

PLETHYSTIC SUBSTITUTION

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1. INTRODUCTION

In [Mac79, p 135], a new type of product on symmetric functions is introduced called “plethysm,” which allows one to take $f \in \Lambda^m$ and $g \in \Lambda^n$ to get a product $f[g] \in \Lambda^{mn}$ (denoted $f \circ g$ in [Mac79]). This notion has become increasingly prevalent in algebraic combinatorics research, and this monograph seeks to give an outline of some of the essentials.

2. DEFINITION AND PROPERTIES

Departing from [Mac79], we define the following.

2.1. Definition. Given a Laurent series A in indeterminates a_1, a_2, a_3, \dots , we define $p_n[A]$ to be the series where each a_i is changed to a_i^n . In other words, each indeterminate is raised to the n th power. In particular, given a symmetric function $g \in \Lambda$, $p_n[g(x_1, x_2, \dots)] = g(x_1^n, x_2^n, \dots)$.

Furthermore, it is a common convention to let $X = x_1 + x_2 + x_3 + \dots$ and then write things such as

$$p_n[X] = p_n(x_1, x_2, x_3, \dots)$$

2.2. Example. (a) If $A = a_1 + a_2 + a_3 + \dots$, then $p_n[A] = a_1^n + a_2^n + a_3^n + \dots$.
(b) In particular, $p_n[p_m] = (x_1^n)^m + (x_2^n)^m + \dots = p_{nm} = p_m[p_n]$. Thus, $p_n[1] = 1$.

2.3. Proposition. [Mac79, p 135] *For $n \geq 1$, the mapping $g \mapsto p_n[g]$ is an endomorphism of the ring Λ .*

Next, since any $f \in \Lambda$ can be written as a (rational) linear combination of p_λ ’s and each p_λ is a product of p_n ’s, we extend the definition of plethysm to say

2.4. Definition. Given a Laurent series A ,

- (a) we say $p_\lambda[A] = p_{\lambda_1}[A]p_{\lambda_2}[A] \cdots p_{\lambda_\ell}[A]$ and
- (b) $(f + g)[A] = f[A] + g[A]$ for any $f, g \in \Lambda$, and

Thus, we can compute $f[A]$ for any symmetric function $f \in \Lambda$ by writing it as a linear combination of p_λ ’s and evaluating the plethysm on each term.

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2.5. Example. (a) Given $A = 1 + t + t^2 + t^3 + \dots = \frac{1}{1-t}$, we get $f[\frac{1}{1-t}] = f(1, t, t^2, t^3, \dots)$ since

$$p_n \left[\frac{1}{1-t} \right] = 1 + t^n + t^{2n} + \dots = p_n(1, t, t^2, \dots)$$

- (b) Recall $p_1(x) = x_1 + x_2 + \dots =: X$. Then, $f[X + a]$ adds a variable a to our set of variables. Similarly, $f[X - x_i]$ removes x_i from the set of variables.
- (c) Combining the ideas above, $f[X - (1-t)x_i]$ removes variable x_i but replaces it with variable tx_i .
- (d) Finally, $f[\frac{1}{1-t}] = f(1, t, t^2, \dots)$ and $f[\frac{X}{1-t}] = f(x_1, tx_1, t^2x_1, \dots, x_2, tx_2, t^2x_2, \dots)$ since $\frac{X}{1-t} = x_1 + tx_1 + t^2x_1 + \dots + x_2 + tx_2 + t^2x_2 + \dots$.

2.6. Proposition. Given $c \in \mathbb{Q}$, we get, by definition, that $f[cA] = cf[A]$ for all $f \in \Lambda$ and Laurent series A . However, given an indeterminate t , we get $p_n[tA] = t^n p_n[A]$. In other words, plethysm and variable evaluation do not commute.

Proof. This follows since plethysm affects indeterminates but not constants. \square

2.7. Remark. The proposition above can be the source of much confusion. One way to distinguish between these two different kinds of values is to call the constants *binomial variables*. So, in the proposition above, we say that c is a binomial variable but t is not.

2.8. Definition. It can be convenient to introduce a minus sign to each variable in the plethystic substitution. So, we define the variable ϵ such that

$$p_r[\epsilon X] := p_r(-x_1, -x_2, -x_3, \dots) = (-1)^r p_r[X]$$

where $X = x_1 + x_2 + x_3 + \dots$.

