Written HW 10 (due 4/8)
7.1: 4, 6, 12, 16, 18

Diagonal matrices are great!

$$A=\begin{pmatrix}2&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix} \qquad A^3=\begin{pmatrix}2^3&0&0&0\\&0^3&0&0\\&&1^3&0\\&&&3^3\end{pmatrix}$$

$\text{rank}(A)=3$

$\det(A)=2\cdot 0\cdot 1\cdot 3=0$

Basis for $\ker(A)=\left\{\underbrace{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\vec{e}_2}\right\}$

<u>Thm</u> A diagonalizable $\Longleftrightarrow$ A has an eigenbasis.
(A is similar to a diagonal matrix)

<u>Def</u> For a linear transformation $T:\mathbb{R}^n\to\mathbb{R}$ given by $T(\vec{x})=A\vec{x}$,
$\vec{v}\in\mathbb{R}^n$ for $\vec{v}\neq\vec{0}$ is called an <u>eigenvector</u> of A or T if

$$A\vec{v}=\lambda\vec{v} \quad \text{for some scalar } \lambda.$$

λ is the <u>eigenvalue</u> associated to $\vec{v}$.

A basis $\{\vec{v}_1,\dots,\vec{v}_n\}$ is called an <u>eigenbasis</u> of A or T if every $\vec{v}_i$ is an eigenvector of A or T.

<u>Eg</u> a) $A=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$ has eigenvectors $\vec{v}_1=\begin{pmatrix}1\\-1\end{pmatrix}$ with eigenvalue $\lambda_1=1$

$\vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$ with eigenvalue $\lambda_2=-1$

Together $\{\vec{v}_1,\vec{v}_2\}$ is an eigenbasis of A.

$$\Longrightarrow \quad A=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}, \quad S=\begin{pmatrix}1&1\\-1&1\end{pmatrix}, \quad B=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$$

$\vec{v}_1$ $\vec{v}_2$ Satisfies $S^{-1}AS=B$

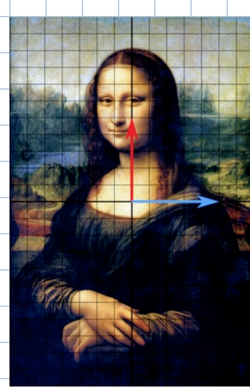
b) $A=\begin{pmatrix}1&0\\k&1\end{pmatrix}$:

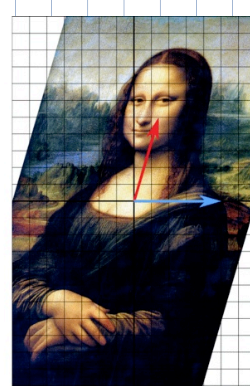
(Wikipedia)

X-axis is not rotated

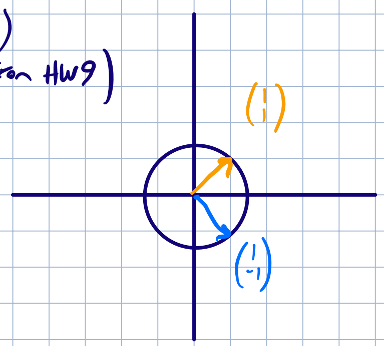
$\Updownarrow$

$\vec{v}=\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector

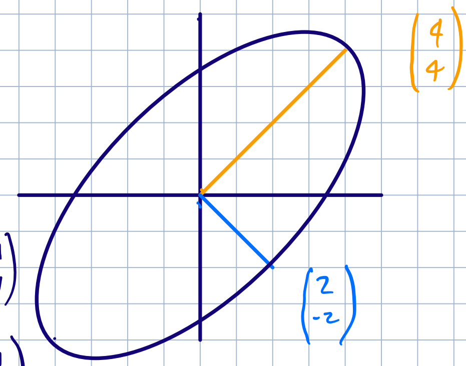




c) (From HW9)



$$A = \begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$$



$$\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\vec{v}_1} = \begin{pmatrix}4\\4\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\vec{v}_2} = \begin{pmatrix}2\\-2\end{pmatrix} = 2\begin{pmatrix}1\\-1\end{pmatrix}$$

$\vec{v}_1$ is an eigenvector with eigenvalue 4.
$\vec{v}_2$ is an eigenvector with eigenvalue 2.

Finding Eigenvalues

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\iff A\vec{v} - \lambda I_n\vec{v} = \vec{0}$$
$$\iff (A - \lambda I_n)\vec{v} = \vec{0}$$
$$\iff \vec{v} \in \ker(A - \lambda I_n)$$

If $\vec{v} \neq \vec{0}$, this can only happen if $\underbrace{\overbrace{\det(A - \lambda I_n)}^{\text{Polynomial in } \lambda \text{ called "characteristic polynomial"}} = 0}_{\text{"Characteristic equation"}}$.

