# INTRODUCTION TO ALGEBRAIC COMBINATORICS: (INCOMPLETE) NOTES FROM A COURSE TAUGHT BY JENNIFER MORSE

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These are a set of incomplete notes from an introductory class on algebraic combinatorics I took with Dr. Jennifer Morse in Spring 2018. Especially early on in these notes, I have taken the liberty of skipping a lot of details, since I was mainly focused on understanding symmetric functions when writing. Throughout I have assumed basic knowledge of the group theory of the symmetric group, ring theory of polynomial rings, and familiarity with set theoretic constructions, such as posets. A reader with a strong grasp on introductory enumerative combinatorics would probably have few problems skipping ahead to symmetric functions and referring back to the earlier sections as necessary.

I want to thank Matthew Lancellotti, Mojdeh Tarighat, and Per Alexandersson for helpful discussions, comments, and suggestions about these notes. Also, a special thank you to Jennifer Morse for teaching the class on which these notes are based and for many fruitful and enlightening conversations.

In these notes, we use French notation for Ferrers diagrams and Young tableaux, so the Ferrers diagram of $(5, 3, 3, 1)$ is
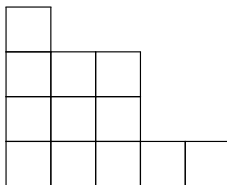
We also frequently use one-line notation for permutations, so the permutation $\sigma = (4, 3, 5, 2, 1) \in \mathfrak{S}_5$ has

$$\sigma(1) = 4, \sigma(2) = 3, \sigma(3) = 5, \sigma(4) = 2, \sigma(5) = 1$$

0. Preliminaries

This section is an introduction to some notions on permutations and partitions. Most of the arguments are given in brief or not at all. A familiar reader can skip this section and refer back to it as necessary.

0.1. **Some Statistics.**

0.1. Definition. A *statistic* on permutations is a function $\mathrm{stat}\colon \mathfrak{S}_n \to \mathbb{N}$.

0.2. Definition. Given a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ in one-line notation, we have the following definitions.

   (a) The *length* of σ, denoted $\ell(\sigma)$, is the number of pairs (i, j) such that $i < j$ and $\sigma_i > \sigma_j$.
   (b) The *sign* of σ is $(-1)^{\ell(\sigma)}$.
   (c) The *inversion statistic* on σ is defined by

$$\mathrm{inv}(\sigma) = \ell(\sigma)$$

   (d) A *descent* of σ is an index $i \in [n-1]$ such that $\sigma_i > \sigma_{i+1}$. The set of all descents on σ is denoted $\mathrm{Des}(\sigma)$.
   (e) The *major statistic* or *major index* is given by

$$\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i$$

0.3. Example. Let $\sigma = (3, 1, 4, 2, 5)$. Then, it has

- Inversions $(1, 2), (1, 4), (3, 4)$. Thus, $\mathrm{inv}(\sigma) = 3 = \ell(\sigma)$.
- Descents $\{1, 3\}$ and thus $\mathrm{maj}(\sigma) = 1 + 3 = 4$.

0.4. Remark. Some readers may be more familiar with the definition of $\ell(\sigma)$ to be the length of the reduced word of σ in terms of simple transpositions of $\mathfrak{S}_n$. These two notions are equivalent. This can be made explicit using "Rothe diagrams."

0.5. Definition. Any statistic which is equidistributed over $\mathfrak{S}_n$ with the inversion statistic, inv, is called *Mahonian*.

0.6. Theorem. *The major index,* maj*, is Mahonian.*

0.7. Example. Given that

$$\mathfrak{S}_3 = \{(123), (213), (132), (231), (312), (321)\} \text{ in one-line notation}$$

we have the corresponding lengths $\{0, 1, 1, 2, 2, 3\}$ and corresponding descent sets $\{\{\}, \{1\}, \{2\}, \{1\}, \{2\}, \{1, 2\}\}$, giving major indexes $\{0, 1, 2, 1, 2, 3\}$. Thus, on $\mathfrak{S}_3$, inv and maj are equidistributed.

   To prove that maj is Mahonian in general, we will use the characterization of Mahonian given below.

0.8. Theorem. *A statistic* stat *is Mahonian if and only if it satisfies the identity*

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{stat}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)}$$

0.9. Remark. Note that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q! := (1)(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

which can be proved using an inductive argument by noting

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_{n-1}} q^{\mathrm{inv}(\sigma)} + \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n) \neq n}} q^{\mathrm{inv}(\sigma)} = [n-1]_q! + \sum_{j=1}^{n-1} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n)=j}} q^{\mathrm{inv}(\sigma)}$$

but we will not need this in what follows.

To prove maj is Mahonian, we will use the following theorem, although other bijections exist, such as the Carlitz bijection.

0.10. Theorem (Foata Bijection). *There exists a bijection* $\psi \colon \mathfrak{S}_n \to \mathfrak{S}_n$ *such that* $\mathrm{maj}(\sigma) = \mathrm{inv}(\psi(\sigma))$.

Proof. Let $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ in one-line notation. Then, set $\psi(w_1) = w_1$. Next, if $\psi(w_1 \cdots w_{i-1}) = v_1 \ldots v_{i-1}$, then, we define $\psi(w_1 \cdots w_i)$ by

   (a) Add w_i on the right.
   (b) Place vertical bars in $w_1 \cdots w_i$ following the rules
      (i) If $w_i > v_{i-1}$, then place a bar to the right of any v_k with $w_i > v_k$.
      (ii) If $w_i < v_{i-1}$, then place a bar to the right of any v_k with $w_i < v_k$.
   (c) Cyclically permute one step to the right in any run between two bars.

0.11. Example. Let $\sigma = 31426875$ which has $\mathrm{maj}(\sigma) = 17$. Then, we iterate

$$3$$
$$|3|1$$
$$|3|1|4$$
$$3|14|2 \to 3412$$
$$3|4|1|2|6$$
$$3|4|1|2|6|8$$
$$|341268|7 \to 8341267$$
$$|8|34126|7|5 \to 86341275 = \psi(\sigma)$$

Indeed, $\mathrm{inv}(\psi(\sigma)) = 17$

To invert the Foata bijection, one does the following

   (a) For $w = w_1 \cdots w_i$, place a dot between w_{i-1} and w_i.
   (b) Place vertical bars in $w_1 \ldots w_i$ following the rules
      (i) If $w_i > w_1$, place a bar to the left of any w_k with $w_k < w_i$ and also before w_i.
      (ii) If $w_i < w_1$, place a bar to the left of any w_k with $w_k > w_i$ and also before w_i.
   (c) Cyclically permute every run between two bars to the left.

(d) w_i will be the ith entry of the resulting permutation.

0.12. **Example.** For our example, if we start with 86341275, we carry out

$$
\begin{array}{ll}
|8|63412|7.5 \rightarrow 8341267 & 5 \\
|834126.7 \rightarrow 341268 & 7 \\
|3|4|1|2|6.8 \rightarrow 34126 & 8 \\
|3|4|1|2.6 \rightarrow 3412 & 6 \\
|3|41.2 \rightarrow 314 & 2 \\
|3|1.4 \rightarrow 31 & 4 \\
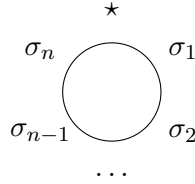3.1 \rightarrow 3 & 1 \\
.3 & 3
\end{array}
$$

and thus we recover $\sigma = 31426875$.

$\square$

0.13. **Proposition.** *The Foata bijection preserves the inverse descent set of* σ.

0.14. **Definition.** The *charge statistic* on a permutation σ is given by

(a) Write the permutation in one-line notation counterclockwise around a circle with a $\star$ between the beginning and the end.
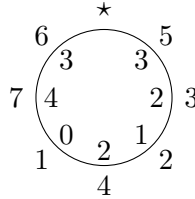

(b) To each entry σ_k, assign an "index" I_k recursively as follows

$$
I_k = \begin{cases} I_{k-1} + 1 & \text{if } \star \text{ is passed going from } \sigma_{k-1} \text{ to } \sigma_k \text{ in a clockwise path around the circle.} \\ I_{k-1} & \text{otherwise.} \end{cases}
$$

(c) Define charge$(\sigma) = \sum_{k=1}^{n} I_k$, that is, the sum of the indices.

0.15. **Example.** The permutation $(6, 7, 1, 4, 2, 3, 5)$ has

charge $3 + 4 + 0 + 2 + 1 + 2 + 3 = 15$.

0.16. Definition. We define the *cocharge* of a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ to be

$$\text{cocharge}(\sigma) := \binom{n}{2} - \text{charge}(\sigma)$$

We also define

$$\text{comaj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} (n - i)$$

0.17. Remark. We can also show cocharge can be computed like charge, but with the "opposite" recurrence relation of charge, which is what we will use to show the proposition below.
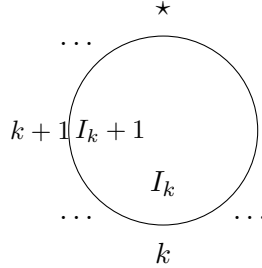
0.18. Proposition. *Given a permutation* σ,

$$\text{cocharge}(\sigma) = \text{comaj}(\sigma^{-1})$$

Idea of Proof. Let σ be a permutation with a descent k, that is, $\sigma_k > \sigma_{k+1}$. This adds $n - k$ to $\text{comaj}(\sigma)$. Then, observe

$$\sigma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ \sigma_1 & \cdots & \sigma_k & \sigma_{k+1} & \cdots & \sigma_n \end{pmatrix} \xrightarrow{-1} \sigma^{-1} = \begin{pmatrix} \cdots & \sigma_{k+1} & \cdots & \sigma_k & \cdots \\ \cdots & k+1 & \cdots & k & \cdots \end{pmatrix}$$

That is, $k+1$ is to the left of k in σ^{-1}. So, our cocharge diagram will look like

Therefore, the $k + 1$ occurring before the k adds 1 to the index for every number greater than k, which adds $n - k$ to the $\text{cocharge}(\sigma^{-1})$. $\square$

0.2. Dominance Order on Partitions.

0.19. Definition. Let S be the set of partitions of degree n, ie whose parts sum to n. Then, for $\lambda, \mu \in S$, the *dominance order* on S is given by

$$\lambda \trianglelefteq \mu \iff \sum_{i=1}^{\ell} \lambda_i \leq \sum_{i=1}^{\ell} \mu_i \quad \forall \ell$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_k)$.

0.20. Example. $(1, 1, 1, 1) \trianglelefteq (2, 2) \trianglelefteq (4)$ and $(2, 2, 1) \trianglerighteq (2, 1, 1, 1)$. Furthermore, $(3, 3)$ and $(4, 1, 1)$ are incomparable since $3 < 4$ but $3 + 3 = 6 > 5 = 4 + 1$.

0.21. Definition. An partition μ *covers* λ, denoted $\mu \gtrdot \lambda$, when $\mu \rhd \lambda$ (that is, $\mu \trianglerighteq \lambda, \mu \neq \lambda$) and there is no partition ν such that $\mu \rhd \nu \rhd \lambda$.

0.22. **Remark.** Covering relations are sufficient to define a partial order in general.

0.23. **Proposition.** *Given partitions λ, μ of degree n,*

$$\lambda \trianglelefteq \mu \iff \lambda' \trianglerighteq \mu'$$

where λ' is the conjugate partition to λ, and similarly for μ'.

Proof. See [Mac79, 1.11]. □

0.24. **Definition.** The *weak order* on $\mathfrak{S}_n$ is defined by, for $\sigma, \gamma \in \mathfrak{S}_n$,

$$\sigma \lessdot_w \gamma \iff \gamma = \sigma s_i \text{ and } \ell(\gamma) = \ell(\sigma) + 1$$

where s_i is the transposition switching i and $i + 1$.

0.25. **Example.** In one-line notation,

$$(123) \lessdot (213) = (123)s_1 \lessdot (231) = (213)s_2 = s_1 s_2$$

Note that

$$s_1 = (213) \not\lessdot (312) = s_2 s_1$$

0.26. **Definition.** The *strong (Bruhat) order* on $\mathfrak{S}_n$ is defined by, for $\sigma, \gamma \in \mathfrak{S}_n$,

$$\sigma \lessdot_s \gamma \iff \gamma = \sigma \tau_{ij} \text{ and } \ell(\gamma) = \ell(\sigma) + 1$$

where τ_{ij} is the transposition switching i and j.

0.27. **Theorem** (Strong Exchange Property). *Given $\sigma \lessdot_s \gamma$ and $\gamma = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, then $\sigma = s_{i_1} s_{i_2} \cdots \hat{s}_{i_k} \cdots s_{i_\ell}$ where the hat means s_{i_k} is deleted from the expression.*

0.3. **Young's Poset on Partitions.**

0.28. **Definition.** The *Young order* on partitions is given by

$$\lambda \leq \mu \iff \lambda \subseteq \mu$$

when viewed as Ferrers diagrams. Equivalently,

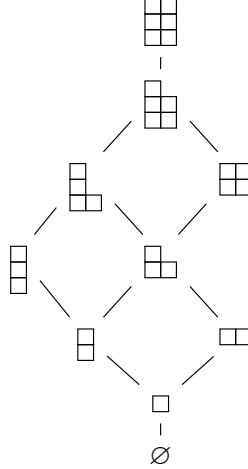$$\lambda \lessdot \mu \iff \mu = \lambda + \text{ an addable corner}$$

0.29. **Definition.** Given a rectangle (m^n), the *order ideal generated by* (m^n) is $\{\lambda \subseteq (m^n)\}$ with induced containment order. The *rank generating function* is given by

$$\sum_{\lambda \subseteq (m^n)} q^{|\lambda|}$$

where $|\lambda| = \sum \lambda_i$.

0.30. **Example.** The induced sub-poset of $(2^3) = (2, 2, 2)$ is given by

and thus the rank generating function is

$$1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$

0.31. **Definition.** We define the *quantum binomial* or *Gaussian polynomial* to be

$$\left[\begin{array}{c} m + n \\ n \end{array} \right]_q := \frac{[m+n]_q!}{[m]_q![n]_q!}$$

0.32. **Example.**

$$\left[\begin{array}{c} 2 \\ 1 \end{array} \right]_q = \frac{1+q}{1 \cdot 1} = 1 + q$$

0.33. **Proposition.** *Given* $m, n \in \mathbb{N}$*, we have*

$$\sum_{\lambda \subseteq (m^n)} q^{|\lambda|} = \left[\begin{array}{c} m + n \\ n \end{array} \right]_q$$

Proof. The proof is given by an inductive argument on $m + n$. For the inductive step, one breaks up the set $\{\lambda \subseteq (m^n)\}$ into the set of all λ's containing a top left cell and the set of all λ's that do not, thus giving

$$\sum_{\lambda \subseteq (m^n)} q^{|\lambda|} = \sum_{\lambda \subseteq ((m-1)^n)} q^n q^{|\lambda|} + \sum_{\lambda \subseteq (m^{n-1})} q^{|\lambda|}$$

$\square$

1. Tableaux

1.1. **Tableaux Basics.**

1.1. **Proposition.** *A saturated chain in Young's Poset is given by*

$$\varnothing \lessdot \lambda^1 \lessdot \lambda^2 \lessdot \cdots \lessdot \lambda^\ell$$

where λ^i/λ^{i-1} *is one cell.*

1.2. Definition. A *standard Young tableau* of shape λ is given by a saturated chain in Young's poset where each λ^i/λ^{i-1} is filled with the number i.
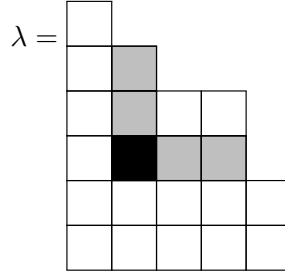
Alternatively, one can define a standard Young tableau of shape $\lambda \vdash n$ to be a filling of λ with $\{1, 2, \ldots, n\}$ such that rows and columns are strictly increasing.

1.3. Example.

1.4. Definition. Given a partition λ, the *hook length* of cell $x = (i, j) \in \lambda$, denoted $h_\lambda(x)$, is given by the number of cells in row i easy of x plus the number of cells in column j north of x plus 1.

1.5. Example.

The black shaded cell above has hook length $2 + 2 + 1 = 5$. The following diagram is filled in with the hook length of each cell.

1.6. Proposition. *Let f^λ be the number of standard Young tableaux of shape $\lambda \vdash n$. Then,*

$$f^\lambda = \frac{n!}{\prod_{x \in \lambda} h_\lambda(x)}$$

1.2. Robinson-Schensted-Knuth (RSK) Correspondence. We will discuss a bijection between matrices with entries in $\mathbb{N}_0$ with finite support and pairs of semistandard Young tableaux (P, Q) of the same shape.

1.7. Definition. Let $\mathsf{T} = (\mathsf{T}_{i,j})$ be a tableau and x be a positive integer. We define *row insertion*, denoted $\mathsf{T} \leftarrow x$ to be the algorithm, that takes T and adds a cell labeled x as follows.

(1) Go to row 1. Pick r_1 such that $\mathsf{T}_{1,r_1-1} \leq x$ (if $\mathsf{T}_{1,1} > x, r_1 = 1$).
(2) If T_{1,r_1} does not exist, add cell labeled x to the end of the row.
(3) Otherwise, replace T_{1,r_1} with x to get T'. Then, do $\mathsf{T}' \leftarrow \mathsf{T}_{1,r_1}$ on row 2.

(4) Repeat this process until either the bumped entry can be put into the end of the row into which it is bumped or until it is bumped out of the final row, in which case it will form a new row with one entry.

1.8. **Example.** Let

$$\mathsf{T} = \begin{array}{|c|c|c|c|}\hline 8 \\ \hline 5 & 6 & 6 \\ \hline 1 & 1 & 2 & 4 \\ \hline \end{array}$$

Then,

$$\mathsf{T} \leftarrow 3 = \begin{array}{|c|c|c|}\hline 8 \\ \hline 5 \\ \hline 4 & 6 & 6 \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array} \qquad \text{and } \mathsf{T} \leftarrow 5 = \begin{array}{|c|c|c|c|c|}\hline 8 \\ \hline 5 & 6 & 6 \\ \hline 1 & 1 & 2 & 4 & 5 \\ \hline \end{array}$$

Explicitly, in the first case, we get the steps

Notice that, in both cases, we ended up with a semistandard Young tableau.

1.9. **Lemma.** *If* T *is a semistandard Young tableau, then so is* $\mathsf{T} \leftarrow x$.

Proof. From the algorithm, it is immediate that the rows will still be weakly increasing, so we need only check that the columns are strictly increasing. To do this, we will define the notion of an "insertion path."

