

# Schubert calculus and $K$ -theoretic Catalan functions

George H. Seelinger (joint with J. Blasiak and J. Morse)

UVA Graduate Seminar

*ghs9ae@virginia.edu*

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- ① An overview of Schubert calculus
- ② Catalan functions: shedding new light on old problems
- ③  $K$ -theoretic Catalan functions

# Overview of Schubert Calculus Combinatorics

## Geometric problem

Find  $c_{\lambda\mu}^\nu = \#$  of points in intersection of subvarieties in a variety  $X$ .

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Special basis of polynomials  $\{f_\lambda\}$  such that  $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

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Combinatorial study of  $\{f_\lambda\}$  enlightens the geometry (and cohomology).

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## Goal

Identify  $\{f_\lambda\}$  in explicit (simple) terms amenable to calculation and proofs.

# Classical Example

$$X = \operatorname{Gr}_m(\mathbb{C}^{m+n}) = \{\text{all } m\text{-dimensional subspaces of } \mathbb{C}^{n+m}\}.$$



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What are the structure constants  $c_{\lambda\mu}^\nu$ ?

# Classical Example (cont.)

$\Lambda_m = \mathbb{C}[x_1, \dots, x_m]^{S_m}$  is the ring of symmetric polynomials in  $m$  variables and has bases indexed by partitions.

$$\underbrace{12x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{symmetric}}$$

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There exists a basis of  $\Lambda_m$  denoted  $\{s_\lambda\}_\lambda$  and a surjection of rings such that

$$\begin{aligned} \Lambda_m &\rightarrow H^*(\text{Gr}(m, n)) \\ s_\lambda &\mapsto \begin{cases} \sigma_\lambda & \lambda \subseteq (n^m) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$



# Classical Example (cont.)

Cohomology structure:  $\sigma_\lambda \leftrightarrow s_\lambda$  when  $\lambda \subseteq (n^m)$ .

$$s_\lambda s_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu s_\nu + \sum_{\nu \not\subseteq (n^m)} c_{\lambda\mu}^\nu s_\nu \leftrightarrow \sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu \sigma_\nu$$

## Example

*Semistandard tableaux*: columns increasing and rows non-decreasing.

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
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1	2	5	6

standard = no repeated letters

# Schur functions $s_\lambda$

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Schur function  $s_\lambda$  is a “weight generating function” of semistandard tableaux:

2	3	3	2	3	3	2	3
1	1	1	1	1	2	3	1
1	1	2	2	3	3	3	2

$$s_{\square\square}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

# Schur functions $s_\lambda$ (cont.)

## Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}}$$

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Since  $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$ , subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients*  $c_{\lambda\mu}^\nu$ .

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- Structure constants  $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$  are combinatorially unknown.

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Theory	$f_\lambda$
(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- $P$ and $Q$ functions
(Co)homology of affine Grassmannian	(dual) $k$ -Schur functions
$K$ -theory of Grassmannian	Grothendieck polynomials
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And many more!



- $QH^*(Fl_{k+1})$  quantum deformation of  $H^*(Fl_{k+1})$  ( $q \rightarrow 0$ ).

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$$\Psi: QH^*(Fl_{k+1}) \rightarrow H_*(Gr_{SL_{k+1}})_{loc}$$

$$\mathfrak{S}_w^Q \mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}$$

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## Upshot

Computations for Schubert polynomials can be moved into symmetric functions.

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- Definition with  $t$  important for Macdonald polynomials.
- Many definitions. A new one makes proofs easier!

