HW 5 due 2/18
3.4: 2, 4, 20, 26, 44

HW 6 due 2/25
4.1: 47, 48
4.2: 10, 14, 22, 54

Quiz on 2/23
3.4, 4.1, 4.2

Linear Transformations between linear spaces

Def For V, W linear spaces, a function $T: V \to W$ is a linear transformation
if a) $T(f+g) = T(f) + T(g)$ for all $f, g \in V$
b) $T(kf) = kT(f)$ for all $f \in V$ and scalars k.

Eg $P_2 = \{ax^2 + bx + c : a, b, c \text{ scalars}\}$
Basis $\mathcal{B} = \{x^2, x, 1\}$
$L_{\mathcal{B}}: P_2 \longrightarrow \mathbb{R}^3$ $\qquad L_{\mathcal{B}}(ax^2+bx+c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
is a linear transformation.

Check a) $L_{\mathcal{B}}((ax^2+bx+c) + (px^2+qx+r)) = [(a+p)x^2 + (b+q)x + (c+r)]_{\mathcal{B}}$

$$= \begin{pmatrix} a+p \\ b+q \\ c+r \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$= L_{\mathcal{B}}(ax^2+bx+c) + L_{\mathcal{B}}(px^2+qx+r) \checkmark$$

b) Similar because $\begin{pmatrix} ka \\ kb \\ kc \end{pmatrix} = k \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\Rightarrow L_{\mathcal{B}}: P_2 \to \mathbb{R}^3$ is a linear transformation. $\checkmark$

Fact For any linear space V with $\dim(V) = n$ and basis $\mathcal{B}$, $L_{\mathcal{B}}: V \to \mathbb{R}^n$ is a linear transformation.

Def $T: V \to W$ a linear transformation

$\text{im}(T) = \{T(f) : f \text{ in } V\}$ is a linear subspace of W and

$\ker(T) = \{f \text{ in } V : T(f) = 0\}$ is a linear subspace of V.

If $\dim(\text{im}(T)) < \infty$, then $\text{rank}(T) := \dim(\text{im}(T))$

$\dim(\ker(T)) < \infty$, then $\text{nullity}(T) := \dim(\ker(T))$

Note If $\dim(V) < \infty$, then $\dim(V) = \text{rank}(T) + \text{nullity}(T)$.

Eg. Infinitely differentiable functions $f: \mathbb{R} \to \mathbb{R}$

"Smooth functions" denoted as C^∞

$V = C^\infty$ and consider $D(f) = f'$

Is D a linear transformation?

a) $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$

b) $D(kf) = (kf)' = k \cdot f' = k D(f)$ ✓

$\ker(D)$? $\iff$ What are all $f \in C^\infty$ satisfy $D(f) = 0$?

$\ker(D) =$ all constant functions
$= \text{Span}\{1\}$ so $\dim(\ker(D)) = 1$.

$\text{im}(D)$? $\iff$ What are all $g \in C^\infty$ such that there exists an f satisfying $D(f) = g$?

Fund Thm Calc $\Rightarrow$ All C^∞ functions have an antiderivative,

so $\text{im}(D) = C^\infty$

So, $D: C^\infty \to C^\infty$ is surjective but <u>not</u> injective $\left(\begin{array}{c} D(f+c) \\ = D(f) \\ \text{for all constants } c \end{array} \right)$

Def An invertible linear transformation is called an <u>isomorphism</u> (of linear spaces)

If $T: V \to W$ is an isomorphism, then V and W are <u>isomorphic</u>.

Eg. a) $D: C^\infty \to C^\infty$ is <u>not</u> an isomorphism

b) $\text{Id}: C^\infty \to C^\infty$, $f \mapsto f$ is an isomorphism, so C^∞ is isomorphic to itself
(Any V is isomorphic to itself)

c) If $\mathcal{B} = \{f_1, \ldots, f_n\}$ is a basis for V, then $L_{\mathcal{B}} : V \to \mathbb{R}^n$ is an isomorphism:

$$\overset{V}{\underset{\Downarrow}{}} \quad g = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n \underset{L_{\mathcal{B}}^{-1}}{\overset{L_{\mathcal{B}}}{\rightleftarrows}} [g]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$$

for unique scalars $c_1, \ldots, c_n$

So, V is isomorphic to $\mathbb{R}^n$.

<u>Thm</u> A linear transformation $T: V \to W$ is an isomorphism if and only if $\ker T = \{0\}$ and $\operatorname{im} T = W$
(injective) (surjective)

<u>Eg</u> Is $T: P \to P$ for $T(f(x)) = x f(x)$ an isomorphism?

No! $f(x) = 1$ is not in $\operatorname{im}(T)$, so $\operatorname{im}(T) \neq P$ and so, by our thm, T is not an isomorphism.

<u>Thm</u> For finite dimensional linear spaces V, W

a) $\dim(V) = \dim(W) \iff V$ is isomorphic to W.

$(n = \dim(V))$

$(\Leftarrow)$ by rank-nullity

$$V \xrightarrow{L_{\mathcal{B}}} \mathbb{R}^n \xrightarrow{L_{\mathcal{B}'}^{-1}} W$$

$(\Rightarrow)$

gives an isomorphism $V \to W$

$\mathcal{B}$ a basis for V
$\mathcal{B}'$ a basis for W

b) If $T: V \to W$ is a linear transformation with $\ker T = \{0\}$ and $\dim(V) = \dim(W)$, then T is an isomorphism.

c) If $T: V \to W$ is a linear transformation with $\operatorname{im} T = W$ and $\dim(V) = \dim(W)$, then T is an isomorphism.

__Eg__ $D: P_2 \longrightarrow P_2$ given by $D(f) = f'$

$\mathcal{B} = \{x^2, x, 1\}$

Can you find a matrix B such that $B[f]_{\mathcal{B}} = [f']_{\mathcal{B}}$?

__Picture__

$$\begin{tikzcd} P_2 \arrow[r, "D"] \arrow[d, "L_{\mathcal{B}}"'] & P_2 \arrow[d, "L_{\mathcal{B}}"] \\ \mathbb{R}^3 \arrow[r, dashed, "B"'] & \mathbb{R}^3 \end{tikzcd}$$

$$D(ax^2 + bx + c) = 2ax + b$$

$\rightsquigarrow$ Matrix B such that $B\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix}$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

__Next time__ Make finding B more systematic