

**Artin-Wedderburn Theory**  
**Notes inspired by a class taught by Brian**  
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## 1. Krull-Schmidt

We start with an underrated lemma.

1.1. THEOREM (Fitting's Lemma). *Let  $0 \neq M$  be a left  $R$ -module with a composition series and  $f: M \rightarrow M$  a module homomorphism. Since  $M$  has a composition series, there exists an  $n$  such that*

$$\operatorname{im} f \supseteq \operatorname{im} f^2 \supseteq \cdots \supseteq \operatorname{im} f^n = \operatorname{im} f^{n+1} = \cdots$$

and

$$\ker f \subseteq \ker f^2 \subseteq \cdots \subseteq \ker f^n = \ker f^{n+1} = \cdots$$

Define  $f^\infty(M) := \operatorname{im} f^n$  and  $f^{-\infty}(M) := \ker f^n$ . Then,

$$M \cong f^\infty(M) \oplus f^{-\infty}(M)$$

PROOF. Let  $x \in f^\infty(M) \cap f^{-\infty}(M)$ . Then,  $x = f^n(y) = f^{2n}(z)$  and  $f^n(x) = 0$ . Then,  $0 = f^n(x) = f^{2n}(y)$ . Thus,  $y \in \ker f^{2n} = \ker f^n$ , so  $x = f^n(y) = 0$ .

Now, let  $x \in M$ . Then,  $x = [x - f^n(y)] + f^n(y)$  where  $f^n(y) \in f^\infty(M)$  and  $y$  such that  $f^{2n}(y) = f^n(x)$ . Then,

$$f^n(x - f^n(y)) = f^n(x) - f^{2n}(y) = f^n(x) - f^n(x) = 0$$

Thus,  $x - f^n(y) \in f^{-\infty}(M)$  and  $f^n(y) \in f^\infty(M)$ . □

Now, we seek to prove the Krull-Schmidt theorem.

1.2. THEOREM (Krull-Schmidt Theorem). *Let  $0 \neq M$  be an  $R$ -module which is both Artinian and Noetherian. Suppose*

$$\begin{aligned} M &= M_1 \oplus \cdots \oplus M_h \\ &= N_1 \oplus \cdots \oplus N_k \end{aligned}$$

where  $M_i, N_j$  are indecomposable. Then,  $h = k$  and, up to rearranging terms,  $M_i \cong N_i$  for  $i = 1, \dots, h$ .

To prove this theorem, we need the following lemmas.

1.3. LEMMA. *Let  $M, N$  be  $R$ -modules with  $N$  decomposable and homomorphisms  $\nu: M \rightarrow N$  and  $\mu: N \rightarrow M$ . If  $\mu\nu: M \rightarrow M$  is an automorphism, then  $\mu$  and  $\nu$  are isomorphisms.*

PROOF. Let  $\tau = \mu\nu$  be an automorphism. Now, replace  $\mu$  with  $\tau^{-1}\mu$ . Then, we have split short exact sequence

$$0 \longrightarrow M \xrightarrow[\nu]{\mu} N \longrightarrow N/M \longrightarrow 0$$

and so  $N \cong M \oplus N/M$ . However,  $N$  is indecomposable, so  $N = M$ . □

1.4. LEMMA. *Let  $0 \neq M$  be indecomposable, Artinian, and Noetherian. Let  $\tau_1, \dots, \tau_r \in \text{End}_R M$  satisfy  $\tau_1 + \dots + \tau_r \in \text{Aut}_R(M)$ . Then, at least one  $\tau_i$  is an automorphism of  $M$*

PROOF. Since  $\tau = \tau_1 + (\tau_2 + \dots + \tau_r)$ , it suffices to prove for the sum of 2 endomorphisms. Let  $\psi = \tau_1 + \tau_2$  be an automorphism. Then,  $\phi_1 = \psi^{-1}\tau_1$  and  $\phi_2 = \psi^{-1}\tau_2$ . We then have  $\phi_1 + \phi_2 = 1_M$  and so, since  $\phi_1 = 1_M - \phi_2$ , we get  $\phi_1\phi_2 = \phi_2\phi_1$ . Now, we can apply the binomial theorem to get

$$0 \neq 1_M^m = (\phi_1 + \phi_2)^m = \sum_{k=1}^m \binom{m}{k} \phi_1^k \phi_2^{m-k}$$

So, at least one of the  $\phi_i$ 's cannot be nilpotent. Without loss of generality, assume  $\phi_1$  is not nilpotent. Since  $\phi_1$  is not nilpotent and  $M$  has a composition series, we can apply Fitting's Lemma to get

$$M \cong \phi_1^\infty(M) \oplus \phi_1^{-\infty}(M)$$

However,  $M$  is indecomposable and  $\phi_1^\infty(M) \neq 0$ , so  $M = \phi_1^\infty(M)$  and  $\ker \phi_1 \subseteq \phi_1^{-\infty}(M) = 0$ , so  $\phi_1$  is an automorphism. Therefore,  $\tau_1$  is an automorphism.  $\square$

We are now ready to prove the Krull-Schmidt theorem.

PROOF OF KRULL-SCHMIDT. Let  $0 \neq M$  be an Artinian and Noetherian  $R$ -module. Suppose

$$\begin{aligned} M &= M_1 \oplus \dots \oplus M_h \\ &= N_1 \oplus \dots \oplus N_k \end{aligned}$$

where the  $M_i, N_j$  are indecomposable. We will proceed by induction on  $h$ . If  $h = 1$ , then  $M$  is already indecomposable and we are done. Now, for the inductive step, let  $\mu: M \rightarrow M_i$  be the projection onto the  $i$ th summand and, similarly, let  $\nu_j: M \rightarrow N_j$  be a projection. Then, we have  $\mu_i\mu_j = \delta_{ij}\mu_i$  and  $\nu_i\nu_j = \delta_{ij}\nu_j$ . Now, we have the following identity

$$\mu_1 = \mu_1 \circ 1_M = \mu_1(\nu_1 + \dots + \nu_k)$$

Now, let  $\bar{\nu}_j = \nu_j|_{M_i}: M_i \rightarrow M$ . If we restrict the above identity to  $M_1$ , we get

