Math 115
Worksheet Section 4.1

Warm-up

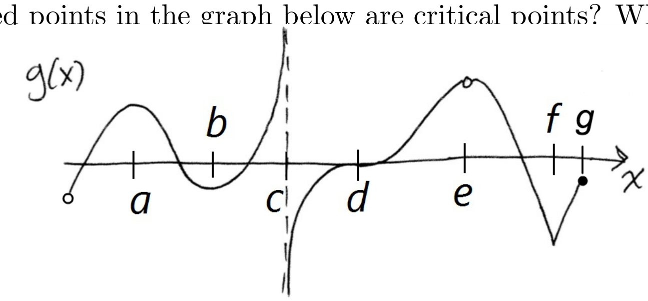
A **critical point** of g is a point p in the domain of g where ____________ or ____________.

Note: "Critical point" may refer to just the x-value, or to the coordinate pair.

A point p in the domain of g where $g(p) \leq g(x)$ for all x near p is a ____________.

A point p in the domain of g where $g(p) \geq g(x)$ for all x near p is a ____________.

Which of the labeled points in the graph below are critical points? Which are local extrema?

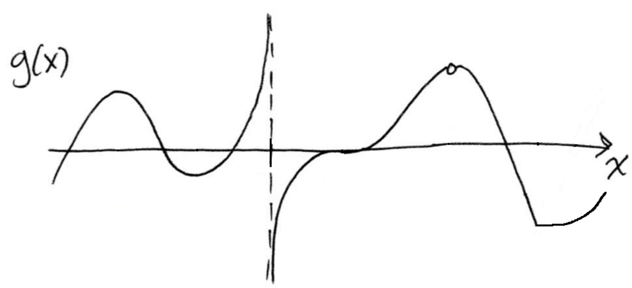


But what if we didn't know what the graph looked like? How would we identify these points?

1st Derivative Test: If p is a critical point of f and f is continuous at p, then, moving from left to right, if:

- f' changes from positive to negative at p, then ____
- f' changes from negative to positive at p, then ____

An **inflection point** of a function g is a point in the domain of g at which g is continuous and where ______________________________. Identify any inflection points in the graph below.



But what if we did not know what the graph looked like? How would we identify these points?

Problem 1. (a) Find the location of all local maxima and minima of $f(x) = (x^2 - 4)^7$. Justify using calculus – *this means you cannot just use a graph of f.* You may use that the derivative is $f'(x) = 14x(x^2 - 4)^6$.

(b) Find the location of any inflection points of $f(x) = (x^2 - 4)^7$. Justify using calculus. You may use that $f''(x) = 14(13x^2 - 4)(x^2 - 4)^5$.

Problem 2. Give an example of

(a) a function f and a critical point a for f such that f *does not* have a local maximum or a local minimum at $x = a$.

(b) a function f and a value of a such that $f'(a) = f''(a) = 0$ but f *does not* have an inflection point at a.

Problem 3. Find and justify using calculus the location of all local maxima and minima of

$$h(x) = \begin{cases} e^x & x \leq 0 \\ x^4 - 4x + 1 & 0 < x \end{cases}$$

You may use the facts that $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}(x^4 - 4x + 1) = 4x^3 - 4$.

Problem 4. Find all critical points of

$$p(x) = x^3(x^2 - 5x + 5), \quad \text{given that}$$

$$p'(x) = 5x^2(x-1)(x-3) \quad \text{and} \quad p''(x) = 10x(2x^2 - 6x + 3).$$

Then **use concavity** to determine whether the p has a local max or min at each critical point, if possible. This is called the **2nd Derivative Test**.

Problem 5. Suppose that the *derivative* of an everywhere differentiable function $k(x)$ is given by

$$k'(x) = (x-a)(x-b)(x-c),$$

where $a < b < c$ are constants. Using calculus, find the location of all local maxima and minima.

Problem 6. (Winter 2018 Exam 2) In the following question, use calculus to justify your answers and show enough evidence to demonstrate that you have found them all. Determine your answers algebraically.

