A Raising Operator Formula for Macdonald Polynomials (and The Journey There)

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Outline

1. **Background on symmetric functions and Macdonald polynomials**
2. Shuffle theorems, combinatorics, and LLT polynomials
3. A new formula for Macdonald polynomials

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q,t)[x_1,\ldots,x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

Generators

$$e_r = \sum_{i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} \text{ or } h_r = \sum_{i_1\leq i_2\leq\cdots\leq i_r} x_{i_1}x_{i_2}\cdots x_{i_r}$$

- E.g. for $n = 3$,

$$\begin{aligned} e_1 = x_1 + x_2 + x_3 =& h_1 \\ e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad & h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ e_3 = x_1x_2x_3 \quad & h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \cdots \end{aligned}$$

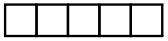
- Let $\Lambda = \mathbb{Q}(q,t)[e_1,e_2,\ldots] = \mathbb{Q}(q,t)[h_1,h_2,\ldots]$. Call these "symmetric functions."
- Λ is a $\mathbb{Q}(q,t)$-algebra.

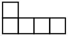
Bases for symmetric functions

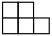
Dimension of degree d symmetric functions? Number of partitions of d.

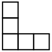
Definition

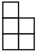
$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

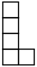
$5 \to$ 

$4+1 \to$ 

$3+2 \to$ 

$3+1+1 \to$ 

$2+2+1 \to$ 

$2+1+1+1 \to$ 

$1+1+1+1+1 \to$ 

$\Longrightarrow$ any basis of symmetric functions is indexed by partitions.

Young Tableaux

Definition

Filling of partition diagram of λ with numbers such that

1. strictly increasing up columns
2. weakly increasing along rows

Collection is called $\mathrm{SSYT}(\lambda)$.

For $\lambda = (2, 1)$,

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 2 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$$

Polynomials from tableaux

Associate a polynomial to SSYT(λ).

$$\begin{ytableau} 2 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 1 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 2 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 2 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 2 & 3 \end{ytableau}, \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}, \begin{ytableau} 3 \\ 1 & 2 \end{ytableau}$$

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Definition

For λ a partition

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \boldsymbol{x}^T \text{ for } \boldsymbol{x}^T = \prod_{i \in T} x_i$$

- s_λ is a symmetric function.
- $\{s_\lambda\}_\lambda$ forms a basis for $\Lambda_{\mathbb{Q}}$.

Representation theory and Schur functions

Frobenius charactersitc, $\mathrm{Frob}\colon Rep(S_n) \to \Lambda$.

- Irreducible representations of S_n are labeled by partitions of n.
- Irreducible S_n-representation V_λ has $\mathrm{Frob}(V_\lambda) = s_\lambda$
- $U \cong V \oplus W \implies \mathrm{Frob}(U) = \mathrm{Frob}(V) + \mathrm{Frob}(W)$
- $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}(V \times W) \mapsto \mathrm{Frob}(V) \cdot \mathrm{Frob}(W)$
- Upshot: S_n-representations go to symmetric functions in structure preserving way.

Hidden Guide: Schur Positivity

"Naturally occurring" symmetric functions which are non-negative (coefficients in $\mathbb{N}$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

Harmonic polynomials

Harmonic polynomials

$M =$ polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

M is the vector space given by

$$\begin{aligned} M &= \mathrm{sp}\left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c\right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\qquad x_3 - x_1, x_2 - x_3, 1\} \end{aligned}$$

Harmonic polynomials

$$\mathrm{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

1. Break M up into irreducible S_n-representations.

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\ydiagram{2,1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

2. How many times does an irreducible S_n-representation occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\ydiagram{1,1,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_+^{S_3})$ is a "regular representation."

