

Quiz on 1/28 (15 min)

→ ch 1, 2.1, 2.3, 2.4

HW 3 due 2/4

2.4: 2, 6, 34, 40, 42

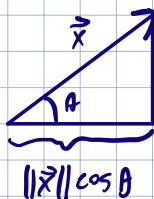
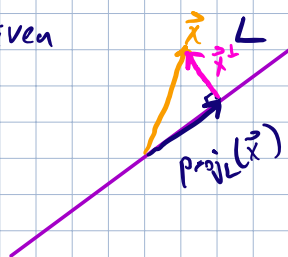
2.2: 20, 32

2.3: 30

Scaling by a factor of $k \rightarrow \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Projection onto a line L :

Given



$$\vec{x} = \text{proj}_L(\vec{x}) + \vec{x}^\perp$$

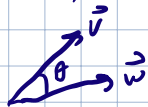
$$\text{proj}_L(\vec{x}) \cdot \vec{x}^\perp = 0$$

1) Take a unit vector \vec{u} parallel to L ,

$$\text{then } \text{proj}_L(\vec{x}) = \underbrace{(\vec{x} \cdot \vec{u})}_{\text{number}} \vec{u} \quad \leftarrow \text{the direction of } L$$

$$\text{In general, } \vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos \theta$$

$$= v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

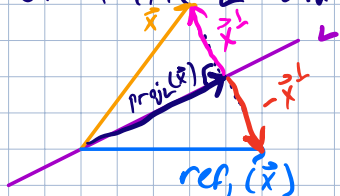


$$\text{so, } \vec{x} \cdot \vec{u} = \|\vec{x}\| \|\vec{u}\| \cos \theta$$
$$= \|\vec{x}\| \cos \theta$$

$$2) \text{proj}_L(\vec{x}) = \begin{pmatrix} u_1 & u_2 \\ u_1 & u_2 \end{pmatrix} \vec{x}$$

Reflections

Given line L and a vector \vec{x} , reflect \vec{x} across L



$$\vec{x} = \text{proj}_L(\vec{x}) + \vec{x}^\perp$$

$$\Rightarrow \text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \vec{x}^\perp$$

$$\Rightarrow \vec{x} + \text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \vec{x}^\perp + \vec{x}$$

$$= 2 \text{proj}_L(\vec{x})$$

$$\Rightarrow \text{ref}_L(\vec{x}) = 2 \text{proj}_L(\vec{x}) - \vec{x}$$

$$\Rightarrow \text{ref}_L(\vec{x}) = \left[2 \begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \vec{x}$$

$$= \begin{pmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{pmatrix} \vec{x}$$

So, this is the matrix for ref_L

$$\det \begin{pmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{pmatrix} = \underbrace{(2u_1^2 - 1)(2u_2^2 - 1) - 4u_1^2 u_2^2}_{4u_1^2 u_2^2 - 2u_1^2 - 2u_2^2 + 1} = -2u_1^2 - 2u_2^2 + 1$$

$$= -2(u_1^2 + u_2^2) + 1$$

$$= -2 \cdot 1 + 1 = -1$$

\Rightarrow reflections have $\det = -1$

$$u_1^2 + u_2^2 = 1 \Rightarrow 2u_1^2 + 2u_2^2 = 2 \Rightarrow 2u_1^2 - 1 = 1 - 2u_2^2 = -(2u_2^2 - 1)$$

$$\text{Set } a = 2u_1^2 - 1, b = 2u_1 u_2 \Rightarrow$$

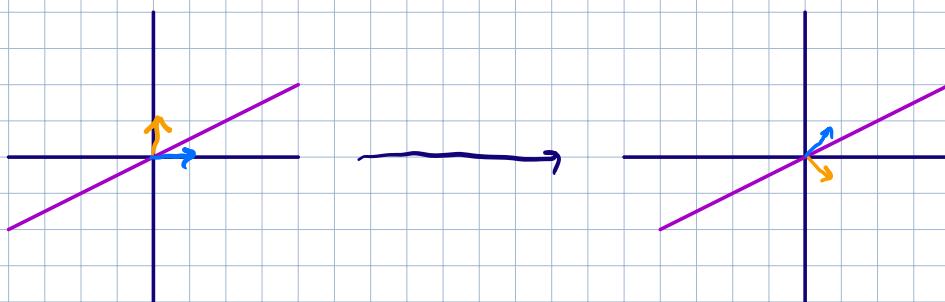
$$\text{reflection is of the form } \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \text{ with } -a^2 - b^2 = -1 \Rightarrow a^2 + b^2 = 1$$

