HW 6 due 2/25
4.1: 47, 48
4.2: 10, 14, 22, 54
4.3: 23, 24, 46

Reflection 2 due 2/25 (Canvas)

HW 7 due 3/11
5.1: 16, 26, 28
5.2: 4, 6, 18, 29
5.3: 2, 6, 8, 10

Break 2/28 – 3/4

Thm (Gram-Schmidt Process) For basis $\{\vec{v}_1, \dots, \vec{v}_m\}$ of subspace V of $\mathbb{R}^n$, we can construct an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_m\}$ by decomposing $\vec{v}_i = \vec{v}_i^{\parallel} + \vec{v}_i^{\perp}$ $(i \geq 2)$ with respect to $\text{span}\{\vec{v}_1, \dots, \vec{v}_{i-1}\}$

and setting

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$$
$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp}$$
$$\vdots$$
$$\vec{u}_j = \frac{1}{\|\vec{v}_j^{\perp}\|}\vec{v}_j^{\perp}$$
$$\vdots$$
$$\vec{u}_m = \frac{1}{\|\vec{v}_m^{\perp}\|}\vec{v}_m^{\perp}$$

where $\vec{v}_j^{\perp} = \vec{v}_j - (\vec{v}_j \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_j \cdot \vec{u}_2)\vec{u}_2 - \cdots - (\vec{v}_j \cdot \vec{u}_{j-1})u_{j-1}$

(§ 5.1)

Eg $\mathcal{B} = \left\{ \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{\vec{v}_3} \right\}$ Want orthonormal $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{1+1+0}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\left(1\cdot 1 + 0\cdot 1 + 1\cdot 0\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\0\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{1/4+1/4+1}}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix} = \frac{1}{\sqrt{6/4}}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix} = \frac{2}{\sqrt{6}}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{pmatrix}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{v}_3\cdot\vec{u}_1)\vec{u}_1 - (\vec{v}_3\cdot\vec{u}_2)\vec{u}_2$$

$$= \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\left(0\cdot 1 + 1\cdot 1 + 1\cdot 0\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} - \underbrace{\left(0\cdot \tfrac{1}{\sqrt{6}} + 1\cdot\left(-\tfrac{1}{\sqrt{6}}\right) + 1\cdot\left(\tfrac{2}{\sqrt{6}}\right)\right)}_{1/\sqrt{6}}\begin{pmatrix}1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\0\end{pmatrix} - \begin{pmatrix}1/6\\-1/6\\2/6\end{pmatrix} \qquad 1 - 1/2 + 1/6$$

$$= \begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix}$$

$$\vec{u}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\vec{v}_3^{\perp} = \frac{1}{\sqrt{4/9+4/9+4/9}}\begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix} = \frac{3}{2\sqrt{3}}\begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix} = \begin{pmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$\vec{u}_1 = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix}1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

Sanity check: (To make sure we got it right)

$$\vec{u}_1\cdot\vec{u}_2 = \tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{6}} + \left(\tfrac{1}{\sqrt{2}}\right)\left(-\tfrac{1}{\sqrt{6}}\right) + 0\cdot\tfrac{2}{\sqrt{6}} = 0$$

$$\vec{u}_1\cdot\vec{u}_3 = \left(\tfrac{1}{\sqrt{2}}\right)\left(-\tfrac{1}{\sqrt{3}}\right) + \tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{3}} + 0\cdot\tfrac{1}{\sqrt{3}} = 0$$

$$\vec{u}_2\cdot\vec{u}_3 = \left(\tfrac{1}{\sqrt{6}}\right)\left(-\tfrac{1}{\sqrt{3}}\right) + \left(-\tfrac{1}{\sqrt{6}}\right)\left(\tfrac{1}{\sqrt{3}}\right) + \left(\tfrac{2}{\sqrt{6}}\right)\left(\tfrac{1}{\sqrt{3}}\right) = 0$$

✓

Question Change of basis matrix from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \to \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$? $\mathcal{B}'$

b/c $\mathcal{B}'$ is orthonormal

For ex above

$$\begin{pmatrix} | & | & | \\ [\vec{v}_1]_{\mathcal{B}'} & [\vec{v}_2]_{\mathcal{B}'} & [\vec{v}_3]_{\mathcal{B}'} \\ | & | & | \end{pmatrix} = \begin{pmatrix} \vec{v}_1\cdot\vec{u}_1 & \vec{v}_2\cdot\vec{u}_1 & \vec{v}_3\cdot\vec{u}_1 \\ \vec{v}_1\cdot\vec{u}_2 & \vec{v}_2\cdot\vec{u}_2 & \vec{v}_3\cdot\vec{u}_2 \\ \vec{v}_1\cdot\vec{u}_3 & \vec{v}_2\cdot\vec{u}_3 & \vec{v}_3\cdot\vec{u}_3 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix} \leftarrow \text{upper triangular!}$$

