

**Spin Representation Theory of Symmetric  
Groups and Related Combinatorics  
Notes from a reading course in Fall 2018**

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## 1. Introduction (presented by Jinkui Wan)

When discussing the representation theory of the symmetric group, one considers *linear representations* which are group homomorphisms

$$\mathfrak{S}_n \rightarrow GL(V)$$

In 1911, Schur started considering projective representations

$$\mathfrak{S}_n \rightarrow PGL(V) = GL(V)/\mathbb{C}^*$$

leading to the projective representation theory of  $\mathfrak{S}_n$ . It turns out that this corresponds to the linear representation theory of an extension of  $\mathfrak{S}_n$ , denoted  $\tilde{\mathfrak{S}}_n$  and referred to as the *double cover of the symmetric group*, fitting into the short exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n \rightarrow 1$$

where, if  $\mathbb{Z}/2\mathbb{Z} = \{1, z\}$ , then  $z$  is central in  $\tilde{\mathfrak{S}}_n$ , which gives us that  $z = 1$  or  $z = -1$ .

When  $z = 1$ , we have the representation theory of  $\mathfrak{S}_n$ . When,  $z = -1$ , we have the representation theory of the *spin symmetric group algebra*

$$\mathbb{C}\mathfrak{S}_n^- = \mathbb{C}\mathfrak{S}_n / \langle z + 1 \rangle = \left\langle t_1, \dots, t_n \mid \begin{array}{l} t_i^2 = 1 \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ t_i t_j = -t_j t_i \text{ when } |i - j| > 1 \end{array} \right\rangle$$

which is equipped with a  $\mathbb{Z}/2\mathbb{Z}$ -grading. So, when we discuss spin representations of  $\mathfrak{S}_n$ , we are discussing linear representations of  $\mathbb{C}\mathfrak{S}_n^-$ . Our program to establish these ideas is as follows.

### Part I

- (1) Basics of associative superalgebras
- (2) Connection to Hecke-Clifford (or Sergeev) algebra,  $\mathcal{H}_n$
- (3) Split conjugacy classes in a finite supergroup
- (4) Characteristic map
- (5) Schur- $Q$  functions
- (6) Schur-Sergeev duality
- (7) Seminormal form of irreducible representations

### Part II

- (1) Centers of  $\mathbb{C}\mathfrak{S}_n^-$  (analog of Farahat-Higman theory for  $\mathbb{C}\mathfrak{S}_n$ )
- (2) Coinvariant theory for  $\mathbb{C}\mathfrak{S}_n^-$
- (3) Spin Kostka polynomials
- (4) Quantum deformation (in particular, Olshanki-Sergeev duality)

## 2. Generalities for Associative Superalgebras (presented by Jinkui Wan)

### 2.1. Definitions and Examples.

- 2.1. DEFINITION. (a) A *vector superspace* (over  $\mathbb{C}$ ) is a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space  $V = V_{\bar{0}} \oplus V_{\bar{1}}$ , where elements of  $V_{\bar{0}}$  are called *even* and elements of  $V_{\bar{1}}$  are called *odd*. For  $v \in V_i$ ,  $i \in \mathbb{Z}/2\mathbb{Z}$ , we say  $|v| = i$ .
- (b) If  $V$  is a vector superspace with  $\dim V_{\bar{0}} = m$  and  $\dim V_{\bar{1}} = n$ , we say the *graded dimension* of  $V$  is  $(m, n)$ , denoted  $\mathbf{dim} V = (m, n)$ .
- (c) A *superalgebra* is a  $\mathbb{C}$ -algebra  $A$  with a  $\mathbb{Z}/2\mathbb{Z}$ -grading  $A = A_{\bar{0}} \oplus A_{\bar{1}}$  such that  $A_i A_j \subseteq A_{i+j}$  for all  $i, j \in \mathbb{Z}/2\mathbb{Z}$ .
- (d) A *superalgebra ideal* is a homogeneous ideal, that is, a subset  $I \subseteq A$  such that  $I = I_{\bar{0}} \oplus I_{\bar{1}} = (I \cap A_{\bar{0}}) \oplus (I \cap A_{\bar{1}})$  as vector spaces and  $A_i I_j \subseteq I_{i+j}$  for all  $i, j \in \mathbb{Z}/2\mathbb{Z}$ .
- (e) A superalgebra that has no non-trivial ideals is called *simple*.
- (f) A *superalgebra homomorphism*  $\theta: A \rightarrow B$  is an even algebra homomorphism, that is, an algebra homomorphism sending  $A_i \rightarrow B_i$  for all  $i \in \mathbb{Z}/2\mathbb{Z}$ .
- (g) Given superalgebras  $A$  and  $B$ , the tensor product  $A \otimes B$  is a superalgebra with multiplication

$$(a \otimes b)(a' \otimes b') = (-1)^{|a||b|} aa' \otimes bb'$$

for homogeneous elements and extended by linearity.

- (h) A *commutative superalgebra* is one that is graded commutative, that is

$$yx = (-1)^{|x||y|}xy$$

Thus, the *supercommutator* of a superalgebra is given by

$$[x, y] = xy - (-1)^{|x||y|}yx$$

and the *supercenter* is given by

$$Z(A) = \{a \in A \mid [a, x] = 0 \text{ for all } x \in A\}$$

which is different than the center of an ungraded algebra.

- (i) Given a superalgebra  $A$ , we let  $|A|$  be the associative algebra where we forget the grading on  $A$ .
- 2.2. EXAMPLE. (a) Let  $V = V(m|n)$ , the vector superspace with  $\mathbf{dim} V = (m, n)$ . Then,  $\text{End}_{\mathbb{C}}(V)$  is a superalgebra and is isomorphic to the matrix superalgebra

$$M(m|n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{array}{l} a \text{ is an } m \times m \text{ matrix} \\ b \text{ is an } m \times n \text{ matrix} \\ c \text{ is an } n \times m \text{ matrix} \\ d \text{ is an } n \times n \text{ matrix} \end{array} \right\}$$

or, in other words,  $M(m|n)$  consists of all  $m|n$ -block matrices and has  $\dim M(m|n) = (m^2 + n^2, 2mn)$ . Furthermore,  $M(m|n)$  is a simple superalgebra since  $|M(m|n)|$  is simple as a  $\mathbb{C}$ -algebra.

- (b) Let  $V = V(n|n)$  and  $p \in \text{End}_{\mathbb{C}}(V)$  be an odd involution (that is, it sends  $V_i \rightarrow V_{i+1}$  for  $i \in \mathbb{Z}/2\mathbb{Z}$ ). Then, we define

$$\mathcal{Q}(V) := \{f \in \text{End}_{\mathbb{C}}(V) \mid fp = (-1)^{|f|}pf\} = \mathcal{Q}(V)_{\bar{0}} \oplus \mathcal{Q}(V)_{\bar{1}}$$

$\mathcal{Q}(V)$  is also a superalgebra. Moreover, if we pick a basis  $\{v_1, \dots, v_n\}$  of  $V_{\bar{0}}$  and let  $v'_i = p(v_i)$  for  $1 \leq i \leq n$ , we have that, with respect to the basis  $\{v_1, \dots, v_n, v'_1, \dots, v'_n\}$ ,  $\mathcal{Q}(V)$  is isomorphic to

$$\mathcal{Q}(n) := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M(n|n) \right\}$$

and it is simple.

- (c) The Clifford algebra  $\mathcal{Cl}_n$  is the superalgebra generated by the odd elements  $c_1, \dots, c_n$  subject to the relations

$$\begin{cases} c_i^2 = 1 \\ c_i c_j = -c_j c_i \quad \forall 1 \leq i \neq j \leq n \end{cases}$$

2.3. LEMMA. *There exist isomorphisms of superalgebras*

- (a)  $M(m|n) \otimes M(k|l) \cong M(mk + nl|mk + nl)$
- (b)  $M(m|n) \otimes \mathcal{Q}(k) \cong \mathcal{Q}((m+n)k)$
- (c)  $\mathcal{Q}(m) \otimes \mathcal{Q}(n) \cong M(mn|mn)$

PROOF. For part (a), we note that

$$\text{End}_{\mathbb{C}}(V(m|n)) \otimes \text{End}_{\mathbb{C}}(V(k|l)) \cong \text{End}_{\mathbb{C}}(V(mk + ml|mk + nl))$$

under the isomorphism sending  $f \otimes g$  to the endomorphism of  $V(mk + ml|mk + nl)$  mapping  $v \otimes w$  to  $(-1)^{|g||v|}f(v) \otimes g(w)$ .

For part (b), we have

$$\text{End}(V(m|n)) \otimes \mathcal{Q}(V(k|k), p) \cong \mathcal{Q}(V(m|n) \otimes V(k|k), id \otimes p)$$

For (c), one explicitly checks that  $\mathcal{Q}(1) \otimes \mathcal{Q}(1) \cong M(1|1)$  and then inductively applies (a) and (b) above.  $\square$

2.4. COROLLARY. *Since  $\mathcal{Cl}_{m+n} \cong \mathcal{Cl}_m \otimes \mathcal{Cl}_n$  under the isomorphism sending generators  $c_1, \dots, c_n$  to  $c_1 \otimes 1, \dots, c_n \otimes 1$  and  $c_{n+1}, \dots, c_{n+m}$  to  $1 \otimes c_1, \dots, 1 \otimes c_m$ , we have the corollaries*

- (a)  $\mathcal{Cl}_1 \cong \mathcal{Q}(1)$  under the isomorphism  $c_1 \mapsto p(v_1)$
- (b)  $\mathcal{Cl}_2 \cong M(1|1)$  since  $\mathcal{Cl}_2 \cong \mathcal{Cl}_1 \otimes \mathcal{Cl}_1 \cong \mathcal{Q}(1) \otimes \mathcal{Q}(1) \cong M(1|1)$
- (c)  $\mathcal{Cl}_{2^k} \cong M(2^{k-1}|2^{k-1})$
- (d)  $\mathcal{Cl}_{2^{k-1}} \cong \mathcal{Q}(2^{k-1})$
- (e) and thus,  $\mathcal{Cl}_n$  is simple by parts (c) and (d).

