# Yangians
# Notes from a reading course in Fall 2019

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## 1. Yangians for $\mathfrak{gl}_N$

For some motivation, consider the standard basis elements $E_{ij}$ of $\mathfrak{gl}_N$ satisfy the relation

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}$$

Now, if we consider an $N \times N$ matrix $E$ whose $ij$th entry is $E_{ij}$, ie

$$E := \sum_{i,j} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathcal{U}(\mathfrak{gl}_N)$$

we have the less well known relation for any $r, s \in \mathbb{Z}_{\geq 0}$,

$$[(E^{r+1})_{ij}, (E^s)_{kl}] - [(E^r)_{ij}, (E^{s+1})_{kl}] = (E^r)_{kl}(E^s)_{il} - (E^s)_{kj}(E^r)_{il}$$

1.1. Definition. We define the *Yangian* for $\mathfrak{gl}_N$, denoted $Y(\mathfrak{gl}_N)$ or $Y_N$ for short, to be the associative algebra generated by elements $t_{ij}^{(r)}$ for $r \in \mathbb{Z}_{\geq 0}$ and $1 \leq i, j \leq N$ with relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)} \tag{1a}$$

where $r, s \in \mathbb{Z}_{\geq 0}$ and $t_{ij}^{(0)} = \delta_{ij}$.

If we define the formal power series $t_{ij}(u) := \sum_{r \geq 0} t_{ij}^{(r)} u^{-r} = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in Y_N[[u^{-1}]]$, then

$$(u - v)[t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u) \tag{1b}$$

where the indeterminants $u, v$ commute with each other. One then recovers 1a by taking the coefficient of $u^{-r}v^{-s}$.

1.2. Definition. Let us define the following elements.

(a) $T(u) := \sum_{i,j} e_{ij} \otimes t_{ij}(u) \in \operatorname{End} \mathbb{C}^N \otimes Y_N[[u^{-1}]]$
(b) $P = \sum_{i,j} e_{ij} \otimes e_{ji} \in (\operatorname{End} \mathbb{C}^N)^{\otimes 2}$, ie $P$ is the endomorphism given by
$$P(x \otimes y) = y \otimes x, \quad x, y \in \mathbb{C}^N$$
(c) $P_{ab} := \sum 1^{a-b} \otimes e_{ij} \otimes a^{b-a-1} \otimes e_{ji} \otimes 1^{m-b} \in (\operatorname{End} \mathbb{C}^N)^{\otimes m}$
(d) $R(u) := 1 - P u^{-1}$
(e) $R_{ab}(u) := 1 - P_{ab} u^{-1}$

1.3. Proposition (Yang-Baxter Equation). *$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$ in $(\operatorname{End} \mathbb{C}^N)^{\otimes 3}(u, v)$.*

Proof. Multiply the desired formula above by $uv(u+v)$ to get

$$(u - P_{12})(u + v - P_{13})(v - P_{23}) = (v - P_{23})(u + v - P_{13})(u - P_{12})$$

Then, it suffices to show that $P_{12}P_{13}P_{23} = P_{23}P_{13}P_{12}$ which we can do by regarding these as elements of $\mathfrak{S}_3$ by their action on $(\mathbb{C}^N)^{\otimes 3}$. In particular, $P_{ij}$ acts as $(i, j)$, and so the result is immediate from the relations in $\mathbb{C}[\mathfrak{S}_3]$. $\square$

1.4. Theorem (RTT Relations). *We have that*

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

*in* $(\operatorname{End}\mathbb{C}^N)^{\otimes 2} \otimes Y_N[[u^{-1}]]$ *where* $T_1(u) = \sum e_{ij} \otimes 1 \otimes t_{ij}(u)$ *and* $T_2(v) = \sum 1 \otimes e_{ij} \otimes t_{ij}(v)$. *Furthermore, this relation is equivalent to the defining relations of the Yangian 1a.*

Proof. First, note that $P \circ (e_{ij} \otimes e_{kl}) = e_{kj} \otimes e_{il} \in (\operatorname{End}\mathbb{C}^N)^{\otimes 2}$ by just observing where this endomorphism would send an $i$th coordinate vector and a $k$th coordinate vector. Similarly, $(e_{ij} \otimes e_{kl}) \circ P = e_{il} \otimes e_{kj}$.

