HW 3 due 2/4
2.4: 2, 6, 34, 40, 42
2.2: 20, 32
2.3: 30

Midterm 1, Wed 2/9
Tentatively Ch 1 – 3.3

HW 4 due Fri 2/11
3.1: 6, 24, 32, 34, 37, 38
3.2: 26, 34

$T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$, we looked at $\text{im}(T)$

$\text{im}(T)$ is an example of a <u>subspace</u>.

<u>Def</u> A subset W of $\mathbb{R}^n$ is called a <u>subspace</u> if

a) $\vec{0} \in W$,

b) $\vec{u} \in W$ and $\vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$, and

c) $\vec{u} \in W \Rightarrow k\vec{u} \in W$ for any scalar k.

<u>Eg</u> a) $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$, then $\text{span}\{\vec{v}_1, \dots, \vec{v}_m\}$ is a subspace of $\mathbb{R}^n$

b) Is $S = \left\{ \begin{pmatrix} x \\ x^2 \end{pmatrix} : x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$ but $2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$? $\quad 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} x \\ x^2 \end{pmatrix}$ for any x

so S violates property c and thus S is <u>not</u> a subspace!

c) Is the plane, P, $x + 2y + 3z = 0$ a subspace of $\mathbb{R}^3$?

i) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in P$ because $0 + 2\cdot 0 + 3\cdot 0 = 0$

ii) $\vec{u} \in P$ and $\vec{v} \in P \Leftrightarrow \begin{matrix} u_1 + 2u_2 + 3u_3 = 0 \\ v_1 + 2v_2 + 3v_3 = 0 \end{matrix} \Rightarrow (u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3) = 0$

$\Rightarrow \vec{u} + \vec{v} \in P$

iii) $\vec{u} \in P$ so $k(u_1 + 2u_2 + 3u_3) = 0 \Rightarrow k\vec{u} \in P$.

Yes! The plane P is a subspace of $\mathbb{R}^3$.

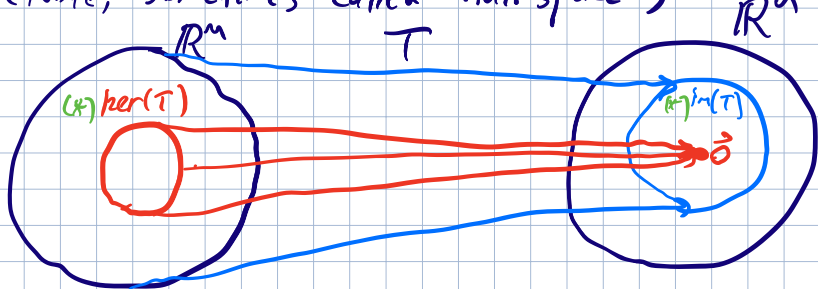
In fact, $P = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \right\}$ (Any choice of 2 non-parallel vectors in P would work)

Another subspace: kernel.

Def The kernel of a linear transformation $T(\vec{x}) = A\vec{x}$ is the set of all solutions to $T(\vec{x}) = \vec{0} \Leftrightarrow A\vec{x} = \vec{0}$

Denoted $\ker(T)$ or $\ker(A)$.

(Note, sometimes called "null space")



both ker(T) & im(T) are subspaces

Thm For $T: \mathbb{R}^m \to \mathbb{R}^n$

$\ker(T)$ is a subspace of $\mathbb{R}^m \leftarrow$ the domain

Note $\text{im}(T)$ is a subspace of $\mathbb{R}^n \leftarrow$ the target

Eg $\ker\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$? $\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 4R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right) \xrightarrow{R_2 \leftarrow -\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -2t \\ t \end{pmatrix}$$

So $\ker\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is the line spanned by $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

Thm If $A\vec{v} = \vec{b}$ and $\vec{w} \in \ker A$, then $A(\vec{v} + k\vec{w}) = \vec{b}$ for any scalar k.

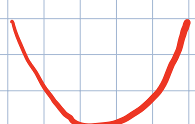
Eg $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \in \ker\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1+t \\ 1-2t \\ 1+t \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}$ for any choice of t.

Compare $\frac{d}{dx} f = x$ is satisfied by $f(x) = \frac{x^2}{2}$, but $\frac{d}{dx}(C) = 0$ for any constant C, so $\frac{d}{dx} f = x$ is satisfied by $f(x) = \frac{x^2}{2} + C$ for any constant C. (Think: $g(x) = C$ is in "the kernel of $\frac{d}{dx}$")

Aside

Subspaces of $\mathbb{R}^2$:
1) $\mathbb{R}^2$ is a subspace of itself
2) Any line going through the origin is a subspace of $\mathbb{R}^2$ $\Leftrightarrow$ $\operatorname{span}\left\{\begin{pmatrix} t \\ kt \end{pmatrix}\right\}$ or $\operatorname{span}\left\{\begin{pmatrix} kt \\ t \end{pmatrix}\right\}$
3) $\{\vec{0}\}$ is a subspace of $\mathbb{R}^2$.



is not a subspace even though it is a subset of $\mathbb{R}^2$.

Thm $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is injective (one-to-one) if and only if $\ker(T) = \{\vec{0}\}$.

Why?
$$\begin{aligned} T(\vec{u}) = T(\vec{v}) &\Leftrightarrow T(\vec{u}) - T(\vec{v}) = \vec{0} \\ &\Leftrightarrow T(\vec{u} - \vec{v}) = \vec{0} \\ &\Leftrightarrow \vec{u} - \vec{v} \in \ker(T). \end{aligned}$$

Thm For $n \times m$ matrix A,

a) $\ker(A) = \{\vec{0}\} \iff \text{rank}(A) = m$

b) $\ker(A) = \{\vec{0}\} \implies m \leq n$

b′) $m > n \implies \ker(A)$ has non-zero vector elements

c) Square matrix $(m=n)$ A has $\ker(A) = \{\vec{0}\} \iff A$ is invertible.

Bases and linear independence

Consider $\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$. Can we describe this set as a span of fewer vectors?

Yes! $\mathbb{R}^2 = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

since $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \text{rref}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix} = \text{rref}\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$

Def For a list of vectors $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$, we say $\vec{v}_i$ is <u>redundant</u> if it is a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}$.

Eg For $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ we have $\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} - 2\begin{pmatrix}1\\0\end{pmatrix}$, so $\begin{pmatrix}0\\1\end{pmatrix}$ is redundant.

Def $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ are <u>linearly independent</u> if none of them are redundant.

— " ——————— <u>linearly dependent</u> if at least one of them is redundant.

Eg $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ are linearly dependent.

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$ is linearly independent.

As are $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$.

Def $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ form a <u>basis</u> in a subspace V if

a) $\text{span}\{\vec{v}_1, \dots, \vec{v}_m\} = V$ & b) $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.