

# SCHUR $Q$ -FUNCTIONS

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## 1. INTRODUCTION

These notes are a companion for [Mac79, III, Sections 2,4,8]. All results and proofs are from [Mac79], **usually verbatim or very close** with some extra detail added for my own sake.

Recall the Hall-Littlewood functions given by

$$P_\lambda(x_1, \dots, x_n; t) = \frac{1}{V_\lambda(t)} \sum_{w \in \mathfrak{S}_n} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right)$$

or alternatively

$$P_\lambda(x_1, \dots, x_n; t) = \sum_{w \in \mathfrak{S}_n / \mathfrak{S}_n^\lambda} w \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{\lambda_i > \lambda_j} \frac{x_i - tx_j}{x_i - x_j} \right)$$

One typically learns some remarkable specializations, namely

$$P_\lambda(x; 0) = s_\lambda(x) \text{ and } P(x; 1) = m_\lambda(x)$$

However, there exist other “variants” of the Hall-Littlewood functions, and we will explore one of these variants in this section.

**1.1. Definition.** For  $r \geq 1$ , we define

$$q_r(x; t) = (1 - t)P_{(r)}(x; t)$$

and set  $q_0(x; t) = 1$ .

**1.2. Remark.** Notice that when  $t = 0$ , we have

$$q_r(x; 0) = P_{(r)}(x; 0) = s_{(r)}(x) = h_r(x)$$

**1.3. Proposition.** *In  $n$  variables,*

$$q_r(x_1, \dots, x_n; t) = (1 - t) \sum_{i=1}^n x_i^r \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}$$

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*Proof.* We observe that  $\mathfrak{S}_n^{(r)}$  is all permutations that fix 1, giving us  $\mathfrak{S}_n/\mathfrak{S}_n^{(r)} \cong \{(1, j)\mathfrak{S}_n^{(r)} | 1 \leq j \leq n\}$ . Let us say  $\tau_{1,j} = (1, j)$ . From the definition,

$$q_r(x_1, \dots, x_n; t) = (1-t) \sum_{w \in \mathfrak{S}_n/\mathfrak{S}_n^{(r)}} w \left( x_1^r \prod_{\lambda_i > \lambda_j} \frac{x_i - tx_j}{x_i - x_j} \right) = (1-t) \sum_{w \in \mathfrak{S}_n/\mathfrak{S}_n^{(r)}} w \left( x_1^r \prod_{j \neq 1} \frac{x_1 - tx_j}{x_1 - x_j} \right)$$

Next, we break up the sum based on where the permutation sends 1.

$$q_r(x_1, \dots, x_n; t) = (1-t) \sum_{i=1}^n x_i^r \tau_{1,i} \left( \prod_{j \neq 1} \frac{x_1 - tx_j}{x_1 - x_j} \right)$$

which then yields the result.  $\square$

**1.4. Example.** We compute using 2 variables

$$\begin{cases} q_1(x_1, x_2) = (1-t) \left( x_1 \left( \frac{x_1 - tx_2}{x_1 - x_2} \right) + x_2 \left( \frac{x_2 - tx_1}{x_2 - x_1} \right) \right) = (1-t) \frac{x_1^2 - tx_1 x_2 - x_2^2 + tx_1 x_2}{x_1 - x_2} \\ = (1-t)(x_1 + x_2) = (1-t)m_1 \\ q_2(x_1, x_2) = (1-t) \frac{x_1^3 - tx_1^2 x_2 - x_2^3 + tx_1 x_2^2}{x_1 - x_2} = \frac{1-t}{x_1 - x_2} (x_1^3 - x_2^3 + t(x_1 x_2^2 - x_1^2 + x_2)) = \\ = (1-t)(x_1^2 + x_1 x_2 + x_2^2 - t(x_1 x_2)) = (1-t)(m_2 + (1-t)m_{11}) \end{cases}$$

Naturally, such computations are not much different than computing Hall-Littlewood polynomials.

**1.5. Proposition.** *The generating function for the  $q_r$  is given by*

$$\sum_{r=0}^{\infty} q_r(x; t) u^r = \prod_i \frac{1 - x_i tu}{1 - x_i u} = \frac{H(u)}{H(tu)}$$

*Proof.* When using a finite number of variables,

$$\begin{aligned} \sum_{r=1}^{\infty} q_r(x; t) u^r &= \sum_{r=1}^{\infty} (1-t) \sum_{i=1}^n x_i^r u^r \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \\ &= (1-t) \sum_{i=1}^n \left( \sum_{r=1}^{\infty} x_i^r u^r \right) \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \\ &= (1-t) \sum_{i=1}^n \frac{x_i u}{1 - x_i u} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} \end{aligned}$$

