http://timbaumann.info/svd-image-compression-demo/

Written HW 11 due 4/15
7.5: 14, 20
7.1: 68, 70
7.4: 4, 34

Reflection 3 due 4/15
Canvas

Course Evaluations!

Optional Suggested Problems
8.1: 4, 14, 16
8.3: 4, 6

Singular Values

For 2x2 matrix A, can we find orthogonal vectors $\vec{v}_1, \vec{v}_2$ such that $A\vec{v}_1$ and $A\vec{v}_2$ are also orthogonal?

$(\vec{v}_1 \cdot \vec{v}_2 = 0 \implies (A\vec{v}_1)\cdot(A\vec{v}_2) = 0\ ?)$

a) If A is orthogonal, then A preserves angles
so if $\vec{v}_1 \cdot \vec{v}_2 = 0 \implies A\vec{v}_1 \cdot A\vec{v}_2 = 0$ in this case

b) If A is symmetric, choose two orthogonal eigenvectors $\vec{v}_1, \vec{v}_2$
then $\vec{v}_1 \cdot \vec{v}_2 = 0$ and $(A\vec{v}_1)\cdot(A\vec{v}_2) = \lambda_1 \vec{v}_1 \cdot \lambda_2 \vec{v}_2 = 0$.

In general?

Consider: For any 2x2 matrix A, then A^TA is always symmetric.

$\implies A^TA$ has an orthonormal eigenbasis $\{\vec{v}_1, \vec{v}_2\}$

$$(A\vec{v}_1)\cdot(A\vec{v}_2) = (A\vec{v}_1)^T(A\vec{v}_2) = v_1^T A^T A \vec{v}_2 = v_1^T \lambda_2 \vec{v}_2$$
$$= \lambda_2(\vec{v}_1 \cdot \vec{v}_2)$$
$$= 0$$

Eg $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$ $\quad A^TA = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$

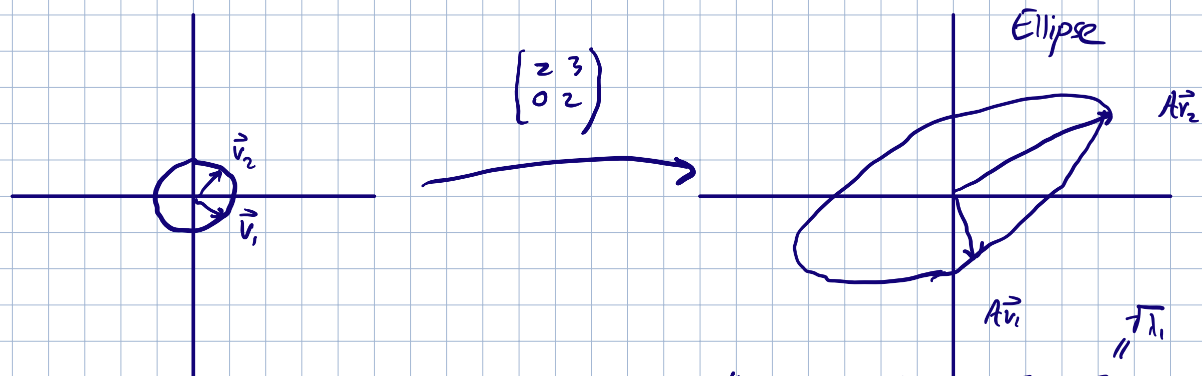
$$\lambda^2 - 17\lambda + 16 = (\lambda - 16)(\lambda - 1)$$

$$\lambda = 1, 16$$

$$E_1 = \ker \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \rightsquigarrow \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$E_{16} = \ker \begin{pmatrix} -12 & 6 \\ 6 & -3 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \rightsquigarrow \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\rightsquigarrow A\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ & $A\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 8 \\ 4 \end{pmatrix}$ are also orthogonal



and $\|A\vec{v}_1\| = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1 = \sigma_2$ $\quad (\sqrt{1} = \sqrt{\lambda_1})$

$\|A\vec{v}_2\| = \sqrt{\frac{64}{5} + \frac{16}{5}} = \sqrt{16} = 4 = \sigma_1$ $\quad (\sqrt{16} = \sqrt{\lambda_2})$, $\sigma_1 \geq \sigma_2$

Remember $\vec{v}_1, \vec{v}_2$ are eigenvectors of A^TA not of A.

Def The <u>singular values</u> of $m \times n$ matrix A are the square roots of the eigenvalues of A^TA, listed with algebraic multiplicity.

