HW 7 due 3/11
5.1: 16, 26, 28
5.2: 4, 6, 18, 29
5.3: 2, 6, 8, 10

HW 8 due 3/18
5.4: 20, 36, 38
6.1: 12, 14, 24, 26, 40, 44

Determinants

Recall $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Def For a **3x3 matrix** $A = \begin{pmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{pmatrix}$, $\det A = \vec{u} \cdot \underbrace{(\vec{v} \times \vec{w})}_{\text{a vector}}$

Eg $A = \begin{pmatrix} 0 & -3 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, columns $\vec{u}, \vec{v}, \vec{w}$

$\vec{v} \times \vec{w} = \begin{pmatrix} 0\cdot 0 - 1\cdot 1 \\ -2\cdot 1 - (-3)\cdot 0 \\ -3\cdot 1 - 0\cdot(-2) \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$

Made a mistake with − sign in class. Correct now!

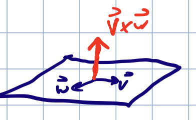
$\det A = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = 0\cdot(-1) + 1\cdot(-2) + 1\cdot(-3) = -5$

Note A 3x3: $\det A = 0 \iff \vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$

$\iff \vec{u} \in \operatorname{Span}\{\vec{v}, \vec{w}\}$

$\iff \operatorname{im} A \neq \mathbb{R}^3$

$\iff A$ is not invertible.



Trick

$$\left(\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right)$$

$- \quad - \quad - \quad + \quad + \quad +$

$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

Notice

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

All ways of circling exactly 1 entry in every row and column.

Def A <u>pattern</u> P of an $n \times n$ matrix A is a choice of n entries in A each with a unique row and column.

We say $\text{prod}(P)$ is the product of the entries in P.

Two entries in a pattern form an <u>inversion</u> if one is above and to the right of the other in A, i.e.,

a_{ij} and $a_{k\ell}$ form an inversion if both $i > k$ and $j < \ell$.

Eg
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Prod(P)	$a_{11}a_{22}a_{33}$	$a_{12}a_{23}a_{31}$	$a_{13}a_{21}a_{32}$	$a_{13}a_{22}a_{31}$	$a_{11}a_{23}a_{32}$	$a_{12}a_{21}a_{33}$
Inversions	0	2	2	3	1	1

Def For any $n \times n$ matrix A, $\det A = \sum_{\substack{\text{patterns } P \\ \text{of } A}} \underbrace{(-1)^{(\#\text{ of inversions in } P)}}_{\text{sgn}(P)} \text{prod}(P)$

Eg a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has patterns $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det A = (-1)^0 ad + (-1)^1 bc$$
$$= ad - bc$$

b) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$ $\qquad \det A = ?$

Only one pattern has $\text{prod}(P) \neq 0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} \implies \det A = (-1)^2\, 1 \cdot 2 \cdot 3 \cdot 4$$
$$= 24$$

c) $A = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\qquad \det A = ?$

$$\begin{pmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$prod(P) = 0$

In fact, only one pattern has $prod(P) \neq 0$

$$P = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \implies \det A = 2 \cdot 2 \cdot 3 \cdot 1 = 12$$

Works for any triangular matrix!

Say we know $\det A$, what happens if we swap rows of A?

Adjacent rows: $a_{i+1,k} \to a_{ij} \quad \longleftrightarrow \quad a_{i+1,k} \quad a_{ij}$ So $\operatorname{sgn} P$ becomes $-\operatorname{sgn} P$

Non-adjacent rows: $\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \to \begin{matrix} r_3 \\ r_2 \\ r_1 \end{matrix}$ is the same $\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix} \to \begin{matrix} r_2 \\ r_1 \\ r_3 \end{matrix} \to \begin{matrix} r_2 \\ r_3 \\ r_1 \end{matrix} \to \begin{matrix} r_3 \\ r_2 \\ r_1 \end{matrix}$

Thm If square matrices A and B are related by swapping 2 rows, then $\det A = -\det B$.

Eg

$$\det \begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underset{\text{swap } 1 \leftrightarrow 3}{=} -\det \begin{pmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -12$$

How are $\det(A)$ and $\det(A^T)$ related?

Eg

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

Example pattern of A

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \text{ goes to } \begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{pmatrix} \text{ of } A^T$$

Thm For square matrix A, $\det(A^T) = \det(A)$.

Combining last 2 theorems:

Thm For square matrices A and B related by swapping 2 columns,
$\det A = -\det B$.

Pf A^T and B^T are related by swapping 2 rows, so
$\det A = \det A^T = -\det B^T = -\det B$ $\square$

Thm

a) $$\det\begin{pmatrix} \text{---} & \vec{v}_1 & \text{---} \\ \text{---} & \vec{v}_2 & \text{---} \\ & \vdots & \\ \text{---} & \vec{x}+\vec{y} & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{pmatrix} = \det\begin{pmatrix} \text{---} & \vec{v}_1 & \text{---} \\ \text{---} & \vec{v}_2 & \text{---} \\ & \vdots & \\ \text{---} & \vec{x} & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{pmatrix} + \det\begin{pmatrix} \text{---} & \vec{v}_1 & \text{---} \\ \text{---} & \vec{v}_2 & \text{---} \\ & \vdots & \\ \text{---} & \vec{y} & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{pmatrix}$$

b) $$\det\begin{pmatrix} \text{---} & \vec{v}_1 & \text{---} \\ & \vdots & \\ \text{---} & k\vec{x} & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{pmatrix} = k \det\begin{pmatrix} \text{---} & \vec{v}_1 & \text{---} \\ & \vdots & \\ \text{---} & \vec{x} & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{pmatrix}$$

c) For fixed $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n$

$$\vec{x} \longmapsto \det\begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{pmatrix} \leftarrow \text{row } i$$

is a linear transformation.