Now, we can directly compute

$$\begin{aligned}
R(u-v)T_1(u)T_2(v) &= \left(1 - \frac{1}{u-v}P\right)\left(\sum_{i,j} e_{ij} \otimes 1 \otimes t_{ij}(u)\right)\left(\sum_{k,l} 1 \otimes e_{kl} \otimes t_{kl}(v)\right) \\
&= \sum_{i,j,k,l} e_{ij} \otimes e_{kl} \otimes t_{ij}(u)t_{kl}(v) - \frac{1}{u-v}\sum_{i,j,k,l} e_{kj} \otimes e_{il} \otimes t_{ij}(u)t_{kl}(v) \\
&= \sum_{i,j,k,l} e_{ij} \otimes e_{kl} \otimes t_{ij}(u)t_{kl}(v) - \frac{1}{u-v}\sum_{k,l,i,j} e_{il} \otimes e_{kj} \otimes t_{kl}(u)t_{ij}(v) \\
&= \left(\sum_{k,l} 1 \otimes e_{kl} \otimes t_{kl}(u)\right)\left(\sum_{i,j} e_{ij} \otimes 1 \otimes t_{ij}(v)\right)\left(1 - \frac{1}{u-v}P\right) \\
&= T_2(v)T_1(u)R(u-v)
\end{aligned}$$

∎

1.5. Proposition.

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} t_{kj}^{(a-1)}t_{il}^{(r+s-1)} - t_{kj}^{(r+s-a)}t_{il}^{(a-1)}$$

Proof. Multiply $\sum_{p=0}^{\infty} u^{-p-1}v^p$ to **??** to get

$$[t_{ij}(u), t_{kl}(v)] = (t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u))\left(\sum_{p=0}^{\infty} u^{-p-1}v^p\right)$$

Then, taking coefficients of $u^{-r}v^{-s}$ on both sides gives

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{r}(t_{kj}^{(a-1)}t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)}t_{il}^{(a-1)})$$

which gives our desired formula if $r \leq s$. However, if $r > s$, then

$$\sum_{a=s+1}^{r}(t_{kj}^{(a-1)}t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)}t_{il}^{(a-1)}) = 0$$

since, for instance, at $a = s+1$, we get the term

$$t_{kj}^{(s)} t_{il}^{(r-1)} - t_{kj}^{(r-1)} t_{il}^{(s)}$$

and at $a = r$ we get

$$t_{kj}^{(r-1)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r-1)}$$

the sum of which is zero. □

1.6. Definition. Let us define the *natural filtration* on $Y_N$ to be given by $\deg t_{ij}^{(r)} = r$.

1.7. Lemma. $\operatorname{gr} Y(\mathfrak{gl}_N)$ *is commutative.*

Proof. This is immediate from a degree argument based on the relations **??**. □

In fact, we will later see that $\operatorname{gr} Y(\mathfrak{gl}_N) \cong \mathbb{C}[t_{ij}^{(r)}]$ as algebras.

1.8. Theorem (PBW Theorem). *Given an arbitrary linear order of the* $t_{ij}^{(r)}$*, the order monomials form a basis of* $Y(\mathfrak{gl}_N)$*.*

Idea of Proof. If we let $\bar{t}_{ij}^{(r)}$ be the image of $t_{ij}^{(r)}$ in the $r$th component of $\operatorname{gr} Y(\mathfrak{gl}_N)$, then it suffised to show that the $\bar{t}_{ij}^{(r)}$ are algebraically independent. Then, one constructs a homomorphism for any $M \geq 0$,

$$\bar{\zeta}_M \colon \operatorname{gr} Y(\mathfrak{gl}_N) \to \mathfrak{S}(\mathfrak{gl}_{N+M})$$

such that $\bar{\zeta}_M(\bar{t}_{ij}^{(r)}) = p_{ij}^{(r)}$ such that $p_{ij}^{(r)}(X) = (X^r)_{ij}$ for any $X \in \mathfrak{gl}_{M+N}$. Then, if one considers a finite family of elements $\bar{t}_{ij}^{(r)}$, there is some $R$ such that $1 \leq r \leq R$ and one needs to demonstrate that the parameter $M$ can be chosen so that the polynomials $p_{ij}^{(r)}$ are algebraically independent. □

1.9. Corollary. *(a)* $\operatorname{gr} Y(\mathfrak{gl}_N) = \mathbb{C}[\bar{t}_{ij}^{(r)}]$.

*(b) There exists an embedding* $\mathcal{U}(\mathfrak{gl}_N) \hookrightarrow Y(\mathfrak{gl}_N)$ *such that* $E_{ij} \mapsto t_{ij}^{(1)}$.
*(c) There exists an "evaluation" map* $Y(\mathfrak{gl}_N) \twoheadrightarrow \mathcal{U}(\mathfrak{gl}_N)$ *induced by sending the power series* $t_{ij}(u) \mapsto \delta_{ij} + E_{ij}u^{-1}$.
*(d)*

1.10. Proposition. *There exists three families of automorphisms of* $Y(\mathfrak{gl}_N)$ *induced by sending the power series* $T(u)$ *to the following:*

*(a)* $T(u) \mapsto f(u)T(u)$ *for any* $f(u) := 1 + f_1 u^{-1} + \cdots \in \mathbb{C}[[u^{-1}]]$,
*(b)* $T(u) \mapsto T(u-c)$ *for any* $c \in \mathbb{C}$,
*(c)* $T(u) \mapsto BT(u)B^{-1}$ *for any* $B \in GL_N\mathbb{C}$.