1.10. **Definition.** The *insertion path* or *bumping route* of the row-insertion $\mathsf{T} \leftarrow x$ is the collection of the cells $(1, r_1), (2, r_2), \ldots, (k, r_k)$ where $(r_1, \ldots, r_k)$ come from the row-bumping algorithm.

1.11. **Example.** In

$$\begin{array}{|c|c|c|c|}\hline 8 \\ \hline 5 & 6 & 6 \\ \hline 1 & 1 & 2 & 4 \\ \hline \end{array} \leftarrow 3 = \begin{array}{|c|c|c|}\hline 8 \\ \hline 5 \\ \hline 4 & 6 & 6 \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array}$$

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the bumping path is

$$
\begin{array}{|c|c|c|c|}
\hline
8 \\
\hline
5 \\
\hline
4 & 6 & 6 \\
\hline
1 & 1 & 2 & 3 \\
\hline
\end{array}
$$

or $((1,4),(2,1),(3,1),(4,1))$.

We want to show that the insertion path (starting at row 1) always moves in a weakly westward direction.

Let us say an entry b, originally in $\mathsf{T}_{n,m}$, is bumped into $\mathsf{T}_{n+1,m'}$. If $m' > m$, then m' is the largest integer such that $\mathsf{T}_{n+1,m'-1} \leq b$ by the construction of the algorithm. So, we have the following picture

$$
\begin{array}{|c|c|c|}
\hline
\mathsf{T}_{n+1,m} & \cdots & \mathsf{T}_{n+1,m'} \\
\hline
\mathsf{T}_{n,m} & \cdots & \mathsf{T}_{n,m'} \\
\hline
\end{array}
\quad \leftrightarrow \quad
\begin{array}{|c|c|c|}
\hline
> b & \cdots & > b \\
\hline
b & \cdots & \geq b \\
\hline
\end{array}
$$

where we have $\mathsf{T}_{n,m'} \geq b$ by the semistandardness of T, which also gives $\mathsf{T}_{n+1,m'} > b$. However, this would force $\mathsf{T}_{n+1,m'-1} > b$, which contradicts the inequality $\mathsf{T}_{n+1,m'-1} \leq b$ above. $\qquad\square$

1.12. Lemma. *Let T be a semistandard Young tableau. If $j \leq k$, then the insertion path $\mathsf{T} \leftarrow j$ is strictly to the left of $(\mathsf{T} \leftarrow j) \leftarrow k$.*

1.13. Remark. This lemma can be strengthened, but we are content with the above. See [Ful97, p 9] for a more complete version. Our proof is adapted from the proof in [Ful97].

Proof of Lemma. Suppose $j \leq k$ and j bumps an element x from the first row. The element y bumped by k from the first row must lie strictly to the right of the cell where j bumped x by the semistandardness of T and the construction of the row-bumping algorithm. Furthermore, we get $x \leq y$, and so the argument continues from row to row.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
\color{red}\blacksquare & & \color{blue}\blacksquare \\
\cline{1-4}
& & x & y & \\
\cline{1-6}
& & j & & k & \\
\hline
\end{array}
$$

$\qquad\square$

1.14. Remark. An important point that underlies the RSK correspondence is that this row-bumping process is invertible provided one knows the insertion path. For instance,

$$
\begin{array}{|c|c|c|c|}
\hline
8 \\
\cline{1-1}
5 \\
\cline{1-1}
4 & 6 & 6 \\
\hline
1 & 1 & 2 & 3 \\
\hline
\end{array}
$$

Can be undone by simply shifting every entry in the path down.

$$
\begin{array}{|c|c|c|}
\hline
\; \\
\cline{1-1}
8 \\
\cline{1-1}
5 & 6 & 6 \\
\hline
1 & 1 & 2 & 4 & 3 \\
\hline
\end{array}
\quad \rightarrow \quad
\begin{array}{|c|c|c|}
\hline
8 \\
\cline{1-1}
5 & 6 & 6 \\
\hline
1 & 1 & 2 & 4 \\
\hline
\end{array}, 3
$$

In fact, one can do this only knowing the location of the cell that has been added to the diagram, simply running the algorithm backwards

$$
\begin{array}{|c|c|c|}
\hline
8 \\
\cline{1-1}
5 \\
\cline{1-1}
4 & 6 & 6 \\
\hline
1 & 1 & 2 & 3 \\
\hline
\end{array},
\begin{array}{|c|c|c|}
\hline
5 \\
\cline{1-1}
4 & 6 & 6 \\
\hline
1 & 1 & 2 & 3 \\
\hline
\end{array}
\leftarrow 8,
\begin{array}{|c|c|c|}
\hline
8 \\
\cline{1-1}
4 & 6 & 6 \\
\hline
1 & 1 & 2 & 3 \\
\hline
\end{array}
\leftarrow 5,
\begin{array}{|c|c|c|}
\hline
8 \\
\cline{1-1}
5 & 6 & 6 \\
\hline
1 & 1 & 2 & 3 \\
\hline
\end{array}
\leftarrow 4,
\begin{array}{|c|c|c|}
\hline
8 \\
\cline{1-1}
5 & 6 & 6 \\
\hline
1 & 1 & 2 & 4 \\
\hline
\end{array}, 3
$$

In RSK, we will use another tableau to record this information about when boxes were added.

1.15. Definition (Robinson-Schensted). Given a word $w = w_1 w_2 \ldots w_\ell$, let $\mathsf{P}(w)$ be the tableau

$$
P(w) := ((\ldots ((\boxed{w_1} \leftarrow w_2) \leftarrow w_3) \leftarrow \cdots) \leftarrow w_{\ell-1}) \leftarrow w_\ell
$$

and let $\mathsf{Q}(w)$ be the *recording tableau* or *insertion tableau* whose entries are integers $1, \ldots, \ell$ and where k is placed in the box that is added at the kth step in the construction of $P(w)$.

1.16. Example. Let $w = 5482$. Then, we get successive pairs

$$
\boxed{5} \; \boxed{1} \qquad
\begin{array}{|c|}
\hline
5 \\
\hline
4 \\
\hline
\end{array}
\begin{array}{|c|}
\hline
2 \\
\hline
1 \\
\hline
\end{array} \qquad
\begin{array}{|c|c|}
\hline
5 \\
\cline{1-1}
4 & 8 \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
2 \\
\cline{1-1}
1 & 3 \\
\hline
\end{array} \qquad
\begin{array}{|c|c|}
\hline
5 \\
\cline{1-1}
4 \\
\cline{1-1}
2 & 8 \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
4 \\
\cline{1-1}
2 \\
\cline{1-1}
1 & 3 \\
\hline
\end{array}
$$

where the last pair is $(\mathsf{P}(w), \mathsf{Q}(w))$.

  The fact that this is a correspondence between words and pairs of tableaux comes from the reversibility of the row-bump combined with knowing the inserted cell. In the example above, Q is a standard tableaux. We will prove a more general correspondence, which will get us what we need about Q.

1.17. Definition. Given a (possibly infinite) matrix A with entries in $\mathbb{N}_0$ with finite support and nonzero entries , we define

$$w_A := \begin{pmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix}$$

in which, for any pair (i,j) that indexes an entry $A_{i,j}$ of A, there are $A_{i,j}$ columns equal to $\binom{i}{j}$, and all columns satisfy the following two conditions.

- $i_1 \leq \cdots \leq i_m$ and
- If $i_r = i_s$ and $r \leq s$, then $j_r \leq j_s$.

1.18. Example.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$$

1.19. Definition (Robinson-Schensted-Knuth)**.** Given a matrix A with entries in $\mathbb{N}_0$ with finite support, construct

$$w_A = \begin{pmatrix} i_1 & \cdots & i_m \\ j_1 & \cdots & j_m \end{pmatrix}$$

Then, we will construct a sequence of pairs of tableau were

$$(\mathsf{P}^0, \mathsf{Q}^0) = (\varnothing, \varnothing), \mathsf{P}^{t+1} = \mathsf{P} \leftarrow j_{t+1},$$

and $\mathsf{Q}^{t+1} = \mathsf{Q}^t$ with i_{t+1} added in the cell that appeared by doing the insertion for P^{t+1}. Then, we say that A corresponds to $(\mathsf{P}^m, \mathsf{Q}^m)$.

1.20. Example. Continue with A, w_A as in the example above. Then,

$$(\mathsf{P}^7, \mathsf{Q}^7) = \left(\begin{array}{c} \begin{array}{|c|c|}\hline 3 & 3 \\\hline\end{array} \\ \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 3 \\\hline\end{array} \end{array} \;,\; \begin{array}{c} \begin{array}{|c|c|}\hline 2 & 2 \\\hline\end{array} \\ \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\\hline\end{array} \end{array} \right)$$

$$(\mathsf{P}^8, \mathsf{Q}^8) = \left(\begin{array}{c} \begin{array}{|c|c|}\hline 3 & 3 \\\hline\end{array} \\ \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 2 & 2 & 3 & 3 \\\hline\end{array} \end{array} \;,\; \begin{array}{c} \begin{array}{|c|c|}\hline 2 & 2 \\\hline\end{array} \\ \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 3 \\\hline\end{array} \end{array} \right)$$

1.21. Lemma. Q^t *constructed above is a semistandard Young tableau.*

Proof. The rows of Q^t are certainly weakly increasing since the first row of w_A is weakly increasing by construction. For columns strictly increasing, we observe that if $i_k = i_{k+1}$, then $j_k \le j_{k+1}$, and so by lemma 1.12, we get that the insertion path $M \leftarrow j_k$ is strictly to the left of $(M \leftarrow j_k) \leftarrow j_{k+1}$ and so the columns must be strictly increasing. $\qquad\square$

1.22. Theorem. *The RSK correspondence is a actual correspondence between matrices with entries in $\mathbb{N}_0$ with finite support and pairs of semistandard Young tableaux of the same shape.*

Proof. To show we have a correspondence, we will construct an inverse algorithm. Given $(\mathsf{P}, \mathsf{Q}) = (\mathsf{P}^m, \mathsf{Q}^m)$, we do the following.

(1) Consider $\mathsf{Q}_{r,s}$, the rightmost entry of the largest element in Q (since Q can have repeated entries). Then, set $\mathsf{Q}^{m-1} = \mathsf{Q}^m \setminus \mathsf{Q}_{r,s}$.
(2) $\mathsf{P}_{r,s}$ is the last spot in the insertion path of $\mathsf{P}^{m-1} \leftarrow j_m$.
(3) We need $\mathsf{P}_{r-1,t}$, the rightmost element of row $r-1$ that is smaller than $\mathsf{P}_{r,s}$. Then, do reverse bumping until you get j_m.
(4) Repeat this process until w_A is recovered.

1.23. Example. (a)

$$\mathsf{Q} = \begin{array}{c} \begin{array}{|c|c|}\hline 2 & 2 \\\hline\end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\\hline\end{array} \end{array} \implies i_5 = 2$$

(b)

$$\mathsf{P} = \begin{array}{c} \begin{array}{|c|c|}\hline 3 & 3 \\\hline\end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\\hline\end{array} \end{array}$$

(c)

$$\begin{array}{c} \begin{array}{|c|}\hline 3 \\\hline\end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\\hline\end{array} \end{array} \leftarrow 3, \quad \begin{array}{c} \begin{array}{|c|}\hline 3 \\\hline\end{array} \\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\\hline\end{array} \end{array} \text{ and } j_5 = 2$$

We also want to show that this inverse is well-defined, that is, if $i_k = i_{k+1}$, then $j_k \le j_{k+1}$. If $i_k = \mathsf{Q}_{r,s}$ and $i_{k+1} = \mathsf{Q}_{u,v}$, then $r \ge u$ and $s < v$. Then, it must be that the inverse insertion path of $\mathsf{P}_{r,s}$ is strictly to the left of the insertion path of $\mathsf{P}_{u,v}$. Thus, we get, in the first row,

$$\begin{array}{|c|c|c|}\hline j_k & \cdots & j_{k+1} \\\hline\end{array} \implies j_k \le j_{k+1}$$

□

2. Symmetric Polynomials and Symmetric Functions

2.1. Introduction to Symmetric Polynomials. Let R be a commutative ring with unity and let $x_1, x_2, \ldots, x_n$ be indeterminates.

2.1. **Definition.** The ring of polynomials in n variables with coefficients in R is

$$R[x_1, x_2, \ldots, x_n] = \left\{ \sum c_\alpha x^\alpha \mid \alpha \in \mathbb{N}^n \right\} \text{ where } x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

with multiplication $x^\alpha x^\beta = x^{\alpha+\beta}$.

2.2. **Definition.** We say $f \in R[x_1, \ldots, x_n]$ has *homogeneous degree d* if

$$f = \sum_{|\alpha|=d} c_\alpha x^\alpha$$

2.3. **Example.** $3x_1^2 + 19x_1x_3$ has degree 2.

2.4. **Proposition.** *Let*

$$R^d[x_1, \ldots, x_n] := \{f \in R[x_1, \ldots, x_n] \mid f \text{ has degree } d\}$$

Then, $R[x_1, \ldots, x_n]$ is a graded ring with

$$R[x_1, \ldots, x_n] = \bigoplus_{d \geq 0} R^d[x_1, \ldots, x_n]$$

Proof. Let $f \in R^a[x_1, \ldots, x_n]$ and $g \in R^b[x_1, \ldots, x_n]$. Then, $fg \in R^{a+b}[x_1, \ldots, x_n]$.
□

2.5. **Proposition.** *There is a degree preserving $\mathfrak{S}_n$-action on $R[x_1, \ldots, x_n]$ given by*

$$\sigma.f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for $\sigma \in \mathfrak{S}_n$.

2.6. **Example.** Let $\sigma = (1, 3, 2)$ in one-line notation. Then, for

$$g(x) = 3x_1^2 + 19x_1x_3 - 7x_1 + 18x_2x_3^4 \in \mathbb{Z}[x_1, x_2, x_3]$$

we have

$$\sigma g = 3x_1^2 + 19x_1x_2 - 7x_1 + 18x_2^4x_3$$

2.7. **Definition.** The *ring of symmetric polynomials* on n indeterminates is

$$\Lambda_n := \{f \in R[x_1, \ldots, x_n] \mid \sigma f = f \ \forall \sigma \in \mathfrak{S}_n\}$$

This is a subring of $R[x_1, \ldots, x_n]$ and is sometimes denoted $R[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ (meaning $\mathfrak{S}_n$-action fixed points).

2.8. **Example.** $x_1 + x_2 + x - 3$ and $3x_1^2 + 3x_2^2 + 3x_3^2 - 9x_1x_2x_3$ are both in Λ_3. However, $x_1 + x_2$ is not in Λ_3 since the permutation $(1, 3, 2)$ (one-line) yields $x_1 + x_3$.

2.9. Corollary.

$$\Lambda_n = \bigoplus_{d \geq 0} \Lambda_n^d \ \text{where} \ \Lambda_n^d := \{f \in \Lambda_n \mid f \ \text{has degree } d\}$$

Furthermore, if R is a field, say $\Bbbk$, then Λ_n is a graded algebra since $\Bbbk[x_1, \ldots, x_n]$ is a graded algebra.

2.10. Remark. In particular, this makes Λ_n a vector space over $\Bbbk$. Thus, we can ask about various $\Bbbk$-bases of Λ_n.

2.11. Definition. For $\lambda \vdash d$, the monomial symmetric polynomial is

$$m_\lambda(x_1, \ldots, x_n) := \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha \sim \lambda}} x^\alpha$$

2.12. Example.
- $m_1(x_1, x_2) = x_1 + x_2$ since $(1, 0) \sim (0, 1)$.
- $m_{211}(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2$ since $(211) \sim (121) \sim (112)$.
- $m_{211}(x_1, x_2) = 0$ because there is no partition with 2 parts that can be rearranged to $(2, 1, 1)$.

2.13. Proposition. $\{m_\lambda(x_1, \ldots, x_n)\}_{\lambda \vdash d}$ *is a basis for* Λ_n^d.

Proof. Each element of our set is symmetric since every monomial coefficient is 1 and every permutation of x^λ occurs in m_λ. Also, linear independence is immediate; each m_λ has unique monomials. Thus, it suffices to show that monomial symmetric polynomials span Λ_n^d.

Given $f \in \Lambda_n^d$, we write

$$f = \sum_{|\alpha| = d} c_\alpha x^\alpha \ \text{where} \ c_\alpha = c_{\sigma(\alpha)}, \forall \sigma \in \mathfrak{S}_n$$

In particular, $c_\alpha = c_\lambda$ for $\alpha \sim \lambda$. In otherwords, λ is a partition rearrangement of α Thus,

$$f = \sum_{\lambda \vdash d} c_\lambda \left(\sum_{\alpha \sim \lambda} x^\alpha \right) = \sum_{\lambda \vdash d} c_\lambda m_\lambda(x_1, \ldots, x_n)$$

$\square$

2.14. Corollary. *An immediate corollary to this result is that any basis for* Λ_n^d *can be indexed by* $\lambda \vdash d$.

2.15. Definition. For α a partition, we establish the convention

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

2.16. Definition. For $r \geq 0$, *elementary symmetric polynomial*

$$e_r(x_1, x_2, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

and, for $\lambda \vdash d$ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$$e_\lambda(x_1, \ldots, x_n) = e_{\lambda_1}(x_1, \ldots, x_n) e_{\lambda_2}(x_1, \ldots, x_n) \cdots e_{\lambda_\ell}(x_1, \ldots, x_n)$$

2.17. Example.

$$
\begin{aligned}
e_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 = m_1(x_1, x_2, x_3) \\
e_2(x_1, x_2, x_3) &= x_1 x_2 + x_1 x_3 + x_2 x_3 = m_{11}(x_1, x_2, x_3) \\
e_{21}(x_1, x_2, x_3) &= (x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3) \\
&= x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + \\
&\quad x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + \\
&\quad x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2 \\
&= m_{21}(x_1, x_2, x_3) + 3 m_{111}(x_1, x_2, x_3)
\end{aligned}
$$

2.18. Proposition. $e_r = m_{1^r}$ *and* $\{e_r(x_1, \ldots, x_n)\}_{r \in \mathbb{N}}$ *cannot form a basis of* Λ.

Proof. By definition, the elementary symmetric polynomials e_r are symmetric polynomials with degree r monomials with no variable occurring more than once in each monomial. Thus, it must be that $e_r = m_{1^r}$. Furthermore, by the corollary to the monomial symmetric polynomials forming a basis, it must be that $\{e_r\}_{r \in \mathbb{N}}$ cannot form a basis because each $e_r \in \Lambda_n^r$ and so there simply are not enough in each Λ_n^d to form a basis. $\square$

2.19. Proposition. *The set* $\{e_\lambda(x_1, \ldots, x_n)\}_{\lambda \vdash d}$ *is a basis for* Λ_n^d.

Proof. Since this set is indexed by $\lambda \vdash d$, it suffices to show that the set spans Λ_n^d. Since $\{m_\lambda\}$ is a basis, it suffices to show

$$m_\lambda = \sum c_{\lambda, \mu} e_\mu$$

for some $c_{\lambda, \mu}$. Instead, we will show that

$$e_\eta = \sum M_{\eta, \mu} m_\mu \text{ and } (M_{\eta, \mu})_{\eta, \mu} \text{ is a unitriangular matrix}$$

To do this, we will need the combinatorics of $(0, 1)$-matrices.

