

A raising operator formula for Macdonald polynomials via LLT polynomials in the elliptic Hall algebra

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Loyola University Chicago TACO Seminar

Based on arXiv:2112.07063 and arXiv:2307.06517

October 4, 2023

Glad to be back



Graduation May 2015

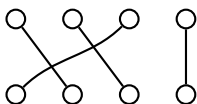
- ① **Background on symmetric functions and Macdonald polynomials**
- ② A new formula for Macdonald polynomials
- ③ LLT polynomials in the elliptic Hall algebra

Symmetric Group

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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} =$$


The diagram illustrates the permutation σ as a mapping between two rows of four nodes. The top row represents the domain $\{1, 2, 3, 4\}$ and the bottom row represents the codomain $\{1, 2, 3, 4\}$. Lines connect the nodes as follows: a line from the first node to the second, from the second to the third, from the third to the first, and a vertical line from the fourth to the fourth. This represents the permutation $\sigma(1)=2, \sigma(2)=3, \sigma(3)=1, \sigma(4)=4$.

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- For $f \in \mathbb{Q}[x_1, \dots, x_n]$ multivariate polynomial, $\sigma \in S_n$ acts as $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

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Bases for symmetric functions

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$n \in \mathbb{Z}_{>0}$, a *partition of n* is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$.

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$$4 + 1 \rightarrow$$


$3 + 2 \rightarrow$ 

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|} \hline \square & & \\ \hline \square & & \\ \hline \square & \square & \square \\ \hline \end{array}$$

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A base ten block diagram consisting of one ten rod (a vertical rectangle) and four one units (small horizontal rectangles) placed to the right of the ten rod.

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\implies any basis of degree d symmetric functions can be indexed by partitions of d .

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$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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For λ a partition, set

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- $\{s_\lambda\}_\lambda$ forms a basis for Λ .

Symmetric functions and Schur functions

- Convention: $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$, set

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Precisely, for $\rho = (n-1, n-2, \dots, 1, 0)$,

$$s_\gamma = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma + \rho) - \rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\operatorname{sort}(\beta)$ = weakly decreasing sequence obtained by sorting β ,
- $\operatorname{sgn}(\beta)$ = sign of the shortest permutation taking β to $\operatorname{sort}(\beta)$.

Example: $s_{201} = 0$, $s_{2-11} = -s_{200}$.

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Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in \mathbb{N}) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \square \\ \hline \end{array}} + s_{\begin{array}{|c|} \hline \square \\ \hline \square \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_{+}^{S_3})$.

Getting more information

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Break M up into irreducible representations.

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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Solution: irreducible S_n -representation of polynomials of degree $d \mapsto q^d s_\lambda$
(graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \\ \hline \square & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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Answer: Hall-Littlewood polynomial $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$ generalizing Hall-Littlewood polynomials (and many other famous bases).

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$.
- Does there exist a family of S_n -regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$ with $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_j) = y_{\sigma(j)}$.

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$$\tilde{H}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}} = qts \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + ts \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + qs \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + s \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

Garsia-Haiman modules

Theorem (Haiman, 2001)

The Garsia-Haiman module M_λ has bigraded Frobenius characteristic given by $\tilde{H}_\lambda(X; q, t)$.

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

- ① Background on symmetric functions and Macdonald polynomials
- ② **A new formula for Macdonald polynomials**
- ③ LLT polynomials in the elliptic Hall algebra

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n , where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

	(12)	(13)	(14)	(15)
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A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.

	(12)	(13)	(14)	(15)
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$\Psi = \text{Roots above Dyck path}$

Weyl symmetrization

Define the *Weyl symmetrization operator* $\sigma: \mathbb{Q}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \rightarrow \Lambda(X)$ by linearly extending

$$\mathbf{z}^\gamma \mapsto s_\gamma(X)$$

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$$H(\Phi; \gamma) = \sigma \left(\frac{\mathbf{z}^\gamma}{\prod_{(i,j) \in \Psi} (1 - tz_i/z_j)} \right)$$

Denominator factors are understood as geometric series
 $(1 - tz_i/z_j)^{-1} = 1 + tz_i/z_j + t^2(z_i/z_j)^2 + \cdots$

