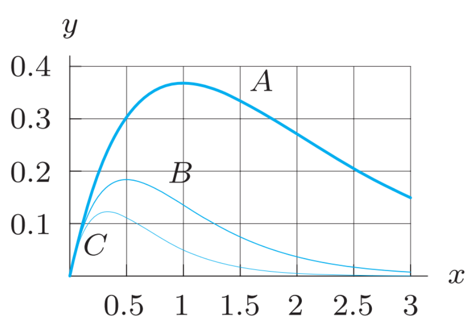
Math 115
Worksheet Section 4.4

Problem 1. Consider the family of functions $\quad f(x) = e^{-(x-a)^2/b}$, for $b > 0$.

Find

1. $f(0)$,
2. $\lim\limits_{x\to\infty} f(x)$,
3. $\lim\limits_{x\to-\infty} f(x)$, and
4. any local maxima and minima.
5. Sketch f.

Problem 2. The graphs of $f(x) = xe^{-ax}$ for $a = 1, 2, 3$ are below. Find the critical point(s) of f, and use this information to identify which graph is which.



Problem 3. Find a formula for a function of the form $y = ae^{-x} + bx$ with a global minimum at $(1, 2)$. Make sure you justify why $(1, 2)$ is a **global** minimum for your function, not just a local minimum.

Problem 4. Find a formula for a cubic polynomial with a critical point at $x = 2$, an inflection point at $(1, 4)$, and a leading coefficient of 1.

9. [8 points] Consider the family of functions given by

$$I(t) = \frac{At^2}{B + t^2}$$

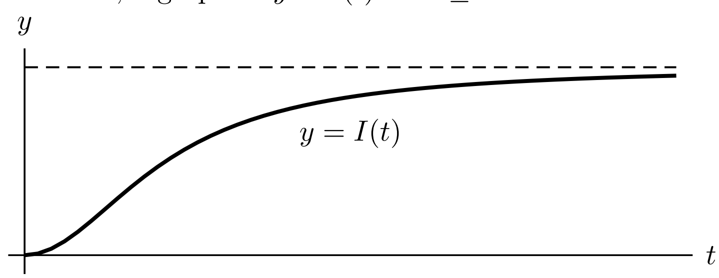
where A and B are positive constants. Note that the first and second derivatives of $I(t)$ are

$$I'(t) = \frac{2ABt}{(B + t^2)^2} \quad \text{and} \quad I''(t) = \frac{2AB(B - 3t^2)}{(B + t^2)^3}.$$

a. [2 points] Find $\lim_{t\to\infty} I(t)$. *Your answer may include the constants A and/or B.*

Answer: $\lim_{t\to\infty} I(t) =$ ____________________

A researcher studying the ice cover over Lake Michigan throughout the winter proposes that for appropriate values of A and B, the function $I(t)$ is a good approximation for the number of thousands of square miles of Lake Michigan covered by ice t days after the start of December. For such values of A and B, a graph of $y = I(t)$ for $t \geq 0$ is shown below.



Based on observations, the researcher chooses values of the parameters A and B so that the following are true.

- $y = 21$ is a horizontal asymptote of the graph of $y = I(t)$.
- $I(t)$ is increasing the fastest when $t = 25$.

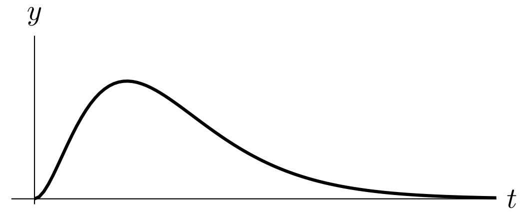
b. [6 points] Find the values of A and B for the researcher's model.
Remember to show your work carefully.

Answer: $A =$ ____________ and $B =$ ____________

4. [7 points] Lin inflates a balloon using a helium pump. When she turns off the pump, the balloon immediately begins to deflate. Lin believes that she can model the balloon's volume, in cubic feet (ft^3), by the function

$$V(t) = \frac{at^2}{e^{bt}},$$

where t is the time, in seconds, after she begins inflating the balloon, and where a and b are positive constants. As an example, this function is shown to the right for one choice of the constants a and b. Note that the derivative of $V(t)$ is given by



$$V'(t) = -\frac{at(bt-2)}{e^{bt}}.$$

a. [4 points] The function $V(t)$ appears to have a local maximum at some time $t > 0$. Find the time at which this local maximum occurs. Use calculus to find your answer, and be sure to give enough evidence that the point you find is indeed a local maximum. Your answer may be in terms of a and/or b.

Answer: local max at $t =$ ______________

b. [3 points] Lin knows that it took 8 seconds to inflate the balloon, and that its volume at that time was 1.5 ft^3. Find the **exact** values of a and b for Lin's model. Show your work.

Answer: $a =$ ______________ and $b =$ ______________

1. [14 points] You are online playing the Facebook-based game, FarmVille, and you receive land with 5 stalks of corn on it. You decide that you would like to model the corn population on this patch of land using your calculus skills, so you recall that a good model for population growth is the logistic model

$$P(t) = \frac{L}{1 + Ae^{-kt}} \qquad L > 0, \quad A > 0, \quad k > 0.$$

a. [5 points] Using the ***limit definition of the derivative***, write an explicit expression for the derivative of the function $P(t)$ at $t = 1$. Do not evaluate this expression.

b. [5 points] Using the definition of the logistic model above, compute the following in terms of L, k, and A, showing your work or providing an explanation for each part:

i. [1 points] $\lim_{t\to\infty} P(t)$

ii. [1 points] $\lim_{t\to-\infty} P(t)$

iii. [1 points] $P(0)$

iv. [2 points] $P'(0)$

c. [4 points] Your farmland satisfies the following conditions:

$$P(0) = 5, \ P'(0) = 1, \ \lim_{t\to\infty} P(t) = 100.$$

Based on your answers in part (b), compute the correct values of L, k, and A for the logistic equation modeling corn population on your land.

$L =$ ________________ $A =$ ________________ $k =$ ________________

7. [14 points] For positive A and B, the force between two atoms is a function of the distance, r, between them:

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3} \qquad r > 0.$$

a. [2 points] Find the zeroes of f (in terms of A and B).

b. [7 points] Find the coordinates of the critical points and inflection points of f in terms of A and B.

c. [5 points] If f has a local minimum at $(1, -2)$ find the values of A and B. Using your values for A and B, justify that $(1, -2)$ is a local minimum.

(4.) (12 points) Consider the function:

$$f(x) = e^{\frac{-(ax)^2}{2}}, \qquad \text{for } a \text{ a positive constant.}$$

The graph of $y = f(x)$ is the (in)famous "bell curve," which occurs frequently in statistics, and occasionally in heated political debates as well.

(a) Compute $f''(x)$. Show your work.

(b) For which value of a does the function f have an inflection point at $x = 3$?

4. (10 points) Every year pesticides used on adjacent agricultural land drain off into Lake Michigan. Eventually, scientists predict that the lake will become saturated with pesticides. As a result, the amount of pesticides in the lake $P(t)$ (in parts per million) is given as a function of time, t, in years since 2000, by

$$P(t) = a(1 - e^{-kt}) + b$$

where a, b and k are positive constants. Assume the saturation level of the lake for pesticides is 50 parts per million.

(a) If in the year 2000 the pesticide level of Lake Michigan was 5 parts per million, find a and b.

(b) Find k if the pesticide level was increasing at a rate of 3 parts per million per year in the year 2000.

(c) When the pesticide level reaches 30 parts per million, fish from the lake cannot be consumed by humans. In what year will the pesticide level in the lake reach 30 parts per million?