A Catalanimal formula for Macdonald polynomials

George H. Seelinger
joint work with J. Blasiak, M. Haiman, J. Morse, and A. Pun

ghseeli@umich.edu

FPSAC 2023

Based on arXiv:2307.06517

July 17, 2023

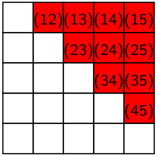
Root ideals

$R_+ = \left\{\alpha_{ij} \mid 1 \leq i < j \leq n\right\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.

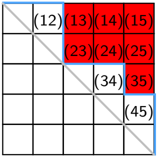
	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

Root ideals

$R_+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$ denotes the set of positive roots for GL_n, where $\alpha_{ij} = \epsilon_i - \epsilon_j$.



A root ideal $\Psi \subseteq R_+$ is an upper order ideal of positive roots.



Ψ = Roots above Dyck path

Symmetric functions and Schur functions

- Let $\Lambda(X)$ be the ring of symmetric functions in $X = x_1, x_2, \dots$
- $h_d = h_d(X) = \sum_{i_1 \le \cdots \le i_d} x_{i_1} \cdots x_{i_d}$ with $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$,

$$s_\gamma = s_\gamma(X) = \det(h_{\gamma_i + j - i}(X))_{1 \le i,j \le n}$$

Symmetric functions and Schur functions

- Let $\Lambda(X)$ be the ring of symmetric functions in $X = x_1, x_2, \ldots$
- $h_d = h_d(X) = \sum_{i_1 \leq \cdots \leq i_d} x_{i_1} \cdots x_{i_d}$ with $h_0 = 1$ and $h_d = 0$ for $d < 0$.
- For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$,

$$s_\gamma = s_\gamma(X) = \det(h_{\gamma_i + j - i}(X))_{1 \leq i,j \leq n}$$

Then,

$$s_\gamma = \begin{cases} \mathrm{sgn}(\gamma + \rho) s_{\mathrm{sort}(\gamma+\rho)-\rho} & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

- $\mathrm{sort}(\beta) =$ weakly decreasing sequence obtained by sorting β,
- $\mathrm{sgn}(\beta) =$ sign of the shortest permutation taking β to $\mathrm{sort}(\beta)$.

Weyl symmetrization

Define the *Weyl symmetrization operator* $\boldsymbol{\sigma}\colon \mathbb{Q}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \to \Lambda(X)$ by linearly extending

$$\boldsymbol{z}^{\gamma} \mapsto s_{\gamma}(X)$$

where $\boldsymbol{z}^{\gamma} = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$.

Modified Macdonald polynomials

The *modified Macdonald polynomials* $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ are Schur positive symmetric functions in $X = x_1, x_2, \ldots$ over $\mathbb{Q}(q, t)$.

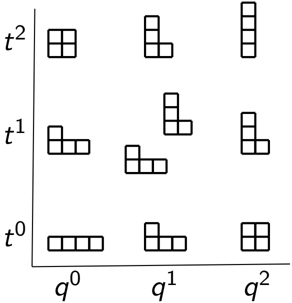
They differ from the *integral form Macdonald polynomials* by $\tilde{H}_\mu(X; q, t) = t^{\mathrm{n}(\mu)} J_\mu[\frac{X}{1-t^{-1}}; q, t^{-1}]$.

Modified Macdonald polynomials

The *modified Macdonald polynomials* $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ are Schur positive symmetric functions in $X = x_1, x_2, \ldots$ over $\mathbb{Q}(q, t)$.

They differ from the *integral form Macdonald polynomials* by $\tilde{H}_\mu(X; q, t) = t^{\mathrm{n}(\mu)} J_\mu[\frac{X}{1-t^{-1}}; q, t^{-1}]$.

