HW 3 due today, 5pm
2.4: 2, 6, 34, 40, 42
2.2: 20, 32
2.3: 30

Midterm 1, Wed 2/9
Ch 1 – 3.3
Some practice on Canvas

HW 4 due Fri 2/11
3.1: 6, 24, 32, 34, 37, 38
3.2: 26, 34
3.3: 30 38
Cool exercise but not collected:
3.3: 90

Rank-Nullity Theorem For a $n \times m$ matrix A

a) $\dim(\text{im} A) = \text{rank}(A) \leftarrow$ # of leading 1's in rref(A)

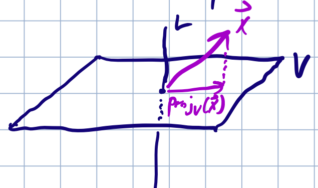
b) $\underbrace{\dim(\ker A)}_{\text{“nullity of } A\text{”}} + \underbrace{\dim(\text{im}(A))}_{\text{rank } A} = m$

b') (nullity of A) + (rank A) $= m$

Eg a) For $A = \begin{pmatrix} 1 & 2 & 2 & -3 \\ 1 & 1 & 2 & -1 \end{pmatrix}$, $\dim(\ker A) = 2$, $\dim(\text{im} A) = 2$ (Last time)

$2 + 2 = 4 =$ # of columns of A (m)

b) Consider projection $\text{proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$ for plane V in $\mathbb{R}^3$



$\text{im}(\text{proj}_V) = V$, $\dim(V) = 2$

$\ker(\text{proj}_V) = L$, $\dim(L) = 1$

$2 + 1 = 3 = m$

Questions a) Can you find a 3×3 matrix A such that $\text{im}(A) = \ker(A)$?

$\dim(\ker A) + \dim(\text{im} A) = 3$ so no! Because $\dim(\ker A) \neq \dim(\text{im} A)$ for A a 3×3 matrix.

b) If 3×3 matrix A satisfies $A = BC$ for B – 3×2 matrix, C – 2×3 matrix

Can A be invertible? No!

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "T_C"] \arrow[rr, bend right, dashed, "T_A"'] & \mathbb{R}^2 \arrow[r, "T_B"] & \mathbb{R}^3 \end{tikzcd}$$

$im(BC)$ is contained in $im(B)$.

Why?

Take $\vec{z} \in im(BC)$, then there is some vector $\vec{x} \in \mathbb{R}^3$ such that

$BC\vec{x} = \vec{z}$ but $\vec{y} = C\vec{x}$, then $BC\vec{x} = B\vec{y} \in im(B)$

So every vector in $im(BC)$ is also in $im(B)$

$\Rightarrow \dim(im(BC)) \leq \dim(im(B))$

By rank-nullity, $\dim(im(B)) + \dim(ker(B)) = 2$

$\Rightarrow \dim(im(B)) \leq 2$

$$\Rightarrow \underbrace{\dim(im(BC))}_{\substack{\shortparallel \\ \dim(im(A))}} \leq 2$$

Since $2 \geq \dim(im(A)) = 3 - \dim(ker(A))$

$\Rightarrow \dim(ker A) \geq 1 \Rightarrow ker A \neq \{\vec{0}\}$

so A is not invertible.

Eg
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thm $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$ if and only if

$$A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \text{ is invertible.}$$

Why?
$$im\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} = Span\{\vec{v}_1, \dots, \vec{v}_n\}$$

$\vec{v}_1, \dots, \vec{v}_n$ is a basis of $\mathbb{R}^n \Leftrightarrow \dim(Span\{\vec{v}_1, \dots, \vec{v}_n\}) = n$

$\Leftrightarrow \dim(im(A)) = n$

$\Leftrightarrow \dim(ker(A)) = 0$ (Rank-Nullity)

$\Leftrightarrow ker(A) = \{\vec{0}\}$

$\Leftrightarrow A$ is invertible.

Eg For which values k is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ k^2-5 \end{pmatrix} \right\}$ a basis of $\mathbb{R}^3$?

Check $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & k^2-5 \end{pmatrix}$ invertible $\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & k^2-4 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & k^2-4 \end{pmatrix}$

We can finish row reduction only when $k \neq \pm 2$

So, we have a basis when $k \neq \pm 2$.

Coordinates

Consider a subspace $V = \text{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}}_{\vec v_2} \right\}$

Is $\vec u = \begin{pmatrix} 2 \\ 5 \\ 4 \\ 3 \end{pmatrix}$ in V? $\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 5 \\ 1 & 2 & 4 \\ 0 & 3 & 3 \end{array}\right) \xrightarrow[\text{rref}]{\cdots} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So, $\vec u = 2\vec v_1 + 1 \cdot \vec v_2$

Since $\dim V = 2$, $\vec u$ can be described by just two numbers.