## 2.2. Classification of Simple Superalgebras.

2.5. THEOREM. *There are two types of finite dimensional simple associative superalgebras over  $\mathbb{C}$ :*

- (a)  $M(m|n)$
- (b)  $\mathcal{Q}(n)$

## 2.3. Wedderburn Theorem and Schur's Lemma.

- 2.6. DEFINITION. (a) A *(super)module* over a superalgebra  $A$  is a vector space  $M = M_{\bar{0}} \oplus M_{\bar{1}}$  with a left action of  $A$  on  $M$  such that  $A_i M_j \subseteq M_{i+j}$  for all  $i, j \in \mathbb{Z}/2\mathbb{Z}$ .
- (b) A homomorphism between  $A$ -modules  $M$  and  $N$  is a linear map  $f: M \rightarrow N$  such that

$$f(am) = (-1)^{|f||a|} a f(m) \text{ for all } a \in A, m \in M$$

and

$$\text{Hom}_A(M, N) := \text{Hom}_A(M, N)_{\bar{0}} \oplus \text{Hom}_A(M, N)_{\bar{1}}$$

where  $f \in \text{Hom}_A(M, N)_{\bar{1}} \subseteq \text{Hom}_{\mathbb{C}}(M, N)$  is such that  $f(M_i) \subseteq N_{i+1}$  for all  $i \in \mathbb{Z}/2\mathbb{Z}$ .

2.7. DEFINITION. An  $A$ -module is said to be *simple* if it is nonzero and has no proper  $A$ -submodules. An  $A$ -module  $M$  is said to be *semisimple* if every  $A$ -submodule of  $M$  is a direct summand of  $M$ .

2.8. THEOREM (Super Wedderburn Theorem). *The following are equivalent for a finite dimensional superalgebra  $A$ .*

- (a) *Every  $A$ -module is semisimple*
- (b)  *$A$  is a finite direct sum of left simple superideals*
- (c)  *$A$  is a direct product of a finite number of simple algebras*

2.9. DEFINITION. Thus, we say a superalgebra  $A$  is *semisimple* if it satisfies one of the three conditions.

- 2.10. EXAMPLE. (a)  $M(m|n) = I_1 \oplus I_2 \oplus \cdots \oplus I_m \oplus I_{m+1} \oplus \cdots \oplus I_{m+n}$  where  $I_k = M(m|n)E_{k,k}$  for  $1 \leq k \leq m+n$ .
- (b)  $\mathcal{Q}(n) = J_1 \oplus \cdots \oplus J_n$  where

$$J_k = \mathcal{Q}(n)(E_{k,k} + E_{n+k,n+k})$$

- (c)  $\text{Hom}_{M(m|n)}(I_k, I_k) \cong \mathbb{C}$  and  $\text{Hom}_{\mathcal{Q}(n)}(J_k, J_k) \cong \mathbb{C} \oplus \mathbb{C}p$ . Importantly, the latter space is not 1-dimensional despite  $J_k$  being 1-dimensional!

2.11. COROLLARY. *A finite dimensional semisimple superalgebra  $A$  is isomorphic to*

$$A \cong \bigoplus_{i=1}^m M(r_i|s_i) \oplus \bigoplus_{j=1}^n \mathcal{Q}(n_j)$$

Check this.  
Most likely depends on your choice of basis and involution.

where  $m = m(A)$  and  $q = q(A)$  are invariants of  $A$ .

2.12. DEFINITION. A simple  $A$ -module  $V$  is said to be of type  $M$  (resp. type  $Q$ ) if it is annihilated by all but one summand of the form  $M(r_i|s_i)$  (resp.  $Q(n_j)$ ).

2.13. COROLLARY. (a) The number of non-isomorphic simple  $A$ -modules is given by  $m(A) + q(A) = \dim(Z(|A|) \cap A_{\bar{0}})$ .

(b) The number of non-isomorphic simple  $A$ -modules of type  $Q$  is given by  $q(A) = \dim(Z(|A|) \cap A_{\bar{1}})$ .

2.14. THEOREM (Schur's Lemma). If  $M$  and  $L$  are simple  $A$ -modules, then

$$\dim \operatorname{Hom}_A(M, L) = \begin{cases} 1 & \text{if } M \cong L \text{ of type } M \\ 2 & \text{if } M \cong L \text{ of type } Q \\ 0 & \text{otherwise} \end{cases}$$

2.15. REMARK. (a) A simple  $A$ -module  $M$  is of type  $M$  if and only if  $|M|$  is a simple  $|A|$ -module

(b) A simple  $A$ -module  $M$  is of type  $Q$  if and only if  $|M|$  is a direct sum of two non-isomorphic simple  $|A|$ -modules.

### 3. Split Conjugacy Classes in a Finite Supergroup (presented by Jinkui Wan)

Throughout, let  $G$  be a finite group with index 2 subgroup  $G_0 \leq G$ .

3.1. DEFINITION. (a) We say that the elements of  $G_0$  are *even elements* and the elements of  $G_1 := G \setminus G_0$  are *odd elements*,

(b)  $\mathbb{C}G$  is a superalgebra, which we will denote  $\mathbb{C}[G, G_0]$

3.2. THEOREM (Super MASchke's Theorem).  $\mathbb{C}[G, G_0]$  is semisimple

3.3. PROPOSITION. (a) If  $g \in G_i$ ,  $h \in G$ , then  $hgh^{-1} \in G_i$  for all  $i \in \mathbb{Z}/2\mathbb{Z}$ .

(b) The number of non-isomorphic simple  $\mathbb{C}[G, G_0]$ -modules is equal to the number of even conjugacy classes in  $G$ .

(c) The number of non-isomorphic simple  $\mathbb{C}[G, G_0]$ -module of type  $Q$  is equal to the number of odd conjugacy classes in  $G$ .

PROOF. We note that, by the usual Artin-Wedderburn theorem,  $\mathbb{C}G$  decomposes into a direct sum of simple matrix algebras, each of which has a 1-dimensional center and can be indexed by a conjugacy class of  $G$  via  $c_i = \sum_{g \in \mathcal{C}_i} g$  where  $\mathcal{C}_i$  is a conjugacy class of  $G$ . In fact, this shows in the classical theory that the number of conjugacy classes of  $G$  equal the number of irreducible representations.

Since conjugacy classes are either even or odd by (a), which is left as an exercise, (b) follows because  $\dim(Z(\mathbb{C}G) \cap \mathbb{C}[G, G_0]_{\bar{0}})$  is equal to the number of non-isomorphic simple  $\mathbb{C}[G, G_0]$ -modules and (c) follow from the fact that  $\dim(Z(\mathbb{C}G) \cap \mathbb{C}[G, G_0]_{\bar{1}})$  gives the number of those of type  $Q$ .  $\square$

Now, consider the following situation. Let  $\tilde{G}$  be a group such that there exists an index 2 subgroup  $\tilde{G}_0 \leq \tilde{G}$  and there exists a short exact sequence

$$1 \rightarrow \{1, z\} \rightarrow \tilde{G} \xrightarrow{\theta} G \rightarrow 1$$

where  $z^2 = 1$  and  $z$  is central in  $\tilde{G}$ . Then,

3.4. PROPOSITION. *For  $C$  a conjugacy class of  $G$ , the preimage*

$$\theta^{-1}(C) = \{g, gz \mid g \in C\} \subseteq \tilde{G}$$

*has that*

- (a)  $\theta^{-1}(C)$  is a single conjugacy class in  $\tilde{G}$  if  $g$  is conjugate to  $zg$  in  $\tilde{G}$   
or
- (b)  $\theta^{-1}(C)$  splits into two conjugacy classes in  $\tilde{G}$  if there exists a  $g \in C$  such that  $g$  is not conjugate to  $zg$ . In this case, we call  $C$  split.

3.5. DEFINITION. We set

$$\mathbb{C}\tilde{G}^- := \mathbb{C}[G, G_0] / \langle z + 1 \rangle$$

and call a  $\mathbb{C}\tilde{G}^-$ -module a *spin*  $\mathbb{C}[\tilde{G}, \tilde{G}_0]$ -module.

3.6. PROPOSITION. (a) *We have the isomorphism of superalgebras*

$$\mathbb{C}[\tilde{G}, \tilde{G}_0] \cong \underbrace{\mathbb{C}[G, G_0]}_{(z=1)} \oplus \underbrace{\mathbb{C}\tilde{G}^-}_{(z=-1)}$$

- (b) *The number of non-isomorphic simple spin  $\mathbb{C}[\tilde{G}, \tilde{G}_0]$ -modules is equal to the number of even split conjugacy classes of  $G$ .*
- (c) *The number of non-isomorphic simple spin  $\mathbb{C}[\tilde{G}, \tilde{G}_0]$ -modules of type  $Q$  is equal to the number of odd split conjugacy classes of  $G$ .*

PROOF. Part (a) follows from the semisimplicity of  $\mathbb{C}[\tilde{G}, \tilde{G}_0]$ .

Now, (a) tells us that

$$Z(|\mathbb{C}\tilde{G}^-|) = \{a \in Z(|\mathbb{C}[\tilde{G}, \tilde{G}_0]|) \mid za = -a\}$$

So let

$$\underbrace{D_1, zD_1, D_2, zD_2, \dots, D_r, zD_r}_{\text{split}}, \underbrace{D_{r+1}, \dots, D_{r+s}}_{\text{non-split}}$$

be the conjugacy classes of  $\tilde{G}$  where  $r$  is the number of split conjugacy classes in  $G$ ,  $D_i \cap zD_i = \emptyset$  for  $1 \leq i \leq r$  and  $zD_j = D_j$  for  $r+1 \leq j \leq r+s$ . Then,

$$Z(|\mathbb{C}\tilde{G}|) \cap \mathbb{C}\tilde{G}_0 = \{a \in Z(|\mathbb{C}\tilde{G}|) \mid a \text{ is even and } za = -a\}$$

has basis  $d_{i_1} - zd_{i_1}, d_{i_2} - zd_{i_2}, \dots, d_{i_k} - zd_{i_k}$  for  $d_{i_i}$  even and

$$Z(|\mathbb{C}\tilde{G}|) \cap \mathbb{C}\tilde{G}_1 = \{a \in Z(|\mathbb{C}\tilde{G}|) \mid a \text{ is odd and } za = -a\}$$

has basis  $d_{j_1} - zd_{j_1}, d_{j_2} - zd_{j_2}, \dots, d_{j_\ell} - zd_{j_\ell}$  for  $d_{j_j}$  odd.  $\square$

Check this part

3.7. EXAMPLE. We have

$$1 \rightarrow \{1, z\} \rightarrow \tilde{\mathfrak{S}}_n \xrightarrow{\theta_n} \mathfrak{S}_n \rightarrow 1$$

where  $z \in \tilde{\mathfrak{S}}_n$  is even and central, the subgroup of index 2 is  $\tilde{A}_n$ , and

$$\mathbb{C}\mathfrak{S}_n^- := \mathbb{C}\tilde{\mathfrak{S}}_n / \langle z + 1 \rangle$$

is the spin symmetric group algebra.

3.8. DEFINITION. Throughout the remainder of these notes, we define  $\theta_n: \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n$  to be the double covering map above.

