Math 115
Worksheet Section 5.3

Warm-up question

What does the Fundamental Theorem of Calculus say?

Problem 1.

(a) Differentiate $x^3 + x$.

(b) Compute $\int_0^2 (3x^2 + 1)dx$

(c) Differentiate e^{x^2}.

(d) Compute $\int_0^1 xe^{x^2}\,dx$

Problem 2. Explain in words what these integrals represent, with units.

(a) $\int_0^6 a(t)dt$ where $a(t)$ is acceleration in km/hr^2 and t is time in hours.

(b) $\int_{2005}^{2011} f(t)dt$ where $f(t)$ is the rate at which the world's population is growing in year t, in billion people per year.

Problem 3. Pollution is removed from a lake on day t at a rate of $f(t)$ kg/day.

(a) Explain the meaning of the statement $f(12) = 500$.

(b) If $\int_5^{15} f(t)dt = 4000$, give the units of the bounds of integration (5 and 15), and the units for the result of the integral.

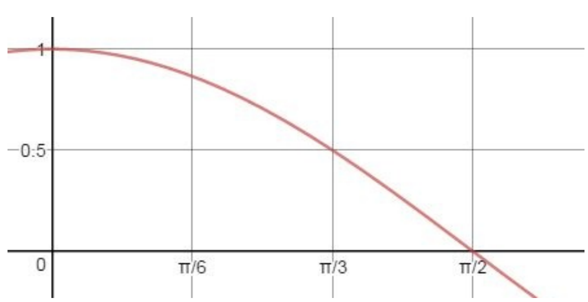
(c) Give the meaning of $\int_5^{15} f(t)dt = 4000$.

Problem 4.

(a) What is the derivative of $\sin t$?

(b) The velocity of a particle at time t is $v(t) = \cos t$. Use the Fundamental Theorem of Calculus to find the total distance traveled by the particle between $t = 0$ and $t = \pi/2$

(c) Does your answer make sense with the graph of $\cos t$?



Problem 5. Use the given graph to answer the following:

(a) Which is larger, $f(0)$ or $f(1)$?

(b) List the following in increasing order: $\dfrac{f(4) - f(2)}{2}$, $f(3) - f(2)$, $f(4) - f(3)$.

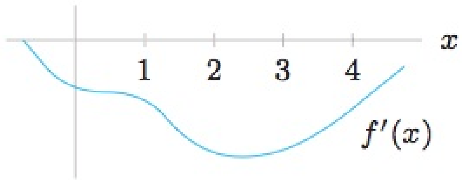


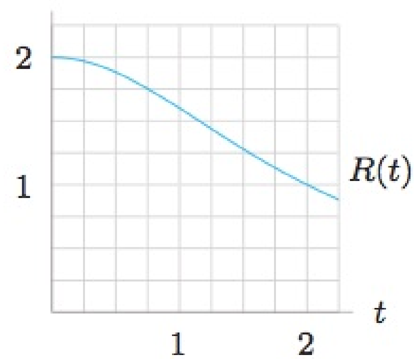
Figure 5.46: Graph of f', not f

Problem 6. Water is leaking out of a tank at a rate of $R(t)$ gallons/hour, graphed below, where t is measured in hours.

(a) Write a definite integral that expresses the total amount of water that leaks out in the first two hours.

(b) In the figure, shade the region whose area represents the total amount of water that leaks out in the first two hours.

(c) Give an upper and lower estimate of the total amount of water that leaks out in the first two hours.



Problem 7. Consider the integral $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ and $G(x) = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin(x)\right)$.

(a) Sketch the graph of the function $y = \sqrt{1-x^2}$ on the interval $[-1, 1]$.
(b) Find the exact value of the integral using your knowledge of areas.
(c) Show that $G'(x) = \sqrt{1-x^2}$.
(d) Does your answer in part (b) agree with the fundamental theorem of calculus?

Problem 8. Below is a table with some values of a twice differentiable function $f(x)$ and its derivative.

x	-1	0	3	5
$f(x)$	3	5	7	3
$f'(x)$	1	-3	6	2

(a) Find $\int_{-1}^{3} f'(x)\,dx$.
(b) If $\int_{0}^{2} f''(x)\,dx = 5$, find $f'(2)$.
(c) Assume f is concave up on the interval $[0, 3]$. What is the area between the graph of f'' and the x-axis on the interval $[0, 3]$?
(d) Find $\int_{0}^{\frac{\pi}{2}} (f'(x)\cos(x) - f(x)\sin(x))\,dx$.

Problem 9. (Winter 2017 Final Exam) Virgil, Duncan, Jasper and Zander are all watching a toy wind-up mouse move across the floor. The toy is placed on the floor 2.3 meters away from Virgil, and it moves in a straight line directly away from Virgil at a strictly decreasing velocity. Below are some values of $v(t)$, the velocity of the toy mouse (m/s), t seconds after the toy is placed on the floor, where a positive velocity corresponds to the toy moving away from Virgil.

t	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$v(t)$	3.19	2.39	1.86	1.43	1.11	0.86	0.54	0.42	0.11

(a) Estimate the value of $\displaystyle\int_{0.25}^{1.75} v(t)\,dt$ using a left-hand Riemann sum with $\Delta t = 0.5$. Be sure to write down all the terms. Is your estimation an overestimate or an underestimate?