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Example:

$$\Psi = \begin{array}{c} \blacksquare \\ \square \\ \square \end{array} \quad \gamma = (1, 1, 1)$$


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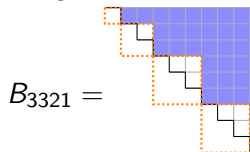
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$\Psi =$  $\gamma = (1, 1, 1)$

$$\begin{aligned} H(\Psi; \gamma) &= \sigma \left(\left(1 + t \frac{z_1}{z_2} + t^2 \frac{z_1^2}{z_2^2} + \dots\right) \left(1 + t \frac{z_1}{z_3} + t^2 \frac{z_1^2}{z_3^2} + \dots\right) z_1 z_2 z_3 \right) \\ &= s_{111} + t(s_{201} + s_{210}) + t^2(s_{3-10} + s_{300} + s_{31-1}) + \dots \\ &= s_{111} + ts_{210} \end{aligned}$$

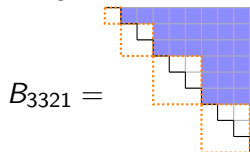
A Catalan function for modified Hall-Littlewoods

B_μ = set of roots above block diagonal matrix with block sizes $\mu_{\ell(\mu)}, \dots, \mu_1$



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Theorem (Weyman, Shimozono-Weyman)

$$\tilde{H}_\mu(X; 0, t) = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - t z^\alpha)} \right),$$

where $z^\alpha = z_i / z_j$.

$\omega(s_\lambda) = s_{\lambda'}$ for λ' the transpose partition of λ .

Catalan functions for modified Hall-Littlewoods

b_1		
b_2	b_3	
b_4	b_5	b_6
b_7	b_8	b_9

row reading order

$$b_1 \prec b_2 \prec \cdots \prec b_n$$

$$R_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \preceq b_j \}.$$

$$R_{3321} =$$

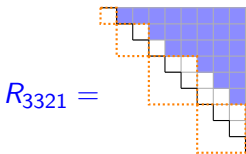
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$$\begin{aligned} \tilde{H}_\mu(X; 0, t) &= \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu} (1 - t z^\alpha)} \right), \\ &= \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \end{aligned}$$

A formula for $\tilde{H}_\mu(X; q, t)$

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$$\hat{R}_\mu := \{ \alpha_{ij} \in R_+ \mid \text{south}(b_i) \prec b_j \}.$$

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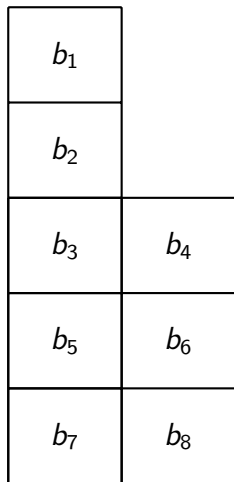
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Theorem (Blasiak-Haiman-Morse-Pun-S.)

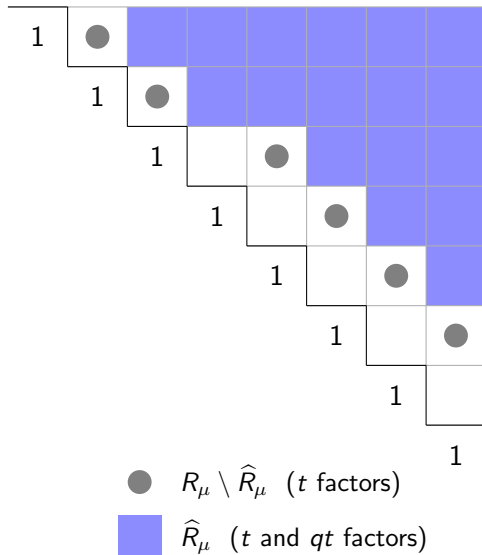
The modified Macdonald polynomial $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ is given by

$$\tilde{H}_\mu = \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right).$$