$$1_{M_1} = \mu_1\bar{\nu}_1 + \dots + \mu_1\bar{\nu}_k \in \text{Aut } M_1$$

Thus, we can use the lemma above to assume that  $\mu_1\bar{\nu}u_1 \in \text{Aut } M_1$ . Furthermore, by the other lemma, since  $\mu_1\bar{\nu}_1$  is an automorphism, we get that  $\bar{\nu}u_1$  is an isomorphism, as is  $\mu_1: N_1 \rightarrow M_1$ . Now, consider

$$N_1 + M_2 + \dots + M_n \leq M$$

We wish to show this sum is actually direct. Assume there is  $n_1 \in N_1$  and  $m_2 \in M_2, \dots, m_h \in M_h$  such that

$$n_1 + m_2 + \dots + m_h = 0.$$

We can just apply  $\mu_1$  to this sum above to get

$$0 = \mu_1(n_1 + m_2 + \cdots + m_h) = \mu_1(n_1)$$

However,  $\mu_1$  is an isomorphism between  $N_1$  and  $M_1$ , so  $\mu_1(n_1) = 0 \implies n_1 = 0$ . So, furthermore, it must be that  $m_2 = \cdots = m_h = 0$  by the direct sum construction, so

$$M' := N_1 + M_2 + \cdots + M_h = N_1 \oplus M_2 \oplus \cdots \oplus M_h \leq M$$

Thus, we wish to finally show that  $M' = M$ . Let

$$\rho: M_1 \oplus \cdots \oplus M_h \rightarrow M'$$

be given by  $\rho = \bar{\nu}_1 + \mu_2 + \cdots + \mu_h$ . Note that  $\rho$  is an isomorphism since  $\bar{\nu}_1$  is an isomorphism from  $M_1 \rightarrow M$ . Since  $M$  is Artinian, there is some  $a \gg 0$  such that

$$\rho^{a+1}(M) = \rho^a(M)$$

Hence, given  $m \in M$ , there is some  $m' \in M$  such that

$$\begin{aligned} \rho^{a+1}(m') &= \rho^a(m) \implies \rho^a(m - \rho(m')) = 0 \\ &\implies m - \rho(m') \in \ker \rho^a = \{0\} \end{aligned}$$

Thus,  $M' = M$  and  $\rho$  is an automorphism of  $M$ . Furthermore,  $\rho(M_i) = M_i$  for  $i = 2, \dots, h$ ,  $\rho(M_1) = N_1$ , and

$$N_1 \oplus M_2 \oplus \cdots \oplus M_h = \rho(M) = M = M_1 \oplus \cdots \oplus M_h$$

Hence,

$$\begin{aligned} M_2 \oplus \cdots \oplus M_h &\cong M/M_1 \\ &\cong \rho(M)/\rho(M_1) \\ &\cong \rho(M)/N_1 \\ &\cong (N_1 \oplus \cdots \oplus N_k)/N_1 \\ &\cong N_2 \oplus \cdots \oplus N_k \end{aligned}$$

Thus, by induction,  $h - 1 = k - 1 \implies h = k$  and  $M_i \cong N_i$  for  $i = 2, \dots, h$ , and we already know  $M_1 \cong N_1$  by above.  $\square$

**1.5. COROLLARY.** *Let  $N$  be a direct summand of  $M$  as in the Krull-Schmidt theorem. Then, there exist a subset of  $\{M_i\}$  indecomposable submodules  $M_{i_1}, \dots, M_{i_t}$  such that*

$$N \cong M_{i_1} \oplus \cdots \oplus M_{i_t}$$

## 2. Nakayama's Lemma: A Hard to Remember Lemma

While one statement of Nakayama's Lemma may be easy to remember, there are actually many special cases of this lemma and so it has a reputation of being hard to remember. To understand and prove Nakayama's Lemma, we must first lay the groundwork of Jacobson radicals.

2.1. DEFINITION. The *Jacobson radical* (often just called the radical)  $R$  of a ring  $A$  is

$$R = \{x \in A \mid xM = 0, \forall \text{ simple } A\text{-modules } M\}$$

2.2. DEFINITION. A ring  $A$  is called *semisimple* when its radical  $R = 0$ .

2.3. THEOREM. *The radical  $R$  of  $A$  is a two-sided ideal of  $A$  and  $A/R$  is semisimple.*

PROOF. If  $M$  is a simple  $A$ -module, then  $M$  is also an  $A/R$ -module. Conversely, any  $A/R$ -modules are naturally  $A$ -modules. Thus, if there were an  $x + R \in A/R$  such that  $(x + R)M = xM = 0$  for all simple  $A/R$ -modules  $M$ , then  $x$  would also annihilate all simple  $A$ -modules and thus  $x \in R$ , so  $x + R = 0 + R$  and thus the radical of  $A/R$  is trivial.  $\square$

2.4. THEOREM. *Let  $R$  be the radical of a ring  $A$ . Then,*

$$R = \bigcap_{M \text{ simple}} \text{Ann}_A(M) = \bigcap_{\substack{I \triangleleft A \\ \text{Maximal left ideal}}} I$$

*That is,  $R$  is the intersection of all maximal left ideals of  $A$ .*

PROOF. Let  $x \in R$  and let  $I$  be a maximal left ideal. We want to show that  $x \in I$ . Since  $I$  is maximal,  $A/I$  is a simple left  $A$ -module. thus,  $x \in \text{Ann}(A/I)$ . Thus,

$$\begin{aligned} 0 &= x(1 + I) \implies x \in I \\ &\implies x \in \bigcap I \\ &\implies R \subseteq \bigcap I \end{aligned}$$

Conversely, let  $x \in \bigcap I$ . Let  $M$  be a simple module  $0 \neq m \in M$ . We show  $xm = 0$ . We have

$$\begin{aligned} A &\xrightarrow{\theta} M \\ a &\mapsto am \end{aligned}$$

Then,  $M \cong A/I$  where  $I = \text{Ann}_A(m)$ . Thus,  $xm = 0$ .  $\square$

2.5. THEOREM. *Given a ring  $A$ ,*

$$\text{rad } A = \{x \in A \mid 1 - axb \in A^\times, \forall a, b \in A\}$$

*where  $A^\times$  is the group of units of  $A$ .*

PROOF. Let  $x \in \text{rad } A$  with  $a, b \in A$ . Then,  $\text{rad } A \trianglelefteq A$  by 2.3, so  $axb \in \text{rad } A$ . Now, write  $y := axb$ . We wish to show  $1 - y$  is a unit.

