

# Dens, nests, and Catalananimals: a walk through the zoo of shuffle theorems

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- $\sigma \in S_n$  acts as  $\sigma.f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$

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$$e_2 = x_1x_2 + x_1x_3 + x_2x_3 \quad h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$

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- $\{f \in \mathbb{Q}[x_1, \dots, x_n] \mid \sigma.f = f \forall \sigma \in S_n\}$  forms a vector space,  $\Lambda_{\mathbb{Q}}$ .

# Combinatorics of Symmetric Polynomials

## Generators

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$



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Symmetric functions are polynomials in the  $e_1, e_2, \dots$ , or in the  $h_1, h_2, \dots$

$$3h_2h_1^2 - h_2^2 + 6h_3h_1 = 3h_{(211)} - h_{(22)} + 6h_{(31)}$$

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Basis of  $\Lambda_{\mathbb{Q}}$ ?

## Definition

$n \in \mathbb{Z}_{>0}$ , a *partition of  $n$*  is  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$  such that  $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$ .

# Partitions

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$$5 \rightarrow \begin{array}{|c|c|c|c|c|}\hline\hline\hline\hline\hline\hline\end{array}$$

$$4 + 1 \rightarrow \begin{array}{|c|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 2 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$3 + 1 + 1 \rightarrow \begin{array}{|c|c|c|}\hline\hline\hline\hline\hline\end{array}$$

$$2 + 2 + 1 \rightarrow \begin{array}{|c|c|}\hline\hline\hline\hline\hline\end{array}$$

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Collection is called  $\text{SSYT}(\lambda)$ .



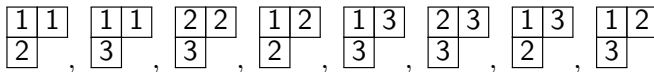
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For  $\lambda = (2, 1)$ ,

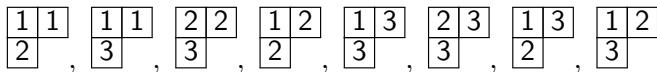


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Associate a polynomial to  $\text{SSYT}(\lambda)$ .

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Weight: 

1	1
2	

, 

1	1
3	

, 

2	2
3	

, 

1	2
2	

, 

1	3
3	

, 

2	3
3	

, 

1	3
2	

, 

1	2
3	

  
(2,1,0) (2,0,1) (0,2,1) (1,2,0) (1,0,2) (0,1,2) (1,1,1) (1,1,1)

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$$s_{(21)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

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$$s_\lambda = \sum_{T \in \text{SSYT}(\lambda)} x^T \text{ for } x^T = \prod_{i \in T} x_i$$

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- Schur functions form a basis for  $\Lambda_{\mathbb{Q}}$



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## Harmonic polynomials

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$$\begin{aligned} M &= \text{sp} \left\{ \left( \partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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①  $S_3$  action on  $M$  fixes vector subspaces!

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$$e_1^3 = (x_1 + x_2 + x_3)^3 = s_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} + s_{\begin{array}{|c|c|} \hline \square & \square \\ \hline & \square \\ \hline \end{array}} + s_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}}$$

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Schur basis expansion counts multiplicity of irreducible  $S_n$  fixed subspaces!

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Via Frobenius characteristic map, questions about  $S_n$ -representations get translated to questions about Schur expansion coefficients in symmetric functions.

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Does a symmetric function expand into Schur basis with nonnegative coefficients? Is there a combinatorial description for coefficients?

# Getting more information



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Solution: minimal  $S_n$ -fixed subspace of degree  $d \mapsto q^d s_\lambda$  (graded Frobenius)

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Answer: "Hall-Littlewood polynomial"  $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$ .

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- $\tilde{H}_\lambda(X; 1, 1) = e_1^{|\lambda|}$ .
- Does there exist a family of  $S_n$ -representations whose (bigraded) Frobenius characteristics equal  $\tilde{H}_\lambda(X; q, t)$ ?