Getting more information

Break M up into smallest S_n fixed subspaces

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\substack{\ydiagram{1,2} \\ \text{deg}=2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1, x_2-x_3\}}_{\substack{\ydiagram{1,2} \\ \text{deg}=1}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Solution: irreducible S_n-representation of polynomials of degree $d \mapsto q^d s_\lambda$ (graded Frobenius)

$$?? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{1,2}} + q s_{\ydiagram{1,2}} + s_{\ydiagram{3}}$$

Answer: Hall-Littlewood polynomial $H_{\ydiagram{1,1,1}}(X; q)$.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q,t)$ generalizing Hall-Littlewood polynomials (and many other famous bases).
- Defined by orthogonality and triangularity under a certain inner-product.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X;q,t) = \sum_\mu \tilde{K}(q,t)s_\mu \text{ conjecturally satisfies } \tilde{K}(q,t) \in \mathbb{N}[q,t]$$

- $\tilde{H}_\lambda(X;1,1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n-regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X;q,t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.
- Garsia-Haiman (1993): M_μ = span of partial derivatives of $\Delta_\mu = \det_{(i,j)\in\mu, k\in[n]}(x_k^{i-1} y_k^{j-1})$

$$\Delta_{(2,1)} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$$

$$M_{2,1} = \underbrace{\mathrm{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\mathrm{sp}\{1\}}_{\deg=(0,0)}$$

Irreducible S_n-representation with bidegree $(a,b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{(2,1)} = qt s_{(1,1,1)} + t s_{(2,1)} + q s_{(2,1)} + s_{(3)}$$

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X;q,t)$

- Proved via connection to the Hilbert Scheme $Hilb^n(\mathbb{C}^2)$.

Corollary

$\tilde{H}_\lambda(X;q,t) = \sum_\mu \tilde{K}_{\lambda\mu}(q,t)s_\mu$ *satisfies* $\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

- No combinatorial description of $\tilde{K}_{\lambda\mu}(q,t)$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\mathrm{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??

Garsia-Haiman modules

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \mathrm{sp}\{f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \mid \left(\sum_{j=1}^{n} \partial_{x_j}^r \partial_{y_j}^s\right) f = 0, \forall r + s > 0\}$$

Question

What symmetric function is the bigraded Frobenius characteristic of DH_n?

∇e_n

Frobenius characteristic of DH_3

$$= \frac{t^3\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q^2t+qt^2+qt)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{q^3\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2+t^3+q^2-qt} - \frac{(q+t+1)\tilde{H}_{2,1}}{-q^2t^2+q^3+t^3-qt} - \frac{\tilde{H}_3}{-q^3+q^2t+qt-t^2}$$

Operator ∇

$$\nabla\tilde{H}_\lambda(X;q,t) = q^{n(\lambda)}t^{n(\lambda')}\tilde{H}_\lambda(X;q,t)\,,$$

where $n(\mu) = \sum_i (i-1)\mu_i$.

Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	$\mathrm{SSYT}(\lambda)$
$\tilde{H}_\lambda(X; q, t)$	Garsia-Haiman M_λ	??
∇e_n	DH_n	Shuffle theorem

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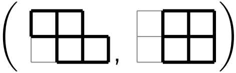
1. Background on symmetric functions and Macdonald polynomials
2. **Shuffle theorems, combinatorics, and LLT polynomials**
3. A new formula for Macdonald polynomials

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$\boldsymbol{\nu} =$ 

-4	-3	-2	-1	0 b_3	1 b_6
-3	-2	-1	0	1 b_5	2 b_8
-2	-1	0	1	2	3
-1 b_1	0 b_2	1	2	3	4
0	1 b_4	2 b_7	3	4	5

Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

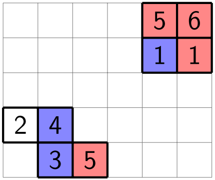
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



inversion

non-inversion

$$\mathrm{inv}(T) = 4, \quad \boldsymbol{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

LLT Polynomials $\mathcal{G}_{\nu}(X;q)$

- $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$ is a symmetric function
- $\mathcal{G}_{\boldsymbol{\nu}}(X;1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_{\boldsymbol{\nu}}$ were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of $U_q(\hat{\mathfrak{sl}}_r)$
- When $\nu^{(i)}$ are partitions, the Schur-expansion coefficients are essentially parabolic Kazdhan-Luzstig polynomials.
- $\mathcal{G}_{\boldsymbol{\nu}}$ is Schur-positive for any tuple of skew shapes $\boldsymbol{\nu}$ [Grojnowski-Haiman, 2007].

