

Schubert calculus and K -theoretic Catalan functions

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UVA Graduate Seminar

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- ① An overview of Schubert calculus
- ② Catalan functions: shedding new light on old problems
- ③ K -theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

Geometric problem

Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of subvarieties in a variety X .

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

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Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

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Goal

Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Classical Example

$$X = \operatorname{Gr}_m(\mathbb{C}^{m+n}) = \{\text{all } m\text{-dimensional subspaces of } \mathbb{C}^{n+m}\}.$$

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What are the structure constants $c_{\lambda\mu}^\nu$?

Classical Example (cont.)

$\Lambda_m = \mathbb{C}[x_1, \dots, x_m]^{S_m}$ is the ring of symmetric polynomials in m variables and has bases indexed by partitions.

$$\underbrace{12x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{symmetric}}$$

$$\underbrace{5x_1^2 + 12x_2^2 - 7x_1x_2}_{\text{not symmetric}}$$

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There exists a basis of Λ_m denoted $\{s_\lambda\}_\lambda$ and a surjection of rings such that

$$\begin{aligned} \Lambda_m &\rightarrow H^*(\text{Gr}(m, n)) \\ s_\lambda &\mapsto \begin{cases} \sigma_\lambda & \lambda \subseteq (n^m) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Classical Example (cont.)

Cohomology structure: $\sigma_\lambda \leftrightarrow s_\lambda$ when $\lambda \subseteq (n^m)$.

$$s_\lambda s_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu s_\nu + \sum_{\nu \not\subseteq (n^m)} c_{\lambda\mu}^\nu s_\nu \leftrightarrow \sigma_\lambda \cup \sigma_\mu = \sum_{\nu \subseteq (n^m)} c_{\lambda\mu}^\nu \sigma_\nu$$

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

5			
3	4		
2	3		
1	2	2	5

8			
7	9		
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1	2	5	6

standard = no repeated letters

Schur functions s_λ

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Schur function s_λ is a “weight generating function” of semistandard tableaux:

2	3	3	2	3	3	2	3
1	1	1	1	1	2	3	2

$$s_{\square\square}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_{\square} s_{\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}} = s_{\begin{smallmatrix} \square & \square & \square \\ \square & \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} + s_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$$

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Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \dots, \mu_r)} + \text{lower order terms}$, subtract to get

$$s_{(\mu_1, \dots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu$.

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Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$ for Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$.

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$

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- Structure constants $\mathfrak{S}_w \mathfrak{S}_u = c_{wu}^v \mathfrak{S}_v$ are combinatorially unknown.

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Schubert Calculus Variations

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(Co)homology of Grassmannian	Schur functions
(Co)homology of flag variety	Schubert polynomials
Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur- P and Q functions
(Co)homology of affine Grassmannian	(dual) k -Schur functions
K -theory of Grassmannian	Grothendieck polynomials
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And many more!

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- Peterson isomorphism

$$\Psi: QH^*(Fl_{k+1}) \rightarrow H_*(Gr_{SL_{k+1}})_{loc}$$

$$\mathfrak{S}_w^Q \mapsto \frac{s_\lambda^{(k)}}{\prod_{i \in Des(w)} \tau_i}$$

where $s_\lambda^{(k)}$ is a k -Schur function.

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Upshot

Computations for Schubert polynomials can be moved into symmetric functions.

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- Has geometric meaning for embedding of affine Grassmannians.
- Definition with t important for Macdonald polynomials.
- Many definitions. A new one makes proofs easier!

Raising Operators on Symmetric Functions

- Raising operators $R_{i,j}$ act on diagrams

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$$s_\lambda = \prod_{i < j} (1 - R_{ij}) h_\lambda$$

$$s_{22} = (1 - R_{12})h_{22} = h_{22} - h_{31}$$

$$\begin{aligned} s_{211} &= (1 - R_{12})(1 - R_{23})(1 - R_{13})h_{211} \\ &= h_{211} - h_{301} - h_{220} - \cancel{h_{310}} + \cancel{h_{310}} + h_{32-1} + h_{400} - h_{41-1} \end{aligned}$$

some terms cancel

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Advantage: gives definition for Schur function indexed by any integer vector $\alpha \in \mathbb{Z}^\ell$. Amazingly:

$$s_\alpha = \prod_{i < j} (1 - R_{ij}) h_\alpha = \begin{cases} \pm s_\lambda & \text{for a partition } \lambda \\ 0 & \end{cases}$$

For $\langle s_{1^r}^\perp s_\lambda, s_\mu \rangle = \langle s_\lambda, s_{1^r} s_\mu \rangle$,

$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1, \ell], |S|=r} s_{\lambda - \epsilon_S}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path above the diagonal.