# Garsia-Haiman modules

- $\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$  satisfying  $\sigma(x_i) = x_{\sigma(i)}$ ,  $\sigma(y_j) = y_{\sigma(j)}$ .



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## Theorem (Haiman, 2001)

*The Garsia-Haiman module  $M_\lambda$  has bigraded Frobenius characteristic given by  $\tilde{H}_\lambda(X; q, t)$*

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- No combinatorial description of  $\tilde{K}_{\lambda\mu}(q, t)$ . (Still open!)

## Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left( \sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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## Question

What symmetric function is the bigraded Frobenius characteristic of  $DH_n$ ?

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Theorem (Haiman, 2002)

*The bigraded Frobenius characteristic of  $DH_n$  is given by  $\nabla e_n$ .*



# A Combinatorial Connection: Shuffle Theorem

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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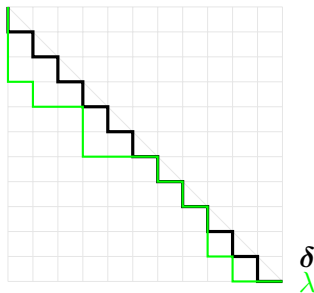
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# Dyck paths

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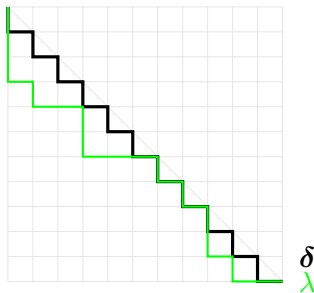
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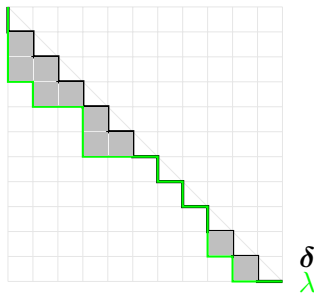


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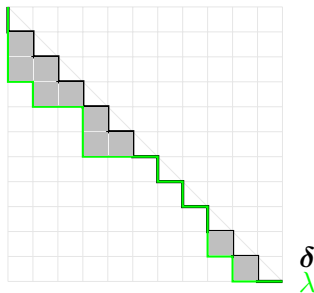
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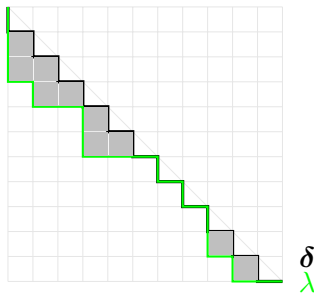


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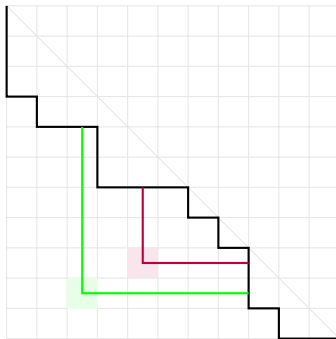
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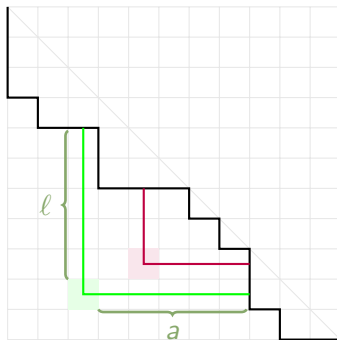


