

# Dens, nests, and Catalananimals: a walk through the zoo of shuffle theorems

George H. Seelinger

*ghseeli@umich.edu*

joint with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun

Michigan Combinatorics Seminar

17 March 2023

- $f \in \mathbb{Q}[x_1, \dots, x_n]$  multivariate polynomial

- $f \in \mathbb{Q}[x_1, \dots, x_n]$  multivariate polynomial

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

- $f \in \mathbb{Q}[x_1, \dots, x_n]$  multivariate polynomial

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} (5x_1^2 + 5x_2^2 + 8x_3^2) = 8x_1^2 + 5x_2^2 + 5x_3^2$$

- $\sigma \in S_n$  acts as  $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$ ?

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$ ?
- Symmetric polynomials ( $n = 3$ )

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

# Symmetric Polynomials

- Polynomials  $f \in \mathbb{Q}[x_1, \dots, x_n]$  satisfying  $\sigma.f = f$ ?
- Symmetric polynomials ( $n = 3$ )

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

$$e_3 = x_1x_2x_3 \quad h_3 = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + \dots$$

- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$  forms a vector space,  $\Lambda_{\mathbb{Q}}$ .

# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$



# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

Basis of  $\Lambda_{\mathbb{Q}}$ ?

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

# Partitions

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition* of  $n$  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\hline\hline\hline\hline\hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$2 + 1 + 1 + 1 \rightarrow \begin{array}{|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$1 + 1 + 1 + 1 + 1 \rightarrow \begin{array}{|c|}\hline\hline\hline\hline\hline\end{array}$$

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- 1 strictly increasing down columns

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Collection is called  $\text{SSYT}(\lambda)$ .



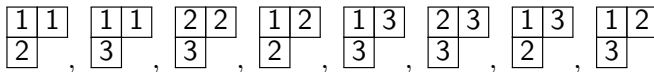
## Definition

Filling of partition diagram of  $\lambda$  with numbers such that

- ① strictly increasing down columns
- ② weakly increasing along rows

Collection is called  $\text{SSYT}(\lambda)$ .

For  $\lambda = (2, 1)$ ,

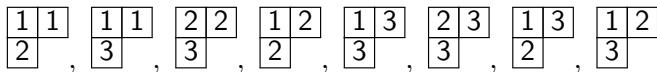


# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .



# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

Weight: 

1	1
2	

, 

1	1
3	

, 

2	2
3	

, 

1	2
2	

, 

1	3
3	

, 

2	3
3	

, 

1	3
2	

, 

1	2
3	

  
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

Weight:  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$   
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

Weight:  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$   
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

## Definition

For  $\lambda$  a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

Weight:  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$   
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

## Definition

For  $\lambda$  a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

- $s_\lambda$  is a symmetric function

# Schur functions

Associate a polynomial to  $\text{SSYT}(\lambda)$ .

Weight:  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$   
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

## Definition

For  $\lambda$  a partition

$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

- $s_\lambda$  is a symmetric function
- Schur functions form a basis for  $\Lambda_{\mathbb{Q}}$



# Why Schur functions?

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

# Why Schur functions?

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

# Why Schur functions?

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$M$  is the vector space given by

# Why Schur functions?

## Harmonic polynomials

$M$  = polynomials killed by all symmetric differential operators.

Explicitly, for

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

$M$  is the vector space given by

$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

# Harmonic polynomials

①  $S_3$  action on  $M$  fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

# Harmonic polynomials

- ①  $S_3$  action on  $M$  fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break  $M$  up into smallest  $S_n$  fixed subspaces

# Harmonic polynomials

- 1  $S_3$  action on  $M$  fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

# Harmonic polynomials

- 1  $S_3$  action on  $M$  fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- 2 Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- 3 How many times does an  $S_n$  fixed subspace occur?



# Harmonic polynomials

- ①  $S_3$  action on  $M$  fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ③ How many times does an  $S_n$  fixed subspace occur? Frobenius:

# Harmonic polynomials

- ①  $S_3$  action on  $M$  fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ③ How many times does an  $S_n$  fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

# Harmonic polynomials

- ①  $S_3$  action on  $M$  fixes vector subspaces!

$$\text{sp}\{\Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, x_3 - x_1, x_2 - x_3, 1\}$$

- ② Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_2 - x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

- ③ How many times does an  $S_n$  fixed subspace occur? Frobenius:

$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Schur basis expansion counts multiplicity of irreducible  $S_n$  fixed subspaces!

# Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.

# Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible  $S_n$ -fixed vector subspaces (representations).

# Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible  $S_n$ -fixed vector subspaces (representations).

## Upshot

Via Frobenius characteristic map, questions about  $S_n$ -representations get translated to questions about Schur expansion coefficients in symmetric functions.

# Recap so far

- Combinatorics: Schur functions are weight generating functions of semistandard tableaux.
- Algebra: Schur functions count multiplicity of irreducible  $S_n$ -fixed vector subspaces (representations).

## Upshot

Via Frobenius characteristic map, questions about  $S_n$ -representations get translated to questions about Schur expansion coefficients in symmetric functions.

Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

# Getting more information



# Getting more information

Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}_{\text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}_{\text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

# Getting more information

Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal  $S_n$ -fixed subspace of degree  $d \mapsto q^d s_\lambda$  (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

# Getting more information

Break  $M$  up into smallest  $S_n$  fixed subspaces

$$\underbrace{\text{sp}\{\Delta\}}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} \oplus \underbrace{\text{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2, 2x_2(x_3-x_1)-x_3^2+x_1^2\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=2} \oplus \underbrace{\text{sp}\{x_3-x_1, x_2-x_3\}}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \text{deg}=1} \oplus \underbrace{\text{sp}\{1\}}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Solution: minimal  $S_n$ -fixed subspace of degree  $d \mapsto q^d s_\lambda$  (graded Frobenius)

$$?? = q^3 s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + q^2 s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + q s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

Answer: "Hall-Littlewood polynomial"  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials.

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .

# A Problem

- In 1988, Macdonald introduces a family of symmetric polynomials with coefficients in  $\mathbb{Q}(q, t)$  generalizing Hall-Littlewood polynomials.
- Garsia modifies these polynomials so

$$\tilde{H}_\lambda(X; q, t) = \sum_{\mu} \tilde{K}(q, t) s_{\mu} \text{ conjecturally satisfies } \tilde{K}(q, t) \in \mathbb{N}[q, t]$$

- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .
- Does there exist a family of  $S_n$ -representations whose (bigraded) Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .



# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of}$   
 $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of}$   
 $\Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \square & & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\deg=(0,0)}$$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\text{deg}=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\text{deg}=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\text{deg}=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\text{deg}=(0,0)}$$

Irreducible  $S_n$ -representation with bidegree  $(a, b) \mapsto q^a t^b s_\lambda$

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .
- Garsia-Haiman (1993):  $M_\mu = \text{span of partial derivatives of } \Delta_\mu = \det_{(i,j) \in \mu, k \in [n]} (x_k^{i-1} y_k^{j-1})$

$$\Delta_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = \det \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix} = x_3 y_2 - y_3 x_2 - y_1 x_3 + y_1 x_2 + y_3 x_1 - y_2 x_1$$

$$M_{2,1} = \underbrace{\text{sp}\{\Delta_{2,1}\}}_{\deg=(1,1)} \oplus \underbrace{\text{sp}\{y_3 - y_1, y_1 - y_2\}}_{\deg=(0,1)} \oplus \underbrace{\text{sp}\{x_3 - x_1, x_1 - x_2\}}_{\deg=(1,0)} \oplus \underbrace{\text{sp}\{1\}}_{\deg=(0,0)}$$

Irreducible  $S_n$ -representation with bidegree  $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} = qts_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + ts_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + qs_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

- Proved via geometric connection to the Hilbert Scheme  $\text{Hilb}^n(\mathbb{C}^2)$ .

## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

- Proved via geometric connection to the Hilbert Scheme  $\text{Hilb}^n(\mathbb{C}^2)$ .

## Corollary

$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}_{\lambda\mu}(q, t) s_\mu$  satisfies  $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$ .



## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

- Proved via geometric connection to the Hilbert Scheme  $\text{Hilb}^n(\mathbb{C}^2)$ .

## Corollary

$\tilde{H}_\lambda(X; q, t) = \sum_\mu \tilde{K}_{\lambda\mu}(q, t) s_\mu$  satisfies  $\tilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$ .

- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ . (Still open!)

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

## Question

What symmetric function is the bigraded Frobenius characteristic of  $DH_n$ ?

Frobenius characteristic of  $DH_3$

Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Operator  $\nabla$

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

Frobenius characteristic of  $DH_3$

$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Compare to

$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

Operator  $\nabla$

$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

Theorem (Haiman, 2002)

*The bigraded Frobenius characteristic of  $DH_n$  is given by  $\nabla e_n$ .*



# A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

# A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.

# A Combinatorial Connection: Shuffle Theorem

## Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all  $k$ -by- $k$  Dyck paths.

# A Combinatorial Connection: Shuffle Theorem

## Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all  $k$ -by- $k$  Dyck paths.
- $\text{area}(\lambda)$  and  $\text{dinv}(\lambda)$  statistics of Dyck paths.

# A Combinatorial Connection: Shuffle Theorem

## Theorem (Carlsson-Mellit, 2018)

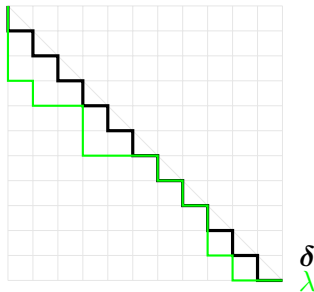
$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).
- Combinatorial RHS: Combinatorics of Dyck paths.
- Summation over all  $k$ -by- $k$  Dyck paths.
- $\text{area}(\lambda)$  and  $\text{dinv}(\lambda)$  statistics of Dyck paths.
- $\mathcal{G}_{\nu(\lambda)}(X; q)$  a symmetric LLT polynomial indexed by a tuple of offset rows.

# Dyck paths

## Dyck paths

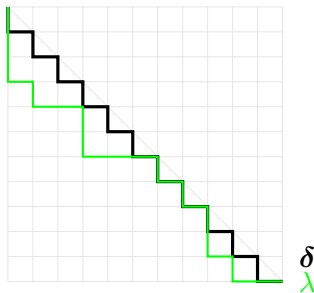
A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .



# Dyck paths

## Dyck paths

A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .

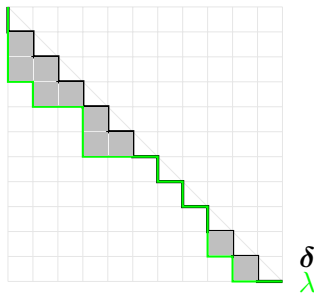


- $\text{area}(\lambda) =$  number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.

# Dyck paths

## Dyck paths

A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .



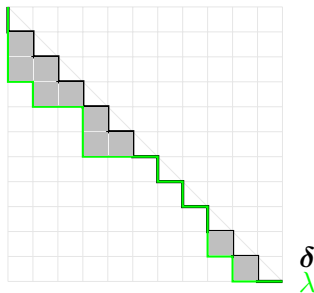
- $\text{area}(\lambda)$  = number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.
- E.g., above  $\text{area}(\lambda) = 10$ .



# Dyck paths

## Dyck paths

A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .

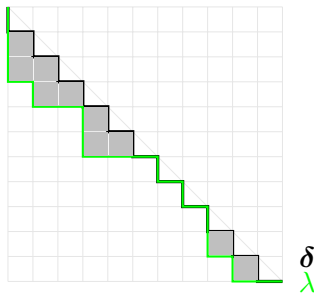


- $\text{area}(\lambda)$  = number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.
- E.g., above  $\text{area}(\lambda) = 10$ .
- Catalan-number many Dyck paths for fixed  $k$ .

# Dyck paths

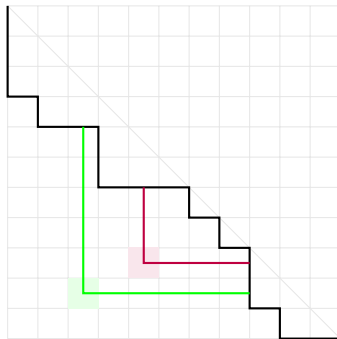
## Dyck paths

A Dyck path  $\lambda$  is a south-east lattice path lying below the line segment from  $(0, k)$  to  $(k, 0)$ .

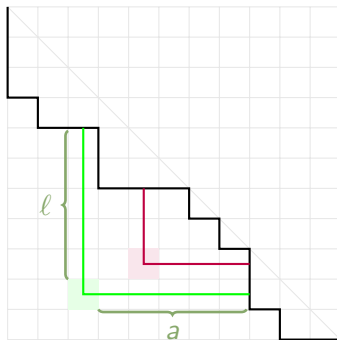


- $\text{area}(\lambda)$  = number of squares above  $\lambda$  but below the path  $\delta$  of alternating S-E steps.
- E.g., above  $\text{area}(\lambda) = 10$ .
- Catalan-number many Dyck paths for fixed  $k$ .  $(1, 2, 5, 14, 42, \dots)$

$\text{dinv}(\lambda) = \#$  of balanced hooks in diagram below  $\lambda$ .



$\text{dinv}(\lambda) = \#$  of balanced hooks in diagram below  $\lambda$ .



Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

Defined in general for a tuple of skew shapes  $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

Defined in general for a tuple of skew shapes  $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function

Defined in general for a tuple of skew shapes  $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$

Defined in general for a tuple of skew shapes  $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_\nu$  were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of  $U_q(\hat{\mathfrak{sl}}_r)$



Defined in general for a tuple of skew shapes  $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_\nu$  were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of  $U_q(\hat{\mathfrak{sl}}_r)$
- When  $\nu^{(i)}$  are partitions, the Schur-expansion coefficients are essentially parabolic Kazhdan-Luzstig polynomials.

Defined in general for a tuple of skew shapes  $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$

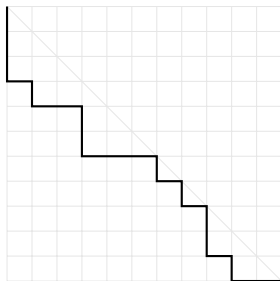
- $\mathcal{G}_\nu(X; q)$  is a symmetric function
- $\mathcal{G}_\nu(X; 1) = s_{\nu^{(1)}} \cdots s_{\nu^{(r)}}$
- $\mathcal{G}_\nu$  were originally defined by Lascoux, Leclerc, and Thibon to explore connections to Fock space representations of  $U_q(\hat{\mathfrak{sl}}_r)$
- When  $\nu^{(i)}$  are partitions, the Schur-expansion coefficients are essentially parabolic Kazhdan-Luzstig polynomials.
- $\mathcal{G}_\nu$  is Schur-positive for any tuple of skew shapes  $\nu$  [Grojnowski-Haiman, 2007].

# LLT Polynomials

$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$ .

