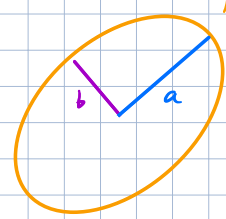
HW 8 due 3/18
5.4: 20, 36, 38
6.1: 12, 14, 24, 26, 40, 44

HW 9 due 3/23 (wed)
6.2: 2, 12, 14, 38, 42
6.3: 2, 18 (*)

Midterm 2 3/25

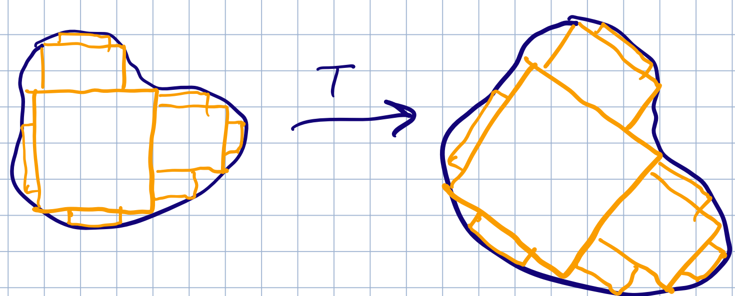
(*) Area of ellipse $= \pi a b$



Thm For $n \times n$ matrix A, $V(A\vec{v}_1, \dots, A\vec{v}_n) = |\det A|\, V(\vec{v}_1, \dots, \vec{v}_n)$ for all vectors $\vec{v}_1, \dots, \vec{v}_n$.

Upshot If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation $T(\vec{x}) = A\vec{x}$, then for any n-parallelepiped Ω, the n-volume of $T(\Omega)$ is $|\det A|$ times the n-volume of Ω.

Consider



Thm For linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{x}) = A\vec{x}$, and Ω any region in the plane, $|\det A| = \dfrac{\text{Area}(T(\Omega))}{\text{Area}(\Omega)}$.

Cramer's Rule

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightsquigarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \quad (\text{for } A \text{ invertible})$$

So $A\vec{x} = \vec{b} \leftrightarrow \underbrace{\vec{x} = A^{-1}\vec{b}}_{\parallel}$ $\quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$x_1 = \frac{1}{\det(A)} \underbrace{(a_{22} b_1 - a_{12} b_2)} = \det \overbrace{\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}^{A_{\vec{b},1}}$$

$$x_2 = \frac{1}{\det(A)} \underbrace{(-a_{21} b_1 + a_{11} b_2)} = \det \overbrace{\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}^{A_{\vec{b},2}}$$

Upshot For 2×2 matrix A, then $A\vec{x}=\vec{b}$ is solved by

$$x_1 = \frac{\det(A_{\vec{b},1})}{\det(A)} \qquad x_2 = \frac{\det(A_{\vec{b},2})}{\det(A)}$$

Thm (Cramer's Rule) For an invertible $n\times n$ matrix A, the system $A\vec{x}=\vec{b}$ satisfies

$$x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}$$

where $A_{\vec{b},i}$ is "replace column i by $\vec{b}$."

Pf

$$\det(A_{\vec{b},i}) = \det\begin{array}{c} \\ \begin{pmatrix} | & & | & & | \\ \vec{w}_1 & \cdots & \vec{b} & \cdots & \vec{w}_n \\ | & & | & & | \end{pmatrix} \\ \text{column } i \end{array} = \det\begin{pmatrix} | & & | & & | \\ \vec{w}_1 & \cdots & A\vec{x} & \cdots & \vec{w}_n \\ | & & | & & | \end{pmatrix}$$

$$= \det\begin{pmatrix} | & & | & & | \\ \vec{w}_1 & \cdots & (x_1\vec{w}_1+\cdots+x_n\vec{w}_n) & \cdots & \vec{w}_n \\ | & & | & & | \end{pmatrix}$$

$$= \det\begin{pmatrix} | & & | & & | \\ \vec{w}_1 & \cdots & x_i\vec{w}_i & \cdots & \vec{w}_n \\ | & & | & & | \end{pmatrix}$$

$$= x_i \det(A) \qquad \square$$

Eg $\begin{cases} x_1 + 2x_2 = 5 \\ 3x_1 + 4x_2 = 7 \end{cases}$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

$$x_1 = \frac{\det\begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} = \frac{5\cdot 4 - 7\cdot 2}{1\cdot 4 - 2\cdot 3} = -3$$

$$x_2 = \frac{\det\begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}}{\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} = \frac{1\cdot 7 - 5\cdot 3}{1\cdot 4 - 2\cdot 3} = 4$$