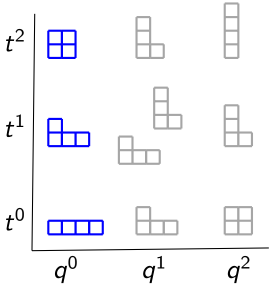
$$\tilde{H}_{22} = s_4 + (q + t + qt)s_{31} + (q^2 + t^2)s_{22} + (qt + q^2t + qt^2)s_{211} + q^2t^2s_{1111}$$



Modified Hall-Littlewood polynomials

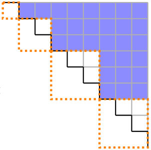
When $q = 0$, the modified Macdonald polynomials reduce to the *modified Hall-Littlewood polynomials* $\tilde{H}_\mu(X; 0, t)$.

$\tilde{H}_{22}(X; 0, t) = s_4 + ts_{31} + t^2 s_{22}$



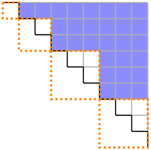
A Catalan function for modified Hall-Littlewoods

B_μ = set of roots above block diagonal matrix with block sizes $\mu_{\ell(\mu)}, \ldots, \mu_1$

$B_{3321} =$ 

A Catalan function for modified Hall-Littlewoods

B_μ = set of roots above block diagonal matrix with block sizes $\mu_{\ell(\mu)}, \ldots, \mu_1$

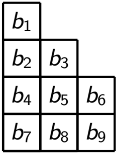
$B_{3321} =$ 

Theorem (Weyman, Shimozono-Weyman)

$$\tilde{H}_\mu(X;0,t) = \omega\boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in B_\mu}(1 - t\boldsymbol{z}^\alpha)}\right),$$

where $\boldsymbol{z}^\alpha = z_i/z_j$.

Catalan functions for modified Hall-Littlewoods

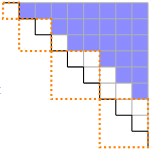


row reading order

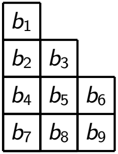
$b_1 \prec b_2 \prec \cdots \prec b_n$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\}.$$

$R_{3321} =$

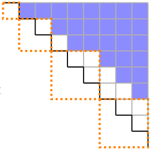


Catalan functions for modified Hall-Littlewoods



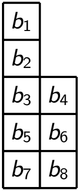
row reading order
$b_1 \prec b_2 \prec \cdots \prec b_n$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\}.$$

$R_{3321} =$ 

$$\tilde{H}_\mu(X;0,t) = \omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in B_\mu}(1-t\boldsymbol{z}^\alpha)}\right),$$
$$= \omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_\mu}(1-t\boldsymbol{z}^\alpha)}\right)$$

A Catalanimal formula for $\tilde{H}_\mu(X; q, t)$



row reading order

$b_1 \prec b_2 \prec \cdots \prec b_n$

$$R_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\},$$
$$\widehat{R}_\mu := \{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \prec b_j\}.$$

A Catalanimal formula for $\tilde{H}_\mu(X; q, t)$

$$\begin{array}{|c|c|}\hline b_1 & \\ \cline{1-1} b_2 & \\ \hline b_3 & b_4 \\ \hline b_5 & b_6 \\ \hline b_7 & b_8 \\ \hline \end{array}$$

row reading order
$b_1 \prec b_2 \prec \cdots \prec b_n$

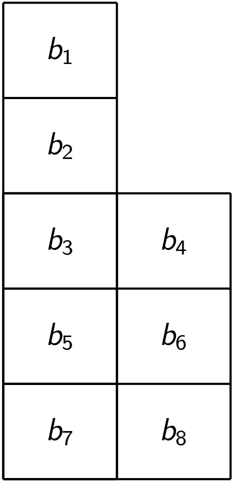
$$R_\mu := \left\{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \preceq b_j\right\},$$
$$\widehat{R}_\mu := \left\{\alpha_{ij} \in R_+ \mid \mathrm{south}(b_i) \prec b_j\right\}.$$

Theorem (Blasiak-Haiman-Morse-Pun-S.)