Eg a) $A = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$ $\quad \det(A - \lambda I_2) = \det\begin{pmatrix}-\lambda & -1\\ -1 & -\lambda\end{pmatrix}$

$A - \lambda I_2 = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ $\quad = (-\lambda)^2 - (-1)^2$
$\quad = \lambda^2 - 1$

Solve $\lambda^2 - 1 = 0 \Rightarrow (\lambda-1)(\lambda+1) = 0$
$\Rightarrow \lambda = 1, -1.$

So, 1 and -1 are the eigenvalues of A.

b) $A = \begin{pmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$ $\quad \det(A - \lambda I_3) = \det\begin{pmatrix}1-\lambda & 2 & 3\\ 0 & 4-\lambda & 5\\ 0 & 0 & 6-\lambda\end{pmatrix}$

$$= (1-\lambda)(4-\lambda)(6-\lambda)$$

Solve $(1-\lambda)(4-\lambda)(6-\lambda) = 0 \Rightarrow \lambda = 1, 4, 6$ are the eigenvalues of A.

Thm The eigenvalues of a triangular matrix are its diagonal entries.

For any 2×2 matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A-\lambda I_2) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

$$= (a-\lambda)(d-\lambda) - bc$$
$$= ad - \lambda(a+d) + \lambda^2 - bc$$
$$= \underbrace{ad-bc}_{\det(A)} - \lambda\underbrace{(a+d)}_{\substack{\text{sum of the} \\ \text{diagonal entries} \\ \text{"trace of } A,\ tr(A)\text{"}}} + \lambda^2$$

Def For an $n\times n$ matrix A, $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$, sum of diagonal entries

Eg a) $tr\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 1+4 = 5$ b) $tr\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} = 1+4+6 = 11$

Thm a) For 2×2 matrix A,

$$\det(A-\lambda I_2) = \lambda^2 - (tr(A))\lambda + \det(A)$$

b) For $n\times n$ matrix A, $\det(A-\lambda I_n)$ is a degree n polynomial of the form $(-\lambda)^n + tr(A)(-\lambda)^{n-1} + \cdots + \det(A)$

$$= (-1)^n\lambda^n + (-1)^{n-1}tr(A)\lambda^{n-1} + \cdots + \det(A).$$

Eg $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ $\det A = 0+1+1-0-0-0$
$= 2$
$tr A = 0$

Thm (b) $\Longrightarrow$ Characteristic poly: $-\lambda^3 + 0\cdot\lambda^2 + ??\lambda + 2$

$$\det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = (-\lambda)^3 + 1 + 1 - (-\lambda) - (-\lambda) - (-\lambda)$$

$= -\lambda^3 + 3\lambda + 2$ (matches thm ✓)
$= (\lambda+1)(-\lambda^2+\lambda+2)$
$= -(\lambda+1)^2(\lambda-2)$

So, eigenvalues are -1 and 2.

Notice -1 is a root of characteristic polynomial twice.

Def An eigenvale λ_0 of $n\times n$ matrix A has <u>algebraic multiplicity</u> k if characteristic polynomial $f_A(\lambda) = (\lambda_0-\lambda)^k g(\lambda)$ for some polynomial $g(\lambda)$ such that $g(\lambda_0)\neq 0$.

Eg -1 is an eigenvalue of $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ with alg mult 2.

Eg $A = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix} \rightsquigarrow \det(A - \lambda I_3) = \det \begin{pmatrix} 4-\lambda & 0 & -1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda)(2-\lambda) + 0 + 0$

$$- (-1)(3-\lambda)(1) - 0 - 0$$

$$= 24 - (8+6+12)\lambda + 9\lambda^2 - \lambda^3 + 3 - \lambda$$

$$= 27 - 27\lambda + 9\lambda^2 - \lambda^3$$

$$= (\lambda - 3)(-\lambda^2 + 6\lambda - 9)$$

$$= (\lambda - 3)^3$$

So eigenvalue 3 occurs with alg mult 3.

Eg $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \det \begin{pmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = \lambda^2(1-\lambda) + 0 + 0 - 0 - 0 - (1-\lambda)(-1)(1)$

$$= \lambda^2(1-\lambda) + 1 - \lambda$$

$$= (\lambda^2 + 1)(1 - \lambda)$$

$\underbrace{(\lambda^2+1)}_{\text{no real solutions}}(1-\lambda) = 0$ $\lambda = 1$ is a solution.

So, only eigenvalue is $\lambda = 1$ with alg mult 1.

Thm a) An $n \times n$ matrix A has at most n real eigenvalues (counted with multiplicity)

b) If n is odd, an $n \times n$ matrix has at least one real eigenvalue.

Why? Zeroes of deg n polynomials.