If  $A(1 - y) \neq A$ , let  $I$  be a maximal left ideal containing  $A(1 - y)$ . Then, by 2.4,  $\text{rad } A \subseteq I$ , so  $y \in \text{rad } A \subseteq I$  and  $1 - y \in A$ . This would give that  $1 = y + (1 - y) \in I \implies I = A$ , which is a contradiction of the maximality

of  $I$ . So,  $A(1 - y) = A$  for all  $y \in \text{rad } A$ .

Now, since  $A(1 - y) = A$ , there is a  $t$  such that  $t(1 - y) = 1$ . So, we wish to show  $(1 - y)t = 1$ . Rearranging our expression  $t(1 - y) = 1$ , we get

$$\begin{aligned}
1 - t = -ty \in \text{rad } A &\implies A(1 - (1 - t)) = A && \text{by above since } A(1 - y) = A, \forall y \in \text{rad } A \\
&\implies At = A \\
&\implies \exists u \in A \text{ such that } ut = 1 \\
&\implies u = ut(1 - y) = 1 - y && \text{since } t(1 - y) = 1 \text{ and } ut = 1 \\
&\implies 1 - y \in A^\times
\end{aligned}$$

Thus, we have shown one containment.

Now, let  $x \in A$  be such that  $1 - axb \in A^\times$  for all  $a, b \in A$ . We want to prove  $x \in \text{rad } A$ . We show  $xM = 0$  if  $M$  is a simple  $A$ -module. Let  $0 \neq m \in M$ . Assume  $xm \neq 0$ . Then,  $M = Axm$  since  $M$  is simple so  $(xm) = M$ . Thus,  $m = axm$  for some  $a \in A$  and so  $(1 - ax)m = 0$ . However,  $1 - ax \in A^\times$ , so  $m = 0$ , which is a contradiction. Thus,  $xM = 0$  and so  $x \in \text{rad } A$ .  $\square$

**2.6. THEOREM** (Nakayama's Lemma). [[Jac89](#), p 415] *Let  $A$  be a ring and let  $M$  be a finitely-generated  $A$ -module. If  $(\text{rad } A)M = M$ , then  $M = 0$ .*

**PROOF.** Let  $m_1, m_2, \dots, m_k$  be a minimum generating set of  $M$ . Given  $m \in M$ , there exists  $a_1, \dots, a_k \in A$  such that

$$m = a_1m_1 + \dots + a_km_k$$

Take  $m = m_1$ . Then,

$$(1 - a_1)m_1 = a_2m_2 + \dots + a_km_k$$

If  $(\text{rad } A)M = M$ , we can assume each  $a_i \in \text{rad } A$ , we can assume each  $a_i \in \text{rad } A$  since  $\text{rad } A$  is a 2-sided ideal by theorem 2.3. Thus, by 2.5,  $1 - a_1$  is a unit, so

$$m_1 = (1 - a_1)^{-1}a_2m_2 + \dots + (1 - a_2)^{-1}a_km_k \in \langle m_2, \dots, m_k \rangle$$

However, this contradicts the minimality of the the generating set.  $\square$

**2.7. COROLLARY.** *Let  $M$  be a finitely generated left  $A$ -module and let  $N \leq M$  be a submodule such that*

$$N + (\text{rad } A)M = M$$

*Then,  $N = M$*

**PROOF.** Given the above,  $\text{rad } A(M/N) = M/N$  simply by quotienting both sides of the equality by  $N$ . Thus, by Nakayama's lemma,  $M/N = 0$  and so  $M = N$ .  $\square$

2.8. COROLLARY. *Let  $J$  be a maximal ideal in a ring  $A$ . Then,  $\text{rad } A \subseteq J$ .*

PROOF. Assume not. Then  $\text{rad } A + J = A \implies J = A$  by the corollary above, which is a contradiction to  $J$  being a maximal ideal.  $\square$

### 3. Completely Reducible Modules

Understanding all the modules of a ring, in general, is an incredibly difficult problem. However, usually the first step in such a program is understanding the simple (or irreducible) modules. Understanding these modules would then give a complete understanding of completely reducible modules.

3.1. DEFINITION. Let  $A$  be a ring. A left  $A$ -module  $M$  is called *completely reducible* if, given any submodule  $N \subseteq M$ , there exists a submodule  $N'$  such that

$$M = N \oplus N'$$

However, there are equivalent notions of completely reducible.

3.2. THEOREM. *Assume  $M$  is a left  $A$ -module. Then, the following are equivalent.*

- (a)  *$M$  is completely reducible.*
- (b)  *$M$  is a direct sum of irreducible submodules, that is*

$$M \cong \bigoplus_{i \in I} L_i$$

*where  $I$  is some indexing set and  $L_i$  is simple.*

- (c)  *$M$  is a sum of irreducible submodules, that is*

$$M \cong \sum_{i \in I} L_i$$

*where  $I$  is some indexing set and  $L_i$  is simple.*

To prove this result, we use the following lemma.

3.3. LEMMA. *If an  $A$ -module  $M$  is completely reducible, so is any submodule  $N \leq M$ .*

PROOF OF LEMMA. Consider submodule  $N$  of completely reducible  $A$ -module  $M$ . Then, by definition, there exists an  $N'$  such that

$$M \cong N \oplus N'.$$

Now, let  $S \leq N$ . Then, we also have  $S \leq M$  and so there is a submodule  $T$  such that

$$M \cong S \oplus T.$$

Thus, we have

$$N = (S \oplus T) \cap N$$



$$= S \oplus (T \cap N)$$

since  $S \leq N$ . Since  $T \cap N \leq N$ , then  $N$  is completely irreducible.  $\square$

3.4. LEMMA. *Let  $M$  be a completely reducible  $A$ -module. Then, every nontrivial submodule  $N$  of  $M$  contains an irreducible submodule.*

PROOF OF LEMMA. Let  $N \leq M$  and  $0 \neq n \in N$ . Consider the collection

$$S = \{N' \leq N \mid n \notin N'\}$$

We note that  $S$  is nonempty since  $(0) \in S$ . Thus, since  $S$  is nonempty, it must contain a maximal element, say  $N_0$ . By the lemma above, we know that  $N$  is completely reducible and so  $N = N_0 \oplus N_1$  for some submodule  $N_1 \leq N$ .  $N_1$  must be irreducible because, if not, then there would be a proper submodule  $N_2 \leq N_1$  and  $N_1 = N_2 \oplus N_3$  for some submodule  $N_3 \leq N_1$ . However, this would give us

$$N = N_0 \oplus N_2 \oplus N_3 \implies \text{either } n \notin N_0 + N_2 \text{ or } n \notin N_0 + N_3$$

since  $(N_0 + N_2) \cap (N_0 + N_3) = N_0$ . Such a result contradicts the maximality of  $N_0$  in  $S$ , and so it must be that  $N_1$  is irreducible.  $\square$

PROOF OF THEOREM. We first assume that  $M \neq (0)$ , otherwise the theorem is immediately true.

