

Diagonal Harmonics and Shuffle Theorems

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on joint work with Jonah Blasiak, Mark Haiman, Jennifer Morse, and Anna Pun
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OIST Representation Theory Seminar

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- Symmetric polynomials and diagonal harmonics
- The Shuffle Theorem and its generalizations
- Proof techniques and new progress

Symmetric Polynomials

- Polynomials $f \in \mathbb{Q}(q, t)[x_1, \dots, x_n]$ satisfying $\sigma.f = f$ for all $\sigma \in S_n$.

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Generators

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \text{ or } h_r = \sum_{i_1 \leq i_2 \leq \dots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

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- E.g. for $n = 3$,

$$e_1 = x_1 + x_2 + x_3 = h_1$$

$$e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad h_2 = x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2$$

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- Λ is a $\mathbb{Q}(q, t)$ -algebra.

Schur Polynomials

Distinguished basis of Schur polynomials

$$s_{\mu}(x_1, \dots, x_l) = \sum_{w \in S_l} w \left(\frac{x_1^{\mu_1} \cdots x_l^{\mu_l}}{\prod_{i < j} (1 - x_j/x_i)} \right)$$

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Hidden Guide: Schur Positivity

“Naturally occurring” symmetric functions which are non-negative (coefficients in $\mathbb{N}[q, t]$) linear combinations in Schur polynomial basis are interesting since they could have representation-theoretic models.

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$$\begin{aligned} M &= \text{sp} \left\{ \left(\partial_{x_1}^a \partial_{x_2}^b \partial_{x_3}^c \right) \Delta \mid a, b, c \geq 0 \right\} \\ &= \text{sp} \{ \Delta, 2x_1(x_2 - x_3) - x_2^2 + x_3^2, 2x_2(x_3 - x_1) - x_3^2 + x_1^2, \\ &\quad x_3 - x_1, x_2 - x_3, 1 \} \end{aligned}$$

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Remark: $M \cong \mathbb{C}[x_1, x_2, x_3]/(\mathbb{C}[x_1, x_2, x_3]_+^{S_3})$.

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Solution: irreducible S_n -representation of polynomials of degree $d \mapsto q^d s_\lambda$
(graded Frobenius)

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Answer: Hall-Littlewood polynomial $H_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}}(X; q)$.

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- Does there exist a family of S_n -regular representations whose bigraded Frobenius characteristics equal $\tilde{H}_\lambda(X; q, t)$?

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Irreducible S_n -representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

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Irreducible S_n -representation with bidegree $(a, b) \mapsto q^a t^b s_\lambda$

$$\tilde{H}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} = qts \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} + ts \begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} + qs \begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} + s \begin{smallmatrix} \square & \square & \square \end{smallmatrix}$$

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- No combinatorial description of $\tilde{K}_{\lambda\mu}(q, t)$.

Observation

All of these Garsia-Haiman modules are contained in the module of diagonal harmonics:

$$DH_n = \text{sp}\{f \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \mid \left(\sum_{j=1}^n \partial_{x_j}^r \partial_{y_j}^s \right) f = 0, \forall r + s > 0\}$$

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Question

What symmetric function is the bigraded Frobenius characteristic of DH_n ?

Frobenius characteristic of DH_3

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$$= \frac{t^3 \tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q^2 t + qt^2 + qt) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{q^3 \tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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$$e_3 = \frac{\tilde{H}_{1,1,1}}{-qt^2 + t^3 + q^2 - qt} - \frac{(q+t+1) \tilde{H}_{2,1}}{-q^2 t^2 + q^3 + t^3 - qt} - \frac{\tilde{H}_3}{-q^3 + q^2 t + qt - t^2}$$

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$$\nabla \tilde{H}_\lambda(X; q, t) = q^{n(\lambda)} t^{n(\lambda')} \tilde{H}_\lambda(X; q, t)$$

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Theorem (Haiman, 2002)

The bigraded Frobenius characteristic of DH_n is given by ∇e_n .

