HW 8 due 3/18
5.4: 20, 36, 38
6.1: 12, 14, 24, 26, 40, 44

HW 9 due 3/23 (wed)
6.2: 2, 12, 14, 38, 42

Midterm 2 3/25

<u>Thm</u>

a) $$\det \underbrace{\begin{pmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{x}+\vec{y} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{pmatrix}}_{n\times n \text{ matrix}} = \det \begin{pmatrix} - & v_1 & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{x} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{pmatrix} + \det \begin{pmatrix} - & v_1 & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{y} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{pmatrix}$$

b) $$\det \underbrace{\begin{pmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & k\vec{x} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{pmatrix}}_{n\times n \text{ matrix}} = k \det \begin{pmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{x} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{pmatrix}$$

c) For fixed $\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n$, the map

$$\vec{x} \longmapsto \det \begin{pmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_2 & - \\ & \vdots & \\ - & \vec{v}_{i-1} & - \\ - & \vec{x} & - \\ - & \vec{v}_{i+1} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{pmatrix} \begin{array}{l} \\ \\ \\ \\ \leftarrow \text{row } i \\ \\ \\ \\ \end{array}$$

is a linear transformation.

<u>Eg</u>

$$\det \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}}_{A}$$

Find patterns

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

prod(P)	24	16
sgn(P)	$(-1)^0 = 1$	$(-1)^1 = -1$
$\Rightarrow \det A$	24 $-$	16 $= 8$

$$\det\begin{pmatrix}1&0&0&0\\0&2&4&0\\0&1&3&0\\0&0&0&4\end{pmatrix} \underset{(i=2)}{\overset{(a)}{=}} \det\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&1&3&0\\0&0&0&4\end{pmatrix} + \det\begin{pmatrix}1&0&0&0\\0&0&4&0\\0&1&3&0\\0&0&0&4\end{pmatrix}$$

both only have one pattern with $\text{prod}(P) \neq 0$

$$\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&1&3&0\\0&0&0&4\end{pmatrix} \qquad \begin{pmatrix}1&0&0&0\\0&0&4&0\\0&1&3&0\\0&0&0&4\end{pmatrix}$$

$\text{sgn}(P)\text{prod}(P) = 24$ $\qquad$ $\text{sgn}(P)\text{prod}(P) = -16$

$$= 24 - 16 = 8$$

Determinants and row-reduction

Recall row-reduction has three "elementary row operations"

1. Multiply row by scalar k $\longrightarrow$ $A \to B$ via $R_i \leftarrow kR_i$ then $\det B = k \det A$
2. Swap two rows $\longrightarrow$ $A \to B$ via $R_i \leftrightarrow R_j$ then $\det B = -\det A$
3. Add multiple of one row to another $\longrightarrow$??

Eg Compute determinants of $A = \begin{pmatrix}1&0&0\\0&1&2\\0&1&1\end{pmatrix}$ and $B = \begin{pmatrix}1&0&0\\0&1&2\\0&0&-1\end{pmatrix}$

$$\begin{pmatrix}1&0&0\\0&1&2\\0&1&1\end{pmatrix} \qquad \begin{pmatrix}1&0&0\\0&1&2\\0&1&1\end{pmatrix}$$

prod(P)	1	2
sgn(P)	1	-1

$\det A = 1 - 2 = -1$

$\det B = 1 \cdot 1 \cdot -1 = -1$

Observe $A \to B$ via $R_i \leftarrow R_i + kR_j$, then

$$A = \begin{pmatrix}- \vec{r}_1 - \\ \vdots \\ - \vec{r}_i - \\ \vdots \\ - \vec{r}_n -\end{pmatrix}, \quad B = \begin{pmatrix}- \vec{r}_1 - \\ \vdots \\ - \vec{r}_i + k\vec{r}_j - \\ \vdots \\ - \vec{r}_n -\end{pmatrix} = \det B = \underbrace{\det\begin{pmatrix}- \vec{r}_1 - \\ \vdots \\ - \vec{r}_i - \\ \vdots \\ - \vec{r}_n -\end{pmatrix}}_{= A} + k \underbrace{\overbrace{\det\begin{pmatrix}- \vec{r}_1 - \\ \vdots \\ - \vec{r}_j - \\ \vdots \\ - \vec{r}_n -\end{pmatrix}}^{C\ =}}_{= ?}$$

$\xrightarrow{\text{row } i}$

C has two equal rows (row $i = \vec{r}_j$, row $j = \vec{r}_j$)

so if we swap $R_i \leftrightarrow R_j$ in C, we get $\det C = -\det C$

$\Rightarrow \det C = 0$

$\Rightarrow \det A = \det B$ $\square$

Thm If $n\times n$ matrix A is put into rref by s row-swaps and multiplying rows by scalers $k_1, \dots, k_r$, then

$$\det(\operatorname{rref}(A)) = (-1)^s (k_1 \cdots k_r) \det A.$$

Eg Compute $\det \begin{pmatrix} 2 & 2 & 0 & 16 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix}$

$$\begin{pmatrix} 2 & 2 & 0 & 16 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \begin{pmatrix} 1 & 1 & 0 & 8 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & 0 & 8 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_2]{} \begin{pmatrix} 1 & 1 & 0 & 8 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 6 \end{pmatrix}$$

$$\xrightarrow[R_4 \leftarrow R_4 - R_3]{R_1 \leftarrow R_1 - R_3} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow[R_4 \leftarrow \frac{1}{2}R_4]{} \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\rightsquigarrow \det(\operatorname{rref}(A)) = (-1)^1 \left(\frac{1}{2} \cdot \frac{1}{2}\right) \det A$

$\Rightarrow \quad 1 = (-1)^1 \frac{1}{4} \det A \quad \Rightarrow \det A = -4$

Thm An $n\times n$ matrix A is invertible $\Leftrightarrow \det A \neq 0$.

Thm a) If A and B are $n\times n$ matrices, then $\det(AB) = (\det A)(\det B)$

b) If A and B are similar $n\times n$ matrices, $\det(A) = \det(B)$

Since $AS = SB$ for invertible S, so $(\det A)(\det S) = (\det S)(\det B)$

$\Rightarrow \det A = \det B$.

c) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det A}$. $\quad (\det(A))(\det(A^{-1})) = \det(I_n))$