HW 2 due Friday, 1/21

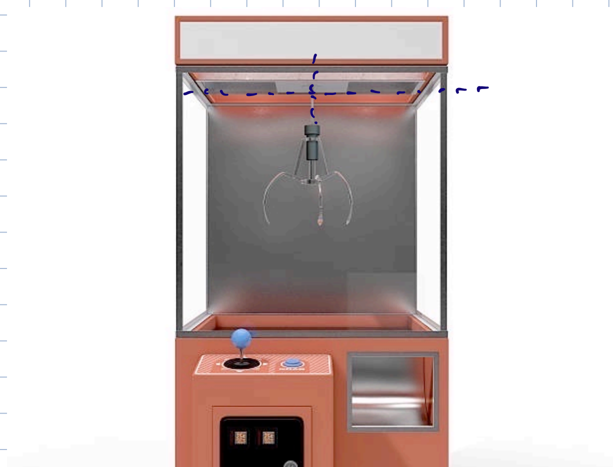
1.3: 4, 48ab, 58
2.3: 2, 4, 8
2.1: 8, 12, 26, 28

Reflection Quiz on Canvas due 1/21

Injectivity, Surjectivity, and Invertibility

Surjectivity

Consider a "Claw machine" with 3 buttons.



red: moves claw in $\begin{pmatrix}1\\3\end{pmatrix}$ direction

blue: — " — $\begin{pmatrix}2\\-1\end{pmatrix}$ direction

green: — " — $\begin{pmatrix}-1\\4\end{pmatrix}$ direction

If claw starts at $\begin{pmatrix}0\\0\end{pmatrix}$, can I get it to $\begin{pmatrix}8\\3\end{pmatrix}$?

Solve $\begin{pmatrix}1 & 2 & -1\\3 & -1 & 4\end{pmatrix}\begin{pmatrix}r\\b\\g\end{pmatrix} = \begin{pmatrix}8\\3\end{pmatrix}$

$$\left(\begin{array}{ccc|c}1 & 2 & -1 & 8\\3 & -1 & 4 & 3\end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left(\begin{array}{ccc|c}1 & 2 & -1 & 8\\0 & -7 & 7 & -21\end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{7}R_2} \left(\begin{array}{ccc|c}1 & 2 & -1 & 8\\0 & 1 & -1 & 3\end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c}1 & 0 & 1 & 2\\0 & 1 & -1 & 3\end{array}\right)$$

$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}r\\b\\g\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

$r + g = 2$

$b - g = 3$

$$\begin{pmatrix}r\\b\\g\end{pmatrix} = \begin{pmatrix}2 - t\\3 + t\\t\end{pmatrix}$$

Infinite solutions: One example

$$\begin{pmatrix}r\\b\\g\end{pmatrix} = \begin{pmatrix}2\\3\\0\end{pmatrix}$$

If buttons were switches instead (so claw could go backwards), then, this would work for any choice of location!

Def A linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is <u>surjective</u> (onto) if, for every $\vec{v} \in \mathbb{R}^n$, there is a $\vec{u} \in \mathbb{R}^m$ such that $T(\vec{u}) = \vec{v}$
(at least one)

Eg a) The linear transformation given by $\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 4 \end{pmatrix}$ is surjective.

b) Another claw machine

r: move $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ b: move $\begin{pmatrix} 4 \\ 12 \end{pmatrix}$ r: move $\begin{pmatrix} -3 \\ -9 \end{pmatrix}$

Can claw go anywhere?

$$\begin{pmatrix} 1 & 4 & -3 \\ 3 & 12 & -9 \end{pmatrix}\begin{pmatrix} r \\ b \\ g \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \left(\begin{array}{ccc|c} 1 & 4 & -3 & x \\ 3 & 12 & -9 & y \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - 3R_1]{} \left(\begin{array}{ccc|c} 1 & 4 & -3 & x \\ 0 & 0 & 0 & y-3x \end{array}\right)$$

So, this claw only moves along the line $0 = y - 3x$

$\Rightarrow$ map $\begin{pmatrix} 1 & 4 & -3 \\ 3 & 12 & -9 \end{pmatrix}$ is <u>not</u> surjective.

Thm An $n \times m$ matrix A gives a surjective linear transformation if and only if $\operatorname{rank}(A) = n$ ($\operatorname{rref}(A)$ has a leading 1 in each row).

Injectivity

Friend has two kinds of pets: birds and cats

Says "my pets have 14 legs, 10 eyes, and 5 tails."

How many birds and cats?

$$\begin{pmatrix} 2 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 4 & 14 \\ 2 & 2 & 10 \\ 1 & 1 & 5 \end{array}\right) \xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left(\begin{array}{ccc} 1 & 2 & 7 \\ 2 & 2 & 10 \\ 1 & 1 & 5 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{array}\right) \xrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left(\begin{array}{ccc} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 + R_2]{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

3 birds and 2 cats

However, what if friend says "5 heads, 10 eyes, and 5 tails"?

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

All we can conclude is 5 pets. Not enough info!

__Def__ A linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ is __injective__ (one-to-one) if, for every $\vec{v} \in \mathbb{R}^n$, there is at most one $\vec{u} \in \mathbb{R}^m$ such that $T(\vec{u}) = \vec{v}$.

__Eg__ $\begin{pmatrix} 2 & 4 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ gives an injective linear transformation,

but $\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{pmatrix}$ does not!

__Thm__ An $n \times m$ matrix A gives an injective linear transformation if and only if $\text{rref}(A)$ has a leading 1 in every column.

__Invertibility__

__Def__ A linear transformation is __invertible__ (bijective) if it is both injective and surjective.

i.e. $T: \mathbb{R}^m \to \mathbb{R}^n$, for every $\vec{v} \in \mathbb{R}^n$, there is __exactly one__ $\vec{u} \in \mathbb{R}^m$ such that $T(\vec{u}) = \vec{v}$.