#### 4. A Morita Superequivalence (presented by Jinkui Wan)

Since  $\mathfrak{S}_n$  acts on  $\mathcal{C}\ell_n$  via  $\sigma.c_i = c_{\sigma(i)}$ , we can define the semidirect product  $\mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$  with multiplication

$$(x, \sigma)(y, \tau) = (x\sigma(y), \sigma\tau)$$

4.1. DEFINITION. We define the *Hecke-Clifford superalgebra* as

$$\mathcal{H} := \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$$

with the  $c_i$  having odd parity and the  $s_j$  having even parity.

4.2. LEMMA. *There exists a superalgebra isomorphism*

$$\begin{aligned} \mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}\ell_n &\xrightarrow{\sim} \mathcal{H}_n \\ c_i &\mapsto c_i \\ t_j &\mapsto \frac{1}{\sqrt{-2}} s_j (c_j - c_{j+1}) \end{aligned}$$

Check this last map.

Recall that  $\mathcal{C}\ell_n$  is a simple superalgebra and it has a unique simple module  $U_n$ . If  $n$  is even,  $U_n$  is of type M and if  $n$  is odd,  $U_n$  is of type Q. This leads us to define two functors

$$\begin{aligned} F_n &:= - \otimes U_n: \mathbb{C}\mathfrak{S}_n\text{-Mod} \rightarrow \mathcal{H}_n\text{-Mod} \\ G_n &:= \text{Hom}_{\mathcal{C}\ell_n}(U_n, -): \mathcal{H}_n\text{-Mod} \rightarrow \mathbb{C}\mathfrak{S}_n^-\text{-Mod} \end{aligned}$$

4.3. LEMMA. [Kle05, Prop 13.2.2]

- (a) If  $n$  is even, then  $F_n \circ G_n \cong \text{id}$  and  $G_n \circ F_n \cong \text{id}$ .
- (b) If  $n$  is odd, then  $F_n \circ G_n \cong \text{id} \oplus \pi$  and  $G_n \circ F_n \cong \text{id} \oplus \pi$  where  $\pi(M)_i = M_{i+1}$  for all  $i \in \mathbb{Z}/2\mathbb{Z}$ .

Thus, because  $(F_n \circ G_n)(M) = \text{Hom}_{\mathcal{C}\ell_n}(U_n, M) \otimes U_n$ , we have a (super)Morita equivalence between  $\mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}\ell_n$  and  $\mathcal{H}_n$ .



## 5. A Double Cover $\tilde{B}_n$ (presented by Jinkui Wan)

Recall that  $B_n = \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$ . We define

5.1. DEFINITION.

$$\Pi_n := \left\langle z, a_1, \dots, a_n \mid \begin{array}{l} z^2 = a_i^2 = 1, \forall 1 \leq i \leq n \\ a_i a_j = z a_j a_i, i \neq j \end{array} \right\rangle$$

Then,  $\mathfrak{S}_n$  acts on  $\Pi_n$  via  $\sigma(z) = z$  and  $\sigma(a_i) = a_{\sigma(i)}$ . This gives us the short exact sequence

$$1 \rightarrow \{1, z\} \rightarrow \Pi_n \rtimes \mathfrak{S}_n \rightarrow B_n \rightarrow 1$$

$$a_i \rightarrow b_i$$

and so we define

The flow here is not great.

5.2. DEFINITION. Let  $\tilde{B}_n$  be the supergroup on  $\Pi_n \rtimes \mathfrak{S}_n$  with the  $a_i$  odd,  $z$  even, and  $\sigma \in \mathfrak{S}_n$  even.

Since  $\mathbb{C}\tilde{B}_n / \langle z + 1 \rangle = \mathcal{H}_n$ , we wish to understand conjugacy classes in  $B_n$ . We will do so by example.

5.3. EXAMPLE. Consider

$$x = ((++++-+-), (1234)(567)(89)) \in B_{10}$$

As an element of  $\mathfrak{S}_{10}$ ,  $(1234)(567)(89)$  has cycle type  $(4, 3, 2, 1)$ , but we wish to assign a parity to each of these cycles. To do so, we look at the  $(+, -)$ -array in the first coordinate and take the product of the entries corresponding to the cycle. So,  $(1234)$  gets cycle type  $+\times+\times+\times- = -$  since those are entries 1, 2, 3, and 4 in the array. This gives the cycle type as a tuple of partitions  $\rho = (\rho^+, \rho^-)$  and so  $\rho(x) = ((3), (4, 2, 1))$ . Similarly, if

$$y = ((+----+----+-), (1386)(279)(45)) \in B_{10}$$

then the first cycle has parity  $+\times-\times-\times- = -$  since those are the 1, 3, 6, and 8 entries of the array. One can check that  $y$  has the same cycle type as  $x$ .

5.4. LEMMA. *Two elements of  $B_n$  are conjugate if and only if their cycle types are the same.*

5.5. COROLLARY. *The number of conjugacy classes in  $B_n$  is*

$$\#\{(\rho^+, \rho^-) \mid |\rho^+| + |\rho^-| = n\}$$

Now, the conjugacy class  $\mathcal{C}_{\rho^+, \rho^-}$  is even if  $k$  is even for  $\underbrace{b_{i_1} b_{i_2} \dots b_{i_k}}_{\in \mathbb{Z}_2^n} \sigma \in$

$\mathcal{C}_{\rho^+, \rho^-}$ .

5.6. THEOREM (Read). [CW12, Theorem 3.31]

- (a) Even  $\mathcal{C}_{\rho^+, \rho^-}$  splits if and only if  $\rho^+ \in \mathcal{OP}_n$  and  $\rho^- = \emptyset$
- (b) Odd  $\mathcal{C}_{\rho^+, \rho^-}$  splits if and only if  $\rho^+ = \emptyset$  and  $\rho^- \in \mathcal{SP}_n^-$

where  $\mathcal{SP}_n^-$  is all partitions of  $n$  with strict parts and odd length.

5.7. DEFINITION. For  $\alpha \in \mathcal{OP}_n$ , let  $\mathcal{C}_\alpha^+$  be the split conjugacy class in  $\tilde{B}_n$  satisfying

- (a)  $\mathcal{C}_\alpha^+ = \theta_n^{-1}(\mathcal{C}_{\alpha, \emptyset})$
- (b) There exists  $\sigma \in \mathcal{C}_\alpha^+$  such that  $\sigma \in \mathfrak{S}_n$  with cycle type  $\alpha$ .

## 6. A ring structure on $R^-$ (presented by Jinkui Wan)

6.1. DEFINITION. We give the following definition

- (a) Let  $R_n^- := [\mathcal{H}_n\text{-}\mathbf{Mod}]$ , the Grothendieck group of  $\mathcal{H}_n\text{-}\mathbf{Mod}$ .
- (b) Let  $R^- := \bigoplus_{n=0}^\infty R_n^-$  where  $R_0^- = \mathbb{Z}$
- (c) Let  $R_{\mathbb{Q}}^- = \mathbb{Q} \otimes_{\mathbb{Z}} R^-$
- (d) Let  $\mathcal{H}_{m,n}$  be the subalgebra of  $\mathcal{H}_{m+n}$  generated by  $\mathcal{C}\ell_{m+n}$  and  $S_m \times S_n$ . Note that  $\mathcal{H}_{m,n} \cong \mathcal{H}_m \otimes \mathcal{H}_n$  as a superalgebra.
- (e) Given  $M \in \mathcal{H}_m\text{-}\mathbf{Mod}$  and  $N \in \mathcal{H}_n\text{-}\mathbf{Mod}$ , we define

$$[M] \cdot [N] := [\mathrm{Ind}_{\mathcal{H}_m \otimes \mathcal{H}_n}^{\mathcal{H}_{m+n}} M \otimes N]$$

6.2. PROPOSITION.  $R^-$  is commutative with respect to the above multiplication.

6.3. DEFINITION. Define a bilinear form via

$$\langle [M], [N] \rangle := \dim \mathrm{Hom}_{\mathcal{H}_n}(M, N)$$

for  $M, N \in \mathcal{H}_n$

6.4. LEMMA. For  $\phi \in R_n^-$  (viewed as a character of  $\tilde{B}_n$ ), set  $\phi_\alpha := \phi(x)$  for any  $x \in \mathcal{C}_\alpha$ . Then,

- (a) For  $\phi \in R_m^-, \psi \in R_n^-$ , and  $\gamma \in \mathcal{OP}_{m+n}$ ,

$$(\phi \cdot \psi)_\gamma = \sum_{\substack{\alpha \in \mathcal{OP}_m, \beta \in \mathcal{OP}_n \\ \alpha \cup \beta = \gamma}} \frac{z_\gamma}{z_\alpha z_\beta} \phi_\alpha \psi_\beta$$

where  $z_\alpha$  is the order of the centralizer of  $\sigma$  of cycle type  $\alpha$  in  $\mathfrak{S}_n$ .

- (b)

$$\langle \phi, \psi \rangle = \sum_{\alpha \in \mathcal{OP}} 2^{-\ell(\alpha)} z_\alpha^{-1} \phi_\alpha \psi_\alpha$$

6.5. PROPOSITION. The character value vanishes unless you are in an even split conjugacy class.

## 7. The ring $\Gamma$ (presented by Jinkui Wan)

7.1. DEFINITION. Let  $x = \{x_1, x_2, \dots\}$ .

- (a) Define  $q_r = q_r(x)$  via the generating function

$$Q(t) = \sum_{r \geq 0} q_r(x) t^r = \prod_{i \geq 1} \frac{1 + tx_i}{1 - tx_i}$$

(b) Let  $\Gamma$  be the  $\mathbb{Z}$ -subring of the ring of symmetric functions generated by  $q_r$ ,  $r \geq 0$ .

(c)  $\Gamma_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$

7.2. PROPOSITION. (a)  $\sum_{r+s=n} (-1)^r q_r q_s = 0$  because  $Q(t)Q(-t) = 1$

(b)  $q_n = \sum_{\alpha \in \mathcal{OP}_n} 2^{\ell(\alpha)} z_{\alpha}^{-1} p_{\alpha}$  where  $p_{\alpha} = P_{\alpha_1} \cdots P_{\alpha_{\ell}}$  because  $\ln Q(t) = \sum_{r \text{ odd}} \frac{2p_r(x)t^r}{r}$ .

7.3. THEOREM. (a)  $\Gamma_{\mathbb{Q}}$  is a polynomial algebra with polynomial generators  $p_{2r-1}$  for  $r \geq 1$ .