Example



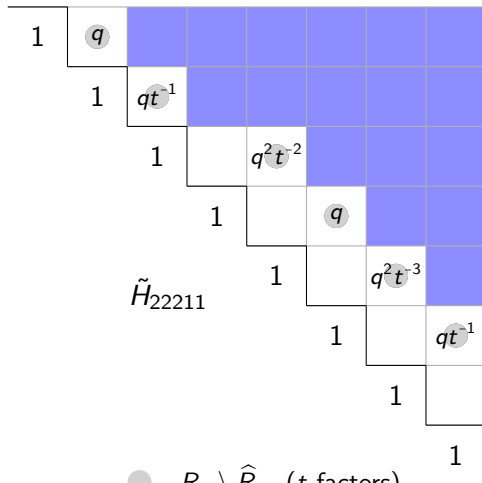
partition $\mu = 22211$



Example

$1 - q^{\frac{z_1}{z_2}}$	
$1 - qt^{-1} \frac{z_2}{z_3}$	
$1 - q^2 t^{-2} \frac{z_3}{z_5}$	$1 - q^{\frac{z_4}{z_6}}$
$1 - q^2 t^{-3} \frac{z_5}{z_7}$	$1 - qt^{-1} \frac{z_6}{z_8}$

numerator factors $1 - q^{\text{arm}+1} t^{-\text{leg}} z_i/z_j$



● $R_\mu \setminus \hat{R}_\mu$ (t factors)

■ \hat{R}_μ (t and qt factors)

$q = t = 1$ specialization

$$\begin{aligned}
 & \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - q t z^\alpha)}{\prod_{\alpha \in R_+} (1 - q z^\alpha) \prod_{\alpha \in R_\mu} (1 - t z^\alpha)} \right) \\
 & \xrightarrow{q=t=1} \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha \in R_\mu \setminus \hat{R}_\mu} (1 - z^\alpha) \prod_{\alpha \in \hat{R}_\mu} (1 - z^\alpha)}{\prod_{\alpha \in R_+} (1 - z^\alpha) \prod_{\alpha \in R_\mu} (1 - z^\alpha)} \right) \\
 & = \omega \sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_+} (1 - z^\alpha)} \right) \\
 & = \omega h_1^n \\
 & = e_1^n
 \end{aligned}$$

$q = 0$ specialization

$$\begin{aligned}
 & \omega\sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \widehat{R}_\mu} (1 - qtz^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right) \\
 & \xrightarrow{q=0} \omega\sigma \left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right) \\
 & = \tilde{H}_\mu(X; 0, t)
 \end{aligned}$$

Proof of formula for \tilde{H}_μ

Definition

∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

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- Start with the Haglund-Haiman-Loehr formula for \tilde{H}_μ as a sum of LLT polynomials $\mathcal{G}_\nu(X; q)$.

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- Start with the Haglund-Haiman-Loehr formula for \tilde{H}_μ as a sum of LLT polynomials $\mathcal{G}_\nu(X; q)$.
- Apply $\omega \nabla$ to both sides.

Proof of formula for \tilde{H}_μ

Definition

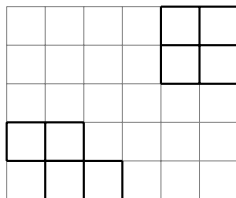
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- Start with the Haglund-Haiman-Loehr formula for \tilde{H}_μ as a sum of LLT polynomials $\mathcal{G}_\nu(X; q)$.
- Apply $\omega \nabla$ to both sides.
- Use Catalan-like (“Catalanimal”) formula for $\omega \nabla \mathcal{G}_\nu(X; q)$ and collect terms.

LLT Polynomials

Let $\nu = (\nu_{(1)}, \dots, \nu_{(k)})$ be a tuple of skew shapes.

$$\nu = \left(\begin{array}{|c|c|c|} \hline \square & \square & \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right)$$



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-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

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Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

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- A *semistandard tableau* on ν is a map $T: \nu \rightarrow \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes ν is

$$\mathcal{G}_\nu(\mathbf{x}; q) = \sum_{T \in \text{SSYT}(\nu)} q^{\text{inv}(T)} \mathbf{x}^T,$$

where $\text{inv}(T)$ is the number of attacking inversions in T and $\mathbf{x}^T = \prod_{a \in \nu} x_{T(a)}$.