Then, using the Heaviside cover-up method of partial sum decomposition, we have

$$\prod_{i=1}^n \frac{z - tx_i}{z - x_i} = 1 + \frac{\prod_{i=1}^n (z - tx_i) - \prod_{i=1}^n (z - x_i)}{\prod_{i=1}^n z - x_i} = 1 + \sum_{i=1}^n \frac{A_i}{z - x_i}$$

where

$$\begin{aligned}
A_i &= \text{Res}_{z=x_i} \left( \frac{\prod_{j=1}^n (z - tx_j) - \prod_{j=1}^n (z - x_j)}{\prod_{j=1}^n z - x_j} \right) \\
&= \frac{\prod_{j=1}^n (x_i - tx_j) - \prod_{j=1}^n (x_i - x_j)}{\prod_{j \neq i} x_i - x_j} \\
&= (x_i - tx_i) \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}
\end{aligned}$$

or, in other words,

$$\prod_{i=1}^n \frac{z - tx_i}{z - x_i} = 1 + \sum_{i=1}^n (1-t) \frac{x_i}{z - x_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}$$

Now, taking  $z = u^{-1}$ , we get

$$\prod_{i=1}^n \frac{1 - tux_i}{1 - ux_i} = 1 + \sum_{i=1}^n (1-t) \frac{ux_i}{1 - ux_i} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}$$

And thus,

$$\sum_{r=0}^{\infty} q_r u^r = 1 + (1-t) \sum_{i=1}^n \frac{x_i u}{1 - x_i u} \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j} = \prod_{i=1}^n \frac{1 - x_i tu}{1 - x_i u}$$

□

**1.6. Definition.** Given a partition  $\lambda$ , we define

$$Q_\lambda(x; t) := b_\lambda(t) P_\lambda(x; t)$$

where

$$b_\lambda(t) := \prod_{i \geq 1} \phi_{m_i(\lambda)}(t)$$

with  $m_i(\lambda)$  being the number of times  $i$  occurs as a part of  $\lambda$  and

$$\phi_r(t) = (1-t)(1-t^2) \cdots (1-t^r)$$

These  $Q_\lambda$ 's are also called Hall-Littlewood functions.

**1.7. Proposition.** Since  $b_{(r)}(t) = (1-t)$

$$Q_{(r)}(x; t) = (1-t) P_{(r)}(x; t) = q_r(x; t)$$

One can also prove that

$$Q_\lambda = \prod_{i < j} \frac{1 - R_{ij}}{1 - tR_{ij}} q_\lambda$$

which also gives that the transition matrix from  $\{Q_\lambda\}_\lambda$  to  $\{q_\mu\}_\mu$  is lower unitriangular. Thus, since Hall-Littlewood functions form a  $\mathbb{Q}(t)$ -basis of  $\Lambda[t]$ , we get

**1.8. Proposition.** The set  $\{q_\lambda\}_\lambda$  forms a  $\mathbb{Q}(t)$ -basis of  $\Lambda[t]$ .

## 2. WHEN $t = -1$

Above, we have mainly focused on the very basics of this “ $Q$ ” class of Hall-Littlewood functions. However, in 1911, Schur published a paper on projective representations that realized the so-called “Schur  $Q$ -functions” as irreducible spin characters of  $\mathfrak{S}_n$ . In this section, we will define

$$q_r = q_r(x; t = -1), P_\lambda = P_\lambda(x; -1), Q_\lambda = Q_\lambda(x; -1)$$

We call such  $Q_\lambda$ ’s *Schur Q-function*. We will now lay some groundwork for  $q_r(x; -1)$ .

**2.1. Example.** Now we have

$$q_1(x; -1) = 2m_1 \quad q_2(x; -1) = 2m_2 + 4m_{11}$$

**2.2. Corollary.** Given  $Q(t) = \sum_{r \geq 0} q_r t^r$ , then,

$$Q(t) = \prod_i \frac{1 + tx_i}{1 - tx_i} = E(t)H(t)$$

and thus, since  $H(t)E(-t) = 1$ , we get

$$Q(t)Q(-t) = E(t)H(t)E(-t)H(-t) = 1$$

*Proof.* The first part comes from specializing  $t = -1$  in the formula  $Q(u)$  in the section above (and then replacing  $u$  with  $t$  since  $t$  is now available as a variable). The second part follows from the exposition.  $\square$

**2.3. Proposition.** For  $n = 2m$ , we have the formula

$$q_{2m} = \sum_{r=1}^{m-1} (-1)^{r-1} q_r q_{2m-r} + \frac{1}{2} (-1)^{m-1} q_m^2$$

and thus  $q_{2m} \in \mathbb{Q}[q_1, q_2, \dots, q_{2m-1}]$ .

*Proof.* Since  $Q(t)Q(-t) = 1$ , we have

$$\sum_{r+s=n} (-1)^r q_r q_s = 0$$

and so, setting  $n = 2m$ , we get

$$0 = \sum_{r=0}^{2m} (-1)^r q_r q_{2m-r} = (-1)^m q_m^2 + 2 \sum_{r=0}^{m-1} (-1)^r q_r q_{2m-r} \implies 0 = q_{2m} + \frac{1}{2} (-1)^m q_m^2 + \sum_{r=1}^{m-1} (-1)^r q_r q_{2m-r}$$

and so the result is obtained by isolating  $q_{2m}$ .  $\square$

**2.4. Corollary.**  $q_{2m} \in \mathbb{Q}[q_1, q_3, q_5, \dots, q_{2m-1}]$ .

*Proof.* This follows by induction on  $m$  using the proposition above.  $\square$

**2.5. Corollary.** Given a partition  $\lambda \vdash n$ , then either  $\lambda$  is strict or  $q_\lambda = q_{\lambda_1} \dots q_{\lambda_\ell}$  is a  $\mathbb{Z}$ -linear combination of the  $q_\mu$  such that  $q_\mu$  is strict and  $\mu \trianglerighteq \lambda$ .