2.20. Definition. Given a matrix $(0, 1)$-matrix A, let the *row sum* be

$$\mathrm{row}(A) := (r_1, f_2, \ldots, r_n), r_i = \text{number of 1's in row } i$$

and *column sum* be

$$\mathrm{col}(A) := (c_1, \ldots, c_n), c_j = \text{number of 1's in row } j$$

Finally, let $B_{\alpha \beta}$ be the number of $(0, 1)$ matrices with row sum vector α and column sum vector β.

2.21. **Example.** Some matrices with $\mathrm{row}(A) = (2,1,1,0)$

$$
\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

The second and fourth matrices have $\mathrm{col}(A) = (2,2,0,0)$.

Then, we claim

$$
e_{\nu'} = m_\nu + \sum_{\nu \rhd \mu} B_{\nu'\mu} m_\mu
$$

To do this, first consider one term. Let A^* be the $(0,1)$-matrix with row sum ν' where all the one's are left justified.

2.22. **Example.** If $\nu' = (2,1,1,0)$, then

$$
A^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

Note that $\mathrm{col}(A^*) = (3,1) = \nu$

Such an element has $\mathrm{col}(A^*) = \nu$ and it is the unique $(0,1)$-matrix with $\mathrm{row}(A) = \nu'$ and $\mathrm{col}(A) = \nu$. In particular, if A has ν_1 ones in column one and $\mathrm{row}(A) = \nu;$, the first $\ell(\nu') = \nu_1$ rows have ones. Thus, all ones in column 1 are north justified. Iterating this process gives the uniqueness result. Any other matrix A with $\mathrm{row}(A) = \nu'$ will have column sum smaller in dominance than ν' since the ones in A^* are pushed all the way to the left.

Thus, let $\eta = \nu'$. Then,

$$
e_\eta = e_{\eta_1}(x_1,\ldots,x_n) e_{\eta_2}(x_1,\ldots,x_n) \cdots e_{\eta_\ell}(x_1,\ldots,x_n)
$$

$$
= \left(\sum_{i_1 < i_2 < \cdots < i_{\eta_1}} x_{i_1} \cdots x_{i_{\eta_1}} \right) \left(\sum_{j_1 < j_2 < \cdots < j_{\eta_2}} x_{j_1} \cdots x_{j_{\eta_2}} \right) \cdots \left(\sum_{z_1 < \cdots < z_{\eta_\ell}} x_{z_1} \cdots x_{z_{\eta_\ell}} \right)
$$

$$
= \sum_{\substack{\alpha \text{ obtained by choosing} \\ i_1 < i_2 < \cdots < i_{\eta_1} \\ j_1 < j_2 < \cdots < j_{\eta_2} \\ \vdots \\ z_1 < z_2 < \cdots < z_{\eta_\ell}}} x^{\mathrm{col}(X)} \text{ where } X = \begin{pmatrix} x_1 & \textcircled{x_2} & \cdots & \textcircled{x_n} \\ \textcircled{x_1} & \textcircled{x_2} & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_1 & \textcircled{x_2} & \cdots & x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix}
$$

where the circled entries are the $i_1, i_2, \ldots, i_{\eta_1}$ entries of row 1, the $j_1, j_2, \ldots, j_{\eta_2}$ entries of row 2, etc. and thus

$$e_\eta = \sum_{\substack{A \in \{(0,1)\text{-matrices with} \\ \text{row}(A)=\eta\}}} x^{\text{col}(A)} = \sum_{\mu \vdash d} \sum_{\substack{(0,1)\text{-matrices } A \\ \text{with row}(A)=\eta, \\ \text{col}(A)=\mu}} m_\mu = \sum_{\mu \vdash d} B_{\eta,\mu} m_\mu$$

2.23. Example.

$$e_{211} = (x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3) = \sum_{\alpha = \text{col}(A)} x^\alpha$$

So for instance,

$$\begin{pmatrix} x_1 & \boxed{x_2} & \boxed{x_3} \\ \boxed{x_1} & x_2 & x_3 \\ \boxed{x_1} & x_2 & x_3 \end{pmatrix} \longrightarrow x^{\text{col}(A)} = x_1^2 x_2 x_3$$

$$\updownarrow$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\square$

2.24. Corollary (Fundamental Theorem of Symmetric Polynomials). *Λ_n with coefficients in R is a polynomial ring in the n elementary symmetric polynomials, that is, $\Lambda_n \cong R[e_1, \ldots, e_n]$ as commutative rings.*

Proof. By the above, for $f \in \Lambda_n^d$, we write

$$f = \sum_{\mu \vdash d} c_\mu e_\mu (x_1, \ldots, x_n)$$

However, since $e_{\mu_1}, e_{\mu_2}, \ldots, e_{\mu_n}$ commute, define $e^\alpha := e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n}$.

2.25. Example. $e_{3222} = e_3 e_2 e_2 e_2 = e_1^0 e_2^3 e_3^1 = e^{(031)}$.

Then, we get that

$$f = \sum c_\mu e^\alpha$$

where α_1 is the number of 1's in μ, α_2 is the number of 2's in μ, etc. This shows that $f \in R[e_1, \ldots, e_n]$. Thus, $\Lambda_n^d \subseteq R[e_1, \ldots, e_n]$. However, since $e_1, \ldots, e_n$ are all symmetric, it must be that $R[e_1, \ldots, e_n] \subseteq \Lambda_n$. Thus, we get the desired isomorphism. $\square$

2.26. Definition. For $r \in \mathbb{N}$, the *complete symmetric polynomial* is

$$h_r(x_1, \ldots, x_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} x_{i_1} \cdots x_{i_r}$$

and, for λ a partition, the *homogeneous symmetric polynomials* are

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$$

2.27. Example.

$$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$$
$$h_2(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_2^2 + x_2 x_3 + x_3^2 + x_1 x_3$$
$$= m_2(x_1, x_2, x_3) + m_{11}(x_1, x_2, x_3)$$

2.28. Proposition. $h_r(x_1, \ldots, x_n) = \sum_{\lambda \vdash r} m_\lambda$

Proof. This follows immediately by grouping the terms of h_r appropriately. $\qquad\square$

We now wish to show the following.

2.29. Proposition. $\{h_\lambda(x_1, \ldots, x_n)\}_{\lambda \vdash d}$ *is a basis for* Λ_n^d.

We will give a proof by generating functions.

2.30. Lemma. *We have the following generating functions*

$$E(t) = \prod_i (1 + x_i t) = \sum_{k \geq 0} t^k e_k$$

$$H(t) = \frac{1}{\prod_i (1 - x_i t)} = \sum_{k \geq 0} t^k h_k$$

Proof of Lemma. Observe first

$$E(t) = \prod_i (1 + x_i t)$$
$$= (1 + x_1 t)(1 + x_2 t) \cdots (1 + x_n t)$$
$$= \sum_{1 \leq i_1 < \cdots < i_k \leq n} t^j x_{i_1} x_{i_2} \cdots x_{i_k}$$
$$= \sum_{k \geq 0} t^k e_k$$

For $H(t)$, recall the geometric series identity

$$\frac{1}{1 - x_i t} = \sum_{k \geq 0} x_i^k t^k$$

and thus

$$\prod \frac{1}{1 - x_i t} = \prod_i \left(\sum_{k \geq 0} x_i^k t^k \right) = \sum_{k \geq 0} t^k h_k$$

$\qquad\square$

Proof of Proposition 2.29. Since $H(t) = \frac{1}{E(-t)}$, we get

$$H(t)E(-t) = 1 \implies \left(\sum_i t^i h_i\right)\left(\sum_j (-t)^j e_j\right) = 1 + \sum_{k\geq 1} 0 \cdot t^k$$

$$\implies \sum_{j=0}^{n} (-1)^j h_{n-j} e_j = 0, \forall n > 0$$

So, we check

$$n = 1: h_1 - e_1 = 0 \implies e_1 = h_1$$

$$n = 2: h_2 - e_1 h_1 + e_2 = 0 \implies e_2 = e_1 h_1 - h_2 = h_1^2 - h_2$$

and so we can iterate to show that e_r is a sum of h_μ's. Thus, we get that $\{h_\mu\}$ spans Λ_n. $\qquad\square$

2.2. Symmetric Functions.

Notice that we can always get a homomorphism of rings

$$\Lambda_{n+1} \twoheadrightarrow \Lambda_n$$

by setting $x_{n+1} = 0$.

2.31. Example.

$$e_{21}(x_1, x_2, x_3, x_4) = (x_1 x_2 + x_1 x_3 + x_1 x_4 + \cdots)(x_1 + x_2 + x_3 + x_4)$$

$$= m_{21}(x_1, x_2, x_3, x_4) + 3m_{111}(x_1, x_2, x_3, x_4)$$

$$\mapsto e_{21}(x_1, x_2, x_3) = (x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3)$$

$$= m_{21}(x_1, x_2, x_3) + 3m_{111}(x_1, x_2, x_3)$$

In fact, one can check that $m_\lambda(x_1, \ldots, x_{n+1}) \mapsto m_\lambda(x_1, \ldots, x_n)$ which is nontrivial provided $\ell(\lambda) > n$. Thus,

2.32. Proposition. $\Lambda_n^d \cong \Lambda_{n+1}^d$ *as R-modules provided $n \geq d$.*

Thus, the number of variables we work in is largely irrelevant provided there are sufficiently many. In order to avoid this formal annoyance, we will often work with "infinite symmetric polynomials", which we will construct in this following section.

2.33. Definition. Let R be a commutative ring with unity. Then, for an infinite collection of indeterminates $\{x_1, x_2, \ldots\}$, we say $R[[x_1, x_2, \ldots]]$ is the *ring of formal series over R* with elements given as (formal) infinite sums of monomials in the indeterminates, addition given by componentwise addition, and multiplication given by extension of polynomial multiplication.

Note that multiplication is well-defined since, given a product of infinite series

$$\left(\sum_{\alpha \in \mathbb{N}^\infty} c_\alpha x^\alpha\right)\left(\sum_{\beta \in \mathbb{N}^\infty} d_\beta x^\beta\right),$$

the coefficient at x^γ for $\gamma \in \mathbb{N}^\infty$ has only finitely many contributions from the c_α and d_β's.

2.34. Definition. Given $\alpha, \beta \in \mathbb{N}^\infty$, we say $\alpha \sim \beta$ if the underlying multisets of entries are equal (not counting 0).

2.35. Example. $(2, 0, 1, 0) \sim (1, 2, 0)$

2.36. Definition. We say $f \in R[[x_1, x_2, \ldots]]$ is a *symmetric infinite series* if the coefficient of x^α is equal to the coefficient of x^β when $\alpha \sim \beta$.

2.37. Example. The element $\displaystyle\sum_{d \geq 0} \sum_{i \geq 1} x_i^d$ is a symmetric infinite series.

2.38. Definition. A *symmetric polynomial in infinitely many variables* (usually called a *symmetric function*) is a symmetric infinite series with bounded degree. The set of these symmetric polynomials forms a ring of symmetric functions, denoted Λ.

2.39. Example. Let $b \in \mathbb{N}$. Then, $\displaystyle\sum_{d \geq 0}^{b} \sum_{i \geq 1} x_i^d$ is a symmetric function.

2.40. Remark. For the categorically inclined, the symmetric functions can also be constructed as an inverse limit by taking

$$\Lambda^d = \varprojlim_n \Lambda_n^d$$

in the category of $\mathbb{Z}$-modules and then setting

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$

This requires setting up the necessary maps, but this is not so hard given the above for those who wish to use inverse limits. Note, however, that Λ is *not* the inverse limit of Λ_n in the category of rings; such an inverse limit would be the ring of symmetric infinite series. However, Λ would be the inverse limit of Λ_n in the category of graded rings.

We wish to extend our definitions for bases of symmetric polynomials to our ring of symmetric functions.

2.41. Definition. Let $\overline{x} = (x_1, x_2, \ldots)$. Then, we define

(a) for λ a partition,
$$m_\lambda(\overline{x}) = \sum_{\alpha \sim \lambda} x^\alpha$$

(b) for $r \in \mathbb{N}$,
$$e_r(\overline{x}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

(c) for $r \in \mathbb{N}$,

$$h_r(\bar{x}) = \sum_{|\lambda|=r} m_\lambda = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

2.42. Example. $h_2(\bar{x}) = m_{(2)}(x) + m_{(1,1)}(x) = x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots + x_2 x_3 + \cdots$

Finally, we wish to introduce one more family of symmetric functions.

2.43. Definition. For any $r \in \mathbb{N}$, we define the *power symmetric functions*

$$p_r := \sum_i x_i^r$$

and, for any partition λ, we define

$$p_\lambda = \prod_i p_{\lambda_i}$$

2.44. Proposition. *The generating function for the p_r is*

$$P(t) = \sum \frac{x_i}{1 - x_i t}$$

and also satisfies the identities

$$P(t) = \frac{H'(t)}{H(t)} \quad P(-t) = \frac{E'(t)}{E(t)}$$

Proof. By direct computation, we see

$$P(t) := \sum_{r \geq 1} p_r t^{r-1} = \sum_{i \geq 1} \sum_{r \geq 1} x_i^r t^{r-1} = \sum_{i \geq 1} \frac{x_i}{1 - x_i t}$$

Furthermore, if we are clever at manipulating generating functions, we see

$$P(t) = \sum_{i \geq 1} \frac{d}{dt} \log\left(\frac{1}{1 - x_i t}\right) = \frac{d}{dt} \log\left(\prod_{i \geq 1} \frac{1}{1 - x_i t}\right) = \frac{d}{dt} \log(H(t)) = \frac{H'(t)}{H(t)}$$

and

$$P(-t) = \sum_{i \geq 1} \frac{d}{dt} \log(1 + x_i t) = \frac{d}{dt} \log\left(\prod_{i \geq 1}(1 + x_i t)\right) = \frac{d}{dt} \log E(t) = \frac{E'(t)}{E(t)}$$

$\square$

2.45. Corollary (Newton's Identities)**.** *We have the following identities*

$$nh_n = \sum_{r=1}^n p_r h_{n-r}, \quad ne_n = \sum_{r=1}^n (-1)^{r-1} p_r e_{n-r}$$

Proof. These come from the proposition above by rearranging and expanding. For instance,

$$H'(t) = P(t)H(t) \implies \sum_{i \geq 1} ih_i t^{i-1} = \left(\sum_{r \geq 1} p_r t^{r-1} \right) \left(\sum_{j \geq 1} h_j t^j \right)$$

and so, fixing a specific n, we have

$$nh_n = \sum_{r+j=n} p_r h_j$$

giving us the desired identity. $\qquad\qquad\square$

2.46. Theorem. *The set $\{p_\lambda\}_{\text{all partitions } \lambda}$ is a basis for Λ over a field of characteristic 0.*

Proof. By Newton's identities,

$$p_n = \sum_{j=1}^{n} e_j p_{n-j}$$

if we rewrite this recurrence, we get

$$e_n = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} e_{n-j} p_j$$

Thus, each e_n can be written as a rational linear combination of the power sums, giving that $\Lambda_n \subseteq \mathbb{Q}[p_1, \ldots, p_n]$. Therefore, the p_λ's form a basis and because every power sum polynomial is symmetric, $\Lambda_n \cong \mathbb{Q}[p_1, \ldots, p_n]$. $\quad\square$

2.47. Remark. An alternative proof is just to let $C = (c_{\lambda,\mu})$ be a matrix expressing p_λ in terms of the monomial basis. One argues that

$$p_\lambda = c_{\lambda\lambda} m_\lambda + \sum_{\mu \rhd \lambda} c_{\lambda\mu} m_\mu$$

with $c_{\lambda\lambda} \neq 0$. Then, if $x_1^{\mu_1} \cdots x_m^{\mu_m}$ appears in

$$p_\lambda = (x_1^{\lambda_1} + x_2^{\lambda_2} + \cdots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \cdots) + \cdots$$

then each μ_i must be a sum of λ_j's, thus making μ larger in dominance order than λ. This proof is probably simpler than the one above but has the disadvantage of not introducing Newton's identities.

Finally, we note how p_n's are explicitly related to the h_n's and e_n's.

2.48. Proposition. *Given $\lambda \vdash n$, let m_i be the multiplicity of i in λ and set $z_\lambda = \prod_i i^{m_i} m_i!$. Then,*

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$$

and

$$e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{p_\lambda}{z_\lambda}$$

Proof. We show this by writing each h_r as a linear combination of p_λ's. Observe,

$$
\begin{aligned}
\sum_{k=0}^{\infty} h_k(\vec{x}) t^k &= \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \\
&= \prod \exp\left(-\log(1 - x_i t)\right) \\
&= \prod \exp\left(\sum_{k=1}^{\infty} \frac{(x_i t)^k}{k}\right) \\
&= \exp\left(\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{(x_i t)^k}{k}\right) \\
&= \exp\left(\sum_{k=1}^{\infty} \frac{p_k(x) t^k}{k}\right) \\
&= \prod_{k=1}^{\infty} \exp\left(\frac{p_k(x) t^k}{k}\right) \\
&= \prod_{k=1}^{\infty} \sum_{m_k=0}^{\infty} \frac{(p_k(x) t^k)^{m_k}}{m_k! \cdot k^{m_k}} \\
&= \sum_{\lambda} \frac{1}{z(\lambda)} p_\lambda t^{|\lambda|}
\end{aligned}
$$

Thus, we see that $h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$. The relation for e_n is a similar manipulation of the generating function $E(t)$. $\qquad \square$

2.49. **Remark.** Note that $z(\lambda)$ is equal to the cardinality of the centralizer of an element of $\mathfrak{S}_n$ with cycle type λ. Indeed, an i-cycle $(1, 2, \ldots, i)$ is fixed by i elements and a product of m_i disjoint i-cycles can be permuted in $m_i!$ ways.

So then, why does this number show up here?

