Schubert basis $\{\sigma_\lambda\}_{\lambda\subseteq(n^m)}$ for $H^*(X)$ with property $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^{\nu}_{\lambda\mu}\sigma_\nu$

$$\downarrow$$

Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu$ for Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$.

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$
\begin{array}{|c|cccc}
\cline{1-1}
5 \\
\cline{1-2}
3 & \multicolumn{1}{c|}{4} \\
\cline{1-2}
2 & \multicolumn{1}{c|}{3} \\
\hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\
\hline
\end{array}
\qquad\qquad
\underbrace{\begin{array}{|c|cccc}
\cline{1-1}
8 \\
\cline{1-2}
7 & \multicolumn{1}{c|}{9} \\
\cline{1-2}
3 & \multicolumn{1}{c|}{4} \\
\hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\
\hline
\end{array}}_{\text{standard} = \text{no repeated letters}}
$$

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$
\begin{array}{|c|c|cc}
\hline
5 & \multicolumn{3}{c}{} \\
\hline
3 & 4 & \multicolumn{2}{c}{} \\
\hline
2 & 3 & \multicolumn{2}{c}{} \\
\hline
1 & 2 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\
\hline
\end{array}
\qquad\qquad
\underbrace{\begin{array}{|c|c|cc}
\hline
8 & \multicolumn{3}{c}{} \\
\hline
7 & 9 & \multicolumn{2}{c}{} \\
\hline
3 & 4 & \multicolumn{2}{c}{} \\
\hline
1 & 2 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\
\hline
\end{array}}_{\text{standard} \,=\, \text{no repeated letters}}
$$

Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$
\begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array},\;
\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array},\;
\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array},\;
\begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array},\;
\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array},\;
\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array},\;
\begin{array}{|c|c}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array},\;
\begin{array}{|c|c}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}
$$

$$
s_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3
$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\ydiagram{1}} s_{\ydiagram{1,2}} = s_{\ydiagram{1,3}} + s_{\ydiagram{2,2}} + s_{\ydiagram{1,1,2}}$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\ydiagram{1}} s_{\ydiagram{2,1}} = s_{\ydiagram{3,1}} + s_{\ydiagram{2,2}} + s_{\ydiagram{2,1,1}}$$

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\#\text{ known tableaux}) s_\nu$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{(1)} s_{(2,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$$

Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c^\nu_{\lambda\mu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu}$.

Next Step: Flag Variety

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by $w \in S_n$.

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by $w \in S_n$.
- $H^*(Fl_n(\mathbb{C}))$ supported by Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$ (Not necessarily symmetric!)

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$
- Decomposes into Schubert varieties indexed by $w \in S_n$.
- $H^*(Fl_n(\mathbb{C}))$ supported by Schubert polynomials $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_n]$ (Not necessarily symmetric!)

$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$ have no tableaux description.

Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	f_λ
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomimals
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
K-theory of Grassmannian	Grothendieck polynomials
K-homology of affine Grassmannian	K-k-Schur functions

Schubert Calculus Variations

There are many variations on classical Schubert calculus of the Grassmannian (Type A).

Theory	f_λ
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomimals
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
K-theory of Grassmannian	Grothendieck polynomials
K-homology of affine Grassmannian	K-k-Schur functions

And many more!

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ by $q_1, \ldots, q_k$.

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ by $q_1, \ldots, q_k$.
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$ (Fomin et al., 1997).

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ by $q_1, \ldots, q_k$.
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$ (Fomin et al., 1997). ($\mathfrak{S}_w^Q \to \mathfrak{S}_w$ when $q_i = 0$.)

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ by $q_1, \ldots, q_k$.
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$ (Fomin et al., 1997). ($\mathfrak{S}_w^Q \to \mathfrak{S}_w$ when $q_i = 0$.)
- Peterson isomorphism

$$\Phi\colon QH^*(Fl_{k+1}) \quad \to H_*(Gr_{SL_{k+1}})_{loc}$$

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ by $q_1, \ldots, q_k$.
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$ (Fomin et al., 1997). ($\mathfrak{S}_w^Q \to \mathfrak{S}_w$ when $q_i = 0$.)
- Peterson isomorphism

$$\begin{aligned} \Phi\colon QH^*(Fl_{k+1}) &\to H_*(Gr_{SL_{k+1}})_{loc} \\ \mathfrak{S}_w^Q &\mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i} \end{aligned}$$

where $s_\lambda^{(k)}$ is a k-Schur symmetric function and $\mathrm{Gr}_{SL_{k+1}}$ is the "affine Grassmannian."