(b)  $\{p_{\mu} \mid \mu \in \mathcal{OP}\}$  is a basis for  $\Gamma_{\mathbb{Q}}$

7.4. DEFINITION. Let us define inner product on  $\Gamma_{\mathbb{Q}}$  via

$$\langle p_{\alpha}, p_{\beta} \rangle := 2^{-\ell(\alpha)} z_{\alpha} \delta_{\alpha\beta}, \forall \alpha, \beta \in \mathcal{OP}$$

7.5. DEFINITION. Define the (spin) characteristic map to be

$$\begin{aligned} \text{ch}^{-} : R_{\mathbb{Q}}^{-} &\rightarrow \Gamma_{\mathbb{Q}} \\ \phi &\mapsto \sum_{\alpha \in \mathcal{OP}_n} z_{\alpha}^{-1} \phi_{\alpha} p_{\alpha} \end{aligned}$$

7.6. PROPOSITION. (a)  $\text{ch}^{-}$  is an algebra isomorphism.

(b)  $\text{ch}^{-}$  is an isometry (that is,  $\langle \phi, \psi \rangle = \langle \text{ch}^{-}(\phi), \text{ch}^{-}(\psi) \rangle$ ).

Now, we seek to construct the basic spin module.

7.7. PROPOSITION.  $\mathcal{H}_n = \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$  acts on  $\mathcal{C}\ell_n = \text{span}\{c_I \mid I \subseteq \{1, 2, \dots, n\}\}$  via

$$\begin{cases} c_i \cdot (c_{i_1} \cdots c_{i_k}) = c_i c_{i_1} \cdots c_{i_k} \\ \sigma \cdot (c_{i_1} \cdots c_{i_k}) = c_{\sigma(i_1)} \cdots c_{\sigma(i_k)} \end{cases}$$

where  $c_i \in \mathcal{C}\ell_n$ ,  $\sigma \in \mathfrak{S}_n$ , and the action is extended by linearity.

7.8. PROPOSITION. Let  $\sigma = \sigma_1 \cdots \sigma_{\ell}$  be a cycle decomposition of  $\sigma$ . Then

$$\sigma c_I = \begin{cases} \pm c_I & \text{if } I \text{ is a union of some supports of } \sigma_1, \dots, \sigma_{\ell} \\ \pm c_J (J \neq I) & \text{otherwise} \end{cases}$$

7.9. EXAMPLE. Let  $\sigma = (134)(25) \in \mathfrak{S}_5$ . Then,

$$\sigma c_3 c_5 = c_4 c_2$$

but

$$\sigma c_1 c_3 c_4 = c_3 c_4 c_1 = c_1 c_3 c_4$$

Thus, the character of this action, say  $\xi^n$ , satisfies

$$\xi^n(\alpha) = 2^{\ell(\alpha)}, \alpha \in \mathcal{OP}_n$$

and thus  $\text{ch}^{-}(\xi^n) = \sum_{\alpha \in \mathcal{OP}_n} z_{\alpha}^{-1} 2^{\ell(\alpha)} p_{\alpha} = q_n$ .

7.10. DEFINITION. For  $\lambda \in \mathcal{SP}$ , we define  $\xi^\lambda$  via the recursive formulas

$$\xi^{(\lambda_1, \lambda_2)} = \xi^{\lambda_1} \xi^{\lambda_2} + 2 \sum_{i=1}^{\lambda_2} (-1)^i \xi^{\lambda_1+i} \xi^{\lambda_2-i}$$

$$\xi^\lambda = \begin{cases} \sum_{j=2}^k (-1)^j \xi^{(\lambda_1, \lambda_j)} \xi^{(\lambda_2, \dots, \hat{\lambda}_j, \dots, \lambda_k)} & k = \ell(\lambda) \text{ is even} \\ \sum_{j=1}^k (-1)^{j-1} \xi^{\lambda_j} \xi^{(\lambda_1, \dots, \hat{\lambda}_j, \dots, \lambda_k)} & k = \ell(\lambda) \text{ is odd} \end{cases}$$

- 7.11. THEOREM. (a)  $\text{ch}^-(\xi^\lambda) = Q_\lambda$ , the Schur- $Q$  function (to be defined in the next lecture).  
(b)  $\left\{ \zeta^\lambda := 2^{-\frac{\ell(\lambda) - \delta(\lambda)}{2}} \xi^\lambda \mid \lambda \in \mathcal{SP}_n \right\}$ , where  $\delta(\lambda) = \chi\{\ell(\lambda) \text{ is odd}\}$ , is a complete list of simple characters.  
(c)  $\zeta^\lambda$  is of type  $M$  if  $\ell(\lambda)$  is even and of type  $Q$  if  $\ell(\lambda)$  is odd.  
(d) The degree of  $\zeta^\lambda$  is

$$2^{n - \frac{\ell(\lambda) - \delta(\lambda)}{2}} \frac{n!}{\lambda_1! \cdots \lambda_\ell!} \left( \prod_{i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right)$$

## 8. Schur- $Q$ functions and related combinatorics (presented by George H. Seelinger)

## 9. Center of Symmetric Group Algebras and Spin Symmetric Group Algebras (presented by Jinkui Wan)

9.1. Farahat-Higman's Construction for  $\mathfrak{S}_n$ . Given a permutation  $\sigma$ , we note that its cycle type is not stable under inclusion from  $\mathfrak{S}_n \hookrightarrow \mathfrak{S}_{n+1}$ .

9.1. EXAMPLE. Let  $\sigma = (134)(2576) \in \mathfrak{S}_8$ . Then,  $\sigma$  has cycle type

$$(4, 3, 1) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}$$

but, when included into  $\mathfrak{S}_9$ ,  $\sigma$  has cycle type

$$(4, 3, 1, 1) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array}$$

9.2. DEFINITION. Given a cycle  $\sigma \in \mathfrak{S}_n$ , its *modified cycle type*,  $\lambda$ , is given by removing the first column from its cycle type.

9.3. EXAMPLE. The modified cycle type of  $\sigma = (134)(2576) \in \mathfrak{S}_8$  is  $\lambda = (3, 2)$ . Note that this is stable with respect to  $\mathfrak{S}_8 \hookrightarrow \mathfrak{S}_9$ .

9.4. PROPOSITION. (a) If  $\sigma$  is of modified type  $\lambda$ , then  $|\lambda|$  is the minimal length for  $\sigma$  as a product of (not necessarily simple) transpositions.

(b) If

$$\begin{cases} \sigma \text{ is of modified type } \lambda \\ \tau \text{ is of modified type } \mu \\ \sigma\tau \text{ is of modified type } \nu \end{cases}, \text{ then } |\nu| \leq |\lambda| + |\mu|$$

9.5. EXAMPLE.  $(134) = (13)(34)$  and has modified type (2).

9.6. DEFINITION. (a) Let  $\mathcal{C}_\lambda(n)$  be the conjugacy class of  $\mathfrak{S}_n$  of modified type  $\lambda$ . Note  $\mathcal{C}_\lambda(n) = \emptyset$  if  $n < |\lambda| + \ell(\lambda)$ .

(b) Let

$$C_\lambda(n) := \begin{cases} \text{Class sum of } \mathcal{C}_\lambda(n) & \text{if } n \geq |\lambda| + \ell(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

9.7. EXAMPLE.  $\mathcal{C}_0(n) = \{id\}$  and  $\mathcal{C}_{(1)}(n)$  contains all transpositions of  $\mathfrak{S}_n$ . Thus,  $C_{(1)}(n) = \sum_{1 \leq i < j \leq n} (ij)$ .

9.8. PROPOSITION.  $\{C_\lambda(n) \mid |\lambda| + \ell(\lambda) \leq n\}$  is a basis for  $Z(\mathbb{Z}\mathfrak{S}_n)$ .

9.9. DEFINITION. Write

$$C_\lambda(n)C_\mu(n) = \sum A_{\lambda\mu}^\nu(n)C_\nu(n)$$

9.10. EXAMPLE.

$$C_{(1)}(n)C_{(1)}(n) = 3C_{(2)}(n) + 2C_{(1,1)}(n) + \frac{1}{2}n(n-1)C_0(n)$$

since  $C_{(1)}(n)^2 = \sum (ij)(kl)$  for all transpositions  $(ij), (kl)$  in  $\mathfrak{S}_n$ .

9.11. THEOREM (Farahat-Higman). Let  $\lambda, \mu, \nu$  be partitions. Then,

- (a) There is a unique polynomial  $f_{\lambda\mu}^\nu(x) \in \mathbb{Q}[x]$  such that  $a_{\lambda\mu}^\nu(n) = f_{\lambda\mu}^\nu(n)$  for all  $n \geq |\nu| + \ell(\nu)$ .
- (b)  $f_{\lambda\mu}^\nu(x) = 0$  unless  $|\nu| \leq |\lambda| + |\mu|$
- (c) If  $|\nu| = |\lambda| + |\mu|$ , then  $f_{\lambda\mu}^\nu(x)$  is a constant. In other words,  $a_{\lambda\mu}^\nu(n)$  is independent of  $n$ .

PROOF IDEA. Let  $\Gamma = \{(\sigma, \tau) \mid \sigma \in \mathcal{C}_\lambda(n), \tau \in \mathcal{C}_\mu(n), \sigma\tau \in \mathcal{C}_\nu(n)\}$ . Then, once computes

$$a_{\lambda\mu}^\nu(n) = \frac{\#\Gamma}{\#\mathcal{C}_\lambda(n)}$$

If we let  $\mathfrak{S}_n$  act on  $\Gamma$  by conjugation, that is  $\gamma \cdot (\sigma, \tau) = (\gamma\sigma\gamma^{-1}, \gamma\tau\gamma^{-1})$ , then we get that

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$$

Without loss of generality, take  $(\sigma_1, \tau_1) \in \Gamma_1$ . Then,

$$\#\Gamma_1 = \frac{n!}{\# \text{ of centralizers of } (\sigma_1, \tau_1) \text{ in } \mathfrak{S}_n}$$

Suppose  $\gamma \cdot (\sigma_1, \tau_1) = (\sigma_1, \tau_1)$ . Then,

$$\gamma \sigma_1 \gamma^{-1} = \sigma_1 \text{ and } \gamma \tau_1 \gamma^{-1} = \tau_1$$

Thus,

$$\gamma \in \mathfrak{S}_{\text{Supp}(\sigma_1, \tau_1)} \times \mathfrak{S}_{\{1, \dots, n\} \setminus \text{Supp}(\sigma_1, \tau_1)}$$

where  $\text{Supp}(\sigma_1, \tau_1) = \{j \in \{1, \dots, n\} \mid \sigma_1(j) \neq j \text{ or } \tau_1(j) \neq j\}$ . So,

$\# \text{ centralizers of } (\sigma_1, \tau_1) \text{ in } \mathfrak{S}_n = \# \text{ centralizers of } (\sigma_1, \tau_1) \text{ in } \mathfrak{S}_{\text{Supp}(\sigma_1, \tau_1)} \times (n - \# \text{Supp}(\sigma_1, \tau_1))!$

Thus, using our formula above for  $a_{\lambda\mu}^\nu$ , we arrive at

Finish this formula

$$a_{\lambda\mu}^\nu(n) = \frac{\sum_i \#\Gamma_i}{\#\mathcal{C}_\lambda(n)}$$

□

9.12. DEFINITION. (a) Let  $\mathbb{B}$  be the ring of polynomials  $f(x) \in \mathcal{Q}[x]$  such that  $f(n) \in \mathbb{Z}$  for all  $n \in \mathbb{Z}$ .

