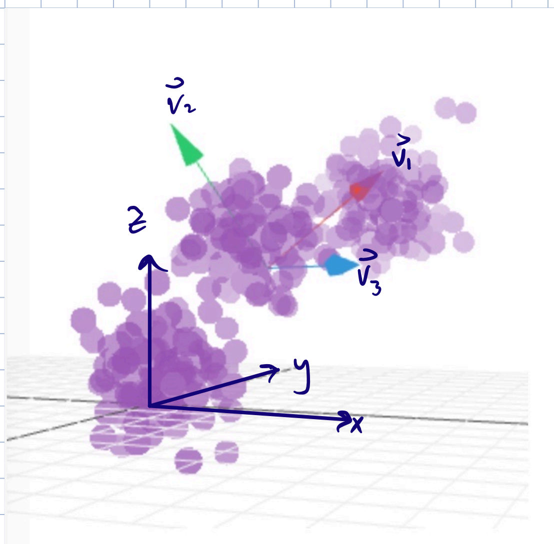
HW 4 due today, 5pm
3.1: 6, 24, 32, 34, 37, 38
3.2: 26, 34
3.3: 30 38
Cool exercise but not collected:
3.3: 90

HW 5 due 2/18
3.4: 2, 4, 20, 26, 44

Coordinates



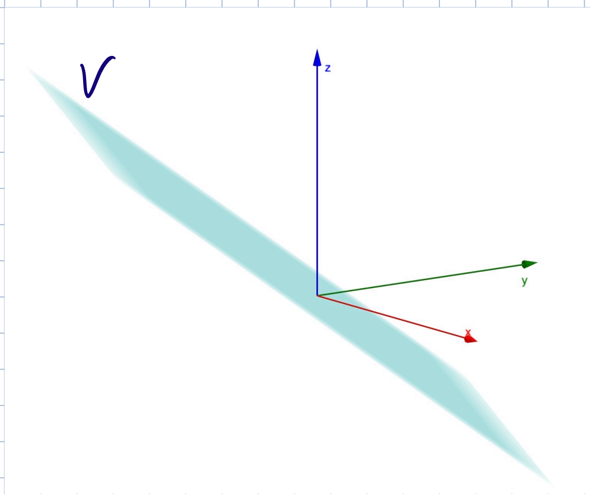
Highly correlated data $\mathbb{R}^N$
Want to describe using fewer dimensions

Data in $\mathbb{R}^N$

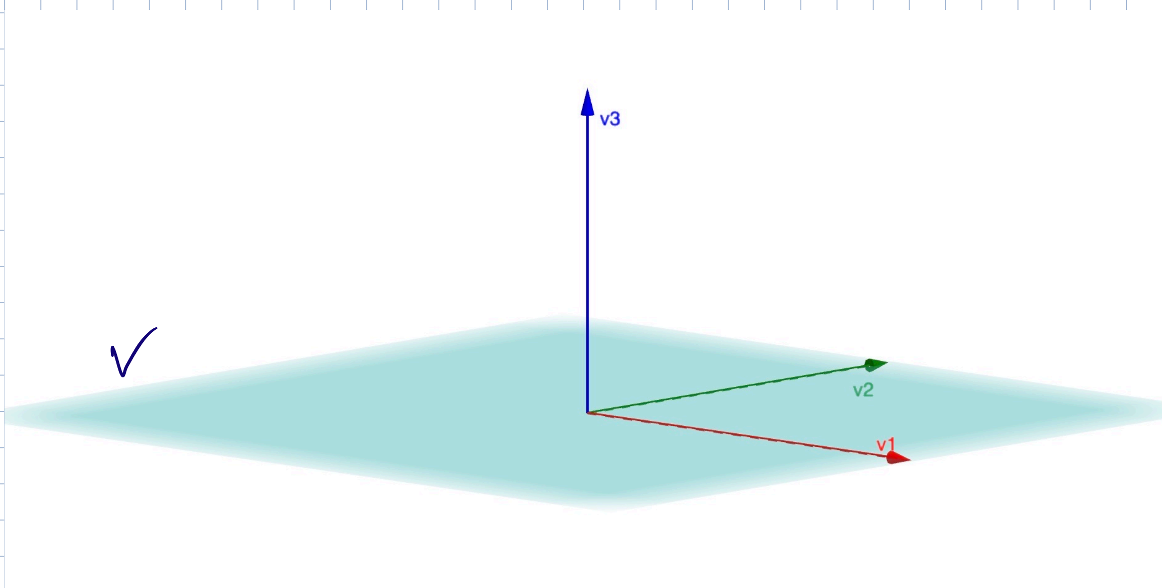
$\vec{v} = \begin{pmatrix} \vdots \end{pmatrix} \in$ subspace V with basis $\mathcal{B}$ $\longrightarrow$ Coordinates in $\mathcal{B}$

