Written HW 10 (due 4/8)
7.1: 4, 6, 12, 16, 18
7.2: 8, 12, 18, 38
7.3: 8, 10, 24

Quiz on 7.1–7.3 4/11

Written HW 11 (due 4/15)
7.5: 14, 20

Will post Midterm 2 grades today
- Adding +15 points to scores
- Grade on final can replace one of two midterm grades if final score is better

Complex Eigenvalues (7.5)

Some matrices have no real eigenvalues:

Rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - (-1) = \lambda^2 + 1$

has no real solutions

$\lambda^2 + 1 = 0$
$\Rightarrow \lambda^2 = -1$

However, over complex numbers, $\mathbb{C}$, $\lambda = \pm i$ are solutions ($i^2 = -1$)

A complex number z is expressed as $a + bi$ for a, b real numbers and i is the "imaginary root" satisfying $i^2 = -1$

Eg $\underset{(0+i)}{i}$, $\underset{(0-i)}{-i}$, $2+3i$, $-5-5i$, $4-1.5i$, $\underset{(7+0i)}{7}$, $\underset{(-4+0i)}{-4}$ are all in $\mathbb{C}$

Facts a) $(a+bi) + (c+di) = (a+c) + (b+d)i$

b) $(a+bi)(c+di) = ac + bci + adi + bdi^2$
$= (ac - bd) + (bc + ad)i$

λ just a generic variable here.

Fundamental Theorem of Algebra Any polynomial $p(\lambda)$ of degree n can be written as a product of n linear factors over $\mathbb{C}$:

$p(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ for complex numbers $\lambda_1, \dots, \lambda_n, k$.

Eg $x^2+1 = (x+i)(x-i)$

Eg Eigenvectors of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (will have complex entries)

$\underline{\lambda = i}$

$$\ker(A - iI_2) = \ker\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

Solve $x_1\begin{pmatrix} -i \\ 1 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 = i$: $\quad \begin{pmatrix} 1 \\ i \end{pmatrix} + x_2\begin{pmatrix} -1 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(-i)i = -i^2$
$= -(-1)$
$= 1$ $\quad \Rightarrow x_2 = 1$

So, $E_i = \operatorname{span}\left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\}$ and, similarly $E_{-i} = \operatorname{span}\left\{\begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$.

Eg Find all complex eigenvalues of

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$. $\quad \operatorname{tr} A = 3$, $\det A = 5$

$$\rightsquigarrow \det\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 5 & -7 & 3-\lambda \end{pmatrix} = (-\lambda)^2(3-\lambda) + 5 + 0 - 0 - (-7)(1)(-\lambda) - 0$$
$$= (-\lambda)^3 + 3\lambda^2 - 7\lambda + 5$$
$$= -\lambda^3 + 3\lambda^2 - 7\lambda + 5$$

$\underline{\text{Recall}}$ 3×3 matrix must have at least 1 real eigenvalue.

$$= -(\lambda - 1)(\lambda^2 - 2\lambda + 5)$$

$$\frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

So, eigenvalues are $\lambda = 1,\ 1+2i,\ 1-2i$.

Thm If A is a 2×2 real matrix with eigenvalues $a \pm ib$, then

a) A is similar to $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ (rotation-scaling) and

b) If $\vec{v} + i\vec{w}$ is an eigenvector for A ($\vec{v}, \vec{w} \in \mathbb{R}^2$), then

$S^{-1}AS = B$ where $S = \begin{pmatrix} | & | \\ \vec{w} & \vec{v} \\ | & | \end{pmatrix}$. <u>Note</u> Entries of S are real.

<u>Eg</u> For $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, find an invertible matrix S such that $S^{-1}AS$ is a rotation-scaling matrix.

$$\det\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = 5 - 2\lambda + \lambda^2$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= 1 \pm 2i \implies S^{-1}AS = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$E_{1+2i} = \ker(A - (1+2i)I_2) = \ker\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}$$

<u>Useful trick</u> $(a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2$
$= a^2 + b^2 \longleftarrow$ real number

Solve $x_1 \begin{pmatrix} 2-2i \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ -2-2i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Note $(-2+2i)\begin{pmatrix} -2 \\ -2-2i \end{pmatrix} = \begin{pmatrix} 4-4i \\ 8 \end{pmatrix}$

So $x_2 = (-2+2i) \rightsquigarrow$ solve $x_1 \begin{pmatrix} 2-2i \\ 4 \end{pmatrix} + \begin{pmatrix} 4-4i \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $x_1 = -2$

$$\implies E_{1+2i} = \text{span}\left\{ \begin{pmatrix} -2 \\ -2+2i \end{pmatrix} \right\} \implies S = \begin{pmatrix} 0 & -2 \\ 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ -2+2i \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ -2 \end{pmatrix}}_{=\vec{v}} + i\underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{=\vec{w}}$$

$$\rightsquigarrow \begin{pmatrix} 0 & -2 \\ 2 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Thm For $n \times n$ matrix A with complex eigenvalues $\lambda_1, \dots, \lambda_n$ listed with algebraic multiplicity,

$$\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n$$

Eg $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \operatorname{tr} A = 0 = i + (-i)$ $\quad \lambda_1 = i,\ \lambda_2 = -i.$

$\det A = 1 = i(-i)$

Rmk Complex numbers $\mathbb{C}$ form a linear space of dimension 2.

For instance, $\mathcal{B} = \{1, i\}$ is a basis of $\mathbb{C}$ (when scalars $= \mathbb{R}$)

$$\rightsquigarrow [a + bi]_{\mathcal{B}} = \begin{pmatrix} a \\ b \end{pmatrix}.$$