(b) Let  $\mathcal{K}$  be the  $\mathbb{B}$ -algebra with basis  $\{c_\lambda \mid \lambda \in \mathcal{P}\}$  such that

$$c_\lambda c_\mu := \sum_{\nu \in \mathcal{P}} f_{\lambda\mu}^\nu(x) c_\nu$$

We call this ring the *Farahat-Higman ring*.

9.13. PROPOSITION. *We have the following facts*

(a)  $\mathcal{K}$  is commutative and associative.

(b)  $\mathcal{K}$  is filtered via  $\deg(c_\lambda) = |\lambda|$  for all  $\lambda \in \mathcal{P}$ .

9.14. REMARK.  $\mathcal{K}$  is not graded because  $\sum_{\nu \in \mathcal{P}} f_{\lambda\mu}^\nu(x) c_\nu$  is not homogeneous. However, if we say  $\mathcal{K}_r = \text{span}\{c_\lambda \mid |\lambda| \leq r\}$ , then  $\mathcal{K}_r \mathcal{K}_s \subseteq \mathcal{K}_{r+s}$ , making  $\mathcal{K}$  filtered.

9.15. DEFINITION. Let  $\text{gr } \mathcal{K}$  be the associated graded algebra, that is,  $\text{gr } \mathcal{K}$  is defined by  $(\text{gr } \mathcal{K})_r = \mathcal{K}_r / \mathcal{K}_{r-1}$  and then  $\text{gr } \mathcal{K} = \bigoplus_{r \geq 0} (\text{gr } \mathcal{K})_r$ .

9.16. LEMMA. *We have the following facts.*

(a) If  $|\lambda| + s = m$ , then

$$a_{\lambda, (s)}^{(m)} = \begin{cases} \frac{(m+1)s!}{\prod_{i \geq 0} m_i(\lambda)!} & \text{if } \ell(\lambda) \leq s+1 \\ 0 & \text{otherwise} \end{cases}$$

where  $m_0(\lambda) = r+1 - \ell(\lambda)$ .

(b) If  $|\lambda| + s = |\nu|$ , then

$$a_{\lambda, (s)}^\nu = \sum_{\substack{(i, \mu) \in \mathbb{N} \times \mathcal{P} \\ 1 \leq i \leq \ell(\lambda) \\ \mu \cup \nu = \lambda \cup (\nu_i)}} a_{\mu, (s)}^{(\nu_i)}$$

9.17. PROPOSITION. Let  $\lambda$  be a partition and let  $m_i(\lambda)$  be the number of  $i$ 's in  $\lambda$  such that  $\lambda = (i^{m_i(\lambda)})_{i \geq 1}$ . Then, the top degree of  $c_\lambda c_{(s)}$  is given by

$$(c_\lambda c_{(s)})^* = \sum_{|\nu|=|\lambda|+s} a_{\lambda,(s)}^\nu c_\nu = \sum_{\mu \subseteq \lambda, \ell(\mu) \leq s+1} \frac{(m_{s+|\mu|}(\lambda) + 1)(s + |\mu| + 1)s!}{(s + 1 - \ell(\mu))! \prod_{i \geq 1} m_i(\mu)!} c_{\lambda \cup (s+|\mu|) - \mu}$$

9.18. REMARK.  $\mu = (i^{m_i(\mu)}) \subseteq \lambda \iff m_i(\mu) \leq m_i(\lambda) \ \forall i \geq 1$ .

9.19. EXAMPLE. To illustrate the formula  $\lambda \cup (s + |\mu|) - \mu$ , consider cycles  $\sigma = (134)(2567)(8)$  and  $\tau = (28)$ . Then,  $\lambda = (3, 2)$  is the modified cycle type of  $\sigma$  and  $s = 1$  gives the modified cycle type of  $\tau$ . Then,

$$\sigma\tau = (134)(28567)$$

which has modified cycle type  $(4, 2) = (3, 2) \cup (1 + |(3)|) - (3)$ .

9.20. COROLLARY.

$$c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_\ell} = \sum_{\mu \supseteq \lambda} d_{\lambda\mu} c_\mu$$

and  $d_{\lambda\lambda} > 0$  in  $\text{gr } \mathcal{K}$ . Thus,  $c_1, c_2, \dots$  are algebraically independent elements of  $\text{span}_{\mathbb{Z}}\{c_\lambda\}$ .

9.21. PROPOSITION.  $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gr } \mathcal{K}$  is a polynomial algebra generated by  $c_1, c_2, \dots$  (Note that  $\mathbb{Z} \hookrightarrow \mathbb{B}$  as constants.)

9.22. REMARK. (a) There exists a ring isomorphism  $\Lambda \xrightarrow{\sim} \text{gr } \mathcal{K}$  sending duals of  $h_\lambda^*$  (images of  $h_\lambda$  under a certain automorphism), called  $g_\lambda$ , to  $c_\lambda$ . See [Mac79, p 132–3].  
(b)  $\text{gr } Z(\mathbb{Z}\mathfrak{S}_n) \cong H^*(\text{Hilb}^n(\mathbb{C}^2); \mathbb{Z})$ , the cohomology ring of the Hilbert Scheme of points on  $\mathbb{C}^2$ , as a  $\mathbb{Z}$ -algebra.

9.23. THEOREM. The homomorphism given by

$$\begin{aligned} \Pi_n : \mathcal{K} &\rightarrow Z(\mathbb{Z}\mathfrak{S}_n) \\ \sum f_\lambda(x) c_\lambda &\mapsto \sum f_\lambda(n) c_\lambda(n) \end{aligned}$$

is a surjective homomorphism.

9.24. PROPOSITION.  $\mathcal{K}$  is generated by  $K_m := \sum_{|\lambda|=m} c_\lambda$  for  $m \geq 0$ .

This tells us that

$$\Pi_m(\mathcal{K}) = \sum_{|\lambda|=m} c_\lambda(n) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \# \text{ of cycles in } \sigma = n-m}} \sigma$$

and so  $Z(\mathbb{Z}\mathfrak{S}_n)$  is generated by  $\Pi_n(\mathcal{K}_0), \Pi_n(\mathcal{K}_1), \dots, \Pi_n(\mathcal{K}_{n-1})$ .

## 10. Double cover of $\tilde{\mathfrak{S}}_n$ and even split conjugacy classes

Recall the short exact sequence

$$1 \rightarrow \{1, z\} \rightarrow \tilde{\mathfrak{S}}_n \rightarrow \mathfrak{S}_n \rightarrow 1$$

where

$$\tilde{\mathfrak{S}}_n = \left\langle z, t_1, t_2, \dots, t_{n-1} \mid \begin{cases} z \text{ is central} \\ z^2 = 1, t_i^2 = z \\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \\ t_i t_j = z t_j t_i \quad |i - j| > 1 \end{cases} \right\rangle$$

10.1. DEFINITION. Define the element

$$x_i := t_i t_{i+1} \cdots t_{n-1} t_n t_{n-1} \cdots t_{i+1} t_i$$

which gets mapped to the transposition  $(i, n) \in \mathfrak{S}_n$  under the map  $\theta_n$ . Then, we let

$$[i_1 i_2 \cdots i_m] := \begin{cases} z & \text{if } m = 1 \\ x_{i_1} x_{i_m} x_{i_{m-1}} \cdots x_{i_2} x_{i_1} & \text{if } m \geq 2 \end{cases}$$

10.2. PROPOSITION. *Every element of  $\tilde{\mathfrak{S}}_n$  is of the form*

$$z^q \underbrace{[i_1 i_2 \cdots i_m][j_1 j_2 \cdots j_k] \cdots}_{\text{disjoint}}$$

where  $q = 0, 1$ .

10.3. LEMMA. *For  $\lambda$  such that  $|\lambda| + \ell(\lambda) \leq n$ ,  $\theta_n^{-1}(\mathcal{C}_\lambda(n))$  splits if and only if*

- (a)  $\lambda$  has only even parts or
- (b)  $\lambda \in \mathcal{SP}$ ,  $|\lambda|$  odd,  $|\lambda| + \ell(\lambda) = n$  or  $n - 1$ .

10.4. PROPOSITION.  $\sigma \in \mathfrak{S}_n$  of modified type  $\lambda$  is even if and only if  $|\lambda|$  is even.

10.5. DEFINITION. Let  $\mathcal{D}_\lambda(n)$  be the even split conjugacy class in  $\tilde{\mathfrak{S}}_n$  containing  $[1, 2, \dots, \lambda_1 + 1][\lambda_1 + 2, \dots, \lambda_1 + \lambda_2 + 2] \dots$

- 10.6. PROPOSITION. (a)  $\theta^{-1}(\mathcal{C}_\lambda(n)) = \mathcal{D}_\lambda(n) \cup z\mathcal{D}_\lambda(n)$ .
- (b)  $\{d_\lambda(n) \mid \lambda \in \mathcal{EP}, |\lambda| + \ell(\lambda) \leq n\}$  is a basis for the even center of  $\mathbb{Z}\tilde{\mathfrak{S}}_n^- = \mathbb{Z}\tilde{\mathfrak{S}}_n / \langle z + 1 \rangle$ .

10.7. DEFINITION. Define  $b_{\lambda\mu}^\nu(n)$  by

$$d_\lambda(n) d_\mu(n) = \sum_{\nu \in \mathcal{EP}} b_{\lambda\mu}^\nu(n) c_\nu(n)$$

10.8. EXAMPLE.

$$d_{(4)}(8) d_{(2)}(8) = 13d_{(4)}(8) - 35d_{(2)}(8) - 18d_{(2,2)}(8) - 7d_{(6)}(8) + 2d_{(4,2)}(8) \in Z(\mathbb{Z}\tilde{\mathfrak{S}}_8^-)$$

10.9. THEOREM (Tysse-Wang). *Let  $\lambda, \mu, \nu \in \mathcal{EP}$ .*



- (a) There exists a unique  $g_{\lambda\mu}^\nu(x) \in \mathcal{Q}[x]$  such that  $b_{\lambda\mu}^\nu(n) = g_{\lambda\mu}^\nu(n)$  for all  $n \geq |\nu| + \ell(\nu)$ .
- (b)  $g_{\lambda\mu}^\nu(x) = 0$  unless  $|\nu| \leq |\lambda| + |\mu|$
- (c) If  $|\nu| = |\lambda| + |\mu|$ , then  $g_{\lambda\mu}^\nu(x)$  is a constnat.