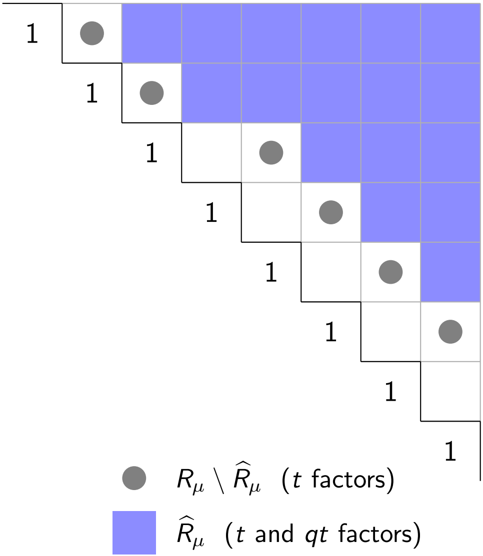
The modified Macdonald polynomial $\tilde{H}_\mu = \tilde{H}_\mu(X; q, t)$ is given by

$$\tilde{H}_\mu = \omega \boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod\limits_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod\limits_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+}\left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu}\left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

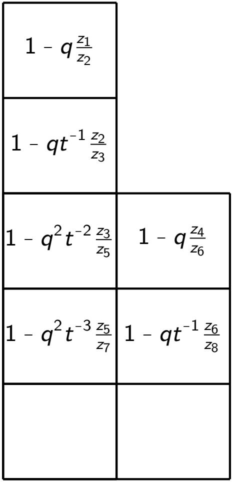
Example



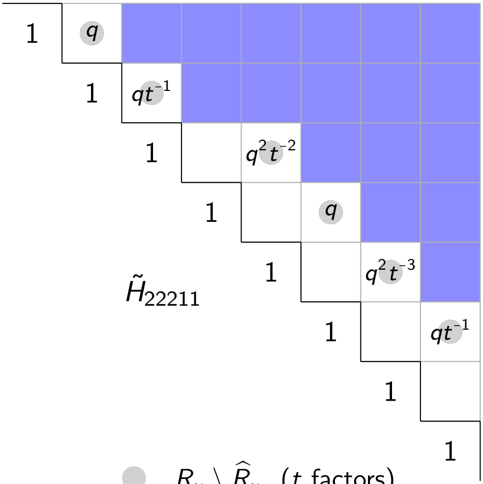
partition $\mu = 22211$



Example



numerator factors $1 - q^{\mathrm{arm}+1}t^{-\mathrm{leg}}z_i/z_j$



$R_\mu \setminus \widehat{R}_\mu$ (t factors)

$\widehat{R}_\mu$ (t and qt factors)

$q = t = 1$ specialization

$$
\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\displaystyle\prod_{\alpha_{ij}\in R_\mu\setminus\widehat{R}_\mu}\left(1-q^{\mathrm{arm}(b_i)+1}t^{-\mathrm{leg}(b_i)}z_i/z_j\right)\prod_{\alpha\in\widehat{R}_\mu}\left(1-qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1-q\boldsymbol{z}^\alpha\right)\prod_{\alpha\in R_\mu}\left(1-t\boldsymbol{z}^\alpha\right)}\right)\\
\overset{q=t=1}{\to}&\omega\boldsymbol{\sigma}\left(z_1\cdots z_n\frac{\prod_{\alpha\in R_\mu\setminus\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in\widehat{R}_\mu}(1-\boldsymbol{z}^\alpha)}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)\prod_{\alpha\in R_\mu}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{\alpha\in R_+}(1-\boldsymbol{z}^\alpha)}\right)\\
=&\omega h_1^n\\
=&e_1^n
\end{aligned}
$$

$q = 0$ specialization

$$
\begin{aligned}
&\omega\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\displaystyle\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right) \\
\overset{q=0}{\to} &\omega\boldsymbol{\sigma}\left(\frac{z_1 \cdots z_n}{\prod_{\alpha \in R_\mu} (1 - t\boldsymbol{z}^\alpha)} \right) \\
= &\tilde{H}_\mu(X; 0, t)
\end{aligned}
$$

Proof of formula for $\tilde{H}_\mu$

Definition

∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

Proof of formula for $\tilde{H}_\mu$

Definition

∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

- Start with the Haglund-Haiman-Loehr formula for $\tilde{H}_\mu$ as a sum of LLT polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$.