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Balanced hook is given by a cell below  $\lambda$  satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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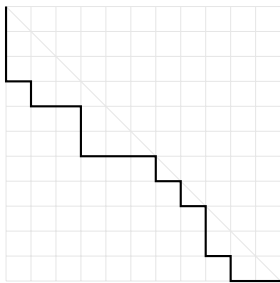
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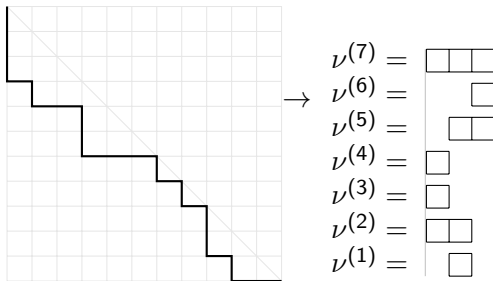
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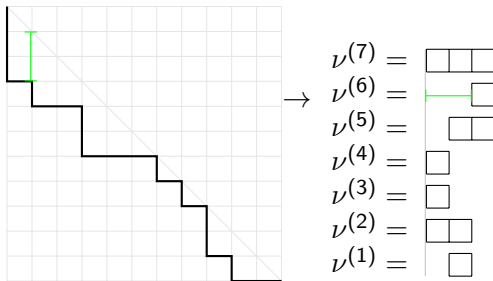
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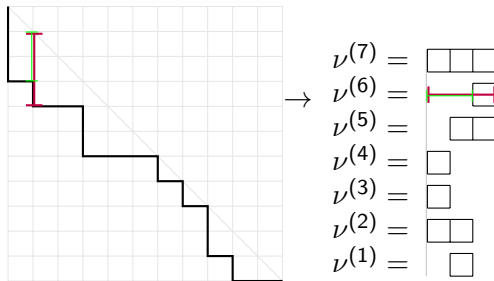
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$G_{\nu(\lambda)}(X; q)$  is an LLT polynomial for a tuple of rows,  
 $\nu(\lambda) = (\nu^{(1)}, \dots, \nu^{(r)})$ .



$$\mathcal{G}_\nu(X; q) = \sum_{T \in \text{SSYT}(\nu)} q^{i(T)} x^T$$

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$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

## Example $\nabla e_3$






$$\lambda \rightarrow q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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




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




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- Entire quantity is  $q, t$ -symmetric
- Coefficient of  $s_{1,1,1}$  in sum is a “ $(q, t)$ -Catalan number”  
 $(q^3 + q^2t + qt + qt^2 + t^3)$ .

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For  $m, n$  coprime, the operator  $e_k[-MX^{m,n}]$  acting on  $\Lambda$  satisfies

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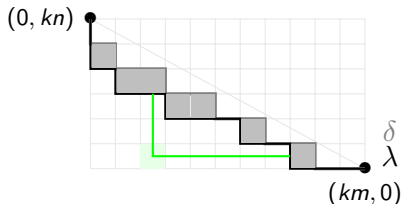
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- In general,  $\mathcal{E}$ -action can be a pain to compute in a nice way, but sometimes it is nice!



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- Can also be thought of as an infinite series of virtual  $GL_l$ -characters.
- We can take “polynomial part” (restrict to only polynomial  $GL_l$ -characters) to get a symmetric function.

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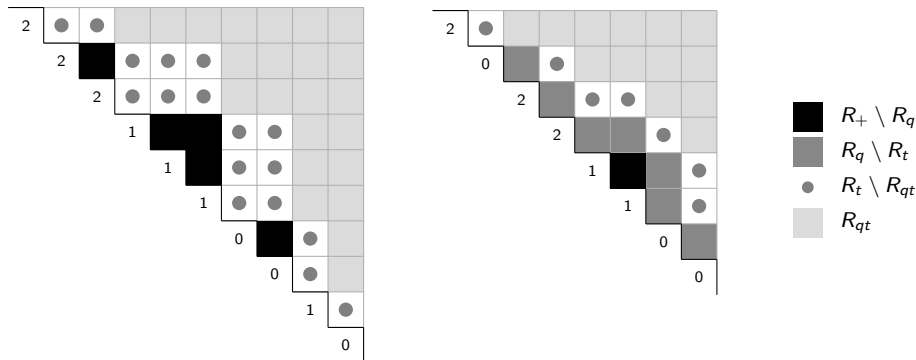
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- In this case, we set  $\operatorname{cub}(H) = f$ .
- The cuddly conditions allow a nice coproduct formula for  $f[X + Y]$  in terms of cubs of “restrictions” of  $H$ .