((a)  $\implies$  (b)). Let  $\{M_i \mid i \in I\}$  be the collection of all irreducible submodules of  $M$  and let

$$T = \left\{ J \subseteq I \mid \sum_{j \in J} M_j \text{ is direct} \right\}$$

$T$  is nonempty since it contains at least singleton sets are in  $T$  and any union of an ascending chain of elements in  $T$  is in  $T$ . Thus, we can apply Zorn's lemma to get a maximal element of  $T$ , say  $J_0$ . Now, let

$$M' := \bigoplus_{j \in J_0} M_j$$

be properly contained in  $M$ . Then, by complete reducibility of  $M$ , we get

$$M = M' \oplus M''$$

for  $(0) \neq M'' \leq M$ . However, since  $M''$  must be completely reducible, by the lemma above,  $M''$  must contain an irreducible submodule, say  $M_{i_0}$ . Then,

$$\left( \bigoplus_{j \in J_0} M_j \right) + M_{i_0} \text{ is direct}$$

and thus we have violated the maximality of  $J_0$ .

((b)  $\implies$  (c)). This result is immediate by the definition of direct sum.

((c)  $\implies$  (a)). Let  $N \leq M$  and let  $N'$  be a maximal submodule with respect to the property that  $N \cap N' = (0)$ . Assume  $M \neq N \oplus N'$ . Then, there is an  $m \in M, m \notin N \oplus N'$ . However, since  $M$  is a sum of irreducible submodules,  $m = m_1 + \cdots + m_k$  where each  $m_i$  belongs to an irreducible summand. Thus, at least one  $m_i \notin N \oplus N'$  since  $m \notin N \oplus N'$  and thus  $M_i \not\subseteq N \oplus N'$ . However,  $M_i$  is irreducible and so  $M_i \cap (N \oplus N') = (0)$ . Thus,  $N' \subsetneq N' + M_i$  and  $(N' + M_i) \cap N = (0)$ , thus violating the maximality of  $N'$ . Thus,  $M = N \oplus N'$ .  $\square$

**3.5. PROPOSITION.** *Let  $M$  be a completely reducible module for a ring  $A$ . Then,  $M$  satisfies the ascending chain condition if and only if  $M$  satisfies the descending chain condition.*

#### 4. Nilpotent and non-Nilpotent Ideals in Artinian Rings

To get to the Artin-Wedderburn Theorem, we must have an understanding of idempotents in Artinian rings, which are intimately (non)-related to nilpotent ideals. We wish to culminate in a theorem that says any non-nilpotent left ideal in a left-artinian ring must contain an idempotent. First, we present a theorem similar to Schur's Lemma, but for Noetherian rings.

**4.1. THEOREM.** *Let  $M$  be a (left) Noetherian  $A$ -module for ring  $A$ . If  $f \in \text{End}_A(M)$  is surjective, then  $f$  is an isomorphism*

**PROOF.** If  $f$  is surjective, then  $\text{im } f = M$  and so  $f$  is not nilpotent. Thus, by Fitting's Lemma

$$M = f^\infty(M) \oplus f^{-\infty}(M)$$

we seek to show  $\ker f = 0$ . For some integer  $n$ ,  $\ker f^n = \ker f^{n+1}$ . Let  $x \in \ker f$ . Then, since  $f$  is surjective,

$$f^n(M) = f^{n+1}(M) = M$$

Thus,  $x = f^n(y)$  for some  $y \in M$ . We then see

$$0 = f(x) = f^{n+1}(y) \implies y \in \ker f^{n+1} = \ker f^n \implies 0 = f^n(y) = x$$

So, we have that  $\ker f = 0$  and thus  $f$  is injective.  $\square$

**4.2. THEOREM.** *Let  $A$  be a left Noetherian ring. If  $a, b \in A$  be such that  $ab = 1$ , then  $ba = 1$  and  $a, b \in A^\times$ .*

**PROOF.** Let  $M = {}_A A$ . Then,  $M$  is a (left) Noetherian  $A$ -module. Then, since  $ab = 1$ ,

$$A = Aab \subseteq Ab \subseteq A$$

and so  $Ab = A$ . Thus, we have a map

$$\begin{aligned} f: M &\twoheadrightarrow M \\ x &\mapsto xb \end{aligned}$$

and so, by the theorem above,  $f$  is an isomorphism. However,

$$\begin{aligned} f(1 - ba) &= (1 - ba)b \\ &= b - bab \\ &= b - b && \text{since } ab = 1 \\ &= 0 \end{aligned}$$

Thus,  $1 - ba = 0 \implies ba = 1 = ab$ .  $\square$

4.3. DEFINITION. An *idempotent* of a ring  $A$  is an element  $0 \neq e \in A$  such that  $e^2 = e$ .

One advantage of idempotents is that they allow us to “project” the ring onto “orthogonal” components, that is, given a ring  $A$  with idempotent  $e$ , then, as a module over itself

$${}_A A \cong {}_A A e \oplus {}_A A(1 - e)$$

This also tells us the following

4.4. REMARK.  ${}_A A e$  is projective as an  $A$ -module since it is the direct summand of a free  $A$ -module.

4.5. DEFINITION. We call a left or right ideal,  $I$ , *nilpotent* if there is an  $m \in \mathbb{N}$  such that  $I^m = \{0\}$ .

4.6. EXAMPLE. Consider  $R = \mathbb{Z}/p^n\mathbb{Z}$  where  $p$  is a prime number. Then, since  $R$  is a PID, every proper ideal is generated by some  $p^k + (p^n)$  and is nilpotent. The only idempotent of  $R$  is  $1 + (p^n)$ .

