HW due Friday, 1/14

1.1: 14, 20, 30
1.2: 10, 44, 45 (use calculator for 45)

HW due Friday, 1/21

1.3: 4, 48ab, 58

Clarifications from 1/7

1) $\begin{cases} x_1 + 2x_2 \quad + 3x_4 = 2 \\ \qquad x_3 - 2x_4 = 5 \end{cases} \Rightarrow \begin{cases} x_1 = 2 - 2x_2 - 3x_4 \\ x_3 = 5 + 2x_4 \end{cases}$

x_1 and x_3 are <u>leading variables</u>

x_2 and x_4 are <u>free variables</u>

So, if $x_2 = t$, $x_4 = r$, then $x_1 = 2 - 2t - 3r$
and $x_3 = 5 + 2r$

Represent as a vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 - 2t - 3r \\ t \\ 5 \quad + 2r \\ r \end{pmatrix} \Longrightarrow$$

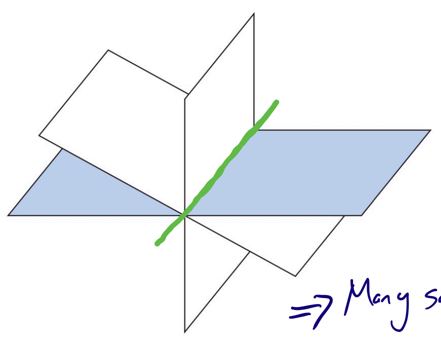
Infinitely many solutions! (A whole 2D plane)

Eg $t = 0$, $r = 0$ $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 5 \\ 0 \end{pmatrix}$ is a solution

but also $t = 42$, $r = -1$ $\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -79 \\ 42 \\ 3 \\ -1 \end{pmatrix}$

or any pair of numbers for (r, s).

Like



$\Rightarrow$ Many solutions

Figure 2(a) Three planes having a line in common.

but in 4D with green intersection a plane instead of a line.

2) **Def** A matrix is in <u>row-reduced echelon form</u> (rref) if it satisfies ...

c) If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

$$\begin{matrix} 1 & & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 & \dots \end{matrix}$$

Necessary to make rref unique!

Def <u>Inconsistent System of linear equations</u> has no solutions

<u>Consistent System</u> — " — has at least one solution

- Infinitely many solutions (when free variables are present)
- One solution (all variables are leading)

Def The <u>rank</u> of a matrix A is the number of leading 1's in its rref, denoted $\text{rank}(A)$.

Thm a) A linear system of n equations in n variables has a unique solution if and only if the rank of its coefficient matrix is n.

$n=3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

b) A linear system of n equations and m unknowns with $n < m$ has no solutions or infinitely many solutions

$$\underbrace{\left(\begin{array}{ccccc|c} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & & 1 & * & * \end{array}\right)}_{\text{Infinitely many}} \qquad \underbrace{\left(\begin{array}{ccccc|c} 1 & & & & & \\ & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right)}_{\text{no solutions}}$$

Matrix Operations

Denote $A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$

Sums $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 8 \\ 6 & 9 & 12 \end{pmatrix}$$

Scalar products k – a number

$$k(a_{ij}) = (k a_{ij})$$

$$2\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -4 & 6 \end{pmatrix}$$

Dot product of vectors $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 4 + 10 + 18 = 32$$

Note, works with row vectors or a mix between column & row vector

Matrix – Vector Product

A – $n \times m$ matrix with row vectors $A = \begin{pmatrix} — \vec{w}_1 — \\ — \vec{w}_2 — \\ \vdots \\ — \vec{w}_n — \end{pmatrix}$

$\vec{x} \in \mathbb{R}^m$

$$A\vec{x} = \begin{pmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{pmatrix} \in \mathbb{R}^n$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot -1 \\ 4\cdot 1 + 5\cdot 2 + 6\cdot -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

Note We can use this to describe a system of linear eqs.

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\begin{pmatrix} x_1 + 2x_2 + x_3 + x_4 \\ x_2 + 2x_2 + 2x_3 - x_4 \\ 2x_2 + 4x_2 \quad + 6x_4 \end{pmatrix}} = \begin{pmatrix} 7 \\ 12 \\ 4 \end{pmatrix} \longleftrightarrow \begin{cases} x_1 + 2x_2 + x_3 + x_4 = 7 \\ x_1 + 2x_2 + 2x_3 + x_4 = 12 \\ 2x_1 + 4x_2 \quad + 6x_4 = 4 \end{cases}$$

Warning Must have # cols in A = # components in $\vec{x}$

Otherwise undefined

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix}$ is undefined!

Trick $(n \times m)(m \times 1) = (n \times 1)$

Thm If A $n \times m$ matrix, $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^m$, k a scalar, then

a) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ ⟵ i^{th} component on each side

b) $A(k\vec{x}) = k(A\vec{x})$

LHS: $(\vec{w}_i \cdot (\vec{x}+\vec{y}))$

RHS: $\vec{w}_i \cdot \vec{x} + \vec{w}_i \cdot \vec{y}$

Column form for matrix-vector product

$$A\vec{x} = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}$$

i^{th} component of $\vec{b}$: $x_1 a_{i1} + x_2 a_{i2} + \cdots + x_m a_{im} = \vec{x} \cdot \vec{w}_i$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

Linear Combinations

Def A vector $\vec{b}$ is called a linear combination of vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ if there exist scalars $x_1, \dots, x_m$ such that $\vec{b} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$

Eg a) Is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ a linear combination of $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$?

Yes! $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) Is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ a linear combination of $\left\{\begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$? Yes!

Need $x_1\begin{pmatrix}5\\1\end{pmatrix}+x_2\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix} \rightsquigarrow \begin{cases}5x_1+x_2=2\\ x_1+2x_2=1\end{cases}$

Use rref of augmented matrix!

$$\left(\begin{array}{cc|c}5&1&2\\1&2&1\end{array}\right)\xrightarrow{R_1\leftarrow\frac{1}{5}R_1}\left(\begin{array}{cc|c}1&\frac{1}{5}&\frac{2}{5}\\1&2&1\end{array}\right)\xrightarrow[R_2\leftarrow R_2-R_1]{}\left(\begin{array}{cc|c}1&\frac{1}{5}&\frac{2}{5}\\0&\frac{9}{5}&\frac{3}{5}\end{array}\right)\xrightarrow[R_2\leftarrow\frac{5}{9}R_2]{}\left(\begin{array}{ccc}1&\frac{1}{5}&\frac{2}{5}\\0&1&\frac{1}{3}\end{array}\right)$$

$$\xrightarrow{R_1\leftarrow R_1-\frac{1}{5}R_2}\left(\begin{array}{cc|c}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{array}\right)$$

$\frac{2}{5}-\frac{1}{3}=\frac{6}{15}-\frac{1}{15}$

So, $\begin{pmatrix}2\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}5\\1\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix}$ ✓

c) Is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ a linear combination of $\left\{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$? No!