Proof of formula for $\tilde{H}_\mu$

Definition

∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

- Start with the Haglund-Haiman-Loehr formula for $\tilde{H}_\mu$ as a sum of LLT polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$.
- Apply $\omega\nabla$ to both sides.

Proof of formula for $\tilde{H}_\mu$

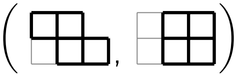
Definition

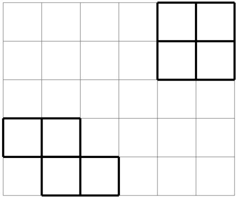
∇ is the linear operator on symmetric functions satisfying $\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu$, where $n(\mu) = \sum_i (i-1)\mu_i$.

- Start with the Haglund-Haiman-Loehr formula for $\tilde{H}_\mu$ as a sum of LLT polynomials $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$.
- Apply $\omega\nabla$ to both sides.
- Use Catalanimal formula for $\omega\nabla\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ and collect terms.

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

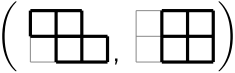
$\boldsymbol{\nu} =$ 



LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.

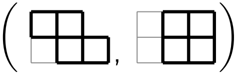
$\boldsymbol{\nu} =$ 

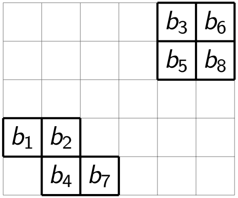
-4	-3	-2	-1	0	1
-3	-2	-1	0	1	2
-2	-1	0	1	2	3
-1	0	1	2	3	4
0	1	2	3	4	5

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.

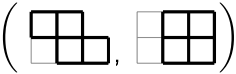
$\boldsymbol{\nu} =$ 

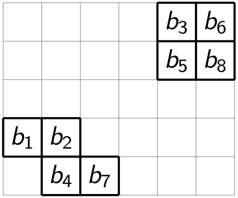


LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$\boldsymbol{\nu} =$ 

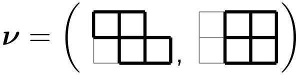


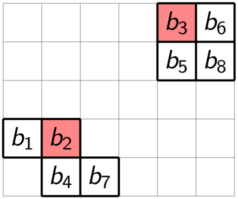
Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.



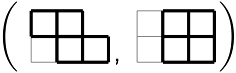


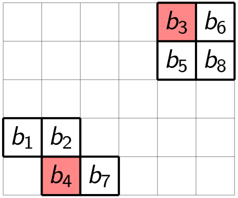
Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$\boldsymbol{\nu} =$ 

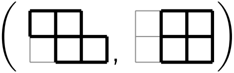


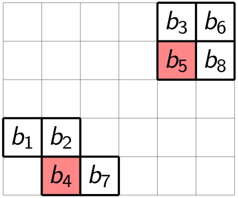
Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$\boldsymbol{\nu} =$ 

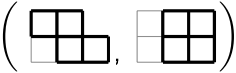


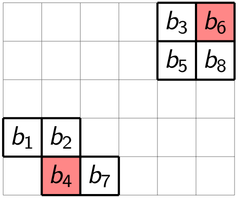
Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
    - $\mathrm{content}(b) = \mathrm{content}(a)$, or
    - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$\boldsymbol{\nu} =$ 

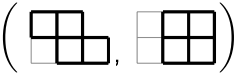


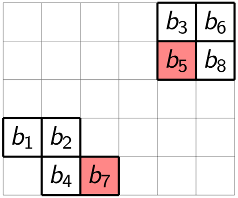
Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$\boldsymbol{\nu} =$ 

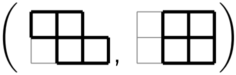


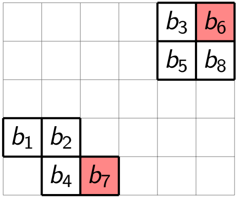
Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), {\color{red}(b_5, b_7)}, (b_6, b_7), (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
    - $\mathrm{content}(b) = \mathrm{content}(a)$, or
    - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.