4.7. THEOREM. Let  $N$  be a nilpotent left ideal of a ring  $A$ . Let  $x \in A$  be non-nilpotent such that  $x^2 - x \in N$ . Then, the left ideal  $Ax$  has an idempotent  $y$ .

The idea to proving this theorem is to take  $A$  and factor out  $N$ . Then, one can find an idempotent  $y$  such that, under the quotient map  $q: A \rightarrow A/N$ ,  $q(x) = q(y)$ .

PROOF. Assume  $N^k = 0$  for some positive integer  $k$ . Let  $m_1 := x^2 - x \in N$ . If  $m_1 = 0$ , then  $x^2 - x = 0$  so  $x$  is an idempotent itself and we can take  $y = x$ .

Assume  $m_1 \neq 0$ . Then, let

$$x_1 := x + m_1 - 2xm_1 \in Ax \text{ since } m_1 \in Ax$$

Note that  $x_1, x, m_1$  all commute. Then, note that  $x_1$  is not nilpotent as well and  $x^2 = x + m_1$ . Consider

$$\begin{aligned} x_1^2 - x_1 &= (x + m_1 - 2xm_1)^2 - (x + m_1 - 2xm_1) \\ &= x^2 + xm_1 - 2x^2m_1 + m_1x + m_1^2 - 2xm_1^2 - 2x^2m_1 - 2xm_1^2 + 4x^2m_1^2 - x - m_1 + 2xm_1 \end{aligned}$$

$$\begin{aligned}
&= 4x^2m_1^2 - 4x^2m_1 - 4xm_1^2 + x^2 + 4xm_1 + m_1^2 - x - m_1 \\
&= 4x^2m_1^2 - 4x^2m_1 - 4xm_1^2 + 4xm_1 + m_1^2 \\
&= (4x^2 - 4x)m_1^2 + (-4x^2 + 4x)m_1 + m_1^2 \\
&= (4x^2 - 4x - 4m_1)m_1^2 + (-4x^2 + 4x + 4m_1)m_1 + m_1^2 + 4m_1^3 - 4m_1^2 \\
&= 4m_1^3 - 3m_1^2
\end{aligned}$$

Then, take

$$m_2 = 4m_1^3 - 3m_1^2 \quad x_2 = x_1 + m_2 - 2x_1m_2.$$

and note that  $m_2$  contains  $m_1^2$  as a factor. Thus, we can successively construct non-nilpotent elements  $x_1, x_2, \dots$  in  $Ax$  such that  $x_i^2 - x_i$  contains  $m_1^{2^i}$  as a factor and commutes with  $x$ . Since  $m_1$  is nilpotent, then  $m_1^{2^i} = 0$  for sufficiently large  $i$  and so  $x_i^2 - x_i = 0$  for some sufficiently large  $i$ . Therefore, for that sufficiently large  $i$ ,  $x_i$  is nilpotent.  $\square$

4.8. REMARK. Note that any nilpotent ideal cannot contain an idempotent element. However, the following theorem gives us a (useful!) converse to that fact.

4.9. THEOREM. *Let  $L$  be a non-nilpotent left ideal in a left-artinian ring  $A$ . Then,  $L$  contains an idempotent  $e$ .*

PROOF. We seek to use the theorem above to get such an idempotent by finding a non-nilpotent  $x \in A$  such that  $x^2 - x$  is in a nilpotent ideal.

Choose a minimal left ideal  $L_1 \subseteq L$  which is not nilpotent. Then,  $0 \neq L_1L_1 \subseteq L_1$  is not nilpotent, so  $L_1L_1 = L_1$  by the minimality of  $L_1$ .

Let  $I$  be a left ideal contained in  $L_1$  such that  $L_1I \neq 0$  and minimal with respect to this property. Let  $a \in I$  be such that  $L_1a \neq 0$ . Then,

$$L_1L_1a = L_1^2a = L_1a \neq 0 \text{ and } L_1a \subseteq I$$

Hence,  $I = L_1a$  by the minimality of  $I$ . Thus, there is an  $x \in L_1$  such that  $a = xa$ . Hence,

$$0 \neq a = xa = x^2a = \dots = x^ka = \dots$$

Therefore,  $x$  is not nilpotent.

Let  $N = \{b \in L_1 \mid ba = 0\}$ . This is a left ideal contained in  $L_1$ . Since  $xa = x^2a$ ,

$$(x - x^2)a = 0 \implies x - x^2 \in N$$

Also,  $L_1a \neq 0$  so  $N \subsetneq L_1$ . Hence,  $N$  is nilpotent by the minimality of  $L_1$  as a non-nilpotent ideal. Thus, we now have  $x, N$  as in the theorem above and so there is an idempotent  $e \in Ax \subseteq L_1 \subseteq L$ .  $\square$

4.10. THEOREM. *Let  $A$  be a left artinian ring. Then  $\text{rad } A$  is the largest nilpotent left ideal.*

PROOF. □

4.11. PROPOSITION. *If  $f: A \rightarrow B$  is a surjective homomorphism of rings, then*

$$f(\text{rad } A) \subseteq \text{rad } B$$

## 5. The Radical of an Artinian Ring

In this section, we seek to understand the radical of an artinian ring  $A$  and show that when it is trivial, we gain a huge amount of insight into  $A$ -modules. Namely, for semisimple  $A$ , every  $A$ -module is completely reducible and every irreducible  $A$ -module is isomorphic to a non-zero minimum left ideal of  $A$ . Perhaps most surprisingly of all, this tells us that a semisimple  $A$  has only a finite number of irreducible modules, which is not a typical phenomenon! We start by finding a deeper insight into the relationship between  $\text{rad } A$  and the nilpotent ideals of  $A$ .

5.1. THEOREM. *Let  $A$  be a left artinian ring. Then, the sum of all left nilpotent ideals of  $A$  is a nilpotent ideal, say  $N$ . It contains every nilpotent right ideal of  $A$ . Also, the quotient ring  $A/N$  has no nontrivial nilpotent ideal.*

To show this, we use the following lemma

5.2. LEMMA. *Any finite sum of nilpotent ideals is nilpotent.*

PROOF OF LEMMA. If we assume the result is true for a sum of  $r - 1$  nilpotent ideals, then it is true for  $r$  nilpotent ideals. Consider nilpotent ideals  $N_1, \dots, N_r$ . Then,

$$N_1 + \dots + N_r = \underbrace{(N_1 + \dots + N_{r-1})}_{\text{Nilpotent ideal by assumption}} + N_r \text{ is nilpotent.}$$

Thus, it suffices to show the result is true for the sum of two nilpotent ideals.

