K-theoretic Catalan functions

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Overview

1. Schubert calculus
2. Catalan functions
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Classical Schubert Calculus Example

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$ for combinatorially understood Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$.

Schur polynomials and raising operators

- Complete homogeneous symmetric function: for $r \in \mathbb{Z}$,
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- Schur function $s_\lambda = \prod_{i<j}(1 - R_{ij})h_\lambda$ (Jacobi-Trudi)

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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
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Simulatenously generalizes K-theory of Grassmannian and (co)homology of affine Grassmannian.

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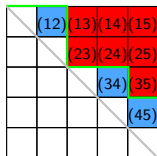
   for k_γ an inhomogeneous analogue of h_γ.

2. Homology classes of affine Grassmannian represented by *k*-Schur functions ($t = 1$).

3. (Lam et al., 2010) leave open the question: what is a direct formulation of the *K*-homology representatives of the affine Grassmannian (*K*-*k*-Schur functions)?

Goal

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Identify K-k-Schur functions in explicit (simple) terms amenable to calculation and proofs.
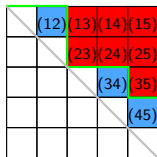
Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).

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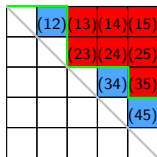$\Delta_\ell^+ \backslash \Psi$ = Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_\ell^+ \backslash \Psi} (1 - R_{ij}) h_\gamma(x)$$

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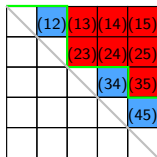
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- $\Psi = $ all roots $\implies H(\Psi; \gamma) = h_\gamma$

Catalan functions

k-Schur root ideal for λ

For $k \in \mathbb{Z}_{\geq 0}$ and $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell) \in \mathbb{Z}^\ell$,

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
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$$\Delta^4(3,3,2,2,1,1) = \boxed{\begin{array}{cccccc} 3 & & & & & \\ & 3 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}} \qquad \leftarrow \text{row } i \text{ has } 4 - \lambda_i \text{ non-roots}$$

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k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

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Remark

(Blasiak et al., 2019) show results for k-Schur functions with parameter t, but $t = 1$ specialization is necessary for Schubert calculus.

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- Extra ingredient: lowering operators $L_j(h_\lambda) = h_{\lambda-\epsilon_j}$

$$L_3\left(\begin{array}{c}\text{diagram}\end{array}\right) = \begin{array}{c}\text{diagram}\end{array} \quad L_1\left(\begin{array}{c}\text{diagram}\end{array}\right) = \begin{array}{c}\text{diagram}\end{array}$$

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Example

non-roots of Ψ in blue, roots of $\mathcal{L}$ marked with $\bullet$

$K(\Psi; \mathcal{L}; 54332)$

$= (1 - L_4)^2 (1 - L_5)^2$

$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$

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Example

$$g_{332111}^{(4)} = \quad \Delta^+ \setminus \Psi = \Delta_6^+ \setminus \Delta^{(4)}(332111), \mathcal{L} = \Delta^{(5)}(332111)$$

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Theorem (Blasiak-Morse-S., 2022)

The $g_\lambda^{(k)}$ "branching coefficients" are alternating by degree, i.e. the $b_{\lambda\mu}^{(k)}$ in

$$g_\lambda^{(k)} = \sum_\mu b_{\lambda\mu}^{(k)} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} b_{\lambda\mu}^{(k)} \in \mathbb{Z}_{\geq 0}$.

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Under the Peterson Isomorphism, the "quantum Grothendieck polynomials" $\mathfrak{G}_w(z; Q)$ get sent to "closed K-k-Schur functions", $\mathfrak{g}_\lambda^{(k)} = K(\Delta^{(k)}; \Delta^{(k)}; \lambda)$ with suitable localization.

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Proved using "Katalan function description."

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Future Directions

For $G_\lambda^{(k)}$ an affine Grothendieck polynomial (dual to $g_\lambda^{(k)}$),

1. Combinatorially describe dual "Pieri rule":
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4. Answer same questions for "closed K-k-Schur's."

Other results using Catalan function methods

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3. New formulas for Macdonald polynomials using raising operators (Blasiak-Haiman-Morse-Pun-S.)

References

Thank you!

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