A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

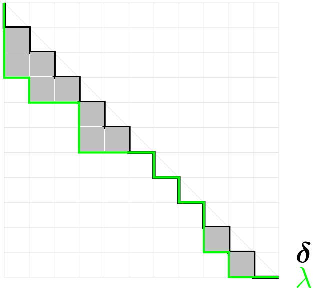
$$\nabla e_k(X) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all k-by-k Dyck paths.
- $\mathrm{area}(\lambda)$ and $\mathrm{dinv}(\lambda)$ statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.
- ω an automorphism of symmetric functions: $\omega(s_\lambda) = s_{\lambda^*}$

Dyck paths

Dyck paths

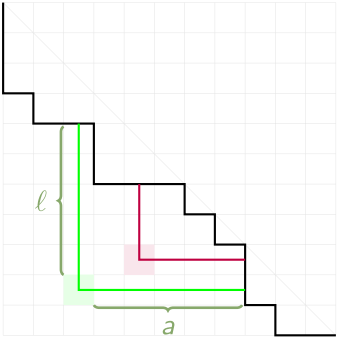
A Dyck path λ is a south-east lattice path lying below the line segment from $(0, k)$ to $(k, 0)$.



- area(λ) = number of squares above λ but below the path δ of alternating S-E steps.
- E.g., above area$(\lambda) = 10$.
- Catalan-number many Dyck paths for fixed k. (1,2,5,14,42,...)

dinv

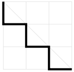
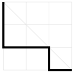
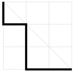
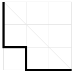
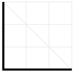
dinv$(\lambda) = \#$ of balanced hooks in diagram below λ.



Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1-\epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Example ∇e_3

λ	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)}$	$q^{\mathrm{dinv}(\lambda)} t^{\mathrm{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + q s_{2,1} + q^2 s_{2,1} + q^3 s_{1,1,1}$
	$q^2 t$	$qt s_{2,1} + q^2 t s_{1,1,1}$
	qt	$t s_{2,1} + qt s_{1,1,1}$
	qt^2	$t^2 s_{2,1} + qt^2 s_{1,1,1}$
	t^3	$t^3 s_{1,1,1}$

- Entire quantity is q, t-symmetric
- Coefficient of $s_{1,1,1}$ in sum is a "(q, t)-Catalan number" $(q^3 + q^2 t + qt + qt^2 + t^3)$.

Generalizing Shuffle Theorem

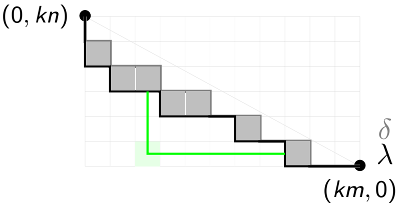
When a problem is too difficult, try generalizing!

Algebraic Expression		Combinatorial Expression
$\nabla e_k(X)$	$=$	$\sum q, t$-weighted Dyck paths

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For $m, n > 0$ coprime, the operator $e_k[-MX^{m,n}]$ acting on Λ satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$



Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra $\mathcal{E}$ of the Hall algebra of coherent sheaves on an elliptic curve over $\mathbb{F}_p$.

The *elliptic Hall algebra* $\mathcal{E}$ is generated by subalgebras $\Lambda(X^{a,b})$ isomorphic to the ring of symmetric functions Λ over $\Bbbk = \mathbb{Q}(q,t)$, one for each coprime pair $(a,b) \in \mathbb{Z}^2$, along with an additional central subalgebra.