Let  $N_1, N_2$  be nilpotent left ideals. Then, there exist positive integers  $q, r$  such that

$$N_1^q = N_2^r = 0.$$

And element in  $(N_1 + N_2)^{q+r}$  is a finite sum of products

$$y_1 y_2 \dots y_{q+r}, \quad y_i \in N_1 \cup N_2, s \in \mathbb{N}$$

We can assume at least  $q$  of the  $y_i$  belong to  $N_1$  or at least  $r$  of the  $y_i$  belong to  $N_2$ . Without loss of generality, we assume the former. Then, the product above may be written

$$(x_1 x_2 \dots x_{i_1})(x_{i_1+1} \dots x_{i_2}) \dots (x_{i_{s-1}+1} \dots x_{i_s}) \dots,$$

where  $x_{i_1}, x_{i_2}, \dots, x_{i_s} \in N_1$  and  $s \geq q$ . Since  $N_1$  is a left ideal, each “segment” belongs to  $N_1$ . Thus,

$$y_1 y_2 \dots y_{q+r} \in N_1^q = 0 \implies y_1 y_2 \dots y_{q+r} = 0$$

However,  $y_1 y_2 \cdots y_{q+r}$  was an arbitrary element of  $(N_1 + N_2)^{q+r}$ , so  $(N_1 + N_2)^{q+r} = 0$  and thus the sum of 2 nilpotent (left) ideals is nilpotent.  $\square$

PROOF OF THEOREM. Let  $\{N_i\}$  be a set of nilpotent left ideals. Then, the elements of  $N$  are finite sums, i.e.

$$n \in N \implies n = x_1 + \cdots + x_r, x_i \in \text{some } N_i$$

Since the sum of left ideals is a left ideal,  $N$  is a left ideal.

Assume  $N$  is not nilpotent, then since  $A$  is left-artinian,  $N$  contains an idempotent,  $e$  (see 4.9). Thus, there is a finite sum of nilpotent ideals such that

$$e \in N_1 + \cdots + N_r$$

However,  $N_1 + \cdots + N_r$  is nilpotent by the lemma above, and so there is a  $k$  such that

$$e = e^k \in (N_1 + \cdots + N_r)^k = 0$$

which is a contradiction.

If  $N_i$  is a nilpotent left ideal, then  $N_i A$  is a nilpotent (two-sided) ideal since, for large enough  $t \in \mathbb{N}$ ,

$$(N_i A)^t = N_i (A N_i) \cdots (A N_i) A \subseteq N_i (N_i)^{t-1} A = 0$$

Thus,  $N$  is an ideal. If  $I$  is a nilpotent right ideal,  $AI$  is a nilpotent left ideal by the same argument. Thus,  $I \subseteq AI \subseteq N$ .

How does this follow exactly?

Since  $N$  is a two-sided ideal, then every left ideal of  $R/N$  is of the form  $I/N$  for some left ideal  $I$  of  $R$ , by the correspondance theorem. Thus,

$$(I/N)^k = 0 \iff I^k \subseteq N \iff I \text{ is nilpotent} \iff I \subseteq N$$

since  $N$  is the sum of all nilpotent left ideals. Thus,  $R/N$  contains no nilpotent left ideal except 0, and by the discussion above, also contains no nontrivial nilpotent right or two-sided ideals.  $\square$

5.3. THEOREM. *Let  $A$  be a left artinian ring. If  $N$  is the sum of all left nilpotent ideals, then  $N = \text{rad } A$ .*

PROOF. We already have that  $N \subseteq \text{rad } A$  by (4.10) since  $N$  is nilpotent by the above theorem. However, consider the sequence of ideals

$$\text{rad } A \supseteq (\text{rad } A)^2 \supseteq \cdots \supseteq (\text{rad } A)^n \supseteq \cdots$$

Since  $A$  is artinian, there is some  $k$  such that

$$(\text{rad } A)^k = (\text{rad } A)^{k+1} = \cdots$$

and thus  $(\text{rad } A)(\text{rad } A)^n = (\text{rad } A)^n$ . Thus, by Nakayama's lemma (2.6),  $(\text{rad } A)^n = 0$ . Thus,  $\text{rad } A$  is nilpotent.  $\square$

Is this not the same result as the theorem we are proving? Did we show something weaker, earlier?

5.4. THEOREM. *Let  $A$  be a left artinian ring. Then,  ${}_A A$  is completely reducible if and only if  $A$  is semisimple.*

PROOF. ( $\Leftarrow$ ). Let  $A$  be semisimple. Then,  $\text{rad } A = 0$ . Now, Let  $I_1$  be minimal among nontrivial non-nilpotent left ideals of  $A$ . Then,  $I_1$  has an idempotent, say  $e_1$ . Thus,  $e_1 \in I_1 \implies Ae_1 \subseteq I_1$ , but  $I_1$  is minimal among non-nilpotent ideals, so  $I_1 = Ae_1$ . Now, notice that

$$A = Ae_1 \oplus A(1 - e_1) \implies {}_A A = I_1 \oplus I'_1,$$

where  $I'_1 = A(1 - e_1)$  is a non-nilpotent left ideal if  $I'_1 \neq 0$ . Then, we can repeat this process with  $I'_1$  to get

$${}_A A = I_1 \oplus I_2 \oplus I'_2, \quad I_2 = Ae_2$$

Why is  $I'_1$  minimal?

Continuing, we get

$${}_A A = I_1 \oplus I_2 \oplus \cdots \oplus I_r$$

There is more work to do here.

where  $I_i = Ae_i$ ,  $e_i$  idempotent. Each  $I_i$  is irreducible as a left  $A$ -module since  $A$  is semisimple.