Proof. One needs only show that these maps are actually homomorphisms since any map from a given family also contains its own inverse.

To show that the maps are homomorphisms, one shows they satisfy the RTT relations 1.4. The first two are more or less immediate. The third is a little more work, but is still straightforward. □

1.11. PROPOSITION. *There exist antiautomorphisms of $Y(\mathfrak{gl}_N)$ induced by sending the power series $T(u)$ to the following:*

*(a)* $\sigma(T(u)) = T(-u)$,
*(b)* $t(T(u)) = T^t(u) := \sum e_{ji} \otimes t_{ij}(u)$, *the transpose of* $T(u)$,
*(c)* $S(T(u)) = T^{-1}(u)$

1.12. THEOREM. *The Yangian $Y(\mathfrak{gl}_N)$ has a Hopf algebra structure with comultiplication induced by*

$$\Delta(t_{ij}(u)) = \sum_{k=1}^{N} t_{ik}(u) \otimes t_{kj}(u)$$

*antipode given by $S$ from 1.10(c), and counit induced by $\epsilon(T(u)) = 1$.*

Write down proof

1.13. DEFINITION. We define the *loop filtration* on $Y(\mathfrak{gl}_N)$ to be given by $\deg' t_{ij}^{(r)} = r - 1$.

1.14. PROPOSITION. $\mathcal{U}(\mathfrak{gl}_N[z]) \cong \mathrm{gr}'(Y(\mathfrak{gl}_N))$ *as Hopf algebras via the map sending*

$$E_{ij} z^{r-1} \mapsto t_{ij}^{(r)}$$

*for $r \geq 1$*

PROOF. In $\mathrm{gr}' Y_N$, we have the relation

$$[\bar{t}_{ij}^{(r)}, \bar{t}_{kl}^{(s)}] = \delta_{kj} \bar{t}_{il}^{(r+s-1)} - \delta_{il} \bar{t}_{kj}^{(r+s-1)}$$

which follows from 1.5. Thus, the map gives a surjective homomorphism of algebras with trivial kernel by the PBW Theorem **??**. Then, one needs only compare the comultiplication structures to conclude this is a morphism of Hopf algebras. □

1.15. REMARK. Consider $Y(\mathfrak{gl}_N, h)$ for any $h \in \mathbb{C}$ which is an associative algebra generated by $t_{ij}^{(r)}$ subject to the relations

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \delta_{kj} t_{il}^{(r+s-1)} - \delta_{il} t_{kl}^{(r+s-1)} + h \left( \sum_{a=2}^{\min(r,s)} t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \right)$$

with comultiplication

$$\Delta(t_{ij}^{(r)}) = t_{ij}^{(r)} \otimes 1 + 1 \otimes t_{ij}^{(r)} + h \sum_{k=1}^{N} \sum_{s=1}^{r-1} t_{ik}^{(s)} \otimes t_{kj}^{(r-s)}$$

Then,

- $Y(\mathfrak{gl}_N, 0) \cong \mathcal{U}(\mathfrak{gl}_N[z])$,

- $Y(\mathfrak{gl}_N, 1) = Y(\mathfrak{gl}_N)$,
- For all $h \neq 0$, $Y(\mathfrak{gl}_N, h) \cong Y(\mathfrak{gl}_N)$ as Hopf algebras via $t_{ij}^{(r)} \mapsto t_{ij}^{(r)} h^{r-1}$.

**1.1. Quantum Determinant.** The quantum determinant will help us get a concrete description of the center of $Y(\mathfrak{gl}_N)$.

1.16. Definition. Let us define the rational function in variables $u_1, \ldots, u_m$ with values in $(\operatorname{End} \mathbb{C}^N)^{\otimes m}$ by

$$Q(u_1, \ldots, u_m) := (Q_{m-1,m})(Q_{m-2,m}Q_{m-2,m-1}) \cdots (\underbrace{Q_{m-a,m} \cdots Q_{m-a,m-a+1}}_{a \text{ terms}}) \cdots (Q_{1,m} \cdots Q_{1,2})$$

where $Q_{i,j} = R_{ij}(u_i - u_j) = 1 - \frac{P_{ij}}{u_i - u_j}$.

