Math 115
Worksheet Section 4.5 continued

The basics

- The **cost function** $C(q)$ gives the cost of producing a quantity q of a certain good.
- The **revenue function** $R(q)$ gives the revenue received from selling a quantity q of some good.
- The **profit** $\pi(q) = R(q) - C(q)$ gives the total profit from producing and selling q of that good.
- To decide whether a company's profit would increase or decrease if the company increased or decreased production of a certain good, we might look at the marginal cost and marginal revenue:
- The **marginal cost** is given by $\text{MC}(q) = C'(q) \approx C(q+1) - C(q)$
- The **marginal revenue** is given by $\text{MR}(q) = R'(q) \approx R(q+1) - R(q)$

When can maximum profit occur?

Where marginal cost = marginal revenue, other critical points where MR or MC do not exist, and at endpoints

How do we identify the fixed cost of producing a certain good?

The value of $C(0)$.

6. [11 points] Ben has recently acquired a cabbage press and is opening a business selling cabbage juice. Let $R(x)$ and $C(x)$ be the revenue and cost, in dollars, of selling and producing x cups of cabbage juice. Ben only has resources to produce up to a hundred cups. After some research, Ben determines that

$$R(x) = 6x - \frac{1}{40}x^2 \qquad \text{for} \qquad 0 \le x \le 100$$

and

$$C(x) = \begin{cases} 60 + 2x & 0 \le x \le 20 \\ 70 + 1.5x & 20 < x \le 100. \end{cases}$$

a. [3 points] What is the smallest quantity of juice Ben will need to sell in order for his profit to not be negative? Round your answer to the nearest hundredth of a cup. Show your work.

Solution: We consider values of x such that $R(x) = C(x)$. We first look in $[0, 20]$

$$60 + 2x = 6x - \frac{1}{40}x^2 \qquad \text{or} \qquad \frac{1}{40}x^2 - 4x + 60 = 0.$$

Using the quadratic formula we get $x = 80 \pm 20\sqrt{10}$. Only one of these two solutions, $x = 80 - 20\sqrt{10} \approx 16.75$, is in the interval $[0, 20]$.

The last step is to verify that $R(x) - C(x)$ is negative on the interval $[0, \quad 80 - 20\sqrt{10})$ and positive on the interval $(80 - 20\sqrt{10}, \quad 20]$. We can test this by picking points in each interval. For example, $R(0) - C(0) = -60$ and $R(20) - C(20) = 10$. **Answer:** 16.75 cups.

For the following parts, determine how many cups of cabbage juice Ben needs to sell in order to maximize the given quantity. If there is no such value, write NONE. Use calculus to find and justify your answers.

b. [3 points] Ben's revenue.

Solution: The critical points of $R(x)$ can be found by solving $R'(x) = 6 - \frac{1}{20}x = 0$. This occurs when $x = 120$ which is not in $[0, 100]$. Hence the maximum has to be at one of the endpoints $x = 0$ or $x = 100$. Since $R(0) = 0$ and $R(100) = 350$, the maximum revenue is attained at $x = 100$. **Answer:** 100 cups.

c. [5 points] Ben's profit.

Solution: Since $P(20) = 10$ and

$$\lim_{x \to 20^-} P(x) = \lim_{x \to 20^-} 6x - \frac{1}{40}x^2 - (60 + 2x) = 10$$

and

$$\lim_{x \to 20^+} P(x) = \lim_{x \to 20^+} 6x - \frac{1}{40}x^2 - (70 + 1.5x) = 10.$$

Then $P(x)$ is continuous on $[0, 20]$. The critical points of $P(x)$ can be found by solving $P'(x) = R'(x) - C'(x) = 0$ in the intervals $(0, 20)$ and $(20, 100)$.

- On $(0, 20)$ we need to solve $6 - \dfrac{1}{20}x = 2$. This yields $x = 80$ (outside the interval).
- On $(20, 100)$ we need to solve $6 - \dfrac{1}{20}x = 1.5$. This yields $x = 90$.

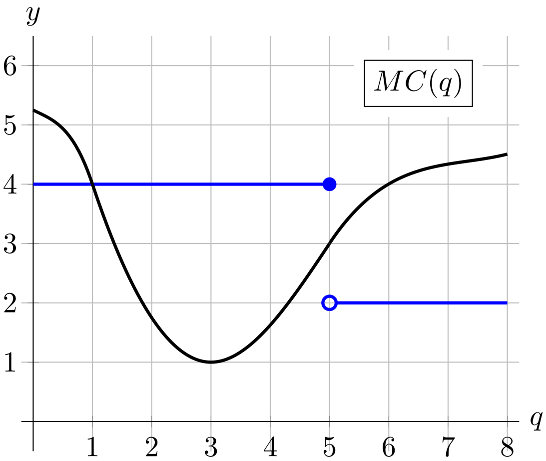
Hence the critical points of $P(x)$ are $x = 0$ and $x = 90$. Since $P(x)$ is continuous then the global maximum must lie on the critical points or in the endpoints.

x	0	20	90	100
$P(x)$	-60	10	132.5	130

Answer: 90 cups.

