Quiz on 2/23
3.4, 4.1, 4.2

HW 6 due 2/25
4.1: 47, 48
4.2: 10, 14, 22, 54
4.3: 23, 24, 46

Reflection 2 due 2/25 (Canvas)

HW 7 due 3/11
5.1: 16, 26, 28

Orthogonal Projections and Orthonormal Bases

Def a) $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$ ← $v_1w_1 + v_2w_2 + \cdots + v_nw_n$

b) $\vec{v} \in \mathbb{R}^n$ has length $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

c) $\vec{u}$ is a unit vector if $\|\vec{u}\| = 1$.

d) $\vec{x} \in \mathbb{R}^n$ is orthogonal to a subspace V of $\mathbb{R}^n$ if $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v} \in V$.

Consequence If subspace V has basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$, then $\vec{x}$ is orthogonal to $V \iff \vec{x} \cdot \vec{v}_i = 0$ for all $1 \le i \le m$.

Recall $\vec{v} \in \mathbb{R}^n \rightsquigarrow \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector.

Def Vectors $\vec{u}_1, \dots, \vec{u}_m$ in $\mathbb{R}^n$ are orthonormal if

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$(\|\vec{u}_i\| = 1)$

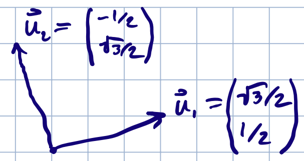
($\vec{u}_i$ and $\vec{u}_j$ are orthogonal if $i \neq j$)

Eg a) $\vec{e}_1, \dots, \vec{e}_n$ are orthonormal since

$$\vec{e}_i \cdot \vec{e}_i = \sqrt{0^2 + 0^2 + \cdots + \underset{i\text{th}}{1^2} + 0^2 + \cdots + 0^2} = 1$$

$$\vec{e}_i \cdot \vec{e}_j = 0 \quad \text{if } i \neq j$$

b)



are orthonormal since

$$\vec{u}_1 \cdot \vec{u}_1 = \tfrac{\sqrt{3}}{2} \cdot \tfrac{\sqrt{3}}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{3}{4} + \tfrac{1}{4} = 1$$
$$\vec{u}_2 \cdot \vec{u}_2 = \left(-\tfrac{1}{2}\right) \cdot \left(-\tfrac{1}{2}\right) + \left(\tfrac{\sqrt{3}}{2}\right)^2 = \tfrac{1}{4} + \tfrac{3}{4} = 1$$
$$\vec{u}_1 \cdot \vec{u}_2 = \tfrac{\sqrt{3}}{2}\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{\sqrt{3}}{2}\right) = 0$$

In fact, $\vec{u}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ & $\vec{u}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ are orthonormal for any θ.

Thm a) Orthonormal vectors are linearly independent.

b) If $\{\vec{u}_1, \dots, \vec{u}_n\}$ are orthonormal vectors in $\mathbb{R}^n$, then they form a basis of $\mathbb{R}^n$.

Orthogonal Projections

For $\vec{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$, then we can write $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ such that

$\vec{x}^{\parallel} \in V$ and
$\vec{x}^{\perp}$ is orthogonal to V.

Such a decomposition is unique.

We call $\vec{x}^{\parallel}$ the orthogonal projection of $\vec{x}$ onto V.

Denote this as $\text{proj}_V \vec{x}$. Note $\text{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation

Thm If V has an orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_m\}$, then
$$\vec{x}^{\parallel} = \text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$
for all $\vec{x} \in \mathbb{R}^n$.