Upshot

Peterson Isomorphism

- $QH^*(Fl_{k+1})$ quantum deformation of $H^*(Fl_{k+1})$ by $q_1, \ldots, q_k$.
- Supported by quantum Schubert polynomials $\mathfrak{S}_w^Q$ (Fomin et al., 1997). ($\mathfrak{S}_w^Q \to \mathfrak{S}_w$ when $q_i = 0$.)
- Peterson isomorphism

$$\begin{aligned} \Phi\colon QH^*(Fl_{k+1}) &\to H_*(Gr_{SL_{k+1}})_{loc} \\ \mathfrak{S}_w^Q &\mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i} \end{aligned}$$

where $s_\lambda^{(k)}$ is a k-Schur symmetric function and $\mathrm{Gr}_{SL_{k+1}}$ is the "affine Grassmannian."

Upshot

Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Branching with positive coefficients (Lam et al., 2010):

$$s^{(2)}_{\ydiagram{2,2}} = \underbrace{s_{\ydiagram{2,2}}}_{s^{(3)}_{\ydiagram{2,2}}} + \underbrace{s_{\ydiagram{3,1}} + s_{\ydiagram{4}}}_{s^{(3)}_{\ydiagram{3,1}}}$$

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Branching with positive coefficients (Lam et al., 2010):

$$s^{(2)}_{(2,2)} = \underbrace{s_{(2,2)}}_{s^{(3)}_{(2,2)}} + \underbrace{s_{(3,1)} + s_{(4)}}_{s^{(3)}_{(3,1)}}$$

  - (Lam et al., 2010) gives geometric interpretation,

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Branching with positive coefficients (Lam et al., 2010):

$$s^{(2)}_{\ydiagram{2,2}} = \underbrace{s_{\ydiagram{2,2}}}_{s^{(3)}_{\ydiagram{2,2}}} + \underbrace{s_{\ydiagram{1,3}} + s_{\ydiagram{4}}}_{s^{(3)}_{\ydiagram{1,3}}}$$

  - (Lam et al., 2010) gives geometric interpretation,
  - but no combinatorial interpretation of branching coefficients.

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Branching with positive coefficients (Lam et al., 2010):

$$s^{(2)}_{\ydiagram{2,2}} = \underbrace{s_{\ydiagram{2,2}}}_{s^{(3)}_{\ydiagram{2,2}}} + \underbrace{s_{\ydiagram{1,3}} + s_{\ydiagram{4}}}_{s^{(3)}_{\ydiagram{1,3}}}$$

  - (Lam et al., 2010) gives geometric interpretation,
  - but no combinatorial interpretation of branching coefficients.
  - Branching with t important for Macdonald polynomial positivity.

k-Schur functions

- $s_\lambda^{(k)}$ for $\lambda_1 \leq k$ a basis for $\mathbb{Z}[s_1, s_2, \ldots, s_k]$ (Lapointe et al., 2003).
- Schubert representatives for $H_*(Gr_{SL_{k+1}})$ (Lam, 2008).
- Has a tableaux formulation and Pieri rule: $s_{1^r} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$
- $s_\lambda^{(k)} = s_\lambda$ as $k \to \infty$.
- Branching with positive coefficients (Lam et al., 2010):

$$s^{(2)}_{(2,2)} = \underbrace{s_{(2,2)}}_{s^{(3)}_{(2,2)}} + \underbrace{s_{(3,1)} + s_{(4)}}_{s^{(3)}_{(3,1)}}$$

  - (Lam et al., 2010) gives geometric interpretation,
  - but no combinatorial interpretation of branching coefficients.
  - Branching with t important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

Overview

- Schubert calculus
- **Catalan functions: a new approach to old problems**
- K-theoretic Catalan functions

Why a new definition of k-Schur?

Why a new definition of k-Schur?

Answer

1. (Blasiak et al., 2019) gives a new definition of $s_\lambda^{(k)}$ and shows it is equivalent to many other previous definitions.

Why a new definition of k-Schur?

Answer

1. (Blasiak et al., 2019) gives a new definition of $s_\lambda^{(k)}$ and shows it is equivalent to many other previous definitions.
2. From a new definition, (Blasiak et al., 2019) shows the branching coefficients $b_{\lambda\mu}$ in the expansion $s_\lambda^{(k)} = \sum_\mu b_{\lambda\mu} s_\mu^{(k+1)}$ have combinatorial interpretation!