2.3. **Schur functions.**

2.50. **Definition.** The *Vandermonde determinant* is

$$
\Delta := \det \begin{pmatrix}
x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\
x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\
\vdots & \vdots & & \vdots & \vdots \\
x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1
\end{pmatrix}
$$

2.51. **Theorem.** $\Delta = \displaystyle\prod_{1 \leq i < j \leq n} (x_i - x_j)$

Proof. Note first that $\deg \Delta = \binom{n}{2}$ and $\deg \prod = \binom{n}{2}$ for $\prod$ the right hand side of the equality. Also, observe that if $x_i = x_j$, then $\Delta = 0$ since then

two rows of the matrix would be identical. Therefore, $(x_i - x_j)$ divides Δ, so

$$\Delta = c \cdot \prod_{1 \le i < j \le n} (x_i - x_j)$$

for some c in our base field. Without too much more work, one can verify that $c = 1$. $\qquad\square$

2.52. Definition. For $\alpha \in \mathbb{N}^n$, we define *Jacobi's generalized Vandermonde* as

$$\Delta_\alpha := \det \begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_n} \\ \vdots & \vdots & & \vdots \\ x_n^{\alpha_1} & x_n^{\alpha_2} & \cdots & x_n^{\alpha_n} \end{pmatrix}$$

2.53. Remark. We let $\rho := (n-1, n-2, \ldots, 1, 0)$. Then, $\Delta_\rho = \Delta$, the Vandermonde determinant.

2.54. Proposition. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. *Then, for Jacobi's generalized Vandermonde*

    (a) $\alpha_i = \alpha_j \implies \Delta_\alpha = 0$
    (b) If α *has distinct parts, then* Δ_α *is "alternating," that is* $\sigma\Delta_\alpha = (-1)^{\ell(\sigma)}\Delta_\alpha$ *for any permutation* σ *of* $[n]$.
    (c) Δ_ρ *divides* Δ_α.

Proof.   (a) If $\alpha_i = \alpha_j$, then our matrix has two equal columns, so its determinant is 0.
    (b) This is immediate since $s_i\Delta_\alpha$ swaps the ith and $i + 1$st rows.
    (c) If $x_i = x_j$, then $\Delta_\alpha = 0$ since our matrix has two equal rows. Thus, $(x_i - x_j)$ must divide Δ_α. However, since i, j were arbitrary, we get that every such difference must divide Δ_α, so Δ_ρ divides Δ_α.

$\qquad\square$

Thus, now our following definition makes sense

2.55. Definition. Given a partition λ, *Jacobi's bialternant formula* for a Schur function is

$$s_\lambda(x_1, \ldots, x_n) := \Delta_{\lambda+\rho}/\Delta_\rho$$

which is a polynomial by the above proposition.

2.56. Remark. Both $\Delta_{\lambda+\rho}$ and Δ_ρ are alternating, so $\Delta_{\lambda+\rho}/\Delta_\rho$ is symmetric.

2.57. Theorem (Littlewood's Combinatorial Definition)**.** *For a partition* $\lambda \vdash n$,

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda)\ in\ alphabet\ [n]} x^{\mathrm{wt}(\mathsf{T})}$$

In infinite variables,

$$s_\lambda(x_1, x_2, \ldots) = \sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})}$$

Proof. Postponed. □

One can instead take the statement of the theorem to be the definition of the s_λ, if so inclined, and prove that it implies the bialternate formula; in fact, we will do this. See 2.74. We wish to show, using this construction, that the Schur functions form a basis for Λ.

2.58. Example. Consider $s_{21}(x_1, x_2, x_3)$. The possible semistandard Young tableaux are

and so

$$s_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

2.59. Proposition. *Given λ a partition of n,*

$$\sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})}$$

is a symmetric function.

To prove this fact, we will need the following definition.

2.60. Definition. Let the *Kostka number*, $K_{\lambda\alpha}$ be the number of semistandard Young tableaux of shape λ with weight α. In other words, $|\mathrm{SSYT}(\lambda, \alpha)| = K_{\lambda\alpha}$.

Proof of Proposition. For s_i a simple reflection in S_n, we will show that

$$\sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})} = s_i \sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})}$$

We observe that

$$\sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})} = \sum_{\alpha \in \mathbb{N}^\infty} \sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda, \alpha)} x^\alpha = \sum_{\alpha \in \mathbb{N}^\infty} K_{\lambda\alpha} x^\alpha$$

Therefore, we need only show that $K_{\lambda,\alpha} = K_{\lambda, s_i\alpha}$. To show this, we will use the *Bender-Knuth Involution*, defined below. Namely, there is an involution (thus a bijection) between $\mathrm{SSYT}(\lambda, \alpha)$ and $\mathrm{SSYT}(\lambda, s_i\alpha)$, so

$$K_{\lambda,\alpha} = |\mathrm{SSYT}(\lambda, \alpha)| = |\mathrm{SSYT}(\lambda, s_i\alpha)| = K_{\lambda,s_i\alpha}$$

□

2.61. Definition. The *Bender-Knuth Involution* is a map $\phi_i \colon \mathrm{SSYT}(\lambda, \alpha) \to \mathrm{SSYT}(\lambda, s_i\alpha)$ given as follows:

(a) We define that entries i and $i+1$ in $\mathsf{T} \in \mathrm{SSYT}(\lambda, \alpha)$ are *married* if they are in the same column. Otherwise, we say they are *single*.

2.62. **Example.** For $i = 3$, in

$$\mathsf{T} = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
3 & 4 & 4 & \textcolor{red}{4} \\
\hline
2 & 2 & 2 & \textcolor{red}{3} & 3 & 4 & 4 \\
\hline
1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
\hline
\end{array}$$

the red entries are married and the pink entries are married. All other entries are single.

(b) $\phi_i(\mathsf{T})$ is obtained by replacing single entries

$$\underbrace{iii\cdots i}_{a}\underbrace{i+1\cdots i+1}_{b}$$

in each row by

$$\underbrace{iii\cdots i}_{b}\underbrace{i+1\cdots i+1}_{a}$$

2.63. **Example.** If T is the example above, then

$$\phi_3(\mathsf{T}) = \begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
3 & 3 & 4 & \textcolor{red}{4} \\
\hline
2 & 2 & 2 & \textcolor{red}{3} & 3 & 4 & 4 \\
\hline
1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 \\
\hline
\end{array}$$

2.64. **Proposition.** $\phi_i(\mathsf{T}) \in \mathrm{SSYT}(\lambda, s_i\alpha)$.

Proof. We note that the shape λ is preserved by construction, so we need only show that

(a) Columns strictly increase
(b) Rows weakly increase
(c) The weight is $s_i\alpha$

Note that single entries in a row are contiguous and between married entries, so rows necessarily must be weakly increasing. That is, in a semistandard tableau T, for ϕ_x, $y = x+1$, and $z > y > x > w$, a sufficiently generic row might look like

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\textcolor{red}{y} & y & z & z & \cdots \\
\hline
\textcolor{red}{x} & x & x & x & y & y & \textcolor{blue}{y} & y \\
\hline
\multicolumn{4}{c}{} & \cdots & w & w & \textcolor{blue}{x} & x \\
\end{array}$$

Thus, if x is single, it lies below $z > x+1$ and if y is single, it lies above $w < x$. Therefore, the replacement is still strictly column increasing. Finally, there are an equal number of married i and $i+1$ entries, and the number of single i's in each row is replaced by the number of single $i+1$'s in that row, and vice-versa. Thus, the weight is permuted. $\qquad\square$

Thus, we have shown that Littlewood's combinatorial definition of Schur functions gives a symmetric polynomial. To see the equivalence with Jacobi's bialternate formula, we will actually prove the more general result

2.65. Theorem. [Ste02, p 2]

$$\Delta_{\lambda+\rho} s_{\mu/\nu} = \sum_{\substack{\mathsf{T} \in \mathrm{SSYT}(\mu/\nu) \\ \mathsf{T} \; is \; \lambda-\mathit{Yamanouchi}}} \Delta_{\lambda+\mathrm{wt}(\mathsf{T})+\rho}$$

where $s_{\mu/\nu}$ and λ-Yamanouchi are defined below.

2.66. Definition. A *skew Schur function* for skew-shape μ/ν is given by

$$\sum_{\mathsf{T} \in \mathrm{SSYT}(\mu/\nu)} x^{\mathrm{wt}(\mathsf{T})}$$

and is symmetric by the Bender-Knuth involution.

2.67. Definition. We say a (skew) semistandard tableau T is a *Yamanouchi tableau* if, for all j, $\mathsf{T}_{\geq j}$ (only columns east of $j-1$) has partition weight. This is also sometimes called a *reverse lattice*.

2.68. Example. (a)

is not Yamanouchi; it fails at the second column since $\mathrm{wt}(T_{\geq 4}) = (2, 0, 1)$ is not a partition.

(b) The unique semistandard tableau of shape λ and weight λ is Yamanouchi. In fact, it is the only semistandard "straight shape" that can be Yamanouchi, sometimes called the *super-semistandard tableau of shape λ*.

(c) Here are some Yamanouchi skew-tableau of the following shape.

2.69. Definition. A *λ-Yamanouchi tableau* T has, for all j, $\mathrm{wt}(T_{\geq j}) + \lambda$ is a partition. (Addition is given by vector addition.)

2.70. **Example.** Take $\lambda = (3, 2)$. Then,

2	2	3	
1	1	1	1

is λ-Yamanouchi since

$$\mathrm{wt}(T_{\geq 4}) + \lambda = (4, 2)$$
$$\mathrm{wt}(T_{\geq 3}) + \lambda = (5, 2, 1)$$
$$\mathrm{wt}(T_{\geq 2}) + \lambda = (6, 3, 1)$$
$$\mathrm{wt}(T_{\geq 1}) + \lambda = (7, 4, 1)$$

Proof of 2.65; [Ste02]. We observe that

$$
\Delta_{\lambda+\rho} s_{\mu/\nu} = \sum_{\substack{T \in \mathrm{SSYT}(\lambda) \\ T \ \lambda\text{-Yamanouchi}}} \Delta_{\lambda + \mathrm{wt}(T) + \rho}
$$

$$
= \sum_{w \in \mathfrak{S}_n} (-1)^{\ell(w)} x^{w(\lambda+\rho)} \sum_{T \in \mathrm{SSYT}(\mu/\nu)} x^{\mathrm{wt}(T)}
$$

$$
= \sum_{w \in \mathfrak{S}_n} \sum_{T \in \mathrm{SSYT}(\mu/\nu)} (-1)^{\ell(w)} x^{w(\lambda+\rho+\mathrm{wt}(T))}
$$

$$
= \sum_{T \in \mathrm{SSYT}(\mu/\nu)} \Delta_{\lambda+\rho+\mathrm{wt}(T)}
$$

However, we want to show the result is true summing only over T that are λ-Yamanouchi. To do this, we will show that all the non λ-Yamanouchi tableaux cancel each other out by finding an appropriate sign-reversing involution on them.

To that effect, say T is a "Bad Guy" if T is not λ-Yamanouchi. For such a T, there exists a $j, k \in [n]$ such that

$$\lambda_k + \mathrm{wt}_k(T_{\geq j}) < \lambda_{k+1} + \mathrm{wt}_{k+1}(T_{\geq j})$$

Pick the largest such j and then pick the pair (j, k) with the smallest such k. Then, $\lambda + \mathrm{wt}(T_{>j})$ is a partition. Since, by column strictness, $\mathrm{wt}_k(T_{\geq j}) - \mathrm{wt}_{k+1}(T_{\geq j})$ can change by at most one when j is incremented or decremented, there must be a $k + 1$ in column j of T, but no k.

2.71. **Example.** For an easy example, take $\lambda = \varnothing$ and

$T =$

3	4	4	4			
2	2	2	3	3	4	4
		1	1	2	2	3	3	3
						2	2	2
						1	1	1

The shaded part of the tableau is Yamanouchi, but it fails to be Yamanouchi at $T_{\geq 6}$. Furthermore, the minimal such k so that our inequality fails is 1, that is

$$3 = 0 + 3 = \lambda_1 + \mathrm{wt}_1(T_{\geq 6}) < \lambda_2 + \mathrm{wt}_2(T_{\geq 6}) = 0 + 4 = 4$$

and $\lambda + \mathrm{wt}(T_{> 6}) = (3, 3, 3, 1)$ is a partition. Indeed also, in column 6, there is a $k + 1 = 2$ entry, but not a $k = 1$ entry.

Thus, we have

$$\lambda_k + \mathrm{wt}_k(T_{\geq j}) + 1 = \lambda_{k+1} + \mathrm{wt}_{k+1}(T_{\geq j})$$

which, since $\rho_k = \rho_{k+1} + 1$, also gives

$$s_k(\lambda + \mathrm{wt}(T_{\geq j}) + \rho) = \lambda + \mathrm{wt}(T_{\geq j}) + \rho$$

where $s_k \in \mathfrak{S}_n$ swaps k and $k + 1$. Now, we apply the Bender-Knuth involution ϕ_k to $T_{<j}$ and leave the remainder of T fixed. Let us call the resulting tableau T^*.

2.72. **Example.**

We apply the Bender-Knuth involution ϕ_1 to the red part only to get

Note that the blue part has an equal number of 1's and 2's.

T^* is still an SSYT since the Bender-Knuth involution only swaps some ks and $k + 1$s in $T_{<j}$ in a semistandard way and column j has a $k + 1$, but contains no ks. Thus, by construction, we have the following two properties

$$\begin{cases} (T^*)_{\geq j} = T_{\geq j} & \text{since left unchanged by involution} \\ \mathrm{wt}(T^*_{<j}) = s_k(\mathrm{wt}(T_{<j})) & \text{by construction of Bender-Knuth.} \end{cases}$$

So, T^* is a Bad Guy and, in fact, the map $T \mapsto T^*$ is an involution on Bad Guys (Note, however, it still has fixed points, but we need not worry about those. See 2.73). Finally,

$$\lambda + \mathrm{wt}(T^*) + \rho = \lambda + \mathrm{wt}(T^*_{\geq j}) + \mathrm{wt}(T^*_{<j}) + \rho$$

$$= \lambda + \mathrm{wt}(\mathsf{T}_{\geq j}) + s_k(\mathrm{wt}(\mathsf{T}_{<j})) + \rho$$
$$= s_k(\lambda + \mathrm{wt}(\mathsf{T}_{\geq j}) + \rho) + s_k(\mathrm{wt}(\mathsf{T}_{<j}))$$
$$= s_k(\lambda + \mathrm{wt}(\mathsf{T}) + \rho)$$

and

$$\Delta_{\lambda + \mathrm{wt}(\mathsf{T}^*) + \rho} = \Delta_{s_k(\lambda + \mathrm{wt}(\mathsf{T})) + \rho} = -\Delta_{\lambda + \mathrm{wt}(\mathsf{T}) + \rho}$$

since s_k acts by swapping the k and $k+1$st rows of Δ, which changes the determinant by a sign. Thus, we can cancel the contributions of the Bad Guys from our desired sums.

2.73. Remark. Note, there is still some work to do. A concern is that some "Bad Guys" will actually have $\mathsf{T}_{<j}$ fixed by the BK involution. Consider, for $\lambda = \varnothing$,

$$\mathsf{T} = \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\\hline \end{array}$$

$$\mathsf{T} = \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\\hline \end{array}$$

(The tableau T with rows: top row $2\ 3\ 3$; middle row $2\ 2\ 2$; bottom row $1\ 1\ 1$.)

which fails to be Yamanouchi at column 1. However, such a Bad Guy will have

$$\mathrm{wt}(\mathsf{T}) + \rho = (3, 4, 2) + (2, 1, 0) = (5, 5, 2)$$

and so $\Delta_{(5,5,2)} = 0$ by 2.54(a). In general, if the BK involution fixes $\mathsf{T}_{<j}$, then $\mathsf{T}_{<j}$ necessarily has an equal number of cells labeled k and $k+1$, that is

$$\mathrm{wt}_k(\mathsf{T}_{<j}) = \mathrm{wt}_{k+1}(\mathsf{T}_{<j})$$

Furthermore, since T is a Bad Guy, our choice of (j, k) is such that

$$\lambda_k + \mathrm{wt}_k(\mathsf{T}_{\geq j}) + 1 = \lambda_{k+1} + \mathrm{wt}_{k+1}(\mathsf{T}_{\geq j})$$

So, adding these two expressions together, we get

$$\lambda_k + \mathrm{wt}_k(\mathsf{T}) + 1 = \lambda_{k+1} + \mathrm{wt}_{k+1}(\mathsf{T})$$

and thus, importantly,

$$(\lambda + \mathrm{wt}(\mathsf{T}) + \rho)_k = (\lambda + \mathrm{wt}(\mathsf{T}) + \rho)_{k+1} \implies \Delta_{\lambda + \mathrm{wt}(\mathsf{T}) + \rho} = 0$$

$\square$

2.74. Corollary (Corollaries to Theorem 2.65)**.** *In the following special cases, we get the following corollaries.*

(a) *If $\nu = \varnothing, \lambda = \varnothing$, then*

$$\Delta_\rho s_\mu = \sum_{\mathsf{T} \in \mathrm{SSYT}(\mu), \mathsf{T}\ Yamanouchi} \Delta_{\mathrm{wt}(\mathsf{T}) + \rho}$$

$$\implies \Delta_\rho s_\mu = \Delta_{\mu + \rho} \qquad \textit{Since there is a unique Yamanouchi SSYT of shape } \mu \text{ by 2.68}$$

$$\implies s_\mu = \Delta_{\mu + \rho} / \Delta_\rho$$

(b) If $\nu = \varnothing$, then

$$\Delta_{\lambda+\rho} s_\mu = \sum_{\substack{\mathsf{T} \in \mathrm{SSYT}(\mu) \\ \mathsf{T} \ \lambda\text{-}Yamanouchi}} \Delta_{\lambda+\mathrm{wt}(\mathsf{T})+\rho}$$

$$\implies s_\lambda s_\mu = \sum_{\substack{\mathsf{T} \ of \ shape \ \mu \\ \mathsf{T} \ \lambda\text{-}Yamanouchi}} s_{\lambda+\mathrm{wt}(\mathsf{T})} \qquad \text{\textit{from dividing by} } \Delta_\rho$$

2.75. Remark. The corollaries above have significance outside of combinatorics.

(a) In representation theory, the product rule for Schur functions gives the decomposition of tensor products of Specht modules for $\mathfrak{S}_n$ into irreducibles.

(b) In Schubert calculus, this is useful for understanding the intersection of Grassmanian subvarieties. [Make this more precise.]

We can rephrase the corollary above to get the following major theorem.