( $\Rightarrow$ ). Let  ${}_A A$  be completely reducible. Consider that  $N = \text{rad } A$  is a left ideal of  $A$  and thus also a submodule of  ${}_A A$  and thus

$${}_A A = N \oplus N'$$

for some left ideal  $N'$  of  $A$ . Then,

$$1 = x + x'$$

for some  $x \in N, x' \in N'$ , which then yields

$$x \cdot 1 = x(x + x') = x^2 + xx' \implies x - x^2 = xx' \in N \cap N' = 0$$

Thus,  $x - x^2 = 0 \implies x = x^2 = \cdots = 0$  since  $N$  is nilpotent. Thus,  $x' = 1$  and  $N' = A \implies \text{rad } A = N = 0$  and thus  $A$  is semisimple.  $\square$

5.5. THEOREM. *Let  $A$  be a (left) artinian ring. Then,  $A$  is semisimple if and only if every  $A$ -module is completely reducible.*

PROOF. To prove ( $\Leftarrow$ ), we simply note that  ${}_A A$  is an  $A$ -module, and so the previous theorem tells us that  $A$  is semisimple.

For ( $\Rightarrow$ ), assume  $A$  is semisimple and  $M$  is an  $A$ -module. Then, it suffices to show  $M$  is a (not-necessarily direct) sum of irreducible  $A$ -modules by 3.2. By the theorem above, we know

$${}_A A \cong L_1 \oplus \cdots \oplus L_n$$

Now, given  $0 \neq m \in M$ , consider  $\phi: A \rightarrow M$  given by  $a \mapsto a.m$ . Then,  $\phi|_{L_i}: L_i \rightarrow M$  is either injective or the 0 map. Thus, we get

$$A.M = M \implies (L_1 \oplus \cdots \oplus L_n).M = M \implies \phi(L_1) + \cdots + \phi(L_n) = M$$

but since each  $\phi(L_i)$  is irreducible or 0, it must be that  $M$  is completely reducible.  $\square$

5.6. THEOREM. *Let  $A$  be artinian and semisimple. Then, every irreducible  $A$ -module  $L$  is isomorphic to a non-zero minimum left ideal of  $A$ . Thus,  $A$  has only a finite number of irreducible modules.*

PROOF. Similar to above, we know

$${}_AA \cong L_1 \oplus \cdots \oplus L_n$$

and, for irreducible  $A$ -module  $M$  and  $0 \neq m \in M$ , we have a map  $\phi: A \rightarrow M$  given by  $a \mapsto a.m$ . Then,

$$M = \phi(L_1) + \cdots + \phi(L_n)$$

and one of the summands must be nonzero, say  $\phi(L_i)$ . However, since  $M$  is irreducible, this means that  $M = \phi(L_i)$  and thus  $\phi|_{L_i}$  is an isomorphism. Since  $A$  has only a finite number of ideals up to isomorphism by the Krull-Schmidt theorem, this means that  $A$  has only a finite number of irreducible modules.  $\square$

5.7. THEOREM. *Let  $A$  be artinian and semisimple. Let  $Ae$  be a left ideal for idempotent  $e$ . Then,  $Ae$  is irreducible if and only if  $eAe$  is a division ring.*

PROOF. Let us assume  $Ae$  is irreducible and let  $L$  be any nonzero left ideal of  $eAe$ . Then,

$$AL \subseteq A \cdot eAe \subseteq Ae$$

and so  $AL$  is a left ideal of  $A$  contained in  $Ae$ , but  $Ae$  is irreducible, so  $AL = Ae$ . Furthermore, since  $e$  acts as the identity on  $eAe$ , then  $eL = L$  and

$$eAe = eAL = eA(eL) = eAe \cdot L \subseteq L \implies eAe = L$$

and so  $eAe$  has only  $(0)$  and itself as left ideals. Thus,  $eAe$  is a division ring.

Let us assume  $eAe$  is a division ring. Since  $A$  is semisimple, if  $Ae$  is not irreducible, then

$$Ae = L_1 \oplus L_2$$

where  $L_1, L_2$  are proper submodules. Then, this gives us the decomposition  $e = e_1 + e_2$  for idempotents  $e_1 \in L_1, e_2 \in L_2$ . Note that

$$\begin{aligned} e &= e^2 \\ &= (e_1 + e_2)(e_1 + e_2) \\ &= e_1^2 + e_2e_1 + e_1e_2 + e_2^2 \\ \implies 0 &= e_1^2 - e_1 + e_2e_1 + e_1e_2 + e_2^2 - e_2 \end{aligned}$$

However,  $e_1e_2 + e_2^2 - e_2 \in L_2$  and  $-e_2e_1 - e_1^2 + e_1 \in L_1$  tells us that  $e_1e_2 + e_2^2 - e_2 = -e_2e_1 - e_1^2 + e_1 = 0$  since  $L_1 \cap L_2 = 0$ . Furthermore,  $e_1, e_2 \in eAe$  since  $e_1, e_2 \in Ae \implies e_1e = e_1$  and  $e_2e = e_2$ , as well as

$$ee_2 = (e_1 + e_2)e_2$$



$$\begin{aligned}
&= e_1 e_2 + e_2^2 \\
&= e_2
\end{aligned}$$

and similarly  $ee_1 = e_1$ . Thus,  $e_1 = ee_1e \in eAe$  and  $e_2 = ee_2e \in eAe$ . Finally,

$$\begin{aligned}
e_2^2 + e_1 e_2 &= e_2 \\
&= e_2(e_1 + e_2) \\
&= e_2 e_1 + e_2^2 \\
\implies e_1 e_2 &= e_2 e_1
\end{aligned}$$

But  $e_1 e_2 \in L_2$  and  $e_2 e_1 \in L_1$  and thus  $e_1 e_2 = 0$ , which tells us that  $eAe$  has zero divisors. Thus,  $eAe$  is not a division ring. □

## 6. The Structure of Semisimple (Left) Artinian Rings

6.1. DEFINITION. A left artinian ring  $A$  is called *simple* if its only two-sided ideals are 0 and  $A$ .

6.2. DEFINITION. A set of idempotents  $\{e_1, \dots, e_n\}$  of a ring  $A$  is a *complete set of orthogonal idempotents* if

- Each  $e_i$  is an idempotent,
- $e_1 + \dots + e_n = 1$
- and  $e_i e_j = 0$  if  $i \neq j$ .

Note that, for an artinian ring  $A$ , we have

$${}_A A = L_1 \oplus \dots \oplus L_n$$

for  $L_i$  irreducible  $A$ -modules, also seen as minimal left ideals. Thus,

$$1 = e_1 + \dots + e_n, \quad e_i \in L_i$$

with  $e_j e_i \in L_i$ . Thus,  $e_j^2 = e_j$  and  $e_i e_j = 0$  if  $i \neq j$ , so  $\{e_1, \dots, e_n\}$  is a complete set of orthogonal idempotents for  $A$ .

