

A Raising Operator Formula for Macdonald Polynomials and other related families

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- ① **Background on symmetric functions and Macdonald polynomials**
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ A new formula for Macdonald polynomials

Symmetric Polynomials

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- E.g. for $n = 3$,

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- Λ is a $\mathbb{Q}(q, t)$ -algebra.

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\implies any basis of symmetric functions is indexed by partitions.

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Collection is called $\text{SSYT}(\lambda)$.

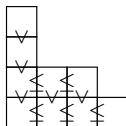
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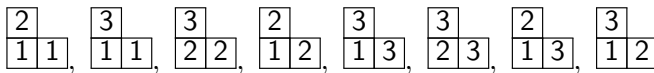
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For $\lambda = (2, 1)$,



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1	1	1	1	2	2	1	2	1	3	2	3	1	3	1	2

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- $\{s_\lambda\}_\lambda$ forms a basis for $\Lambda_{\mathbb{Q}}$.

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Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in \mathbb{N}) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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- ① Break M up into irreducible S_n -representations.

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$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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Remark: M is a “regular representation.”

Getting more information

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Break M up into smallest S_n fixed subspaces

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Solution: irreducible S_n -representation of polynomials of degree $d \mapsto q^d s_\lambda$
(graded Frobenius)

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Answer: Hall-Littlewood polynomial $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n -regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}_{\lambda\mu}(q, t) s_{\mu}$ satisfies $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\text{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n ?

Frobenius characteristic of DH_3

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Frobenius characteristic of DH_3

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Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

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Operator ∇

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda^*)} \tilde{H}_\lambda(X; q, t),$$

where $n(\lambda) = \sum_i (i-1)\lambda_i$ and λ^* is the transpose partition to λ .

Frobenius characteristic of DH_3

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n .

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??
∇e_n	DH_n	Shuffle theorem

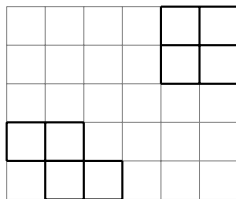
Outline

- ① Background on symmetric functions and Macdonald polynomials
- ② **Shuffle theorems, combinatorics, and LLT polynomials**
- ③ A new formula for Macdonald polynomials

Key Object: LLT Polynomials

Let $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$ be a tuple of skew shapes. (Skew shape = $\lambda \setminus \mu$)

$$\nu = \left(\begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$



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- The *content* of a box in row y , column x is $x - y$.

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-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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- The *content* of a box in row y , column x is $x - y$.
- *Reading order*: label boxes b_1, \dots, b_n by scanning each diagonal from southwest to northeast, in order of increasing content.

$$\nu = \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$

				b_3	b_6
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- A *semistandard tableau* on ν is a map $T: \nu \rightarrow \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} \mathbf{x}^T,$$

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$T =$

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- \mathcal{G}_ν is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} (q, t \text{ monomial})(LLT \text{ polynomial})$$

- Summation over all k -by- k Dyck paths.

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} (LLT \text{ polynomial})$$

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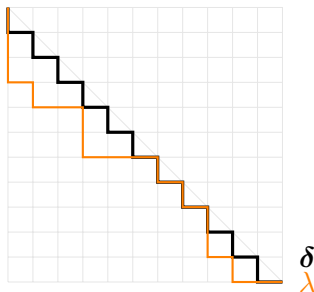
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- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

Dyck paths

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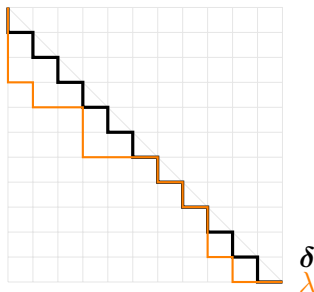
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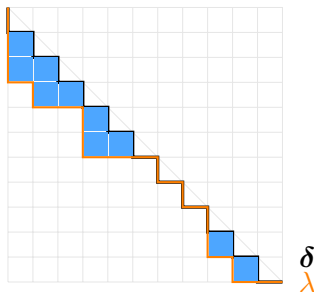


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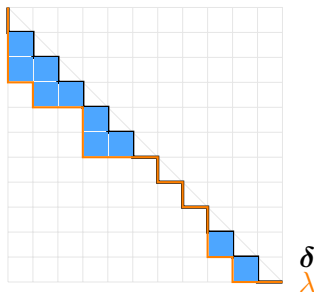