"$[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \vdots \end{pmatrix}$

Eg Data in $\mathbb{R}^3$ on left has little variation in $\vec{v}_3$ direction:
describe data points only using $\{\vec{v}_1, \vec{v}_2\}$



Change coordinate system



Lots of examples in §3.4.

Eg Consider a subspace $V = \text{span}\left\{ \begin{pmatrix}1\\2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\end{pmatrix} \right\}$

$\begin{pmatrix}1\\2\\1\\0\end{pmatrix} = \vec{v}_1$, $\begin{pmatrix}0\\1\\2\\3\end{pmatrix} = \vec{v}_2$

Is $\vec{u} = \begin{pmatrix}2\\5\\4\\3\end{pmatrix}$ in V?

$$\left(\begin{array}{cc|c}1&0&2\\2&1&5\\1&2&4\\0&3&3\end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{cc|c}1&0&2\\0&1&1\\0&0&0\\0&0&0\end{array}\right)$$

So, $\vec{u} = 2\vec{v}_1 + \vec{v}_2$

Since $\dim V = 2$, $\vec{u}$ can actually be described by just two numbers!

Def For basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ of subspace V in $\mathbb{R}^n$, for $\vec{x} \in V$, let $c_1, \dots, c_m$ be the unique scalars such that $\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$.

Then, $c_1, \dots, c_m$ are the $\mathcal{B}$-coordinates of $\vec{x}$ and

$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix}c_1\\\vdots\\c_m\end{pmatrix}$ is the $\mathcal{B}$-coordinate vector of $\vec{x}$.

Eg $\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\end{pmatrix} \right\}$, $\vec{u} = \begin{pmatrix}2\\5\\4\\3\end{pmatrix}$ then $[\vec{u}]_{\mathcal{B}} = \begin{pmatrix}2\\1\end{pmatrix}$

Note $\vec{x} = \begin{pmatrix}| & & |\\ \vec{v}_1 & \cdots & \vec{v}_m\\ | & & |\end{pmatrix} [\vec{x}]_{\mathcal{B}} = \begin{pmatrix}| & & |\\ \vec{v}_1 & \cdots & \vec{v}_m\\ | & & |\end{pmatrix}\begin{pmatrix}c_1\\\vdots\\c_m\end{pmatrix} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$

Eg $\mathcal{B} = \left\{ \begin{pmatrix}2\\-1\\0\end{pmatrix}, \begin{pmatrix}3\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$ is a basis of $\mathbb{R}^3$

For $\vec{u} = \begin{pmatrix}6\\1\\2\end{pmatrix}$, what is $[\vec{u}]_{\mathcal{B}}$?