We now seek to prove the Artin-Wedderburn theorem, that will tell us that a semisimple left artinian ring is isomorphic to a direct sum of matrix rings over division rings.

6.3. LEMMA. *Let  $L, L'$  be minimal left ideals. Then,  $L \cong L'$  as left  $A$ -modules if and only if there is an  $a' \in L'$  such that  $L' = La'$ .*

PROOF. For  $(\implies)$ , let  $\phi: L \xrightarrow{\sim} L'$ . Then, for  $x \in L = (e)$ ,

$$\phi(x) = \phi(xe) = x\phi(e) = xa' \quad \text{where } a' = \phi(e)$$

Then,  $L' = La'$  as required.

For ( $\Leftarrow$ ), assume there is an  $a' \in L'$  such that  $L' = La'$ . Define  $\phi: L \rightarrow L'$  by  $\phi(x) = xa'$ . Thus,  $\phi$  is surjective. Since  $L$  is irreducible,  $\ker \phi = \{0\}$  and so  $\phi$  is injective. Thus,  $\phi$  is an isomorphism.  $\square$

6.4. THEOREM (Weddurburn). *Let  $A$  be a semisimple artinian ring. Let  $L$  be a minimal left ideal. The sum of all minimal left ideals  $\{L' \trianglelefteq A \mid L' \cong L\}$ , say  $B_L$ , is a simple subring and a 2-sided ideal in  $A$ . Also,*

$$A = \bigoplus_{i=1}^r B_{L_i}$$

where  $L_1, \dots, L_r$  are representatives of distinct isomorphism classes of left ideals.

PROOF. Let

$$A = L_1 \oplus \dots \oplus L_r$$

where the  $L_i$  are minimal left ideals of  $A$ . There is an equivalence relation on  $\{L_1, \dots, L_r\}$  given by

$$L_i \sim L_j \iff L_i \cong L_j$$

Thus, if  $L_1, \dots, L_s$  is an equivalence class, then for  $i, j \leq s$ ,  $L_i L_j = L_j$ . Thus, let us define

$$B_{L_1} := L_1 \oplus \dots \oplus L_s \subseteq A$$

Such a subset is closed under multiplication and is a 2-sided ideal since  $A$ . Let us do this analogously for every equivalence class to get  $B_1, B_2, \dots, B_t$ . Then,

$$i \neq j \implies B_i B_j = 0$$

and also,

$$B_1 \oplus B_2 \oplus \dots \oplus B_t = A$$

For a given  $B_i$ , we have

$$B_i = L_{i,1} \oplus L_{i,2} \oplus \dots \oplus L_{i,s_i}$$

where  $L_{i,j} = L_\ell$  for some  $\ell$ . We note

$$1_i := e_{i,1} + \dots + e_{i,s_i}$$

is the multiplicative identity in  $B_i$ . Thus,  $B_i$  is a subring of  $A$ . Also,  $B_i$  is a left ideal, so it is artinian.

It still remains to show that  $B_i$  is simple subring and a 2-sided ideal. Without loss of generality, consider

$$B_1 = L_1 \oplus \dots \oplus L_r$$

$$1_B = e_1 + \dots + e_r$$

and let  $0 \neq D \trianglelefteq B_1$ , that is, let  $D$  be a nontrivial 2-sided ideal of  $B_i$ . Then,  $D$  is a left-ideal and thus contains a minimal left ideal  $L$  of  $A$ . Thus,

$$L \subseteq D \subseteq B_1 \implies L \cong L_1$$

It is not clear to me where we show  $B_i$  is a 2-sided ideal.

and so we conclude that  $L \cong L_1$  since  $L \subseteq B_1$ . However, we also know that

$$Lx \subseteq D, \quad x \in A$$

since  $D$  is a right ideal, too. Thus, by 6.3,  $\{Lx \mid x \in A\}$  gives all minimal left ideals of  $A$  isomorphic to  $L$ . Therefore, it must be that  $D = B_1$ , so  $B_1$  contains no non-trivial two-sided ideals and is thus a simple subring.  $\square$

6.5. THEOREM (Wedderburn). *Let  $A$  be a simple artinian ring. Then, there exists a unique division ring and a unique positive integer  $n$  such that*

$$A \cong M_n(D)$$

*Conversely, if  $D$  is a division ring, then  $M_n(D)$  is a simple artinian ring.*

Thus, combining our results, we arrive at the landmark theorem.

6.6. THEOREM (Artin-Wedderburn Theorem). *Let  $A$  be a semisimple left artinian ring. Then, up to reordering, there is a unique decomposition*

$$A \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_r}(D_r)$$

*where each  $D_i$  is a division ring and each  $n_i$  is a positive integer.*

## 7. The Double Centralizer Property

In order to finish our proofs, however, we will appeal to some other facts that make use of the following language.

7.1. DEFINITION. Let  $A$  be a ring and  $M$  be a left  $A$ -module. Furthermore, let  $A_L \subseteq \text{End}_A(M)$  be the endomorphisms given by applying elements of  $A$  to  $M$ . If  $A_L = \text{End}_A(M)$ , we say that the pair  $(A, M)$  has the *double centralizer property*.

7.2. REMARK. More generall, this notion can be extended to any ring  $R$  with subring  $S$ . We say  $(R, S)$  has the double centralizer property if  $C_R(C_R(S)) = S$ .

Thus, Wedderburn's theorem above can be reprhased as follows

7.3. THEOREM (Wedderburn). *Let  $A$  be a simple Artinian ring. Then, for some minimal left ideal  $M \trianglelefteq A$ ,  $(A, M)$  has the double centralizer property.*

7.4. LEMMA. *Let  $A$  be a ring. Then  $(A, {}_A A)$  has the double centralizer property.*

7.5. LEMMA. *Let  $V = M^{\oplus k}$  for a left  $A$ -module  $M$ ,  $k \in \mathbb{N}$ . If  $(A, V)$  has the double centralizer property, then  $(A, M)$  also has the double centralizer property.*

7.6. LEMMA. *Let  $M = Ae$  for  $e \in A$  an idempotent. Then,*

$$\text{End}_A(M) \cong eAe$$

*via isomorphism  $f: \text{End}_A(M) \rightarrow eAe$  and  $md = m \cdot f(d)$  for all  $m \in M, d \in \text{End}_A(M)$ .*

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