# Cuddly Catalananimals with cub $e_k$

- $H(R_+, R_+, [R_+, R_+], (1^k))$  is  $(1, 1)$ -cuddly with cub  $e_k$ .

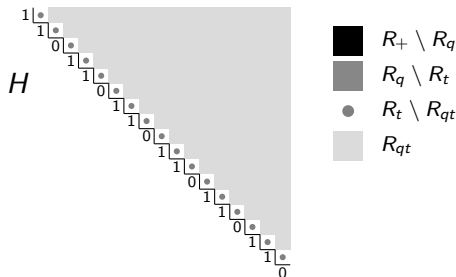
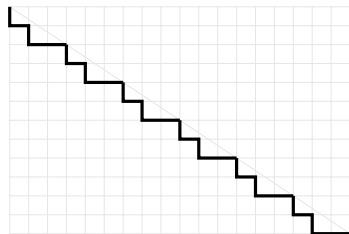
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$\delta = (1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0)$  and  $e_6[-MX^{3,2}] \cdot 1 = \omega \operatorname{pol}_X H$

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 $\rightarrow$ 

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


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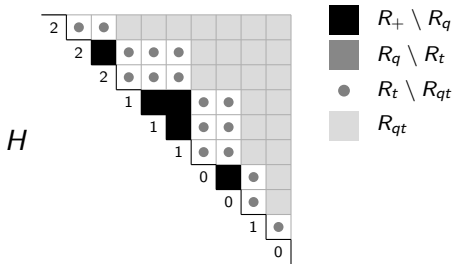
1	0	1	0
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## Theorem (Blasiak-Haiman-Morse-Pun-S. (2021<sup>+</sup>))

For every partition  $\mu$  and coprime positive integers  $m, n$ , we have

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- Combinatorial RHS: Over all *nests*  $\pi$  in a *den* associated to  $\mu$  and  $m, n$ .
- Conjectured by Loehr-Warrington (2008) when  $n = 1$  with different combinatorics (but bijectively related).

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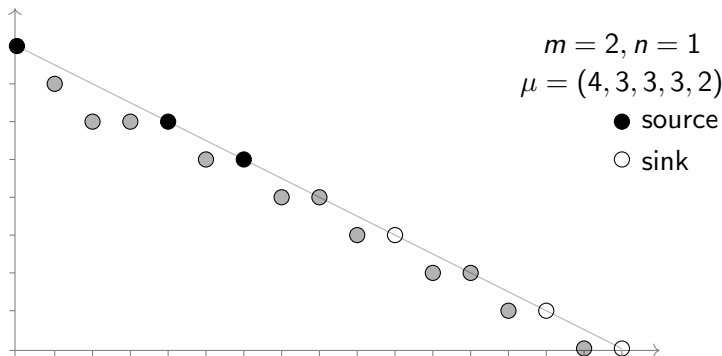
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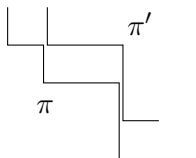
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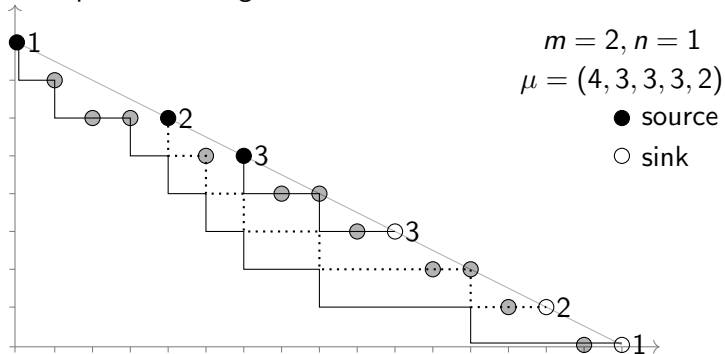
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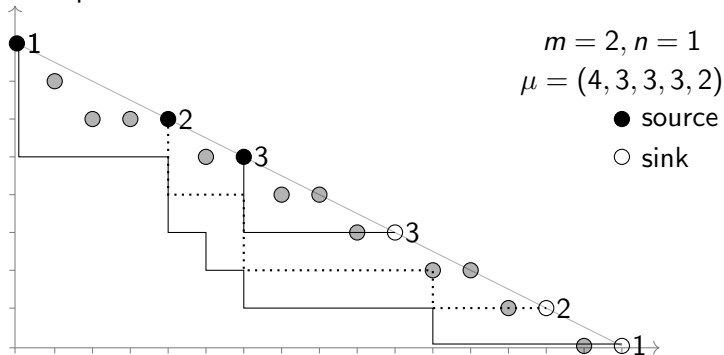
# Dens and nests

