K-theoretic Catalan functions

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UVA Algebra Seminar

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- Schubert calculus
- Catalan functions: a new approach to old problems
- K-theoretic Catalan functions

Overview of Schubert Calculus Combinatorics

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Special basis of polynomials $\{f_\lambda\}$ such that $f_\lambda \cdot f_\mu = \sum_\nu c_{\lambda\mu}^\nu f_\nu$

Overview of Schubert Calculus Combinatorics (cont.)

Combinatorial study of $\{f_\lambda\}$ enlightens the geometry (and cohomology).

Goal
Identify $\{f_\lambda\}$ in explicit (simple) terms amenable to calculation and proofs.

Classical Schubert Calculus

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Find $c_{\lambda\mu}^\nu = \#$ of points in intersection of Schubert varieties $\{X_\lambda\}_{\lambda \subseteq (n^m)}$ in variety $X = \mathsf{Gr}(m, n)$.

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Representatives

Special basis of Schur polynomials $\{s_\lambda\}$ such that $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu$ for Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$.

Schur functions s_λ

Example

Semistandard tableaux: columns increasing and rows non-decreasing.

standard = no repeated letters

Schur functions s_λ

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Semistandard tableaux: columns increasing and rows non-decreasing.

$$\begin{array}{l} \boxed{5} \\ \boxed{3}\,\boxed{4} \\ \boxed{2}\,\boxed{3} \\ \boxed{1}\,\boxed{2}\,\boxed{2}\,\boxed{5} \end{array} \qquad\qquad \begin{array}{l} \boxed{8} \\ \boxed{7}\,\boxed{9} \\ \boxed{3}\,\boxed{4} \\ \boxed{1}\,\boxed{2}\,\boxed{5}\,\boxed{6} \end{array}$$

$$\underbrace{}$$
standard = no repeated letters

Schur function s_λ is a "weight generating function" of semistandard tableaux:

$$\begin{array}{c}\boxed{2}\\\boxed{1}\,\boxed{1}\end{array}, \begin{array}{c}\boxed{3}\\\boxed{1}\,\boxed{1}\end{array}, \begin{array}{c}\boxed{3}\\\boxed{2}\,\boxed{2}\end{array}, \begin{array}{c}\boxed{2}\\\boxed{1}\,\boxed{2}\end{array}, \begin{array}{c}\boxed{3}\\\boxed{1}\,\boxed{3}\end{array}, \begin{array}{c}\boxed{3}\\\boxed{2}\,\boxed{3}\end{array}, \begin{array}{c}\boxed{2}\\\boxed{1}\,\boxed{3}\end{array}, \begin{array}{c}\boxed{3}\\\boxed{1}\,\boxed{2}\end{array}$$

$$s_{\boxplus}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + 2x_1 x_2 x_3$$

Schur functions s_λ (cont.)

Pieri rule

Determines multiplicative structure:

$$s_r s_\lambda = \sum (1 \text{ or } 0) s_\nu$$

$$s_\square s_{\square\square} = s_{\square\square\blacksquare} + s_{\square\blacksquare} + s_\blacksquare$$

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Iterate Pieri rule

$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\# \text{ known tableaux}) s_\nu$$

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$$s_{\mu_1} \cdots s_{\mu_r} s_\lambda = \sum (\text{\# known tableaux}) s_\nu$$

Since $s_{\mu_1} \cdots s_{\mu_r} = s_{(\mu_1, \ldots, \mu_r)} +$ lower order terms, subtract to get

$$s_{(\mu_1, \ldots, \mu_r)} s_\lambda = \sum c_{\lambda\mu}^\nu s_\nu$$

for well-understood *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu$.

Next Step: Flag Variety

- $X = Fl_n(\mathbb{C}) = \{V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \mid \dim V_i = i\}$

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$$\mathfrak{S}_{s_i} = x_1 + \cdots + x_i$$

Open Problem

Structure constants $\mathfrak{S}_w \mathfrak{S}_u = \sum_v c_{wu}^v \mathfrak{S}_v$ have no tableaux description.

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(Co)homology of Grassmannian	Schur functions
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Quantum cohomology of flag variety	Quantum Schuberts
(Co)homology of Types BCD Grassmannian	Schur-P and Q functions
(Co)homology of affine Grassmannian	(dual) k-Schur functions
K-theory of Grassmannian	Grothendieck polynomials
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And many more!

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where $s_\lambda^{(k)}$ is a k-Schur symmetric function and $Gr_{SL_{k+1}}$ is the "affine Grassmannian."

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Computations for (quantum) Schubert polynomials can be moved into symmetric functions.

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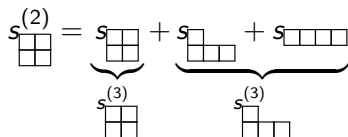
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- Many conjecturally equivalent definitions.

- Schubert calculus
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Key: Catalan functions = large class of symmetric functions.