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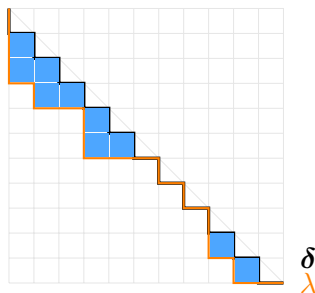


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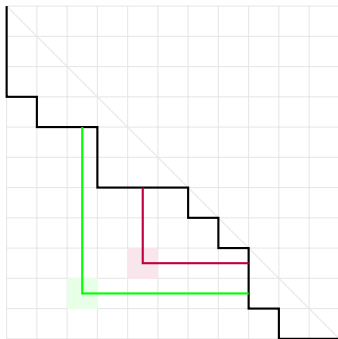
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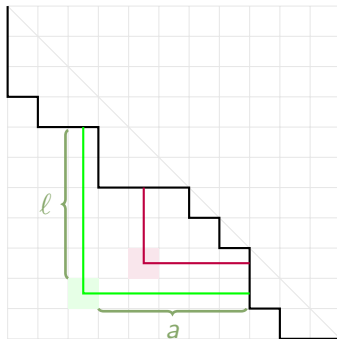
dinv

$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



dinv

$\text{dinv}(\lambda) = \#$ of balanced hooks in diagram below λ .



Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Example ∇e_3

$$\lambda \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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$$q^3$$



$$q^2 t$$



$$qt$$








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






$$t^3$$

Example ∇e_3

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	q^2t	$qts_{2,1} + q^2ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
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- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number” $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

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Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2021)

For $m, n > 0$ coprime, the operator $e_k^{(m,n)}$ acting on Λ satisfies

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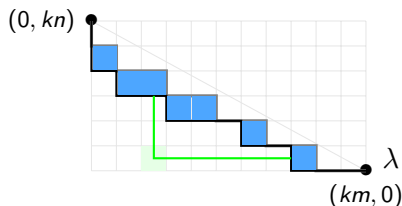
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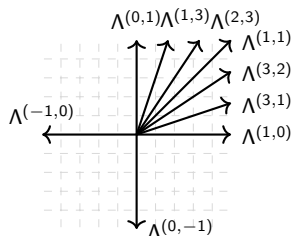
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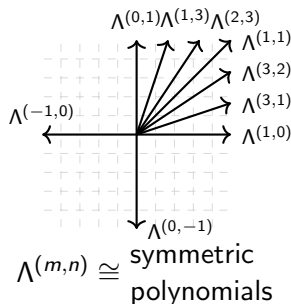


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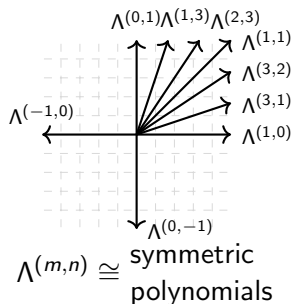


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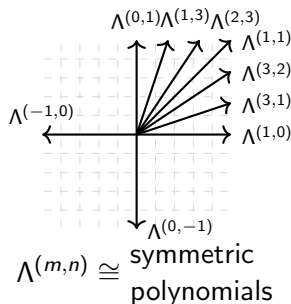
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Can be difficult to work with in general. Can we make it more explicit?

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n , where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

	(12)	(13)	(14)	(15)
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A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$\Psi = \text{Roots above Dyck path}$

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$, set

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Precisely, for $\rho = (n-1, n-2, \dots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β ,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\operatorname{sort}(\beta)$.

Example: $s_{201} = 0$, $s_{2-11} = -s_{200}$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$ by linearly extending

$$\mathbf{z}^\gamma \mapsto s_\gamma(X)$$

where $\mathbf{z}^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

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Example

$$\sigma(\mathbf{z}^{111} + \mathbf{z}^{201} + \mathbf{z}^{210} + \mathbf{z}^{3-11}) = s_{111} + s_{201} + s_{210} + s_{3-11} = s_{111} + s_{210} - s_{300}$$

Catalanimals

Definition

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where $z^{\alpha_{ij}} = z_i/z_j$ and $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \dots$.

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$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left(\frac{z^{111} (1 - qtz_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - qz_i/z_j) (1 - tz_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$.

