HW 1 due today (right now)

HW 2 due Friday, 1/21

1.3: 4, 48ab, 58
2.3: 2, 4, 8
2.1: 8, 12, 26, 28

Reflection Quiz on Canvas due 1/21

No class Monday, 1/17 (MLK day)

Why multiply matrices?

An $n \times m$ matrix A takes $\vec{v} \in \mathbb{R}^m$ and sends it to $A\vec{v} \in \mathbb{R}^n$.

So, A acts as a function with domain $\mathbb{R}^m$ and target (codomain) $\mathbb{R}^n$

$$\mathbb{R}^m \xrightarrow{A} \mathbb{R}^n$$

If B is a $p \times n$ matrix, it sends $\mathbb{R}^n \to \mathbb{R}^p$

So, BA as a composition

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "A"] & \mathbb{R}^n \arrow[r, "B"] & \mathbb{R}^p \\ \vec{v} \arrow[r, mapsto] \arrow[rr, bend right, dashed] & A\vec{v} \arrow[r, mapsto] & BA\vec{v} \end{tikzcd}$$

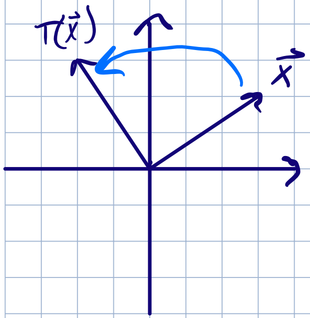
Linear Transformations

Def A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if there exists a matrix A such that

$$T(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^m.$$

Eg a)
$$\underset{\mathbb{R}^4}{T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \underset{\mathbb{R}^3}{\begin{pmatrix} x_1 + 2x_2 + x_3 + x_4 \\ x_1 + 2x_2 + 2x_3 - x_4 \\ 2x_1 + 4x_2 \qquad + 6x_4 \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 2 & 4 & 0 & 6 \end{pmatrix}}_{\text{coefficient matrix of}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

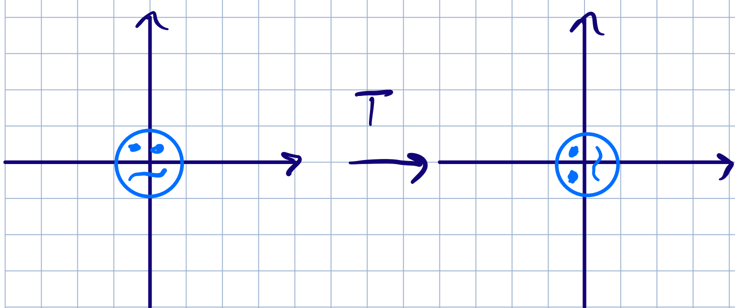
b) Let T take a vector in $\mathbb{R}^2$ and rotate it 90° ccw.



For example, $T\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$

Guess: $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-x_2\\x_1\end{pmatrix} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

coefficient matrix $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ of $\begin{cases} & -x_2 \\ x_1 & \end{cases}$



<u>Thm</u> A linear transformation $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is represented by the matrix $\begin{pmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{pmatrix}$ $\quad \vec{e}_i = \begin{pmatrix}0\\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0\end{pmatrix} \leftarrow i^{th}$ component

<u>Eg</u> b') $T(\vec{e}_1) = T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$T(\vec{e}_2) = T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

So, by the theorem, T is represented by $A = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

<u>Thm</u> A transformation $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is linear if and only if

a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^m$

b) $T(k\vec{v}) = kT(\vec{v})$ for all scalars k and $\vec{v} \in \mathbb{R}^m$.

$\rightarrow$ c) $T(\vec{0}) = \vec{0}$

<u>Eg</u> c) Is $T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1^2\\x_2^2\\x_3^2\end{pmatrix}$ a linear transformation?

$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ but $T\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}4\\4\\4\end{pmatrix} \neq 2T\begin{pmatrix}1\\1\\1\end{pmatrix}$ So <u>no</u>! violates b)

d) Is $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 + x_2 \\ x_1 \end{pmatrix}$ a linear transformation?

$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so <u>no</u>! violates c)

<u>upshot</u> Linear transformations are special.

<u>Note</u> Look at Ch 2.1 Example 9 for application to PageRank

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[d] & 2 \arrow[l, shift left] \arrow[dl] \\ 3 \arrow[r] & 4 \arrow[u] \end{tikzcd}$$

use matrix to describe how users transition between pages $\rightsquigarrow$ "transition matrix"

<u>Thm</u> If $T: \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a linear transformation represented by $p \times m$ matrix A and
$S: \mathbb{R}^p \longrightarrow \mathbb{R}^n$ ——— " ———
——— " ——— $n \times p$ matrix B,
then $S \circ T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is represented by BA,
ie $S(T(\vec{x})) = BA\vec{x}$ for all $\vec{x} \in \mathbb{R}^m$.

<u>Inverses</u>

<u>Eg</u> $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 25 \\ 55 \end{pmatrix} \longrightarrow \begin{pmatrix} 25 \\ 55 \end{pmatrix}$ decode to get back $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$?

solve $\begin{cases} x_1 + 2x_2 = 25 \\ 3x_1 + 4x_2 = 55 \end{cases}$?

Can we find a matrix A such that

$A\begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

Say $\begin{cases} x_1 + 2x_2 = y_1 \\ 3x_1 + 4x_2 = y_2 \end{cases}$ how can we write x_1, x_2 in terms of y_1, y_2?

$$\begin{cases} x_1 + 2x_2 = y_1 \\ 3x_1 + 4x_2 = y_2 \end{cases} \xrightarrow{E_2 \leftarrow E_2 - 3E_1} \begin{cases} x_1 + 2x_2 = y_1 \\ -2x_2 = -3y_1 + y_2 \end{cases} \xrightarrow{E_2 \leftarrow -\frac{1}{2}E_2}$$

$$\begin{cases} x_1 + 2x_2 = y_1 \\ x_2 = \frac{3}{2}y_1 - \frac{1}{2}y_2 \end{cases} \xrightarrow{E_1 \leftarrow E_1 - 2E_2} \begin{cases} x_1 = -2y_1 + y_2 \\ x_2 = \frac{3}{2}y_1 - \frac{1}{2}y_2 \end{cases}$$

$$\rightsquigarrow A = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \underbrace{\begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So,

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \overbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{=} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Identity transformation!

We say $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ are <u>inverses</u>.