

# $K$ -theoretic Catalan functions

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- Schubert calculus
- Catalan functions: a new approach to old problems
- $K$ -theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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## Representatives

Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

# Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of Schubert varieties  $\{X_\lambda\}_{\lambda \subseteq (n^m)}$  in variety  $X = \text{Gr}(m, n)$ .

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## Representatives

Special basis of Schur polynomials  $\{s_\lambda\}$  such that  $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$  for Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$ .

## Example

*Semistandard tableaux*: columns increasing and rows non-decreasing.

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
3	4		
1	2	5	6

standard = no repeated letters

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Schur function  $s_\lambda$  is a “weight generating function” of semistandard tableaux:

2	3	3	2	3	3	2	3
1	1	1	1	1	2	3	1
1	1	2	2	3	3	3	2

$$s_{\square\square}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$$

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$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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Since  $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$ , subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients*  $c_{\lambda\mu}^\nu$ .

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$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

### Open Problem

Structure constants  $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$  have no tableaux description.

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(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
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And many more!

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$$\begin{aligned}\Phi: QH^*(Fl_{k+1}) &\rightarrow H_*(Gr_{SL_{k+1}})_{loc} \\ \mathfrak{S}_w^Q &\mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}\end{aligned}$$

where  $s_\lambda^{(k)}$  is a  $k$ -Schur symmetric function and  $Gr_{SL_{k+1}}$  is the “affine Grassmannian.”

Upshot

# Peterson Isomorphism

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## Upshot

Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

# $k$ -Schur functions

- $s_{\lambda}^{(k)}$  for  $\lambda_1 \leq k$  a basis for  $\mathbb{Z}[s_1, s_2, \dots, s_k]$  (Lapointe et al., 2003).

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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\lambda} + s_{\lambda} + s_{\lambda}$$

The diagram shows the branching rule for  $k$ -Schur functions. The left side is  $s_{\lambda}^{(2)}$ , represented by a 2x2 square. The right side is the sum of three terms:  $s_{\lambda}^{(3)}$  (a 3x2 rectangle),  $s_{\lambda}^{(3)}$  (a 3x1 vertical rectangle), and  $s_{\lambda}^{(3)}$  (a 3x1 vertical rectangle). Brackets indicate that the second and third terms are grouped together as  $s_{\lambda}^{(3)}$ .

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The diagram shows the branching of the 2-partition  $(2,2)$  into 3-partitions. On the left is  $s_{(2,2)}^{(2)}$ , represented by a 2x2 grid. On the right is the sum of three 3-partitions:  $s_{(2,2)}^{(3)}$  (a 2x2 grid),  $s_{(3,1)}^{(3)}$  (a 2x2 grid with an extra cell to the right of the bottom row), and  $s_{(4)}^{(3)}$  (a single row of 4 cells). Brackets below the three terms on the right group them under  $s_{(2,2)}^{(3)}$  and  $s_{(3,1)}^{(3)}$  respectively.

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- Branching with positive coefficients (Lam et al., 2010):

$$s_{\lambda}^{(2)} = s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}}$$

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- (Lam et al., 2010) gives geometric interpretation,
- but no combinatorial interpretation of branching coefficients.
- Branching with  $t$  important for Macdonald polynomial positivity.
- Many conjecturally equivalent definitions.

- Schubert calculus
- **Catalan functions: a new approach to old problems**
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- 2 From a new definition, (Blasiak et al., 2019) shows the branching coefficients  $b_{\lambda\mu}$  in the expansion  $s_{\lambda}^{(k)} = \sum_{\mu} b_{\lambda\mu} s_{\mu}^{(k+1)}$  have combinatorial interpretation!

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Key: Catalan functions = large class of symmetric functions.

# Ingredients for Catalan functions

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# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

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$$s_{22} = (1 - R_{12}) h_{22} = h_{22} - h_{31}$$

$$s_{211} = (1 - R_{12})(1 - R_{23})(1 - R_{13}) h_{211}$$

$$= h_{211} - h_{301} - h_{220} - \text{red } h_{310} + \text{red } h_{310} + \underbrace{h_{32-1}}_{=0} + h_{400} - \underbrace{h_{41-1}}_{=0}$$

some terms cancel

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Simplifies formulas. E.g., for  $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$  (note  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ ),

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Simplifies formulas. E.g., for  $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$  (note  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ ),

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path (lattice path above diagonal).



$\Psi =$  Roots above Dyck path  
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## Intuition

Catalan functions interpolate between  $h_\lambda$  and  $s_\lambda$ .

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## Theorem (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive!

## $k$ -Schur root ideal for $\lambda$

$$\begin{aligned}\psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots

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$k$ -Schur is a Catalan function (Blasiak et al., 2019).

For partition  $\lambda$  with  $\lambda_1 \leq k$ ,

$$s_{\lambda}^{(k)} = H(\Delta^k(\lambda); \lambda).$$

# Key ingredient of branching proof

Dual vertical Pieri rule:  $s_{1^r}^\perp s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$  for  $\langle s_{1^r}^\perp f, g \rangle = \langle f, s_{1^r} g \rangle$ .