Denote singular values as $\sigma_1, \dots, \sigma_m$ such $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m \geq 0$.

Note All eigenvalues of A^TA are ≥ 0.

Thm a) For invertible $T: \mathbb{R}^2 \to \mathbb{R}^2$, the image of unit circle is an ellipse with length of semimajor axis $= \sigma_1$, semiminor axis $= \sigma_2$

b) More generally, for $T: \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$, there exists an orthonormal basis $\vec{v}_1, \dots, \vec{v}_m$ of $\mathbb{R}^m$ such that

i) $T(\vec{v}_1), \dots, T(\vec{v}_m)$ are orthogonal

ii) $\|T(\vec{v}_i)\| = \sigma_i$ for all $1 \leq i \leq m$ where $\sigma_1, \dots, \sigma_m$ are singular values of A.

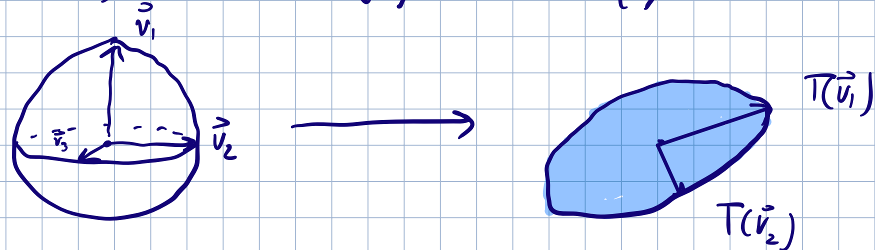
In fact, $\vec{v}_1, \dots, \vec{v}_m$ is an orthonormal eigenbasis of A^TA.

Eg $T(\vec{x}) = A\vec{x}$ $\qquad A = \underset{2\times 3}{\begin{pmatrix} 4 & 11 & 14 \\ 4 & 7 & -2 \end{pmatrix}}$

$$A^TA = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix} \rightsquigarrow \text{Eigenvalues } \lambda_1 = 360,\ \lambda_2 = 90, \text{ and } \lambda_3 = 0.$$

$$E_{360} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}}_{= \vec{v}_1} \right\} \quad E_{90} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}}_{= \vec{v}_2} \right\} \quad E_0 = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}}_{= \vec{v}_3} \right\}$$

$T(\vec{v}_1) = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$, $T(\vec{v}_2) = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$, $T(\vec{v}_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are orthogonal.



Thm If A is an $n \times m$ matrix of rank r, then singular values $\sigma_1, \dots, \sigma_r$ are non-zero; $\sigma_{r+1}, \dots, \sigma_m$ are zero.

Setup in terms of matrices

For $n \times m$ matrix A of rank r, pick $\vec{v}_1, \dots, \vec{v}_m$ as above.

Set $\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1, \dots, \vec{u}_r = \frac{1}{\sigma_r} A\vec{v}_r$ and $\vec{u}_{r+1}, \dots, \vec{u}_n$ so that $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

So $A\vec{v}_1 = \sigma_1 \vec{u}_1, \dots, A\vec{v}_r = \sigma_r \vec{u}_r$

$$\Rightarrow \underset{n\times m}{A} \underbrace{\begin{pmatrix} | & & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \vec{v}_{r+1} & \cdots & \vec{v}_m \\ | & & | & | & & | \end{pmatrix}}_{\substack{V \\ \text{orthogonal } m\times m}} = \begin{pmatrix} | & & | & | & & | \\ \sigma_1 \vec{u}_1 & \cdots & \sigma_r \vec{u}_r & \vec{0} & \cdots & \vec{0} \\ | & & | & | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | & | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r & \vec{0} & \cdots & \vec{0} \\ | & & | & | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} | & & | & | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_n \\ | & & | & | & & | \end{pmatrix}}_{\substack{U \\ \text{orthogonal } n\times n}} \underbrace{\begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}}_{\substack{\Sigma \\ n\times m}}$$

$$\Rightarrow AV = U\Sigma \Rightarrow A = U\Sigma V^T$$

__Thm__ (Singular Value Decomposition) For A an $n\times m$ matrix of rank r, A can be written as $A = U\Sigma V^T$ for

- U orthogonal $n\times n$ matrix
- V is orthogonal $m\times m$ matrix
- Σ is an $n\times m$ matrix satisfying $\Sigma_{ii} = \sigma_i$ for $1 \le i \le r$ and all other entries 0.

In other words $A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$