(a) Let $f(x)$ be a continuous function defined for all real numbers with derivative given by

$$f'(x) = \frac{(2x+1)(x-2)^2}{(x+3)^{\frac{1}{3}}}.$$

Find the x-coordinate(s) of the local maximum(s) and local minimum(s) of the function $f(x)$. Write *none* if the function has no local maximum(s) and/or local minimum(s).

(b) Let $g(x)$ be a continuous function defined for all real numbers with second derivative given by

$$g''(x) = (2^x - 4)(x^2 - 4).$$

Find the x-coordinates of the inflection points of the function $g(x)$. Write *none* if the function has no inflection points.

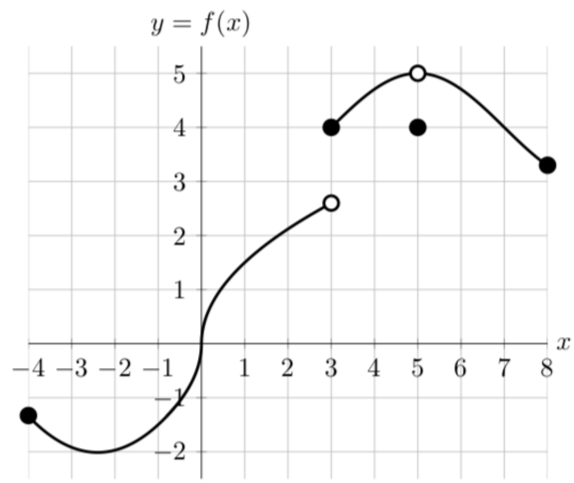
Problem 7. (Fall 2017 Final Exam) Let A and B be positive constants and $f(x) = \frac{A(x^2-B)}{\sqrt{x-3}}$ for all $x > 3$. Note that

$$f'(x) = \frac{A(x^2 - 12x + B)}{2(x-3)^{\frac{3}{2}}} \qquad \text{and} \qquad f''(x) = \frac{2A(x^2 - 8x + 24 - B)}{4(x-3)^{\frac{5}{2}}}.$$

Find all the values A and B such that f has an inflection point at $(8, 2)$. Use calculus to justify that $(8, 2)$ is an inflection point. If there are no such values, write *none*.

Problem 8. (Winter 2018 Exam 2) The graph of the function f with domain $-4 \leqslant x \leqslant 8$ is shown below. The function $f(x)$ satisfies:

- $f(x) = 1.5x^{\frac{1}{3}}$ for $-1 < x < 1$, and
- $f(x) = 4 + \sin\left(\frac{\pi}{4}(x-3)\right)$ for $3 \leqslant x < 5$ and $5 < x \leqslant 8$.

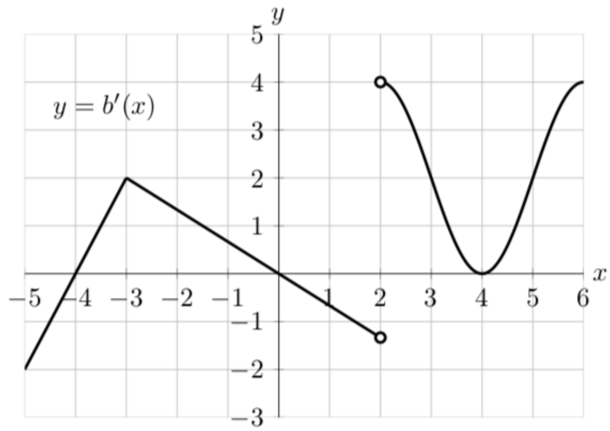


(a) Estimate the x-coordinate(s) of all the local minimum(s) of $f(x)$ in $-4 < x < 8$. Write *none* if $f(x)$ does not have any local minima.

(b) Find the value(s) of b in $-4 < x < 8$ for which the limit $\lim_{h \to 0} \frac{f(b+h) - f(b)}{h}$ does not exist. Write *none* if there are no such values of b.

(c) Estimate the x-coordinate(s) of all critical points of $f(x)$ in $-4 < x < 8$. Write *none* if $f(x)$ does not have any critical points.