Row-reduce $\left(\begin{array}{ccc|c} 2 & 3 & 1 & 6 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \end{array}\right) \xrightarrow{\quad \cdots \quad} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$ so $[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Thm $\mathcal{B}$ a basis of subspace V of $\mathbb{R}^n$, then

a) $[\vec{x}+\vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ for all $\vec{x}, \vec{y}$ in V

b) $[k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}}$ for all $\vec{x}$ in V and scalars k

Eg $\vec{v} = \begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix}$ in $\mathbb{R}^3$ with basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

$[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and

$$[\vec{v}+\vec{u}]_{\mathcal{B}} = \left[\begin{pmatrix} 5 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} \right]_{\mathcal{B}} = \left[\begin{pmatrix} 11 \\ 0 \\ 1 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = [\vec{v}]_{\mathcal{B}} + [\vec{u}]_{\mathcal{B}}$$

check $2\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \\ 1 \end{pmatrix}$ ✓

Question If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ a basis of $\mathbb{R}^n$,

What is a matrix B such that $B[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}$?

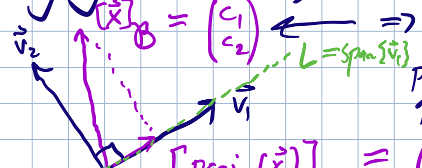
Thm In this situation,

$$B = \begin{pmatrix} | & & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & \cdots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$ is the unique matrix such that $$[T(\vec{x})]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$$

"Changing coordinates of linear transformation"

Eg

$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Longleftrightarrow \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$



$\text{Proj}_L(\vec{x}) = c_1\vec{v}_1$

$[\text{proj}_L(\vec{x})]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

So, $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{B} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$

Check $\quad \mathrm{proj}_L(\vec{v}_1) = \vec{v}_1$, $\quad \mathrm{proj}_L(\vec{v}_2) = \vec{0}$ $\Rightarrow$ $[\mathrm{proj}_L(\vec{v}_1)]_{\mathcal{B}} = \begin{pmatrix}1\\0\end{pmatrix}$, $\quad [\mathrm{proj}_L(\vec{v}_2)]_{\mathcal{B}} = \begin{pmatrix}0\\0\end{pmatrix}$

So this gives columns of $B = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$.

Diagram For basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$

$$S = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & T(\vec{x}) \\ {[\vec{x}]_{\mathcal{B}}} \arrow[u, "S"] \arrow[r, "B"'] & {[T(\vec{x})]_{\mathcal{B}}} \arrow[u, "S"'] \end{tikzcd}$$

$$\Rightarrow \qquad AS = SB \iff B = S^{-1}AS \ \ \& \ \ A = SBS^{-1}$$

Def Two $n \times n$ matrices are <u>similar</u> if there exists an invertible matrix S such that $AS = SB \iff B = S^{-1}AS$

Eg Is $\underbrace{\begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}}_{A}$ similar to $\underbrace{\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}}_{B}$?

$\iff$

Does $\begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}\underbrace{\begin{pmatrix}x & y\\ z & t\end{pmatrix}}_{S} = \begin{pmatrix}x & y\\ z & t\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$ for some x, y, z, t?

$$\begin{pmatrix}x+4z & y+4t\\ z & t\end{pmatrix} = \begin{pmatrix}x+2y & y\\ z+2t & t\end{pmatrix}$$

$$\begin{cases} x+4z = x+2y \\ y+4t = y \\ z = z+2t \\ t = t \end{cases} \Rightarrow \begin{cases} 4z = 2y \\ 4t = 0 \\ 0 = 2t \\ t = t \end{cases} \Rightarrow \begin{array}{l} t = 0,\ y = 2z \\ x, y, z \text{ are free} \end{array}$$

⚠ but! $\quad xt - yz \neq 0$

$\Rightarrow yz \neq 0$

So, $\begin{pmatrix}x & y\\ z & t\end{pmatrix} = \begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}$ works and A is therefore similar to B.

Thm A $n \times n$ matrix

a) A is similar to itself $\quad (S = Id)$
b) A is similar to $B \iff B$ is similar to A (set S to S^{-1})
c) A is similar to B and B is similar to $C \Rightarrow A$ is similar to C.