$\boldsymbol{\nu} =$ 

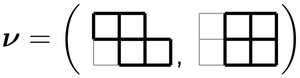


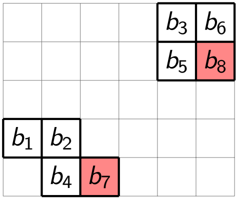
Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), {\color{red}(b_6, b_7)}, (b_7, b_8)$

LLT Polynomials

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew shapes.

- The *content* of a box in row y, column x is $x - y$.
- *Reading order*: label boxes $b_1, \ldots, b_n$ by scanning each diagonal from southwest to northeast, in order of increasing content.
- A pair $(a, b) \in \boldsymbol{\nu}$ is *attacking* if a precedes b in reading order and
  - $\mathrm{content}(b) = \mathrm{content}(a)$, or
  - $\mathrm{content}(b) = \mathrm{content}(a) + 1$ and $a \in \nu_{(i)}, b \in \nu_{(j)}$ with $i > j$.





Attacking pairs: $(b_2, b_3), (b_3, b_4), (b_4, b_5), (b_4, b_6), (b_5, b_7), (b_6, b_7), (b_7, b_8)$

LLT Polynomials

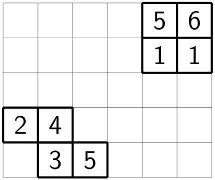
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



LLT Polynomials

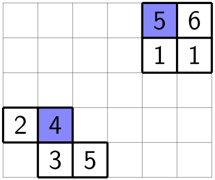
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



non-inversion

LLT Polynomials

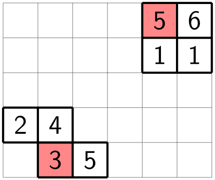
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T \colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



inversion

LLT Polynomials

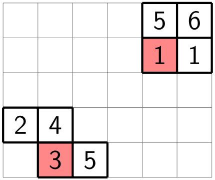
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



inversion

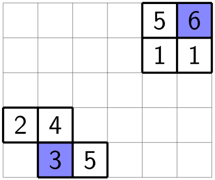
LLT Polynomials

- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$ 

non-inversion

LLT Polynomials

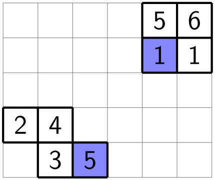
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T : \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



non-inversion

LLT Polynomials

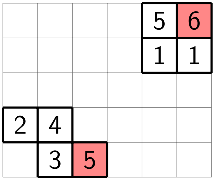
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



inversion

LLT Polynomials

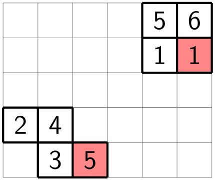
- A *semistandard tableau* on $\boldsymbol{\nu}$ is a map $T\colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard tableau on each $\nu_{(i)}$.
- An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$.

The *LLT polynomial* indexed by a tuple of skew shapes $\boldsymbol{\nu}$ is

$$\mathcal{G}_{\boldsymbol{\nu}}(\boldsymbol{x}; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \boldsymbol{x}^T,$$

where $\mathrm{inv}(T)$ is the number of attacking inversions in T and $\boldsymbol{x}^T = \prod_{a \in \boldsymbol{\nu}} x_{T(a)}$.

$T \quad =$



inversion

$\mathrm{inv}(T) = 4, \quad \boldsymbol{x}^T = x_1^2 x_2 x_3 x_4 x_5^2 x_6$

Catalanimals

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha \in R_{qt}} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_q} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_t} \left(1 - t\boldsymbol{z}^\alpha\right)}\right).$$

Catalanimals

The *Catalanimal* indexed by $R_q, R_t, R_{qt} \subseteq R_+$ and $\lambda \in \mathbb{Z}^n$ is

$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda \prod_{\alpha \in R_{qt}} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_q} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_t} \left(1 - t\boldsymbol{z}^\alpha\right)}\right).$$

With $n = 3$,

$$\begin{aligned} H(R_+, R_+, \{\alpha_{13}\}, (111)) &= \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{111}(1 - qtz_1/z_3)}{\prod_{1 \le i < j \le 3}(1 - qz_i/z_j)(1 - tz_i/z_j)}\right) \\ &= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2 t + qt^2 + t^3)s_3 \\ &= \omega \nabla e_3. \end{aligned}$$