Key:

Why a new definition of k-Schur?

Answer

1. (Blasiak et al., 2019) gives a new definition of $s_\lambda^{(k)}$ and shows it is equivalent to many other previous definitions.
2. From a new definition, (Blasiak et al., 2019) shows the branching coefficients $b_{\lambda\mu}$ in the expansion $s_\lambda^{(k)} = \sum_\mu b_{\lambda\mu} s_\mu^{(k+1)}$ have combinatorial interpretation!

Key: Catalan functions = large class of symmetric functions.

Ingredients for Catalan functions

- Raising operators

Ingredients for Catalan functions

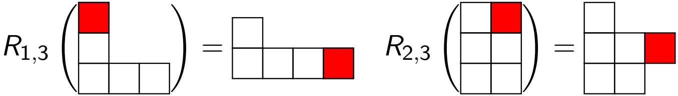
- Raising operators
- Symmetric functions indexed by integer vectors

Ingredients for Catalan functions

- Raising operators
- Symmetric functions indexed by integer vectors
- Root ideals

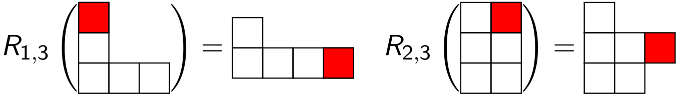
Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams



Raising Operators on Symmetric Functions

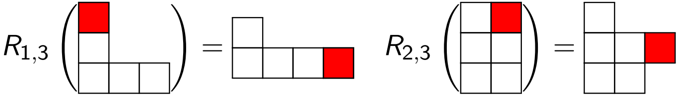
- Raising operators $R_{i,j}$ act on diagrams



- Extend action to a symmetric function f_λ by $R_{i,j}(f_\lambda) = f_{\lambda+\epsilon_i-\epsilon_j}$.

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

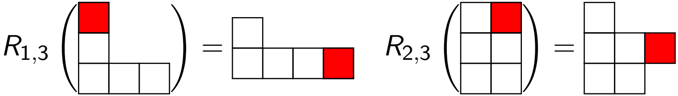


- Extend action to a symmetric function f_λ by $R_{i,j}(f_\lambda) = f_{\lambda+\epsilon_i-\epsilon_j}$.
- For $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$, we have the *Jacobi-Trudi identity*

$$s_\lambda = \prod_{i<j} (1 - R_{ij}) h_\lambda$$

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams



- Extend action to a symmetric function f_λ by $R_{i,j}(f_\lambda) = f_{\lambda+\epsilon_i-\epsilon_j}$.
- For $h_\lambda = s_{\lambda_1} \cdots s_{\lambda_r}$, we have the *Jacobi-Trudi identity*

$$s_\lambda = \prod_{i<j} (1 - R_{ij}) h_\lambda$$

$$\begin{aligned} s_{22} &= (1 - R_{12}) h_{22} = h_{22} - h_{31} \\ s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211} \\ &= h_{211} - h_{301} - h_{220} - h_{310} + h_{310} + \underbrace{h_{32\,\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\,\text{-}1}}_{=0} \end{aligned}$$

some terms cancel

Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$.

Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$. Straightening:

$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$. Straightening:

$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

Simplifies formulas. E.g., for $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda =$$

Raising Operators on Symmetric Functions

Upside: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$. Straightening:

$$s_\alpha = \prod_{i<j}(1 - R_{ij})h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

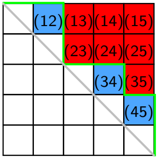
Simplifies formulas. E.g., for $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

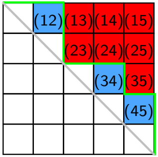
A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



Ψ = Roots above Dyck path
$\Delta_\ell^+ \backslash \Psi$ = Non-roots below

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



Ψ = Roots above Dyck path
$\Delta_\ell^+ \setminus \Psi$ = Non-roots below

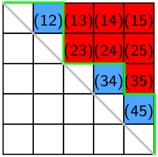
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) h_\gamma(x)$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



Ψ = Roots above Dyck path

$\Delta_\ell^+ \backslash \Psi$ = Non-roots below

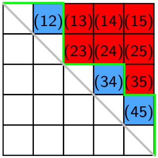
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+\backslash\Psi} (1 - R_{ij}) h_\gamma(x)$$

- $\Psi = \varnothing \Longrightarrow H(\varnothing;\gamma) = s_\gamma$

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).