Theorem (Carlsson-Mellit, 2018)

$$\nabla e_k(X) = \sum_{\lambda} t^{\text{area}(\lambda)} q^{\text{dinv}(\lambda)} \omega \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

- Conjectured by (Haiman-Haglund-Loehr-Remmel-Ulyanov, 2002).

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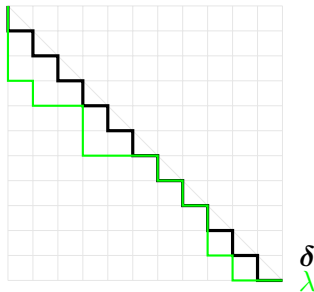
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- $\mathcal{G}_{\nu(\lambda)}(X; q)$ a symmetric LLT polynomial indexed by a tuple of offset rows.

Dyck paths

Dyck paths

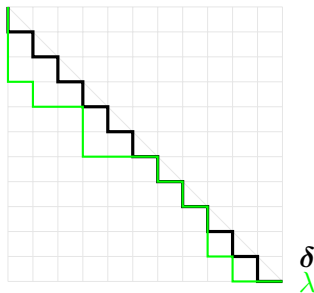
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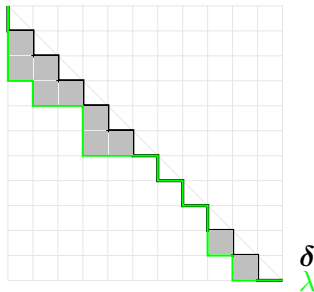


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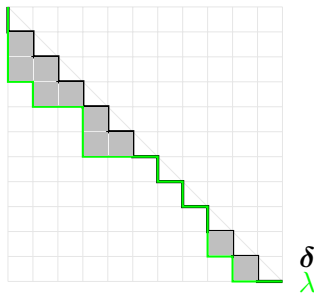


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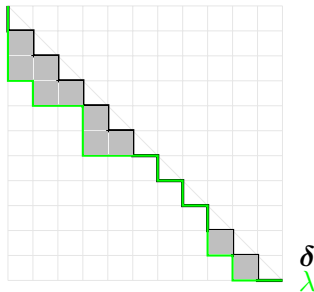


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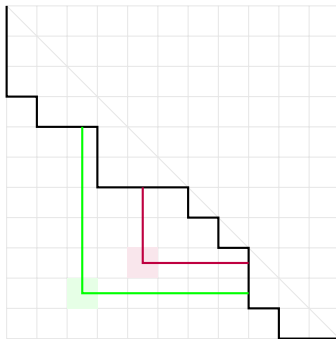
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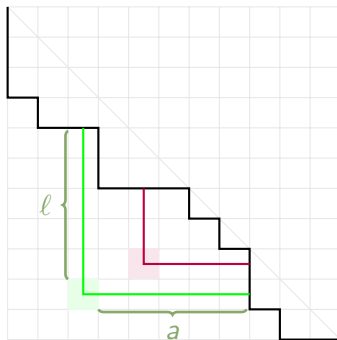
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dinv

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Balanced hook is given by a cell below λ satisfying

$$\frac{\ell}{a+1} < 1 - \epsilon < \frac{\ell+1}{a}, \quad \epsilon \text{ small.}$$

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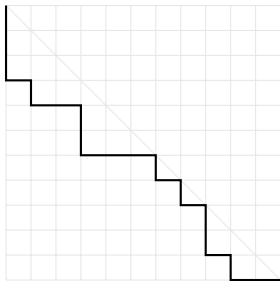
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- \mathcal{G}_ν is Schur-positive for any tuple of skew shapes ν [Grojnowski-Haiman, 2007].