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$$\text{inv}(T) = 4, \quad \mathbf{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$$

Definition

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Catalanimals

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$$H(R_q, R_t, R_{qt}, \lambda) = \sigma \left(\frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right).$$

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With $n = 3$,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \sigma \left(\frac{z^{111} (1 - qt z_1/z_3)}{\prod_{1 \leq i < j \leq 3} (1 - q z_i/z_j) (1 - t z_i/z_j)} \right) \\ &= s_{111} + (q + t + q^2 + qt + t^2) s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3) s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

LLT Catalanimals

For a tuple of skew shapes ν , the *LLT Catalanimal* $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,

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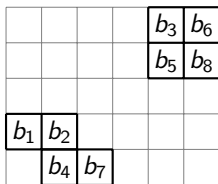
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- $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
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- λ : fill each diagonal D of ν with $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
Listing this filling in reading order gives λ .

LLT Catalanimals

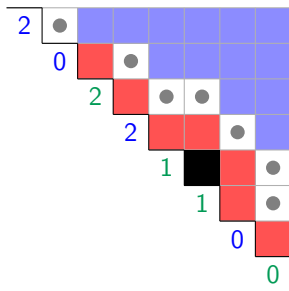
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ν

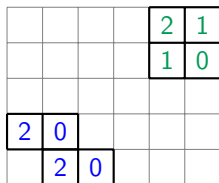


LLT Catalanimals

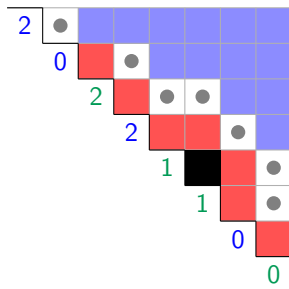
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λ , as a filling of ν



Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let ν be a tuple of skew shapes and let $H_\nu = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned}\nabla \mathcal{G}_\nu(X; q) &= c_\nu \omega H_\nu \\ &= c_\nu \omega \sigma \left(\frac{z^\lambda \prod_{\alpha \in R_{qt}} (1 - qt z^\alpha)}{\prod_{\alpha \in R_q} (1 - q z^\alpha) \prod_{\alpha \in R_t} (1 - t z^\alpha)} \right)\end{aligned}$$

for some $c_\nu \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

Haglund-Haiman-Loehr formula

Theorem (Haglund-Haiman-Loehr, 2005)

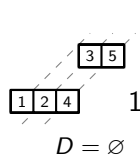
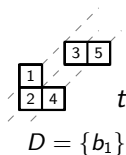
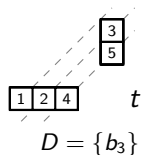
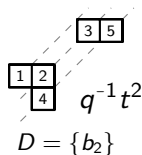
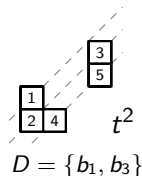
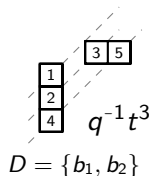
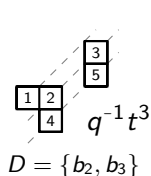
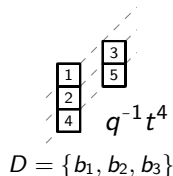
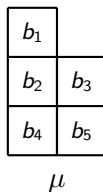
$$\tilde{H}_\mu(X; q, t) = \sum_D \left(\prod_{u \in D} q^{-\text{arm}(u)} t^{\text{leg}(u)+1} \right) \mathcal{G}_{\nu(\mu, D)}(X; q),$$

where

- the sum runs over all subsets $D \subseteq \{(i, j) \in \mu \mid j > 1\}$, and
- $\nu(\mu, D) = (\nu^{(1)}, \dots, \nu^{(k)})$ where $k = \mu_1$ is the number of columns of μ , and $\nu^{(i)}$ is a ribbon of size μ_i^* , i.e., box contents $\{-1, -2, \dots, -\mu_i^*\}$, and descent set $\text{Des}(\nu^{(i)}) = \{-j \mid (i, j) \in D\}$.

Haglund-Haiman-Loehr formula example

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Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\nu(\mu,D)}$ and apply $\omega \nabla$.