2.76. Theorem (Littlewood-Richardson Rule). *Given shapes λ, μ,*

$$s_\lambda s_\mu = \sum_\nu c_{\lambda,\mu}^\nu s_\nu$$

where $c_{\lambda,\mu}^\nu$, called the Littlewood-Richardson coefficients, *counts the number tableaux of skew shape ν/λ of weight μ which are Yamanouchi .* [Check this removal of λ]

2.77. Remark. $c_{\lambda,\mu}^\nu$ is also the number of tableaux of the form

$\mu \to$
$\leftarrow \lambda$
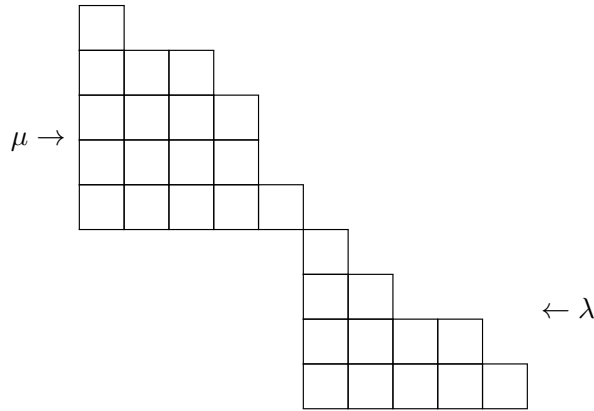
where the bottom right connected component must be super semi-standard and the whole skew shape is Yamanouchi of weight ν.

2.78. Example. Naturally, we expect that, since $s_\varnothing = 1$, we get $s_\varnothing s_\mu = s_\mu$. Indeed, $c_{\varnothing,\mu}^\nu$ is the number of tableaux of shape ν and weight μ, so

$$c_{\varnothing,\mu}^\nu = |\mathrm{SSYT}(\nu,\mu)| = K_{\nu,\mu}$$

recovering s_μ from the Littlewood-Richardson rule.

For something less trivial, take $\lambda = (2, 1)$ and $\mu = (1)$. Then,

$$c_{(2,1),(1)}^{(3,1)} = 1, c_{(2,1),(1)}^{(2,2)} = 1, c_{(2,1),(1)}^{(2,1,1)} = 1$$

since these are the only skew-shapes such that $\nu/(2,1)$ can have one box. Therefore, all the other coefficients are 0 and we get

$$s_{21}s_1 = s_{31} + s_{22} + s_{211}$$

2.79. **Theorem** (Pieri Rule). *We have*

$$h_r s_\lambda = \sum_{\nu = \lambda + \; horizontal \; r\text{-}strip} s_\nu$$

2.80. **Remark.** If one already has the Littlewood-Richardson rule, the Pieri rule is actually a specialization since $s_{(r)} = h_r$ and it turns out that $c_{(r),\mu}^{\nu}$ is always 0 or 1.

Proof of 2.79. Recall

$$h_r s_\lambda = \left(\sum_{\mathsf{T} \; of \; shape \; (r)} x^{\mathrm{wt}(\mathsf{T})} \right) \left(\sum_{\mathsf{S} \; of \; shape \; \lambda} x^{\mathrm{wt}(\mathsf{S})} \right)$$

and we want to get a sum of the form

$$\sum_{\nu = \lambda + \; horizontal \; r\text{-}strip} s_\nu = \sum_{\substack{\nu = \lambda + \; horizontal \\ r\text{-}strip}} \sum_{\substack{tableaux \; \mathsf{I} \\ of \; shape \; \nu}} x^{\mathrm{wt}(\mathsf{I})}$$

In other words, we want to find a bijection ϕ_λ^r from pairs of tableaux (T, S) where T has shape (r) and S has shape $\lambda \leftrightarrow$ Tableaux I of shape λ+horizontal r-strip such that $\mathrm{wt}(\mathsf{T}) + \mathrm{wt}(\mathsf{S}) = \mathrm{wt}(\mathsf{I})$.

Let ϕ_λ^r be RSK. That is, $\phi_\lambda^r(\mathsf{T}, \mathsf{S}) = (\mathsf{I} = \mathsf{S} \leftarrow \mathsf{T})$ with $\mathsf{T} = \boxed{x_1 \; x_2 \; \ldots \; x_r}$. Since $x_1 \le x_2 \le \cdots \le x_r$, then, by Lemma 1.12, the bumping route of x_1 is to the left of the bumping route of x_2 and so on.
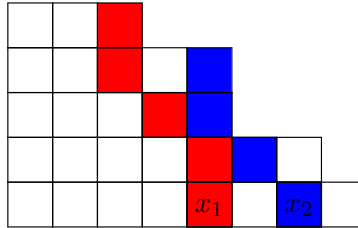
For the inverse, it must be that ν/λ is a horizontal r-strip. Therefore, the inverse is straightforward to compute directly. $\qquad\square$

Finally, another important relationship between s_λ's and h_n's is given below without proof.

2.81. **Theorem** (Jacobi-Trudi Identities)**.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ *be a partition of* n. *Then,*

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq \ell} = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1 + 1} & \cdots & h_{\lambda_1 + \ell - 1} \\ h_{\lambda_2 - 1} & h_{\lambda_2} & \cdots & \vdots \\ \vdots & & \ddots & \\ h_{\lambda_\ell - (\ell - 1)} & \cdots & & h_{\lambda_\ell} \end{pmatrix}$$

where $h_n = 0$ *for all* $n < 0$. *Similarly,*

$$s_{\lambda'} = \det(e_{\lambda_i - i + j})_{1 \leq i,j \leq \ell}$$

2.82. **Example.** If $\lambda = (4, 3, 3, 1, 1)$, then

$$s_{43311} = \det \begin{pmatrix} h_4 & h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 & h_4 & h_5 \\ 0 & 0 & h_0 & h_1 & h_2 \\ 0 & 0 & 0 & h_0 & h_1 \end{pmatrix}$$

2.4. Inner Product Structure of Λ.

2.83. **Proposition.** Λ *is an inner-product space with Hall-inner product*

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda, \mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

2.84. **Remark.** This is a symmetric form, although this is not obvious from the definition.

A natural question about this inner product is how might it behave with respect to other bases we have encountered. One significant basis with this form is the Schur basis.

2.85. **Theorem.** *Schur functions are an orthonormal basis with respect to* $\langle \cdot, \cdot \rangle$. *In other words,*

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$$

There are multiple ways to show this, each of which reveals a different perspective.

2.86. **Lemma.** *Given a shape* λ,

$$h_\lambda = \sum_\mu K_{\mu, \lambda} s_\mu$$

where $K_{\lambda, \alpha} = |\operatorname{SSYT}(\lambda, \alpha)|$ *is the Kostka number (see 2.60).*

Proof. From the Pieri rule, we see that

$$h_{r_1} s_\varnothing = s_{(r_1)}$$

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Then,
$$h_{r_2}h_{r_1}s_\varnothing = h_{r_2}s_{(r_1)} = \sum_{\nu=(r_1)+\text{horizontal } r_2\text{-strip}} s_\nu$$

2.87. Example. For instance, if $r_1 = 5, r_2 = 3$, we get
$$h_{r_2}s_{(r_1)} = s_{\square} + s_{\square} + s_{\square} + s_{\square}$$
which, if you fill the gray boxes with 1 and the white boxes with 2, you get an SSYT of weight $(5,3)$. Thus, the sum is precisely
$$h_{(5,3)} = \sum_{\mu,(5,3)} K_{\mu,(5,3)}s_\mu$$

In general, we see that the process of multiplying many h_{r_i}'s to $s_\varnothing$ will be equivalent to building different SSYTs of *partition* weight $(r_1, \ldots, r_k)$. In fact, since the weight is partition weight, this will exactly get us the desired equality. $\square$

> I know this is not the best proof; the example really illustrates the point, though.

2.88. Remark. Note, if we encode the Kostka numbers is a matrix, called the *Kostka matrix*, then we have change of bases given by
$$(K_{\lambda,\mu})_{\mu,\lambda}(s_\mu)_\mu = (h_\lambda)_\lambda$$
and thus also the other change of basis given by taking the inverse of the Kostka matrix.

A proof of 2.85. Based on our lemma above and Littlewood's combinatorial description of Schur functions, we observe that
$$s_\lambda = \sum_\mu K_{\mu,\lambda}^{-1}h_\mu \text{ and } s_\lambda = \sum_\mu K_{\lambda,\mu}m_\mu$$
where $K_{\mu,\lambda}^{-1}$ is the (μ,λ)-entry of inverse Kostka matrix $(K_{\alpha,\beta})_{\alpha,\beta}^{-1}$. Therefore,

$$\langle s_\lambda, s_\mu \rangle = \langle \sum_\alpha K_{\lambda,\alpha}m_\alpha, \sum_\beta K_{\beta,\mu}^{-1}h_\beta \rangle$$
$$= \sum_\alpha K_{\lambda,\alpha} \sum_\beta K_{\beta,\mu}^{-1}\langle m_\alpha, h_\beta \rangle$$
$$= \sum_\alpha K_{\lambda,\alpha}K_{\alpha,\mu}^{-1} \qquad\qquad \text{since } \langle m_\alpha, h_\beta \rangle = \delta_{\alpha,\beta}$$
$$= (Id)_{\lambda,\mu}$$
$$= \delta_{\lambda,\mu}$$

$\square$

2.89. Corollary. *The Schur functions can be characterized as the unique orthonormal basis for Λ that is unitriangularly related to the basis $\{m_\lambda\}_\lambda$.*

In fact, one could take the corollary to be the definition of the Schur functions.

2.90. **Example.** Imagine one knows only the above corollary. Then, we have by the unitriangularity that

$$\begin{cases} s_{111} = m_{111} \\ s_{21} = m_{21} + u_{21,111}m_{111} \\ s_3 = m_3 + u_{3,21}m_{21} + u_{3,111}m_{111} \end{cases}$$

Then, since

$$s_{111} = m_{111} = h_{111} - 2h_{21} + h_3$$

we get

$$\langle s_{111}, s_{21} \rangle = 0 \implies 0 = \langle h_{111} - 2h_{21} + h_3, m_{21} + u_{21,111}m_{111} \rangle = u_{21,111} - 2$$

and so

$$s_{21} = m_{21} + 2m_{111}$$

One could then continue this process to get the unknown coefficients in s_3, but this would be quite tedious.

Another proof technique to see the orthonormality of Schur functions is using "Cauchy kernels." This process requires us to know more about how the power sum basis behaves with respect to the Hall inner product.

2.91. **Definition.** The *Cauchy kernel* in $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ is defined by

$$\Omega(x, y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_i}$$

2.92. **Proposition.** *We have the following facts about the Cauchy kernel.*

(a) $\Omega(x, y) = \sum_\lambda h_\lambda(x) m_\lambda(y)$
(b) $\Omega(x, y) = \sum_\lambda s_\lambda(x) s_\lambda(y)$

> Add that $\Omega(x,y) = \sum_\lambda p_\lambda(x) p_\lambda(y)$. Only problem is that I do not know how to prove this yet...

Proof. To see part (a), observe

$$\prod_{i,j \geq 1} \frac{1}{1 - x_j y_j} = \prod_j \left(\sum_n h_n(\vec{x}) y_j^n \right)$$

and so, expanding out our product and collecting terms, we see

$$\prod_j \left(\sum_n h_n(\vec{x}) y_j^n \right) = 1 + \underbrace{h_1(\vec{x})(y_1 + y_2 + \cdots)}_{\vdash 1} + \underbrace{h_2(\vec{x}) \cdot (y_1^2 + y_2^2 + \cdots) + h_1(\vec{x})h_1(\vec{x}) \cdot (y_1 y_2 + y_1 y_3 + \cdots + y_2 y_3 + \cdots)}_{\vdash 2} + \cdots$$

Therefore, we have our desired result.

To see part (b), we make use of the RSK correspondence. Note that

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \prod_{i,j \geq 1} \sum_{k=0}^{\infty} (x_i y_j)^k$$

when expanded out into an infinite series has monomials (each with coefficient 1) indexed by matrices with nonnegative integer entries. On the righthand side, if we expand

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \sum_\lambda \left(\sum_{\mathsf{T}\in\mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})} \right) \left(\sum_{\mathsf{S}\in\mathrm{SSYT}(\lambda)} y^{\mathrm{wt}(\mathsf{S})} \right) = \sum_\lambda \sum_{\mathsf{T},\mathsf{S}\in\mathrm{SSYT}(\lambda)} x^{\mathrm{wt}(\mathsf{T})} y^{\mathrm{wt}(\mathsf{S})}$$

the monomials are indexed by pairs of SSYTs of the same shape, each with coefficient 1. Thus, by the RSK correspondence, these two infinite sums are equal. $\square$

2.93. Theorem. *Two bases $\{b_\lambda\}$ and $\{\hat{b}_\lambda\}$ for Λ are dual if and only if $\sum_\lambda b_\lambda(x)b_\lambda(y) = \Omega(x,y)$.*

Proof. Write $b_\lambda(x) = \sum_\mu a_{\lambda,\mu} m_\mu$ and $\hat{b}_\gamma(x) = \sum_\nu c_{\gamma,\nu} h_\nu$. Then,

$$\langle b_\lambda, \hat{b}_\gamma \rangle = \sum_\mu \sum_\nu a_{\lambda,\mu} c_{\gamma,\nu} \langle m_\mu, h_\nu \rangle = \sum_\mu a_{\lambda,\mu} c_{\gamma,\mu}$$

Thus, the two bases are orthonormal if and only if

$$\sum_\mu a_{\lambda,\mu} c_{\gamma,\mu} = \delta_{\lambda,\gamma} \iff AC^T = I \iff C^T A = I \iff A^T C = I \iff \sum_\mu a_{\mu,\lambda} c_{\mu,\gamma} = \delta_{\lambda,\gamma}$$

where $A = (a_{\lambda,\mu})$ and $C = (c_{\gamma,\nu})$. However, this happens if and only if

$$\sum_{\lambda,\gamma} \left(\sum_\mu a_{\mu,\lambda} c_{\mu,\gamma} \right) h_\lambda(x) m_\gamma(y) = \sum_{\lambda,\gamma} \delta_{\lambda,\gamma} h_\lambda(x) m_\gamma(y) = \sum_\lambda h_\lambda(x) m_\lambda(y) = \Omega(x,y)$$

$\square$

Alternative proof of 2.85. Since $\Omega(x,y) = \sum_\lambda s_\lambda(x)s_\lambda(y)$, then the Schur basis is self-dual by the theorem above. $\square$

3. LLT Polynomials

3.1. Some Notions on Semistandard Young Tableaux. Note that we can consider a semistandard Young tableau T of shape λ on n letters as a function

$$\mathsf{T}\colon \{\text{boxes of } \lambda\} \to \{1,\ldots,n\}$$

and it T is a standard Young tableau of shape λ on n letters, then this function is bijective.

3.1. Definition. Let T be a standard Young tableau and let $\mathsf{T}(x) = a$ and $\mathsf{T}(y) = a+1$. If x is strictly south of y and weakly to the east of y (that is, $c(x) > c(y)$), we say that a is a *descent* of T and we say that

$$\mathrm{des}(\mathsf{T}) := \{a \mid a \text{ is a descent of } \mathsf{T}\} \subseteq \{1,\ldots,n-1\}$$

3.2. **Example.**

$$
\mathsf{T} = \begin{array}{|c|c|c|c|}
\hline
7 & 9 & & \\
\hline
2 & 5 & 8 & \\
\hline
1 & 3 & 4 & 6 \\
\hline
\end{array}
$$

has $\mathrm{des}(\mathsf{T}) = \{1, 4, 6, 8\}$

3.3. **Proposition.** *Every semistandard Young tableau* T *of shape* λ *has a unique "standardization"* S*, that is, a unique standard Young tableau* S *such that*

(a) $\mathsf{T} \circ \mathsf{S}^{-1}$ *is a weakly increasing function and*

(b) *The contents of the "standardization" corresponding to the same letter have increasing contents. Formally, if* $\mathsf{T} \circ \mathsf{S}^{-1}(j) = \mathsf{T} \circ \mathsf{S}^{-1}(j+1) = \cdots = \mathsf{T} \circ \mathsf{S}^{-1}(k) = a$*, then* $\{j, \ldots, k-1\} \cap \mathrm{des}(\mathsf{S}) = \varnothing$*.*

Proof. Consider that if λ is a horizontal strip, there is a unique labeling of the cells of λ to form a standard tableau (with no descents). Similarly, if λ is a vertical strip, there is a unique standard tableau on λ with descents at every position. To get a weakly increasing sequence, one must simply replace the numbers of the SSYT in increasing order. To resolve the multiple ways one can relabel boxes with the same number, one starts from lowest content to highest content. One then gets the desired standardization.

$$
\mathsf{T} = \begin{array}{|c|c|c|c|c|}
\hline
3 & 4 & & & \\
\hline
2 & 2 & 4 & & \\
\hline
1 & 1 & 3 & 5 & 6 \\
\hline
\end{array}
\quad \text{has standardization } \mathsf{S} = \begin{array}{|c|c|c|c|c|}
\hline
5 & 7 & & & \\
\hline
3 & 4 & 8 & & \\
\hline
1 & 2 & 6 & 9 & 10 \\
\hline
\end{array}
\quad \text{since}
$$

$$
\begin{cases}
(\mathsf{T} \circ \mathsf{S}^{-1})(1) = 1 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(2) = 1 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(3) = 2 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(4) = 2 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(5) = 3 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(6) = 3 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(7) = 4 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(8) = 4 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(9) = 5 \\
(\mathsf{T} \circ \mathsf{S}^{-1})(10) = 6
\end{cases}
\quad \text{and } \mathrm{des}(\mathsf{S}) = \{2, 4, 6\}
$$

$\square$

3.4. **Definition.** The *content* of the cell in row i and column j in λ is $j - i$.

3.5. Example. The diagram is filled such that the number in the cell indicates its content.

-3				
-2	-1	0		
-1	0	1	2	
0	1	2	3	4

3.6. Remark. A semistandard Young tableau of weight α can be created from a standard Young tableau on $|\alpha|$ letters where the sequences of cells $(1, 2, \ldots, \alpha_1)$, $(\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_2), \ldots$ each have increasing content by replacing the labels $\{1, 2, \ldots, \alpha_1\}$ by 1, $\{\alpha_1 + 1, \alpha_1 + 2 \ldots, \alpha_2\}$ by 2, etc. This is essentially inverse to the process of "standardization."