LLT Catalanimals

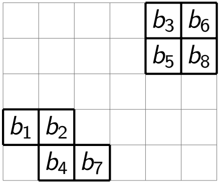
For a tuple of skew shapes $\boldsymbol{\nu}$, the *LLT Catalanimal* $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ is determined by

- $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$,
- $R_+ \setminus R_q =$ pairs of boxes in the same diagonal,
- $R_q \setminus R_t\ =$ the attacking pairs,
- $R_t \setminus R_{qt} =$ pairs going between adjacent diagonals,
- λ: fill each diagonal D of $\boldsymbol{\nu}$ with
  $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.
  Listing this filling in reading order gives λ.

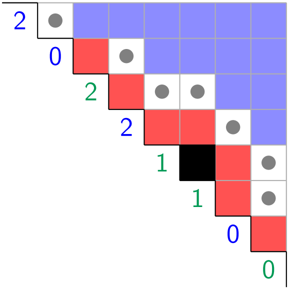
LLT Catalanimals

- ■ $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
- ■ $R_q \setminus R_t$ = the attacking pairs,
- ● $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- ■ R_{qt} = all other pairs,

λ: fill each diagonal D of $\boldsymbol{\nu}$ with
$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.



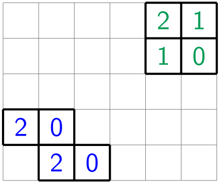
$\boldsymbol{\nu}$



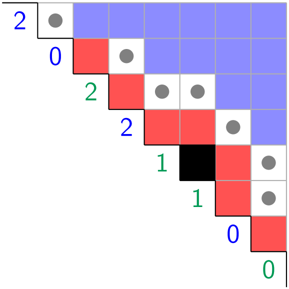
LLT Catalanimals

- $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
- $R_q \setminus R_t$ = the attacking pairs,
- $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- R_{qt} = all other pairs,

λ: fill each diagonal D of $\boldsymbol{\nu}$ with
$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.



λ, as a filling of $\boldsymbol{\nu}$



LLT Catalanimals

Theorem (Blasiak-Haiman-Morse-Pun-S.)

Let $\boldsymbol{\nu}$ be a tuple of skew shapes and let $H_{\boldsymbol{\nu}} = H(R_q, R_t, R_{qt}, \lambda)$ be the associated LLT Catalanimal. Then

$$\begin{aligned} \nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) &= c_{\boldsymbol{\nu}}\, \omega\, \mathrm{pol}_X(H_{\boldsymbol{\nu}}) \\ &= c_{\boldsymbol{\nu}}\, \omega\, \mathrm{pol}_X\, \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{\lambda} \prod_{\alpha \in R_{qt}} \left(1 - qt\, \boldsymbol{z}^{\alpha}\right)}{\prod_{\alpha \in R_q} \left(1 - q\, \boldsymbol{z}^{\alpha}\right) \prod_{\alpha \in R_t} \left(1 - t\, \boldsymbol{z}^{\alpha}\right)} \right) \end{aligned}$$

for some $c_{\boldsymbol{\nu}} \in \pm q^{\mathbb{Z}} t^{\mathbb{Z}}$.

Haglund-Haiman-Loehr formula

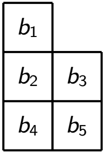
Theorem (Haglund-Haiman-Loehr, 2005)

$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u \in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1} \right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X;q)\,,$$