Ψ = Roots above Dyck path

$\Delta_\ell^+ \setminus \Psi =$ Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) h_\gamma(x)$$

- $\Psi = \varnothing \implies H(\varnothing;\gamma) = s_\gamma$
- $\Psi = \text{all roots} \implies H(\Psi;\gamma) = h_\gamma$

Catalan functions

Intuition

Catalan functions interpolate between h_λ and s_λ.

Catalan functions

Intuition

Catalan functions interpolate between h_λ and s_λ.

Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive!

Catalan functions

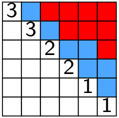
k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

Catalan functions

k-Schur root ideal for λ

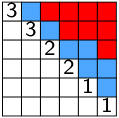
$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

$\Delta^4(3,3,2,2,1,1) =$  $\leftarrow$ row i has $4 - \lambda_i$ non-roots

Catalan functions

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
$$= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i$$

$\Delta^4(3,3,2,2,1,1) =$ 

$\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)\,.$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = s_{\lambda}^{(k)} .$$

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = s_\lambda^{(k)} .$$

Proof: $k - \lambda_i = (k+1) - (\lambda_i + 1)$

Key ingredient of branching proof

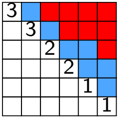
Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

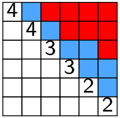
Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = s_{\lambda}^{(k)} .$$

Proof: $k - \lambda_i = (k+1) - (\lambda_i + 1)$

$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Key ingredient of branching proof

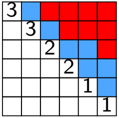
Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

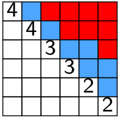
Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = s_{\lambda}^{(k)}.$$

Proof: $k - \lambda_i = (k+1) - (\lambda_i + 1)$

$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Pieri:

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

Key ingredient of branching proof

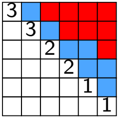
Dual vertical Pieri rule: $s_{1^r}^{\perp} s_{\lambda}^{(k)} = \sum_{\mu} a_{\lambda\mu} s_{\mu}^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

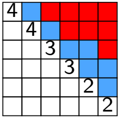
Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

$$s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = s_{\lambda}^{(k)} .$$

Proof: $k - \lambda_i = (k+1) - (\lambda_i + 1)$

$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

$$s_{\lambda}^{(k)} = s_{1^\ell}^{\perp} s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s_{\mu}^{(k+1)}$$

Overview

- Schubert calculus
- Catalan functions: a new approach to old problems
- **K-theoretic Catalan functions**

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.

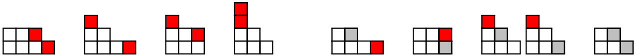
Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
- Satisfies Pieri rule on "set-valued strips"

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
- Satisfies Pieri rule on "set-valued strips"

$$g_{1^2} g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{31}$$

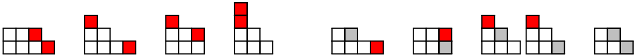


Add (addable) or mark (removable) in any combination of r boxes, but only once per row.

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
- Satisfies Pieri rule on "set-valued strips"

$$g_{1^2} g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{31}$$



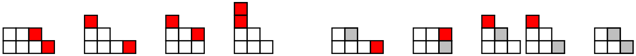
Add (addable) or mark (removable) in any combination of r boxes, but only once per row.

- $g_\lambda = \prod_{i<j}(1 - R_{ij})k_\lambda$ for k_λ and inhomogeneous analogue of h_λ.

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.
- Satisfies Pieri rule on "set-valued strips"

$$g_{1^2} g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{31}$$



Add (addable) or mark (removable) in any combination of r boxes, but only once per row.

- $g_\lambda = \prod_{i<j}(1 - R_{ij})k_\lambda$ for k_λ and inhomogeneous analogue of h_λ.
- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m,n))$

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)}$+lower degree terms

K-k-Schur functions

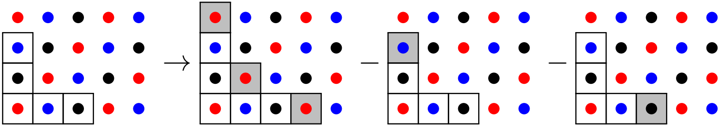
- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)}$+lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)}$+lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions $\leftrightarrow$ 3-cores