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$$s_\lambda = \prod_{i<j}(1-R_{ij})h_\lambda$$

$$s_{22} = (1-R_{12})h_{22} = h_{22} - h_{31}$$
$$s_{211} = (1-R_{12})(1-R_{23})(1-R_{13})h_{211}$$
$$= h_{211} - h_{301} - h_{220} \underbrace{-h_{310} + h_{310}}_{} + \underbrace{h_{32\text{-}1}}_{=0} + h_{400} - \underbrace{h_{41\text{-}1}}_{=0}$$

some terms cancel

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Simplifies formulas. E.g., for $\langle s_{1^r}^{\perp} s_{\lambda}, s_{\mu} \rangle = \langle s_{\lambda}, s_{1^r} s_{\mu} \rangle$ (note $\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda\mu}$),

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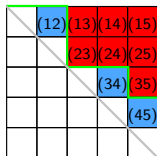
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$$s_{1^r}^\perp s_\lambda = \sum_{S \subseteq [1,\ell], |S|=r} s_{\lambda - \epsilon_S}$$

$$s_{1^2}^\perp s_{333} = s_{322} + s_{232} + s_{223}$$

Root Ideals

A root ideal Ψ of type $A_{\ell-1}$ positive roots: given by Dyck path (lattice path above diagonal).

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Catalan Function (Chen, 2010; Panyushev, 2010; Blasiak et al., 2019)

For Ψ and $\gamma \in \mathbb{Z}^\ell$

$$H(\Psi; \gamma)(x) = \prod_{(i,j) \in \Delta_\ell^+ \backslash \Psi} (1 - R_{ij}) h_\gamma(x)$$

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- $\Psi = $ all roots $\implies H(\Psi; \gamma) = h_\gamma$

Catalan functions

Intuition

Catalan functions interpolate between h_λ and s_λ.

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Theorem (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive!

Catalan functions

k-Schur root ideal for λ

$$\Psi = \Delta^k(\lambda) = \{(i,j) : j > k - \lambda_i\}$$
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$$\Delta^4(3,3,2,2,1,1) = \begin{array}{}\end{array}$$ $\leftarrow$ row i has $4 - \lambda_i$ non-roots

k-Schur is a Catalan function (Blasiak et al., 2019).

For partition λ with $\lambda_1 \leq k$,

$$s_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda).$$

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

Key ingredient of branching proof

Dual vertical Pieri rule: $s_{1^r}^{\perp} s_\lambda^{(k)} = \sum_\mu a_{\lambda\mu} s_\mu^{(k)}$ for $\langle s_{1^r}^{\perp} f, g \rangle = \langle f, s_{1^r} g \rangle$.

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For partition λ of length ℓ with $\lambda_1 \leq k$,

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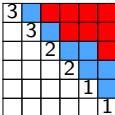
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Proof: $k - \lambda_i = (k+1) - (\lambda_i + 1)$

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For partition λ of length ℓ with $\lambda_1 \le k$,

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Proof: $k - \lambda_i = (k+1) - (\lambda_i + 1)$

$$\Delta^4(3,3,2,2,1,1) = \quad \Delta^5(4,4,3,3,2,2) =$$

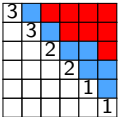# Key ingredient of branching proof

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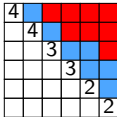
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Branching is a special case of Pieri:

$$s_\lambda^{(k)} = s_{1^\ell}^\perp s_{\lambda+1^\ell}^{(k+1)} = \sum_\mu a_{\lambda+1^\ell,\mu} s_\mu^{(k+1)}$$

- Schubert calculus
- Catalan functions: a new approach to old problems
- **K-theoretic Catalan functions**

Dual Grothendieck polynomials

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- Dual to Grothendieck polynomials G_λ: Schubert representatives for $K^*(Gr(m, n))$

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No direct formula for $g_\lambda^{(k)}$

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Find a formula for $g_\lambda^{(k)}$ analogous to raising operator formula for $s_\lambda^{(k)}$.

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Requires an inhomogeneous refinement of Catalan functions.

Lowering Operators $L_j(f_\lambda) = f_{\lambda - \epsilon_j}$

Affine K-Theory Representatives with Raising Operators

K-theoretic Catalan function

Let $\Psi, \mathcal{L} \subseteq \Delta_\ell^+$ be order ideals of positive roots and $\gamma \in \mathbb{Z}^\ell$, then

$$K(\Psi; \mathcal{L}; \gamma) := \prod_{(i,j) \in \mathcal{L}} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma$$

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Example

non-roots of Ψ, roots of $\mathcal{L}$

	(12)	(13)	(14)	(15)
		(23)	(24)	(25)
			(34)	(35)
				(45)

$K(\Psi; \mathcal{L}; 54332)$

$= (1 - L_4)^2 (1 - L_5)^2 (1 - R_{12})(1 - R_{34})(1 - R_{45}) k_{54332}$

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Example

$$g_{332111111}^{(4)} = \quad \Delta_9^+/\Delta^4(332111111), \Delta^5(332111111)$$

Pieri Rule Illustrated (Recurrences)

A "graphical calculus."

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3-core perspective:

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- k-Rectangle Property fails for $g_\lambda^{(k)}$.

Positivity of Katalan functions

Recall (Blasiak et al., 2020)

For Ψ any root ideal and λ a partition, $H(\Psi; \lambda)$ is Schur positive.

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Future Directions

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References

Thank you!

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Details

$$k_m^{(r)} = \sum_{i=0}^{m} \binom{r+i-1}{i} h_{m-i} = s_m(X+r),$$

a specialization of "multiSchur functions." See, e.g., Lascoux-Naruse (2014).

$$k_\gamma = k_{\gamma_1}^{(0)} k_{\gamma_2}^{(1)} \cdots k_{\gamma_\ell}^{(\ell-1)}$$