Example of the “highest nest”  $\pi^0$



# Dens and nests

Example of another nest.

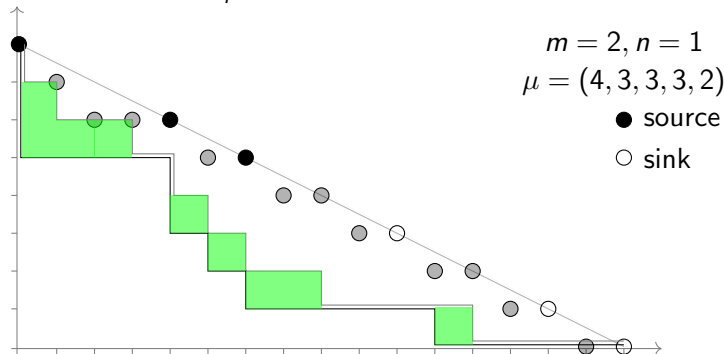




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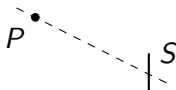


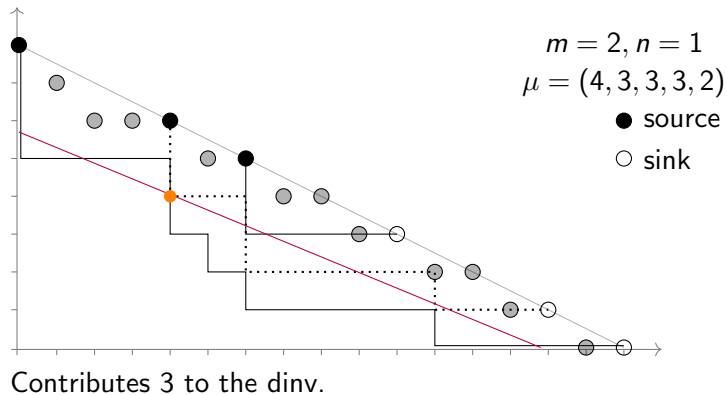
$$\text{area}(\pi_1) = 9$$

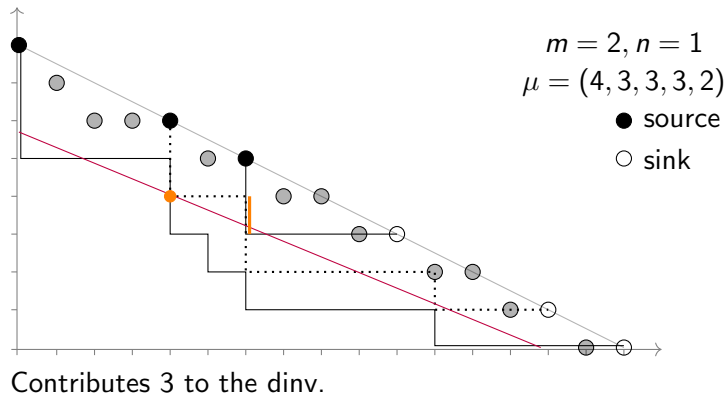
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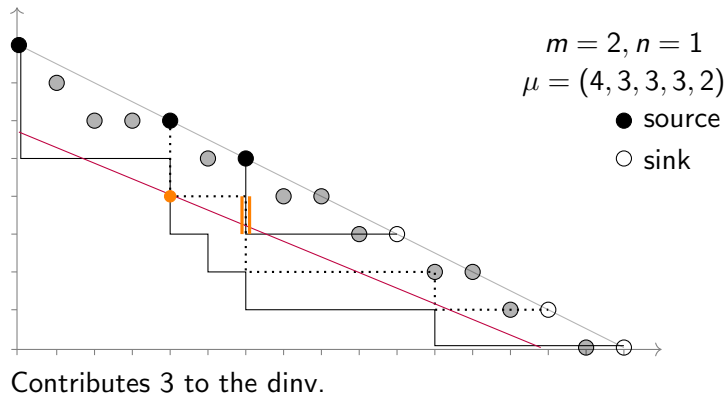
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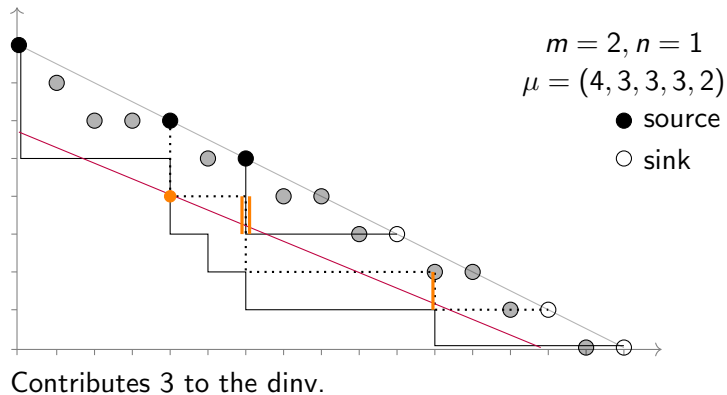












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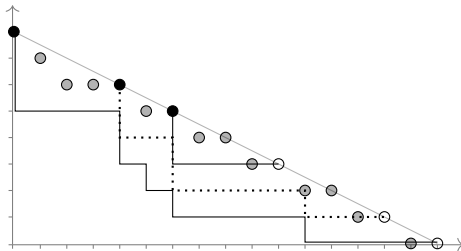
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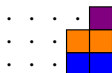