**2.6. Definition.** We define

$$\Gamma := \mathbb{Z}[q_1, q_2, q_3, \dots] \subseteq \Lambda$$

and  $\Gamma^n := \Gamma \cap \Lambda^n$ . We also denote

$$\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q} = \mathbb{Q}[q_1, q_2, q_3, \dots]$$

**2.7. Lemma.**

$$\frac{Q'(t)}{Q(t)} = \frac{E'(t)H(t) + E(t)H'(t)}{E(t)H(t)} = \frac{E'(t)}{E(t)} + \frac{H'(t)}{H(t)} = P(t) + P(-t)$$

and thus

$$\frac{Q'(t)}{Q(t)} = 2 \sum_{r \geq 0} p_{2r+1} t^{2r}$$

**2.8. Proposition.**

$$rq_r = 2(p_1 q_{r-1} + p_3 q_{r-3} + \dots)$$

*Proof.* Rearranging the results of our lemma above, we get

$$Q'(t) = Q(t) \left( 2 \sum_{r \geq 0} p_{2r+1} t^{2r} \right)$$

and so, looking at the  $t^{r-1}$  coefficient on both sides, we get

$$rq_r = \sum_{2s+u=r-1} p_{2s+1} q_u = \sum_{s=0}^{r-1} p_{2s+1} q_{r-1-2s}$$

where we take  $q_u = 0$  if  $u < 0$ .  $\square$

**2.9. Corollary.** *Thus,*

$$\Gamma_{\mathbb{Q}} = \mathbb{Q}[p_r \mid r \text{ is odd}] = \mathbb{Q}[q_r \mid r \text{ is odd}]$$

and the  $q_r$  for  $r$  odd are algebraically independent over  $\mathbb{Q}$

*Proof.* The formula above allows us to express odd power  $p_r$ 's in terms of  $q$ 's and vice versa. Since the odd  $p_r$ 's are algebraically independent over  $\mathbb{Q}$ , we get the result.  $\square$

**2.10. Lemma.** *The number of odd partitions on  $n$  is equal to the number of strict partitions on  $n$ .*

*Proof.* Consider the generation function

$$\sum_{\lambda \text{ is odd}} t^{|\lambda|}$$

which would have a  $t^n$  term for every odd partition of  $n$ . Since every odd partition is some combination of odd terms, we can rewrite this sum as

$$(1 + t^{\text{wt}(1)} + t^{\text{wt}(1,1)} + \dots) (1 + t^{\text{wt}(3)} + t^{\text{wt}(3,3)} + \dots) \dots = (1 + t + t^2 + \dots) (1 + t^3 + t^6 + \dots) \dots = \prod_{r=1}^{\infty} \frac{1}{1 - t^{2r-1}}$$

However, using some clever algebra, we rewrite our fraction to have every  $1 - t^{2r}$  term in the numerator and all  $1 - t^r$  terms in the denominator (so after cancellation, only  $1 - t^{2r-1}$  terms remain in the denominator)

$$\prod_{r \geq 1} \frac{1}{1 - t^{2r-1}} = \prod_{r \geq 1} \frac{1 - t^{2r}}{1 - t^r} = \prod_{r \geq 1} (1 + t^r)$$

Finally, we observe by multiplying out the terms that

$$\prod_{r \geq 1} (1 + t^r) = \sum_{\lambda \text{ distinct parts}} t^{|\lambda|}$$

□

**2.11. Remark.** The proof above is originally due to Euler. In fact, one can give an explicit bijection using a method by Sylvester relying on the fact that every number can be expressed uniquely as a power of 2 multiplied by an odd number.

**2.12. Proposition.** (a) *The  $q_\lambda$  for  $\lambda$  odd form a  $\mathbb{Q}$ -basis of  $\Gamma_{\mathbb{Q}}$ .*  
(b) *The  $q_\lambda$  for  $\lambda$  strict form a  $\mathbb{Z}$ -basis of  $\Gamma$ .*

*Proof.* The first part follows immediately from the previous proposition. By 2.5, we have that the  $q_\lambda$  for  $\lambda$  strict span  $\Gamma^n$  (and thus also  $\Gamma_{\mathbb{Q}}^n$ ). Furthermore, since, by the lemma above, the number of strict partitions is equal to the number of odd partitions, they must form a  $\mathbb{Q}$  basis of  $\Gamma_{\mathbb{Q}}^n$ . Therefore, they are linearly independent over  $\mathbb{Q}$  and thus also  $\mathbb{Z}$ , giving the second part. □

