

# KAZHDAN-LUSZTIG BASIS FOR HECKE ALGEBRAS A CLASS PRESENTATION FOR QUANTUM GROUPS

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## 1. INTRODUCTION

The Kazhdan-Lusztig basis was introduced in [KL79]. We will define the basis and give a proof of its existence and uniqueness, although we will mainly follow the proof in [Soe97]. Since their introduction, the so-called Kazhdan-Lusztig polynomials, which appear in the definition of the basis, have appeared in many other fields of mathematics. For a more detailed overview of connections, see [Bre03, p 5].

## 2. PRELIMINARIES

We work with the Hecke algebra, for which we will give two presentations.

**2.1. Definition.** [Hum90, Section 7.4] Let  $A = \mathbb{Z}[q, q^{-1}]$ . Then, the *Hecke algebra*  $\mathcal{H}$  associated to a Weyl group  $\mathcal{W}$  has a basis  $\{T_w \mid w \in \mathcal{W}\}$  with relations

- (a)  $T_x T_y = T_{xy}$  if  $\ell(x) + \ell(y) = \ell(xy)$  and
- (b)  $T_s^2 = (q - 1)T_s + qT_{id}$  for all simple reflections  $s \in \mathcal{W}$ .

**2.2. Remark.** We need not restrict  $\mathcal{W}$  to be a Weyl group. In full generality, we can replace  $\mathcal{W}$  with any Coxeter group.

For our purposes, it will also be convenient to work with the Hecke algebra over an enlarged ring. Let  $v := q^{-\frac{1}{2}}$ . Then, we have the following.

**2.3. Proposition.** *The Hecke algebra over  $\mathbb{Z}[v, v^{-1}]$  is given as the associative algebra with generators  $\{H_s\}$  for  $H_s = vT_s$  and relations*

- (a)  $H_s^2 = 1 + (v^{-1} - v)H_s$  and
- (b)  $H_s H_t \cdots H_s = H_t H_s \cdots H_t$  or  $H_s H_t H_s \cdots H_t = H_t H_s H_t \cdots H_s$  if  $st \cdots s = ts \cdots t$  or  $sts \cdots t = tst \cdots s$ , respectively, for simple reflections  $s, t \in \mathcal{W}$

**2.4. Proposition.** *The Hecke algebra over  $\mathbb{Z}[v, v^{-1}]$  has a basis given by  $\{H_w \mid w \in \mathcal{W}\}$  where  $H_w = v^{\ell(w)} T_w$ . Furthermore, this basis has relation  $H_x H_y = H_{xy}$  if  $\ell(x) + \ell(y) = \ell(xy)$ .*

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**2.5. Lemma.** We have  $H_s^{-1} = H_s + (v - v^{-1})$  and so all the  $H_x$  basis elements are units in  $\mathcal{H}$ .

*Proof.*

$$H_s^2 - (v^{-1} - v)H_s = 1 \implies H_s(H_s + (v - v^{-1})) = 1$$

□

**2.6. Lemma.** For simple reflection  $s \in \mathcal{W}$ , if  $\ell(xs) < \ell(x)$ , then

$$H_x H_s = H_{xs} + (v^{-1} - v)H_x.$$

*Proof.* We know that

$$H_x = H_{xs} H_s \implies H_x H_s = H_{xs} H_s^2 = H_{xs}(1 + (v^{-1} - v)H_s) = H_{xs} + (v^{-1} - v)H_x$$

□

### 3. THE KAZHDAN-LUSZTIG BASIS

**3.1. Definition.** Recall that  $A = \mathbb{Z}[q, q^{-1}]$ .

- (a) We define the  $\mathbb{Z}$ -linear map, called the *bar involution*,  $-: A \rightarrow A$  given by sending  $q \mapsto q^{-1}$
- (b) The Hecke algebra  $\mathcal{H}$  admits an extension of the bar involution, say  $\iota: \mathcal{H} \rightarrow \mathcal{H}$ , given by

$$\iota(T_w) := T_{w^{-1}}^{-1}$$

for any  $w \in \mathcal{W}$ . For convenience, we will overload notation and write

$$\overline{T_w} := \iota(T_w)$$

Note that  $\iota(H_s) = v^{-1}T_s^{-1} = v^{-1}(v^2T_s - 1 + v^2) = H_s - v^{-1} + v = H_s^{-1}$  and, similarly,  $\iota(H_w) = H_{w^{-1}}^{-1}$ . Then, we have an  $\iota$ -invariant of the form

$$C_s := q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_{id} = H_s - v^{-1}H_{id}$$