10.10. DEFINITION. Let the *spin Farahat-Higman algebra*  $\mathbb{F}$  be a  $\mathbb{B}$ -algebra with basis  $\{d_\lambda \mid \lambda \in \mathcal{EP}\}$  and

$$d_\lambda d_\mu = \sum_{\nu \in \mathcal{EP}} g_{\lambda\mu}^\nu(x) d_\nu$$

which is filtered with respect to  $\deg(d_\lambda) = |\lambda|$ .

10.11. PROPOSITION. Let  $\lambda$  be a partition and rewrite  $\lambda = (i_{i \geq 1}^{m_i(\lambda)})$ . Let  $s \geq 0$  be event. Then,

$$(d_\lambda d_{(s)})^* = \sum_{\mu} (-1)^{\ell(\mu)} \frac{(m_{s+|\mu|}(\lambda) + 1)(s + |\mu| + 1)s!}{(s + 1 - \ell(\mu))! \prod_{i \geq 1} m_i(\mu)!} d_{\lambda \cup (s+|\mu|) - \mu}$$

What is this summation over?

- 10.12. COROLLARY. (a)  $\mathbb{Q} \otimes_{\mathbb{Z}[\frac{1}{2}]} \text{gr } \mathbb{F}$  is generated by  $d_2, d_4, \dots$   
(b) There exists an injective homomorphism  $\mathbb{Q} \otimes_{\mathbb{Z}[\frac{1}{2}]} \text{gr } \mathbb{F} \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{K}$  via  
(for  $\lambda \in \mathcal{EP}$ )  $d_\lambda \mapsto (-1)^{\ell(\lambda)c_\lambda}$ .

### 10.1. Connections to odd Jucys-Murphy elements.

10.13. DEFINITION. Let us define

$$M_k := \sum_{i=1}^{k-1} [i, k] \in \mathbb{Z}\mathfrak{S}_n^-$$

10.14. PROPOSITION. We have

- (a)  $M_k M_l = -M_l M_k$  for  $k \neq l$
- (b)

$$M_k^2 = -(k-1) - \sum_{1 \leq i \neq j \leq k-1} [i, j, k] \in \mathbb{Z}\mathfrak{S}_n^-$$

10.15. DEFINITION. we define

$$e_{r,n} := \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} M_{i_1}^2 M_{i_2}^2 \dots M_{i_r}^2 \in Z(\mathbb{Z}\mathfrak{S}_n^-)$$

- 10.16. PROPOSITION. (a)  $e_{r,n}$  has top degree  $2r$   
(b)  $e_{r,n} = \sum_{\lambda \in \mathcal{EP}, |\lambda| + \ell(\lambda) \leq n} A_\lambda(n) d_\lambda(n)$  for some  $A_\lambda(n)$ .  
(c)  $A_\lambda(n)$  is the coefficient of  $[1, 2, \dots, \lambda_1 + 1][\lambda_1 + 2, \dots, \lambda_1 + \lambda_2 + 2] \dots$  in  $e_{r,n}$  and is independent of  $n$ .

10.17. DEFINITION. In light of the proposition above, we write  $A_\lambda := A_\lambda(n)$  and define

$$e_r^* := \sum_{\lambda \in \mathcal{EP}, |\lambda| = 2r} A_\lambda d_\lambda \in \mathbb{F}$$

10.18. EXAMPLE.  $e_1^* = -d_2$  and  $e_2^* = d_{(2,2)} - 2d_4$ .

10.19. PROPOSITION.  $A_\lambda = (-1)^{\ell(\lambda)} \prod_{i \geq 1} c_{\frac{\lambda_i}{2}}$  where  $c_0 = 1$  and  $c_r = \frac{1}{r+1} \binom{2r}{r}$  are the Cartan numbers.

10.20. THEOREM.  $\mathbb{B} \left[ \frac{1}{2} \right] \otimes_{\mathbb{B}} \mathbb{F}$  is generated by  $e_1^*, e_2^*, e_3^*, \dots$

10.21. COROLLARY. Via the surjective homomorphism

$$\begin{aligned} \mathbb{B} \left[ \frac{1}{2} \right] \otimes_{\mathbb{B}} \mathbb{F} &\rightarrow Z(\mathbb{Z} \left[ \frac{1}{2} \right] \mathfrak{S}_n^-) \\ \sum_{\lambda \in \mathcal{EP}} f_\lambda(x) d_\lambda &\mapsto \sum_{\lambda \in \mathcal{EP}} f_\lambda(n) d_\lambda(n) \end{aligned}$$

the even center of  $\mathbb{Z} \left[ \frac{1}{2} \right] \mathfrak{S}_n^-$  is generated by (the top degree of)  $e_{r,n}$ .

## 11. Schur-Sergeev duality for $\mathfrak{q}(n)$ (presented by Chris Chung)

## 12. Seminormal form construction for irreducible $\mathcal{H}_n$ -modules (presented by Jinkui Wan)

A review for the symmetric group case was presented, but not written up here yet.

12.1. DEFINITION. We define the *Jucys-Murphy elements* in  $\mathcal{H}_n = \mathcal{C}\ell_n \rtimes \mathbb{C}\mathfrak{S}_n$  as

$$J_k := \sum_{1 \leq j < k \leq n} (1 + c_j c_k)(jk),$$

12.2. PROPOSITION. *The Jucys-Murphy elements have the following properties.*

- (a)  $J_k J_l = J_l J_k$  for  $l \leq k \neq l \leq n$ .
- (b)  $c_k J_k = -J_k c_k$  and  $c_l J_k = J_k c_l$  for  $k \neq l$ .
- (c)  $s_k J_k = J_{k+1} s_k - (1 + c_k c_{k+1})$  for  $1 \leq k \leq n-1$  and  $s_l J_k = J_k s_l$  for  $k \neq l, l+1$ .

12.3. DEFINITION. The *degenerate affine Hecke-Clifford algebra* is given by

$$\hat{\mathcal{H}}_n = \langle s_1, \dots, s_{n-1}, c_1, \dots, c_n, x_1, \dots, x_n \rangle$$

with additional relations

$$\begin{cases} x_k x_l = x_l x_k \\ s_k x_k = x_{k+1} s_k - (1 + c_k c_{k+1}) \\ s_i x_k = x_k s_i & k \neq i, i+1 \\ x_k c_k = -c_k x_k \\ x_k c_l = c_l x_k & l \neq k \end{cases}$$

12.4. PROPOSITION. *There exists a projection  $\pi: \hat{\mathcal{H}}_n \rightarrow \mathcal{H}_n$  such that*

$$\pi(s_i) = s_i, \pi(c_k) = c_k, \pi(x_l) = J_l$$

*In particular,  $\pi(x_1) = J_1 = 0$ , so  $x_1 \in \ker \pi$ .*

In fact,  $\ker \pi = \langle x_1 \rangle$  and so  $\mathcal{H}_n\text{-}\mathbf{Mod}$  can be identified as the subcategory of  $\hat{\mathcal{H}}_n\text{-}\mathbf{Mod}$  on which  $x_1 = 0$ .

12.5. THEOREM (PBW Theorem). *We have that*

$$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} c_1^{\beta_1} \cdots c_n^{\beta_n} w \mid \alpha_i \in \mathbb{Z}_+, \beta_i \in \{0, 1\}, 1 \leq i \leq n, w \in \mathfrak{S}_n\}$$

*is a basis for  $\hat{\mathcal{H}}_n$ .*

12.6. COROLLARY (Corollary of PBW Theorem). *The subalgebra of  $\hat{\mathcal{H}}_n$  generated by  $x_1, \dots, x_n, c_1, \dots, c_n$  is isomorphic to*

$$\mathcal{C}\ell_n \otimes \mathbb{C}[x_1, \dots, x_n] / \langle x_k c_k = -c_k x_k, x_k c_l = c_l x_k, l \neq k \rangle = \underbrace{P_1^c \otimes P_1^c \otimes \cdots \otimes P_1^c}_{n \text{ copies}}$$

*where  $P_1^c = \langle x_1, c_1 \rangle$ .*

12.7. PROPOSITION. *The  $\mathfrak{S}_n$  fixed points  $\mathbb{C}[x_1^2, x_2^2, \dots, x_n^2]^{\mathfrak{S}_n} \subseteq \text{Center of } \hat{\mathcal{H}}_n$ .*

12.8. PROPOSITION. (a) *The eigenvalues of  $x_1^2, \dots, x_n^2$  are of the form  $q(i) := i(i+1)$  for  $i \in \mathbb{Z}_+$ .*

(b) *If all the eigenvalues of  $x_j^2$  on a finite dimensional  $\hat{\mathcal{H}}_n$ -module  $M$  for a fixed  $j$  are of the form  $q(i)$ , then  $M$  is integral.*

The second part of the proposition follows from the intertwining elements.