**2.13. Proposition.** *Given a partition  $\lambda$ , we have*

$$Q_\lambda(x; -1) = \begin{cases} 2^{\ell(\lambda)} P_\lambda & \text{if } \lambda \text{ is strict} \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Recall from 1.6 that

$$Q_\lambda(x; t) = b_\lambda(t) P_\lambda(x; t)$$

where  $b_\lambda(t) = \prod_{i \geq 1} \phi_{m_i(\lambda)}(t)$  and  $\phi_r(t) = (1 - t)(1 - t^2) \cdots (1 - t^r)$ . Then, if  $r = 1$ ,  $\phi_r(-1) = 2$ , but for  $r > 1$ ,  $\phi_r(-1) = 0$ . Thus, if  $\lambda$  does not have distinct parts, there is an  $i$  such that  $\phi_{m_i(\lambda)}(-1) = 0 \implies b_\lambda(-1) = 0$ . When  $\lambda$  has distinct parts, then  $b_\lambda(-1) = 2^{\ell(\lambda)}$ . Thus, we get our result. □

**2.14. Definition.** We call  $Q_\lambda(x; -1)$  the *Schur-Q* function indexed by  $\lambda$ .

**2.15. Proposition.** *The  $\{Q_\lambda\}$  with  $\lambda$  strict form a  $\mathbb{Z}$  basis of  $\Gamma$ .*

*Proof.* We have from the previous section that

$$Q_\lambda(x; -1) = \prod_{i < j} \frac{1 - R_{ij}}{1 + R_{ij}} q_\lambda$$

and thus  $\{Q_\lambda\}$  is unitriangularly related to the  $\{q_\lambda\}$ . This proves the claim. □

**2.1. The Pfaffian Definition of the Schur- $Q$  function (Optional).** Originally, Schur defined the Schur- $Q$  functions using the “Pfaffian” of a matrix (indirectly named after a German mathematician Johann Friedrich Pfaff). From a combinatorial standpoint, this section is optional, but can paint a nice picture and give more intuition for working with Schur- $Q$  functions.

**2.16. Proposition.** *A  $2n \times 2n$  real skew-symmetric matrix  $A$  with eigenvalues  $i\lambda_1, -i\lambda_1, i\lambda_2, -i\lambda_2, \dots, i\lambda_r, -i\lambda_r$  with  $i = \sqrt{-1}$  and  $\lambda_j \in \mathbb{R}$  can be written in the form  $A = QSQ^t$  where  $Q$  is an orthogonal matrix and*

$$S = \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \\ & & 0 & \lambda_2 \\ & & -\lambda_2 & 0 \\ & & & & 0 & \lambda_r \\ & & & & -\lambda_r & 0 \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix}$$

**2.17. Corollary.** *From the above characterization of a  $2n \times 2n$  skew symmetric matrix  $A$ , we have that  $\det A$  is a perfect square.*

**2.18. Definition.** The *Pfaffian* of a  $2n \times 2n$  skew-symmetric matrix  $A = (a_{ij})_{1 \leq i,j \leq 2n}$  is given by

$$\text{Pf}(A) = \sum_{\substack{w \in \mathfrak{S}_{2n} \\ w(2r-1) < w(2r), 1 \leq r \leq n \\ w(2r-1) < w(2r+1), 1 \leq r \leq n-1}} \text{sgn}(w) a_{w(1),w(2)} \cdots a_{w(2n-1),w(2n)}$$

**2.19. Lemma.**

$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{w \in \mathfrak{S}_{2n}} \text{sgn}(w) a_{w(1),w(2)} \cdots a_{w(2n-1),w(2n)}$$

**2.20. Proposition.** *Given a  $2n \times 2n$  skew-symmetric matrix  $A$ ,*

$$(\text{Pf}(A))^2 = \det A$$

2.21. **Lemma.** *Given  $m$  an even positive integer, denote the Pfaffian of the  $m \times m$  matrix*

$$P(t_1, \dots, t_m) = \text{Pf} \left( \frac{t_i - t_j}{t_i + t_j} \right)_{1 \leq i, j \leq m}$$

*Then, from the definition of the Pfaffian, we get*

$$P(t_1, \dots, t_m) = \sum_{i=2}^m (-1)^i P(t_1, t_i) P(t_2, \dots, \hat{t}_i, \dots, t_m)$$

2.22. **Lemma.** *From the formula in the proof above, we have for  $r > s \geq 0$ ,*

$$Q_{(r,s)} = q_r q_s + 2 \sum_{i=1}^s (-1)^i q_{r+i} q_{s-i}$$

*Proof.*

$$\begin{aligned} Q_{(r,s)}(x; -1) &= \prod_{i < j} \frac{1 - R_{ij}}{1 + R_{ij}} q_{(r,s)} \\ &= \prod_{i < j} (1 - R_{ij})(1 - R_{ij} + R_{ij}^2 - R_{ij}^3 + \dots) q_{(r,s)} \\ &= \prod_{i < j} (1 - 2R_{ij} + 2R_{ij}^2 + \dots) q_{(r,s)} \\ &= q_r q_s + 2 \sum_{i=1}^s (-1)^i q_{r+i} q_{s-i} \end{aligned}$$