LLT Polynomials

$G_{\nu(\lambda)}(X; q)$ is an LLT polynomial for a tuple of rows,
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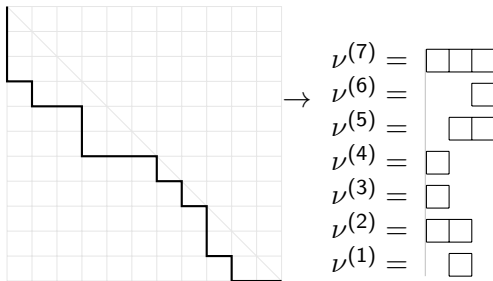
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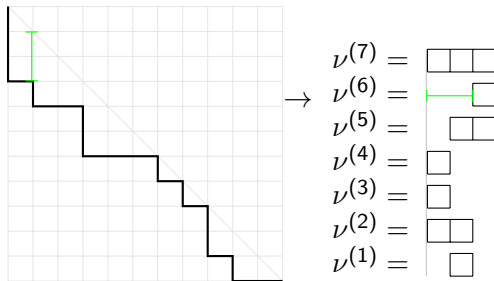
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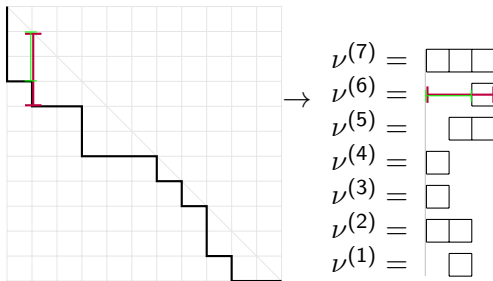
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for T a weakly increasing filling of rows and $i(T)$ the number of attacking inversions:

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$$T = \begin{array}{cccccc} 1 & 2 & 3 & 3 & 5 \\ 2 & 4 & 4 & 7 & 8 & 9 & 9 \\ 1 & 1 & 6 & 7 & 7 & 7 \end{array}$$

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$$\mathcal{G}_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}}(x_1, x_2; q) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + q x_1^2 x_2 + q x_1 x_2^2$$

$$\begin{array}{ccccc} \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} & \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 2 \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}$$

$$= s_3 + q s_{2,1}$$

Example ∇e_3






$$\lambda \mapsto q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$$

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




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Example ∇e_3

λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	
	$q^2 t$	
	$q t$	
	$q t^2$	
	t^3	

Example ∇e_3






λ	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)}$	$q^{\text{dinv}(\lambda)} t^{\text{area}(\lambda)} \mathcal{G}_{\nu(\lambda)}(X; q^{-1})$
	q^3	$s_3 + qs_{2,1} + q^2s_{2,1} + q^3s_{1,1,1}$
	q^2t	$qts_{2,1} + q^2ts_{1,1,1}$
	qt	$ts_{2,1} + qts_{1,1,1}$
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- Entire quantity is q, t -symmetric
- Coefficient of $s_{1,1,1}$ in sum is a “ (q, t) -Catalan number”
 $(q^3 + q^2t + qt + qt^2 + t^3)$.

- Symmetric polynomials and diagonal harmonics
- **The Shuffle Theorem and its generalizations**
- Proof techniques and new progress

Schiffmann's Elliptic Hall Algebra \mathcal{E}

For an abelian category \mathcal{A} , the *Hall algebra* of \mathcal{A} has basis $\{[A]\}_{A \in \text{ob}(\mathcal{A})}$ and product

Schiffmann's Elliptic Hall Algebra \mathcal{E}

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- \mathcal{E} contains, for every coprime $m, n \in \mathbb{Z}$, subalgebra $\Lambda(X^{m,n}) \cong \Lambda$, with relations between them. (Burban-Schiffmann, 2012)

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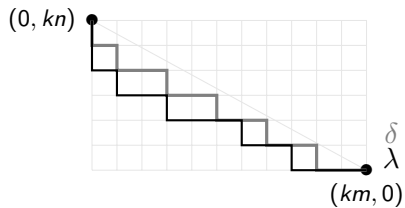
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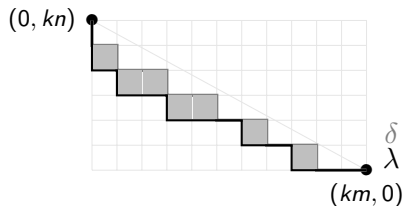
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- Coefficient of $s_{1,\dots,1}$ is “rational (q, t) -Catalan number”

