Quiz on 1/28 (15 min)
↳ Ch 1, 2.1, 2.3, 2.4

HW 3 due 2/4
2.4: 2, 6, 34, 40, 42
2.2: 20, 32
2.3: 30

Midterm 1, Wed 2/9
Tentatively Ch 1 – 3.3

HW 4 due Fri 2/11
3.1: 6, 24, 32, 34, 37, 38

We discussed 5 types of geometric linear transformations in $\mathbb{R}^2$, but we can combine them to get more complex operations via composition of linear transformations ⟷ matrix multiplication

Eg Give a matrix that rotates CCW by $\frac{\pi}{4}$ radians and scales by a factor of 2

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{2}x - \sqrt{2}y \\ \sqrt{2}x + \sqrt{2}y \end{pmatrix}$$

Subspaces

Recall a linear combination $\{\vec{v}_1, \dots, \vec{v}_m\}$ is $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$ for scalars c_i.

Eg $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ is a linear combination of $\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}$

$$\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Def For $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$

$\operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_m\}$ = The set of all linear combinations of $\{\vec{v}_1, \dots, \vec{v}_m\}$

$= \{ c_1\vec{v}_1 + \cdots + c_m\vec{v}_m : c_1, \dots, c_m \text{ are scalars} \}$

Eg a) $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ is in $\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

b) Is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$? $\Leftrightarrow$ Is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ a lin comb of $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$?

Yes! Since $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Not the only way to see:

Solve $c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Updownarrow$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -3+2t \\ 2-t \\ t \end{pmatrix}$$

Recall $T: \mathbb{R}^m \to \mathbb{R}^n$, $\operatorname{im}(T) = \{\vec{v} \in \mathbb{R}^n : \text{there is a vector } \vec{w} \text{ in } \mathbb{R}^m \text{ satisfying } T(\vec{w}) = \vec{v}\}$

Thm The image of a linear transformation $T(\vec{x}) = A\vec{x}$ is the span of the columns of A.

I.e., for $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix}$, $\operatorname{im}(T) = \operatorname{im}(A) = \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_m\}$.

Eg $\operatorname{im}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

$= \left\{ c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \text{Scalars } c_1, \dots, c_3 \right\}$

Properties of $\mathrm{im}(T)$

$T: \mathbb{R}^m \to \mathbb{R}^n$ a linear transformation

1) $\vec{0} \in \mathrm{im}(T)$ $\qquad T(\vec{0}) = A\vec{0} = \vec{0}$

2) If $\vec{u}_1, \vec{u}_2 \in \mathrm{im}(T)$, then $\vec{u}_1 + \vec{u}_2 \in \mathrm{im}(T)$

$$\vec{u}_1 = T(\vec{w}_1), \quad \vec{u}_2 = T(\vec{w}_2) \implies \vec{u}_1 + \vec{u}_2 = T(\vec{w}_1) + T(\vec{w}_2) = T(\vec{w}_1 + \vec{w}_2)$$

Also $\begin{array}{l} \vec{u}_1 = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \\ \vec{u}_2 = d_1\vec{v}_1 + \cdots + d_m\vec{v}_m \end{array} \implies \vec{u}_1 + \vec{u}_2 = (c_1+d_1)\vec{v}_1 + \cdots (c_m+d_m)\vec{v}_m.$

3) If $\vec{u} \in \mathrm{im}(T)$, then $k\vec{u} \in \mathrm{im}(T)$ for any scalar k.

$$\vec{u} = T(\vec{w}) \implies k\vec{u} = kT(\vec{w}) = T(k\vec{w})$$

Upshot If $\vec{u}_1, \ldots, \vec{u}_m \in \mathrm{im}(T)$, then $c_1\vec{u}_1 + \cdots + c_m\vec{u}_m \in \mathrm{im}(T)$ for any scalars $c_1, \ldots, c_m$.

Eg Observe $\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Then, $c_1\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathrm{im}\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$ for any choice of scalars c_1, c_2.

$$\Updownarrow$$

$\begin{cases} x_1 = 3 \\ 2x_1 + x_2 = 5 \\ 3x_1 + 2x_2 = 7 \end{cases}$ and $\begin{cases} x_1 = 1 \\ 2x_1 + x_2 = 1 \\ 3x_1 + 2x_2 = 1 \end{cases}$ both have solutions,

then $\begin{cases} x_1 = 3c_1 + c_2 \\ 2x_1 + x_2 = 5c_1 + c_2 \\ 3x_1 + 2x_2 = 7c_1 + c_2 \end{cases}$ has a solution for any choice of scalars c_1, c_2.