3.7. Example. Take $(123)(456)(78)(9)(10)$, that is, take weight $\alpha = (3, 3, 2, 1, 1)$, so $|\alpha| = 10$. Then, from the standard Young tableau below (which has the desired content increasing condition), we get the following semistandard tableau using this method:

10		
9		
4	5	6
1	2	3	7	8

$\mapsto$

5		
4		
2	2	2
1	1	1	3	3

Below, we will seek to construct so-called "semistandard ribbon tableau" from "standard ribbon tableau" in an analogous manner.

3.2. Rim Hook Tabloids. Warning: In this section, partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ will be in **increasing order**, that is $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$.

3.8. Definition.   (a) A *rim hook* H of a partition λ is a consecutive sequence of cells on F_λ's northeast rim such that any two adjacent cells of H have a common edge and removal of H from F_λ leaves a "legal" Ferrers diagram.
   (b) A *special rim hook* is a rim hook which starts on the first column of λ, that is, it starts in cell $(\lambda_1, 1)$.
   (c) The number of rows of the rim hook is called its *height*, denoted $\mathrm{ht}(H)$.
   (d) The number of cells of the rim hook is called its *size*, denoted $|H|$.

3.9. Example. The green shaded boxes form a rim hook. The one on the right is a special rim hook.

3.10. Definition. A *special rim hook tabloid* T of shape μ and type λ is a filling of F_μ repeatedly with rim hooks of sizes $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ such that each rim hook is special.

Note, in this context, rim hooks may be called "ribbons" since they are not rim hooks of the original diagram.
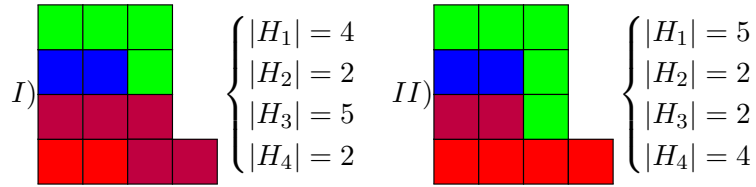
3.11. Definition. An *n-ribbon* R is a connected skew shape of n squares with no subshape of 2×2 squares.

To get a special rim hook tabloid, one follows the following algorithm

(a) Pick a special rim hook H_1 in F_μ and remove the cells of H_1 to get $F_{\mu(1)}$.
(b) Pick a special rim hook H_2 in $F_{\mu(1)}$ and remove it to get $F_{\mu(2)}$.
(c) Proceed analogously until there is nothing to remove.

3.12. Definition. The *type* of a special rim hook tabloid T is λ if $(|H_1|, |H_2|, \ldots, |H_\ell|)$ arranged in weakly increasing order produces partition λ

3.13. Example. Take $\mu = (3, 3, 3, 4)$. The only special rim hook tabloids of type $\lambda = (2, 2, 4, 5)$ are

$$I) \quad \begin{cases} |H_1| = 4 \\ |H_2| = 2 \\ |H_3| = 5 \\ |H_4| = 2 \end{cases} \qquad II) \quad \begin{cases} |H_1| = 5 \\ |H_2| = 2 \\ |H_3| = 2 \\ |H_4| = 4 \end{cases}$$

3.14. Definition. The *sign* of a rim hook H is taken to be $(-1)^{\mathrm{ht}(H)-1}$. The sign of T is the product of signs of rim hooks of T.

3.15. Example. The sign of rim hook tabloid $I)$ in the example above is

$$(-1)^{2-1}(-1)^{1-1}(-1)^{2-1}(-1)^{1-1} = 1$$

> At some point we switch back to standard convention.

3.16. Theorem (Murnaghan-Nakayama Rule)**.** *Given a shape λ and $r \in \mathbb{N}$, we have that*

$$p_r s_\lambda = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)-1} s_\mu$$

where the sum is over all partitions μ such that μ/λ is a rim-hook of size r.

Proof. Let us work in Λ_n. First, consider that

$$\Delta_{\lambda+\rho} p_r = \sum_{i=1}^{n} \Delta_{\lambda + r\epsilon_i + \rho}$$

Then, arrange $\lambda + r\epsilon_i + \rho$ in descending order. If two terms are equal, then $\Delta_{\lambda+r\epsilon_i+\rho} = 0$, otherwise there will be some $p \leq q$ such that

$$\lambda_{p-1} + \rho_{p-1} > \lambda_q + \rho_q + r > \lambda_p + \rho_p$$

$$\Longleftrightarrow$$

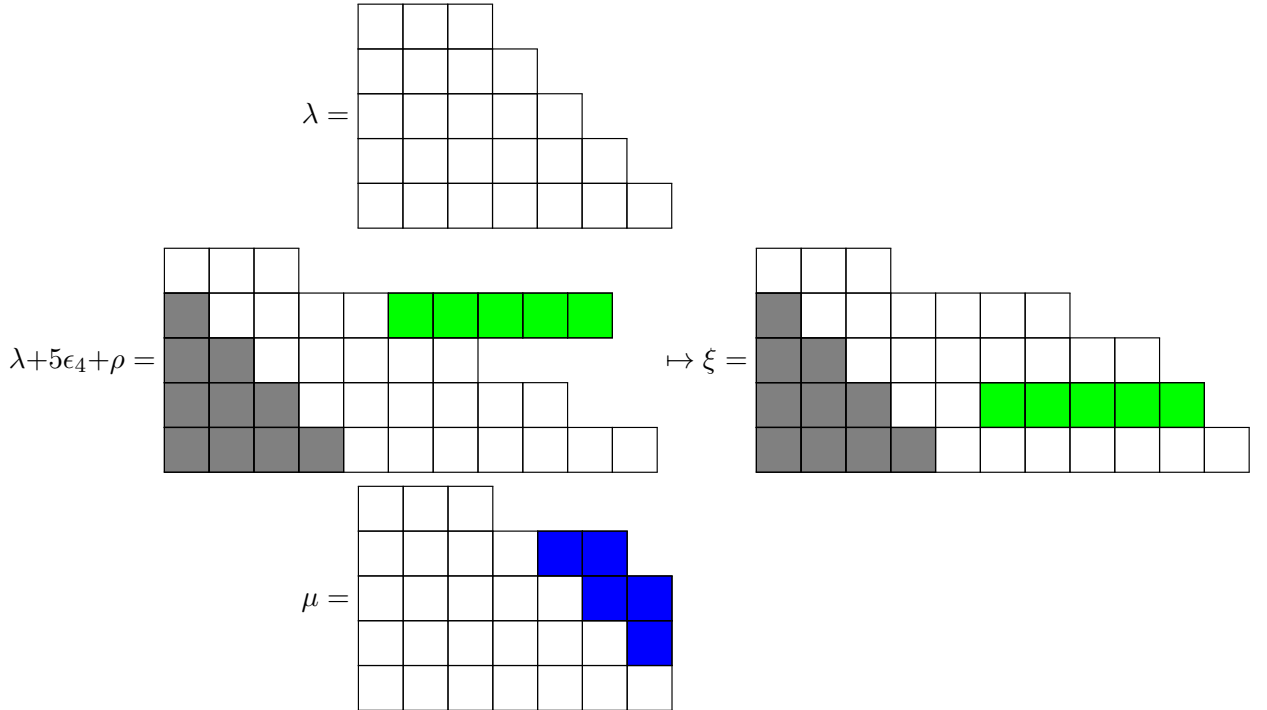$$\lambda_{p-1} + n - p + 1 > \lambda_q + n - q + r > \lambda_p + n - p$$

and so $\Delta_{\lambda+r\epsilon_i+\rho} = (-1)^{q-p}\Delta_\xi$ where

$$\xi = (\lambda_1+n-1, \cdots, \lambda_{p-1}+n-p+1, \underbrace{\lambda_q + n - q + r, \lambda_p + n - p, \ldots, \lambda_{q-1} + n - q + 1}_{q-p \text{ rows}}, \lambda_{q+1}+n-q-1, \ldots, \lambda_n)$$

Thus, $\mu := \xi - \rho$ is a partition of the form

$$\mu = (\lambda_1, \ldots, \lambda_{p-1}, \lambda_p + p - q + r, \lambda_p + 1, \ldots, \lambda_{q-1} + 1, \lambda_{q+1}, \ldots, \lambda_n)$$

Such a partition is exactly such that μ/λ has a rim-hook of size r. In the example below, $r = 5, p = 2, q = 4$

To complete the proof, divide both sides by Δ_ρ and let $n \to \infty$. $\qquad\square$

3.17. Definition. The *type* of a rim hook tabloid is $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ if the ribbon labeled i has α_i cells.

3.18. Definition. A *rim hook tabloid* of type α is a sequence of partitions

$$\lambda^0 \subseteq \lambda^1 \subseteq \cdots \subseteq \lambda^\ell$$

such that λ^i/λ^{i-1} is a rim hook with α_i-cells and the cells of λ^i/λ^{i-1} are labeled with the letter i

3.19. Example. A rim hook tabloid with type $\alpha = (3, 4, 5)$ is

2	3	3	3
2	2	2	3
1	1	1	3

3.20. Definition. Fix $n, a > 0$. A *standard ribbon tableau* (sometimes called an *n-ribbon tableau*) of degree a is a rim hook tabloid of type $(\underbrace{n, \ldots, n}_{a}) = (n^a)$.

3.21. Example. A 3-ribbon tableau of degree 6 is given by

4					
4	4				
3	3				
1	3	5	5	5	6
1	1	2	2	2	6	6

3.22. Definition. A *ribbon head* is the southeastern-most cell in a ribbon R. A *ribbon tail* is the northwestern-most cell.

3.23. Example. A ribbon is given as the colored cells combined. The tail is the dark colored cell (olive), the head is the light colored cell (lime).

3.24. Definition. Fix $n > 0$. A *semistandard ribbon tableau* of weight α is created from an n-ribbon tableau on $\text{len}(\alpha)$ letters such that the contents of the ribbon heads labeled $\{1, 2, \ldots, \alpha_1\}$ are increasing, the contents of the ribbon heads labeled $\{\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_2\}$ are increasing, etc., by replacing the labels $\{1, 2, \ldots, \alpha_1\}$ by 1, $\{\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_2\}$ by 2, etc.

3.25. Example. Let $\alpha = (2, 3, 1, 1)$. Then, the semistandard ribbon tableau is given on the right

7					
7					
7					
3	3	3	4	6	6
1	2	2	4	5	6
1	1	2	4	5	5

4					
4					
4					
2	2	2	2	3	3
1	1	1	2	2	3
1	1	1	2	2	2

Ribbons with same numbers have increasing content heads.

3.26. Definition. Let R be a ribbon. Then, we define the *spin* of a ribbon to be

$$\mathrm{spin}(R) := \mathrm{ht}(R) - 1$$

If T is a ribbon tableau, we define

$$\mathrm{spin}(T) := \sum_{\text{ribbon } R \in T} \mathrm{spin}(R)$$

3.3. p-quotients Of Integer Partitions. For later use in understanding n-ribbon tableaux, we will need to understand a certain quotienting operation on partitions. This exposition will follow problem 8 in chapter 1 of MacDonald's book.

3.27. Proposition. *Let λ, μ be partitions of length $\leq m$ such that $\mu \subseteq \lambda$ and such that $\lambda - \mu$ is a rim hook of length p. Let $\rho = (m-1, m-2, \ldots, 1, 0)$ and set*

$$\xi = \lambda + \rho, \quad \eta = \mu + \rho$$

Then, η is obtained from ξ by subtracting p from some part of ξ_i of ξ and rearranging in descending order.

Proof. Consider that, due to the shift by ρ, ξ and μ both have distinct parts and our rim-hook will no longer be connected vertically.

$$\begin{cases} \lambda = (4, 4, 2, 2) \\ \mu = (4, 1, 1) \end{cases} \mapsto$$
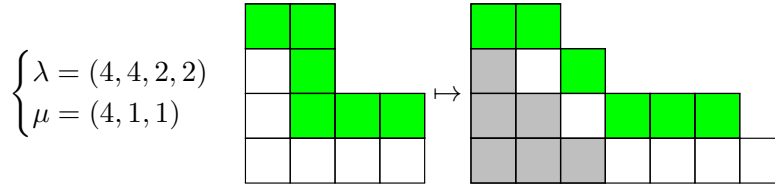
Furthermore, our rim hook (before the shift) can only move down vertically if the two rows have the same length and, when it moves horizontally, it must move to the end of the row. Therefore, after the shift, deletion of the rim-hook corresponds to subtracting p from a part of ξ and shifting, resulting in η either way.

$\square$
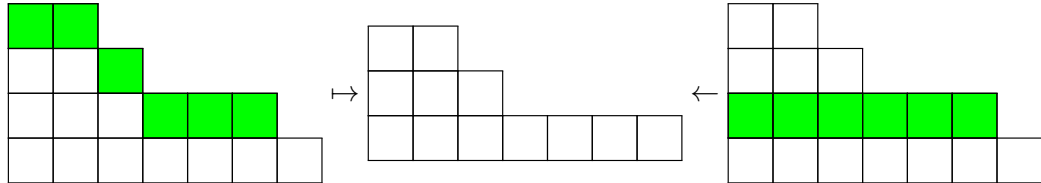
3.28. Definition. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $m_k(\lambda)$ be the number of parts of λ of size k. In other words $m_k(\lambda) = |\{\lambda_i \in \lambda \mid \lambda_i = k\}|$.

3.29. Definition. Let λ be a partition of length $\leq m$, let $\xi = \rho_m + \lambda$ where $\rho_m = (m-1, m-2, \ldots, 1, 0)$, and let $p \geq 2$. Then, we define the *p-quotient* of λ as follows.

Let $m_r = \overline{m}_r(\xi) = |\{\xi_i \in \xi \mid \xi_i \equiv r \mod p\}|$, that is, the number of parts of ξ congruent to $r \mod p$. Then, $\xi_i = p\xi_k^{(r)} + r$ for $1 \leq k \leq m_r$ where

$$\xi_1^{(r)} > \xi_2^{(r)} > \cdots > \xi_{m_r}^{(r)} \geq 0$$

and let $\lambda_k^{(r)} = \xi_k^{(r)} - m_r + k$ so that $\lambda^{(r)} = (\lambda_1^{(r)}, \ldots, \lambda_{m_r}^{(r)})$ is a partition. Then, the collection $\tilde{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(p-1)})$ is called the p-*quotient* of the partition λ.

3.30. **Example.** Let $\lambda = (4, 4, 2, 2)$. Then, $\zeta = \lambda + \rho_4 = (7, 6, 3, 2)$ has distinct parts. If $p = 3$, then

$$m_0 = 2, m_1 = 1, m_2 = 1$$

and

$$\begin{cases} \zeta_2 = 6 = 2 \cdot 3 + 0 \\ \zeta_3 = 3 = 1 \cdot 3 + 0 \\ \zeta_1 = 7 = 2 \cdot 3 + 1 \\ \zeta_2 = 2 = 0 \cdot 3 + 2 \end{cases} \implies \begin{cases} \zeta_1^{(0)} = 2 \\ \zeta_2^{(0)} = 1 \\ \zeta_1^{(1)} = 2 \\ \zeta_1^{(2)} = 0 \end{cases} \implies \begin{cases} \lambda_1^{(0)} = 2 - 2 + 1 = 1 \\ \lambda_2^{(0)} = 1 - 2 + 2 = 1 \\ \lambda_1^{(1)} = 2 - 1 + 1 = 2 \\ \lambda_1^{(2)} = 0 - 1 + 1 = 0 \end{cases}$$

Thus, we get the 3-quotient $\lambda^* = ((1,1), (2), (0))$, ie

Note that, for a different choice of m (here we picked $m = 4$), we would get a cyclic rearrangement of the 3-quotient. Thus, p-quotients are defined up to cyclic rearrangement. This fact leads [Mac79] to suggest that a p-quotient should really be thought of as a "necklace of partitions".

3.31. **Proposition.** *The p-core of a partition is unique and may be obtained by removing border strips of length p in any order until it is no longer possible.*

Proof. By above, removing a border strip of length p corresponds to subtracting p from some part of $\zeta = \lambda + \rho$ and then rearranging the sequence in descending order. Simply subtracting p from parts of ζ until it is no longer possible is certainly independent of choice, and will result in a unique p-core. $\square$

3.4. **LLT Polynomials.**

3.32. **Definition.** Given $n > 0$ and a shape λ, we define the (spin-squared) *LLT polynomial* (named after Lascoux, Leclerc, and Thibon) to be

$$G_\lambda^{(n)}(x, t) = \sum_{\substack{\mathsf{T} \text{ a ss ribbon tableau} \\ \text{of shape } \lambda}} t^{\mathrm{spin}(\mathsf{T})} x^{\mathrm{wt}(\mathsf{T})}$$

44

where every ribbon of T is of size n.

3.33. Example. If we take $n = 3$ and $\lambda = (3,3)$, we have the following possible ribbon tableaux of shape λ:

$$
\begin{cases}
\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\\hline 1 & 1 & 2 \\\hline\end{array} \\
\alpha = (1,1) \\
\text{spin} = 2
\end{cases}
\begin{cases}
\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\\hline 1 & 1 & 1 \\\hline\end{array} \\
\alpha = (1,1) \\
\text{spin} = 0
\end{cases}
\begin{cases}
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\\hline 1 & 1 & 1 \\\hline\end{array} \\
\alpha = (2,0) \\
\text{spin} = 2
\end{cases}
\begin{cases}
\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\\hline 2 & 2 & 2 \\\hline\end{array} \\
\alpha = (0,2) \\
\text{spin} = 2
\end{cases}
$$

Thus giving us

$$G_{(3,3)}^{(3)} = t^2 x_1 x_2 + x_1 x_2 + t^2 x_1^2 + t^2 x_2^2$$

Also useful is to rewrite this in terms of the monomial and Schur bases:

$$G_{(3,3)}^{(3)} = (t^2 + 1)m_{11} + t^2 m_2 = t^2 s_2 + s_{11}$$

Notice, however, our definition is not clear for something like

$$G_{\begin{array}{|c|c|}\hline & \\\hline & \\\hline\end{array}}^{(3)} = ???$$

because we cannot tile $\begin{array}{|c|c|}\hline & \\\hline & \\\hline\end{array}$ with 3-ribbons. To deal with this, we will have to define the following:

3.34. Definition. An n-core is a partition with no removable n-rim-hook. Equivalently, an n-core is a shape where none of its cells have a hook length of n.