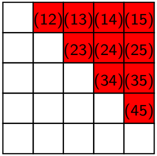
E.g., $e_k[-MX^{m,n}] \in \Lambda(X^{m,n})$.

$\mathcal{E}$ acts on symmetric functions and $e_k[-MX^{1,1}] \cdot 1 = \nabla e_k$.

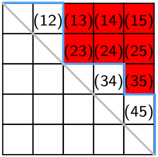
Can be difficult to work with in general. Can we make it more explicit?

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.



A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.



Ψ = Roots above Dyck path

Schur functions revisited

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, set

$$s_\gamma = \det(h_{\gamma_i + j - i})_{1 \leq i,j \leq n}$$

Then, $s_\gamma = \pm s_\lambda$ or 0 for some partition λ.
Precisely, for $\rho = (n-1, n-2, \ldots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma+\rho) s_{\mathrm{sort}(\gamma+\rho)-\rho} & \text{if } \gamma+\rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\mathrm{sort}(\beta) =$ weakly decreasing sequence obtained by sorting β,
- $\operatorname{sgn}(\beta) =$ sign of the shortest permutation taking β to $\mathrm{sort}(\beta)$.

Example: $s_{201} = 0, s_{2\text{-}11} = -s_{200}$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma}\colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$\boldsymbol{z}^{\gamma} \mapsto s_{\gamma}(X)$$

where $\boldsymbol{z}^{\gamma} = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

Example

$\boldsymbol{\sigma}(\boldsymbol{z}^{111} + \boldsymbol{z}^{201} + \boldsymbol{z}^{210} + \boldsymbol{z}^{3\text{-}11}) = s_{111} + s_{201} + s_{210} + s_{3\text{-}11} = s_{111} + s_{210} - s_{300}$

Catalanimals

Definition

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha\in R_{qt}} (1 - qt\boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_q} (1 - q\boldsymbol{z}^\alpha) \prod_{\alpha\in R_t} (1 - t\boldsymbol{z}^\alpha)}\right),$$

where $\boldsymbol{z}^{\alpha_{ij}} = z_i/z_j$ and $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2 z_i^2/z_j^2 + \cdots$.

With $n = 3$,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1\le i<j\le 3}(1 - qz_i/z_j)(1 - tz_i/z_j)}\right) \\ &= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3 \\ &= \omega\nabla e_3. \end{aligned}$$

Why?

Let $R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq l\}$ and $R_+^0 = \{\alpha_{ij} \in R_+ \mid i+1 < j\}$.

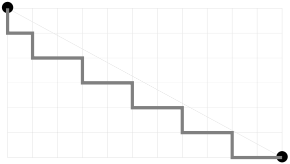
Proposition

For $(m, n) \in \mathbb{Z}^2$ coprime,

$$e_k[-MX^{m,n}] \cdot 1 = H(R_+, R_+, R_+^0, \mathbf{b})$$

for $\mathbf{b} = (b_0, \dots, b_{km-1})$ satisfying $b_i =$ the number of south steps on vertical line $x = i$ of highest lattice path under line $y + \frac{n}{m}x = n$.

$\delta =$ highest Dyck path.



δ

$$\mathbf{b} = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0)$$

Results

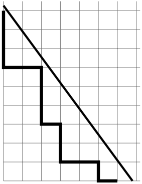
Manipulating Catalanimal $\Longrightarrow$ a proof of the Rational Shuffle Theorem + a generalization.

Theorem (Blasiak-Haiman-Morse-Pun-S., 2023a)

Given $r, s \in \mathbb{R}_{>0}$ such that $p = s/r$ irrational, take $\mathbf{b} = (b_1, \ldots, b_l) \in \mathbb{Z}^l$ to be the south step sequence of highest path δ under the line $y + px = s$.

$$H(R_+, R_+, R_+^0, \mathbf{b}) = \sum_{\lambda} t^{\mathrm{area}(\lambda)} q^{\mathrm{dinv}_p(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

where summation is over all lattice paths under the line $y + px = s$,



area(λ) as before

$\mathrm{dinv}_p(\lambda) = \#p$-balanced hooks $\frac{\ell}{a+1} < p < \frac{\ell+1}{a}$

A Question

Why stop at $e_k[-MX^{m,n}]$?