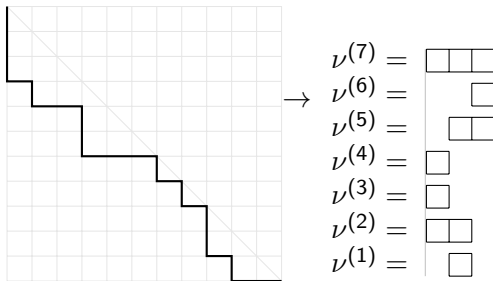
# LLT Polynomials

$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$ .



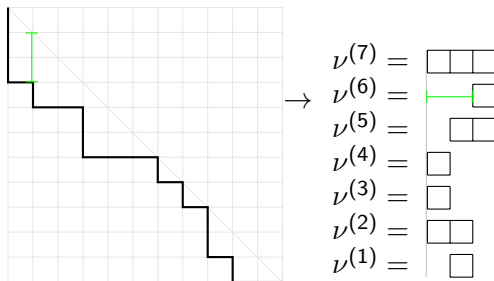
# LLT Polynomials

$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$ .



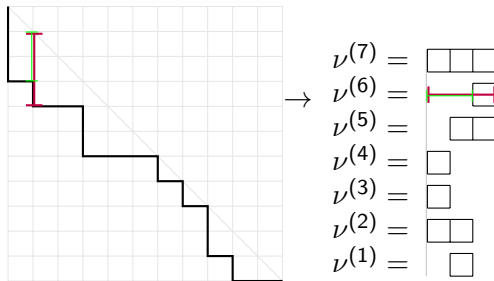
# LLT Polynomials

$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$ .



# LLT Polynomials

$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$ .



$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for  $T$  a weakly increasing filling of rows and  $i(T)$  the number of attacking inversions:



# LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for  $T$  a weakly increasing filling of rows and  $i(T)$  the number of attacking inversions:

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 3 & 5 \\ 2 & 4 & 4 & 7 & 8 & 9 & 9 \\ 1 & 1 & 6 & 7 & 7 & 7 \end{array}$$

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for  $T$  a weakly increasing filling of rows and  $i(T)$  the number of attacking inversions:

1 2 3 3 5

2 4 4 7 8 9 9

$$T = \begin{array}{cccccc} 1 & 1 & 6 & 7 & 7 & 7 \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$

# LLT Polynomials

$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

for  $T$  a weakly increasing filling of rows and  $i(T)$  the number of attacking inversions:

1 2 3 3 5

2 4 4 7 8 9 9

$$T = \begin{array}{ccccccc} 1 & 1 & 6 & 7 & 7 & 7 \\ \end{array} \rightarrow q^{i(T)} x^T = q^{18} x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_7^4 x_8 x_9^2$$



$$\mathcal{G}_{\begin{array}{cc} \square & \square \\ \square & \end{array}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{cc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

## Example $\nabla e_3$






$$\lambda \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

# Example $\nabla e_3$






$$\lambda \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \quad q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$








# Example $\nabla e_3$

$\lambda$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	$q^3$	
	$q^2 t$	
	$qt$	
	$qt^2$	
	$t^3$	

# Example $\nabla e_3$

$\lambda$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	$q^3$	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	$q^2t$	$qts_{2,1} + q^2ts_{1,1,1}$
	$qt$	$ts_{2,1} + qts_{1,1,1}$
	$qt^2$	$t^2s_{2,1} + qt^2s_{1,1,1}$
	$t^3$	$t^3s_{1,1,1}$

# Example $\nabla e_3$

$\lambda$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	$q^3$	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	$q^2t$	$qts_{2,1} + q^2ts_{1,1,1}$
	$qt$	$ts_{2,1} + qts_{1,1,1}$
	$qt^2$	$t^2s_{2,1} + qt^2s_{1,1,1}$
	$t^3$	$t^3s_{1,1,1}$

- Entire quantity is  $q, t$ -symmetric



# Example $\nabla e_3$

$\lambda$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	$q^3$	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	$q^2t$	$qts_{2,1} + q^2ts_{1,1,1}$
	$qt$	$ts_{2,1} + qts_{1,1,1}$
	$qt^2$	$t^2s_{2,1} + qt^2s_{1,1,1}$
	$t^3$	$t^3s_{1,1,1}$

- Entire quantity is  $q, t$ -symmetric
- Coefficient of  $s_{1,1,1}$  in sum is a “ $(q, t)$ -Catalan number”  
 $(q^3 + q^2t + qt + qt^2 + t^3)$ .

# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For  $m, n$  coprime, the operator  $e_k[-MX^{m,n}]$  acting on  $\Lambda$  satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$

# Generalizing Shuffle Theorem

When a problem is too difficult, try generalizing!

Algebraic Expression

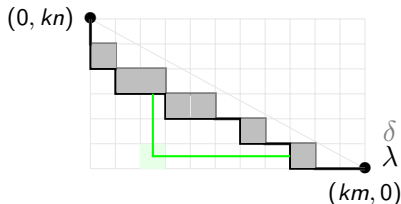
Combinatorial Expression

$$\nabla e_k(X) = \sum q, t\text{-weighted Dyck paths}$$

Rational Shuffle Conjecture (F. Bergeron, Garsia, Sergel Leven, Xin, 2016) (Proved by Mellit, 2016)

For  $m, n$  coprime, the operator  $e_k[-MX^{m,n}]$  acting on  $\Lambda$  satisfies

$$e_k[-MX^{m,n}] \cdot 1 = \sum q, t\text{-weighted } (km, kn)\text{-Dyck paths}$$



- The operators  $e_k[-MX^{m,n}]$  arise from an action of *Schiffmann algebra*  $\mathcal{E}$  on  $\Lambda$ .

# Welcome to the Zoo

- The operators  $e_k[-MX^{m,n}]$  arise from an action of *Schiffmann algebra*  $\mathcal{E}$  on  $\Lambda$ .
- $\mathcal{E}$  contains subalgebra  $\Lambda(X^{m,n}) \cong \Lambda$  for each coprime pair  $(m, n) \in \mathbb{Z}^2$ .

# Welcome to the Zoo

- The operators  $e_k[-MX^{m,n}]$  arise from an action of *Schiffmann algebra*  $\mathcal{E}$  on  $\Lambda$ .
- $\mathcal{E}$  contains subalgebra  $\Lambda(X^{m,n}) \cong \Lambda$  for each coprime pair  $(m, n) \in \mathbb{Z}^2$ .
- In general,  $\mathcal{E}$ -action can be a pain to compute in a nice way, but sometimes it is nice!



# Welcome to the Zoo: Catalananimals

Fix  $l \in \mathbb{Z}_{>0}$ . Let  $R_+ = \{(i, j) \mid 1 \leq i < j \leq l\}$  .

# Welcome to the Zoo: Catalananimals

Fix  $l \in \mathbb{Z}_{>0}$ . Let  $R_+ = \{(i, j) \mid 1 \leq i < j \leq l\}$ .

## Definition

For subsets  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\gamma \in \mathbb{Z}^l$ , a *Catalananimal*  $H = H(R_q, R_t, R_{qt}, \gamma)(z_1, \dots, z_l; q, t)$  is a symmetric rational function

# Welcome to the Zoo: Catalanimals

Fix  $l \in \mathbb{Z}_{>0}$ . Let  $R_+ = \{(i, j) \mid 1 \leq i < j \leq l\}$ .