3.35. Example. The diagram on the left is not a 3-core, the one on the right is.
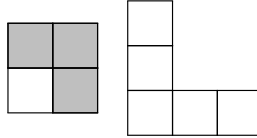
3.36. Proposition. *For any partition γ, successively removing all removable n-rim hooks in any order yields a unique n-core.*

3.37. Definition. Fix $n > 0$. For any shape λ, let $\tilde{\lambda}$ be its n-core. We define

$$G_\lambda^{(n)} := G_{\lambda/\tilde{\lambda}}^{(n)}$$

3.38. Example. Now, we can deal with our issue above. Namely,

$$G_{\begin{array}{|c|c|}\hline & \\\hline & \\\hline\end{array}}^{(3)} = G_{\begin{array}{|c|c|}\hline & \\\hline & \\\hline\end{array}}^{(3)}$$

3.39. Proposition. *For any skew shape λ, $G_\lambda^{(n)}$ is a symmetric function.*

3.40. Proposition. $G_\lambda^{(n)}$ *is "Schur-positive", that is,*

$$G_\lambda^{(n)} = \sum_\mu c_{\mu\lambda}(t) s_\mu$$

such that $c_{\mu\lambda}(t) \in \mathbb{N}[t]$.

3.41. Proposition. *When* $t = 1$, $G_\lambda^{(n)}$ *is a product of Schur functions.*

3.42. Example. From above

$$G_{(3,3)}^{(3)}(x; 1) = s_2 + s_{11} = s_1 \cdot s_2$$

3.43. Definition. We define the statistic $\mathrm{cospin}(\mathsf{T}) = (\mathrm{spin\,max}(\mu) - \mathrm{spin}(\mathsf{T}))/2$ where T is a (semistandard) ribbon tableaux of skew-shape μ and $\mathrm{spin\,max}(\mu) = \max\{\mathrm{spin}(\mathsf{T}) \mid \mathsf{T}$ is a ribbon tableau of skew-shape $\mu\}$.

3.44. Definition. We define the (cospin) *LLT polynomial* to be

$$\mathrm{LLT}_\mu^{(n)}(x, t) = \sum_{\mathsf{T}} x^{\mathrm{wt}(\mathsf{T})} t^{\mathrm{cospin}(\mathsf{T})}$$

3.45. Example. To see the difference (and similarity) with the spin-squared LLT family defined above, we see

$$\mathrm{LLT}_{(3,3)}^{(3)}(x, t) = (t + 1)m_{11} + m_2$$

since, in this case, $\mathrm{spin} = 2 \implies \mathrm{cospin} = 0$ and $\mathrm{spin} = 0 \implies \mathrm{cospin} = 1$.

3.46. Proposition. *For any skew shape* μ, $\mathrm{LLT}_\mu^{(n)}(x, t)$ *is a symmetric function.*

To show this, we will need the another statistic.

3.5. Bylund and Haiman's inversion statistic. We define a quotient map which takes a ribbon tableau T to a tuple $\tilde{\mathsf{T}}$ of SSYT's as follows.

3.47. Definition. Recall the content of a ribbon R is the content of its head. Write the content of each ribbon R in T as $rn + p$ where $0 \le p < n$. Then, send R to a square of content r in the $p + 1$st SSYT.
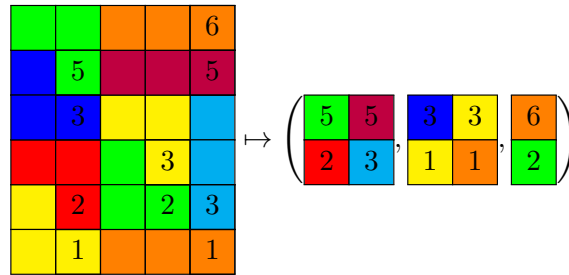
> Why does this work?! Probably follows from the "quotient-ing" aspect. The p-quotient section probably resolves this.

3.48. Example. We send

where the coloring of the SSYT's are to indicate from which ribbon the number comes. For example, the orange 6 has content $-1 = -1 \cdot 3 + 2$, and so it is sent to the 3rd tableau in a cell with content 1.

3.49. Definition. We define the *Bylund and Haiman inversion statistic* on a tuple of SSYTs $\tilde{\mathsf{T}} = (\tilde{\mathsf{T}}^1, \ldots, \tilde{\mathsf{T}}^n)$ to be the number of inversion pairs in $\tilde{\mathsf{T}}$ and denote this $\mathrm{inv}(\tilde{\mathsf{T}})$. We define an *inversion pair* to be a pair (c, d) of squares containing numbers $\sigma(c)$ and $\sigma(d)$ respectively, with $c \in \tilde{\mathsf{T}}^i$ and $d \in \tilde{\mathsf{T}}^j$ where $1 \le i < j \le n$ where either

(a) $\sigma(c) > \sigma(d)$ and c and d have the same content, or

(b) $\sigma(c) < \sigma(d)$ and the content of c is one more than the content of d (that is, c is one diagonal below d).

Alternatively, for $x \in \mathsf{T}^i$, define the *adjusted content*

$$\tilde{c}(x) := nc(x) + (i - 1)$$

where $c(x)$ is the content of x. Then, an inversion pair is a pair (x, y) such that $\sigma(x) < \sigma(y)$ and $0 < \tilde{c}(x) - \tilde{c}(y) < n$.

> Note that this is a modification the definition in [HHL+04]: $\tilde{c}(x) = nc(x) + s_i$.

3.50. Example. Below the tuple above is filled in with numbers representing the adjusted contents.

$$\left(\begin{array}{|c|c|} \hline -3 & 0 \\ \hline 0 & 3 \\ \hline \end{array} \, , \, \begin{array}{|c|c|} \hline -2 & 1 \\ \hline 1 & 4 \\ \hline \end{array} \, , \, \begin{array}{|c|c|} \hline -1 \\ \hline 2 \\ \hline \end{array} \right)$$

To use the first definition of inversion pairs, redraw our tuple like so

so the contents line up. Then, we see the inversion pairs of the first type

$$\left(\boxed{5} , \boxed{3} \right), \left(\boxed{5} , \boxed{6} \right), \left(\boxed{5} , \boxed{1} \right), \left(\boxed{5} , \boxed{3} \right), \left(\boxed{5} , \boxed{2} \right), \left(\boxed{2} , \boxed{1} \right), \left(\boxed{3} , \boxed{1} \right)$$

that is to say, the 5 in the upper left hand corner of $\tilde{\mathsf{T}}^1$ forms an inversion pair with the 3 in the upper left hand corner of $\tilde{\mathsf{T}}^2$, the 5 in the upper right hand corner of $\tilde{\mathsf{T}}^1$ forms an inversion pair with the 1 and 3 on the same diagonal in $\tilde{\mathsf{T}}^2$ and with the 2 in $\tilde{\mathsf{T}}^3$, etc. The inversion pairs of the second type are

$$\left(\boxed{2} , \boxed{3} \right), \left(\boxed{2} , \boxed{6} \right), \left(\boxed{1} , \boxed{6} \right)$$

so, for instance, the 2 in the bottom left hand corner of $\tilde{\mathsf{T}}^1$ forms an inversion pair with the 3 in the upper left hand corner of $\tilde{\mathsf{T}}^2$.

Thus, $\mathrm{inv}(\tilde{T}) = 9$.

3.51. Lemma (Spin Lemma)**.** *Let $|\mu| = 2n$. Show that if μ can be tiled by two n-ribbons whose cells have at least one content in common, then there are exactly two tilings of μ and their spins differ by 2.*
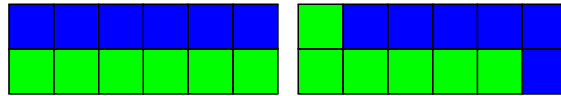
*Proof.* Below is a sufficiently generic example illustrating the situation.
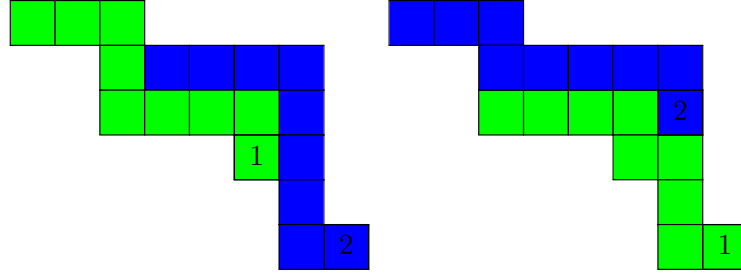
In this situation, there can only be one contiguous "section" of the shape that has common contents, otherwise the shape cannot be filled with two n-ribbons. Consider such a contiguous section:
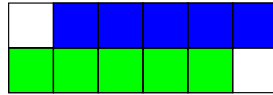
We can fill the white boxes two different ways:

In the left configuration, each ribbon has height 1 and in the right configuration, each ribbon has height 2. Importantly, in the contiguous region, the two ribbons must have the same height since the swap trades a vertical move for a horizontal move on both ribbons.
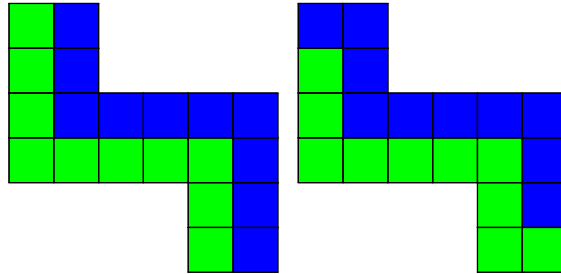
Any addition of cells with unique contents will add to the overall height of the ribbon tableau, but the contributed height will not be affected by the swap. Thus, the total spins will differ by 2. $\qquad\square$

3.52. Lemma (Bylund, Haiman). *Given a skew shape μ and $n > 0$, there is a constant e such that for every standard n-ribbon tableau T of shape μ, we have $\mathrm{cospin}(\mathsf{T}) + e = \mathrm{inv}(\tilde{\mathsf{T}})$*

Proof. We say that standard tableaux S, S' of shape $\tilde{\mu} = (\mu^1, \ldots, \mu^n)$ (that is, S, S' are a disjoint collection of n standard tableaux of shapes $\mu^1, \ldots, \mu^n$) differ by a *switch* if they are identical except for the positions of two consecutive entries $a, a + 1$. Then, two tableaux S, S' differing by a switch can

only have $\mathrm{inv}(\mathsf{S}), \mathrm{inv}(\mathsf{S}')$ differing by 1.

Now, assume that T, T' are standard n-ribbon tableau such that $\tilde{\mathsf{T}}$ and $\tilde{\mathsf{T}}'$ differ by a switch. We wish to show that $\mathrm{cospin}(\mathsf{T}) - \mathrm{cospin}(\mathsf{T}') = \mathrm{inv}(\tilde{\mathsf{T}}) - \mathrm{inv}(\tilde{\mathsf{T}}')$. Since T, T' are identical except for the ribbons labeled a and $a+1$, the problem reduces to the case where $\tilde{\mu}$ has only two non-empty SSYTs and $|\mu| = 2n$; in other words, T, T' are n-ribbon tableaux tiled by only 2 ribbons each with a distinct content. We now reduce to 2 cases.

- $(\mathrm{inv}(\tilde{\mathsf{T}}) - \mathrm{inv}(\tilde{\mathsf{T}}') = 0)$. For this situation, our switched cells x, y cannot have the same content, nor can they be in two distinct SSYTs with one a column higher than the other. Therefore, μ cannot be tiled with two ribbons whose contents differ by less than n. Thus, in this case, the shape μ is the union of two ribbons whose cells have no contents in common, otherwise the shape could not be connected. Therefore, these ribbons are unique and so $\mathrm{spin}(\mathsf{T}) = \mathrm{spin}(\mathsf{T}')$ and thus $\mathrm{cospin}(\mathsf{T}) = \mathrm{cospin}(\mathsf{T}')$.
- $(|\mathrm{inv}(\tilde{\mathsf{T}}) - \mathrm{inv}(\tilde{\mathsf{T}}')| = 1)$. This situation is possible if and only if $|\tilde{c}(y) - \tilde{c}(x)| < n$, which forces the ribbon heads to have contents differing by less than n. Therefore, the two ribbons must have cells with at least one content in common. Thus, by our lemma above, there are exactly 2 tilings of μ whose spin differ by 2, say T, T' with $\mathrm{spin}(\mathsf{T}) = \mathrm{spin}(\mathsf{T}') + 2$ Therefore,

$$\mathrm{cospin}(\mathsf{T}) = \frac{1}{2}(\mathrm{spin\,max}(\mu) - \mathrm{spin}(\mathsf{T})) = \frac{1}{2}(\mathrm{spin\,max}(\mu) - \mathrm{spin}(\mathsf{T}') - 2) = \mathrm{cospin}(\mathsf{T}') - 1$$

$\square$

3.53. **Lemma** (Bylund, Haiman). *The lemma above also holds for semistandard ribbon tableaux.*

3.54. **Definition.** If $\tilde{\mathsf{S}}$ is a tuple of standard tableaux of shape $\tilde{\mu}$, we call a a *descent* of $\tilde{\mathsf{S}}$ if $\tilde{\mathsf{S}}(x) = a$, $\tilde{\mathsf{S}}(y) = a+1$, and $\tilde{c}(x) > \tilde{c}(y)$ for boxes x, y in $\tilde{\mathsf{S}}$.

3.55. **Proposition.** *Given a tuple of semistandard tableaux $\tilde{\mathsf{T}}$, there is a unique "standardization" $\tilde{\mathsf{S}}$ such that $\tilde{\mathsf{T}} \circ \tilde{\mathsf{S}}^{-1}$ is weakly increasing and, if $\tilde{\mathsf{T}} \circ \tilde{\mathsf{S}}^{-1}(j) = \tilde{\mathsf{T}} \circ \tilde{\mathsf{S}}^{-1}(j+1) = \cdots = \tilde{\mathsf{T}} \circ \tilde{\mathsf{S}}^{-1}(k)$, then $\mathrm{des}(\tilde{\mathsf{S}}) \cap \{j, \ldots, k-1\} = \varnothing$ where a is a descent if there is an x, y such that $\tilde{\mathsf{S}}(x) = a$, $\tilde{\mathsf{S}}(y) = a+1$ and $\tilde{c}(x) > \tilde{c}(y)$.*

Proof of Proposition. Observe that if $\tilde{\mu}$ is a horizontal strip, there is a unique standard tableau of shape $\tilde{\mu}$ with no descents, and conversely, such a tableau exists only on a horizontal strip.

1	2	3	4	5	6	7

Thus, each semistandard tableau $\tilde{\mathsf{T}}$ has a unique standardization meeting the properties above.

$$\tilde{\mathsf{T}} = \left(\begin{array}{|c|c|} \hline 5 & 5 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 6 \\ \hline 2 \\ \hline \end{array} \right) \quad \text{has standardization } \tilde{\mathsf{S}} = \left(\begin{array}{|c|c|} \hline 8 & 9 \\ \hline 3 & 7 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 5 & 6 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|} \hline 10 \\ \hline 4 \\ \hline \end{array} \right)$$

with $\operatorname{des}(\tilde{\mathsf{S}}) = \{2, 4, 7, 9\}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of Lemma. By the definition of an inversion pair, equal entries $\tilde{\mathsf{T}}(x) = \tilde{\mathsf{T}}(y) = a$ contribute nothing to $\operatorname{inv}(\tilde{\mathsf{T}})$. In the standardization $\tilde{\mathsf{S}}$, equal entries are replaced with entries labeled in increasing order of adjusted content $\tilde{c}$, contributing nothing to $\operatorname{inv}(\tilde{\mathsf{S}})$. On the other hand, unequal entries of $\tilde{\mathsf{T}}$ give rise to entries ordered in the same way in $\tilde{\mathsf{S}}$, so $\operatorname{inv}(\tilde{\mathsf{T}}) = \operatorname{inv}(\tilde{\mathsf{S}})$.

Given a semistandard n-ribbon tableau T, we define its standardization to be the unique standard n-ribbon tableau S such that $\tilde{\mathsf{S}}$ is the standardization of $\tilde{\mathsf{T}}$. Then, T and S have the same underlying ribbon tiling and so $\operatorname{spin}(\mathsf{T}) = \operatorname{spin}(\mathsf{S})$. Thus, by the previous lemma,

$$\operatorname{inv}(\tilde{\mathsf{T}}) = \operatorname{inv}(\tilde{\mathsf{S}}) = \operatorname{cospin}(\mathsf{S}) + e = \operatorname{cospin}(\mathsf{T}) + e$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3.56. **Theorem** (Bylund, Haiman). *For any skew shape μ that can be tiled by n-ribbons, there is a constant e depending on μ such that*

$$t^e \operatorname{LLT}_\mu^{(n)}(x, t) = \sum_{\tilde{\mathsf{T}}} t^{\operatorname{inv}(\tilde{\mathsf{T}})} x^{\operatorname{wt}(\tilde{\mathsf{T}})}$$

where the sum is over all SSYT of shape $\tilde{\mu}$.

Proof. Since $\operatorname{cospin}(\mathsf{T}) + e = \operatorname{inv}(\tilde{\mathsf{T}})$ for all T of shape μ by the lemma above and $x^{\operatorname{wt}(\mathsf{T})} = x^{\operatorname{wt}(\tilde{\mathsf{T}})}$ by definition, we have that

$$t^e \operatorname{LLT}_\mu^{(n)}(x, t) = \sum_{\mathsf{T}} t^{e + \operatorname{cospin}(\mathsf{T})} x^{\operatorname{wt}(\mathsf{T})} = \sum_{\tilde{\mathsf{T}}} t^{\operatorname{inv}(\mathsf{T})} x^{\operatorname{wt}(\tilde{\mathsf{T}})}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

4. Hall-Littlewood Polynomials

Throughout our discussions on symmetric functions, most of our examples have been stated and worked out for symmetric functions over reasonably simple fields, such as $\mathbb{Q}$, or some other relatively structured ring, such as a PID like $\mathbb{Z}$. However, there is no reason we cannot consider Λ over other rings. In this section, we will primarily be interested in symmetric functions over $\mathbb{Q}(t)$.

4.1. **Symmetric polynomials over $\mathbb{Q}(t)$.**

4.1. **Example.** Consider Λ_2 over $\mathbb{Q}(t)$. The polynomial

$$tx_1^2 + tx_2^2 + \frac{1-3t}{1+t^2}x_1x_2$$

would be in Λ_2.

4.2. **Definition.** Let λ be a partition. We define $m_i(\lambda)$ to be the number of parts in λ of size i. Equivalently, this is the number of rows with i boxes if λ is represented as a Ferrers diagram.

4.3. **Example.** Let $\lambda = (4, 4, 4, 3, 3, 2, 2, 2, 1, 1, 1, 1)$. Then,

$$m_1(\lambda) = 4, \ m_2(\lambda) = 3, \ m_3(\lambda) = 2, \ m_4(\lambda) = 3$$

4.4. **Definition.** Let λ be a partition and let $n = \ell(\lambda)$. Then, we define $\mathfrak{S}_n^\lambda$ to be a subgroup of $\mathfrak{S}_n$ such that

$$\sigma \in \mathfrak{S}_n^\lambda \iff \sigma(\lambda) = \lambda$$

where $\sigma(\lambda_i) = \lambda_{\sigma(i)}$.