Rational Path Combinatorics

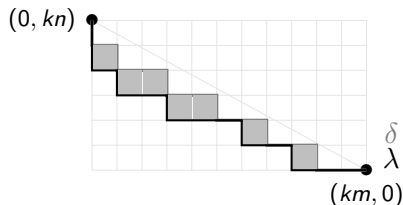


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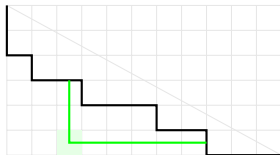


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- $\text{dinv}_p(\lambda) = \text{number of } p\text{-balanced hooks:}$



$$\frac{\ell}{a+1} < p < \frac{\ell+1}{a} \quad p = \frac{n}{m} - \epsilon$$

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For $\mathbf{b} \in \mathbb{Z}^I$, special elements $D_{\mathbf{b}} \in \mathcal{E}$ generalizing $e_k[-MX^{m,n}]$.

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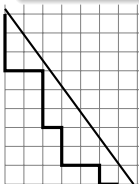
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- Symmetric polynomials and The Shuffle Theorem
- Generalizations of The Shuffle Theorem
- **Proof techniques and new progress**

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- Need an “infinite series” version of LLT polynomials!

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Proof Idea

Let $H_q(f) = \sigma \left(\frac{f}{\prod_{i < j} (1 - qx_i/x_j)} \right)$.

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For $\mathbf{b} \in \mathbb{Z}^l$ corresponding to some choice of highest path under line of slope $-r/s$,

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Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

$$\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda) = r_{i-1}(\lambda) + 1} (1 + zt^{-r_i(\lambda)}).$$

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Same paradigm works to show the following formula.

- $B_\mu = \sum_{(a,b) \in \mu} q^{a-1} t^{b-1}$, e.g., $\mu = \begin{array}{|c|c|c|} \hline 11 & 12 & 13 \\ \hline 21 & 22 & \\ \hline \end{array} \rightarrow B_\mu = 1 + q + q^2 + t + qt$
- $\Delta_f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu$
- $\Delta'_f \tilde{H}_\mu = f[B_\mu - 1] \tilde{H}_\mu$

Extended Delta Theorem (Blasiak-Haiman-Morse-Pun-S., 2021b)

$$\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \langle z^n \rangle \sum_{\lambda, P} q^{\text{dinv}(P)} t^{\text{area}(\lambda)} x^P \prod_{r_i(\lambda) = r_{i-1}(\lambda) + 1} (1 + zt^{-r_i(\lambda)}).$$

- $\Delta_{h_r} \Delta'_{e_{n-1}} e_k = \sum_{\substack{s \in \mathbb{N}^{k+r}: |s|=n-k \\ 1 \in J \subseteq [k+r], |J|=k}} (D_{s+\epsilon_J} \cdot 1)$

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Convex Curve Conjecture (Blasiak-Haiman-Morse-Pun-S., 2021a)

For $\mathbf{b} = (b_1, \dots, b_I)$ the south steps of highest path under a convex curve, the Schur expansion of $D_{\mathbf{b}} \cdot 1$ has coefficients in $\mathbb{N}[q, t]$.

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- Experimental computation suggests this is “tight.”
- Coefficient of $s_{1, \dots, 1}$ coincides with (q, t) -polynomials found in (Gorsky-Hawkes-Schilling-Rainbolt, 2020), (Galashin-Lam, 2021).

Loehr-Warrington Conjecture (2008)

$$\nabla s_\mu = \operatorname{sgn}(\mu) \sum_{(G,R) \in \operatorname{LNDP}_\mu} t^{\operatorname{area}(G,R)} q^{\operatorname{dinv}(G,R)} x^R$$

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- What other rational functions give nice representatives in the Shuffle Algebra? (Catalanimals)
- S_I -representation theory interpretations?

References

Thank you!

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