For which symmetric functions f can we find a Catalanimal such that $f[-MX^{m,n}] \cdot 1 =$ a Catalanimal?

Answer: for f equal to any LLT polynomial!

Special case: $\mathcal{G}_{\boldsymbol{\nu}}[-MX^{1,1}] \cdot 1 = \nabla \mathcal{G}_{\boldsymbol{\nu}}(X; q)$.

LLT Catalanimals

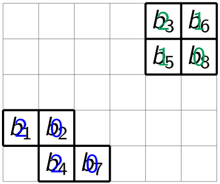
For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal,
- $R_q \setminus R_t \; =$ the attacking pairs,
- $R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,
- λ: fill each diagonal D of $\boldsymbol{\nu}$ with
  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in reading order gives λ.

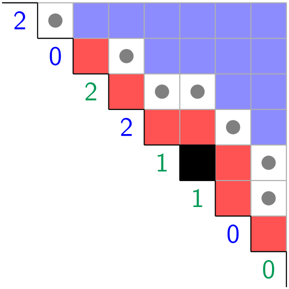
LLT Catalanimals

- (black square) $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
- (red square) $R_q \setminus R_t$ = the attacking pairs,
- (grey dot) $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- (blue square) R_{qt} = all other pairs,

λ: fill each diagonal D of $\boldsymbol{\nu}$ with
$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.



λ, as a filling of $\boldsymbol{\nu}$



LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S., 2021+)

Let $\boldsymbol{\nu}$ be a tuple of skew shapes and let $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) &= c_{\boldsymbol{\nu}}\, \omega H_{\boldsymbol{\nu}} \\ &= c_{\boldsymbol{\nu}}\, \omega \boldsymbol{\sigma} \left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} \left(1 - qt\, \boldsymbol{z}^{\alpha}\right)}{\prod_{\alpha \in R_q} \left(1 - q\, \boldsymbol{z}^{\alpha}\right) \prod_{\alpha \in R_t} \left(1 - t\, \boldsymbol{z}^{\alpha}\right)} \right)\end{aligned}$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

What about Macdonald polynomials?!

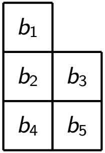
- Remember $\nabla \tilde{H}_\mu = q^{n(\mu)} t^{n(\mu^*)} \tilde{H}_\mu$.
- We have a formula for $\nabla \mathcal{G}_{\boldsymbol{\nu}}$.
- Does there exist formula $\tilde{H}_\mu = \sum_{\boldsymbol{\nu}} a_{\mu\boldsymbol{\nu}}(q,t)\mathcal{G}_{\boldsymbol{\nu}}$? Yes!

Outline

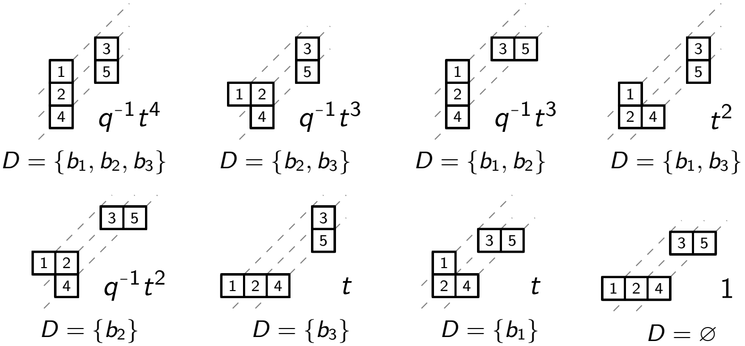
1. Background on symmetric functions and Macdonald polynomials
2. Shuffle theorems, combinatorics, and LLT polynomials
3. **A new formula for Macdonald polynomials**

