Written HW 10 (due 4/8)
7.1: 4, 6, 12, 16, 18
7.2: 8, 12, 18, 38
7.3: 8, 10, 24

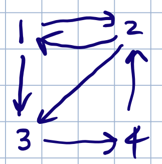
Quiz on 7.1–7.3 4/11

Written HW 11 (due 4/15)
7.5: 14, 20
7.1: 68, 70

Dynamical Systems

Eg Mini-web with 4 webpages

(§2.1 Ex 9)



$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

x_i is the proportion of users on each page

"State vector"

Each minute users move randomly on one link to new page

So at next minute

$$\begin{aligned} y_1 &= \tfrac{1}{2}x_2 \\ y_2 &= \tfrac{1}{2}x_1 + x_4 \\ y_3 &= \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 \\ y_4 &= + x_3 \end{aligned} \quad \rightsquigarrow \quad A = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

State at time t $\xrightarrow{A}$ State at time $t+1$

$$\vec{x}(t+1) = A\vec{x}(t)$$

This is a <u>discrete dynamical system</u>.

<u>Eg</u> Populations of Cyotes $C(t)$ and roadrunners $r(t)$ in desert at year t.
(7.1 ex7)

Model: $\begin{cases} C(t+1) = 0.86\,C(t) + 0.08\,r(t) \\ r(t+1) = -0.12\,C(t) + 1.14\,r(t) \end{cases}$

$$\rightsquigarrow \begin{pmatrix} C(t+1) \\ r(t+1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}}_{A} \begin{pmatrix} C(t) \\ r(t) \end{pmatrix}$$

State vector $\vec{x}(t) = \begin{pmatrix} C(t) \\ r(t) \end{pmatrix}$

$$\rightsquigarrow \vec{x}(t+1) = A\vec{x}(t)$$

$$\Rightarrow \vec{x}(t) = A^t \vec{x}(0)$$

<u>Goal</u> Want to understand long term behavoir of system.

<u>Approach</u> i) If $\vec{x}(0) = \vec{v}$ is an eigenvector of A, (so $A\vec{v} = \lambda\vec{v}$)

$\vec{x}(t) = A^t\vec{x}(0) = A^t\vec{v} = \lambda^t\vec{v}$ for eigenvalue λ of $\vec{v}$.

ii) If A is diagonalizable, then A has eigenbasis $\vec{v}_1, \vec{v}_2$

So, write, for any initial state vector $\vec{x}(0) \in \mathbb{R}^2$,

$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$ for some constants c_1, c_2.

$$\Rightarrow \vec{x}(t) = A^t\vec{x}(0) = A^t(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1A^t\vec{v}_1 + c_2A^t\vec{v}_2$$
$$= c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2$$

Applying approach

Char poly of A: $\lambda^2 - 2\lambda + 0.99 = (\lambda - 1.1)(\lambda - 0.9)$

$A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}$ $\Rightarrow$ A has eigenvalues $\lambda_1 = 1.1$, $\lambda_2 = 0.9$

and A is diagonalizable.

<u>$\lambda = 1.1$</u> $\ker\begin{pmatrix} -0.24 & 0.08 \\ -0.12 & 0.04 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$ so $\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector.

<u>$\lambda = 0.9$</u> $\ker\begin{pmatrix} -0.04 & 0.08 \\ -0.12 & 0.24 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ So $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector

So, if $\vec{x}(0) = \begin{pmatrix}1000\\1000\end{pmatrix} = 200\overset{=\vec{v}_1}{\begin{pmatrix}1\\3\end{pmatrix}} + 400\overset{=\vec{v}_2}{\begin{pmatrix}2\\1\end{pmatrix}}$

$$\vec{x}(t) = A^t\begin{pmatrix}1000\\1000\end{pmatrix} = 200A^t\begin{pmatrix}1\\3\end{pmatrix} + 400A^t\begin{pmatrix}2\\1\end{pmatrix}$$