7. [9 points]
Anna is interested in selling some feed corn to a pair of local farms, and is trying to determine the optimal amount to sell. One farm is willing to buy up to 5000 bushels at a price of \$4 per bushel, while the other is willing to buy up to 3000 bushels at a price of \$2 per bushel. The graph to the right shows the marginal cost $MC(q)$, in thousands of dollars per thousand bushels, of q thousand bushels of corn.

Assume Anna sells as much corn as she can to the farm paying \$4 per bushel before selling any to the farm paying \$2 per bushel.



a. [2 points] On the axes above, carefully sketch the graph of the marginal revenue $MR(q)$, in thousands of dollars per thousand bushels, of q thousand bushels of corn.

b. [1 point] What is the total revenue Anna receives for selling 6000 bushels of corn?

\$20,000 $\boxed{\$22{,}000}$ \$24,000 \$26,000 NONE OF THESE

c. [2 points] Recall that *profit*, $\pi(q)$, equals total revenue minus total cost. Circle all values of q below that are critical points of the profit function $\pi(q)$.

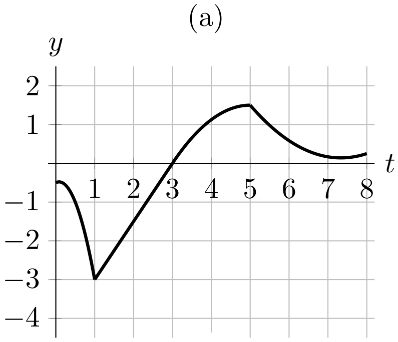
$\boxed{q=1}$ $q=3$ $\boxed{q=5}$ $q=6$ NONE OF THESE

d. [2 points] What production level maximizes profit? Circle all correct answers below.

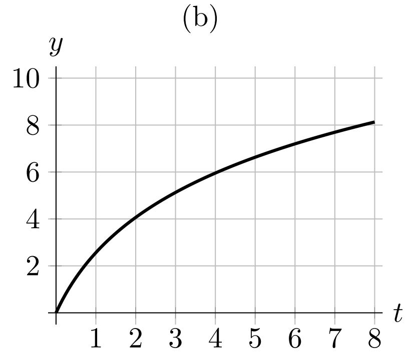
$q=1$ $q=3$ $\boxed{q=5}$ $q=6$ $q=8$ NONE OF THESE

e. [2 points] Which of the following *could* be the graph of the total cost function? Circle the letters of *all* correct answers below.

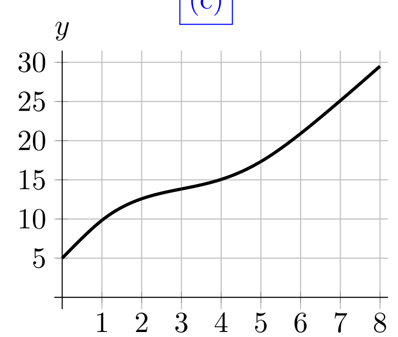
(a)

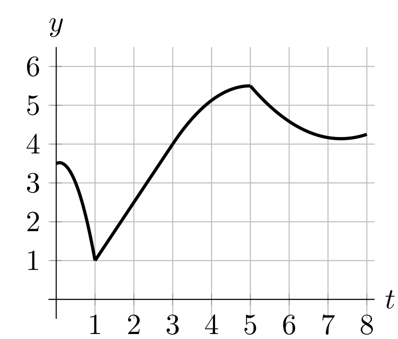


(b)

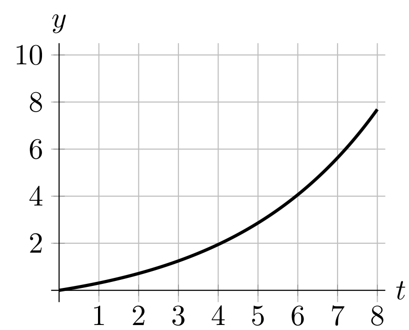


$\boxed{\text{(c)}}$

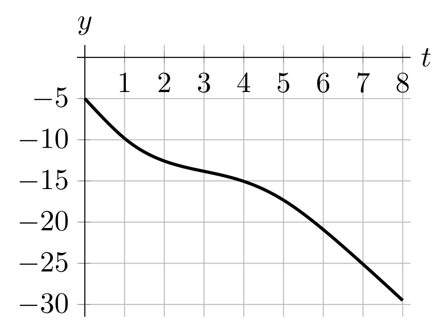




(d)