□

2.23. **Proposition.** *Given  $\lambda$  a strict partition written in the form  $\lambda_1 > \lambda_2 > \dots > \lambda_{2n} \geq 0$ , the  $2n \times 2n$  matrix*

$$M_\lambda := (Q_{(\lambda_i, \lambda_j)})_{i,j}$$

*is skew-symmetric. Then,*

$$Q_\lambda(x; -1) = \text{Pf}(M_\lambda) = \sum_{\substack{w \in \mathfrak{S}_{2n} \\ w(2r-1) < w(2r), 1 \leq r \leq n \\ w(2r-1) < w(2r+1), 1 \leq r \leq n-1}} \text{sgn}(w) Q_{\lambda_{w(1)}, \lambda_{w(2)}} \cdots Q_{\lambda_{w(2n-1)}, \lambda_{w(2n)}}$$

2.24. **Corollary.** *We have the following recursive relations for  $Q_\lambda$*

$$\begin{aligned} Q_\lambda &= \sum_{j=2}^m (-1)^j Q_{(\lambda_1, \lambda_j)} Q_{(\lambda_2, \dots, \hat{\lambda}_j, \dots, \lambda_\ell)} && \text{for } \ell \text{ even} \\ Q_\lambda &= \sum_{j=1}^m (-1)^{j-1} Q_{(\lambda_j)} Q_{(\lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_\ell)} && \text{for } \ell \text{ odd} \end{aligned}$$

Note that this description of Schur- $Q$  functions is somewhat difficult to work with by hand, but is natural with respect to projective characters of the Symmetric group.

### 3. ORTHOGONALITY

When working with the Hall inner product on  $\Lambda$ , one typically encounters the Cauchy kernel

$$\Omega(x, y) = \prod_{i,j \geq 1} (1 - x_i y_j)^{-1}$$

A generalization of this formula can be given by

$$\Omega(x, y; t) = \prod_{i,j \geq 1} \frac{1 - tx_i y_j}{1 - x_i y_j}$$

so that, when  $t = 0$ , the usual Cauchy kernel is recovered. We will then use the  $t$ -generalized Cauchy kernel to show various orthogonality relations with a  $t$ -generalization of the Hall inner product.

**3.1. Theorem.** *We have*

- (a)  $\Omega(x, y; t) = \sum_{\lambda} z_{\lambda}(t)^{-1} p_{\lambda}(x) p_{\lambda}(y)$  where  $z_{\lambda}(t) = z_{\lambda} \prod_{i \geq 1} (1 - t^{\lambda_i})^{-1}$
- (b)  $\Omega(x, y; t) = \sum_{\lambda} q_{\lambda}(x; t) m_{\lambda}(y) = \sum_{\lambda} m_{\lambda}(x) q_{\lambda}(y; t)$
- (c)  $\Omega(x, y; t) = \sum_{\lambda} P_{\lambda}(x; t) Q_{\lambda}(y; t)$

*Proof.* This is the subject of [Mac79, pp 222-224]. In brief,

- (a) The first identity follows from generating function-ology. The key observation is that

$$\log \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} = \sum_{m=1}^{\infty} \frac{1 - t^m}{m} p_m(x) p_m(y)$$

and then exponentiating both sides.

- (b) Since

$$Q(y_j) = \sum_{r=0}^{\infty} q_r(x; t) y_j^r = \prod_i \frac{1 - tx_i y_j}{1 - x_i y_j}$$

we get

$$\prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} = \prod_j Q(y_j) = \prod_j \sum_{r_j=0}^{\infty} q_{r_j}(x; t) y_j^{r_j} = \sum_{\lambda} q_{\lambda}(x; t) m_{\lambda}(y)$$

and similarly for  $x$ 's and  $y$ 's interchanged.

- (c) If we take the linear transformations

$$(A_{\lambda,\mu})(Q_{\lambda})_{\lambda} = (q_{\mu})_{\mu} \quad (B_{\lambda,\mu})(Q_{\lambda})_{\lambda} = (m_{\mu})_{\mu} \quad (C_{\lambda,\mu})(m_{\lambda})_{\lambda} = (q_{\mu})_{\mu}$$

We have that  $A$  is lower unitriangular by ?? and  $B$  is upper triangular because it is a product of upper triangular matrices ( $m \rightarrow s \rightarrow P \rightarrow Q$ ). Thus,  $D = B^t A$  is lower triangular. However,  $D = B^t C B$  and since  $C$  is symmetric,  $D$  must also be symmetric. Thus,  $D$  must be

a diagonal matrix with diagonal entries equal to those of  $B$  since  $A$  is unitriangular. Thus,  $D = \text{diag}(b_\lambda(t)^{-1})$ . This gives us

$$\begin{aligned}\sum_{\lambda} q_{\lambda}(x; t) m_{\lambda}(y) &= \sum_{\lambda, \mu, \nu} A_{\lambda, \mu} B_{\lambda, \nu} Q_{\mu}(x; t) Q_{\nu}(y; t) \\ &= \sum_{\mu} b_{\mu}(t)^{-1} Q_{\mu}(x; t) Q_{\mu}(y; t) \\ &= \sum_{\mu} P_{\mu}(x; t) Q_{\mu}(y; t)\end{aligned}$$