# Raising Operators on Symmetric Functions

- Raising operators  $R_{i,j}$  act on diagrams

$$R_{1,3} \left( \begin{array}{|c|c|c|} \hline \color{red}{\square} & & \\ \hline \square & & \\ \hline \square & \square & \square \\ \hline \end{array} \right) = \begin{array}{|c|c|c|c|} \hline \square & & & \\ \hline \square & \square & \square & \color{red}{\square} \\ \hline \end{array} \quad R_{2,3} \left( \begin{array}{|c|} \hline \color{red}{\square} \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline \square & \color{red}{\square} \\ \hline \square & \\ \hline \end{array}$$

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$$s_{22} = (1 - R_{12})h_{22} = h_{22} - h_{31}$$

$$\begin{aligned} s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13})h_{211} \\ &= h_{211} - h_{301} - h_{220} - \text{red } h_{310} + \text{red } h_{310} + h_{32-1} + h_{400} - h_{41-1} \end{aligned}$$

some terms cancel



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$$s_\alpha = \prod_{i < j} (1 - R_{ij}) h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

For  $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$ ,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

# Root Ideals

A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path above the diagonal.

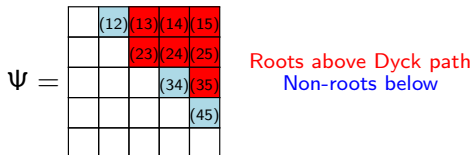
$\Psi =$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

Roots above Dyck path  
Non-roots below

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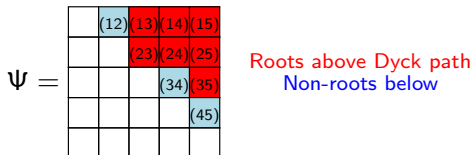
## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For  $\Psi$  and  $\gamma \in \mathbb{Z}^{\ell}$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_{\ell}^{+} \setminus \Psi} (1 - R_{ij}) h_{\gamma}(x)$$

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A root ideal  $\Psi$  of type  $A_{\ell-1}$  positive roots: given by Dyck path above the diagonal.



## Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

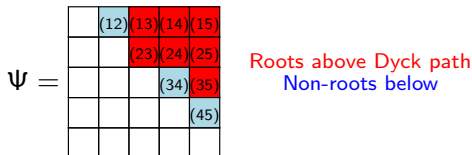
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## $k$ -Schur root ideal for $\lambda$

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$\leftarrow$  row  $i$  has  $4 - \lambda_i$  non-roots



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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

# Catalan functions

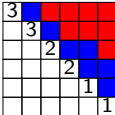
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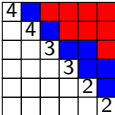
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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

# Dual Grothendieck polynomials

- Inhomogeneous basis:  $g_\lambda = s_\lambda + \text{lower degree terms}$ .

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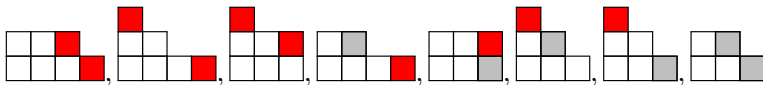




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- Dual to Grothendieck polynomials: Schubert representatives for  $K^*(Gr(m, n))$

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The diagram illustrates the Pieri rule for  $K$ - $k$ -Schur functions. The top row shows the product of a single vertical strip ( $g_1$ ) and a 2-strip ( $g^{(2)}$ ) as the difference of two 2-strips. The bottom row shows the corresponding set-valued strips with colored dots (red, blue, black) and shaded cells, illustrating the same relationship.

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- Conjecture:  $g_\lambda^{(k)}$  have branching into  $g_\mu^{(k+1)}$ .

## Problem

No direct formula for  $g_\lambda^{(k)}$

# An Extra Ingredient: Lowering Operators

Lowering Operators  $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left( \begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$



## $K$ -theoretic Catalan function

Let  $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$  be order ideals of positive roots and  $\gamma \in \mathbb{Z}^\ell$ , then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

# Affine $K$ -Theory Representatives with Raising Operators

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## Example

non-roots of  $\Psi$ , roots of  $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332} \end{aligned}$$

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For  $K$ -homology of affine Grassmannian,

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## Example

$$g_{332111}^{(4)} =$$

3						
	3					
		2				
			1			
				1		
					1	

$$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$$

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## Theorem (Blasiak-Morse-S., 2020)

The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy  $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$ .



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- 4 Describe the image of  $\mathfrak{G}_w^Q$  under Peterson isomorphism for all  $w \in S_{k+1}$ .

Thank you!

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