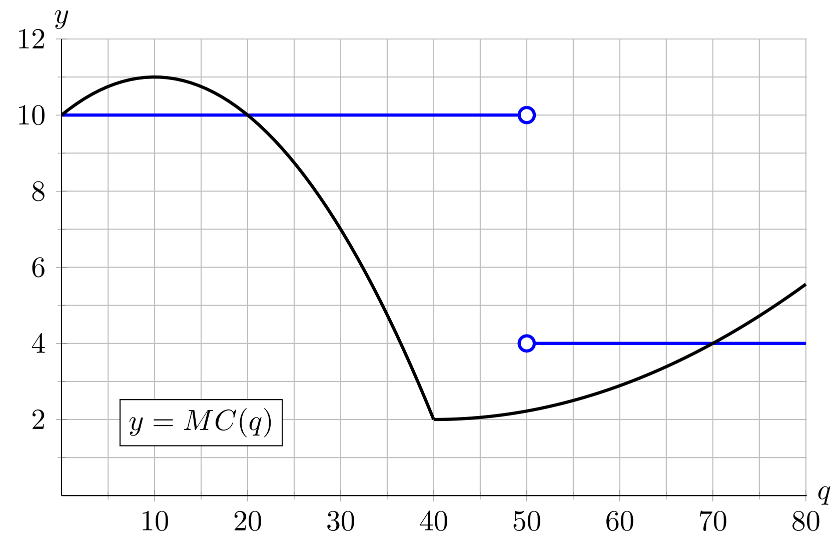


(e)



(f)

5. [10 points] Javier plans to make and sell his own all-natural shampoo. The graph below shows the marginal cost $MC(q)$, in dollars per liter, of q liters of shampoo. In order to start making shampoo, Javier must first spend \$25 on supplies, but he has no other fixed costs.



Javier can sell up to 50 liters of shampoo for \$10 per liter. Any additional shampoo can be sold to a local salon for \$4 per liter. Throughout this problem, you do not need to show work.

a. [2 points] On the axes above, carefully sketch the graph of the marginal revenue $MR(q)$, in dollars per liter, of q liters of shampoo.

Solution: See above.

b. [1 point] At what value(s) of q in the interval $[0, 80]$ is marginal cost maximized?

Answer: 10

c. [1 point] At what value(s) of q in the interval $[0, 80]$ is cost maximized?

Answer: 80

d. [2 points] At which values of q in the interval $[0, 80]$ is profit increasing? Give your answer as one or more intervals.

Answer: $(20, 70)$

e. [1 point] How many liters of shampoo should Javier make in order to maximize his profit?

Answer: 70

f. [3 points] Write an expression involving integrals which represents the company's profit when $q = 45$. Your expression may involve $MC(q)$ and/or $MR(q)$.

Answer: $-25 + \int_0^{45} (MR(q) - MC(q))dq$

7. [9 points] In an unexpected twist, Carson Soltonni also runs a business selling vacuum cleaners out of his house. The cost in hundreds of dollars for him to produce q hundred vacuum cleaners is

$$C(q) = \frac{q^3}{3} - 5q^2 + 59q + 5.$$

Carson sells his vacuum cleaners for 50 dollars each, and he is trying to determine how many to sell in order to maximize profit. Some values of $C(q)$, rounded to the nearest integer, are given below.

q	1	2	3	4	5	6	7	8	9
$C(q)$	59	106	146	182	217	251	287	328	374

a. [1 point] What is the fixed cost of Carson's business?

Answer: 5 hundred dollars.

b. [3 points] Find the marginal revenue function $MR(q)$ and marginal cost function $MC(q)$ of Carson's business, in hundreds of dollars per hundred vacuum cleaners.

Solution: The marginal revenue and marginal cost functions are the derivatives of the revenue and cost functions, respectively. Since Carson sells vacuum cleaners for 50 dollars each, his marginal revenue is \$50 per vacuum cleaner, or, equivalently, 50 hundred dollars per hundred vacuum cleaners. And the marginal cost function will be $MC(q) = C'(q) = q^2 - 10q + 59$.