QR-factorization

Thm For $n\times m$ matrix M with linearly independent columns (so $n \geq m$) there exists $n \times m$ matrix Q with orthonormal columns $\vec{u}_1, \dots, \vec{u}_m$ and upper triangular matrix R with positive diagonal entries such that $M = QR$.

Furthermore, for $M = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$, $R = (r_{ij})$ then

$$r_{11} = \|\vec{v}_1\|$$

$$r_{jj} = \|\vec{v}_j^{\perp}\| \quad j \geq 2 \quad (\text{with respect to } W = \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_{j-1}\}$$

$$r_{ij} = \vec{u}_i \cdot \vec{v}_j \quad \text{for } i < j$$

$$\vec{v}_j = \vec{v}_j^{\parallel} + \vec{v}_j^{\perp}$$

$$\begin{aligned} \vec{v}_j \cdot \vec{u}_j &= (\vec{v}_j^{\parallel} + \vec{v}_j^{\perp}) \cdot \vec{u}_j \\ &= (\vec{v}_j^{\parallel} + \vec{v}_j^{\perp}) \cdot \frac{1}{\|\vec{v}_j^{\perp}\|} \vec{v}_j^{\perp} \\ &= \frac{1}{\|v_j^{\perp}\|} \Big(\underbrace{\vec{v}_j^{\parallel} \cdot v_j^{\perp}}_{=0} + \underbrace{v_j^{\perp} \cdot v_j^{\perp}}_{=\|\vec{v}_j^{\perp}\|^2} \Big) \end{aligned}$$

Eg QR-factorization

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}}_{=Q} \underbrace{\begin{pmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 \\ 0 & \|\vec{v}_2^{\perp}\| \end{pmatrix}}_{=R}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{pmatrix}$$

$$r_{11} = \|\vec{v}_1\| = \sqrt{2}$$

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}} (1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1) = 1/\sqrt{2}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 1/\sqrt{2} \cdot 1/\sqrt{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$r_{22} = \|\vec{v}_2^{\perp}\| = \sqrt{6}/2 \qquad \vec{u}_2 = \frac{2}{\sqrt{6}} \vec{v}_2^{\perp} = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

Process 1st col of R, 1st col of Q
2nd col of R, 2nd col of Q
⋮

Orthogonal Matrices

For $A = QR$, Q is an example of an "orthogonal matrix" when A is square.

Def Linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if

$$\|T(\vec{x})\| = \|\vec{x}\| \text{ for all } \vec{x} \in \mathbb{R}^n.$$

For such a $T(\vec{x}) = A\vec{x}$, we say A is an orthogonal matrix.

Eg a) Rotations and reflections are orthogonal transformations.

Orthogonal projections may not be $\left(\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|\right)$

b) $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$ is orthogonal.

Pythagorean Thm

$\vec{a}, \vec{b}$ orthogonal $\iff \|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$

$$\begin{pmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3$$

and
$$\begin{aligned}\|x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3\|^2 &= \|x_1\vec{u}_1 + x_2\vec{u}_2\|^2 + \|x_3\vec{u}_3\|^2 \\ &= \|x_1\vec{u}_1\|^2 + \|x_2\vec{u}_2\|^2 + \|x_3\vec{u}_3\|^2 \\ &= x_1^2 + x_2^2 + x_3^2 \\ &= \|\vec{x}\|^2\end{aligned}$$

$$\implies \|T(\vec{x})\| = \|\vec{x}\| \checkmark$$

Thm If the columns of $n \times n$ matrix A are an orthonormal basis, then A is an orthogonal matrix.

Thm If $T: \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation,

then $\vec{u} \cdot \vec{v} = 0 \implies T(\vec{u}) \cdot T(\vec{v}) = 0$

Pf By Pythagorean Thm and SSS congruence of triangles.

Thm a) $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if $\{T(\vec{e}_1), \ldots, T(\vec{e}_n)\}$ is an orthonormal basis of $\mathbb{R}^n$.

b) A, B are orthogonal $\implies AB$ is orthogonal

c) A is orthogonal $\implies A^{-1}$ orthogonal (All orthogonal matrices are invertible)