HW 3 due 2/4
2.4: 2, 6, 34, 40, 42
2.2: 20, 32
2.3: 30

Midterm 1, Wed 2/9
Tentatively Ch 1 – 3.3

HW 4 due Fri 2/11
3.1: 6, 24, 32, 34, 37, 38
3.2: 26, 34
3.3: 30 38
Cool exercise but not collected:
3.3: 90

<u>Def</u> For a list of vectors $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$, we say $\vec{v}_i$ is <u>redundant</u> if it is a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}$.

<u>Eg</u> For $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ we have $\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} - 2\begin{pmatrix}1\\0\end{pmatrix}$, so $\begin{pmatrix}0\\1\end{pmatrix}$ is redundant.

<u>Def</u> $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ are <u>linearly independent</u> if none of them are redundant.
— " ———— <u>linearly dependent</u> if at least one of them is redundant.

<u>Eg</u> $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ are linearly dependent.

$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \right\}$ is linearly independent.

As are $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$.

<u>How to check for linear independence?</u>

Take $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ and we check for linear relations:

Solve $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$ for all possible c_i's.

$c_1 = c_2 = \cdots = c_m = 0$ is always a solution.

- That is the only solution $\iff$ $\vec{v}_1, \dots, \vec{v}_m$ are linearly indep
- Other solutions $\iff$ $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent.

Eg $\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix} \right\} \longrightarrow c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}1\\1\\1\end{pmatrix} + c_3\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\rightsquigarrow \left(\begin{array}{ccc|c}1&1&1&0\\2&1&1&0\\3&1&2&0\end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix}1&1&1&0\\0&-1&-1&0\\0&-2&-1&0\end{pmatrix} \xrightarrow{R_2 \leftarrow -R_2} \begin{pmatrix}1&1&1&0\\0&1&1&0\\0&-2&-1&0\end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 + 2R_2]{R_1 \leftarrow R_1 - R_2} \begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} \Longrightarrow \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

so our vectors are linearly independent

Thm For $\underset{n \times m}{A} = \begin{pmatrix}| & & |\\ \vec{v}_1 & \cdots & \vec{v}_m\\ | & & |\end{pmatrix}$, the column vectors $\{\vec{v}_1, \ldots, \vec{v}_m\}$ are linearly indep if and only if $\ker A = \{\vec{0}\} \iff \operatorname{rank}(A) = m$.

Why? $A\vec{x} = \vec{0} \iff x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \vec{0}$

Consequence We can only find at most n linearly independent vectors of $\mathbb{R}^n$.

Thm Consider $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ with $\vec{v}_1 \neq \vec{0}$. If each $\vec{v}_i$ for $i \geq 2$ has non-zero entry in a component where all previous vectors have a 0, then $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent.

Eg a) $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ are linearly independent.

(Works for $\{\vec{e}_1, \ldots, \vec{e}_n\}$ in $\mathbb{R}^n$)

b) $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$ are linearly independent.

Def $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ form a __basis__ in a subspace V if

1) $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = V$
2) $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent.

Think "non-redundant spanning set"

Eg a) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is a basis of $\mathbb{R}^3$

b) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is a basis of $\mathbb{R}^3$

c) $\left\{\begin{pmatrix}2\\-1\\0\end{pmatrix},\begin{pmatrix}3\\0\\-1\end{pmatrix}\right\}$ is a basis of $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x+2y+3z=0\right\}$

__Unique Representation__

__Thm__ $\vec{v}_1,\dots,\vec{v}_m$ in subspace V of $\mathbb{R}^n$ form a basis of V

$\iff$ every vector $\vec{u}\in V$ can be expressed __uniquely__ as a linear combination $\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2+\dots+c_m\vec{v}_m$.

__Think__ Coordinates.

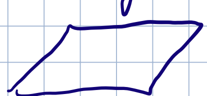
__Eg__ $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ basis of $\mathbb{R}^3$ $\vec{u}=\begin{pmatrix}3\\-2\\7\end{pmatrix}$. then $\vec{u}=3\begin{pmatrix}1\\0\\0\end{pmatrix}-2\begin{pmatrix}0\\1\\0\end{pmatrix}+7\begin{pmatrix}0\\0\\1\end{pmatrix}$

and there is no other way to write $\vec{u}$ as a linear combination of $\{\vec{e}_1,\dots,\vec{e}_3\}$

__Thm__ All bases of a subspace V of $\mathbb{R}^n$ consist of the same number of vectors.

__Def__ For subspace V of $\mathbb{R}^n$, $\dim(V)=\#$ vectors in a basis of V.

__Eg__ a) $V=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x+2y+3z=0\right\}$ has a basis $\left\{\underbrace{\begin{pmatrix}2\\-1\\0\end{pmatrix}}_{\vec{v}_1},\underbrace{\begin{pmatrix}3\\0\\-1\end{pmatrix}}_{\vec{v}_2}\right\}$



So $\dim(V)=2$. Any $\vec{u}$ on our plane has unique decomposition $\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2$

b) $\mathbb{R}^n$ has a basis $\{\vec{e}_1,\dots,\vec{e}_n\}$ so $\dim(\mathbb{R}^n)=n$.

__Computing bases__

__Eg__ a) What is a basis for $\ker\underbrace{\begin{pmatrix}1&2&2&-3\\1&1&2&-1\end{pmatrix}}_{A}$?

First, what is the kernel? $\Longleftrightarrow$ Solve $A\vec{x} = \vec{0}$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -3 & 0 \\ 1 & 1 & 2 & -1 & 0 \end{array}\right) \xrightarrow[R_2 \leftarrow R_2 - R_1]{} \begin{pmatrix} 1 & 2 & 2 & -3 & 0 \\ 0 & -1 & 0 & 2 & 0 \end{pmatrix} \xrightarrow[R_2 \leftarrow -R_2]{} \begin{pmatrix} 1 & 2 & 2 & -3 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t-s \\ 2s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the kernel.

Also Look at Example 1 in §3.3 has much bigger example.

b) What is a basis $\operatorname{im}\begin{pmatrix} 1 & 2 & 2 & -3 \\ 1 & 1 & 2 & -1 \end{pmatrix}$?

Already know $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix} \right\}$ = the image

Just have to find and remove redundant column vectors.

$\operatorname{rref}\begin{pmatrix} 1 & 2 & 2 & -3 \\ 1 & 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -2 \end{pmatrix}$ Free variables $\Longleftrightarrow$ redundant columns

so, $\operatorname{im}\begin{pmatrix} 1 & 2 & 2 & -3 \\ 1 & 1 & 2 & -1 \end{pmatrix}$ has a basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

Note, since $\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $\begin{pmatrix} -3 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

col 1 of rref(A) col 2 col 1 of A col 2