4.5. **Example.** Let $\sigma = (1, 3, 2, 5, 4, 8, 7, 6, 12, 11, 10, 9)$ in one line notation. Then,

$$\mathfrak{S}_n^\lambda \cong \mathfrak{S}_{m_N} \times \cdots \times \mathfrak{S}_{m_1}$$
$$\sigma \mapsto (1, 3, 2) \times (2, 1) \times (3, 2, 1) \times (4, 3, 2, 1)$$

all in one line notation.

4.6. **Remark.** Recall for $\sigma \in \mathfrak{S}_n$, (i, j) is an inversion pair if $i < j$ but $\sigma(i) > \sigma(j)$. Thus, if $\sigma \in \mathfrak{S}_n^\lambda$, say $\sigma = w_1 \times w_2 \times \cdots \times w_N$, then

$$\mathrm{inv}(\sigma) = \sum_{i=1}^{N} \mathrm{inv}(w_i)$$

since blocks do not contribute inversions outside of the block.

4.7. **Definition.** For $n \in \mathbb{N}$, let

$$V_n(t) := \prod_{i=1}^{n} \frac{1-t^i}{1-t} = \prod_{i=1}^{n}(1 + t + \cdots + t^{i-1})$$

and, for a partition λ, let

$$V_\lambda(t) = \prod_{i \geq 0} V_{m_i(\lambda)}(t)$$

4.8. **Remark.** One should exercise caution since, for $a \in \mathbb{N}$,

$$V_{(a)}(t) = 1 \text{ but } V_a(t) = \prod_{i=1}^{a}(1 + t + \cdots + t^{i-1})$$

4.9. Example. Continuing with $\lambda = (4, 4, 4, 3, 3, 2, 2, 2, 1, 1, 1, 1)$, we have

$$V_\lambda(t) = V_3^2(t)V_2(t)V_4(t)$$

4.10. Proposition. *Given* $n \in \mathbb{N}$,

$$V_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}(\sigma)}$$

Proof. The proof is a straightforward application of the quantum factorial.

$\square$

4.11. Corollary. *By the above proposition,*

$$V_\lambda(t) = \sum_{\sigma \in \mathfrak{S}_n^\lambda} t^{\mathrm{inv}(\sigma)}$$

Proof. We check

$$
\begin{aligned}
V_\lambda(t) &= \prod_{i \geq 0} V_{m_i(\lambda)}(t) \\
&= \prod_{i \geq 0} \sum_{w_i \in \mathfrak{S}_{m_i}} t^{\mathrm{inv}(w_i)} && \text{by Proposition} \\
&= \sum_{(w_1, \cdots, w_N) \in \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_N}} t^{\mathrm{inv}(w_1) + \mathrm{inv}(w_2) + \cdots + \mathrm{inv}(w_N)} \\
&= \sum_{\sigma \in \mathfrak{S}_n^\lambda} t^{\mathrm{inv}(\sigma)} && \text{by Remark 4.6}
\end{aligned}
$$

$\square$

4.12. Definition. Let λ be a partition. We define

$$R_\lambda(x_1, \ldots, x_n; t) := \sum_{w \in \mathfrak{S}_n} w \left(x^\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right) \in \mathbb{Q}(t)[x_1, \ldots, x_n]$$

4.13. Remark. There are many things to notice about this R_λ polynomial, and things to be suspicious of, too.

(a) The product should be reminiscent of the Vandermonde determinant. Recall

$$\Delta = \det(x_i^j) = \prod_{i<j}(x_i - x_j) = \sum_{w \in \mathfrak{S}_n} \mathrm{sgn}(w) w(x^\rho)$$

and, more generally, for $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with distinct parts,

$$\Delta_\alpha = \det(x_i^{\alpha_j}) = \sum_{w \in \mathfrak{S}_n} \mathrm{sgn}(w) w(x^\alpha)$$

Thus, this R_λ polynomial is of a similar form.

(b) In the definition of R_λ, we are dividing the expression by $x_i - x_j$'s, but these elements are not in our ring. Thus, we must argue that the numerator is somehow divisible by this term. However, recall that a Schur function s_λ is

$$s_\lambda = \frac{\Delta_{\lambda+\rho}}{\Delta} = \frac{\Delta_{\lambda+\rho}}{\prod_{i<j} x_i - x_j} \in \Lambda$$

and so it is not unreasonable for this idea to work out.

(c) In fact, when $t = 0$,

$$R(x_1, \ldots, x_n; 0) = \sum_{w \in \mathfrak{S}_n} w \left(x^\lambda \prod_{i<j} \frac{x_i}{x_i - x_j} \right)$$

and, since $\prod_{1 \leq i < j \leq n} x_i = x^\rho$ and $w\left(\prod_{i<j} x_i - x_j\right) = \operatorname{sgn}(w)\Delta$, we get

$$\begin{aligned}
R_\lambda(x_1, \ldots, x_n; 0) &= \sum_{w \in \mathfrak{S}_n} \frac{\operatorname{sgn}(w)}{\Delta} w(x^{\lambda+\rho}) \\
&= \frac{1}{\Delta} \sum_{w \in \mathfrak{S}_n} \operatorname{sgn}(w) w(x^{\lambda+\rho}) \\
&= \frac{\Delta_{\lambda+\rho}}{\Delta_\rho} \\
&= s_\lambda
\end{aligned}$$

(d) When $t = 1$, we get

$$R(x_1, \ldots, x_n; 1) = \sum_{w \in \mathfrak{S}_n} w \left(x^\lambda \prod_{i<j} \frac{x_i - x_j}{x_i - x_j} \right) = \sum_{w \in \mathfrak{S}_n} w(x^\lambda)$$

However, since $\sum_{w \in \mathfrak{S}_n} w(x^\lambda) = \sum_{w \in \mathfrak{S}_n} x^{w(\lambda)}$, then we have

$$R(x_1, \ldots, x_n; 1) = |\operatorname{Stab}_{\mathfrak{S}_n}(\lambda)| \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha \sim \lambda}} x^\alpha = |\mathfrak{S}_n^\lambda| m_\lambda(x_1, \ldots, x_n)$$

4.14. **Example.** We compute the small examples of $R_{11}(x_1, x_2)$ and $R_2(x_1, x_2)$. Thus, we are working over $\mathfrak{S}_2 = \{id, (12)\}$ (in cycle notation). First,

$$\begin{aligned}
R_{11}(x_1, x_2) &= x_1 x_2 \left(\frac{x_1 - tx_2}{x_1 - x_2} \right) + x_1 x_2 \left(\frac{x_2 - tx_1}{x_2 - x_1} \right) \\
&= \frac{x_1 x_2}{x_1 - x_2} (x_1 - tx_2 - x_2 + tx_1) \\
&= \frac{x_1 x_2}{x_1 - x_2} (x_1 - x_2 + t(x_1 - x_2)) \\
&= x_1 x_2 (1 + t)
\end{aligned}$$

which is $(1+t)s_{11}$ in terms of Schur functions. Also, note that

$$V_{(11)}(t) = V_2(t) = \frac{1 - t^2}{1 - t} = 1 + t$$

Second,

$$\begin{aligned}
R_2(x_1, x_2) &= x_1^2 \frac{x_1 - tx_2}{x_1 - x_2} + x_2^2 \frac{x_2 - tx_1}{x_2 - x_1} \\
&= \frac{1}{x_1 - x_2}(x_1^3 - tx_1^2 x_2 - x_2^3 + tx_1 x_2^2) \\
&= \frac{1}{x_1 - x_2}(x_1^3 - x_2^3 - tx_1 x_2(x_1 - x_2)) \\
&= x_1^2 + x_1 x_2 + x_2^2 - tx_1 x_2
\end{aligned}$$

which is $s_2 - ts_{11}$ in terms of Schur functions. Thus, we see that R_λ's are not necessarily Schur positive. Also, of note,

$$V_{(2)} = V_1(t) = 1$$

Thus, in both our examples, R_λ has been divisible by $V_\lambda(t)$.

4.15. **Proposition.** $R_\lambda(x_1, \ldots, x_n; t)$ *is a symmetric polynomial.*

Proof. Observe that

$$\prod_{i<j}(x_i - tx_j) = \begin{array}{llll}
(x_1 - tx_2) & (x_1 - tx_3) & (x_1 - tx_4) & \cdots \\
 & (x_2 - tx_3) & (x_2 - tx_4) & \cdots \\
 & & (x_3 - tx_4) & \cdots
\end{array}$$

and so, expanding this product into a sum, each monomial will be of the form

$$\prod_{i<j} x_i^{r_{ij}}(-tx_j)^{s_{ij}}, \quad r_{ij}, s_{ij} \in \mathbb{N}$$

However, there are some restrictions we must place on r_{ij} and s_{ij}. In fact, from each term in initial product above, we are picking either an x_i or a $-tx_j$; we cannot pick both. So, the combinatorics of picking monomials amounts to circling either the (i, j) or (j, i) entry of the following matrix, but not both.

$$\begin{pmatrix}
0 & x_1 & x_1 & x_1 & \cdots \\
-tx_2 & 0 & x_2 & x_2 & \cdots \\
-tx_3 & -tx_3 & 0 & x_3 & \cdots \\
\vdots & & & \ddots & \\
-tx_n & -tx_n & -tx_n & \cdots & 0
\end{pmatrix}$$

Thus, if we take $U = (u_{i,j})$ to be a $(0, 1)$-matrix with $u_{i,i} = 0$ and $u_{i,j} + u_{j,i} = 1$, we get

$$\prod_{i<j}(x_i - tx_j) = \sum_U \prod_{i<j} x_i^{u_{ij}}(-tx_j)^{u_{ji}}$$

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where the sum is over all such $(0, 1)$-matrices U. Thus, the symmetry follows.

□ Elaborate

4.16. Proposition.

$$R_\lambda(x_1, \ldots, x_n; t) = V_\lambda(t) \left(s_\lambda + \sum_{\lambda \rhd \mu} c_{\lambda,\mu}(t) s_\mu \right)$$

for some $c_{\lambda,\mu}(t) \in \mathbb{Q}(t)$.

4.17. Corollary. $\{R_\lambda\}_\lambda$ *is a basis of symmetric polynomials.*

4.18. Remark. If we let

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{V_\lambda(t)} R_\lambda(x_1, \ldots, x_n; t)$$

then

$$s_\lambda(x_1, \ldots, x_n) = \sum_\mu c_{\lambda,\mu}(t) P_\mu(x_1, \ldots, x_n; t)$$

with $c_{\lambda,\mu}(t) \in \mathbb{N}[t]$, that is, each $c_{\lambda,\mu}(t)$ is a polynomial in t with no negative coefficients! This is incredibly remarkable since there are no t's on the left hand side, so they must all cancel somehow. Furthermore, it is known that

$$c_{\lambda,\mu}(t) = K_{\lambda,\mu}(t) = \sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{charge}(\mathsf{T})}$$

where $\mathrm{charge}(\mathsf{T})$ is appropriately defined. Some of this material will be elaborated in the next section.

4.2. t-Generalization of Hall Inner Product. Following the remark concluding the previous section, we define the following.

4.19. Definition. Let λ be a partition. Then, we define the *Hall-Littlewood polynomials* to be

$$P_\lambda(x; t) = \frac{1}{V_\lambda(t)} \sum_{w \in \mathfrak{S}_n} w \left(x^\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

4.20. Proposition.   *(a) $P_\lambda(x; 0) = s_\lambda$*
    (b) $P_\lambda(x; 1) = m_\lambda$
    (c) $\{P_\lambda(x; t)\}$ is a basis for $\Lambda[t]$, that is, Λ over $\mathbb{Q}(t)$.

4.21. Definition. Let $\langle \cdot, \cdot \rangle_t \colon \Lambda[t] \times \Lambda[t] \to \mathbb{Q}(t)$ be given by

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda,\mu} z_\lambda \prod_i (1 - t^{\lambda_i})$$

where $z_\lambda = \prod_i i^{m_i} m_i!$. (See 2.48).

4.22. Proposition. *The set $\{P_\lambda(x; t)\}$ is characterized by*

    (a) Orthogonality with respect to $\langle \cdot, \cdot \rangle_t$.

(b) *Unitriangularity with respect to Schur basis. That is,*

$$P_\lambda(x;t) = s_\lambda + \sum_{\lambda \rhd \mu} c_{\lambda,\mu}(t) s_\mu$$

4.23. Corollary. *If we set $t = 1$, this gives us that*

$$m_\lambda = s_\lambda + \text{ lower order terms}$$

where the lower order terms come from the combinatorics of special rim hook tabloids.

4.24. Definition. Let $\{Q'_\mu(x;t)\}$ in $\Lambda[t]$ be the dual basis to $\{P_\lambda(x;t)\}$ with respect to $\langle \cdot, \cdot \rangle_t$.

4.25. Proposition. *Since $s_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu$ by 4.18, we get*

$$Q'_\mu = \sum_\lambda K_{\lambda,\mu}(t) s_\lambda$$

4.26. Theorem.

$$K_{\lambda,\mu}(t) = \sum_{\mathsf{T} \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{charge}(\mathsf{T})}$$

Of course, to understand this theorem, we must define the charge of a semistandard Young tableau.

4.27. Definition. The *reading word* of a tableau T is the word created from reading the entries of T from left to right, top to bottom.

4.28. Example. Let

$$\mathsf{T} = \begin{array}{|c|c|c|c|}
\cline{1-1}
5 \\
\cline{1-1}
4 \\
\cline{1-3}
3 & 6 & 8 \\
\cline{1-4}
1 & 2 & 7 & 9 \\
\cline{1-4}
\end{array}$$

Then, the reading word is 543681279.

4.29. Definition. Recall that the charge of a permutation σ is given by writing σ counterclockwise on a circle and labeling the letters $1, 2, 3, \ldots$ with an "index" clockwise starting at 0 and increasing if you cross $\star$. Finally, the charge is the sum of the indices. Then, for T a *standard* Young tableau, we define charge(T) to be the charge of the reading word of T. Alternatively, one can assign indices $I = (I_1, I_2, \ldots, I_\ell)$ by setting

$$I_1 = 0, I_x = \begin{cases} I_{x-1} + 1 & \text{if } x \text{ is east of } x - 1 \\ I_{x-1} & \text{else} \end{cases}$$

4.30. **Example.** Let T be as above so it has reading word $\sigma = 543681279$. Then, we compute
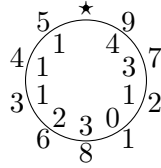


giving charge(T) = 16.

4.31. **Definition.** We define the charge of a semistandard Young tableau as follows. Given a word (tableau) of partition weight,

  (a) Extract the "standard" (permutation) subwords.
  (b) Compute charge on each of these subwords.
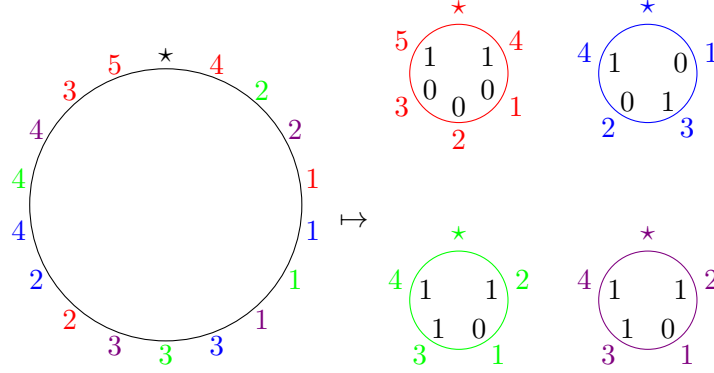 (c) Sum up each of the separate charges to get the charge.

4.32. **Example.** Let

$$
\mathsf{T} =
\begin{array}{|c|c|c|c|c|c|c|}
\hline
5 \\
\cline{1-1}
3 & 4 & 4 & 4 \\
\cline{1-4}
2 & 2 & 3 & 3 & 3 \\
\cline{1-5}
1 & 1 & 1 & 1 & 2 & 2 & 4 \\
\hline
\end{array}
$$

Then, we draw the reading word on a charge circle and find the standard subwords, labeled in different colors and done successively for clarity.

$\rightarrow$ $\rightarrow$

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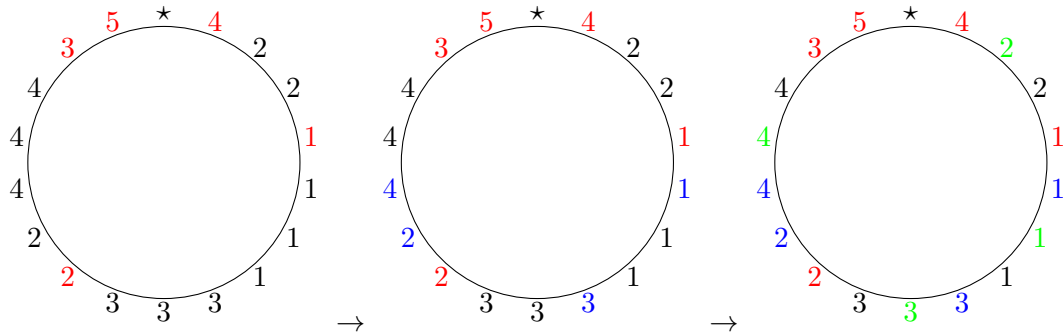
And thus giving us

Thus, the charge would be $2 + 2 + 3 + 3 = 10$.

4.33. Example. Our theorem says that

$$s_{21} = \sum_{\text{partition } \mu} \left(\sum_{\mathsf{T} \in \mathrm{SSYT}((2,1),\mu)} t^{\mathrm{charge}(\mathsf{T})} \right) P_\mu(x;t)$$

$$= \left(\sum_{\mathsf{T} \in \mathrm{SSYT}((2,1),(2,1))} t^{\mathrm{charge}(\mathsf{T})} \right) P_{21}(x;t) + \left(\sum_{\mathsf{T} \in \mathrm{SSYT}((2,1),(1,1,1))} t^{\mathrm{charge}(\mathsf{T})} \right) P_{111}(x;t)$$

Thus, we can now use our theorem to compute

$$s_{21} = t^0 P_{21}(x;t) + (t^2 + t) P_{111}(x;t)$$

since

4.34. Remark. It should be noted that all Hall-Littlewood polynomials are actually special cases of LLT polynomials. _____

However, I do not know how to prove this.

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