Course Evaluations! (58 %)
Close 4/20

Optional Suggested Problems
8.1: 4, 14, 16
8.3: 4, 6

Office Hours
Today 11a – 12p
Wed 4/20 12:30p – 1:30p
Fri 4/22 12:30p – 1:30p
Mon 4/25 12:30p – 2:30p

Final 4/26 1:30p - 3:30p
Here (EH 1068)

1. (20 pts) For the following statements, circle True if the statement is **always** true, and circle False otherwise. Make sure it is completely clear which is your final answer. No explanations are required for this question, and no partial credit. Read the questions very carefully!

All matrices/eigenvalues/eigenvectors are assumed to have coefficients in $\mathbb{R}$ **unless otherwise specified**, and similarly for the property of diagonalizability.

(a) If the reduced row echelon form of a square matrix A is the identity, then A is invertible.

True   False

(b) If $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation whose image is a plane, then the kernel of T is also a plane.

dim = 2

dim = 2

True   False

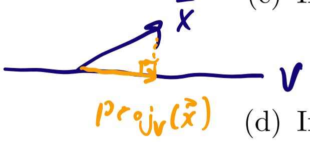
rref $\underbrace{\begin{pmatrix} & & & \end{pmatrix}}$

4 cols

each has a leading 1 or does not have a leading 1 (free variable)

$\operatorname{rank} T = 2 \implies \ker T = 4 - \operatorname{rank} T = 4 - 2 = 2$

(c) If $\vec{x}$ is a vector in $\mathbb{R}^n$ and V is a subspace of $\mathbb{R}^n$, then $\|\operatorname{proj}_V(\vec{x})\| \leq \|\vec{x}\|$.



True   False

(d) If $\vec{v}$ and $\vec{w}$ are linearly independent vectors in $\mathbb{R}^4$, then $\vec{v} \cdot \vec{w} = 0$.

True   False

Eg $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(e) If A is a 100×100 matrix and $AA^T = \mathrm{Id}_{100}$ (the identity matrix), then $\det A = 1$.

$A^{-1} = A^T$

Also, A is orthogonal

True   False

$A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$

$\det(A^T) = \det(A)$

$\implies \det(AA^T) = 1$

$\implies \det(A)^2 = 1$

$\implies \det(A) = \pm 1$

(f) All symmetric $n \times n$ matrices are diagonalizable.

True   False

By spectral Theorem, all sym matrices are orthogonally diagonalizable

(g) All diagonalizable $n \times n$ matrices are symmetric.

True   False

(h) If an $n \times n$ matrix A is diagonalizable, then every vector $v \in \mathbb{R}^n$ may be written as a linear combination of the eigenvectors of A.

diagonalizable $\iff$ eigenbasis $\mathbb{R}^n$

True   False

(i) If $\vec{v}$ is an eigenvector of a matrix A, then $\vec{v}$ is an eigenvector of A^{1000}.

$A\vec{v} = \lambda \vec{v}$

$\implies A^{1000}\vec{v} = \lambda^{1000}\vec{v}$

True   False

(j) The following matrix has negative determinant: $A = \begin{pmatrix} 0 & 1 & 1 & 1 & 99999 \\ 99999 & 0 & 0 & 0 & 0 \\ 2 & 99999 & 1 & 1 & 2 \\ 1 & -2 & 99999 & 1 & 0 \\ 1 & 0 & 0 & 99999 & 2 \end{pmatrix}$

True   False

$\det A = \sum_{\substack{\text{patterns} \\ P}} \operatorname{sgn}(P) \operatorname{prod}(P) = (-1)^4 (99999)^5 + \cdots > 0$

9. (10 pts) Let $A = \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{\text{Symmetric!}}$. Find an **orthogonal** matrix S and a diagonal matrix B such that

$$SAS^{\mathrm{T}} = B.$$

(Hint: at least one of the eigenvalues is -1.)

Normally, $(S')^{-1}AS' = B$

$S = (S')^T \Rightarrow (S')^T A (S')$ where $S' = \begin{pmatrix} \text{eigenvectors} \\ \text{as cols} \end{pmatrix}$, $B = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

So to answer this question, use eigenvectors as rows of S.

Char poly: $\det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = (-\lambda)^3 + 1 + 1 - (-\lambda) - (-\lambda) - (-\lambda)$

$= -\lambda^3 + 3\lambda + 2$

$= -(\lambda^3 - 3\lambda - 2)$

$= -(\lambda+1)(\lambda^2 - \lambda - 2)$

$= -(\lambda+1)(\lambda-2)(\lambda+1)$

$= -(\lambda+1)^2(\lambda-2) = 0$

$\rightsquigarrow$ $\lambda = -1$ with alg mult 2

$\lambda = 2$ w/ alg mult 1

$E_2 = \ker\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \rightarrow \vec{u}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$E_{-1} = \ker\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \left\{\begin{pmatrix} -s-t \\ t \\ s \end{pmatrix} : t, s \in \mathbb{R}\right\} = \operatorname{span}\left\{\underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_2}\right\}$

$\begin{pmatrix} -s-t \\ t \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Gram-Schmidt

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \vec{u}_1$

$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \underbrace{(-1\cdot -1 + 0 \cdot 1 + 1 \cdot 0)\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}}_{\frac{1}{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$

$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\sqrt{1/4 + 1 + 1/4}}\begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} = \frac{1}{\sqrt{6/4}}\begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$

$= \frac{2}{\sqrt{6}}\begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \vec{u}_2$

$\rightsquigarrow S = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$ & $B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

6. Let $A = \begin{pmatrix} 9 & k \\ k & 1 \end{pmatrix}$ for $k \in \mathbb{R}$.

(a) (3 pts) For which values of k is A invertible? For any values of k where A is not invertible, give a basis for the kernel of A.

$\det A \neq 0$

$9 - k^2 \neq 0$

$\Rightarrow (3-k)(3+k) \neq 0$

$\Rightarrow k \neq \pm 3$

$\underline{k=3}$ $\ker \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$

$\underline{k=-3}$ $\ker \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$

(b) (2 pts) For which values of k does A have 2 real eigenvalues? (Explain why.)

I $\lambda^2 - 10\lambda + (9-k^2) = 0$

$\lambda = \dfrac{10 \pm \sqrt{100 - 4(9-k^2)}}{2}$

$\vdots$

II Symmetric matrix

$\Rightarrow$ A always has 2 real e.vals when counted with alg mult.

(c) (2 pts) For which values of k is A diagonalizable? (Explain why.)

All k b/c A is symmetric so diagonalizable by spectral theorem.

(d) (3 pts) What is the quadratic form q associated to A? For which values of k is q positive definite?