Problem 9. (Winter 2017 Exam 2) The graph of a portion of the derivative of $b(x)$ is shown below. Assume that $b(x)$ is defined and continuous on $[-5, 6]$.



(a) At which of the following values of x does $b(x)$ appear to have a critical point?

$x = -4$ $\quad$ $x = -3$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ none of these

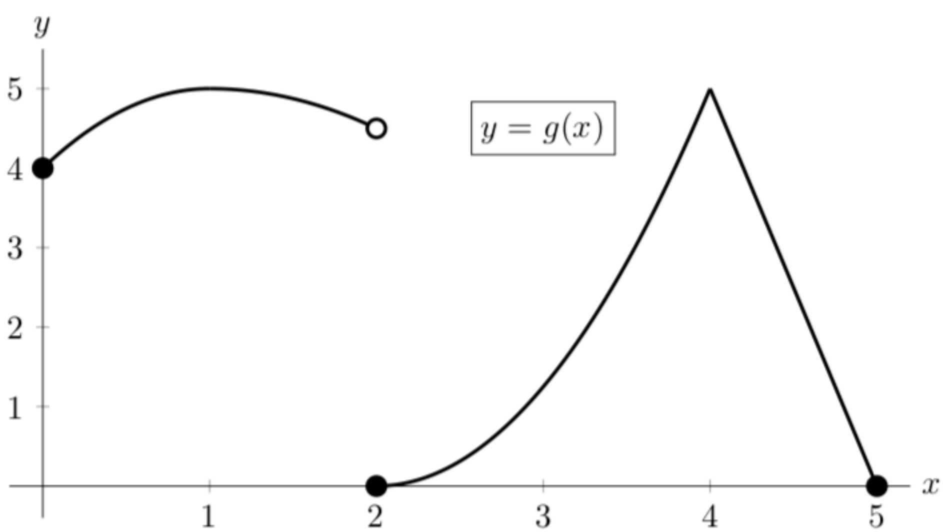
(b) At which of the following values of x does $b(x)$ attain a local minimum?

$x = -4$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ $x = 5$ $\quad$ none of these

(c) At which of the following values of x does $b(x)$ appear to have an inflection point?

$x = -3$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ $x = 5$ $\quad$ none of these

Problem 10. (Fall 2016 Exam 2) The entire graph of a function $g(x)$ is shown below. Note that the graph of $g(x)$ has a horizontal tangent line at $x = 1$ and a sharp corner at $x = 4$.



(a) At which of the following values of x does $g(x)$ appear to have a critical point?

$x = 1$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ $x = 4$ $\quad$ none of these

(b) At which of the following values of x does $g(x)$ attain a local maximum?

$x = 1$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ $x = 4$ $\quad$ none of these

Problem 11. (Winter 2016 Exam 2) Let $h(x)$ be a twice differentiable function defined for all real numbers x. (So h is differentiable and its derivative h' is also differentiable.) Some values of the derivative of h are given in the table below.

x	-8	-6	-4	-2	0	2	4	6	8
$h'(x)$	3	7	0	-3	-5	-4	0	-2	6

(a) Circle all the intervals below in which $h(x)$ must have a critical point.

$-8 < x < -6$ $\quad$ $-6 < x < -2$ $\quad$ $-2 < x < 2$ $\quad$ $2 < x < 6$ $\quad$ $6 < x < 8$ $\quad$ none of these

(b) Circle all the intervals below in which $h(x)$ must have a local extremum (i.e. a local maximum or a local minimum).

$-8 < x < -6$ $\quad$ $-6 < x < -2$ $\quad$ $-2 < x < 2$ $\quad$ $2 < x < 6$ $\quad$ $6 < x < 8$ $\quad$ none of these

(c) Circle all the intervals below in which $h(x)$ must have an inflection point.

$-8 < x < -4$ $\quad$ $-4 < x < 0$ $\quad$ $0 < x < 4$ $\quad$ $2 < x < 6$ $\quad$ $4 < x < 8$ $\quad$ none of these