Quiz on 1/28 (15 min)
↳ ch 1, 2.1, 2.3, 2.4

HW 3 due 2/4
2.4: 2, 6, 34, 40, 42

Last time $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ if $= 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ not invertible

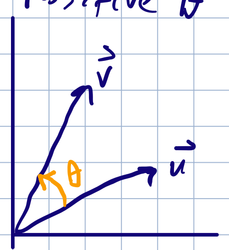
$\neq 0$, then it is.

Geometric interpretation

Def For vector $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, define norm $\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$ (length of $\vec{u}$)

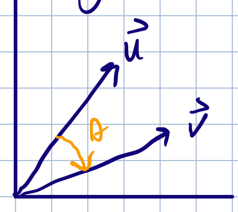
2×2 matrix $A = \begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix}$, $\det A = \|\vec{u}\| \cdot \|\vec{v}\| \sin\theta$ for $-\pi < \theta \leq \pi$

Positive θ



$\Rightarrow \sin\theta > 0$
$\Rightarrow \det A > 0$

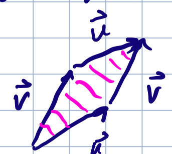
Negative θ



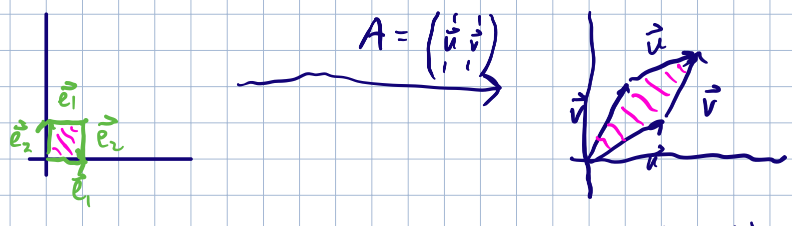
$\Rightarrow \sin\theta < 0$
$\Rightarrow \det A < 0$

$\vec{u} = k\vec{v} \Rightarrow \theta = 0$ or π radians $\Rightarrow \det A = 0$

$|\det A| =$ area of parallelogram



Think of A as a linear transformation, then A sends "unit square"



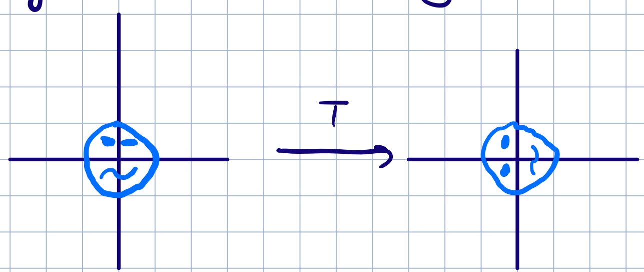
Area 1 square $\longmapsto$ area $|\det A|$ parallelogram

Linear transformations in geometry

Linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ transforms geometric shapes

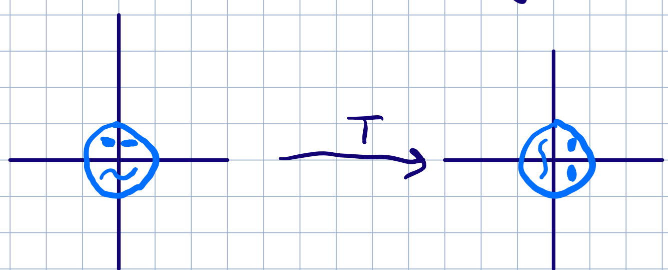
Eg a) Rotate CCW by 90°

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



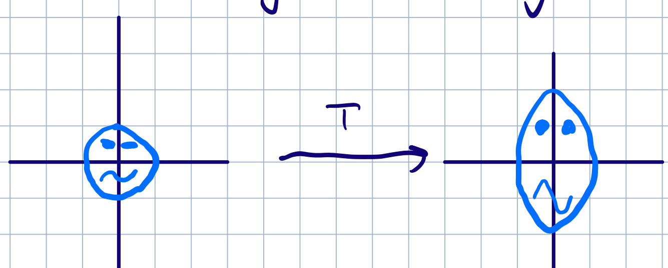
b) Reflect over line $y=x$ (HW 2 2.1:26)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



c) Scale y-direction by 2 (2.1:28)

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$



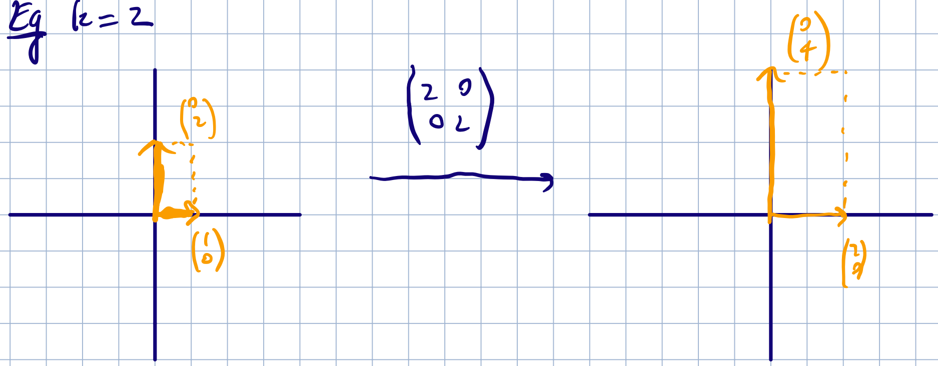
What other simple geometric transformations can we describe using linear transformations?

