HW 2 due right now!

Reflection Quiz on Canvas due 1/21 (tonight)

Quiz on 1/28 (15 min)
↳ ch 1, 2.1, 2.3, 2.4

HW 3 due 2/4
2.4: 2, 6, 34, 40, 42

Invertibility

Def A linear transformation is <u>invertible</u> (bijective) if it is both injective and surjective.

ie $T: \mathbb{R}^m \to \mathbb{R}^n$, (m has to equal n) for every $\vec{v} \in \mathbb{R}^n$, there is <u>exactly one</u> $\vec{u} \in \mathbb{R}^m$ such that $T(\vec{u}) = \vec{v}$.

Eg $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by rotating $\begin{pmatrix} x \\ y \end{pmatrix}$ by $90°$ CCW.

Thm An $n \times n$ matrix A gives an invertible linear transformation if and only if $\text{rref}(A) = I_n$ if and only if $\text{rank}(A) = n$.

Def A square $n \times n$ matrix A is an <u>invertible matrix</u> if there exists a matrix B such that $AB = I_n = BA$.

In this case, we will say $B = A^{-1}$.

Eg $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ so $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is invertible

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Thm An $n \times n$ matrix A gives an invertible map $T: \mathbb{R}^n \to \mathbb{R}^n$ if and only if A is invertible. Furthermore, the inverse map $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ corresponds to the matrix A^{-1}. Ie $T^{-1}(\vec{v}) = A^{-1}\vec{v}$.

Eg Rotate $\begin{pmatrix} x \\ y \end{pmatrix}$ by 90° CCW has matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Inverse $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$ and rotates $\begin{pmatrix} x \\ y \end{pmatrix}$ by 90° CW.

Thm a) Let A be an invertible $n \times n$ matrix
Then, the system $A\vec{x} = \vec{b}$ has unique solution given by $\vec{x} = A^{-1}\vec{b}$.

Why? Invertible $n \times n$ matrix $A \Rightarrow \operatorname{rank}(A) = n \xRightarrow{(\S 1.2)} A\vec{x} = \vec{b}$ has a unique solution.

$A\vec{x} = \vec{b} \rightsquigarrow A(A^{-1}\vec{b}) = I_n\vec{b} = \vec{b}$ so $A^{-1}\vec{b}$ is the unique solution!

a') A an invertible $n \times n$ matrix then $A\vec{x} = \vec{0}$ has unique solution $\vec{x} = \vec{0}$!

b) A a non-invertible $n \times n$ matrix. Then, the system $A\vec{x} = \vec{b}$ has infinitely many solutions or no solutions.

Rephrase §1.2.

b') A non-invertible, then $A\vec{x} = \vec{0}$ has infinitely many solutions. (Since $\vec{x} = \vec{0}$ is a solution.)

Eg $\begin{cases} x_1 + 2x_2 = 4 \\ 3x_1 + 4x_2 = 10 \end{cases} \to \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$

How to find the inverse of a matrix

Remember $\to \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{cases} x_1 + 2x_2 = y_1 \\ 3x_1 + 4x_2 = y_2 \end{cases} \xrightarrow[E_2 \leftarrow E_2 - 3E_1]{} \begin{cases} x_1 + 2x_2 = y_1 \\ -2x_2 = -3y_1 + y_2 \end{cases} \xrightarrow[E_2 \leftarrow -\frac{1}{2}E_2]{} \begin{cases} x_1 + 2x_2 = y_1 \\ x_2 = \frac{3}{2}y_1 - \frac{1}{2}y_2 \end{cases}$

$\downarrow E_1 \leftarrow E_1 - 2E_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leftarrow \begin{cases} x_1 = -2y_1 + y_2 \\ x_2 = \frac{3}{2}y_1 - \frac{1}{2}y_2 \end{cases}$

Want to do algorithm to $n\times n$ matrix A

$\longrightarrow$ row reduce $(A \,|\, I_n)$ to get $(I_n \,|\, A^{-1})$

Eg $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - 3R_1]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow[R_2 \leftarrow -\frac{1}{2}R_2]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

A^{-1}

Thm a) If $\mathrm{rref}(A \,|\, I_n)$ is of the form $(I_n \,|\, B)$, then A is invertible and $A^{-1} = B$.

b) If $\mathrm{rref}(A \,|\, I_n)$ is of another form, then A is not invertible.

Eg
$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - 2R_1]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow[R_2 \leftarrow -\frac{1}{2}R_2]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

Not I_2

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is not invertible!

Some rules about inverses

Thm If A and B are invertible $n\times n$ matrices, then BA is invertible with inverse $(BA)^{-1} = A^{-1}B^{-1}$.

⚠ Careful because order matters!

Pf $\vec{y} = BA\vec{x}$

$$\Rightarrow B^{-1}\vec{y} = B^{-1}BA\vec{x} = I_n A\vec{x} = A\vec{x}$$

$$\Rightarrow A^{-1}B^{-1}\vec{y} = A^{-1}A\vec{x} = I_n\vec{x} = \vec{x} \quad \text{so } (BA)^{-1} = A^{-1}B^{-1}. \quad \square$$

Thm If A, B are $n\times n$ matrices such that $AB = I_n$

Then a) A & B are both invertible

b) $A^{-1} = B$ and $B^{-1} = A$

c) $BA = I_n$

Upshot Enough to check $AB = I_n$.

2x2 matrices

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$?

From HW 2, 2.3: #8

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ assuming $ad-bc \neq 0$.

Def The determinant of a 2x2 matrix is given by

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$$

Thm $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible with inverse $\frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

if and only if $\det(A) \neq 0$.

Eg a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has $\det(A) = 1\cdot 4 - 2\cdot 3 = -2$

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \checkmark$$

b) $\det\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 1\cdot 4 - 2\cdot 2 = 0$, so not invertible!