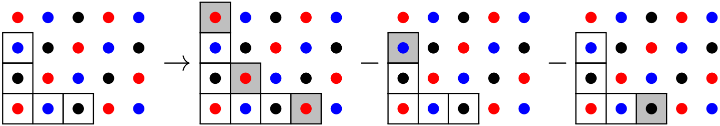


K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)}$+lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions $\leftrightarrow$ 3-cores



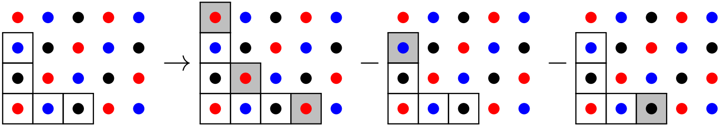
- Conjecture: $g_\lambda^{(k)}$ have positive branching into $g_\mu^{(k+1)}$ (Lam et al., 2010; Morse, 2011).

K-k-Schur functions

- Inhomogeneous basis: $g_\lambda^{(k)} = s_\lambda^{(k)}$+lower degree terms
- Satisfies Pieri rule on "affine set-valued strips"

$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions $\leftrightarrow$ 3-cores



- Conjecture: $g_\lambda^{(k)}$ have positive branching into $g_\mu^{(k+1)}$ (Lam et al., 2010; Morse, 2011).

Problem

No direct formula for $g_\lambda^{(k)}$

K-k-Schur functions

Solution

Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

K-k-Schur functions

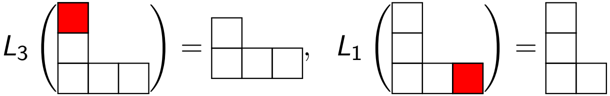
Solution

Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

Requires an inhomogeneous refinement of Catalan functions.

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi;\mathcal{L};\gamma) := \prod_{(i,j)\in\mathcal{L}} (1 - L_j) \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) k_\gamma$$

Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi;\mathcal{L};54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

Affine K-Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

Affine K-Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

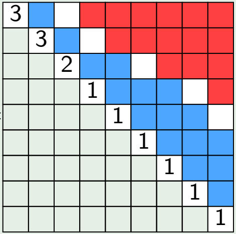
For K-homology of affine Grassmannian, $g_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$ since this family satisfies the Pieri rule.

Affine K-Theory Representatives with Raising Operators

Answer (Blasiak-Morse-S., 2020)

For K-homology of affine Grassmannian, $g_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$ since this family satisfies the Pieri rule.

Example

$g^{(4)}_{332111111} =$ 

$\Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$

Pieri Rule Illustrated (Recurrences)

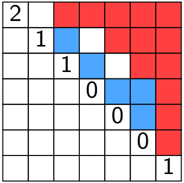
A "graphical calculus."

$$g_1 g_{211}^{(2)}$$

Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

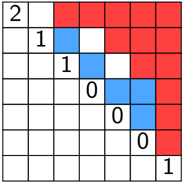
$$g_1 g^{(2)}_{211} =$$



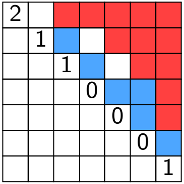
Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

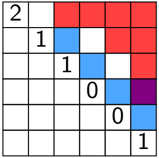
$g_1 g^{(2)}_{211}$ $=$



$=$

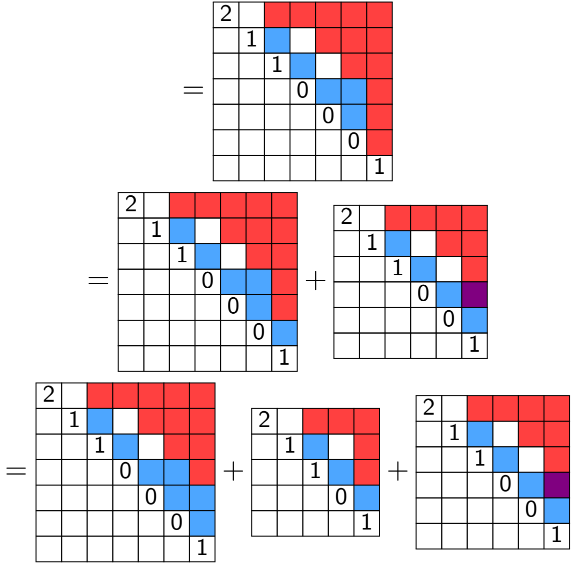


$+$

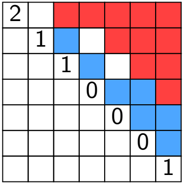


A "graphical calculus."