1.17. Proposition. *We have that*

$$Q(u_1, \ldots, u_m) T_1(u_1) \cdots T_m(u_m) = T_m(u_m) \cdots T_1(u_1) Q(u_1, \ldots, u_m)$$

1.18. Proposition. *(a) If $u_i - u_{i+1} = 1$ for all $1 \leq i < m$, then*

$$Q(u_1, \ldots, u_m) = A_m$$

*where $A_m$ is the natural image of the anti-symmetrizer under the natural action of $\mathfrak{S}_m$ on $(\mathbb{C}^N)^{\otimes m}$.*

*(b) If $u_i - u_{i+1} = -1$ for all $1 \leq i < m$, then*

$$Q(u_1, \ldots, u_m) = H_m$$

*where $H_m$ is the natural image of the symmetrizer under the natural action of $\mathfrak{S}_m$ on $(\mathbb{C}^N)^{\otimes m}$.*

Proof. The proof is done via induction. □

Now, consider that, by 1.17 and 1.18(a), we have the equality

$$A_m T_1(u) \cdots T_m(u - m + 1) = T(u - m + 1) \cdots T_1(u) A_m$$

where $A_m$ is identified with $A_m \otimes 1$. Then, if $m = N$, the space $A_N(C^N)^{\otimes N} = \langle e_1 \otimes \cdots \otimes e_N \rangle$ is one dimensional. Thus,

$$A_N T_1(u) \cdots T_N(u - N + 1) = C A_N$$

for some $C \in Y_N[[u^{-1}]]$ and so we are led to the following definition.

1.19. Definition. We define $\operatorname{qdet} T(u) \in Y_N[[u^{-1}]]$ to be the element such that

$$A_N T_1(u) \cdots T_N(u - N + 1) = A_N \operatorname{qdet} T(u)$$

Furthermore, we define $d_i \in Y_N$ by the formal power series $\operatorname{qdet} T(u) = 1 + d_1 u^{-1} + d_2 u^{-2} + \cdots$

1.20. Proposition. *We have explicit formula*

$$\operatorname{qdet} T(u) = \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn} \sigma \, t_{\sigma(1),1}(u) \cdots t_{\sigma(N),N}(u - N + 1)$$

1.21. Definition. For $m \leq N$, we define *quantum minors* $t^{a_1,\dots,a_m}_{b_1,\dots,b_m}(u)$ to be the numbers given by

$$A_m T_1(u) \cdots t_m(u-m+1) = \sum_{a_i,b_i=1}^{N} e_{a_1 b_1} \otimes \cdots \otimes e_{a_m,b_m} \otimes t^{a_1,\dots,a_m}_{b_1,\dots,b_m}(u)$$

1.22. Remark. When $m = N$, we recover the quantum determinant. In other words, $\operatorname{qdet} T(u) = t^{1\cdots N}_{1\cdots N}(u)$.

1.23. Proposition. *For $\sigma \in \mathfrak{S}_m$, we have $t^{\sigma(a_1)\cdots\sigma(a_m)}_{b_1\cdots b_m}(u) = \operatorname{sgn}\sigma t^{a_1\cdots a_m}_{b_1\cdots b_m}(u)$.*

1.24. Lemma. [**Mol07**, Prop 1.7.1]

$$(u-v)[t_{kl}(u), t^{a_1\cdots a_m}_{b_1\cdots b_m}(v)] = \sum_{i=1}^{m} t_{a_i l}(u) t^{a_1\cdots k\cdots a_m}_{b_1\cdots b_m}(v) - \sum_{i=1}^{m} t^{a_1\cdots a_m}_{b_1\cdots l\cdots b_m}(v) t_{k b_i}(u)$$

*where the indices $k$ and $l$ in the quantum minors replace $a_i$ and $b_i$, respectively.*

1.25. Corollary. [**Mol07**, Corollary 1.7.2] *For any indices $i, j$ we have that $[t_{a_i b_j}(u), t^{a_1\cdots a_m}_{b_1\cdots b_m}(v)] = 0$. In particular, $[t_{kl}(u), t^{1\cdots N}_{1\cdots N}(v)] = 0$ for all $1 \leq k, l \leq N$.*

1.26. Theorem. [**Mol07**, Theorem 1.7.5] *The elements $d_1, d_2, \dots$ generate the center of $Y(\mathfrak{gl}_N)$ and they are algebraically independent.*

Idea of Proof. The centrality of the $d_i$ are given by 1.25.

For independence, we send $d_i$ to an element of $\mathcal{U}(\mathfrak{gl}_N[z]) \cong \operatorname{gr}' Y_N$ and see that the images are algebraically independent, so the $d_i$'s themselves must be. We do not address the fact that the $d_i$ generate the center here. $\square$

# Bibliography

[Mol07] A. Molev, *Yangians and Classical Lie Algebras*, 2007.