12.9. DEFINITION. Let *intertwining element*  $\Phi_k \in \hat{\mathcal{H}}_n$  be given by

$$\begin{aligned} \Phi_k &:= s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}) \\ &= (x_{k+1}^2 - x_k^2)s_k - (x_k + x_{k+1}) - c_k c_{k+1}(x_k - x_{k+1}) \end{aligned}$$

Note, the second equality follows from the fact that

$$s_k x_k^2 = x_{k+1} s_k x_k - (1 + c_k c_{k+1})x_k = x_{k+1}^2 s_k - (1 + c_k c_{k+1})(1 + x_k)$$

and, using conjugation by  $s_k$ ,

$$x_k^2 s_k = s_k x_{k+1}^2 - s_k(1 + c_k c_{k+1})(1 + x_k)s_k \implies -s_k x_{k+1}^2 = -x_k^2 s_k - s_k(1 + c_k c_{k+1})(1 + x_k)s_k$$

Therefore, we get

$$s_k(x_k^2 - x_{k+1}^2) = (x_{k+1}^2 - x_k^2)s_k - (1 + c_k c_{k+1})(1 + x_k) - s_k(1 + c_k c_{k+1})(1 + x_k)s_k$$

12.10. PROPOSITION. *We have the following useful relations for the intertwining elements.*

$$\begin{cases} \Phi_k \Phi_l = \Phi_l \Phi_k & |k - l| > 1 \\ \Phi_k \Phi_{k+1} \Phi_k = \Phi_{k+1} \Phi_k \Phi_{k+1} \\ \Phi_k^2 = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2 \\ \Phi_k x_k = x_{k+1} \Phi_k, \Phi_k x_{k+1} = x_k \Phi_k, \Phi_k x_l = x_l \Phi_k & l \neq k, k+1 \\ \Phi_k c_k = c_{k+1} \Phi_k, \Phi_k c_{k+1} = c_k \Phi_k, \Phi_k c_l = c_l \Phi_k & l \neq k, k+1 \end{cases}$$

12.11. PROPOSITION. *If  $v$  is some eigenvector of  $x_{j+1}^2$ , that is, if  $x_{j+1}^2 v = av$ , then  $x_j^2 \Phi_j v = a \Phi_j v$ , so  $\Phi_j v$  is an  $x_j^2$  eigenvector with the same eigenvalue.*

PROOF. Since  $x_j^2 \Phi_j = \Phi_j x_{j+1}^2$ , we get that

$$x_j^2 \Phi_j v = \Phi_j x_{j+1}^2 v = \Phi_j av = a \Phi_j v$$

□

12.12. DEFINITION. A finite dimensional  $\hat{\mathcal{H}}_n$ -module  $M$  is called *completely splittable (CS)* if  $x_1, x_2, \dots, x_n$  act semisimply, ie the actions of  $x_1, \dots, x_n$  can be diagonalized simultaneously.

12.13. PROPOSITION. *Every integral CS  $\hat{\mathcal{H}}_n$ -module  $M$  can be decomposed as*

$$M = \bigoplus_{\mathbf{i} \in \mathbb{Z}_+^n} M_{\mathbf{i}}$$

where

$$M_{\mathbf{i}} = \{v \in M \mid x_k^2 v = q(i_k) v, 1 \leq k \leq n\}$$

is the common eigenspace of  $x_1^2, x_2^2, \dots, x_n^2$  with eigenvalues  $q(i_1), q(i_2), \dots, q(i_n)$ . Furthermore, define

$$\text{wt}(M) := \{\mathbf{i} \in \mathbb{Z}_+^n \mid M_{\mathbf{i}} \neq 0\}$$

Our goal is to describe  $\text{wt}(M)$  for integral irreducible CS  $\hat{\mathcal{H}}_n$ -modules.

12.14. LEMMA. *If  $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$  is in  $\text{wt}(M)$  for some integral irreducible CS  $\hat{\mathcal{H}}_n$ -module, then  $i_k \neq i_{k+1}$  for all  $1 \leq k \leq n-1$ .*

PROOF. Suppose  $i_k = i_{k+1}$  so that  $x_k^2 v = x_{k+1}^2 v$  for  $v \in M_{\mathbf{i}}$ . Since

$$x_k^4 s_k - 2q(i_k) x_k^2 s_k + q(i_k)^2 s_k = x_k^2 (s_k x_{k+1}^2 - s_k (1 + c_k c_{k+1}) (1 + x_k) s_k) - 2q(i_k) x_k^2 s_k + q(i_k)^2 s_k$$

Somehow we get that  $s_k v \in M_{\mathbf{i}}$ . Then, we get

figure this out

$$\implies x_k^2 s_k v = q(i_k) s_k v$$

$$\implies (x_k^2 - q(i_k)) s_k v = 0$$

$$\implies (s_k x_{k+1}^2 - x_k (1 - c_k c_{k+1}) - (1 - c_k c_{k+1}) x_{k+1}) v - q(i_k) s_k v = 0$$

However,  $x_{k+1}^2 v = q(i_{k+1})v = q(i_k)v$  by assumption, so

$$\begin{aligned}
&\implies (x_k(1 - c_k c_{k+1}) + (1 - c_k c_{k+1})x_k)v = 0 \\
&\implies 2(x_k^2 + x_{k+1}^2)v = 0 \\
&\implies i_k = i_{k+1} = 0 \\
&\implies x_k^2 v = 0 = x_{k+1}^2 v \\
&\implies x_k v = 0 = x_{k+1} v \quad \text{since } M \text{ is CS} \\
&\implies v = 0 \quad \text{since}
\end{aligned}$$

□

Finish this proof.

12.15. LEMMA. Suppose  $\mathbf{i} = (i_1, \dots, i_n) \in \text{wt}(M) \subseteq \mathbb{Z}_+^n$  for some integral, irreducible, CS  $\hat{\mathcal{H}}_n$ -module. Fix  $1 \leq k \leq n-1$ .

- (a) If  $i_k \neq i_{k+1} \pm 1$ , then  $\Phi_k z \neq 0$  for all  $0 \neq z \in M_{\mathbf{i}}$ .
- (b) If  $i_k = i_{k+1} \pm 1$ , then  $\Phi_k = 0$  on  $M_{\mathbf{i}}$ .

PROOF. Since  $\Phi_k^2 = 2(x_k^2 + x_{k+1}^2) - (x_k^2 - x_{k+1}^2)^2$ , then

$$\Phi_k^2 z = (2(q(i_k) + q(i_{k+1})) - q(i_k)^2 + 2q(i_k)q(i_{k+1}) - q(i_{k+1})^2)z = 0$$

if and only if

$$(q(i_k) - q(i_{k+1}))^2 = 2(q(i_k) + q(i_{k+1}))$$

Now, if we write  $i_k = i_{k+1} + c$ , then

$$\begin{cases} q(i_k) = i_{k+1}^2 + 2ci_{k+1} + c^2 + i_{k+1} + c \\ q(i_{k+1}) = i_{k+1}^2 + i_{k+1} \end{cases} \implies \begin{cases} q(i_k) - q(i_{k+1}) = 2ci_{k+1} + c^2 + c \\ q(i_k) + q(i_{k+1}) = 2i_{k+1}^2 + 2ci_{k+1} + c^2 + 2i_{k+1} + c \end{cases}$$

From here, one checks that  $c = \pm 1$  certainly gives solutions independent of  $i_{k+1}$ . Thus,  $i_k = i_{k+1} \pm 1 \implies \Phi_k^2 z = 0$ . Furthermore, there are other formal solutions to these equations, namely  $c = -2i_{k+1}$  and  $c = -2(i_{k+1} + 1)$ , but since  $i_{k+1} \geq 0$ , this would force  $i_k < 0$  unless  $i_k = i_{k+1} = 0$ , which is not admissible by the previous lemma.

So, to prove the second part, since  $\Phi_k^2 z = 0$ , it must be that if  $\Phi_k z \neq 0$  and so  $\Phi_k z \in M_{s_k \mathbf{i}}$ . Then, there exists a minimal sequence  $\Phi_{j_1}, \dots, \Phi_{j_r}$  such that  $\Phi_{j_1} \cdots \Phi_{j_r} \Phi_k z \in M_{\mathbf{i}}$  since  $M$  is irreducible. Then, if  $\sigma = s_{j_1} \cdots s_{j_r} s_k \in \mathfrak{S}_n$ , it must be that  $\sigma \cdot \mathbf{i} = \mathbf{i}$ . If one assumes  $\sigma \neq 1$ , this leads to a violation of Lemma 12.14 with some work. Then, using the exchange condition for Coxeter groups, one shows that  $r = 1$  which gives  $j_1 = k$ , so  $\Phi_k^2 z \neq 0$ , contradicting what we showed above. □

Why is this true?

12.16. REMARK. Suppose  $V$  is an integral, irreducible  $\hat{\mathcal{H}}_n$ -module. Let  $\hat{\mathcal{H}}_{(n-r, 1^k)} = \langle s_1, \dots, s_{n-r-1}, c_1, \dots, c_n, x_1, \dots, x_r \rangle$ . Then,

$$\begin{aligned}
V \text{ is CS} &\iff \forall \mathbf{i} \in \text{wt}(V), 1 \leq k \leq n-1, i_k \neq i_{k+1} \\
&\iff \text{Res}_{\hat{\mathcal{H}}_{(n-r, 1^r)}}^{\hat{\mathcal{H}}_n} V \text{ is semisimple } \forall 1 \leq r \leq n \\
&\iff \text{Res}_{\hat{\mathcal{H}}_{(1^k, n-k-r, 1^r)}}^{\hat{\mathcal{H}}_n} V \text{ is semisimple}
\end{aligned}$$

The second  $\hat{\mathcal{H}}_{(1^k, n-k-r, 1^r)}$  is not defined.

12.17. COROLLARY. Suppose  $\mathbf{i} \in \text{wt}(V)$  for some integral, irreducible, CS  $\hat{\mathcal{H}}_n$ -module  $V$ . If  $i_k = i_{k+2}$  for some  $1 \leq k \leq n-2$ , then  $i_k = i_{k+2} = 0$  and  $i_{k+1} = 1$ .

PROOF. If  $i_k \neq i_{k+1} \pm 1$ , then  $s_k \cdot \mathbf{i} \in \text{wt } V$  by the first part of 12.15, but  $s_k \cdot \mathbf{i} = (\dots, i_{k+1}, i_k, i_{k+2}, \dots)$  and, by assumption,  $i_k = i_{k+2}$  contradicting 12.14. So, it must be that  $i_{k+1} = i_k \pm 1$  and so  $\Phi_k = 0 = \Phi_{k+1}$  on  $V_{\mathbf{i}}$  by the second part of 12.15. Thus, for  $z \in V$ ,

$$\begin{cases} \Phi_k z = 0 & \implies (q(i_k) - q(i_{k+1}))s_k z = -((x_k + x_{k+1}) + c_k c_{k+1}(x_k - x_{k+1}))z \\ \Phi_{k+1} z = 0 & \implies (q(i_{k+1}) - q(i_{k+2}))s_{k+1} z = -((x_{k+1} + x_{k+2}) + c_{k+1} c_{k+2}(x_{k+1} - x_{k+2}))z \end{cases}$$

which gives us the  $s_k$  and  $s_{k+1}$  actions on  $V_{\mathbf{i}}$ . From here, we can use the braid relation  $s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$  to arrive at the equality

$$((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z = 0$$

Now,  $x_k z = \pm \sqrt{q(i_k)}z$ . Furthermore, since  $i_k = i_{k+2}$ , we have that either  $x_k z = x_{k+2} z$  or  $x_k z = -x_{k+2} z$ . Decompose

$$V_{\mathbf{i}} = W_1 \oplus W_2$$

where  $W_1 := \{z \in V_{\mathbf{i}} \mid x_k z = x_{k+2} z\}$  and  $W_2 := \{z \in V_{\mathbf{i}} \mid x_k z = -x_{k+2} z\}$ . Now, we can break up our braid relation equality.

$$\begin{aligned} z \in W_1 \implies 0 &= ((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z \\ &= 2x_k(6x_{k+1}^2 + 2x_k^2)z \\ &= 2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k))z \\ z \in W_2 \implies 0 &= ((x_k + x_{k+2})(6x_{k+1}^2 + 2x_k x_{k+2}) + c_k c_{k+2}(x_k - x_{k+2})(6x_{k+1}^2 - 2x_k x_{k+2}))z \\ &= c_k c_{k+2}2x_k(6x_{k+1}^2 + 2x_k^2)z \\ &= c_k c_{k+2}(2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k)))z \end{aligned}$$

Thus, we obtain that

$$2\sqrt{q(i_k)}(6q(i_{k+1}) + 2q(i_k)) = 0$$

Moreover, we know that  $i_{k+1} = i_k \pm 1$ , so we get

$$\begin{cases} \sqrt{q(i_k + 1)}(6(i_k - 1)i_k + 2(i_k + 1)i_k) = 0 & \text{if } i_{k+1} = i_k - 1 \\ \sqrt{q(i_k + 1)}(6(i_k + 1)(i_k + 2) + 2(i_k + 1)i_k) = 0 & \text{if } i_{k+1} = i_k + 1 \end{cases}$$

There are no solutions to the first equation that give  $i_k$  and  $i_{k+1}$  as non-negative integers and the only such solution for the second equation is  $i_k = 0$  and  $i_{k+1} = 1$ .  $\square$

In conclusion, we have the following.