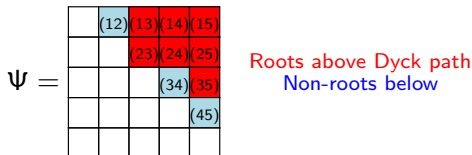
$$\Psi =$$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

Roots above Dyck path
Non-roots below

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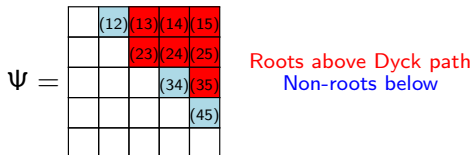
Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) h_\gamma(x)$$

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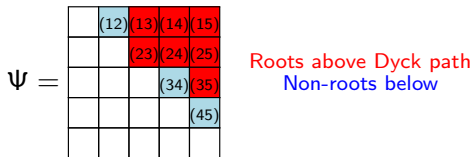
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- $\Psi = \text{all roots} \implies H(\Psi; \gamma) = h_\gamma$

k -Schur root ideal for λ

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

Catalan functions ($t = 1$)

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

\leftarrow row i has $4 - \lambda_i$ non-roots

Catalan functions ($t = 1$)

k -Schur root ideal for λ

$$\begin{aligned}\Psi = \Delta^k(\lambda) &= \{(i, j) : j > k - \lambda_i\} \\ &= \text{root ideal with } k - \lambda_i \text{ non-roots in row } i\end{aligned}$$

$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

\leftarrow row i has $4 - \lambda_i$ non-roots

- For partition λ with $\lambda_1 \leq k$, $s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda)$.

Catalan functions

Key ingredient of branching proof:

Catalan functions

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Shift Invariance (Blasiak et al., 2019)

For partition λ of length ℓ with $\lambda_1 \leq k$,

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where $\langle s_{1^\ell}^\perp f, g \rangle = \langle f, s_{1^\ell} g \rangle$.

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$$\Delta^4(3, 3, 2, 2, 1, 1) =$$

3					
	3				
		2			
			2		
				1	
					1

$$\Delta^5(4, 4, 3, 3, 2, 2) =$$

4					
	4				
		3			
			3		
				2	
					2

Catalan functions

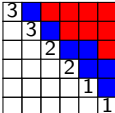
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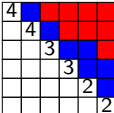
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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_{\mu} a_{\lambda+1^\ell, \mu} s_\mu^{(k+1)}$$

Dual Grothendieck polynomials

- Inhomogeneous basis: $g_\lambda = s_\lambda + \text{lower degree terms}$.

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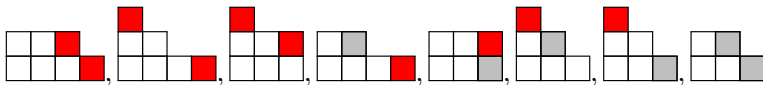
$$g_{1^2}g_{3,2} = g_{43} + g_{421} + g_{331} - g_{42} - g_{33} - 2g_{321} + g_{31}$$



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- Dual to Grothendieck polynomials: Schubert representatives for $K^*(Gr(m, n))$

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The diagram illustrates the Pieri rule for K - k -Schur functions. The top row shows the product of a single vertical strip (g_1) and a 2-strip ($g^{(2)}$) as the difference of two 2-strips. The bottom row shows the corresponding set-valued strips with colored dots (red, blue, black) and shaded cells, demonstrating the same relationship.

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Problem

No direct formula for $g_{\lambda}^{(k)}$

An Extra Ingredient: Lowering Operators

Lowering Operators $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

$$L_3 \left(\begin{array}{|c|c|c|c|} \hline \text{red} & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad L_1 \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & \text{red} & \\ \hline \end{array} \right) = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$$

K -theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

Affine K -Theory Representatives with Raising Operators

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Example

non-roots of Ψ , roots of \mathcal{L}

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$$\begin{aligned} K(\Psi; \mathcal{L}; 54332) &= (1 - L_4)^2 (1 - L_5)^2 \\ &\cdot (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332} \end{aligned}$$

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Example

$$g_{332111}^{(4)} =$$

3						
	3					
		2				
			1			
				1		
					1	

$$\Delta_6^+ / \Delta^{(4)}(332111), \Delta^{(5)}(332111)$$

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The branching coefficients in

$$g_\lambda^{(k)} = \sum_{\mu} a_{\lambda\mu} g_\mu^{(k+1)}$$

satisfy $(-1)^{|\lambda|-|\mu|} a_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$.

K -theoretic Peterson isomorphism

$$\Phi: QK^*(Fl_{k+1}) \rightarrow K_*(Gr_{SL_{k+1}})_{loc}$$

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- 4 Describe the image of \mathfrak{G}_w^Q under Peterson isomorphism for all $w \in S_{k+1}$.

Thank you!

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