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- By construction, all the LLT Catalan animals $H_{\nu(\mu,D)}$ appearing on the RHS will have the same root ideal data (R_q, R_t, R_{qt}) .
- Collect terms to get $\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j)$ factor.

$$\tilde{H}_\mu = \omega \sigma \left(z_1 \cdots z_n \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i/z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right).$$

Outline

- ① Background on symmetric functions and Macdonald polynomials
- ② A new formula for Macdonald polynomials
- ③ **LLT polynomials in the elliptic Hall algebra**

Elliptic Hall Algebra

Burban and Schiffmann studied a subalgebra \mathcal{E} of the Hall algebra of coherent sheaves on an elliptic curve over \mathbb{F}_p .

The *elliptic Hall algebra* \mathcal{E} is generated by subalgebras $\Lambda(X^{a,b})$ isomorphic to the ring of symmetric functions Λ over $\mathbb{k} = \mathbb{Q}(q, t)$, one for each coprime pair $(a, b) \in \mathbb{Z}^2$, along with an additional central subalgebra.

Shuffle algebra

Define a linear map

$$\sigma_{\Gamma}: \bigoplus_n \mathbb{k}(z_1, \dots, z_n) \rightarrow \bigoplus_n \mathbb{k}(z_1, \dots, z_n)^{S_n}$$

whose graded components σ_{Γ}^n are given by

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$$\text{where } \Gamma(z_i, z_j) = \frac{1 - qtz_i/z_j}{(1 - z_j/z_i)(1 - qz_i/z_j)(1 - tz_i/z_j)}$$

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The *shuffle algebra* \mathcal{S}_{Γ} is the image of $\bigoplus_n \mathbb{k}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ under the map σ_{Γ} , equipped with a variant of the concatenation product.

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Nice fact (up to some modifications of definitions)

Some Catalan animals are elements in \mathcal{S}_{Γ} . (“Tame Catalan animals”)

Shuffle to elliptic Hall isomorphism

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Theorem (Schiffmann-Vasserot)

There is an algebra isomorphism $\psi: \mathcal{S}_\Gamma \rightarrow \mathcal{E}^+$.

Elliptic Hall algebra action

Schiffmann-Vasserot and Feigin-Tsymbaliuk constructed an action of \mathcal{E} on Λ , where $f(X^{0,1})$ acts by multiplication by $f(X)$.

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$$f(X^{1,1}) \cdot 1 = \nabla f(X^{0,1}) \nabla^{-1} \cdot 1 = \nabla f.$$

Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let H be a Catalan number such that $\psi(H) = f(X^{1,1})$. Then

$$\nabla f = \omega H.$$

Shuffle to elliptic Hall summary

$$\begin{array}{ccc}
 & \mathcal{E} \curvearrowright \Lambda & f(X^{1,1}) \cdot 1 = \nabla f \\
 & \uparrow & \\
 \bigoplus_{\substack{a>0 \\ b \in \mathbb{Z} \\ (a,b)=1}} \Lambda(X^{a,b}) & \stackrel{\cong}{\underset{\text{v.sp.}}{\longrightarrow}} & \mathcal{E}^+ \\
 & \uparrow \psi \cong & \\
 \sigma_\Gamma \left(\bigoplus_n \mathbb{k}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \right) & \stackrel{\cong}{\underset{\text{v.sp.}}{\longrightarrow}} \mathcal{S}_\Gamma \ni H & \text{"tame" Catalanimal}
 \end{array}$$

Theorem (Blasiak-Haiman-Morse-Pun-S.)

$$\psi(H) = f(X^{1,1}) \implies f(X^{1,1}) \cdot 1 = \nabla f = \omega H.$$

Proof of $\nabla \mathcal{G}_\nu$ formula

- ① LLT Catalananimals H_ν are tame.
- ② LLT Catalananimals lie in $\psi^{-1}(\Lambda(X^{1,1}))$.
- ③ Describe coproduct Δ on \mathcal{E} explicitly on tame Catalananimals and show ΔH_ν matches $\Delta \mathcal{G}_\nu$.
- ④ Conclude $\psi(H_\nu) = c_\nu^{-1} \mathcal{G}_\nu(X^{1,1}) \in \mathcal{E}$.
- ⑤ Apply previous theorem to conclude $\nabla \mathcal{G}_\nu = c_\nu \omega H_\nu$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega \sigma \left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij} \in R_\mu \setminus \hat{R}_\mu} (1 - q^{\text{arm}(b_i)+1} t^{-\text{leg}(b_i)} z_i / z_j) \prod_{\alpha \in \hat{R}_\mu} (1 - qt z^\alpha)}{\prod_{\alpha \in R_+} (1 - qz^\alpha) \prod_{\alpha \in R_\mu} (1 - tz^\alpha)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s , the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_{\nu} K_{\nu, \mu}^{(s)}(q, t) s_\nu(X)$$

satisfy $K_{\nu, \mu}^{(s)}(q, t) \in \mathbb{N}[q, t]$.