Eg a) Project $\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ onto $V =$ x-y plane
$$= \text{span}\underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}}_{\text{orthonormal basis}}$$

then, $\text{proj}_V(\vec{x}) = (\vec{e}_1 \cdot \vec{x})\vec{e}_1 + (\vec{e}_2 \cdot \vec{x})\vec{e}_2$
$= 3\vec{e}_1 + 2\vec{e}_2$

$$= \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

b) $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Find $\text{proj}_V \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

$\vec{v}_1$, $\vec{v}_2$ Not orthonormal!

$$\vec{v}_1 \cdot \vec{v}_2 = 1\cdot 1 + 1\cdot(-1) + 0\cdot 1 + -1\cdot 0$$
$$= 1 - 1 = 0$$

$$\vec{v}_1 \cdot \vec{v}_1 = 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + -1\cdot -1$$
$$= 3$$
$$\vec{v}_2 \cdot \vec{v}_2 = 3$$

$\vec{u}_1 = \frac{1}{\sqrt{3}}\vec{v}_1$, $\vec{u}_2 = \frac{1}{\sqrt{3}}\vec{v}_2$ are unit vectors and $\vec{u}_1 \cdot \vec{u}_2 = 0$ so $\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal basis of V.

$$\text{proj}_V \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}}_{\vec{x}} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 = \frac{1}{\sqrt{3}}(1\cdot 1 + 2\cdot 1 + 3\cdot 0 + 4\cdot -1)\vec{u}_1 + \frac{1}{\sqrt{3}}(1\cdot 1 + 2\cdot(-1) + 3\cdot 1 + 4\cdot 0)\vec{u}_2$$

$$= \frac{-1}{\sqrt{3}}\vec{u}_1 + \frac{2}{\sqrt{3}}\vec{u}_2 \qquad \vec{u}_1 = \frac{1}{\sqrt{3}}\vec{v}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ -1/3 \\ 0 \\ 1/3 \end{pmatrix} + \begin{pmatrix} 2/3 \\ -2/3 \\ 2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1 \\ 2/3 \\ 1/3 \end{pmatrix}.$$

<u>Consequence</u> For $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$ an orthonormal basis of $\mathbb{R}^n$, then
$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

<u>Upshot</u> Easy to find coordinates for $\vec{x} \in \mathbb{R}^n$ with respect to orthonormal basis $\mathcal{B} = \{\vec{u}_1, \dots, \vec{u}_n\}$. I.e, for $\mathcal{B}$ orthonormal

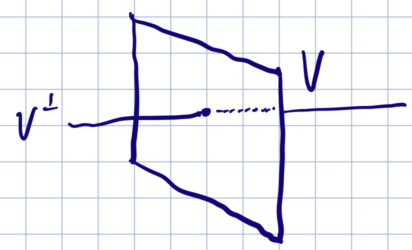
$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} \vec{x}\cdot\vec{u}_1 \\ \vec{x}\cdot\vec{u}_2 \\ \vdots \\ \vec{x}\cdot\vec{u}_n \end{pmatrix}.$$

For $\text{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$, $\text{im}(\text{proj}_V) = V$

What is $\ker(\text{proj}_V)$?

Def For subspace V of $\mathbb{R}^n$, the <u>orthogonal complement</u> of V is $V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n \text{ such that } \vec{x}\cdot\vec{v} = 0 \text{ for all } \vec{v} \text{ in } V\}$.

Thus, $\ker(\text{proj}_V) = V^\perp$



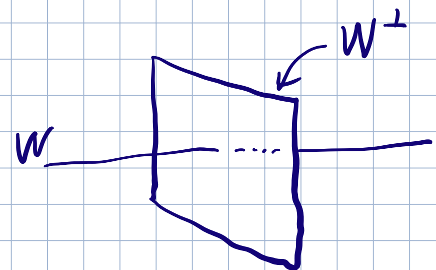
Thm a) $V^\perp$ is a subspace of $\mathbb{R}^n$ since $\ker(\text{proj}_V) = V^\perp$.

b) $V \cap V^\perp = \{\vec{0}\}$ (bc $\vec{0}$ is the only vector orthogonal to itself)

c) By rank-nullity on $T = \text{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$,

$$\dim(V) + \dim(V^\perp) = n$$

d) $(V^\perp)^\perp = V$.



Question How to construct an orthonormal basis?