## Definition

For subsets  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\gamma \in \mathbb{Z}^l$ , a *Catalanimal*  $H = H(R_q, R_t, R_{qt}, \gamma)(z_1, \dots, z_l; q, t)$  is a symmetric rational function

$$\sum_{w \in S_l} w \left( \frac{z_1^{\gamma_1} \cdots z_l^{\gamma_l} \prod_{(i,j) \in R_{qt}} (1 - qtz_i/z_j)}{\prod_{(i,j) \in R_+} (1 - z_j/z_i) \prod_{(i,j) \in R_q} (1 - qz_i/z_j) \prod_{(i,j) \in R_t} (1 - tz_i/z_j)} \right)$$

# Welcome to the Zoo: Catalanimals

Fix  $l \in \mathbb{Z}_{>0}$ . Let  $R_+ = \{(i, j) \mid 1 \leq i < j \leq l\}$ .

## Definition

For subsets  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\gamma \in \mathbb{Z}^l$ , a *Catalanimal*  $H = H(R_q, R_t, R_{qt}, \gamma)(z_1, \dots, z_l; q, t)$  is a symmetric rational function

$$\sum_{w \in S_l} w \left( \frac{z_1^{\gamma_1} \cdots z_l^{\gamma_l} \prod_{(i,j) \in R_{qt}} (1 - qt z_i / z_j)}{\prod_{(i,j) \in R_+} (1 - z_j / z_i) \prod_{(i,j) \in R_q} (1 - q z_i / z_j) \prod_{(i,j) \in R_t} (1 - t z_i / z_j)} \right)$$

- Can also be thought of as an infinite series of virtual  $GL_l$ -characters.

# Welcome to the Zoo: Catalanimals

Fix  $l \in \mathbb{Z}_{>0}$ . Let  $R_+ = \{(i, j) \mid 1 \leq i < j \leq l\}$ .

## Definition

For subsets  $R_q, R_t, R_{qt} \subseteq R_+$  and  $\gamma \in \mathbb{Z}^l$ , a *Catalanimal*  $H = H(R_q, R_t, R_{qt}, \gamma)(z_1, \dots, z_l; q, t)$  is a symmetric rational function

$$\sum_{w \in S_l} w \left( \frac{z_1^{\gamma_1} \cdots z_l^{\gamma_l} \prod_{(i,j) \in R_{qt}} (1 - qtz_i/z_j)}{\prod_{(i,j) \in R_+} (1 - z_j/z_i) \prod_{(i,j) \in R_q} (1 - qz_i/z_j) \prod_{(i,j) \in R_t} (1 - tz_i/z_j)} \right)$$

- Can also be thought of as an infinite series of virtual  $GL_l$ -characters.
- We can take “polynomial part” (restrict to only polynomial  $GL_l$ -characters) to get a symmetric function.

# Welcome to the Zoo: Catalanimals

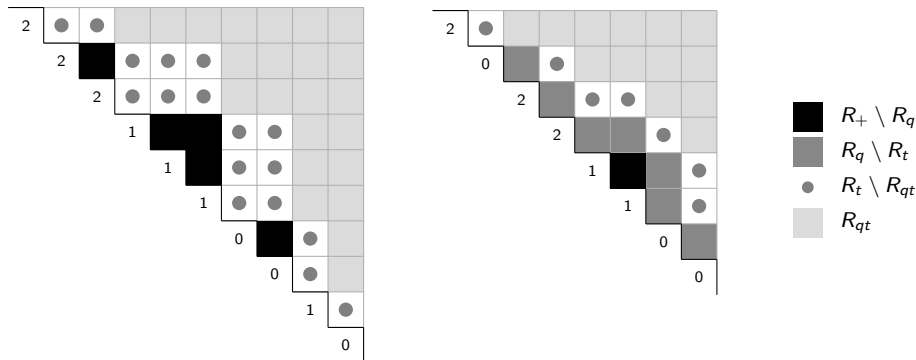
- Visual representations of Catalanimals are less scary.

# Welcome to the Zoo: Catalanimals

- Visual representations of Catalanimals are less scary.
- Assume  $R_{qt} \subseteq R_t \subseteq R_q \subseteq R_+$ :

# Welcome to the Zoo: Catalananimals

- Visual representations of Catalananimals are less scary.
- Assume  $R_{qt} \subseteq R_t \subseteq R_q \subseteq R_+$ :





- Sometimes, there exists  $\xi \in \mathcal{E}$  such that  $\xi \cdot 1 = \omega \operatorname{pol}_X H$ .

- Sometimes, there exists  $\xi \in \mathcal{E}$  such that  $\xi \cdot 1 = \omega \operatorname{pol}_X H$ . (!!!)

# Tame and cuddly Catalananimals

- Sometimes, there exists  $\xi \in \mathcal{E}$  such that  $\xi \cdot 1 = \omega \operatorname{pol}_X H$ . (!!!)
- When  $R_{qt} \subseteq [R_q, R_t]$ , then this happens. (Associated  $H$  is *tame*.)

# Tame and cuddly Catalan animals

- Sometimes, there exists  $\xi \in \mathcal{E}$  such that  $\xi \cdot 1 = \omega \operatorname{pol}_X H$ . (!!!)
- When  $R_{qt} \subseteq [R_q, R_t]$ , then this happens. (Associated  $H$  is *tame*.)
- When,  $H$  is  $(m, n)$ -*cuddly* (a set of inequalities on root sets and weight), there exists an  $f \in \Lambda$  such that  $f[-MX^{m,n}] \cdot 1 = \omega \operatorname{pol}_X H$  (up to  $q, t$ -monomial and sign).

# Tame and cuddly Catalananimals

- Sometimes, there exists  $\xi \in \mathcal{E}$  such that  $\xi \cdot 1 = \omega \operatorname{pol}_X H$ . (!!!)
- When  $R_{qt} \subseteq [R_q, R_t]$ , then this happens. (Associated  $H$  is *tame*.)
- When,  $H$  is  $(m, n)$ -*cuddly* (a set of inequalities on root sets and weight), there exists an  $f \in \Lambda$  such that  $f[-MX^{m,n}] \cdot 1 = \omega \operatorname{pol}_X H$  (up to  $q, t$ -monomial and sign).
- In this case, we set  $\operatorname{cub}(H) = f$ .

# Tame and cuddly Catalananimals

- Sometimes, there exists  $\xi \in \mathcal{E}$  such that  $\xi \cdot 1 = \omega \operatorname{pol}_X H$ . (!!!)
- When  $R_{qt} \subseteq [R_q, R_t]$ , then this happens. (Associated  $H$  is *tame*.)
- When,  $H$  is  $(m, n)$ -*cuddly* (a set of inequalities on root sets and weight), there exists an  $f \in \Lambda$  such that  $f[-MX^{m,n}] \cdot 1 = \omega \operatorname{pol}_X H$  (up to  $q$ ,  $t$ -monomial and sign).
- In this case, we set  $\operatorname{cub}(H) = f$ .
- The cuddly conditions allow a nice coproduct formula for  $f[X + Y]$  in terms of cubs of “restrictions” of  $H$ .

# Cuddly Catalananimals with cub $e_k$

- $H(R_+, R_+, [R_+, R_+], (1^k))$  is  $(1, 1)$ -cuddly with cub  $e_k$ .