Thank you!

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Catalan animals in the shuffle algebra

For $\lambda \in \mathbb{Z}^n$,

$$\begin{aligned}\sigma_{\Gamma}^n(\mathbf{z}^\lambda) &= \sum_{w \in S_n} w \left(\frac{\mathbf{z}^\lambda \prod_{\alpha \in R_+} (1 - qt\mathbf{z}^\alpha)}{\prod_{\alpha \in R_+} ((1 - \mathbf{z}^{-\alpha})(1 - q\mathbf{z}^\alpha)(1 - t\mathbf{z}^\alpha))} \right) \\ &= H(R_+, R_+, R_+, \lambda) \in \mathcal{S}_{\Gamma}.\end{aligned}$$

- Technicality: we have redefined

$\sigma(\mathbf{z}^\gamma) = \sum_{w \in S_n} \left(\frac{\mathbf{z}^\gamma}{\prod_{\alpha \in R_+} (1 - \mathbf{z}^{-\alpha})} \right) = \chi_\gamma$, the irreducible GL_n character.

- Let pol_X send $\chi_\lambda \mapsto s_\lambda$ if $\lambda_n \geq 0$, otherwise $\chi_\lambda \mapsto 0$.
- The σ from before is given by $\sigma_{\mathrm{old}} = \mathrm{pol}_X \sigma_{\mathrm{new}}$.

Catalanimals in the Shuffle algebra

$\sigma_{\Gamma}^n(f)$ can lie in \mathcal{S}_{Γ} even when f is not a Laurent polynomial.

Theorem (Negut)

The following family of Catalanimals lie in the shuffle algebra:

$$\sigma_{\Gamma}^n\left(\frac{z^{\lambda}}{\prod_{i=1}^{n-1}(1 - qtz_i/z_{i+1})}\right) = H(R_+, R_+, R'_+, \lambda) \in \mathcal{S}_{\Gamma},$$

where $R'_+ = \{\alpha_{ij} \in R_+ \mid i + 1 < j\}$.

The wheel condition

- A symmetric Laurent polynomial $g(\mathbf{z})$ satisfies the *wheel condition* if it vanishes whenever any three of the variables z_i, z_j, z_k are in the ratio $(z_i : z_j : z_k) = (1 : q : qt) = (1 : t : qt)$.
- Let $\mathcal{S}_{\check{\Gamma}} \cong \mathcal{S}_{\Gamma}$ for
$$\check{\Gamma}(z_i, z_j) = (1 - z_i/z_j)(1 - qz_j/z_i)(1 - tz_j/z_i)(1 - qtz_i/z_j).$$

Theorem (Negut)

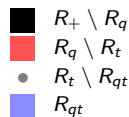
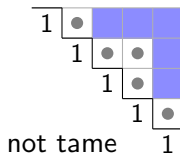
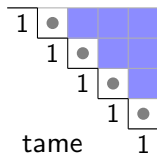
A symmetric Laurent polynomial $g(z_1, \dots, z_n)$ belongs to $\mathcal{S}_{\check{\Gamma}}$ if and only if it satisfies the wheel condition and vanishes whenever $z_i = z_j$ for $i \neq j$.

The wheel condition and tame Catalananimals

A Catalananimal $H(R_q, R_t, R_{qt}, \lambda)$ is *tame* if

$$R_q + R_t \subseteq R_{qt},$$

where $R_q + R_t = \{\alpha + \beta \mid \alpha \in R_q, \beta \in R_t\}$.



The Catalananimals $H(R_+, R_+, R'_+, \lambda)$ and the LLT Catalananimals are tame.

Using Negut's theorem, we show: Tame Catalananimals belong to the shuffle algebra \mathcal{S}_Γ .