Math 417: Matrix Algebra

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Office Hours: MWF 11am - noon (EH 3827)

Systems of equations

Eg $\begin{cases} x + y = 1 \\ x - y = -1 \end{cases}$

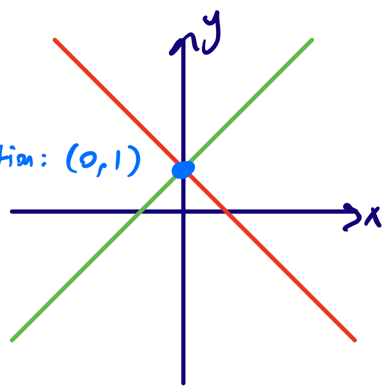
Methods to solve

① Algebraically

$$\begin{cases} x + y = 1 \\ x - y = -1 \end{cases} \to \begin{cases} x + y = 1 \\ 2x = 0 \end{cases}$$

$$\to \begin{cases} y = 1 \\ x = 0 \end{cases}$$

② Geometrically



Eg $\begin{cases} x + y - z = 7 \\ x - y + 2z = 3 \\ 2x + y + z = 9 \end{cases} \longrightarrow \begin{array}{r} \\ -1^{st}\text{eq} \\ \\ \end{array} \begin{cases} x + y - z = 7 \\ -2y + 3z = -4 \\ 2x + y + z = 9 \end{cases}$

Want $\begin{cases} x = \cdots \\ y = \cdots \\ z = \cdots \end{cases}$

$$\to \begin{array}{r} \\ \\ -2 * 1^{st}\text{eq} \end{array} \begin{cases} x + y - z = 7 \\ -2y + 3z = -4 \\ -y + 3z = -5 \end{cases}$$

$$\to \begin{array}{r} \\ * -\frac{1}{2} \\ \\ \end{array} \begin{cases} x + y - z = 7 \\ y - \frac{3}{2}z = 2 \\ -y + 3z = -5 \end{cases}$$

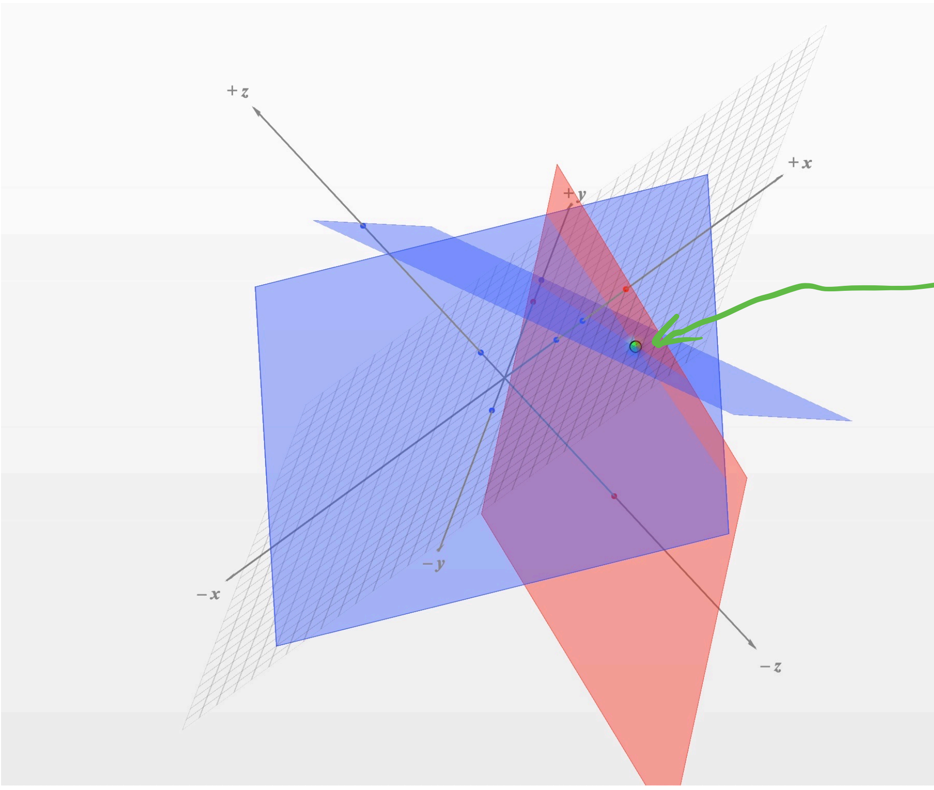
$$\to \begin{array}{r} -2^{nd}\text{Eq} \\ \\ +2^{nd}\text{Eq} \end{array} \begin{cases} x + \frac{1}{2}z = 5 \\ y - \frac{3}{2}z = 2 \\ \frac{3}{2}z = -3 \end{cases}$$

$$\to \begin{array}{r} \\ \\ * \frac{2}{3} \end{array} \begin{cases} x + \frac{1}{2}z = 5 \\ y - \frac{3}{2}z = 2 \\ z = -2 \end{cases}$$

