Midterm 1, Wed 2/9
Ch 1 – 3.3
Some practice on Canvas, more today!
If you are sick, email me!!

HW 4 due Fri 2/11
3.1: 6, 24, 32, 34, 37, 38
3.2: 26, 34
3.3: 30 38
Cool exercise but not collected:
3.3: 90

Post on Piazza

Last time Can 3×3 matrix A have $\mathrm{im}(A) = \ker(A)$. No!

Used Rank-Nullity.

However, 2×2 $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $\mathrm{im}(A) = \ker(A)$ (Check this!)

1. For the following statements, circle True if the statement is **always** true, and circle False otherwise. Make sure it is completely clear which is your final answer. No explanations are required for this question, and no partial credit. Read the questions very carefully!

   (a) (2 points) Let A and B be matrices of the same size. If $\text{rref}(A) = \text{rref}(B)$, then $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions.

   True False

   Row-operations do not change solution to system of equations.

   (b) (2 points) Let A be a $n \times m$ matrix and suppose $A\vec{v} = \vec{b}$ for some $\vec{v} \in \mathbb{R}^m$ and $\vec{b} \in \mathbb{R}^n$. Then $\vec{b}$ is a linear combination of the column vectors of A.

   Then $\vec{b} = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{pmatrix} \vec{v} = v_1\vec{u}_1 + \cdots + v_m\vec{u}_m$

   True False

   (c) (2 points) Let T be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$. If the kernel of T is $\{\vec{0}\}$, then T is invertible.

   This a theorem

   True False

   Sanity-Check: Rank-Nullity: $\dim(\ker(T)) = 0$, then $\dim(\text{im}(T)) = 4$

   $\text{rank}(T) = 4$

   (d) (2 points) Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. If the image of T is $\mathbb{R}^n$, then T is invertible.

   True False

   If $m \neq n$, then T cannot be invertible.

   (e) (2 points) If A is a 3×4 matrix, then $A\vec{x} = \vec{b}$ has infinitely many solutions for any vector $\vec{b} \in \mathbb{R}^3$.

   True False

   Eg

   $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

   There is no $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ for $A\vec{x} = \vec{b}$

2. (a) (5 points) Compute the inverse of $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix}$, if it is invertible. If it is not invertible, explain why not.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & & \\ -1 & 1 & 3 & & 1 & \\ 0 & 1 & 3 & & & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & & \\ 0 & 2 & 5 & 1 & 1 & \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & -1/2 & -1/2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & & & 0 & -1 & 1 \\ & 1 & & 3 & 3 & -5 \\ & & 1 & -1 & -1 & 2 \end{array}\right)$$

Yes, $A^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 3 & 3 & -5 \\ -1 & -1 & 2 \end{pmatrix}$

(b) (5 points) Find all $\vec{x}$ satisfying $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. (You may use your answer from part (a).) Justify why you have found all the solutions $\vec{x}$.

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underbrace{A^{-1}A\vec{x}}_{\vec{x}} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 3 & 3 & -5 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

3. Let A be the 2×2 matrix representing rotation counterclockwise by angle $\frac{\pi}{6}$, and let B be the 2×2 matrix representing reflection about the line $7x + 11y = 0$. Compute the following:

(a) (2 points) $A^3 = A \cdot A \cdot A$ = rotation CCW by $\frac{\pi}{2}$

$$= \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) (2 points) $B^4 = B^2 \cdot B^2 = Id \cdot Id = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$B^2 = Id$ since B is a reflection.

(c) (4 points) $\underbrace{B \cdot A^{18} \cdot B \cdot A^6}$

A = rot CCW by $\frac{\pi}{6}$

$\Rightarrow A^6 =$ rot CCW by $6 \cdot \frac{\pi}{6} = \pi$

$A^{18} =$ rot CCW by $18 \cdot \frac{\pi}{6} = 3\pi = \pi$

$A^6 = A^{18} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= -Id$

$= B \cdot (-Id) \cdot B \cdot (-Id)$

$= (-1)^2 B \cdot Id \cdot B \cdot Id$

$= B^2 = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4. Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11x_1 + 10x_2 + 12x_3 \\ 5x_1 + 4x_2 + 6x_3 \\ 2x_1 + x_2 + 3x_3 \end{pmatrix}.$$

(a) (1 point) Find the 3×3 matrix A representing this linear transformation.

$$A = \begin{pmatrix} 11 & 10 & 12 \\ 5 & 4 & 6 \\ 2 & 1 & 3 \end{pmatrix}$$

(b) (6 points) What is the dimension of the kernel of A and the dimension of the image of A? Find a basis for the kernel of A and a basis for the image of A.

$$\operatorname{rref}(A) = \cdots\cdots = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{im}(A)$ has basis $\left\{ \begin{pmatrix} 11 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 \\ 4 \\ 1 \end{pmatrix} \right\}$

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$\operatorname{Span}\left\{ \begin{pmatrix} 11 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 12 \\ 1 \\ 3 \end{pmatrix} \right\}$

$\ker(A) = \left\{ \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} \right\}$ has basis $\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$

$A\vec{x} = \vec{0}$

(c) (3 points) Find a matrix B (of any size) such that $\ker B = \operatorname{Im} A$.

5. Find 2×2 matrices satisfying the stated properties, if one exists. Either verify the required property or explain why such a matrix does not exist.

   (a) (3 points) Find A with rank $A = 1$ and satisfying $A^2 = A$.

   $$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

   (b) (3 points) Find B with rank $B = 1$ and satisfying $B^2 = 0$.

   $$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   (c) (3 points) Find C such that C is not invertible and all entries of C are distinct and nonzero; that is, if $C = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then a, b, c, d are all different nonzero numbers.

   $$C = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \qquad \det C = 12 - 12 = 0$$

   (d) (3 points) Find D such that $D^2 \neq \mathrm{Id}_2$ but $D^{100} = \mathrm{Id}_2$, where Id_2 denotes the 2×2 identity matrix.

   Think rotation by $\frac{2\pi}{100} = \frac{\pi}{50}$

   $$D = \begin{pmatrix} \cos\frac{\pi}{50} & -\sin\frac{\pi}{50} \\ \sin\frac{\pi}{50} & \cos\frac{\pi}{50} \end{pmatrix}$$