Answer: $MR(q) =$ 50 and $MC(q) =$ $q^2 - 10q + 59$

c. [3 points] How many vacuum cleaners should Carson produce and sell to maximize profit? *Show your work and use calculus.* You do not need to fully justify your answer, but partial credit may be awarded for work shown.

Solution: Carson's profit, $\pi(q)$, is equal to revenue minus cost, that is, $\pi(q) = R(q) - C(q)$. We want to maximize $\pi(q)$ over the interval $[0, \infty)$. Since $\pi(q)$ is differentiable everywhere, its critical points will occur when $\pi'(q) = 0$, that is, when $MR(q) = MC(q)$. We have

$$\pi'(q) = MR(q) - MC(q) = 50 - (q^2 - 10q + 59) = -(q^2 - 10q + 9) = -(q-1)(q-9),$$

so the critical points of $\pi(q)$ occur at $q = 1$ and $q = 9$. Checking the endpoints, we have

$$\pi(0) = -5 \qquad \text{and} \qquad \lim_{q\to\infty} \pi(q) = \lim_{q\to\infty} (50q - C(q)) = -\infty.$$

Plugging the critical points into π, we get

$$\pi(1) = 50 - 59 = -9 \qquad \text{and} \qquad \pi(9) = 450 - 374 > 0.$$

This is enough to show that profit is maximized at $q = 9$, that is, when Carson produces and sells 900 vacuum cleaners.

Answer: 9 hundred vacuum cleaners.

d. [2 points] Unsure how to solve the calculus problem in part **c.**, Carson just decides to produce and sell as many vacuum cleaners as he can. Unfortunately, a court order terminates Carson's business immediately after he had produced and sold 600 vacuum cleaners. At this point, had Carson's business *gained* or *lost* money? How much?

Give your answer by circling GAINED *or* LOST *and writing a* positive *number on the blank.*

Solution: Selling 600 vacuum cleaners at \$50 per unit nets Carson \$30,000. On the other hand, the cost of selling 600 vacuum cleaners is $C(6) = 251$ hundred dollars, or \$25,100. Thus Carson has gained $30000 - 25100 = 4900$ dollars after selling 600 vacuum cleaners.

Answer: Carson's business $\boxed{\text{GAINED}}$ LOST 49 hundred dollars.

7. [10 points] Zerina owns a small business selling custom screen-printed and embroidered apparel.

a. Zerina receives orders for embroidered polo shirts, which she sells for \$11 each. The cost, in dollars, for her to complete an order of q embroidered polo shirts is

$$C(q) = \begin{cases} 6q - \frac{1}{8}q^2 + \frac{56}{9} & 0 \leq q \leq 16 \\ \frac{2}{9}q^{3/2} + 10q - 104 & q > 16. \end{cases}$$

Note that $C(q)$ is continuous for all $q \geq 0$.

i. [1 point] What is the fixed cost, in dollars, of an order of embroidered polo shirts?

Answer: 56/9

ii. [5 points] Find the quantity q of embroidered polo shirts in an order that would result in the most profit for Zerina. Assume that, because of storage constraints, Zerina cannot accept an order for more than 80 embroidered polo shirts. Use calculus to find and justify your answer, and make sure you provide enough evidence to fully justify your answer.

Solution: We are given $C(q)$, and know that $R(q) = 11q$. Then since $\pi(q) = R(q) - C(q)$, any point at which $MR = MC$ is a critical point of $\pi(q)$. Now $MR(q) = 11$ and

$$MC(q) = \begin{cases} 6 - \frac{1}{4}q & 0 \leq q < 16 \\ \frac{1}{3}q^{1/2} + 10 & q > 16. \end{cases}$$

We set $MR = MC$ in both of these cases:

$$6 - \frac{1}{4}q = 11 \qquad\qquad \frac{1}{3}q^{1/2} + 10 = 11$$

$$-\frac{1}{4}q = 5 \qquad\qquad q^{1/2} = 3$$

$$q = -20 \qquad\qquad q = 9$$

but neither critical point falls within the domain of the appropriate formula. So there are no points at which $MR = MC$. However, MC is undefined at $q = 16$, since if we plug 16 in to both pieces of $MC(q)$ we get different values. Therefore $\pi'(q)$ is also undefined at $q = 16$. This is the only critical point.

So the possible locations for the global maximum are the endpoints 0 and 80 and the critical point 16. Since $\pi(0) = -56/9$, $\pi(80) \approx 25$, and $\pi(16) \approx 105.7$, an order of 16 polo shirts would result in the most profit for Zerina.

