HW 8 due 3/18
5.4: 20, 36, 38
6.1: 12, 14, 24, 26, 40, 44

HW 9 due 3/23 (wed)
6.2: 2, 12, 14, 38, 42
6.3: 2, 18

Midterm 2 3/25

Minors and Cofactor Expansion

Def For an $n \times n$ matrix A, let A_{ij} be matrix obtained by omitting the ith row of A and jth column of A.

The determinant of A_{ij} is called a minor of A.

Eg $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ A_{12}: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightarrow A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$

$\det A_{12} = 36 - 42 = -6$ is a minor of A.

For $n \times n$ matrix, what if we collect all patterns with one fixed position?

Eg $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$-1 \cdot 2 \cdot 4 \cdot 9 + (-1)^2 \cdot 2 \cdot 6 \cdot 7 = 12 = -2 \cdot (4 \cdot 9 - 6 \cdot 7)$
$= -2 \cdot \det A_{12}$

Thm (Cofactor expansion) Let A be an $n \times n$ matrix.

a) Pick a column j. Then, $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$.

b) Pick a row i. Then, $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$.

Think

$(-1)^{i+j} \rightarrow \begin{pmatrix} + & - & + & - & + & - \\ - & + & - & + & - & \\ + & - & + & - & & \\ - & + & - & & & \\ + & - & & & & \\ - & & & & & \end{pmatrix}$

Eg $\det\begin{pmatrix} 2 & 2 & 0 & 16 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 & 0 & 16 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 16 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 16 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 16 \\ 1 & 1 & 1 & 14 \\ 0 & 1 & 1 & 10 \\ 0 & 1 & 0 & 6 \end{pmatrix}$

$(i=1, j=1)$ $(i=2, j=1)$ $(i=3, j=1)$

$$= 2\cdot\det\begin{pmatrix} 1 & 1 & 14 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 2 & 0 & 16 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix} + 0\cdot\det A_{31} - 0\cdot\det A_{41}$$

$\begin{pmatrix} 1 & 1 & 14 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 14 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 14 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 2 & 0 & 16 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 & 16 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 & 16 \\ 1 & 1 & 10 \\ 1 & 0 & 6 \end{pmatrix}$

$(i=3, j=1)$ $(i=3, j=2)$ $(i=3, j=3)$ $\quad$ $(i=2, j=2)$

$1\cdot\det\begin{pmatrix} 1 & 14 \\ 1 & 10 \end{pmatrix} - 0\cdot\det A_{32} + 6\cdot\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad$ $0 \quad +1\cdot\det\begin{pmatrix} 2 & 16 \\ 1 & 6 \end{pmatrix} \quad 0$

$1\cdot(1\cdot 10 - 1\cdot 14)$ $\quad 0 \quad 0$ $\quad$ $2\cdot 6 - 16 = -4$

-4

$$= 2\cdot(-4) - 1\cdot(-4) = -8 + 4 = -4.$$

Geometric Interpretations of the Determinant

For a square matrix A, then $A = QR$

invertible $\quad$ orthogonal matrix $\quad$ upper triangular

What is $\det Q$?

$$Q \text{ orthogonal} \Leftrightarrow QQ^T = I_n \implies (\det Q)(\det Q^T) = 1$$
$$\implies (\det Q)^2 = 1$$
$$\implies \det Q = \pm 1$$

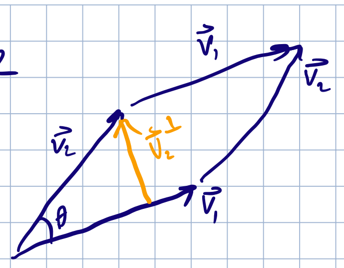
Thm The determinant of an orthogonal matrix is 1 or -1.

If A is orthogonal with $\det A = 1 \implies A$ is a rotation matrix.

Recall (§5.2) $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \implies Q = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix}$ $\quad R = \begin{pmatrix} \|\vec{v}_1\| & & & \\ 0 & \|\vec{v}_2^{\perp}\| & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \|\vec{v}_n^{\perp}\| \end{pmatrix}$

$$\implies \det A = (\det Q)(\det R) = (\pm 1)\, \|\vec{v}_1\|\cdot\|\vec{v}_2^{\perp}\|\cdots\|\vec{v}_n^{\perp}\|$$
$$\implies |\det A| = \|\vec{v}_1\|\cdot\|\vec{v}_2^{\perp}\|\cdots\|\vec{v}_n^{\perp}\|$$

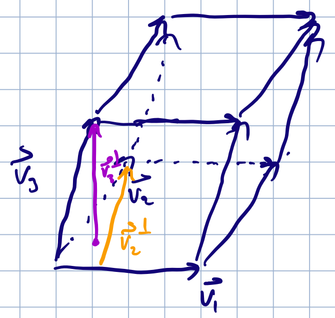
Eg $n=2$



has area $\|\vec{v}_1\|\|\vec{v}_2^{\perp}\|$

Since $\|\vec{v}_2^{\perp}\| = \|\vec{v}_2\|\cdot|\sin\theta|$

$n=3$



has volume $\underbrace{\|\vec{v}_1\|\|\vec{v}_2^{\perp}\|}_{\text{base}} \cdot \underbrace{\|\vec{v}_3^{\perp}\|}_{\text{height}}$

Def The <u>m-parallelepiped</u> defined by $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ is the set of all vectors $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ where $0 \le c_i \le 1$.
The <u>m-volume</u> $V(\vec{v}_1, \dots, \vec{v}_m)$ of this m-parallelepiped is defined as $\|\vec{v}_1\|\|\vec{v}_2^{\perp}\| \cdot \cdots \cdot \|\vec{v}_m^{\perp}\|$.

Thm For $n \times m$ matrix $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix}$, $V(\vec{v}_1, \dots, \vec{v}_m) = \sqrt{\det(A^TA)}$

If $m=n$, $V(\vec{v}_1, \dots, \vec{v}_n) = \det A$.

Eg $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ forms a 2-parallelepiped (parallelogram) in $\mathbb{R}^3$

with area $= \sqrt{\det\left[\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\right]} = \sqrt{\det\begin{pmatrix} 6 & 5 \\ 5 & 6 \end{pmatrix}}$

$= \sqrt{36-25}$

$= \sqrt{11}$

Thm For an $n \times n$ matrix A, $V(A\vec{v}_1, \dots, A\vec{v}_n) = |\det A|\, V(\vec{v}_1, \dots, \vec{v}_n)$ for all vectors $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$.