□

**3.2. Definition.** Given a partition  $\lambda$ , we define

$$S_{\lambda}(x; t) := \det(q_{\lambda_i - i + j}(x; t))$$

**3.3. Proposition.** *We can also express*

$$S_{\lambda}(x; t) = \prod_{i < j} (1 - R_{ij}) q_{\lambda} = \prod_{i < j} (1 - t R_{ij}) Q_{\lambda}$$

*Proof.* The first equality follows using an identical argument for  $s_{\lambda} = \prod_{i < j} (1 - R_{ij}) h_{\lambda}$  from the Jacobi-Trudi identity. For the second equality, we have

$$\prod_{i < j} (1 - t R_{ij}) Q_{\lambda} = \prod_{i < j} (1 - t R_{ij}) \left( \prod_{i < j} \frac{1 - R_{ij}}{1 - t R_{ij}} q_{\lambda} \right) = \prod_{i < j} (1 - R_{ij}) q_{\lambda}$$

□

**3.4. Lemma.** *For  $r \geq 1$ , we have*

$$q_r(x; t) = h_r[(1 - t)p_1]$$

where  $h_r[(1 - t)p_1]$  is a plethystic substitution.

*Proof.* We write

$$\begin{aligned}h_r(x) &= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) \\ \implies h_r[(1 - t)p_1] &= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}[(1 - t)p_1] = \sum_{\lambda \vdash r} z_{\lambda}^{-1} \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i}) p_{\lambda_i} = \sum_{\lambda \vdash r} z_{\lambda}^{-1} p_{\lambda} \prod_{i=1}^{\ell(\lambda)} (1 - t^{\lambda_i})\end{aligned}$$

since  $p_1(x_1^{\lambda_i}, x_2^{\lambda_i}, \dots) = p_{\lambda_i}(x)$ . Now, using some generating function-ology<sup>1</sup>, we get

$$\sum_{r=0}^{\infty} h_r[(1 - t)p_1] y^r = \sum_{\lambda} \frac{\prod_{j=1}^{\ell(\lambda)} (1 - t^{\lambda_j})}{z_{\lambda}} p_{\lambda} y^{|\lambda|}$$

---

<sup>1</sup>The reader may find it helpful to review the proof that  $\sum_{r=0}^{\infty} h_r y^r = \sum_{\lambda} \frac{p_{\lambda}}{z_{\lambda}} y^{|\lambda|}$ . See [See18, 2.2] or [Mac79, p 25]

$$\begin{aligned}
&= \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} \frac{((1-t^k)p_k y^k)^{m_k}}{m_k! k^{m_k}} \\
&= \prod_{k=1}^{\infty} \exp \left( \frac{(1-t^k)p_k y^k}{k} \right) \\
&= \exp \left( \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{(1-t^k)(x_i y)^k}{k} \right) \\
&= \prod_i \exp \left( \sum_{k=1}^{\infty} \frac{(1-t^k)(x_i y)^k}{k} \right) \\
&= \prod_i \exp \left( \sum_{k=1}^{\infty} \frac{(x_i y)^k}{k} - \sum_{k=1}^{\infty} \frac{(tx_i y)^k}{k} \right) \\
&= \prod_i \exp(-\log(1-x_i y) + \log(1-tx_i y)) \\
&= \prod_i \frac{\exp(\log(1-tx_i y))}{\exp(\log(1-x_i y))} \\
&= \prod_i \frac{1-tx_i y}{1-x_i y} \\
&= \sum_{r=0}^{\infty} q_r(x; t) y^r \tag{by 1.5}
\end{aligned}$$

Thus, we have proven the lemma.  $\square$

**3.5. Remark.** The plethystic substitution  $f \mapsto f[(1-t)p_1]$  has explicit inverse given by  $g \mapsto g[\frac{p_1}{1-t}]$ , which can be seen by direct computation of  $p_\lambda[\frac{p_1}{1-t}] = p_\lambda \prod_{i=1}^{\ell(\lambda)} (1-t^{\lambda_i})^{-1}$ .

As some trivia, the plethystic substitutions have a representation theoretic meaning. If  $f(x; t) = \sum_r t^r \text{ch}(\chi_{A_r})$  for  $A = \bigoplus_r A_r$  a graded  $\mathfrak{S}_n$ -module and  $V = \mathbb{C}^n$  the defining representation, then

$$f[(1-t)p_1] = \sum_k (-1)^k t^k \left( \sum_r t^r \text{ch}(\chi_{A \otimes \wedge^k V}) \right)$$

and

$$f[\frac{p_1}{1-t}] = \sum_k t^k \left( \sum_r t^r \text{ch}(\chi_{A \otimes \text{Sym}^k V}) \right)$$

which follows from proving the result for  $f = h_n$  and doing some extra representation theoretic work.