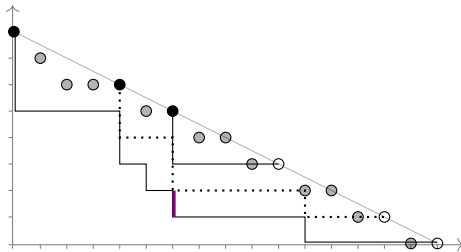
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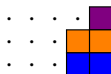
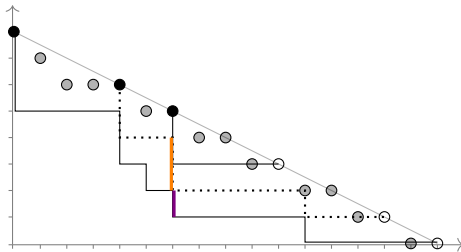
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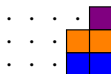
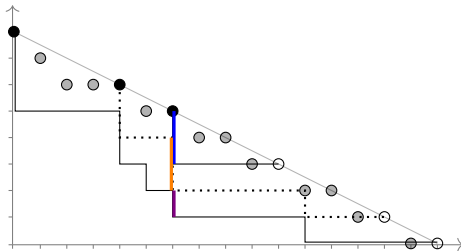
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- To each den we can associate a tame Catalan animal  $H$  and give a corresponding shuffle theorem as a sum over the nests of the den.
- These results hold “stably.” In other words, a stronger result is proven before applying polynomial truncation.
- This allows us to simultaneously generalize the  $s_\lambda[-MX^{m,n}]$  formula and our “shuffle theorem for paths under any line” formula (BHMPs).