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

# Key ingredient of branching proof

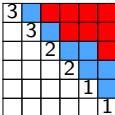
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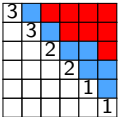
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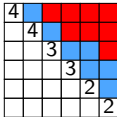
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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

- Schubert calculus
- Catalan functions: a new approach to old problems
- ***K*-theoretic Catalan functions**

# Dual Grothendieck polynomials

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$$g_{1^2}g_{3,2} = g_{43} + g_{421} + g_{331} + g_{3211} - g_{42} - g_{33} - 2g_{321} + g_{31}$$



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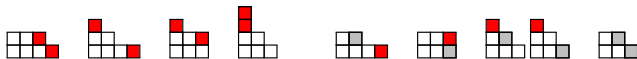
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- $g_\lambda = \prod_{i < j} (1 - R_{ij}) k_\lambda$  for  $k_\lambda$  and inhomogeneous analogue of  $h_\lambda$ .
- Dual to Grothendieck polynomials  $G_\lambda$ : Schubert representatives for  $K^*(Gr(m, n))$

# $K$ - $k$ -Schur functions

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$$g_1 g_{211}^{(2)} = g_{2111}^{(2)} - 2g_{211}^{(2)}$$

2-bounded partitions  $\leftrightarrow$  3-cores

The diagram illustrates the Pieri rule for  $K$ - $k$ -Schur functions. It shows the product of  $g_1$  and  $g_{211}^{(2)}$  as a difference of two 2-bounded partitions. The first partition (left) is a 5x5 grid with a 3-core. The second partition (right) is a 5x5 grid with a 3-core. The difference is shown as a 5x5 grid with a 3-core. The diagram uses red, blue, and black dots to represent the partitions and their cores.

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- Conjecture:  $g_{\lambda}^{(k)}$  have positive branching into  $g_{\mu}^{(k+1)}$  (Lam et al., 2010; Morse, 2011).

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## Problem

No direct formula for  $g_{\lambda}^{(k)}$

## Solution

Find a formula for  $g_{\lambda}^{(k)}$  analogous to raising operator formula for  $s_{\lambda}^{(k)}$ .



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Requires an inhomogeneous refinement of Catalan functions.

# An Extra Ingredient: Lowering Operators

Lowering Operators  $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$

$$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12}) (1 - R_{34}) (1 - R_{45}) k_{54332}$$

Answer (Blasiak-Morse-S., 2020)

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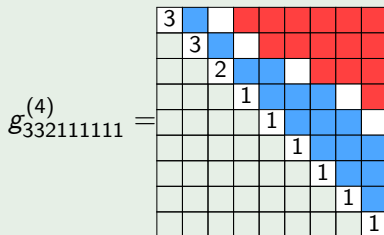
For  $K$ -homology of affine Grassmannian,  $g_{\lambda}^{(k)} = K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda)$  since this family satisfies the Pieri rule.

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Example



$$\Delta_9^+ / \Delta^4(332111111), \Delta^5(332111111)$$

# Pieri Rule Illustrated (Recurrences)

A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$



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$$g_1 g_{211}^{(2)}$$

	2						
		1					
			1				
				0			
					0		
						0	
							1

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A “graphical calculus.”

$$g_1 g_{211}^{(2)}$$

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

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$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$=$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

# Pieri Rule Illustrated (Straightening)

$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$ 

2						
	1					
		1				
			0			
				0		
					0	
						1

 $+$ 

2						
	1					
		1				
			0			
				0		
					0	
						1

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$$g_1 g_{211}^{(2)} =$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			0			
				0		
					0	
						1

$$+$$

2						
	1					
		1				
			1			
				0		
					0	
						1

$$=$$

2			
	1		
		1	
			1

$$-$$

2		
	1	
		1

$$-$$

2		
	1	
		1

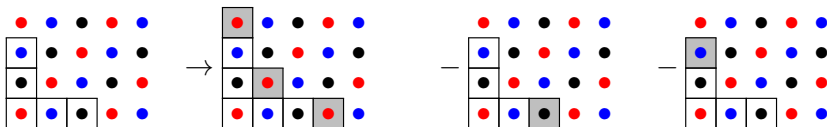
# Pieri Rule Illustrated (Straightening)

$$\begin{aligned}
 g_1 g_{211}^{(2)} &= \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 0 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline 2 & & & & & & \\ \hline & 1 & & & & & \\ \hline & & 1 & & & & \\ \hline & & & 1 & & & \\ \hline & & & & 0 & & \\ \hline & & & & & 0 & \\ \hline & & & & & & 1 \\ \hline \end{array} \\
 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
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 &= \begin{array}{|c|c|c|c|c|} \hline 2 & & & & \\ \hline & 1 & & & \\ \hline & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \\
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3-core perspective:



## Theorem (Blasiak-Morse-S., 2020)



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## Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

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satisfies  $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$  such that  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

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$$\Phi(\mathfrak{G}_w^Q) = \frac{\tilde{g}_w}{\prod_{i \in Des(w)} \tau_i}$$

satisfies  $\tilde{g}_w = g_\lambda^{(k)} + \sum_\mu a_{\lambda\mu} g_\mu^{(k)}$  such that  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .

## Theorem (Blasiak-Morse-S., 2020)

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# $K$ -theoretic Peterson isomorphism

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Theorem (S. (thesis), 2021)

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- $k$ -Rectangle Property fails for  $g_{\lambda}^{(k)}$ .



# Positivity of Catalan functions

Recall (Blasiak et al., 2020)

For  $\Psi$  any root ideal and  $\lambda$  a partition,  $H(\Psi; \lambda)$  is Schur positive.

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## Thank you!

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$$k_m^{(r)} = \sum_{i=0}^m \binom{r+i-1}{i} h_{m-i} = s_m(X+r),$$

a specialization of “multiSchur functions.” See, e.g., Lascoux-Naruse (2014).

$$k_{\gamma} = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \dots k_{\gamma_{\ell}}^{(\ell-1)}$$