Haglund-Haiman-Loehr formula example

$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u\in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}\right) \mathcal{G}_{\nu(\mu,D)}(X;q)$$



μ



Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D}\mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{(b_i,b_j)\in V(\mu)}(1 - q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j)$ factor for $V(\mu)$ the set of vertical dominoes (b_i, b_j) in μ.

$$\tilde{H}_\mu = \omega\boldsymbol{\sigma}\left(z_1\cdots z_n \frac{\prod_{\alpha_{ij}\in V(\mu)}\left(1 - q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1 - q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1 - t\boldsymbol{z}^\alpha\right)}\right).$$

The root ideal R_μ

b_1		
b_2	b_3	
b_4	b_5	b_6
b_7	b_8	b_9

row reading order

$b_1 \prec b_2 \prec \cdots \prec b_n$

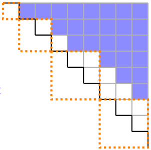
$$R_\mu := \left\{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\right\},$$

$$\widehat{R}_\mu := \left\{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \prec b_j\right\},$$

$$R_\mu \setminus \widehat{R}_\mu \leftrightarrow V(\mu)$$

Example:

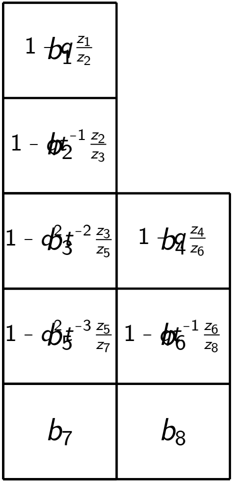
$R_{3321} =$



Remark

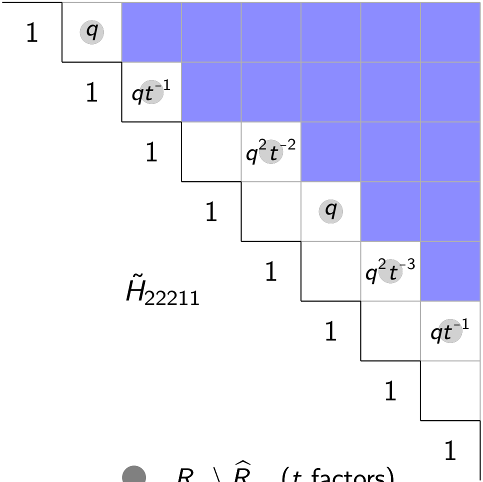
$$\tilde{H}_\mu(X; 0, t) = \omega \boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu}(1 - t\boldsymbol{z}^\alpha)}\right)$$

Example



numerator factors $1 - q^{\mathrm{arm}+1}t^{-\mathrm{leg}}z_i/z_j$

partition $\mu = 22211$



$R_\mu \setminus \widehat{R}_\mu$ (t factors)

$\widehat{R}_\mu$ (t and qt factors)

$q = t = 1$ specialization

$$
\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)\\
\stackrel{q=t=1}{\to}&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\prod_{\alpha\in R_\mu\setminus\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in R_\mu}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega h_1^n\\
=&e_1^n
\end{aligned}
$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega\boldsymbol{\sigma}\left((z_1\cdots z_n)^s \frac{\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_\nu K_{\nu,\mu}^{(s)}(q,t)\, s_\nu(X)$$

satisfy $K_{\nu,\mu}^{(s)}(q,t)\in\mathbb{N}[q,t]$.

Symmetric functions, representation theory, and combinatorics

Symmetric function	Representation theory	Combinatorics
$s_\lambda(X)$	Irreducible V_λ	SSYT(λ)
$\tilde{H}_\lambda(X;q,t)$	Garsia-Haiman M_λ	HHL
∇e_n	DH_n	Shuffle theorem
$\tilde{H}^{(s)}_\lambda(X;q,t)$	??	??

Grazie Mille!

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