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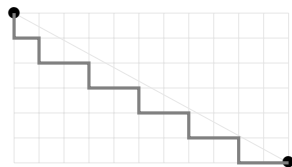
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for $\mathbf{b} = (b_0, \dots, b_{km-1})$ satisfying $b_i =$ the number of south steps on vertical line $x = i$ of highest lattice path under line $y + \frac{n}{m}x = n$.

δ = highest Dyck path.



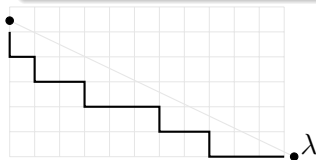
$$\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

Results

Manipulating Catalanimal \implies a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.



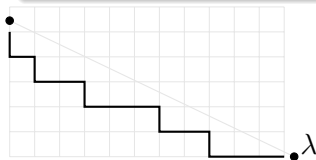
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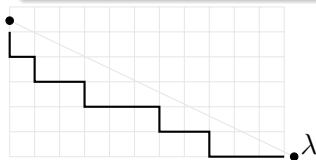
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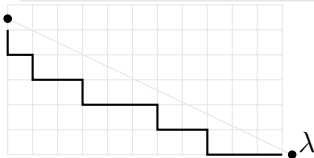
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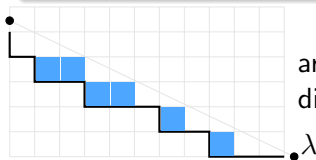
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$\text{area}(\lambda)$ as before

$\text{dinv}_p(\lambda) = \#p\text{-balanced hooks } \frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

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Special case: $\mathcal{G}_\nu^{(1,1)} \cdot 1 = \nabla \mathcal{G}_\nu(X; q)$.

LLT Catalanimals

For a tuple of skew shapes ν , the *LLT Catalanimal* $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

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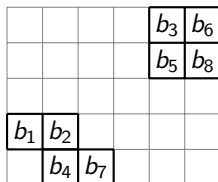
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- λ : fill each diagonal D of ν with $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
Listing this filling in reading order gives λ .

LLT Catalanimals

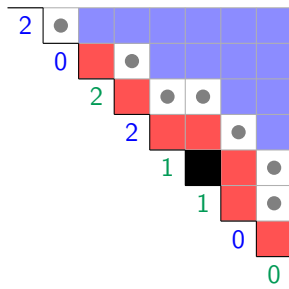
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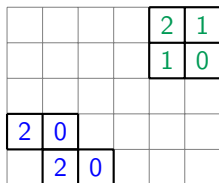


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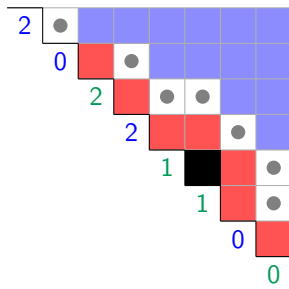
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λ , as a filling of ν



Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let ν be a tuple of skew shapes and let $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega H_\nu \\ &= c_\nu \omega \sigma \left(\frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

What about Macdonald polynomials?!

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- Does there exist formula $\tilde{H}_\mu = \sum_\nu a_{\mu\nu}(q, t) \mathcal{G}_\nu$? Yes!

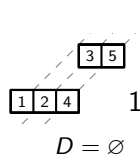
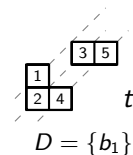
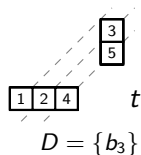
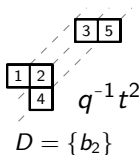
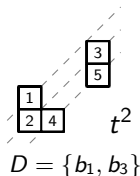
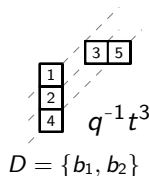
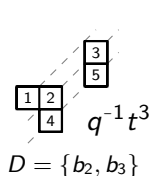
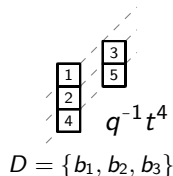
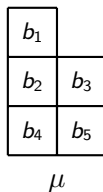
- ① Background on symmetric functions and Macdonald polynomials
- ② Shuffle theorems, combinatorics, and LLT polynomials
- ③ **A new formula for Macdonald polynomials**

Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X; q, t) = \sum_D \left(\prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q)$$

Haglund-Haiman-Loehr formula example

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Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$ and apply $\omega \nabla$.