Answer: $q =$ **16**

b. [3 points] Zerina also receives orders for screen-printed t-shirts. When a customer places such an order, they pay a \$6 setup fee, plus \$9 per t-shirt for the first 20 t-shirts ordered. Any additional t-shirts ordered only cost \$7 per t-shirt. Let $P(s)$ be the total price, in dollars, a customer pays for an order of s screen-printed t-shirts. Find a formula for $P(s)$.

Answer: $$P(s) = \begin{cases} 6 + 9s & \text{if } 0 \leq s \leq 20 \\ 186 + 7(s - 20) & \text{if } s > 20 \end{cases}$$

3. [16 points] Last summer, Brad and Angelina set up a lemonade stand where they sold lemonade charging 60 cents per ounce. In other words, $MR(q) = 0.6$ (in dollars per ounce) where q is the number of ounces of lemonade they sold.

a. [2 points] Find a formula for their revenue $R(q)$ (in dollars) were q is the amount (in ounces) of lemonade sold. Assume their initial revenue is zero dollars.

Solution: **Answer:** $R(q) = 0.6q$.

Since they are using utensils they already have, they have no fixed costs. They can produce at most 120 ounces of lemonade, and the **marginal cost** function, $MC(q)$, is:

- continuous for $0 < q < 120$,
- concave down for $0 < q < 120$,
- increasing for $0 < q < 60$ and decreasing for $60 < q < 120$.

Brad and Angelina recorded some of the values of $MC(q)$ (in dollars per ounce) in the following table:

q	0	15	30	45	60	75	90	105	120
$MC(q)$	0.15	0.45	0.6	0.7	0.75	0.7	0.6	0.45	0.15

b. [3 points] Recall that $C(60) - C(0) = \int_0^{60} MC(q)\, dq$. Estimate $C(60)$ by using a right-hand Riemann sum with 2 equal subdivisions. Make sure to write down all terms in your sum.

Solution: **Answer:** $30(0.6 + 0.75) = 40.5$

c. [1 point] Is your estimate in **b** an overestimate or an underestimate of $C(60)$? Circle your answer.

Solution:
$\boxed{\text{OVERESTIMATE}}$ UNDERESTIMATE NOT ENOUGH INFORMATION

d. [3 points] Suppose Brad and Angelina want to use a Riemann sum to calculate $C(60)$, accurate within 50 cents of the actual value. At least how many times, using equal intervals on $[0, 60]$, should Brad and Angelina have measured $MC(q)$ in order to guarantee this accuracy? Justify your answer.

Solution:

$$\begin{aligned}\text{Difference between righthand and lefthand sums} &= (MC(60) - MC(0))\Delta q \\ &= (0.75 - 0.15)\Delta q = 0.6\Delta q \leq 0.5.\end{aligned}$$

yields $\Delta q \leq \frac{5}{6} \approx .833$. Recall that $\Delta q = \frac{b-a}{n}$ where n is the number of measurements. This gives:

$$n = \frac{60}{\Delta q} = \frac{60}{5/6} = 72.$$

Answer: They should have measured it at least 72 times.

A part of the question has been reproduced for your convenience.

Brad and Angelina have no fixed costs and they can produce at most 120 ounces of lemonade. The **marginal cost** function (in dollars per ounce), $MC(q)$, is:

- continuous for $0 < q < 120$,
- concave down for $0 < q < 120$,
- increasing for $0 < q < 60$ and decreasing for $60 < q < 120$.

and some of its values are:

q	0	15	30	45	60	75	90	105	120
$MC(q)$	0.15	0.45	0.6	0.7	0.75	0.7	0.6	0.45	0.15

The marginal revenue is $MR(q) = 0.6$ (in dollars per ounce).

e. [2 points] Find all the critical points of the profit function $\pi(q)$.

Solution: **Answer:** $q = 30, 90$.

The table below gives some values of $C(q)$ (in dollars):

q	15	30	45	60	75	90	105	120
$C(q)$	4.7	12.7	22.5	33.5	44.5	54.3	62.8	67

Use the values in the table to answer the following question.

f. [5 points] How many ounces of lemonade should Brad and Angelina sell in order to maximize their profit, and what is their maximal profit? Use calculus to fully justify that your answer is a global maximum. Remember to include units in your answer.

Solution: The critical points are $q = 30, 90$ and the end points are $q = 0$ and $q = 120$.

q	0	30	90	120
$R(q)$	0	18	54	72
$C(q)$	0	12.7	54.3	67
$\pi(q) = R(q) - C(q)$	0	5.3	-0.3	5

Answer: Brad and Angelina's profit is maximized when they sell 30 ounces

of lemonade, and their maximal profit is 5.30 dollars.