HW 5 due 2/18
3.4: 2, 4, 20, 26, 44

HW 6 due 2/25
4.1: 47, 48

Linear Spaces

Eg a) Consider equation $f''(x) = -f(x)$.

Some solutions: $f(x) = \sin(x)$, $f(x) = -\cos(x)$

General solution: $c_1 \sin(x) + c_2 \cos(x)$ c_1, c_2 are constants

Looks like "span$\{\sin(x), \cos(x)\}$"

"all linear combinations of $\sin(x)$ and $\cos(x)$"

b) Consider degree ≤ 2 polynomials: $ax^2 + bx + c$ a, b, c – scalars

"linear combinations of $\{x^2, x, 1\}$"

Make this precise!

Def A linear space V is a set with "+" and scalar multiplication such that for all $f, g, h \in V$ and scalars c, k:

1) $(f+g)+h = f+(g+h)$
2) $f+g = g+f$
3) There is an element $0 \in V$ such that $f+0 = f$ for all f.
4) For every $f \in V$, there is a $g \in V$ such that $f+g = 0$
   (Say $g = -f$)
5) $k(f+g) = kf + kg$
6) $(k+c)f = kf + cf$
7) $c(kf) = (ck)f$
8) $1f = f$

Eg a) All these rules work for $\mathbb{R}^n$. Take $f=\vec{u}$, $g=\vec{v}$, $h=\vec{w}$ and all these rules are satisfied.

b) Solutions to $f''(x) = -f(x)$:

Addition of functions satisfies 1 & 2

3) $f(x)=0$ is a solution

4) If f is a solution, $-f$ is a solution and

$$f+(-f)=0$$

5-8) Standard rules for scalar multiplication on functions and if f is a solution, so is kf.

c) Functions $\{f:\mathbb{R}\to\mathbb{R}\}$

If f, g, h are functions, then rules apply.

d) $M_{2\times 2} = \{2\times 2 \text{ matrices}\}$

If $f=A$, $g=B$, $h=C$ are 2×2 matrices, these rules apply.

<u>Upshot</u> Use language of linear algebra in many settings!

<u>Def</u> A subset W is a <u>subspace</u> of linear space V if

a) $0 \in W$

b) $f, g \in W \Rightarrow f+g \in W$

c) $f \in W \Rightarrow cf \in W$ for all scalars c.

Eg V = All functions $f:\mathbb{R}\to\mathbb{R}$

W = deg ≤ 2 polynomials ax^2+bx+c

a) $0 \in W$ $(a=b=c=0)$

b) $ax^2+bx+c + px^2+qx+r = (a+p)x^2 + (b+q)x + (c+r) \in W$

c) $k(ax^2+bx+c) = (ka)x^2 + (kb)x + kc \in W$

So, W is a subspace of V.

More examples in §4.1 pp168–169.

<u>Def</u> $f_1, \dots, f_n \in V$ – linear space

a) $\text{span}\{f_1, \dots, f_n\} = \{c_1 f_1 + c_2 f_2 + \cdots + c_n f_n : \text{scalars } c_1, \dots, c_n\}$

b) f_i is <u>redundant</u> if it is a linear combination of $f_1, \dots, f_{i-1}$

and $f_1, \dots, f_n$ are <u>linearly independent</u> if none are redundant.

c) $\{f_1, \dots, f_n\}$ forms a <u>basis</u> of V if $\text{span}\{f_1, \dots, f_n\} = V$

and $f_1, \dots, f_n$ are linearly independent

$\Rightarrow$ Every $g \in V$ has unique $c_1, \dots, c_n$ such that

$$g = c_1 f_1 + \cdots + c_n f_n$$

So $\mathcal{B} = \{f_1, \dots, f_n\} \Rightarrow [g]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$

The map $L_{\mathcal{B}} : V \to \mathbb{R}^n$ given by $L_{\mathcal{B}}(g) = [g]_{\mathcal{B}}$

is the <u>$\mathcal{B}$-coordinate transformation.</u>

<u>Eg</u> a) $W = P_2$ = deg ≤ 2 polynomials

$= \{ax^2 + bx + 1 : a, b, c - \text{scalars}\}$

$= \text{Span}\{x^2, x, 1\}$

and $x^2, x, 1$ are linearly independent so

$\mathcal{B} = \{x^2, x, 1\}$ is a basis,

$$[ax^2 + bx + c]_{\mathcal{B}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and $\dim W = 3$.

b) Linear space $M_{2\times2}$ has a basis

$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis

$\rightsquigarrow \dim M_{2\times2} = 4$ and $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

Aside $\mathcal{B}' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ is also a basis

c) Solutions to $f''(x) = -f(x)$

has $\mathcal{B} = \{\sin(x), \cos(x)\}$ as a basis

$\Rightarrow$ solutions form a 2-dimensional subspace of the linear space of (twice) differentiable functions.

Thm The solutions to

$$f^{(n)}(x) + a_{n-1} f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) = 0$$

form an n-dimensional subspace of n-fold differentiable functions.

Linear spaces can be infinite dimensional!

Eg Consider $V = P = \{\text{all polynomials}\}$

Cannot have a finite basis because $\deg f = m$, $\deg g = n$

then $\deg(f+g) \leq \max(m,n)$.

Def A linear space V is finite-dimensional if it has a finite basis $\{f_1, \dots, f_n\} \Rightarrow \dim(V) = n$

Otherwise, V is infinite-dimensional.

Rmk We do not have a definition for an "infinite basis."

(Different course! "Functional analysis")