We can also introduce a similar  $\iota$ -invariant of the form

$$C'_s := H_s + vH_{id}$$

This justifies why we introduced the  $H$ -basis in Proposition 2.4. In [Hum90, p 158], it is noted that it could be tempting to construct further  $\iota$ -invariants by taking products of these  $C_s$  elements. However, if one has a word  $sts = tst$  with  $s, t \in \mathcal{W}$  both simple reflections and  $\ell(sts) = 3 = \ell(tst)$ , then one can check that  $C_s C_t C_s \neq C_t C_s C_t$ . However, if we compute (still assuming  $\ell(sts) = 3$ )

$$C_s C_t C_s - C_t = q^{-\frac{3}{2}}(T_{sts} - q(T_{st} - T_{ts}) + q^2(1 + q^{-1})(T_s + T_t) - q^3(1 + 2q^{-1})T_{id})$$

we get an  $\iota$ -invariant expression where the  $s$  and  $t$ 's are interchangeable. Similarly, we can compute

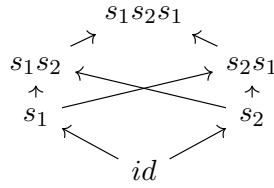
$$C'_s C'_t C'_s - C'_s = H_{sts} + v(H_{ts} + H_{st}) + v^2(H_s + H_t) + v^3H_{id}$$

since  $vH_s^2 = H_s - v^2H_s + vH_{id}$  and so  $vH_s^2 - C'_s = -v^2H_s$ .

This illustrates the problem more generally we wish to solve. For every  $w \in \mathcal{W}$ , we want to associate an  $\iota$ -invariant element,  $C_w$ , which is a linear combination of  $T_x$  for  $x \leq w$ , thus giving us a basis. In order to follow [Soe97], we will actually produce elements  $C'_w$  as a linear combination of  $H_x$ 's, but the idea remains the same. To do this, we first recall a partial ordering on the Weyl group.

**3.2. Definition.** For  $u, v \in \mathcal{W}$ , we say  $u \leq v$  in the (strong) *Bruhat order* on  $\mathcal{W}$  if some substring of some reduced word for  $v$  is a reduced word for  $u$ .

**3.3. Example.** Let  $\mathcal{W} = \mathfrak{S}_3 = \langle s_1, s_2 \rangle$ . Then, the Bruhat order is given by the following poset.



**3.4. Theorem.** [Soe97, Theorem 2.1] *For each  $w \in \mathcal{W}$ , there exists a unique element  $C'_w \in \mathcal{H}$  having the following properties:*

- (a)  $\iota(C'_w) = C'_w$
- (b)  $C'_w \in H_w + \sum_{x < w} v\mathbb{Z}[v]H_x$  where  $x < w$  in the (strong) Bruhat order.

Then, one may wish to construct

**3.5. Example.** (a) From the above, we already see that if  $s \in \mathcal{W}$  is a simple reflection, then it must be that

$$C'_s = H_s + vH_{id}$$

(b) We can compute the basis for  $\mathfrak{S}_3 = \langle s_1, s_2 \rangle$  by hand. We know that the simple reflections must be of the form.

$$\begin{aligned} C'_{s_1} &= H_{s_1} + vH_{id} \\ C'_{s_2} &= H_{s_2} + vH_{id} \end{aligned}$$

Then, to form  $\iota$ -invariants of length 2, we check

$$C'_{s_1} C'_{s_2} = H_{s_1 s_2} + v(H_{s_1} + H_{s_2}) + v^2 H_{id}$$

is  $\iota$ -invariant. If we apply  $\iota$  to this, we get

$$\begin{aligned} \iota(C'_{s_1} C'_{s_2}) &= H_{s_1 s_2} + (v - v^{-1})(H_{s_1} + H_{s_2}) + (v - v^{-1})^2 + v^{-1}(H_{s_1} + H_{s_2} + 2(v - v^{-1})) + v^{-2} \\ &= H_{s_1 s_2} + v(H_{s_1} + H_{s_2}) + (v - v^{-1})^2 + 2(1 - v^{-2}) + v^{-2} \\ &= H_{s_1 s_2} + v(H_{s_1} + H_{s_2}) + v^2 \end{aligned}$$

So, by uniqueness, it must be  $C'_{s_1 s_2} = C'_{s_1} C'_{s_2}$ . A similar computation gives  $C'_{s_2 s_1}$ . For length 3, we already computed above that

$$C'_{s_1 s_2 s_1} = C'_{s_1} C'_{s_2} C'_{s_1} - C'_{s_1} = H_{s_1 s_2 s_1} + v(H_{s_1 s_2} + H_{s_2 s_1}) + v^2(H_{s_1} + H_{s_2}) + v^3$$