**3.6. Corollary.** Let variables  $\xi_i$  be such that  $h_r(\xi) = h_r[(1-t)p_1]$ . Then,

$$q_r(x; t) = h_r(\xi) \text{ and } S_\lambda(x; t) = s_\lambda(\xi)$$

*Proof.* It is a general fact that one can always introduce variables  $\xi_i$  such that  $f(\xi) = f[g]$ . Thus, the first part is immediate from the above lemma. Then, from above, we have

$$S_\lambda(x; t) = \det(q_{\lambda_i-i+j}(x; t)) = \det(h_{\lambda_i-i+j}(\xi)) = s_\lambda(\xi)$$

by the Jacobi-Trudi identity.  $\square$

**3.7. Proposition.** *We have*

$$\prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} = \sum_{\lambda} S_\lambda(x; t) s_\lambda(y) = \sum_{\lambda} s_\lambda(x) S_\lambda(y; t)$$

*Proof.* Since the Cauchy kernel has equality

$$\Omega(x, y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

we compute in the spirit of the proof of the corollary above,

$$\prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} = \prod_{i,j} \frac{1}{1 - \xi_i y_j} = \sum_{\lambda} s_\lambda(\xi) s_\lambda(y) = \sum_{\lambda} S_\lambda(x; t) s_\lambda(y)$$

and similarly for the other equality.  $\square$

**3.8. Definition.** We define a bilinear product on  $\Lambda[t]$

$$\langle q_\lambda(x; t), m_\mu(x) \rangle_t := \delta_{\lambda\mu}$$

**3.9. Remark.** This is a  $t$ -generaliation of the Hall-inner product, but is not the same one used in [See18, Section 4.2].

**3.10. Lemma.** *Given  $\{u_\lambda\}, \{v_\lambda\}$  as  $\mathbb{Q}(t)$ -bases of  $\Lambda[t]$ , the following are equivalent.*

- (a)  $\langle u_\lambda, v_\mu \rangle_t = \delta_{\lambda\mu}$  for all  $\lambda, \mu$
- (b)  $\sum_{\lambda} u_\lambda(x) v_\lambda(y) = \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j}$

*Proof.* If we let

$$u_\lambda = \sum_{\nu} a_{\lambda\nu} q_\nu \quad v_\mu = \sum_{\sigma} b_{\mu\sigma} m_\sigma$$

then

$$\begin{aligned} \sum_{\lambda} u_\lambda(x) v_\lambda(y) &= \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} \\ \iff \sum_{\lambda} u_\lambda(x) v_\lambda(y) &= \sum_{\mu} q_\mu(x; t) m_\mu(y) \\ \iff \sum_{\lambda} a_{\lambda\nu} b_{\lambda\sigma} &= \delta_{\nu\sigma} \\ \iff \langle u_\lambda, v_\mu \rangle_t &= \sum_{\rho} a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu} \end{aligned}$$

□

**3.11. Proposition.** [Mac79, p 225] *Given the definition of  $\langle \cdot, \cdot \rangle_t$  above, we have*

- (a)  $\langle P_\lambda(x; t), Q_\mu(x; t) \rangle_t = \delta_{\lambda, \mu}$
- (b)  $\langle S_\lambda(x; t), s_\mu(x) \rangle_t = \delta_{\lambda, \mu}$
- (c)  $\langle p_\lambda(x), p_\mu(x) \rangle_t = \delta_{\lambda, \mu} z_\lambda \prod_i (1 - t^{\lambda_i})^{-1}$
- (d)  $\langle \cdot, \cdot \rangle_t$  is symmetric.

*Proof.* The first three identites follow by applying lemma 3.10 to 3.1. The symmetry follows from the self-duality of the power-sum basis. □

**3.12. Proposition.** *The  $t$ -generalized Hall inner product is given by*

$$\langle f, g \rangle_t = \langle f, g[(1-t)^{-1}p_1] \rangle$$

*where the inner product on the right is the standard Hall-inner product.*

*Proof.* Let  $\phi, \psi \in \Lambda$  over  $\mathbb{Q}$ . Then, we may write

$$\phi = \sum_{\lambda} a_{\lambda} m_{\lambda} \quad \psi = \sum_{\mu} b_{\mu} h_{\mu}$$

for  $a_{\lambda}, b_{\mu} \in \mathbb{Q}$ . Then,

$$\langle \phi, \psi \rangle = \sum_{\lambda} \sum_{\mu} a_{\lambda} b_{\mu} \langle m_{\lambda}, h_{\mu} \rangle = \sum_{\lambda} a_{\lambda} b_{\lambda} = \langle \phi, \sum_{\mu} b_{\mu} q_{\mu} \rangle_t = \langle \phi, \psi[(1-t)p_1] \rangle_t$$

and thus the result follows by remark 3.5. □

**3.13. Remark.** The  $t$ -generalization of the Hall inner product in [See18] is given by  $\langle p_{\lambda}, p_{\mu} \rangle_t = \langle p_{\lambda}, p_{\mu}[(1-t)p_1] \rangle$ .

