HW due Friday, 1/14

1.1: 14, 20, 30
1.2: 10, 44, 45 (use calculator for 45)

Row Operations

System of Linear Equations

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 7 \\ x_1 + 2x_2 + 2x_3 - x_4 = 12 \\ 2x_1 + 4x_2 \qquad + 6x_4 = 4 \end{cases}$$

Coefficient matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{pmatrix}$$

Augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right)$$

Idea Manipulate the augmented matrix to solve system of equations.

Allowable Operations (Row Operations)

① Multiply/divide a row by a nonzero scalar

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right) \xrightarrow[R_3 \leftarrow \frac{1}{2} * R_3]{} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 1 & 2 & 0 & 3 & 2 \end{array}\right)$$

② Subtract a multiple of a row from another row

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 0 & 0 & -2 & 4 & -10 \end{array}\right)$$

③ Swap two rows

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 12 \\ 1 & 2 & 1 & 1 & 7 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right)$$

Want Each row to be

$$\begin{array}{cccccc} & & \text{above} \left\{ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right. & & & \\ 0 \cdots & 0 & 1 & * & \cdots & * \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & & \end{array}$$

Eg
$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & -2 & 4 & -10 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 + 2R_2]{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 2x_2 \quad + 3x_4 = 2 \\ \quad x_3 - 2x_4 = 5 \end{cases} \Rightarrow \begin{cases} x_1 = 2 - 2x_2 - 3x_4 \\ x_3 = 5 + 2x_4 \end{cases}$$

x_1 and x_3 are <u>leading variables</u>

x_2 and x_4 are <u>free variables</u>

So, if $x_2 = t$, $x_4 = r$, then $x_1 = 2 - 2t - 3r$
and $x_3 = 5 + 2r$

Represent as a vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 2t - 3r \\ t \\ 5 + 2r \\ r \end{pmatrix}$$

<u>Nice properties of desired matrix form for system of equations</u>

(P1) The leading coefficient is 1 in each equation

(P2) Each leading variable appears in only one equation

(P3) Leading variables are in increasing order.

To solve a system of equations, we want to get to this form!

<u>Def.</u> A matrix is in <u>row-reduced echelon form</u> (rref) if it satisfies

a) If a row has nonzero entries, then 1^{st} nonzero entry is a <u>1</u>
(leading 1 or <u>pivot</u>)

$0 \cdots 0 \; 1 \; * \cdots *$

b) If a column has a leading 1, then all other entries in that column are 0.

$$\begin{matrix} & & 0 & & \\ & & \vdots & & \\ & & 0 & & \\ 0 \cdots 0 & & 1 & * \cdots * \\ & & 0 & & \\ & & \vdots & & \\ & & 0 & & \end{matrix}$$

c) If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

$$\begin{array}{cccc} 1 & & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \ \cdots \end{array}$$

Examples

1) $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ 2) $\left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ 3) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$

Non-examples

1) $\left(\begin{array}{cccc|c} 0 & 0 & 1 & -2 & 5 \\ 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ violates c)

2) $\left(\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ Violates b)

3) $\left(\begin{array}{cccc} 1 & 0 & 0 & 6 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right)$ Violates a)

How to put a matrix in rref

Proceed row by row, top to bottom

At row i

I) Take the leftmost non-zero entry and normalize it to 1. Call this column j.

II) Add/Subtract multiples of row i from all the others so column j has zeros everywhere except row i

III) Proceed to row i+1

At the end, swap rows so leading 1's are in increasing order.

Eg → $\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 4 & 2 & 10 \\ 1 & 2 & 7 \\ 3 & 6 & 21 \end{array}\right) \xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left(\begin{array}{ccc} 1 & \frac{1}{2} & \frac{5}{2} \\ 4 & 2 & 10 \\ 1 & 2 & 7 \\ 3 & 6 & 21 \end{array}\right) \xrightarrow[R_4 \leftarrow R_4 - 3R_1]{R_2 \leftarrow R_2 - 4R_1,\ R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{9}{2} \\ 0 & \frac{9}{2} & \frac{27}{2} \end{array}\right) \xrightarrow{R_3 \leftarrow \frac{2}{3}R_3} \left(\begin{array}{ccc} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & \frac{9}{2} & \frac{27}{2} \end{array}\right)$

$R_1 \leftarrow R_1 - \frac{1}{2}R_3$

$R_4 \leftarrow R_4 - \frac{9}{2}R_3$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So, $\begin{cases} 2x_1 + x_2 = 5 \\ 4x_1 + 2x_2 = 10 \\ x_1 + 2x_2 = 7 \\ 3x_1 + 6x_2 = 21 \end{cases}$ has solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \leftrightarrow x_1 = 1,\ x_2 = 3$

When done on equations, this is called <u>Gauss-Jordan Elimination</u>

<u>Caveat</u> A system of equations can be <u>inconsistent</u>

<u>Eg</u> $\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + 4x_2 = 11 \end{cases} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 4 & 11 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 1 \end{array}\right) \leftarrow$ This row says $0 = 1$, which is inconsistent!

$\Rightarrow$ System has no solutions

<u>Def</u> The <u>rank</u> of a matrix A is the number of leading 1's in its rref.