*Proof of Theorem 3.4.* We have already established the formula for  $C'_s$  for  $s$  a simple reflection. Now, we compute

$$H_x C'_s = \begin{cases} H_{xs} + v H_x & \text{if } xs > x; \\ H_{xs} + v^{-1} H_x & \text{if } xs < x \end{cases}$$

where the first case is immediate from the definition of the Hecke algebra and the second case is a straightforward application of Lemma 2.6. To show existence, we proceed by induction on the Bruhat order. Certainly,  $C'_{id} = H_{id} = 1$ . Now, let  $x \in \mathcal{W}$  be given and suppose we know  $C'_y$  exists for all  $y < x$ . If  $x \neq id$ , we can find a simple reflection  $s$  such that  $xs < x$  and by induction, we get

$$C'_{xs} C'_s = H_x + \sum_{y < x} h_y H_y$$

for some  $h_y \in \mathbb{Z}[v]$ . Then, we say

$$C'_x = C'_{xs} C'_s - \sum_{y < x} h_y(0) C'_y.$$

$C'_x$  is  $\iota$ -invariant because it is a sum of  $\iota$ -invariant elements and it lies in  $H_x + \sum_{y < x} v\mathbb{Z}[v]H_y$  since, if  $C'_y = H_y + \sum_{z < y} h_{z,y} H_x$  for  $h_{z,y} \in v\mathbb{Z}[v]$ , then

$$C'_x = H_x + \sum_{y < x} \left( (h_y - h_y(0)) H_y - \sum_{z < y} h_y(0) h_{z,y} H_z \right).$$

For uniqueness, we prove the following.

**3.6. Lemma.** *If  $H \in \sum_y v\mathbb{Z}[v]H_y$  is  $\iota$ -invariant, then  $H = 0$ .*

We have  $H_z \in C'_z + \sum_{y < z} \mathbb{Z}[v, v^{-1}]C'_y$  for the  $C'_x$  elements described earlier in the proof by the unitriangularity condition. Now, if  $H = \sum_y h_y H_y$  and we choose  $z$  maximal such that  $h_z \neq 0$ , then  $\iota(H) = H$  implies that  $\overline{h_z} = h_z$ . However, this contradicts  $h_z \in v\mathbb{Z}[v]$ , so it must be that  $H = 0$ .

Thus, if there were two  $\iota$ -invariant elements  $C'_w$  and  $D'_w$  satisfying the hypotheses of Theorem 3.4, then it must be that  $C'_w - D'_w \in v\mathbb{Z}[v]$  is  $\iota$ -invariant, but the lemma shows that  $C'_w - D'_w = 0$ . Thus, uniqueness is established.  $\square$

**3.7. Definition.** For  $x, y \in \mathcal{W}$ , we define the *Kazhdan-Lusztig polynomials*  $h_{y,x} \in \mathbb{Z}[v, v^{-1}]$  by the equality

$$C'_x = \sum_y h_{y,x} H_y$$

**3.8. Remark.** These polynomials are related to the Kazhdan-Lusztig polynomials in [KL79], denoted  $P_{y,x}$ , by

$$h_{y,x} = v^{\ell(x) - \ell(y)} P_{y,x}$$

**3.9. Proposition.** *Let  $\mathcal{W}$  be finite,  $w_\circ \in \mathcal{W}$  be the longest element, and  $r = \ell(w_\circ)$  its length. Then, we have  $C'_{w_\circ} = \sum_{y \in \mathcal{W}} v^{r-\ell(y)} H_y$ .*

**3.1. Further Properties of Kazhdan-Lusztig Polynomials.** Since their introduction, the Kazhdan-Lusztig polynomials have been an area of intense research. Now, much more is known than when they were first introduced.

**3.10. Proposition.** [KL80] *For any Weyl group  $\mathcal{W}$  and  $x, y \in \mathcal{W}$ , we have that the coefficients  $a_i$  occurring in*

$$P_{y,x}(q) = \sum_i a_i q^i$$

*satisfy  $a_i \in \mathbb{Z}_{\geq 0}$ .*

**3.11. Remark.** This has been proved by [EW14] for general Coxeter systems.

In [KL79], the following was conjectured. It was proven in [BB81] and [BK81].

**3.12. Proposition.** *Given a semisimple Lie algebra  $\mathfrak{g}$  with Weyl group  $\mathcal{W}$ , for each  $w \in \mathcal{W}$ , let  $M_w$  be the Verma module with highest weight  $-w(\rho) - \rho$  and let  $L_w$  be its unique irreducible quotient. Then, we have the equivalent identities*

$$\begin{aligned} (a) \quad \text{ch } L_w &= \sum_{y \leq w} (-1)^{\ell(w)+\ell(y)} P_{y,w}(1) \text{ch } M_y \\ (b) \quad \text{ch } M_w &= \sum_{y \leq w} P_{w_\circ w, w_\circ y}(1) \text{ch } L_y \end{aligned}$$

*where  $w_\circ$  is the longest element of  $\mathcal{W}$ .*

Finally, there exists a geometric interpretation of the Kazhdan-Lusztig polynomials using perverse sheaves.

**3.2. Historical Note.** Kazhdan and Lusztig were originally interested in using the Kazhdan-Lusztig basis to construct representations of the Hecke algebra, but their significance has extended far beyond this goal. Our exposition here does not follow [KL79] and our definitions do not match those in [KL79], although it is straightforward to translate between [KL79] and these notes. The proof given for existence and uniqueness here is simpler; notably, this exposition does not include the  $R$ -polynomials. Such a proof can be found in [Hum90].

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