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- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$ and apply $\omega \nabla$.
- By construction, all the LLT Catalan animals $H_{\nu(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}) .
- Collect terms to get $\prod_{(b_i, b_j) \in V(\mu)} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$ factor for $V(\mu)$ the set of vertical dominoes (b_i, b_j) in μ .

$$\tilde{H}_\mu = \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\substack{\alpha_{ij} \in V(\mu)}} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

The root ideal R_μ

b_1		
b_2	b_3	
b_4	b_5	b_6
b_7	b_8	b_9

row reading order

$$b_1 \prec b_2 \prec \cdots \prec b_n$$

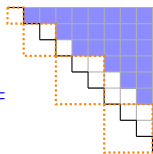
$$R_\mu := \{\alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j\},$$

$$\widehat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j\},$$

$$R_\mu \setminus \widehat{R}_\mu \leftrightarrow V(\mu) = \text{vertical dominoes in } \mu$$

Example:

$$R_{3321} =$$



The root ideal R_μ

b_1		
b_2	b_3	
b_4	b_5	b_6
b_7	b_8	b_9

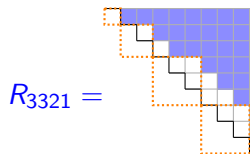
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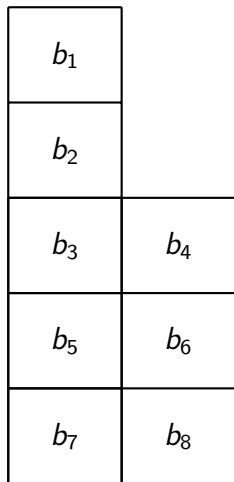
Example:



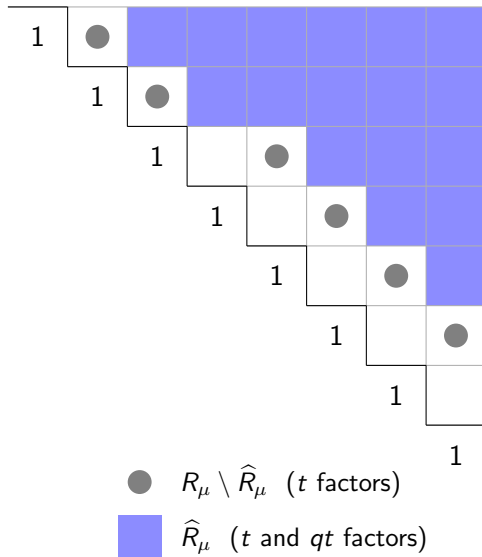
Remark

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right)$$

Example



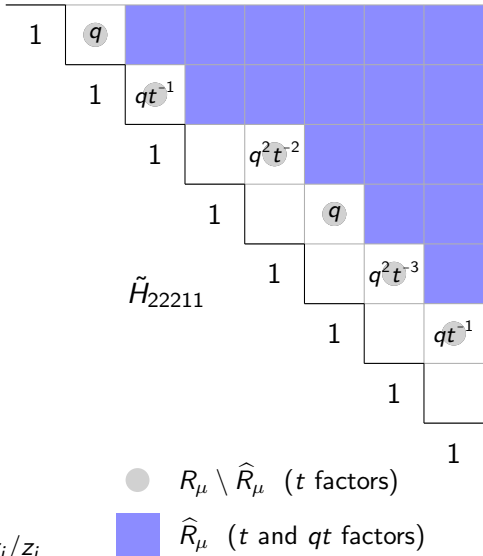
partition $\mu = 22211$



Example

$1 - q^{\frac{z_1}{z_2}}$	
$1 - qt^{-1} \frac{z_2}{z_3}$	
$1 - q^2 t^{-2} \frac{z_3}{z_5}$	$1 - q^{\frac{z_4}{z_6}}$
$1 - q^2 t^{-3} \frac{z_5}{z_7}$	$1 - qt^{-1} \frac{z_6}{z_8}$

numerator factors $1 - q^{\text{arm}+1} t^{-\text{leg}} z_i / z_j$



- $R_\mu \setminus \hat{R}_\mu$ (t factors)

\hat{R}_μ (t and qt factors)

$q = t = 1$ specialization

$$\begin{aligned}
 & \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=t=1} \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \hat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\
 & = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\
 & = \omega h_1^n \\
 & = e_1^n
 \end{aligned}$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega \sigma \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s , the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_\nu(X)$$

satisfy $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	HHL
∇e_n	DH_n	Shuffle theorem
$\tilde{H}_\lambda^{(s)}(X; q, t)$??	??

Thank you!

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