Schubert calculus and K-theoretic Catalan functions

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UVA Graduate Seminar

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Overview

1. An overview of Schubert calculus
2. Catalan functions: shedding new light on old problems
3. K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^{\nu} = \#$ of points in intersection of subvarieties in a variety X.

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c^{\nu}_{\lambda\mu} f_\nu$

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Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

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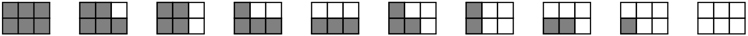

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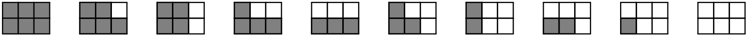

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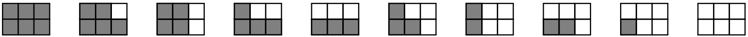

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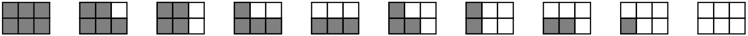

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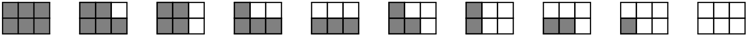

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- $\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu$

$c^\nu_{\lambda\mu}$ =number of points in intersection of Schubert varieties.
What are the structure constants $c^\nu_{\lambda\mu}$?

Classical Example (cont.)

$\Lambda_m = \mathbb{C}[x_1, \ldots, x_m]^{S_m}$ is the ring of symmetric polynomials in m variables and has bases indexed by partitions.

$$\underbrace{12x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{symmetric}} \qquad \underbrace{5x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{not symmetric}}$$

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There exists a basis of Λ_m denoted $\{s_\lambda\}_\lambda$ and a surjection of rings such that

$$\begin{aligned} \Lambda_m &\to H^*(\mathrm{Gr}(m,n)) \\ s_\lambda &\mapsto \begin{cases} \sigma_\lambda & \lambda \subseteq (n^m) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Classical Example (cont.)

Cohomology structure: $\sigma_\lambda \leftrightarrow s_\lambda$ when $\lambda \subseteq (n^m)$.

$$s_\lambda s_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^{\nu} s_\nu + \sum_{\nu \not\subseteq (n^m)} c_{\lambda\mu}^{\nu} s_\nu \leftrightarrow \sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^{\nu} \sigma_\nu$$

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

$$
\begin{array}{|c|cccc}
\cline{1-1} 5 & & & & \\
\cline{1-2} 3 & \multicolumn{1}{c|}{4} & & & \\
\cline{1-2} 2 & \multicolumn{1}{c|}{3} & & & \\
\hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\
\hline
\end{array}
\qquad\qquad
\underbrace{\begin{array}{|c|cccc}
\cline{1-1} 8 & & & & \\
\cline{1-2} 7 & \multicolumn{1}{c|}{9} & & & \\
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\end{array}}_{\text{standard} \,=\, \text{no repeated letters}}
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Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$
\begin{array}{|c|c|}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array},\;
\begin{array}{|c|c|}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array},\;
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$$

$$
s_{\begin{smallmatrix}\square & \\ \square & \square\end{smallmatrix}}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3
$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{(2,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^{\nu} s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^{\nu}$.

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Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$

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- Structure constants $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$ are combinatorially unknown.

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(Co)homology of Grassmannian	Schur functions
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And many more!

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Upshot

Computations for Schubert polynomials can be moved into symmetric functions.

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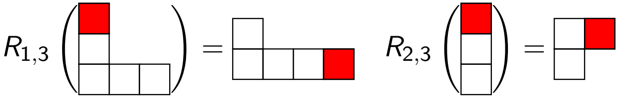
- Many definitions. A new one makes proofs easier!

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams



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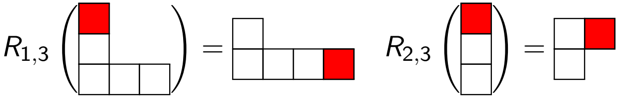

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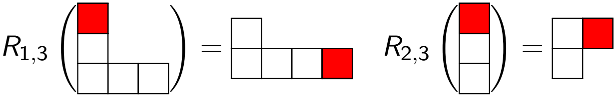

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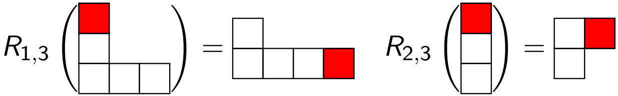

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$$\begin{aligned} s_{22} &= (1-R_{12})h_{22} = h_{22} - h_{31} \\ s_{211} &= (1-R_{12})(1-R_{23})(1-R_{13})h_{211} \\ &= h_{211} - h_{301} - h_{220} \color{red}{- h_{310} + h_{310}} + h_{32\,\text{-}1} + h_{400} - h_{41\,\text{-}1} \end{aligned}$$

some terms cancel

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Advantage: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$.