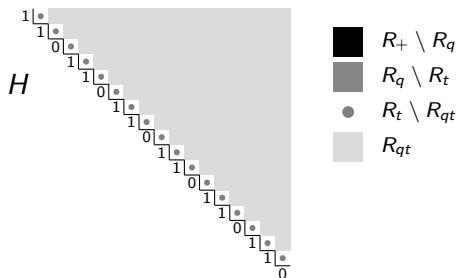
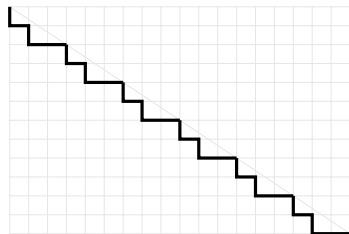
# Cuddly Catalananimals with cub $e_k$

- $H(R_+, R_+, [R_+, R_+], (1^k))$  is  $(1, 1)$ -cuddly with cub  $e_k$ .
- More generally, if  $\delta$  is the sequence of south step runs of highest path under the line through  $(0, kn)$  to  $(km, 0)$ , then 
$$e_k[-MX^{m,n}] \cdot 1 = H(R_+, R_+, [R_+, R_+], \delta).$$



# Cuddly Catalananimals with cub $e_k$

- $H(R_+, R_+, [R_+, R_+], (1^k))$  is  $(1, 1)$ -cuddly with cub  $e_k$ .
- More generally, if  $\delta$  is the sequence of south step runs of highest path under the line through  $(0, kn)$  to  $(km, 0)$ , then  $e_k[-MX^{m,n}] \cdot 1 = H(R_+, R_+, [R_+, R_+], \delta)$ .



$\delta = (1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0)$  and  $e_6[-MX^{3,2}] \cdot 1 = \omega \text{pol}_X H$

# 1, 1-Cuddly Catalananimals with cub $s_\mu$

- Can construct root sets and weight from the content diagonals of  $\mu$ .

# 1, 1-Cuddly Catalananimals with cub $s_\mu$

- Can construct root sets and weight from the content diagonals of  $\mu$ .

•  $\mu =$ 


 $\rightarrow$ 

+	.	.	-
+	.	-	
+	.	-	




 $\rightarrow$ 

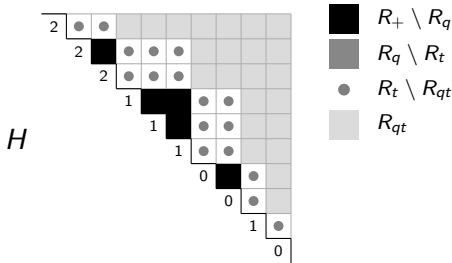
1	0	1	0
2	1	0	
2	2	1	

.

# 1, 1-Cuddly Catalananimals with cub $s_\mu$

- Can construct root sets and weight from the content diagonals of  $\mu$ .

$\mu =$    $\rightarrow$    $\rightarrow$  .



$$s_\mu[-MX^{1,1}] \cdot 1 = \nabla s_\mu = \omega \operatorname{pol}_X H \text{ (up to } q, t\text{-monomial)}$$

## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$\begin{aligned} s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \end{aligned}$$

- Combinatorial RHS: Over all *nests*  $\pi$  in a *den* associated to  $\mu$  and  $m, n$ .

## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$\begin{aligned} s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \end{aligned}$$

- Combinatorial RHS: Over all *nests*  $\pi$  in a *den* associated to  $\mu$  and  $m, n$ .
- Conjectured by Loehr-Warrington (2008) when  $n = 1$  with different combinatorics (but bijectively related).

## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$\begin{aligned} s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \end{aligned}$$

$$\mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \end{array}$$

## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$\begin{aligned} s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \end{aligned}$$

$$\mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \end{array}$$

- $p(\mu)$  = number of boxes with positive content.





## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$\begin{aligned} s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \end{aligned}$$

$$\mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \end{array}$$

- $p(\mu)$  = number of boxes with positive content.



- $h = m(\text{largest hook length in } \mu) = m(\mu_1 + \ell(\mu) - 1).$



## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$\begin{aligned} s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1}) \end{aligned}$$

$$\mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \end{array}$$

- $p(\mu)$  = number of boxes with positive content.



- $h = m(\text{largest hook length in } \mu) = m(\mu_1 + \ell(\mu) - 1)$ .



## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$s_{\mu}[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

$$\mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \end{array}$$

- $\gamma(\mu)$  is the tuple of the sizes of content diagonals.

## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

$$s_\mu[-MX^{m,n}] \cdot 1 \\ = (-1)^{p(\mu)} (qt)^{p(\mu)+m \sum_{i=1}^h \binom{\gamma_i}{2}} \sum_{\pi} t^{\text{area}(\pi)} q^{\text{dinv}_p(\pi)} \omega \mathcal{G}_{\nu(\pi)}(X; q^{-1})$$

$$\mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \end{array}$$

- $\gamma(\mu)$  is the tuple of the sizes of content diagonals.

$$\mu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \end{array} \implies \gamma = (1, 2, 3, 2, 1, 1).$$

For given partition  $\lambda$

- $\delta_i(\lambda) = \chi(\lambda_1 - 1 - i \text{ is the content of the last box of some row of } \lambda)$

For given partition  $\lambda$

- $\delta_i(\lambda) = \chi(\lambda_1 - 1 - i \text{ is the content of the last box of some row of } \lambda)$

- $\mu = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 1 & \\ \hline & & 0 & \\ \hline \end{array} \implies \delta(\mu) = (1, 0, 1, 1, 0, 0, \dots)$

For given partition  $\lambda$

- $\delta_i(\lambda) = \chi(\lambda_1 - 1 - i \text{ is the content of the last box of some row of } \lambda)$
- $\mu = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 1 & \\ \hline & & 0 & \\ \hline \end{array} \implies \delta(\mu) = (1, 0, 1, 1, 0, 0, \dots)$
- $\epsilon_i(\lambda) = \chi(i \geq \lambda_1)$

For given partition  $\lambda$

- $\delta_i(\lambda) = \chi(\lambda_1 - 1 - i \text{ is the content of the last box of some row of } \lambda)$

- $\mu = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 1 & \\ \hline & & 0 & \\ \hline \end{array} \implies \delta(\mu) = (1, 0, 1, 1, 0, 0, \dots)$

- $\epsilon_i(\lambda) = \chi(i \geq \lambda_1)$

- $\mu = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \implies \epsilon(\mu) = (0, 0, 0, 0, 1, 1, \dots)$



# Dens and nests

To construct a (simplified) den,

# Dens and nests

To construct a (simplified) den,

- 1 Draw the line connecting  $(0, \frac{n}{m}h)$  and  $(h, 0)$

# Dens and nests

To construct a (simplified) den,

- 1 Draw the line connecting  $(0, \frac{n}{m}h)$  and  $(h, 0)$
- 2 Relationship between  $\delta$  and  $\epsilon$  tell us where to place a lattice point on each vertical, (weakly) below the line.

# Dens and nests

To construct a (simplified) den,

- 1 Draw the line connecting  $(0, \frac{n}{m}h)$  and  $(h, 0)$
- 2 Relationship between  $\delta$  and  $\epsilon$  tell us where to place a lattice point on each vertical, (weakly) below the line.
- 3 If  $\delta_i(\mu) > \epsilon_i(\mu)$ , lattice point on  $x = im$  is a *source*.

# Dens and nests

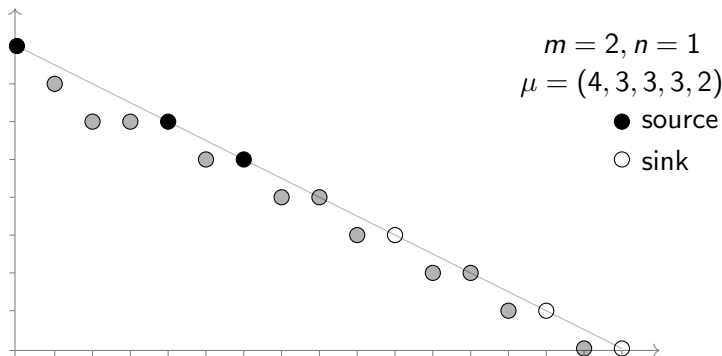
To construct a (simplified) den,

- 1 Draw the line connecting  $(0, \frac{n}{m}h)$  and  $(h, 0)$
- 2 Relationship between  $\delta$  and  $\epsilon$  tell us where to place a lattice point on each vertical, (weakly) below the line.
- 3 If  $\delta_i(\mu) > \epsilon_i(\mu)$ , lattice point on  $x = im$  is a *source*.
- 4 Similarly,  $\delta_i(\mu) < \epsilon_i(\mu) \implies$  point on  $x = im$  is a *sink*.