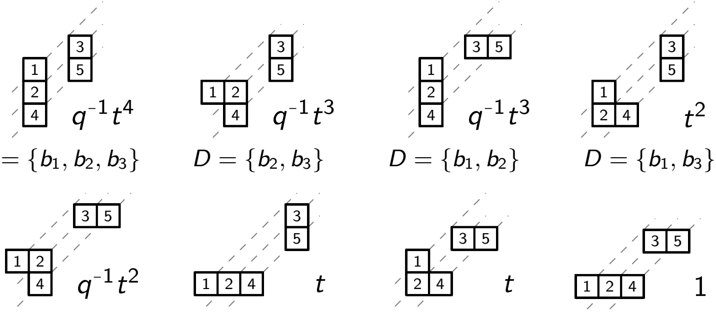
where

- *the sum runs over all subsets* $D \subseteq \{(i,j) \in \mu \mid j > 1\}$, *and*
- $\boldsymbol{\nu}(\mu,D) = (\nu^{(1)}, \ldots, \nu^{(k)})$ *where* $k = \mu_1$ *is the number of columns of* μ, *and* $\nu^{(i)}$ *is a ribbon of size* μ_i^*, *i.e., box contents* $\{-1,-2,\ldots,-\mu_i^*\}$, *and descent set* $Des(\nu^{(i)}) = \{-j \mid (i,j) \in D\}$.

Haglund-Haiman-Loehr formula example



μ



$q^{-1}t^4$ — $D = \{b_1, b_2, b_3\}$

$q^{-1}t^3$ — $D = \{b_2, b_3\}$

$q^{-1}t^3$ — $D = \{b_1, b_2\}$

t^2 — $D = \{b_1, b_3\}$

$q^{-1}t^2$ — $D = \{b_2\}$

t — $D = \{b_3\}$

t — $D = \{b_1\}$

1 — $D = \varnothing$

Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.

Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the LHS will have the same root ideal data (R_q, R_t, R_{qt}).

Putting it all together

- Take HHL formula $\tilde{H}_\mu = \sum_D a_{\mu,D} \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}$ and apply $\omega\nabla$.
- By construction, all the LLT Catalanimals $H_{\boldsymbol{\nu}(\mu,D)}$ appearing on the LHS will have the same root ideal data (R_q, R_t, R_{qt}).
- Collect terms to get $\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} (1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j)$ factor.

$$\tilde{H}_\mu = \omega \boldsymbol{\sigma} \left(z_1 \cdots z_n \frac{\displaystyle\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha \in R_+} \left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu} \left(1 - t\boldsymbol{z}^\alpha\right)} \right).$$

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

A positivity conjecture

What can this formula tell us that other formulas for Macdonald polynomials do not?

$$\tilde{H}_\mu^{(s)} := \omega\boldsymbol{\sigma}\left((z_1 \cdots z_n)^s \frac{\prod_{\alpha_{ij}\in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(b_i)+1} t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{\alpha \in \widehat{R}_\mu} \left(1 - qt\boldsymbol{z}^\alpha\right)}{\prod_{\alpha\in R_+}\left(1 - q\boldsymbol{z}^\alpha\right) \prod_{\alpha \in R_\mu}\left(1 - t\boldsymbol{z}^\alpha\right)} \right)$$

Conjecture (Blasiak-Haiman-Morse-Pun-S.)

For any partition μ and positive integer s, the symmetric function $\tilde{H}_\mu^{(s)}$ is Schur positive. That is, the coefficients in

$$\tilde{H}_\mu^{(s)} = \sum_\nu K_{\nu,\mu}^{(s)}(q,t)\, s_\nu(X)$$

satisfy $K_{\nu,\mu}^{(s)}(q,t) \in \mathbb{N}[q,t]$.

Thank you!

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021. *LLT Polynomials in the Schiffmann Algebra*, arXiv e-prints, arXiv:2112.07063.

———. 2023. *A Raising Operator Formula for Macdonald Polynomials*, arXiv e-prints, arXiv:2307.06517.

Haglund, J., M. Haiman, and N. Loehr. 2005. *A Combinatorial Formula for Macdonald Polynomials* **18**, no. 3, 735–761 (electronic).

Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919

Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sém. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754

Shimozono, Mark and Jerzy Weyman. 2000. *Graded Characters of Modules Supported in the Closure of a Nilpotent Conjugacy Class*, European Journal of Combinatorics **21**, no. 2, 257–288, DOI 10.1006/eujc.1999.0344.

Weyman, J. 1989. *The Equations of Conjugacy Classes of Nilpotent Matrices*, Inventiones mathematicae **98**, no. 2, 229–245, DOI 10.1007/BF01388851.