2.9. Remark. Notice that $p_r[\epsilon X]$ is not necessarily equal to $p_r[-X]$ in our notation. In particular, for a binomial variable $c \in \mathbb{Q}$, we have

$$p_r[cX] = cp_r[X] \text{ but } p_r[\epsilon X] = (-1)^r p_r[X]$$

Furthermore, authors are often not careful with this distinction, so one needs to use context.

2.10. Proposition. [Mac79, p 135] *Plethysm is associative. That is,*

$$(f[g])[h] = f[g[h]]$$

Proof. Because the p_n generate Λ over \mathbb{Q} , we need only verify the associativity for p_n 's, which we already did in 2.2. \square

2.11. Lemma. Given Laurent series A and B , we get

$$p_k[A + B] = p_k[A] + p_k[B]$$

Proof. By definition, $p_k[A + B]$ raises all the indeterminates from A and B to the k th power, which is the same effect as $p_k[A]$ and $p_k[B]$. \square

Now, recall the Cauchy kernel

$$\Omega(x, y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

We seek to generalize this notion as follows. Let us define

$$\Omega := \exp \left(\sum_{k=1}^{\infty} \frac{p_k}{k} \right)$$

which gives us that

2.12. Proposition. (a)

$$\Omega[x] = \exp \left(\sum_{k=1}^{\infty} \frac{x^k}{k} \right) = \exp(\log(x - 1)) = \exp(\log((1 - x)^{-1})) = \frac{1}{1 - x}$$

(b) $\Omega[A + B] = \Omega[A]\Omega[B]$ and $\Omega[-A] = \frac{1}{\Omega[A]}$ for any Laurent series A and B

(c)

$$\Omega[X] = \prod_{i \geq 1} \frac{1}{1 - x_i} \text{ and } \Omega[XY] = \Omega(x, y)$$

for formal power series $X = \sum x_i$ and $Y = \sum y_j$.

Proof. By definition, $p_k[x] = x^k$ and so the first part follows. For part (b), using the lemma above, we have

$$\exp(p_k[A + B]) = \exp(p_k[A] + p_k[B]) = \exp(p_k[A]) \exp(p_k[B])$$

and so $\Omega[A + B] = \Omega[A]\Omega[B]$. Similarly,

$$\exp(p_k[-A]) = \exp(-p_k[A]) = \frac{1}{\exp(p_k[A])}$$

Finally, part (c) follows from repeated iteration of part (a). \square

2.13. Corollary. (a) $e_r[X] = h_r[-\epsilon X] = (-1)^r h_r[-X]$.

(b) The involution on symmetric functions $\omega: \Lambda \rightarrow \Lambda$ corresponds to the plethystic substitution $X \mapsto -\epsilon X$.

Proof. To start, we note $\Omega[tX] = \prod_{i \geq 1} \frac{1}{1 - tx_i} = \sum_{r \geq 0} h_r[X] t^r$ and

$$\sum_{r \geq 0} h_r[-\epsilon X] t^r = \Omega[-t\epsilon X] = \prod_{i \geq 1} 1 + x_i = \sum_{r \geq 0} e_r[X] t^r.$$

Then, the first part follows immediately. The second part follows from the first since one definition of ω is precisely that $\omega(h_r) = e_r$. \square

3. EXAMPLES WITH SCHUR FUNCTIONS

Some examples

3.1. Proposition. [Sta99, Cor 7.21.3] *We have*

$$s_\lambda \left[\frac{1}{1-t} \right] = s_\lambda(1, t, t^2, t^3, \dots) = \frac{t^{n(\lambda)}}{\prod_{x \in \lambda} (1 - t^{h(x)})}$$

for $h(x)$ the hook-length of cell $x \in \lambda$ and $n(\lambda) = \sum_i (i-1)\lambda_i$.