# Dens and nests

To construct a (simplified) den,

- 1 Draw the line connecting  $(0, \frac{n}{m}h)$  and  $(h, 0)$
- 2 Relationship between  $\delta$  and  $\epsilon$  tell us where to place a lattice point on each vertical, (weakly) below the line.
- 3 If  $\delta_i(\mu) > \epsilon_i(\mu)$ , lattice point on  $x = im$  is a *source*.
- 4 Similarly,  $\delta_i(\mu) < \epsilon_i(\mu) \implies$  point on  $x = im$  is a *sink*.



# Dens and nests

- Number the sources left to right and the sinks right to left.

# Dens and nests

- Number the sources left to right and the sinks right to left.
- A *nest* is a collection of east end lattice paths  $(\pi^{(1)}, \dots, \pi^{(r)})$  that lie weakly below the marked lattice points.



# Dens and nests

- Number the sources left to right and the sinks right to left.
- A *nest* is a collection of east end lattice paths  $(\pi^{(1)}, \dots, \pi^{(r)})$  that lie weakly below the marked lattice points.
- Each  $\pi^{(i)}$  begins with a south step, starting at source  $i$ , and ends with an east step into sink  $i$ .

# Dens and nests

- Number the sources left to right and the sinks right to left.
- A *nest* is a collection of east end lattice paths  $(\pi^{(1)}, \dots, \pi^{(r)})$  that lie weakly below the marked lattice points.
- Each  $\pi^{(i)}$  begins with a south step, starting at source  $i$ , and ends with an east step into sink  $i$ .
- Each  $\pi^{(i)}$  is *nested below*  $\pi^{(i+1)}$ .

# Dens and nests

- Number the sources left to right and the sinks right to left.
- A *nest* is a collection of east end lattice paths  $(\pi^{(1)}, \dots, \pi^{(r)})$  that lie weakly below the marked lattice points.
- Each  $\pi^{(i)}$  begins with a south step, starting at source  $i$ , and ends with an east step into sink  $i$ .
- Each  $\pi^{(i)}$  is *nested below*  $\pi^{(i+1)}$ .
  - The interval of  $x$ -coordinates of  $\pi^{(i+1)}$  is contained in the interval of  $x$ -coordinates of  $\pi^{(i)}$ .

# Dens and nests

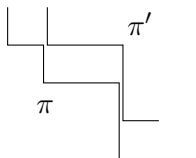
- Number the sources left to right and the sinks right to left.
- A *nest* is a collection of east end lattice paths  $(\pi^{(1)}, \dots, \pi^{(r)})$  that lie weakly below the marked lattice points.
- Each  $\pi^{(i)}$  begins with a south step, starting at source  $i$ , and ends with an east step into sink  $i$ .
- Each  $\pi^{(i)}$  is *nested below*  $\pi^{(i+1)}$ .
  - The interval of  $x$ -coordinates of  $\pi^{(i+1)}$  is contained in the interval of  $x$ -coordinates of  $\pi^{(i)}$ .
  - Top of a south run of  $\pi^{(i+1)}$  strictly above the top of a south run of  $\pi^{(i)}$  on same vertical.

# Dens and nests

- Number the sources left to right and the sinks right to left.
- A *nest* is a collection of east end lattice paths  $(\pi^{(1)}, \dots, \pi^{(r)})$  that lie weakly below the marked lattice points.
- Each  $\pi^{(i)}$  begins with a south step, starting at source  $i$ , and ends with an east step into sink  $i$ .
- Each  $\pi^{(i)}$  is *nested below*  $\pi^{(i+1)}$ .
  - The interval of  $x$ -coordinates of  $\pi^{(i+1)}$  is contained in the interval of  $x$ -coordinates of  $\pi^{(i)}$ .
  - Top of a south run of  $\pi^{(i+1)}$  strictly above the top of a south run of  $\pi^{(i)}$  on same vertical.
  - Bottom of a south run of  $\pi^{(i)}$  strictly below the bottom of a south run of  $\pi^{(i+1)}$  on same vertical.

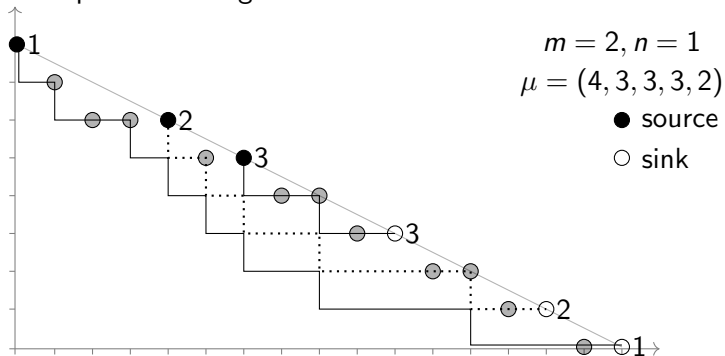
# Dens and nests

- Number the sources left to right and the sinks right to left.
- A *nest* is a collection of east end lattice paths  $(\pi^{(1)}, \dots, \pi^{(r)})$  that lie weakly below the marked lattice points.
- Each  $\pi^{(i)}$  begins with a south step, starting at source  $i$ , and ends with an east step into sink  $i$ .
- Each  $\pi^{(i)}$  is *nested below*  $\pi^{(i+1)}$ .
  - The interval of  $x$ -coordinates of  $\pi^{(i+1)}$  is contained in the interval of  $x$ -coordinates of  $\pi^{(i)}$ .
  - Top of a south run of  $\pi^{(i+1)}$  strictly above the top of a south run of  $\pi^{(i)}$  on same vertical.
  - Bottom of a south run of  $\pi^{(i)}$  strictly below the bottom of a south run of  $\pi^{(i+1)}$  on same vertical.



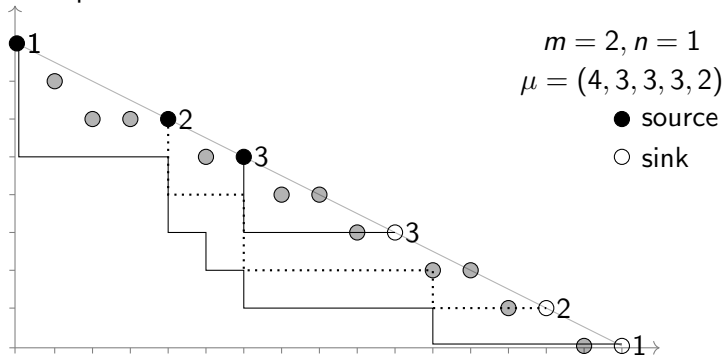
# Dens and nests

Example of the “highest nest”  $\pi^0$



# Dens and nests

Example of another nest.