Any matrix of this form is a reflection

Eg 1) $a=0, b=1$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from HW \rightarrow reflect over $y=x$

2) Reflect \vec{e}_1 and \vec{e}_2 over $L: 2x - y = 0$

$$\vec{u} = \frac{1}{\sqrt{4+1}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$



Two methods to compute

$$1) \text{ref}_L(\vec{e}_1) = 2\text{proj}_L(\vec{e}_1) - \vec{e}_1$$

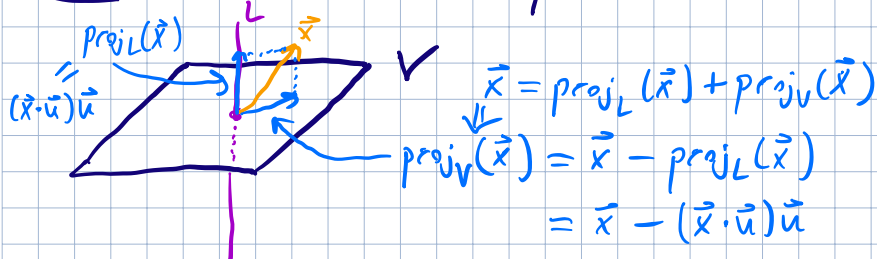
$$= 2(\vec{e}_1 \cdot \vec{u})\vec{u} - \vec{e}_1 = 2 \left(\frac{2}{\sqrt{5}} \right) \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8/5 \\ 4/5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$\text{Similarly, } \text{ref}_L(\vec{e}_2) = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$$

$$2) \leadsto \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \quad \text{Similarly for } \begin{pmatrix} \quad \\ \quad \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \dots$$

Ref 3-dimensions (see p.65)



$$\text{Similarly, } \text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \text{proj}_{L^\perp}(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$$

$$\text{ref}_V(\vec{x}) = \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x}) = \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}.$$

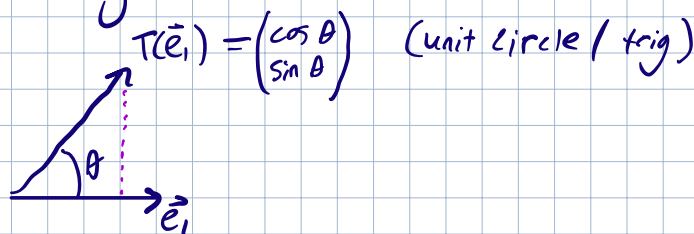
Rotations

T given by rotating $\vec{x} \in \mathbb{R}^2$ by θ CCW.

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$\det A = \cos^2 \theta + \sin^2 \theta = 1$$



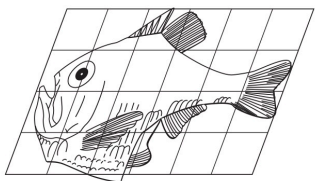
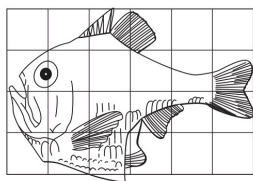
$$T(\vec{e}_2) = \begin{pmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$

Any matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a^2 + b^2 = 1$ is a rotation.

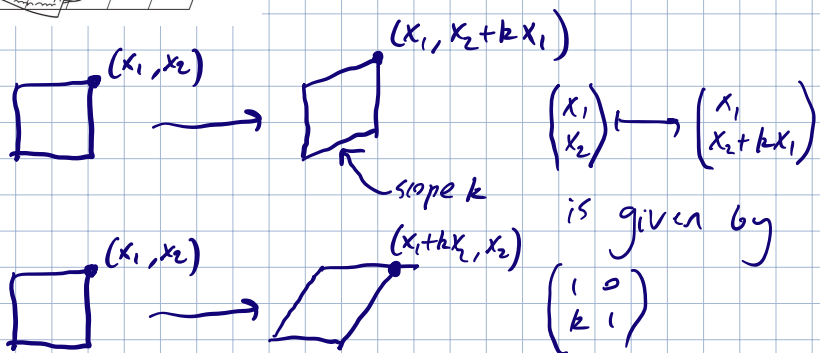
Eg $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is rot CCW by $\frac{\pi}{2}$ radians ($a=0, b=1$)

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is π radians ($a=-1, b=0$)

Shears



Vertical Shear



Horiz Shear