(b) How often should the values of $v(t)$ be measured in order to find upper and lower estimates for $\displaystyle\int_{0.25}^{1.75} v(t)\,dt$ that are within 0.1 m of the actual value?

(c) Find the value of $\displaystyle\int_{0.5}^{1.25} v'(t)\,dt$.

(d) Which of the following represents how much the distance from the toy mouse to Virgil increases during the 2nd second after it has been placed on the floor? Circle the one best answer.

(i) $2.3 - \displaystyle\int_1^2 v(t)\,dt$

(ii) $2.3 - \displaystyle\int_1^2 v'(t)\,dt$

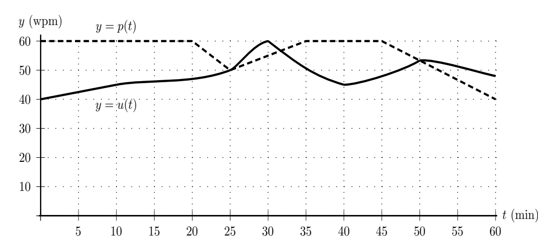
(iii) $\displaystyle\int_1^2 v(t)\,dt - \int_0^1 v(t)\,dt$

(iv) $\displaystyle\int_1^2 v(t)\,dt$

(v) $\displaystyle\int_1^2 v'(t)\,dt$

(vi) $v(2) - v(1)$

Problem 10. (Fall 2014 Final Exam) A professor delivers a 60 minute lecture on the general theory of relativity. One eager undergraduate student takes notes by typing what the professor says, word for word. Unfortunately the student cannot always type as quickly as the professor is speaking. The graph below shows the professor's speaking rate $p(t)$ (dashed) in words per minute (wpm) and the student's typing rate $u(t)$ (solid), also in words per minute (wpm).

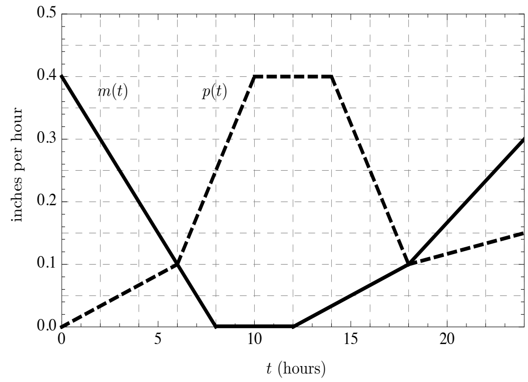


(a) How many minutes after the start of the lecture is the student typing most quickly?

(b) Write a definite integral equal to the number of words the student types between the start of the lecture and the time the professor reaches the 600th **word** of his lecture.

(c) How many minutes after the start of the lecture is the student furthest behind in typing up the lecture? (When is the difference between the total number of words the professor has spoken and the total number of words the student has typed the greatest?)

Problem 11. (Fall 2011 Final Exam) The graph below shows the rate of snow melt and snowfall on Mount Arvon, the highest peak in Michigan, during a day in April of last year. The function $m(t)$ (solid curve) is rate of snow melt, in inches per hour, t hours after the beginning of the day. The function $p(t)$ (dashed curve) is the snowfall rate in inches per hour t hours after beginning of the day. There were 18 inches of snow on the ground at the beginning of the day.



(a) Over what time period(s) was the snowfall rate greater than the snow melt rate?

(b) When was the amount of snow on Mount Arvon increasing the fastest?

(c) When was the amount of snow on Mount Arvon decreasing the fastest?

(d) When was the amount of snow on Mount Arvon the greatest? Explain.

(e) How much snow was there on Mount Arvon at the end of the day (at $t = 24$)?

Problem 12. (Winter 2018 Final Exam) There were 3 trillion trees in the world in the year 2000.

- Since the year 2000, a group of environmentalists have recorded the number of trees lost in the world due to natural causes or due to human activities. Let $C(t)$ be the rate at which the number of trees decreases due to any of these causes, t years after the year 2000, in trillions of trees per year.
- At the same time, some governments and other organizations plant new trees to increase the number of trees in the world. The group is also measuring the rate $P(t)$ at which the trees are being planted, t years after the year 2000, in trillions of trees per year.

Throughout this question, you may assume that the functions $C(t)$ and $P(t)$ describe the only changes to the number of trees in the world.

(a) Find an expression for the total number of trees in the world (in trillions) in the year 2005.

(b) Find an expression for the average rate at which the trees were being planted (in trillions of trees per year) between the years 2002 and 2009.

(c) Write a practical interpretation of the statement $\int_{13}^{17} C(t)\,dt = 0.05$. Your answer must be a complete sentence.