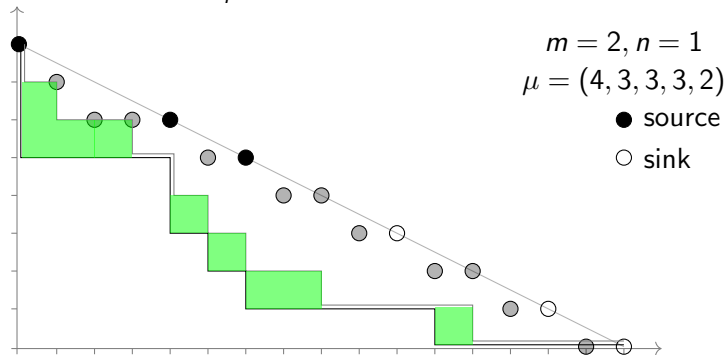




$\text{area}(\pi) = \sum_{i=1}^r \text{area}(\pi_i)$  where  $\text{area}(\pi_i) =$  number of lattice squares between  $\pi_i$  and  $\pi_i^0$ .

# Area

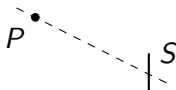
$\text{area}(\pi) = \sum_{i=1}^r \text{area}(\pi_i)$  where  $\text{area}(\pi_i) =$  number of lattice squares between  $\pi_i$  and  $\pi_i^0$ .

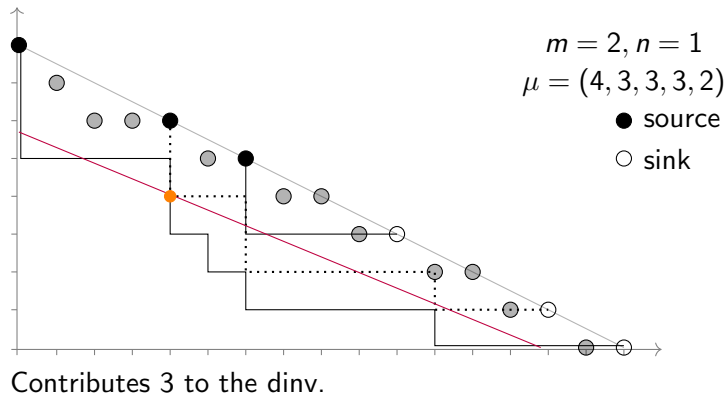


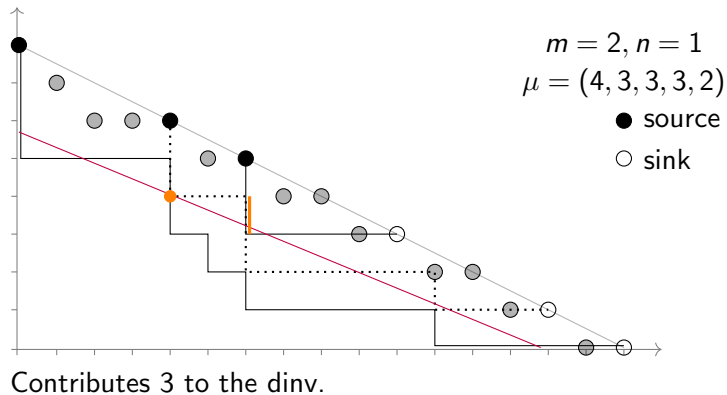
- For  $p = \frac{n}{m} - \epsilon \in \mathbb{R} \setminus \mathbb{Q}$  and  $\epsilon$  small,  $\text{dinv}_p(\pi) = \#\{(P, i, S, j)\}$  where

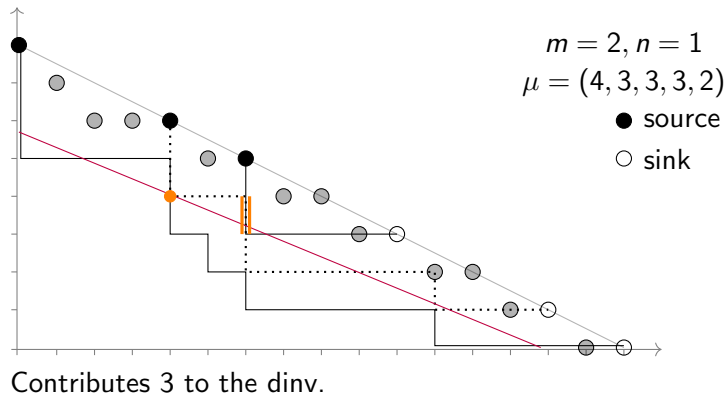
- For  $p = \frac{n}{m} - \epsilon \in \mathbb{R} \setminus \mathbb{Q}$  and  $\epsilon$  small,  $\text{dinv}_p(\pi) = \#\{(P, i, S, j)\}$  where
  - $P$  is a non-sink lattice point in  $\pi_i$
  - $S$  is a south step in  $\pi_j$
  - $P$  is strictly to the left of  $S$
  - A line of slope  $-p$  passing through  $P$  passes through  $S$ .

- For  $p = \frac{n}{m} - \epsilon \in \mathbb{R} \setminus \mathbb{Q}$  and  $\epsilon$  small,  $\text{dinv}_p(\pi) = \#\{(P, i, S, j)\}$  where
  - $P$  is a non-sink lattice point in  $\pi_i$
  - $S$  is a south step in  $\pi_j$
  - $P$  is strictly to the left of  $S$
  - A line of slope  $-p$  passing through  $P$  passes through  $S$ .

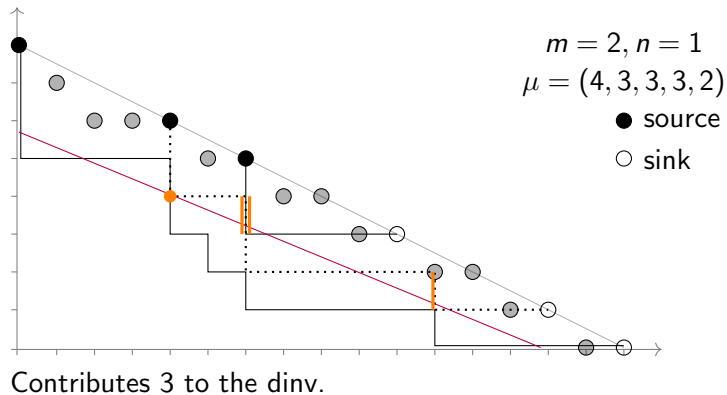












# Associating a tuple of skew partitions to a nest

- Each vertical line  $x = i$  will give a skew partition.

# Associating a tuple of skew partitions to a nest

- Each vertical line  $x = i$  will give a skew partition.
- South steps of each path will contribute a row.

# Associating a tuple of skew partitions to a nest

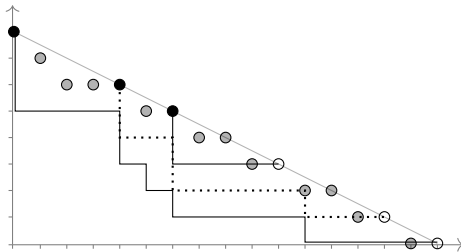
- Each vertical line  $x = i$  will give a skew partition.
- South steps of each path will contribute a row.
- Content determined by how far down south step is from highest lattice point under the line  $+j$  for  $\pi_j$ .

# Associating a tuple of skew partitions to a nest

- Each vertical line  $x = i$  will give a skew partition.
- South steps of each path will contribute a row.
- Content determined by how far down south step is from highest lattice point under the line  $+j$  for  $\pi_j$ .
- Tuple ordered by how far marked lattice points are from slight perturbation of line.

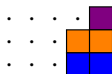
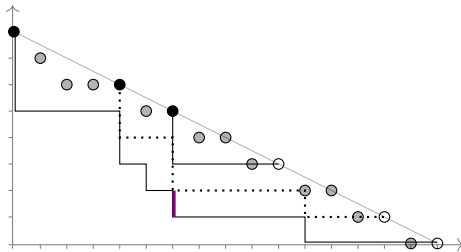
# Associating a tuple of skew partitions to a nest

- Each vertical line  $x = i$  will give a skew partition.
- South steps of each path will contribute a row.
- Content determined by how far down south step is from highest lattice point under the line  $+j$  for  $\pi_j$ .
- Tuple ordered by how far marked lattice points are from slight perturbation of line.