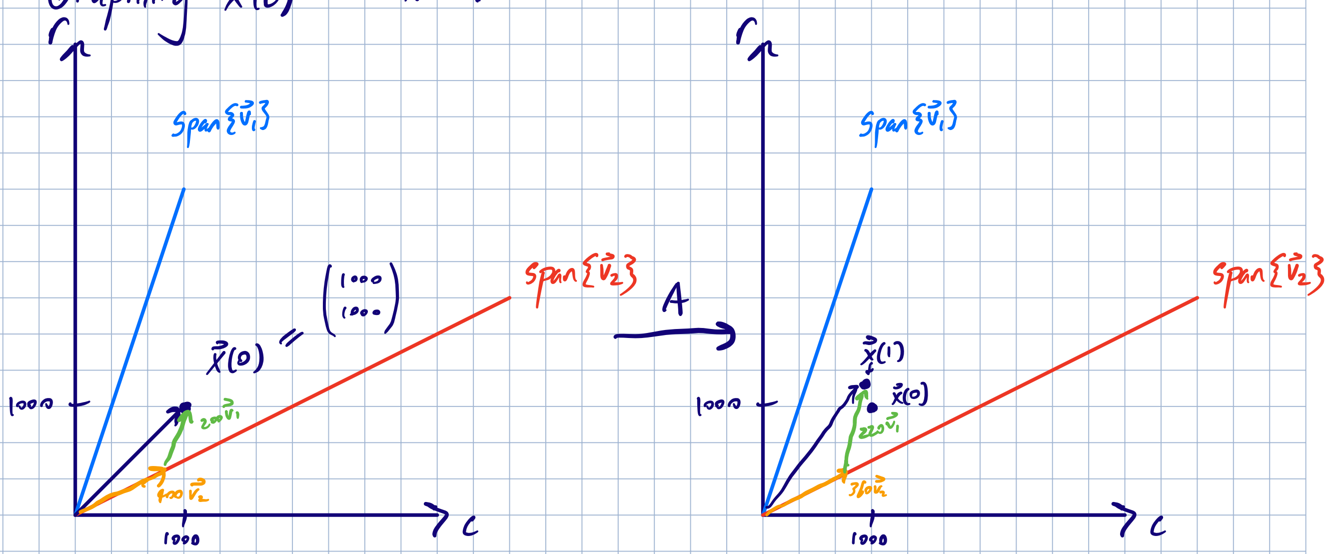
$$= 200(1.1)^t\begin{pmatrix}1\\3\end{pmatrix} + 400(0.9)^t\begin{pmatrix}2\\1\end{pmatrix}$$

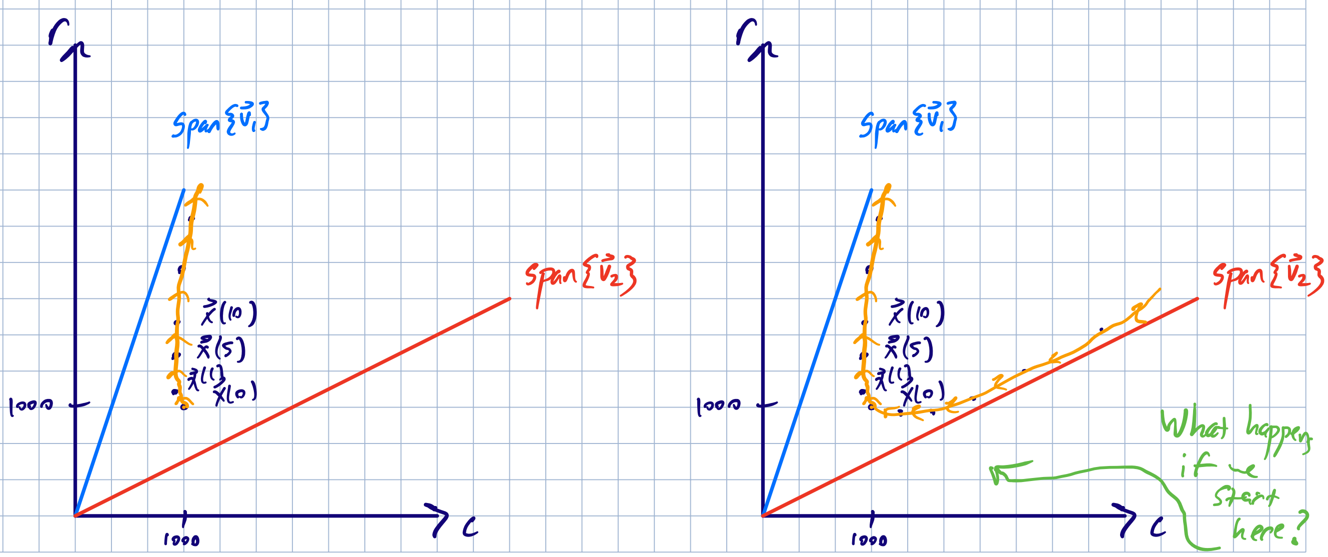
or $\begin{cases} c(t) = 200\cdot(1.1)^t + 800\cdot(0.9)^t \\ r(t) = 600\,(1.1)^t + 400\cdot(0.9)^t \end{cases}$

<u>Note</u> $t\to\infty \implies (0.9)^t \to 0$

So, eventually both populations grow 10% per year and $c(t)/r(t) \to 1/3$

Graphing $\vec{x}(t)$ over time?





This diagram is called a <u>phase portrait</u>.

<u>Eg</u> $\vec{x}(0) = \begin{pmatrix}2000\\500\end{pmatrix} = -200\begin{pmatrix}1\\3\end{pmatrix} + 1100\begin{pmatrix}2\\1\end{pmatrix}$ $\left(\text{rref}\begin{pmatrix}1&2&2000\\3&1&500\end{pmatrix}\to\begin{pmatrix}1&0&-200\\0&1&1100\end{pmatrix}\right)$

$\vec{x}(t) = -200(1.1)^t\begin{pmatrix}1\\3\end{pmatrix} + 1100(0.9)^t\begin{pmatrix}2\\1\end{pmatrix}$

$t\to\infty$

Gets more negative $\qquad$ 0