HW due Friday, 1/14

1.1: 14, 20, 30
1.2: 10, 44, 45 (use calculator for 45)

HW due Friday, 1/21

1.3: 4, 48ab, 58
2.3: 2, 4, 8

<u>Last time</u> Matrix Operations and linear combinations

<u>Putting it all together</u>

<u>Thm</u> Linear system of equations with augmented matrix $(A \vdots \vec{b})$ can be written $A\vec{x} = \vec{b}$ in matrix form.

<u>Eg</u> $\begin{cases} x_1 + x_2 = 5 \\ x_1 - x_2 = 1 \end{cases}$ Augmented matrix $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -1 & 1 \end{array}\right)$

Matrix form: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Is $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ a linear combination of $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$?

In other words, is there an x_1, x_2 such that

$$x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Same as solving linear system!

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & -1 & 1 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - R_1]{} \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -2 & -4 \end{array}\right) \xrightarrow[R_2 \leftarrow -\frac{1}{2}R_2]{} \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

$x_1 = 3$, $x_2 = 2$

So, $3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Matrix Products

Def B – $n \times p$ matrix

A – $p \times m$ matrix

then BA is the $n \times m$ matrix given by

$$BA = B\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{pmatrix}$$

$$(n \times p)(p \times m) = (n \times m)$$

Warning The number of cols of B = The number of rows of A, otherwise this is undefined!

Eg a) $$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 5 \\ 4 & 10 & 7 \\ 6 & 12 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 0\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 7\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 19 \\ 43 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 8 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 8\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 22 \\ 50 \end{pmatrix}$$

What about $\underbrace{\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{\parallel}$?

<u>Check</u>

$$\begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix} \neq \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

<u>Warning!</u> BA and AB are not always equal!

What is the i,j-entry of BA?

i,j-entry of BA = i^{th} component of $B\vec{v}_j = B\begin{pmatrix} a_{1j} \\ \vdots \\ a_{pj} \end{pmatrix}$

$$= (b_{i1} \cdots b_{ip}) \cdot \begin{pmatrix} a_{1j} \\ \vdots \\ a_{pj} \end{pmatrix} = \sum_{k=1}^{p} b_{ik} a_{kj}$$

<u>Thm</u> Let B be an $n \times p$ matrix and A be a $p \times m$ matrix.

Then, the i,j-entry of BA is $\sum_{k=1}^{p} b_{ik} \cdot a_{kj}$.

<u>Trick</u>

$$\begin{array}{cc} & \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} \end{array}$$

$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ component

<u>Some Matrix Product Rules</u>

<u>Thm</u> a) For A an $n \times m$ matrix,

$$A I_m \overset{\checkmark}{=} A = I_n A$$

$I_k = k \times k$ identity matrix

$$= \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}}_{k \text{ columns}} \Big\} k \text{ rows}$$

$$A \cdot \begin{pmatrix} | & & | \\ \vec{e}_1 & \cdots & \vec{e}_m \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ A\vec{e}_1 & \cdots & A\vec{e}_m \\ | & & | \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} a_{11} & & a_{1m} \\ \vdots & \cdots & \vdots \\ a_{n1} & & a_{nm} \end{pmatrix} = A \checkmark$$

$$(*)\ A\vec{e}_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_k \\ | & | & & | \end{pmatrix}$$

b) $(AB)C = A(BC)$, so we can write ABC.

Let $C = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{pmatrix}$, then $(AB)C = \begin{pmatrix} | & & | \\ (AB)\vec{v}_1 & \cdots & (AB)\vec{v}_m \\ | & & | \end{pmatrix}$

and $A(BC) = \begin{pmatrix} | & & | \\ A(B\vec{v}_1) & \cdots & A(B\vec{v}_m) \\ | & & | \end{pmatrix}$ (*)

(*) Check $(AB)\vec{v}_i = A(B\vec{v}_i)$ for $\vec{v}_i = \begin{pmatrix} c_{1i} \\ \vdots \\ c_{pi} \end{pmatrix}$ ✓

$$AB = A\begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ A\vec{u}_1 & \cdots & A\vec{u}_p \\ | & & | \end{pmatrix}$$

So, $(AB)\vec{v}_i = c_{1i}A\vec{u}_1 + \cdots + c_{pi}A\vec{u}_p$

and $A(B\vec{v}_i) = A(c_{1i}\vec{u}_1 + \cdots + c_{pi}\vec{u}_p)$ $\square$

c) If A and B are $n \times p$ matrices and C and D are $p \times m$ matrices, then

$$A(C+D) = AC + AD$$
$$(A+B)C = AC + BC$$

d) If A is an $n \times p$ matrix, B is a $p \times m$ matrix, and k is a scalar,

$$(kA)B = A(kB) = k(AB)$$

<u>Why multiply matrices?</u>

A $n \times m$ matrix

$\vec{v} \in \mathbb{R}^n \longmapsto A\vec{v} \in \mathbb{R}^n$