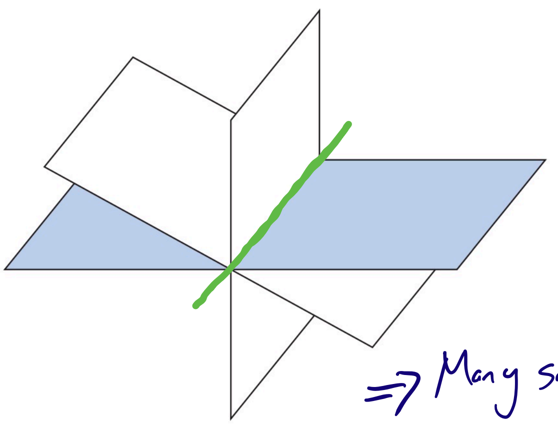
$$\to \begin{array}{r} -\frac{1}{2} * 3^{rd}\text{Eq} \\ +\frac{3}{2} * 3^{rd}\text{Eq} \\ \\ \end{array} \begin{cases} x = 6 \\ y = -1 \\ z = -2 \end{cases}$$

$\Rightarrow x = 6,\ y = -1,\ z = -2$

Geometrically

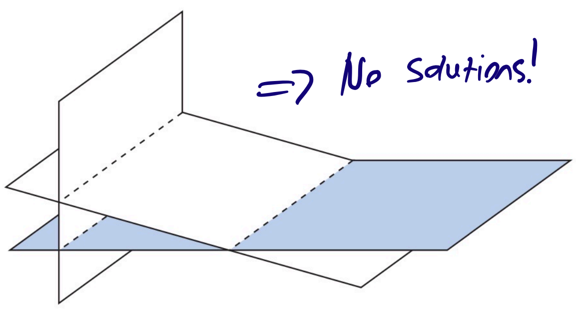


Coordinates: (6, -1, -2)



⇒ Many solutions

Figure 2(a) Three planes having a line in common.



⇒ No solutions!

Figure 2(b) Three planes with no common intersection.

Matrices

$$\begin{cases} x + y - z = 7 \\ x - y + 2z = 3 \\ 2x + y + z = 9 \end{cases} \longrightarrow \begin{array}{cc} & \text{columns} \\ \text{rows} & \begin{pmatrix} 1 & 1 & -1 & 7 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{pmatrix} \end{array}$$

Example of a 3 row by 4 column matrix

General matrix m rows $\times$ n columns

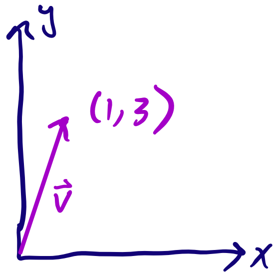
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{for numbers } a_{ij}$$

- $A = B \iff a_{ij} = b_{ij}$ for all entries
- If # rows = # columns of A, then A is a <u>square matrix</u>, and the entries $a_{11}, \dots, a_{nn}$ form the <u>main diagonal</u>. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ main diagonal
- Square matrix A is called
  - <u>diagonal</u> if all entries are zero off of the main diagonal $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
  - <u>upper triangular</u> if all entries are zero below the main diagonal $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$
  - <u>Lower triangular</u> —— " —— —— " —— above the main diagonal $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$
- A matrix with all zero entries is a zero matrix and is denoted by 0 $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Vectors

- An $m \times 1$ matrix is called a <u>column vector</u> $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
- A $1 \times n$ matrix is called a <u>row vector</u> $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
- Entries are called <u>components</u>
- The set of all column vectors with n components is $\mathbb{R}^n$. (vector space)

Eg $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \mathbb{R}^2$



Row Operations

System of Equations

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 7 \\ x_1 + 2x_2 + 2x_3 - x_4 = 12 \\ 2x_1 + 4x_2 \qquad + 6x_4 = 4 \end{cases}$$

Coefficient matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{cccc:c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right)$$

Idea Manipulate the augmented matrix to solve system of equations.