**3.14. Corollary.** *The plethystic substitution of  $(1-t)p_1$  is self-adjoint with the the standard Hall inner product,  $\langle \cdot, \cdot \rangle$ . In other words, for  $f, g \in \Lambda$ ,*

$$\langle f, g[(1-t)p_1] \rangle = \langle f[(1-t)p_1], g \rangle$$

*Proof.* We observe

$$\langle f, g[(1-t)p_1] \rangle = \langle f, g \rangle_t = \langle g, f \rangle_t = \langle g, f[(1-t)p_1] \rangle = \langle f[(1-t)p_1], g \rangle$$

□

#### 4. BERNSTEIN AND JING OPERATORS

First, recall from [Mac79, pp 75–76] that, given a symmetric function  $f \in \Lambda$ , there exists  $f^\perp \in \text{End}(\Lambda)$  given as an adjoint to  $f$  under the Hall inner product, that is

$$\langle f^\perp u, v \rangle = \langle u, fv \rangle$$

In particular, we may express

$$p_n^\perp = n \frac{\partial}{\partial p_n}$$

Then, from [Mac79, p 96], we may define the following.

**4.1. Definition.** The *Bernstein operators* on  $\Lambda$  are given by

$$B_n := \sum_{i \geq 0} (-1)^i h_{n+i} e_i^\perp$$

which are also encoded in the generating function

$$B(t) = \sum_{n \in \mathbb{Z}} B_n t^n = H(t) E^\perp(-t^{-1}) = \exp \left( \sum_{k \geq 1} \frac{t^k}{k} p_k \right) \exp \left( - \sum_{k \geq 1} \frac{t^{-k}}{k} p_k^\perp \right) = \exp \left( \sum_{k \geq 1} \frac{t^k}{k} p_k \right) \exp \left( - \sum_{k \geq 1} t^{-k} \frac{\partial}{\partial p_k} \right)$$

However, we can also rephrase the description using plethysm and the Cauchy kernel. Recall the Cauchy kernel

$$\Omega(x, y) = \prod_{i,j \leq 1} (1 - x_i y_j)^{-1}$$

and let us set

$$\Omega(x) := \prod_{i \leq 1} (1 - x_i)^{-1}$$

We may then we have

**4.2. Proposition.** Given  $f \in \Lambda$ , we have  $B(t)f(x) = f[p_1 - \frac{1}{t}] \Omega(tx)$  and thus  $B_m f$  is the coefficient of the  $m$ th term.

Prove this.

**4.3. Theorem.** Given a Schur function  $s_\lambda$  with  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ , if  $m \geq \lambda_1$ , then

$$B_m s_\lambda = s_{m,\lambda} = s_{m,\lambda_1, \dots, \lambda_\ell}$$

*Proof.* [Mac79] gives a proof using his definition on page 96. Using the proposition above, we observe first that

$$\Omega(tx) = \prod_i \frac{1}{1 - tx_i} = \sum_i \frac{1}{1 - tx_i} \prod_{j \neq i} \frac{1}{1 - \frac{x_j}{x_i}}$$

via partial fraction expansion (similar to the proof of 1.5). Next, we observe that the coefficient of  $t^m$  in  $f(t^{-1})(1 - tx)^{-1}$  is the coefficient of  $t^0$  in  $t^{-m}f(t^{-1})(1 - tx)^{-1}$ , giving us  $x^m f(t^{-1})$ . Thus,

$$Bs_\lambda = s_\lambda[p_1 - t^{-1}] \Omega(tx) = s_\lambda[p_1 - t^{-1}] \sum_i \frac{1}{1 - tx_i} \prod_{j \neq i} \frac{1}{1 - \frac{x_j}{x_i}}$$

Thus, we get that

$$B_m s_\lambda = \sum_i x_i^m \frac{s_\lambda[p_1 - x_i]}{\prod_{j \neq i} (1 - \frac{x_j}{x_i})}$$

How did this happen?

However,  $s_\lambda[p_1 - x_i]$  is simply evaluating  $s_\lambda$  in all the other variables, removing  $x_i$ .  $\square$

**4.4. Definition.** The *Jing operators*  $S^k(t)$  are defined by

$$S^k(t)f(x) := f[p_1 + (k-1)t^{-1}] \Omega(tx)$$

where  $S_n^k(t)f(x)$  is the coefficient of  $t^n$  in  $S^k(t)f(x)$ .

Of course, when  $k = 0$ , we recover the Bernstein operators. However, their general action on Schur functions is slightly more complicated. Namely, we have

**4.5. Lemma.** [Hai03, Lem 3.4.6] *If  $n \geq \mu_1$  and  $\lambda \trianglerighteq \mu$ , then*

$$S_n^k s_\lambda \in \mathbb{Z}[t]\{s_\gamma \mid \gamma \trianglerighteq (m, \mu)\}$$

*Furthermore,  $s_{(m, \mu)}$  occurs with coefficient 1 in  $S_n^k s_\mu$ .*

#### REFERENCES

- [Hai03] M. Haiman, *Combinatorics, Symmetric Functions, and Hilbert Schemes* (2003). <https://math.berkeley.edu/~mhaiman/ftp/cdm/cdm.pdf>.
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