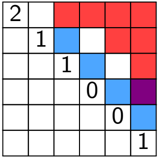
$g_1 g_{211}^{(2)}$



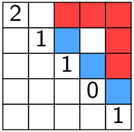
$$g_1 g_{211}^{(2)} =$$



$+$

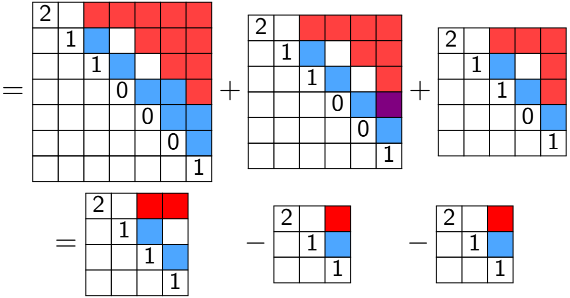


$+$



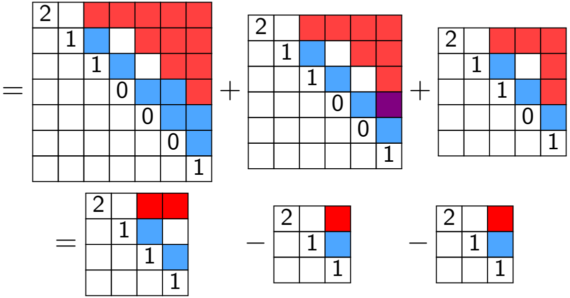
Pieri Rule Illustrated (Straightening)

$g_1 g_{211}^{(2)}$



Pieri Rule Illustrated (Straightening)

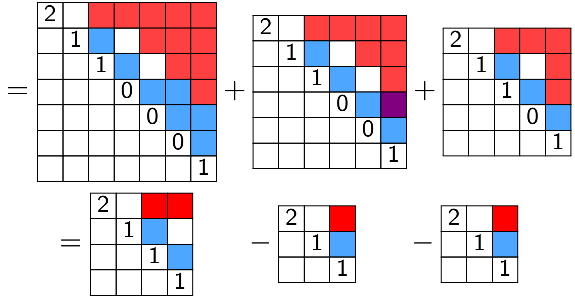
$$g_1 g^{(2)}_{211} =$$



$$= g^{(2)}_{2111} \qquad - g^{(2)}_{211} \qquad - g^{(2)}_{211}$$

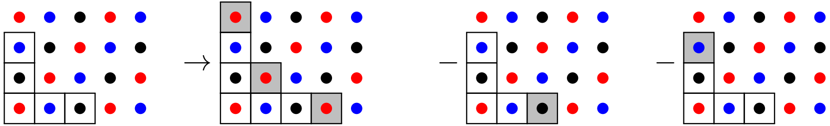
Pieri Rule Illustrated (Straightening)

$g_1 g_{211}^{(2)}$



$= g_{2111}^{(2)} \qquad - g_{211}^{(2)} \qquad - g_{211}^{(2)}$

3-core perspective:



Branching Positivity

Theorem (Blasiak-Morse-S., 2020)

Branching Positivity

Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ are "shift invariant", i.e. for $\ell = \ell(\lambda)$

$$G_{1^\ell}^\perp g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Branching Positivity

Theorem (Blasiak-Morse-S., 2020)

The $g_\lambda^{(k)}$ are "shift invariant", i.e. for $\ell = \ell(\lambda)$

$$G_{1^\ell}^\perp g_{\lambda+1^\ell}^{(k+1)} = g_\lambda^{(k)}$$

Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

K-theoretic Peterson isomorphism

$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

K-theoretic Peterson isomorphism

$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothtendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

K-theoretic Peterson isomorphism

$$\Phi \colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothtendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$ such that $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

K-theoretic Peterson isomorphism

$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothtendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$ such that $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

Theorem (Blasiak-Morse-S., 2020)

If $\lambda \subseteq (d^{k+1-d})$ for some $1 \leq d \leq k$, then $g_\lambda^{(k)} = g_\lambda$. Thus, conjecture is true for w a Grassmannian permutation (i.e. w has only one descent).

K-theoretic Peterson isomorphism

$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

Conjecture (Ikeda et al., 2018)

For $w \in S_{k+1}$ and $\mathfrak{G}_w^Q$ a "quantum Grothtendieck polynomial",

$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$ such that $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

Theorem (Blasiak-Morse-S., 2020)

If $\lambda \subseteq (d^{k+1-d})$ for some $1 \leq d \leq k$, then $g_\lambda^{(k)} = g_\lambda$. Thus, conjecture is true for w a Grassmannian permutation (i.e. w has only one descent).