Proof. As discussed above, $f \left[\frac{1}{1-t} \right] = f(1, t, t^2, t^3, \dots)$ for any symmetric function f . Now, we observe that

$$s_\lambda(1, t, t^2, t^3, \dots, t^{n-1}) = \frac{t^{n(\lambda)+n(n-1)(n-2)/6} \prod_{i < j} (1 - t^{\lambda_i - \lambda_j - i + j})}{t^{n(n-1)(n-2)/6} \prod_{i < j} (1 - t^{-i+j})}$$

However, one can show using the combinatorics of tableaux (see [Man98, Exercise 1.4.9] and proof of [Man98, Proposition 1.4.10]) that

$$\prod_{x \in \lambda} (1 - t^{h(x)}) \prod_{i < j} (1 - t^{\lambda_i - \lambda_j - i + j}) = \prod_{i=1}^n \prod_{k=1}^{\lambda_i + n - i} (1 - t^k)$$

and so, plugging this in, we get

$$s_\lambda(1, t, t^2, \dots, t^{n-1}) = t^{n(\lambda)} \frac{\prod_{i=1}^n \prod_{k=1}^{\lambda_i + n - i} (1 - t^k)}{\prod_{x \in \lambda} (1 - t^{h(x)}) \prod_{i=1}^n \prod_{k=1}^{n-i} (1 - t^k)} = t^{n(\lambda)} \frac{\prod_{i=1}^n \prod_{k=n-i+1}^{\lambda_i + n - i} (1 - t^k)}{\prod_{x \in \lambda} (1 - t^{h(x)})}$$

However, since $\lambda_i = 0$ for $i > \ell(\lambda)$, we can remove one dependence on n to get:

$$s_\lambda(1, t, t^2, \dots, t^{n-1}) = t^{n(\lambda)} \frac{\prod_{i=1}^{\ell} \prod_{k=n-i+1}^{\lambda_i + n - i} (1 - t^k)}{\prod_{x \in \lambda} (1 - t^{h(x)})}$$

At this point, [Sta99] claims that $\lim_{n \rightarrow \infty} (1 - t^n) = 1$, so we are done. \square

3.2. Proposition. *Let λ be a partition. Then,*

$$s_\lambda[X + a] = \sum_k a^k \sum_{\lambda = \nu + \text{horizontal } k\text{-strip}} s_\nu(x)$$

for $X = x_1 + x_2 + x_3 + \dots$.

Proof. Using Littlewood's combinatorial description of Schur functions, we get

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in \text{SSYT}(\lambda)} x^{\text{wt}(T)}$$

However, since semistandard tableaux must have strictly increasing columns, all the boxes labelled n must form a (possibly empty) horizontal strip. Thus,

Of course, I do not understand why this should be true; certainly, it would not work for my calculus students. Perhaps since we are using an expansion where $t \nmid 1$ anyways, this follows.

if we break up the sum based on how many boxes labelled n there are, we get

$$s_\lambda(x_1, \dots, x_n) = \sum_{k \geq 0} x_n^k \sum_{\lambda = \nu + \text{horizontal } k\text{-strip}} \sum_{T \in \text{SSYT}(\nu)} x^{\text{wt}(T)}$$

where $\text{SSYT}(\nu)$ are labelled with letters $\{1, \dots, n-1\}$.

$$= \sum_{k \geq 0} x_n^k \sum_{\lambda = \nu + \text{horizontal } k\text{-strip}} s_\nu(x_1, \dots, x_{n-1})$$

Thus, we see how to write a Schur function in terms of Schur functions with one fewer variable. \square

3.3. Proposition. [Mac79, 8.8] *Given a partition λ and symmetric functions g, h ,*

$$s_\lambda[g + h] = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu[g] s_\nu[h]$$

where $c_{\mu\nu}^\lambda$ are the Littlewood-Richardson coefficients.

Proof. This follows from [Mac79, 5.9] which states

$$s_\lambda(x, y) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu(y) s_\nu(x)$$

following from formal manipulations of skew-Schur functions. \square

3.4. Remark. One can actually take this as the definition of a skew-Schur function. That is, we can define the skew-Schur function $s_{\lambda/\mu}$ to be such that

$$s_\lambda[X + Y] = \sum_{\mu} s_{\lambda/\mu}[X] s_\mu[Y]$$

for $X = x_1 + x_2 + \dots$ and $Y = y_1 + y_2 + \dots$.

Finally, we state without proof

3.5. Theorem. *Given partitions λ, μ , we get*

$$s_\lambda[s_\mu] = \sum_{\nu} a_{\lambda\mu}^\nu s_\nu$$

with $a_{\lambda\mu}^\nu \in \mathbb{Z}_{\geq 0}$.

3.6. Remark. While one can prove that these coefficients are non-negative (see [Mac79, Appendix I.A]), actually describing these coefficients is an old and difficult problem in general, sometimes referred to as the “plethysm problem.”

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