Quiz on 2/23
3.4, 4.1, 4.2

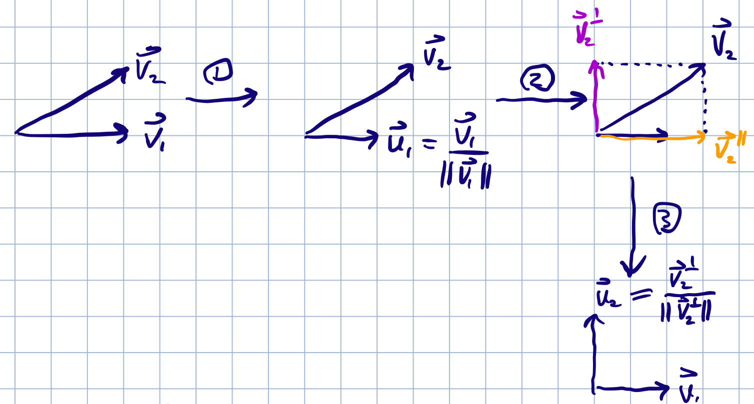
HW 6 due 2/25
4.1: 47, 48
4.2: 10, 14, 22, 54
4.3: 23, 24, 46

Reflection 2 due 2/25 (Canvas)

HW 7 due 3/11
5.1: 16, 26, 28
5.2: 4, 6, 18, 29

<u>Question</u> How to construct an orthonormal basis?

Eg



Eg $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 2y + 3z = 0 \right\}$

$$\mathcal{B} = \left\{ \underbrace{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}}_{\vec{v}_2} \right\}$$

① $\vec{u}_1 = \frac{\vec{v}_1}{\sqrt{4+1}} = \frac{1}{\sqrt{5}} \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

② $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$

$$= \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt{5}}(2 \cdot 3 + -1 \cdot 0 + 0 \cdot -1)\frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} - \frac{6}{5} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ -1 \end{pmatrix}$$

③ $\vec{u}_2 = \frac{1}{\sqrt{9/25 + 36/25 + 1}} \vec{v}_2^{\perp} = \frac{1}{\sqrt{70}/5} \vec{v}_2^{\perp} = \frac{5}{\sqrt{70}} \begin{pmatrix} 3/5 \\ 6/5 \\ -1 \end{pmatrix}$

So $\left\{ \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 3/\sqrt{70} \\ 6/\sqrt{70} \\ -5/\sqrt{70} \end{pmatrix} \right\}$ is an orthonormal basis.

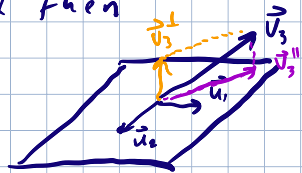
Gram-Schmidt Process

Saw this in 2D above. What about 3D?

I.e., from $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis of $\mathbb{R}^3$, construct an orthonormal basis from it.

Already know 2D, so we can $\{\vec{v}_1, \vec{v}_2\} \longrightarrow \{\vec{u}_1, \vec{u}_2\}$

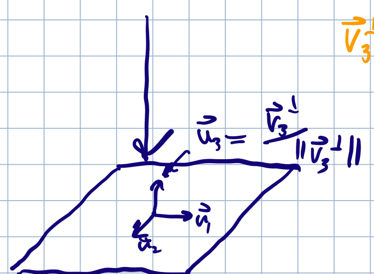
and then



$$\vec{v}_3^{\parallel} = \text{proj}_W \vec{v}_3 = (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

$$W = \text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\{\vec{u}_1, \vec{u}_2\}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \text{proj}_W \vec{v}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$



Process works similarly in n dimensions.

Thm (Gram-Schmidt Process) For basis $\{\vec{v}_1, \dots, \vec{v}_m\}$ of subspace V of $\mathbb{R}^n$, we can construct an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_m\}$ by decomposing $\vec{v}_i = \vec{v}_i^{\parallel} + \vec{v}_i^{\perp}$ $(i \geq 2)$ with respect to $\text{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}\}$

and setting

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$$
$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp}$$
$$\vdots$$
$$\vec{u}_j = \frac{1}{\|\vec{v}_j^{\perp}\|}\vec{v}_j^{\perp}$$
$$\vdots$$
$$\vec{u}_m = \frac{1}{\|\vec{v}_m^{\perp}\|}\vec{v}_m^{\perp}$$

where

$$\vec{v}_j^{\perp} = \vec{v}_j - \underbrace{(\vec{v}_j \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_j \cdot \vec{u}_2)\vec{u}_2 - \cdots - (\vec{v}_j \cdot \vec{u}_{j-1})\vec{u}_{j-1}}_{= \text{proj}_{\text{span}\{\vec{u}_1, \dots, \vec{u}_{j-1}\}} \vec{v}_j}$$

(§ 5.1)

Algorithm First compute $\vec{u}_1$

Then $\vec{v}_2^{\perp}$

Then $\vec{u}_2$

$\vdots$

Then $\vec{v}_j^{\perp}$

Then $\vec{u}_j$

$\vdots$

Then $\vec{v}_n^{\perp}$

Then $\vec{u}_n$