Conjecture (Blasiak-Morse-S., 2020)

$$\tilde{g}_w = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Closed K-k-Schur functions

Definition (Blasiak-Morse-S., 2020)

For any partition λ with $\lambda_1 \leq k$, we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Closed K-k-Schur functions

Definition (Blasiak-Morse-S., 2020)

For any partition λ with $\lambda_1 \leq k$, we set

$$\tilde{g}_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Conjecture (Blasiak-Morse-S., 2020)

These $\tilde{g}_{\mu}^{(k)}$ satisfy the following properties.

Closed K-k-Schur functions

Definition (Blasiak-Morse-S., 2020)

For any partition λ with $\lambda_1 \leq k$, we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Conjecture (Blasiak-Morse-S., 2020)

These $\tilde{g}_\mu^{(k)}$ satisfy the following properties.

- The coefficients in $G_{1^m}^\perp \tilde{g}_\mu^{(k)} = \sum_\nu c_{\mu\nu} \tilde{g}_\nu^{(k)}$ satisfy $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.

Closed K-k-Schur functions

Definition (Blasiak-Morse-S., 2020)

For any partition λ with $\lambda_1 \leq k$, we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Conjecture (Blasiak-Morse-S., 2020)

These $\tilde{g}_\mu^{(k)}$ satisfy the following properties.

- The coefficients in $G_{1^m}^{\perp} \tilde{g}_\mu^{(k)} = \sum_\nu c_{\mu\nu} \tilde{g}_\nu^{(k)}$ satisfy $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.
- The coefficients in $\tilde{g}_\mu^{(k)} = \sum_\nu a_{\mu\nu} \tilde{g}_\nu^{(k+1)}$ satisfy $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.

Definition (Blasiak-Morse-S., 2020)

For any partition λ with $\lambda_1 \leq k$, we set

$$\tilde{g}_\lambda^{(k)} = K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$

Conjecture (Blasiak-Morse-S., 2020)

These $\tilde{g}_\mu^{(k)}$ satisfy the following properties.

- The coefficients in $G_{1^m}^{\perp} \tilde{g}_\mu^{(k)} = \sum_\nu c_{\mu\nu} \tilde{g}_\nu^{(k)}$ satisfy $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.
- The coefficients in $\tilde{g}_\mu^{(k)} = \sum_\nu a_{\mu\nu} \tilde{g}_\nu^{(k+1)}$ satisfy $(-1)^{|\mu|-|\nu|} a_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.
- The coefficients in $\tilde{g}_\mu^{(k)} = \sum_\nu b_{\mu\nu} g_\nu^{(k)}$ satisfy $(-1)^{|\mu|-|\nu|} b_{\mu\nu} \in \mathbb{Z}_{\geq 0}$.

k-Rectangle Property

Theorem (S. (thesis), 2021)

For $1 \leq d \leq k$, *set* $R_d = ((k+1-d)^d)$ *to be the* k*-rectangle partition.*

k-Rectangle Property

Theorem (S. (thesis), 2021)

For $1 \leq d \leq k$, set $R_d = ((k+1-d)^d)$ to be the k-rectangle partition. Then,

$$\tilde{g}^{(k)}_{R_d} \tilde{g}^{(k)}_{\mu} = \tilde{g}^{(k)}_{\mu \cup R_d},$$

where $\mu \cup R_d$ is the partition given by sorting (μ, R_d).

k-Rectangle Property

Theorem (S. (thesis), 2021)

For $1 \leq d \leq k$, set $R_d = ((k+1-d)^d)$ to be the k-rectangle partition. Then,

$$\tilde{g}_{R_d}^{(k)} \tilde{g}_{\mu}^{(k)} = \tilde{g}_{\mu \cup R_d}^{(k)},$$

where $\mu \cup R_d$ is the partition given by sorting (μ, R_d).

- Corresponding result for $s_\lambda^{(k)}$ is known, but this gives a Catalan/Katala-theoretic proof.

k-Rectangle Property

Theorem (S. (thesis), 2021)

For $1 \leq d \leq k$, set $R_d = ((k+1-d)^d)$ to be the k-rectangle partition. Then,

$$\tilde{g}^{(k)}_{R_d} \tilde{g}^{(k)}_{\mu} = \tilde{g}^{(k)}_{\mu \cup R_d},$$

where $\mu \cup R_d$ is the partition given by sorting (μ, R_d).