## Other exhibits for next time

- For each LLT polynomial  $\mathcal{G}_\nu$  and coprime  $(m, n)$  with  $m > 0$ , an  $m, n$ -cuddly Catalan animal with cub  $\mathcal{G}_\nu$  is given. (BHMPs)

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- Special cases include Schur functions and Hall-Littlewood polynomials.
- Unicorn Catalananimals (or Catalan functions) where  $R_t = R_{qt} = \emptyset$  also have a rich (older) results and combinatorics, but served as inspiration. (Chen-Haiman, Blasiak-Morse-Pun-Summers, Blasiak-Morse-Pun)



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- What connections do Catalan animals have with machinery used to prove other shuffle theorems, such as work by Carlsson-Mellit?

# Thank you for visiting!

Bergeron, Francois, Adriano Garsia, Emily Sergel Leven, and Guoce Xin. 2016. *Compositional  $(km, kn)$ -shuffle conjectures*, Int. Math. Res. Not. IMRN **14**, 4229–4270, DOI 10.1093/imrn/rnv272. MR3556418

Blasiak, Jonah, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. 2021a. *A Shuffle Theorem for Paths Under Any Line*, arXiv e-prints, available at arXiv:2102.07931.

———. 2021b. *Dens, nests and the Loehr-Warrington conjecture*, arXiv e-prints, available at arXiv:2112.07070.

———. 2021c. *LLT polynomials in the Schiffmann algebra*, arXiv e-prints, available at arXiv:2112.07063.

Burban, Igor and Olivier Schiffmann. 2012. *On the Hall algebra of an elliptic curve, I*, Duke Math. J. **161**, no. 7, 1171–1231, DOI 10.1215/00127094-1593263. MR2922373

Carlsson, Erik and Anton Mellit. 2018. *A proof of the shuffle conjecture*, J. Amer. Math. Soc. **31**, no. 3, 661–697, DOI 10.1090/jams/893. MR3787405

Garsia, Adriano M. and Mark Haiman. 1993. *A graded representation model for Macdonald's polynomials*, Proc. Nat. Acad. Sci. U.S.A. **90**, no. 8, 3607–3610, DOI 10.1073/pnas.90.8.3607. MR1214091

# References continued

- Grojnowski, Ian and Mark Haiman. 2007. *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, Unpublished manuscript.
- Haglund, J. and Haiman, M. and Loehr. 2005. *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. **126**, no. 2, 195–232, DOI 10.1215/S0012-7094-04-12621-1.
- Haiman, Mark. 2001. *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. **14**, no. 4, 941–1006, DOI 10.1090/S0894-0347-01-00373-3. MR1839919
- . 2002. *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. **149**, no. 2, 371–407, DOI 10.1007/s002220200219. MR1918676
- Lascoux, Alain, Bernard Leclerc, and Jean-Yves Thibon. 1995. *Ribbon tableaux, Hall-Littlewood functions and unipotent varieties*, Sémin. Lothar. Combin. **34**, Art. B34g, approx. 23. MR1399754
- Loehr, Nicholas A. and Gregory S. Warrington. 2008. *Nested quantum Dyck paths and  $\nabla(s_\lambda)$* , Int. Math. Res. Not. IMRN **5**, Art. ID rnm 157, 29, DOI 10.1093/imrn/rnm157. MR2418288
- Mellit, Anton. 2016. *Toric braids and  $(m, n)$ -parking functions*, arXiv e-prints, arXiv:1604.07456, available at arXiv:1604.07456.
- Negut, Andrei. 2014. *The shuffle algebra revisited*, Int. Math. Res. Not. IMRN **22**, 6242–6275, DOI 10.1093/imrn/rnt156. MR3283004