Computing A^{-1} with Cramer's Rule

For $A^{-1} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}$ Consider $A\begin{pmatrix} m_{1,j} \\ \vdots \\ m_{n,j} \end{pmatrix} = \vec{e}_j$

Cramer's Rule $\Rightarrow$ $m_{ij} = \dfrac{\det A_{\vec{e}_j,i}}{\det A}$

$$A_{\vec{e}_j,i} = \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} a_{11} & 0 & a_{1n} \\ a_{12} & 0 & \vdots \\ \vdots & \vdots & \vdots \\ a_{j1} & 1 & a_{jn} \\ \vdots & \vdots & \\ a_{n1} & 0 & a_{nn} \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ \leftarrow \text{row } j \\ \\ \\ \end{array}$$

By cofactor expansion on column i: $\det A_{\vec{e}_j, i} = (-1)^{i+j} \underbrace{\det(A_{ji})}_{(j,i)-\text{minor}}$

$$\Rightarrow m_{ij} = \frac{(-1)^{i+j} \det(A_{ji})}{\det A}$$

Def For ~~invertible~~ $n \times n$ matrix A, the <u>classical adjoint</u> is the $n \times n$ matrix $\operatorname{adj}(A) = \left((-1)^{i+j} \det(A_{ji}) \right)$.

Thm For invertible $n \times n$ matrix A, $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$.

$$\begin{pmatrix} \cancel{1} & 2 & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ \cancel{1} & 2 & 1 \end{pmatrix}$$

Eg $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 1 \end{pmatrix}$ $\quad \operatorname{adj}(A) = \begin{pmatrix} \det A_{11} & -\det A_{21} & \det A_{31} \\ -\det A_{12} & \det A_{22} & -\det A_{32} \\ \det A_{13} & -\det A_{23} & \det A_{33} \end{pmatrix}$

$A_{21} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

$-\det A_{21} = -(2 \cdot 1 - 3 \cdot 2)$

$\det A = 6$

$$= \begin{pmatrix} 5 \cdot 1 - 6 \cdot 2 & -(2 \cdot 1 - 3 \cdot 2) & 2 \cdot 6 - 3 \cdot 5 \\ -(4 \cdot 1 - 6 \cdot 1) & 1 \cdot 1 - 3 \cdot 1 & -(1 \cdot 6 - 3 \cdot 4) \\ 4 \cdot 2 - 5 \cdot 1 & -(1 \cdot 2 - 2 \cdot 1) & 1 \cdot 5 - 2 \cdot 4 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 4 & -3 \\ 2 & -2 & 6 \\ 3 & 0 & 3 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{6} \begin{pmatrix} -7 & 4 & -3 \\ 2 & -2 & 6 \\ 3 & 0 & 3 \end{pmatrix}.$$

Determinant of a linear transformation between linear spaces

Eg $D : P_2 \to P_2$ $\quad$ What is $\det(D)$? $\quad$ Want $\det(D) = 0$.

$D(f) = f'$

For $\mathcal{B} = \{1, x, x^2\}$, the $\mathcal{B}$-matrix D

$$\text{is} \quad B = \begin{pmatrix} | & | & | \\ [D(1)]_{\mathcal{B}} & [D(x)]_{\mathcal{B}} & [D(x^2)]_{\mathcal{B}} \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | & | \\ [0]_{\mathcal{B}} & [1]_{\mathcal{B}} & [2x]_{\mathcal{B}} \\ | & | & | \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \det B = 0.$$

Def For a linear transformation $T: V \to V$ for V a finite-dimensional linear space, $\det T = \det B$ for B the $\mathcal{B}$-matrix of T for some basis $\mathcal{B}$ of V.

Note (a) $\det T$ is independent of choice of $\mathcal{B}$

(b) $\det T \neq 0$ if and only if $T: V \to V$ is an isomorphism
(V is finite dimensional)