1) Scalings
2) Orthogonal projections
3) Reflections
4) Rotations
5) Shears

Scalings

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$$

Eg $k=2$



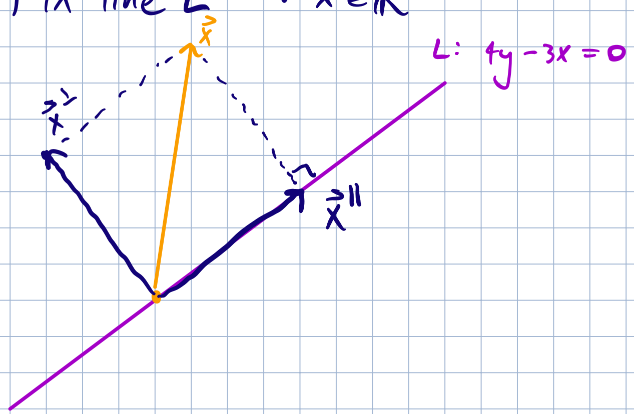
$k > 1 \longrightarrow$ dilation

$0 < k < 1 \longrightarrow$ contraction

<u>Note</u> $\det\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = k^2 - 0^2 = k^2$

Orthogonal Projection

Fix line L and $\vec{x} \in \mathbb{R}^2$



Note $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ and $\vec{x}^{\perp}$ & $\vec{x}^{\parallel}$ are othogonal

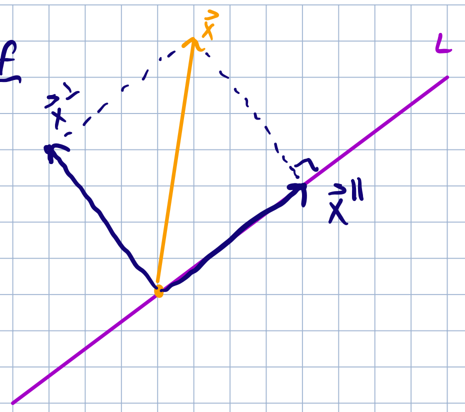
(so $\vec{x}^{\perp} \cdot \vec{x}^{\parallel} = 0$)

<u>Def</u> For line L and $\vec{x} \in \mathbb{R}^2$, define $\text{proj}_L(\vec{x}) = \vec{x}^{\parallel}$

Computing $\text{proj}_L(\vec{x})$

Take $\vec{w} \neq \vec{0}$ parallel to L. Then $\text{proj}_L(\vec{x}) = \left(\dfrac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$

Pf



$\vec{x}^{\parallel} = k\vec{w}$ for some $k \in \mathbb{R}$ b/c they are on the line L.

and $\vec{x} = \vec{x}^{\parallel} + \vec{x}^\perp \implies \vec{x}^\perp = \vec{x} - \vec{x}^{\parallel} = \vec{x} - k\vec{w}$

$\vec{x}^{\parallel} \cdot \vec{x}^\perp = 0 \implies \vec{x}^{\parallel} \cdot (\vec{x} - k\vec{w}) = 0 \implies \vec{x}^{\parallel} \cdot \vec{x} - \vec{x}^{\parallel} \cdot (k\vec{w}) = 0$

$\implies \vec{w} \cdot \vec{x} - \vec{w} \cdot (k\vec{w}) = 0$

$\implies k = \dfrac{\vec{w} \cdot \vec{x}}{\vec{w} \cdot \vec{w}}$

$$\vec{x}^{\parallel} = \left(\frac{\vec{w} \cdot \vec{x}}{\vec{w} \cdot \vec{w}}\right)\vec{w} \qquad \square$$

Tip Pick $\vec{w}$ so $\vec{w} \cdot \vec{w} = 1$ (length 1) makes the formula easier

Thm $proj_L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation

Pf Using formula

i) $proj_L(\vec{u} + \vec{v}) = \left(\dfrac{\vec{w} \cdot (\vec{u} + \vec{v})}{\vec{w} \cdot \vec{w}}\right)\vec{w} = \left(\dfrac{\vec{w} \cdot \vec{u}}{\vec{w} \cdot \vec{w}}\right)\vec{w} + \left(\dfrac{\vec{w} \cdot \vec{v}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$

$= proj_L(\vec{u}) + proj_L(\vec{v})$

Check

ii) $Proj_L(k\vec{u}) = k\, proj_L(\vec{u})$

$\square$

Matrix for $proj_L$?

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be a unit vector (length 1) $\implies (\vec{u} \cdot \vec{u} = 1)$ parallel to L.

Then, $A = \begin{pmatrix} | & | \\ proj_L(\vec{e}_1) & proj_L(\vec{e}_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ (\vec{u} \cdot \vec{e}_1)\vec{u} & (\vec{u} \cdot \vec{e}_2)\vec{u} \\ | & | \end{pmatrix}$

$= \begin{pmatrix} | & | \\ u_1\vec{u} & u_2\vec{u} \\ | & | \end{pmatrix} = \begin{pmatrix} u_1^2 & u_2 u_1 \\ u_1 u_2 & u_2^2 \end{pmatrix}$

Eg Find matrix of orthogonal projection onto $L: 3x-4y=0$.

$\vec{w} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is parallel to L.

Normalize to length 1: $\vec{u} = \frac{1}{\|\vec{w}\|}\vec{w} = \frac{1}{\sqrt{4^2+3^2}}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$= \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$$

Thus, $\begin{pmatrix} (4/5)^2 & (4/5)(3/5) \\ (4/5)(3/5) & (3/5)^2 \end{pmatrix} = \begin{pmatrix} 16/25 & 12/25 \\ 12/25 & 9/25 \end{pmatrix}$ is the matrix of ortho projection onto L

$\det\begin{pmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{pmatrix} = u_1^2u_2^2 - u_1u_2u_1u_2 = 0 \Rightarrow$ Not invertible!