- Corresponding result for $s^{(k)}_\lambda$ is known, but this gives a Catalan/Katala-theoretic proof.
- k-Rectangle Property fails for $g^{(k)}_\lambda$.

Positivity of Katalan functions

Conjecture (Blasiak-Morse-S., 2020)

For Ψ a root ideal and λ a partition,

Positivity of Katalan functions

Conjecture (Blasiak-Morse-S., 2020)

For Ψ a root ideal and λ a partition,

- $K(\Psi; \Psi; \lambda) = \sum_\mu a_\mu g_\mu$ satisfies $(-1)^{|\lambda|-|\mu|} a_\mu \in \mathbb{Z}_{\geq 0}$.

Positivity of Katalan functions

Conjecture (Blasiak-Morse-S., 2020)

For Ψ a root ideal and λ a partition,

- $K(\Psi;\Psi;\lambda) = \sum_\mu a_\mu g_\mu$ satisfies $(-1)^{|\lambda|-|\mu|} a_\mu \in \mathbb{Z}_{\geq 0}$.
- $K(\Psi; RC^a(\Psi); \lambda) = \sum_\mu b_\mu s_\mu$ satisfies $b_\mu \in \mathbb{Z}_{\geq 0}$.

Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule:
   $G_{1^r}^{\perp} g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ??G_\lambda^{(k)}$, $1 \leq r \leq k$.

Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule: $G_{1^r}^\perp g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ?? G_\lambda^{(k)}$, $1 \leq r \leq k$.
2. Combinatorially describe branching coefficients: $g_\lambda^{(k)} = \sum_\mu ?? g_\mu^{(k+1)}$.

Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual Pieri rule:
   $G_{1^r}^{\perp} g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k)} \iff G_{1^r} G_\mu^{(k)} = \sum_\lambda ??G_\lambda^{(k)}$, $1 \leq r \leq k$.
2. Combinatorially describe branching coefficients: $g_\lambda^{(k)} = \sum_\mu ??g_\mu^{(k+1)}$.
3. Combinatorially describe $g_\lambda^{(k)} = \sum_\mu ??s_\mu^{(k)}$.

References

Thank you!

Anderson, David, Linda Chen, and Hsian-Hua Tseng. 2017. *On the quantum K-ring of the flag manifold*, preprint. arXiv: 1711.08414.

Blasiak, Jonah, Jennifer Morse, Anna Pun, and Daniel Summers. 2019. *Catalan Functions and k-Schur Positivity*, J. Amer. Math. Soc. **32**, no. 4, 921–963.

Blasiak, Jonah, Jennifer Morse, and Anna Pun. 2020. *Demazure crystals and the Schur positivity of Catalan functions*, preprint. arXiv: 2007.04952.

Blasiak, Jonah, Jennifer Morse, and George H. Seelinger. 2020. *K-theoretic Catalan functions*, preprint. arXiv: 2010.01759.

Chen, Li-Chung. 2010. *Skew-linked partitions and a representation theoretic model for k-Schur functions*, Ph.D. thesis.

Fomin, Sergey, Sergei Gelfand, and Alexander Postnikov. 1997. *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10**, no. 3, 565–596, DOI 10.1090/S0894-0347-97-00237-3. MR1431829

Ikeda, Takeshi, Shinsuke Iwao, and Toshiaki Maeno. 2018. *Peterson Isomorphism in K-theory and Relativistic Toda Lattice*, preprint. arXiv: 1703.08664.

Lam, Thomas. 2008. *Schubert polynomials for the affine Grassmannian*, J. Amer. Math. Soc. **21**, no. 1, 259–281.

Lam, Thomas, Luc Lapointe, Jennifer Morse, and Mark Shimozono. 2010. *Affine insertion and Pieri rules for the affine Grassmannian*, Mem. Amer. Math. Soc. **208**, no. 977.

Lam, Thomas, Anne Schilling, and Mark Shimozono. 2010. *K-theory Schubert calculus of the affine Grassmannian*, Compositio Math. **146**, 811–852.

Lapointe, Luc, Alain Lascoux, and Jennifer Morse. 2003. *Tableau atoms and a new Macdonald positivity conjecture*, Duke Mathematical Journal **116**, no. 1, 103–146.

Morse, Jennifer. 2011. *Combinatorics of the K-theory of affine Grassmannians*, Advances in Mathematics.

Panyushev, Dmitri I. 2010. *Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles*, Selecta Math. (N.S.) **16**, no. 2, 315–342.