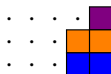
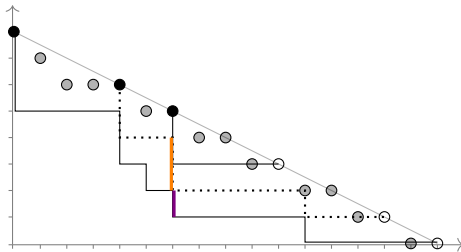
# Associating a tuple of skew partitions to a nest

- Each vertical line  $x = i$  will give a skew partition.
- South steps of each path will contribute a row.
- Content determined by how far down south step is from highest lattice point under the line  $+j$  for  $\pi_j$ .
- Tuple ordered by how far marked lattice points are from slight perturbation of line.



# Associating a tuple of skew partitions to a nest

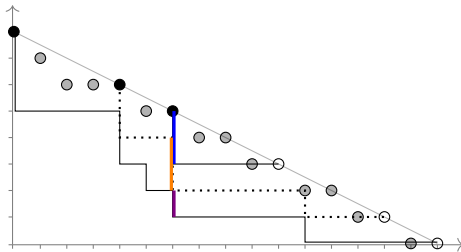
- Each vertical line  $x = i$  will give a skew partition.
- South steps of each path will contribute a row.
- Content determined by how far down south step is from highest lattice point under the line  $+j$  for  $\pi_j$ .
- Tuple ordered by how far marked lattice points are from slight perturbation of line.





# Associating a tuple of skew partitions to a nest

- Each vertical line  $x = i$  will give a skew partition.
- South steps of each path will contribute a row.
- Content determined by how far down south step is from highest lattice point under the line  $+j$  for  $\pi_j$ .
- Tuple ordered by how far marked lattice points are from slight perturbation of line.



- In our paper, we provide a more general definition of den as a tuple of data  $(h, p, d, e) \in \mathbb{Z}_{>0} \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{Z}^{h+1} \times \mathbb{Z}^{h+1}$  subject to some conditions.

- In our paper, we provide a more general definition of den as a tuple of data  $(h, p, d, e) \in \mathbb{Z}_{>0} \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{Z}^{h+1} \times \mathbb{Z}^{h+1}$  subject to some conditions.
- To each den we can associate a tame Catalanimal  $H$  and give a corresponding shuffle theorem as a sum over the nests of the den.

- In our paper, we provide a more general definition of den as a tuple of data  $(h, p, d, e) \in \mathbb{Z}_{>0} \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{Z}^{h+1} \times \mathbb{Z}^{h+1}$  subject to some conditions.
- To each den we can associate a tame Catalanimal  $H$  and give a corresponding shuffle theorem as a sum over the nests of the den.
- These results hold “stably.” In other words, a stronger result is proven before applying polynomial truncation.

- In our paper, we provide a more general definition of den as a tuple of data  $(h, p, d, e) \in \mathbb{Z}_{>0} \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{Z}^{h+1} \times \mathbb{Z}^{h+1}$  subject to some conditions.
- To each den we can associate a tame Catalanimal  $H$  and give a corresponding shuffle theorem as a sum over the nests of the den.
- These results hold “stably.” In other words, a stronger result is proven before applying polynomial truncation.
- This allows us to simultaneously generalize the  $s_\lambda[-MX^{m,n}]$  formula and our “shuffle theorem for paths under any line” formula (BHMPs).

## Other exhibits for next time

- For each LLT polynomial  $\mathcal{G}_\nu$  and coprime  $(m, n)$  with  $m > 0$ , an  $m, n$ -cuddly Catalan animal with cub  $\mathcal{G}_\nu$  is given. (BHMPs)

## Other exhibits for next time

- For each LLT polynomial  $\mathcal{G}_\nu$  and coprime  $(m, n)$  with  $m > 0$ , an  $m, n$ -cuddly Catalanimal with cub  $\mathcal{G}_\nu$  is given. (BHMPs)
- Special cases include Schur functions and Hall-Littlewood polynomials.

# Other exhibits for next time

- For each LLT polynomial  $\mathcal{G}_\nu$  and coprime  $(m, n)$  with  $m > 0$ , an  $m, n$ -cuddly Catalan animal with cub  $\mathcal{G}_\nu$  is given. (BHMPs)
- Special cases include Schur functions and Hall-Littlewood polynomials.
- Unicorn Catalan animals (or Catalan functions) where  $R_t = R_{qt} = \emptyset$  also have a rich (older) results and combinatorics, but served as inspiration. (Chen-Haiman, Blasiak-Morse-Pun-Summers, Blasiak-Morse-Pun)



## Future work: exit through the gift shop

- Is there a representation-theoretic model for  $\nabla s_\mu$ ? For any Catalan animal associated to a den?

## Future work: exit through the gift shop

- Is there a representation-theoretic model for  $\nabla s_\mu$ ? For any Catalan animal associated to a den?
- Any direct combinatorial formula (even a conjecture) for the Schur-expansion coefficients?

# Future work: exit through the gift shop

- Is there a representation-theoretic model for  $\nabla s_\mu$ ? For any Catalan animal associated to a den?
- Any direct combinatorial formula (even a conjecture) for the Schur-expansion coefficients?
- What other families of symmetric functions can be represented by Catalan animals?

# Future work: exit through the gift shop

- Is there a representation-theoretic model for  $\nabla s_\mu$ ? For any Catalan animal associated to a den?
- Any direct combinatorial formula (even a conjecture) for the Schur-expansion coefficients?
- What other families of symmetric functions can be represented by Catalan animals? Upcoming: Macdonald polynomials

# Future work: exit through the gift shop

- Is there a representation-theoretic model for  $\nabla s_\mu$ ? For any Catalan animal associated to a den?
- Any direct combinatorial formula (even a conjecture) for the Schur-expansion coefficients?
- What other families of symmetric functions can be represented by Catalan animals? Upcoming: Macdonald polynomials
- What connections do Catalan animals have with machinery used to prove other shuffle theorems, such as work by Carlsson-Mellit?

# Thank you for visiting!

Bergeron, Francois, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. 2016. *Compositional  $(km, kn)$ -shuffle conjectures*, Int. Math. Res. Not. IMRN **14**, 4229–4270, DOI 10.1093/imrn/rnv272. MR3556418

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021a. *A Shuffle Theorem for Paths Under Any Line*, arXiv e-prints, available at arXiv:2102.07931.

———. 2021b. *Dens, nests and the Loehr-Warrington conjecture*, arXiv e-prints, available at arXiv:2112.07070.

———. 2021c. *LLT polynomials in the Schiffmann algebra*, arXiv e-prints, available at arXiv:2112.07063.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Anton Mellit. 2018. *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31**, no. 3, 661–697, DOI 10.1090/jams/893. MR3787405

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

# References continued

- Grojnowski, Ian and Mark Haiman. 2007. *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, Unpublished manuscript.
- Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.
- Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919
- \_\_\_\_\_. 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676
- Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sémin. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754
- Loehr, Nicholas A. and Gregory S. Warrington. 2008. *Nested quantum Dyck paths and  $\nabla(s_\lambda)$* , Int. Math. Res. Not. IMRN **5**, Art. ID rnm 157, 29, DOI 10.1093/imrn/rnm157. MR2418288
- Mellit, Anton. 2016. *Toric braids and  $(m, n)$ -parking functions*, arXiv e-prints, arXiv:1604.07456, available at arXiv:1604.07456.
- Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004