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$$s_\alpha = \prod_{i<j} (1 - R_{ij}) h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

For $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S| = r} s_{\lambda - \epsilon_S}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$\Psi =$

	(12)	(13)	(14)	(15)
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			(34)	(35)
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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi;\gamma)(x) = \prod_{(i,j)\in\Delta_\ell^+\setminus\Psi} (1 - R_{ij}) h_\gamma(x)$$

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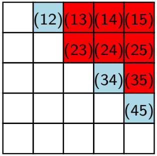

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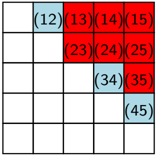
Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

$\Psi =$ 

Roots above Dyck path
Non-roots below

Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

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Catalan functions ($t = 1$)

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
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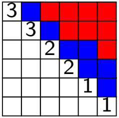

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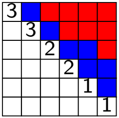
$\Delta^4(3,3,2,2,1,1) =$ 

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Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

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where $\langle s_{1^\ell}^{\perp} f, g\rangle = \langle f, s_{1^\ell} g\rangle$.

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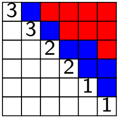
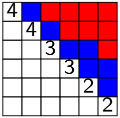
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

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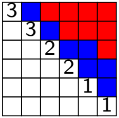
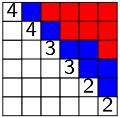
$\Delta^4(3,3,2,2,1,1) =$ 

$\Delta^5(4,4,3,3,2,2) =$ 

Branching is a special case of Pieri:

$$s^{(k)}_{\lambda} = s^{\perp}_{1^\ell} s^{(k+1)}_{\lambda+1^\ell} = \sum_{\mu} a_{\lambda+1^\ell,\mu} s^{(k+1)}_{\mu}$$

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda +$ lower degree terms.

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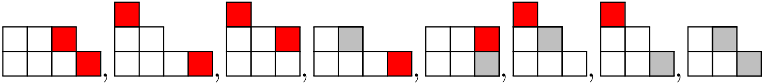

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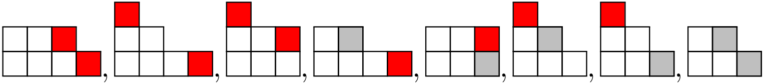

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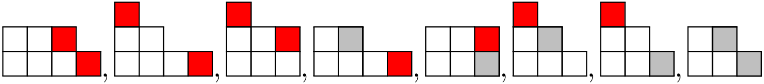

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- Dual to Grothendieck polynomials: Schubert representatives for $K^*(Gr(m, n))$

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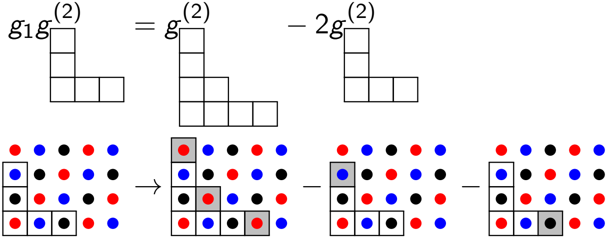

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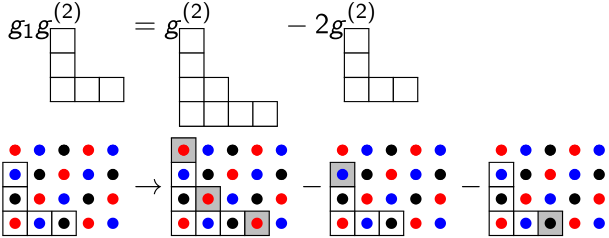

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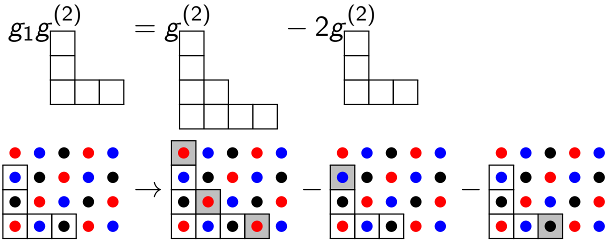

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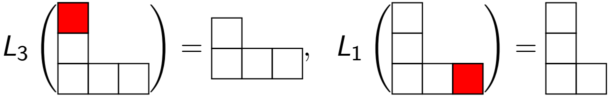
Problem

No direct formula for $g_\lambda^{(k)}$

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda-\epsilon_j}$



Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$K(\Psi; \mathcal{L}; 54332)$$
$$= (1 - L_4)^2 (1 - L_5)^2$$
$$\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$$

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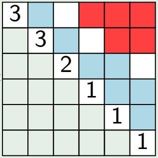
$g_{332111}^{(4)} =$ 

$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$

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satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

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$$\Phi\colon QK^*(Fl_{k+1}) \to K_*(Gr_{SL_{k+1}})_{loc}$$

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4. Describe the image of $\mathfrak{G}_w^Q$ under Peterson isomorphism for all $w \in S_{k+1}$.

References

Thank you!

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