12.18. THEOREM. Let  $\mathcal{W}(n)$  be the set of weights of all integral irreducible CS  $\hat{\mathcal{H}}_n$ -modules and let  $\mathbf{i} \in \mathcal{W}(n)$ . Then,

- (a)  $i_k \neq i_{k+1}$  for all  $1 \leq k \leq n-1$ .
- (b) If  $i_k = i_\ell = 0$  for some  $1 \leq k < \ell \leq n$ , then  $1 \in \{i_{k+1}, \dots, i_{\ell-1}\}$ .
- (c) If  $i_k = i_\ell \geq 1$  for some  $1 \leq k < \ell \leq n$ , then  $\{i_k - 1, i_k + 1\} \subseteq \{i_{k+1}, \dots, i_{\ell-1}\}$ .

PROOF. The first part is just a restatement of 12.14.

For the next part, assume  $1 \notin \{i_{k+1}, \dots, i_{\ell-1}\}$ . Then, we can swap indices using ?? to get new weights until we obtain a weight of the form

$$(\dots, 0, 0, \dots)$$

which is not allowed by the previous part.

For the last part, let  $u = i_k = i_\ell \geq 1$  be such that  $k - \ell$  is minimal among such occurrences. If only one of  $u + 1, u - 1$  appear in between  $i_k$  and  $i_\ell$ , then it must appear twice because, otherwise, we could swap indices using 12.15 to get new weights until we are of the form

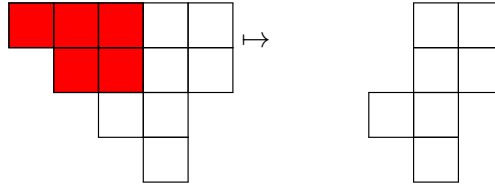
$$(\dots, u, u \pm 1, u, \dots)$$

which is not a weight by 12.17 since  $u \geq 1$ . Thus, we have violated the minimality of our choice of  $u$ .  $\square$

Now, we wish to describe a bijection between  $\mathcal{W}(n)$  and standard skew shifted tableaux of size  $n$ .

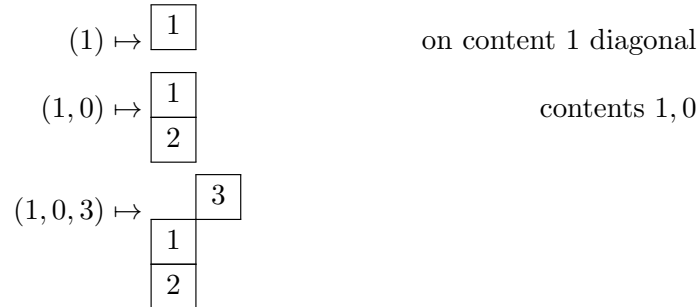
12.19. DEFINITION. Given strict partitions  $\nu \subseteq \xi$ , the *skew shifted Ferrers diagram*  $\xi/\nu$  is given by removing the boxes of  $\nu$  from  $\xi$ .

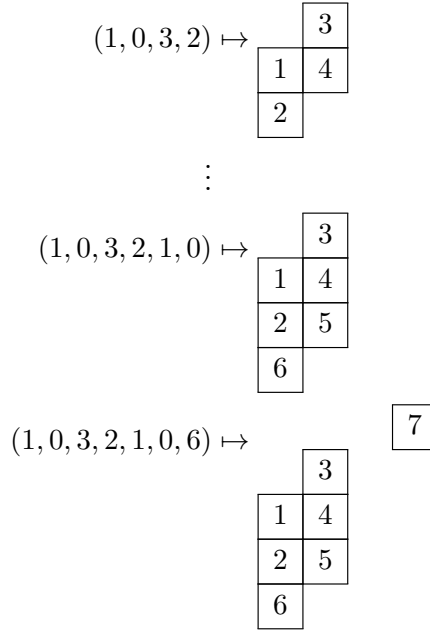
12.20. EXAMPLE. Consider  $(3, 2) \subseteq (5, 4, 2, 1)$ . Then,



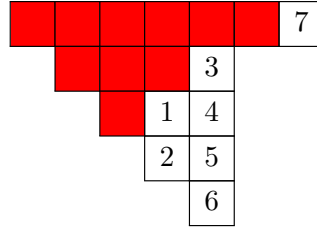
We now illustrate the bijection by example.

12.21. EXAMPLE. Let  $\mathbf{i} = (1, 0, 3, 2, 1, 0, 6)$ . Then, we construct our standard skew shifted tableau in steps by adding a box of content  $i_k$  labelled  $k$  on the  $k$ th step.





So, our final answer has outer shape  $\xi = (7, 4, 3, 2, 1)$  and inner shape  $\nu = (6, 3, 1)$ :



12.22. DEFINITION. Given  $\mathbf{i} \in \text{wt } V$ , let  $T(\mathbf{i})$  be the corresponding standard skew shifted tableau.

12.23. PROPOSITION. If  $\mathbf{i}, \mathbf{j} \in \text{wt}(V)$  for some integral irreducible CS  $\hat{\mathcal{H}}_n$ -module  $V$ , then  $T(\mathbf{i})$  and  $T(\mathbf{j})$  have the same shape.

PROOF. If  $(i_1, \dots, i_k, i_{k+1}, \dots, i_n)$ , then  $(i_1, \dots, i_{k-1}, i_{k+1}, i_k, \dots, i_n)$  is a weight only if  $i_k \neq i_{k+1} \pm 1$  by ??. However, under such a condition, it does not matter in which order we add the boxes corresponding to  $i_k$  and  $i_{k+1}$ . Thus, these two weights will yield the same shape.  $\square$

What is the reference here?

12.24. DEFINITION. Given  $\xi/\nu$  a skew-shifted Ferrer's diagram of size  $n$ , we define

$$\mathcal{F}(\xi/\nu) := \{\mathsf{T} \mid \mathsf{T} \text{ a standard Young tableau of shape } \xi/\nu\}$$

and

$$\hat{D}^{\xi/\nu} := \bigoplus_{\mathsf{T} \in \mathcal{F}(\xi/\nu)} \mathcal{C}\ell_n v_{\mathsf{T}}$$



as a vector space with actions

$$x_k v_{\mathbf{T}} = \sqrt{q(c(\mathbf{T}_k))} v_{\mathbf{T}}$$

where  $c(\mathbf{T}_k)$  is the content of the box labelled by  $k$  in  $\mathbf{T}$  and where  $\mathcal{C}_{\ell_n}$  acts by multiplication on the left.

12.25. PROPOSITION. *We have*

$$s_k v_{\mathbf{T}} = \left( \frac{1}{\sqrt{q(c(\mathbf{T}_{k+1}))} - \sqrt{q(c(\mathbf{T}_k))}} + \frac{1}{\sqrt{q(c(\mathbf{T}_{k+1}))} + \sqrt{q(c(\mathbf{T}_k))}} c_k c_{k+1} \right) v_{\mathbf{T}} + \sqrt{1 - \frac{2(q(c(\mathbf{T}_{k+1}))) + q(c(\mathbf{T}_k))}{(q(c(\mathbf{T}_{k+1})) - q(c(\mathbf{T}_k)))}}$$

PROOF. This fact follows formally from the fact that  $\Phi_k v_{\mathbf{T}} = a v_{s_k \mathbf{T}}$  for some scalar  $a$  if  $s_k \mathbf{T}$  is standard.  $\square$

Actually do this proof.

12.26. COROLLARY.  $\hat{D}^{\xi/\nu}$  is an integral CS  $\hat{\mathcal{H}}_n$ -module.

PROOF. Mainly, one needs to check the Coxeter relations.  $\square$

Fill in this proof.

### 13. Spin Kostka Polynomials

13.1. DEFINITION. Let  $\nu \in \mathcal{SP}$  and  $\mu$  be a partition. Then, the *spin Kostka polynomials* are the transition polynomials from the Hall-Littlewood  $P$ -functions to the  $Q$ -Schur functions. In other words,

$$Q_{\nu}(x) = \sum_{\mu} K_{\nu\mu}^{-}(t) P_{\mu}(x; t)$$

If we also let  $b_{\nu\lambda}$  be such that

$$Q_{\nu}(x) = \sum_{\lambda} b_{\nu\lambda} s_{\lambda}(x)$$

then we get

13.2. PROPOSITION. *For  $\nu \in \mathcal{SP}$  and  $\mu$  a partition,*

$$K_{\nu\mu}^{-}(t) = \sum_{\lambda} b_{\nu\lambda} K_{\lambda\mu}(t)$$

where  $K_{\lambda\mu}(t)$  are the Kostka-Foulkes polynomials.

PROOF. By definition of Kostka-Foulkes, we have

$$s_{\lambda} = \sum_{\mu} K_{\lambda\mu}(t) P_{\mu}(x; t)$$

and so

$$\sum_{\mu} K_{\nu\mu}(t) P_{\mu}(x; t) = Q_{\nu}(x) = \sum_{\lambda} b_{\nu\lambda} s_{\lambda} = \sum_{\lambda, \mu} b_{\nu\lambda} K_{\lambda\mu}(t) P_{\mu}(x; t)$$

$\square$

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