HW 7 due 3/11
5.1: 16, 26, 28
5.2: 4, 6, 18, 29
5.3: 2, 6, 8, 10

HW 8 due 3/18
5.4: 20, 36, 38

Last time

Def Linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>orthogonal</u> if

$$\|T(\vec{x})\| = \|\vec{x}\| \text{ for all } \vec{x} \in \mathbb{R}^n.$$

For such a $T(\vec{x}) = A\vec{x}$, we say A is an <u>orthogonal matrix</u>.

Matrix Transpose

Def For an $n \times m$ matrix $A = (a_{ij})$, the <u>transpose</u> A^T of A is the $m \times n$ matrix given by $A^T = (a_{ji})$.

Eg $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \to A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Def An $n \times n$ matrix A is a) <u>symmetric</u> if $A^T = A$

b) <u>skew-symmetric</u> if $A^T = -A$

Eg a) $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is symmetric.

b) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is skew-symmetric: $A^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -A.$

In general for 2×2: $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ is skew-symmetric.

Thm For $\vec{v}, \vec{w} \in \mathbb{R}^n$, $\underbrace{\vec{v} \cdot \vec{w}}_{\text{dot product}} = \underbrace{\vec{v}^T \vec{w}}_{\text{matrix product}}$

Eg $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\qquad \vec{v}^T \vec{w} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 4 = 11$

$\vec{v} \cdot \vec{w} = 1 \cdot 3 + 2 \cdot 4 = 11$

<u>Aside/Recall</u>

$\vec{r}_i, \vec{c}_i \in \mathbb{R}^s$

$$\begin{pmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_m & - \end{pmatrix}\begin{pmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} \vec{r}_1\cdot\vec{c}_1 & \vec{r}_1\cdot\vec{c}_2 & \cdots \\ \vec{r}_2\cdot\vec{c}_1 & \vec{r}_2\cdot\vec{c}_2 & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

<u>Thm</u> An $n\times n$ matrix A is orthogonal if and only if $A^TA = I_n$
$(\Leftrightarrow A^T = A^{-1})$

<u>Eg</u>
$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Think
$$\begin{pmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \\ - & \vec{u}_3 & - \end{pmatrix}\begin{pmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{pmatrix} = \left(\vec{u}_i\cdot\vec{u}_j\right) = I_3$$

Since $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ orthonormal.

<u>Thm</u> For subspace V of $\mathbb{R}^n$ with orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_m\}$, the matrix $P = QQ^T$ where
$$Q = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{pmatrix}$$

gives orthogonal projection onto V.

<u>Eg</u> Projection onto
$$V = \operatorname{span}\left\{\underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}_{\vec{u}_1}, \underbrace{\begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}}_{\vec{u}_2}\right\} \text{ in } \mathbb{R}^3$$

is given by
$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix} \longleftarrow$$
<u>Note</u> This is symmetric!

<u>Why?</u>
$$\begin{pmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \end{pmatrix}\vec{x} = \begin{pmatrix} \vec{x}\cdot\vec{u}_1 \\ \vec{x}\cdot\vec{u}_2 \end{pmatrix} \qquad \vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2$$

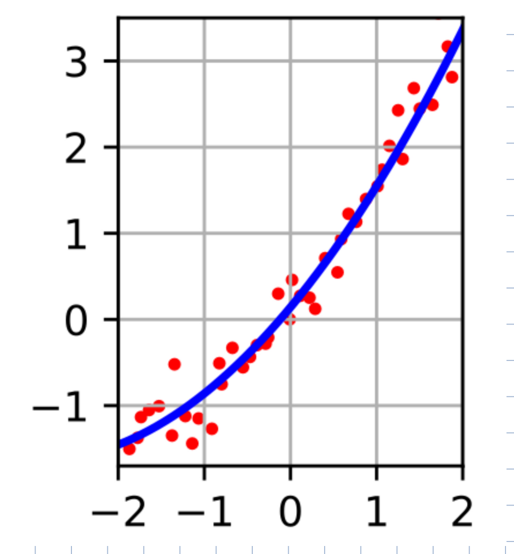
$$\begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}\begin{pmatrix} \vec{x}\cdot\vec{u}_1 \\ \vec{x}\cdot\vec{u}_2 \end{pmatrix} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2$$

<u>Thm</u>
a) $(A+B)^T = A^T + B^T$ $\quad A, B$ $n\times m$ matrices
b) $(kA)^T = k\cdot A^T$ $\quad k$ a scalar
c) $(AB)^T = B^TA^T$ for A $n\times p$ matrix, B $p\times m$ matrix
d) $\operatorname{rank}(A^T) = \operatorname{rank}(A)$ for all matrices A
e) $(A^T)^{-1} = (A^{-1})^T$ for all invertible matrices A.

So for $P = QQ^T$ above, P is always symmetric:

$$P^T = (QQ^T)^T \overset{(c)}{=\!=} (Q^T)^T Q^T = QQ^T = P.$$

Least Squares



Given data (x_i, y_i),
find a polynomial of degree n, f,
such that $f(x_i)$ is as "close"
to y_i as possible.

$\rightarrow f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$

Eg Find deg 2 poly f "fitting" $\{(1.5, 53), (1.6, 58), (1.7, 64), (1.8, 72), \ldots\}$

$$\begin{cases} f(1.5) = c_0 + 1.5c_1 + 2.25c_2 = 53 \\ f(1.6) = c_0 + 1.6c_1 + 2.56c_2 = 58 \\ f(1.7) = \qquad\qquad\qquad\qquad = 64 \\ f(1.8) = \qquad\qquad\qquad\qquad = 72 \\ \qquad\qquad \vdots \end{cases}$$

will (almost surely) be inconsistent, but can we "approximate" it optimally?

Thm For $\vec{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$, $\text{proj}_V \vec{x}$ is the closest vector in V to $\vec{x}$, i.e.,

$$\|\vec{x} - \text{proj}_V \vec{x}\| < \|\vec{x} - \vec{v}\| \text{ for all } \vec{v} \in V \text{ where } \vec{v} \neq \text{proj}_V \vec{x}$$